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Using the Galerkin–Hermite Method to Solve Volterra Integral Equations

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Acknowledgement

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Thanks

Dedication

To my dear father, a symbol of strength and dignity, and my greatest support in life. To the one who taught me that faith in knowledge never fails — I dedicate the fruit of this effort in gratitude for your sacrifices and unconditional love.

To my beloved mother, the source of kindness, patience, and endless prayers. Your words and warmth were my light through every hardship.

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I dedicate this humble work, from the depths of my heart, to all those who made this journey possible.

Introduction

Volterra integral equations of the second kind appear in various fields such as physics, engineering, biology, and mathematical modeling. Due to their integral structure and dependence on the variable upper limit, these equations are often challenging to solve analytically, which makes the development of reliable numerical methods crucial.

In recent years, a wide range of numerical techniques has been developed to solve Volterra integral equations of the second kind. Among the most prominent methods are: Quadrature methods, which approximate the integral using classical formulas such as the Trapezoidal and Simpson rules. These methods are thoroughly discussed in [1] [2].

Iterative methods, such as the Picard iteration, which construct the solution through successive approximations. These methods are covered in detail in [3].

The collocation method, which is based on enforcing the equation at specific points called collocation points. This approach is presented in [4] as an efficient technique for solving integral equations.

In recent years, spectral methods have gained popularity in the numerical solution of integral and differential equations because of their high accuracy and fast convergence. Among these, the Galerkin method stands out for its versatility and theoretical foundation.

This study focuses on applying the Galerkin method using Hermite polynomials as basis functions to approximate the solutions of linear Volterra integral equations of the second kind. Hermite polynomials are chosen for their properties and computational advantages, especially in approximating smooth functions.

The objective of this work is to derive the Galerkin–Hermite method, implement it on a set of test problems, and evaluate its performance by comparing the obtained results with exact solutions and other numerical techniques.

Below is a summary of each chapter in this thesis:

Chapter 1: Basic Concepts This chapter lays the theoretical groundwork for the thesis. It starts with the definition of contraction mappings and presents the Banach-Picard Fixed-Point Theorem, ensuring existence and uniqueness of solutions for specific equation classes. The chapter then introduces Hermite polynomials, covering their definitions, key properties such as orthogonality and recurrence relations, and their applications in function approximation. It concludes with an overview of the Galerkin method, a fundamental numerical technique for approximating solutions of functional and integral equations.

Chapter 2: Integral Equations This chapter introduces the essential concepts and classification of integral equations, focusing particularly on Volterra integral equations. It differentiates between Fredholm and Volterra equations, as well as between first-kind and second-kind equations. The existence and uniqueness of solutions for Volterra equations are discussed, along with their connection to differential equations. The chapter also surveys various numerical methods for solving Volterra equations, including the Adomian decomposition method and the method of successive approximations.

Chapter 3: Solving Volterra Equations via the Galerkin–Hermite Method This chapter is devoted to implementing the Galerkin–Hermite method to solve Volterra integral equations. It details the general approach, emphasizing the use of Hermite polynomials as basis functions within the Galerkin framework. Several illustrative examples are provided to demonstrate the construction of approximate solutions. Numerical experiments assess the accuracy and efficiency of the method, with results compared against exact solutions and alternative numerical approaches, showcasing the method's strengths and limitations.

Chapter 1

Basic Concepts

In this chapter, we study the Hermite polynomials, focusing on their main properties such as orthogonality, recurrence relations, and explicit formulas. These elements are essential for constructing the Galerkin-Hermite method, which will be used later to solve Volterra integral equations.

1.1 Contraction Mappings

Let (X, d) and (Y, δ) be two metric spaces, and let $k \in [0, 1[$. A function $f : X \to Y$ is said to be a contraction with constant k if:

$$\forall (x, y) \in X \times X, \quad \delta(f(x), f(y)) \leq kd(x, y).$$

Such a function f is also called a k-contraction.

1.2 Banach-Picard Fixed-Point Theorem

Let (E, d) be a complete metric space, and let $f : E \to E$ be a contraction mapping, meaning that there exists $k \in [0, 1]$ such that for all $x, y \in E$,

$$d(f(x), f(y)) \leq k\, d(x, y). \tag{1.1}$$

Then, f has a unique fixed point $\ell \in E$ such that $f(\ell) = \ell$.

Moreover, for any initial point $u_0 \in E$, the sequence (u_n) defined by

$$u_0 \in E,$$

$$u_{n+1} = f(u_n), \quad n \geq 0,$$

converges to ℓ, and the following estimates hold for all $n \in \mathbb{N}$:

$$d(u_n, \ell) \leq k^n d(u_0, \ell), \tag{1.2}$$

and

$$d(u_n, \ell) \leq \frac{k}{1-k} d(u_n, u_{n-1}). \tag{1.3}$$

The fixed-point theorem is a fundamental tool in analysis and is used to prove other important results such as the local inversion theorem and the Cauchy-Lipschitz theorem.

Theorem 1.1 *Let A be an operator on a space E, such that A^n is a contraction operator. Then, the equation:*

$$Au = u$$

admits a unique solution in E.

Proof. Since A^n is a contraction, it has a unique fixed point denoted by u_0. Therefore, we have:

$$\|A(u_0) - u_0\| = \|A^n(A(u_0)) - A^n(u_0)\| \leq k\|A(u_0) - u_0\|,$$

which implies that $A(u_0) = u_0$, since $0 < k < 1$. The uniqueness of the fixed point of A follows from the fact that it is also a fixed point of A^n. □

1.3 Definition of Hermite Polynomials

Hermite polynomials, denoted by $H_n(x)$, are a sequence of orthogonal polynomials that play an important role in mathematical analysis, especially in solving differential and integral equations. Hermite polynomials are defined by the reltion :

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right). \tag{1.4}$$

where $n = 0, 1, 2, ...$

1.4 Some Properties of Hermite Polynomials

[5]

1.4.1 Recursive Definition

Hermite polynomials satisfy the following recurrence relation:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n = 1, 2, \ldots \tag{1.5}$$

with the initial conditions:

$$H_0(x) = 1, \qquad H_1(x) = 2x. \tag{1.6}$$

1.4.2 The First Few Hermite Polynomials

The explicit forms of the first few Hermite polynomials are:

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x \\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120
\end{aligned}$$

1.4.3 Orthogonality Property

Hermite polynomials are orthogonal with respect to the weight function e^{-x^2} on the interval $]-\infty, +\infty[$. Specifically,

$$\int_{-\infty}^{+\infty} H_n(x)H_m(x)e^{-x^2}\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ 2^n n! \sqrt{\pi} & \text{if } n = m. \end{cases} \tag{1.7}$$

This figure illustrates the distinct shapes of Hermite polynomials, which are orthogonal and linearly independent functions commonly used in approximation theory and mathematical analysis.

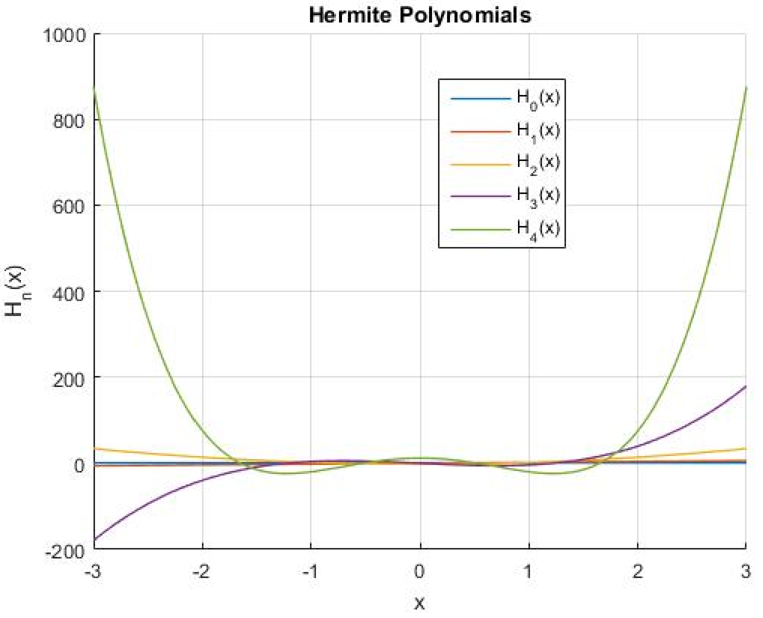

Figure 1.1: Hermite polynomials

1.5 Application of Hermite Polynomials

Example 1.2 (Approximation of $f(x) = e^x$ using Hermite polynomials)

We seek to approximate the function $f(x) = e^x$, where $x \in \mathbb{R}$, using Hermite polynomials as follows:

$$Z_n(x) = \sum_{k=0}^{n} C_k H_k(x)$$

For $n = 2$, the approximation becomes:

$$Z_2(x) = C_0 H_0(x) + C_1 H_1(x) + C_2 H_2(x) \approx e^x$$

To determine the coefficients C_0, C_1, and C_2, we use the orthogonality property of Hermite polynomials. We multiply both sides by $H_j(x)e^{-x^2}$ (for $j = 0, 1, 2$) and integrate over $]-\infty, +\infty[$:

$$\int_{-\infty}^{+\infty} Z_2(x) H_j(x) e^{-x^2} dx = \int_{-\infty}^{+\infty} e^x H_j(x) e^{-x^2} dx$$

By orthogonality, we have:

$$\int_{-\infty}^{+\infty} H_i(x) H_j(x) e^{-x^2} dx = \begin{cases} 0 & \text{if } i \neq j \\ 2^j j! \sqrt{\pi} & \text{if } i = j \end{cases}$$

Therefore, for each j:

$$C_j \cdot 2^j j! \sqrt{\pi} = \int_{-\infty}^{+\infty} e^x H_j(x) e^{-x^2} dx$$

$$C_j = \frac{1}{2^j j! \sqrt{\pi}} \int_{-\infty}^{+\infty} e^x H_j(x) e^{-x^2} dx$$

Now, we compute the coefficients:

For** C_0**:

$$C_0 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^x e^{-x^2} dx$$

Let

$$I_0 = \int_{-\infty}^{+\infty} e^{x-x^2} dx = \int_{-\infty}^{+\infty} e^{-(x-\frac{1}{2})^2+\frac{1}{4}} dx = e^{1/4} \int_{-\infty}^{+\infty} e^{-(x-\frac{1}{2})^2} dx = e^{1/4}\sqrt{\pi}$$

So,

$$C_0 = \frac{e^{1/4}\sqrt{\pi}}{\sqrt{\pi}} = e^{1/4}$$

For** C_1**:

$$C_1 = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} e^x H_1(x) e^{-x^2} dx = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} 2x e^x e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x e^{x-x^2} dx$$

Let $I_1 = \int_{-\infty}^{+\infty} x e^{x-x^2} dx$. Set $u = x - \frac{1}{2}$, so $x = u + \frac{1}{2}$, $dx = du$:

$$I_1 = \int_{-\infty}^{+\infty} (u + \frac{1}{2}) e^{-u^2+\frac{1}{4}} du = e^{1/4} \left(\int_{-\infty}^{+\infty} u e^{-u^2} du + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-u^2} du \right)$$

The first integral is zero (odd function), the second is $\sqrt{\pi}$:

$$I_1 = e^{1/4} \cdot \frac{1}{2}\sqrt{\pi}$$

Thus,

$$C_1 = \frac{1}{\sqrt{\pi}} \cdot e^{1/4} \cdot \frac{1}{2}\sqrt{\pi} = \frac{e^{1/4}}{2}$$

For** C_2**:

$$C_2 = \frac{1}{8\sqrt{\pi}} \int_{-\infty}^{+\infty} e^x H_2(x) e^{-x^2} dx = \frac{1}{8\sqrt{\pi}} \int_{-\infty}^{+\infty} (4x^2 - 2) e^x e^{-x^2} dx$$

$$= \frac{1}{8\sqrt{\pi}} \left(4 \int_{-\infty}^{+\infty} x^2 e^{x-x^2} dx - 2 \int_{-\infty}^{+\infty} e^{x-x^2} dx \right)$$

Let $I_2 = \int_{-\infty}^{+\infty} x^2 e^{x-x^2} dx$. *Using the same substitution as before:*

$$x^2 = (u + \frac{1}{2})^2 = u^2 + u + \frac{1}{4}$$

So,

$$I_2 = e^{1/4} \left(\int_{-\infty}^{+\infty} u^2 e^{-u^2} du + \int_{-\infty}^{+\infty} u e^{-u^2} du + \frac{1}{4} \int_{-\infty}^{+\infty} e^{-u^2} du \right)$$

The second term is zero, the first is $\frac{1}{2}\sqrt{\pi}$, *the third is* $\frac{1}{4}\sqrt{\pi}$*:*

$$I_2 = e^{1/4} \left(\frac{1}{2}\sqrt{\pi} + 0 + \frac{1}{4}\sqrt{\pi} \right) = e^{1/4} \cdot \frac{3}{4}\sqrt{\pi}$$

Now,

$$C_2 = \frac{1}{8\sqrt{\pi}} \left(4 \cdot e^{1/4} \cdot \frac{3}{4}\sqrt{\pi} - 2 \cdot e^{1/4}\sqrt{\pi} \right) = \frac{1}{8\sqrt{\pi}} \left(3e^{1/4}\sqrt{\pi} - 2e^{1/4}\sqrt{\pi} \right) = \frac{1}{8\sqrt{\pi}} \left(e^{1/4}\sqrt{\pi} \right) = \frac{e^{1/4}}{8}$$

Final approximation:

$$e^x \approx e^{1/4} H_0(x) + \frac{e^{1/4}}{2} H_1(x) + \frac{e^{1/4}}{8} H_2(x)$$

1.6 Galerkin Method Definition

The Galerkin method is a numerical technique for approximating the solution of functional equations.For example, let the funtional equation

$$Au = f$$

Where: A is an invertible operator, f $\in C(\omega)$, u is the function to be derived.

Let X be a Hilbert space (for example, $L^2(\Omega)$), and let X_n be a finite-dimensional subspace of X with basis $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$. We seek an approximate solution $u_n(x)$ of the form:

$$u_n(x) = \sum_{j=1}^{N} \alpha_j \varphi_j(x) \qquad (1.8)$$

The Galerkin method requires that the residual $R_n(x)$ is orthogonal to the subspace X_n, that is,

$$\langle R_n, \varphi_i \rangle = 0, \quad \text{for all } i = 1, 2, \ldots, N \qquad (1.9)$$

where $R_n(x)$ is the residual obtained by substituting $u_n(x)$ into the equation.

Chapter 2

Introduction To Volterra Integral Equations

This chapter introduces the basic concepts, classifications, and fundamental properties of integral equations, with a particular focus on Volterra integral equations. Moreover, it discusses both the exact and approximate solutions commonly used in the numerical treatment of such equations.

2.1 Definition of Integral Equations

[6]

An integral equation is an equation in which the unknown function $u(x)$ appears under an integral sign. A general form of an integral equation is given by:

$$h(x)u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)\,dt, \tag{2.1}$$

where $g(x)$ and $h(x)$ define the limits of integration, λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t, referred to as the **kernel** of the integral equation.

The unknown function $u(x)$ appears both inside and outside the integral. The functions $f(x)$ and $K(x,t)$ are given in advance. It is important to note that the limits of integration, $g(x)$ and $h(x)$, may be variables, constants, or a combination of both.

2.2 Classification of Integral Equations

Integral equations are divided into several types, mainly depending on the limits of integration and the kernel of the equation. In this text, we focus on the following main types of integral equations:

2.2.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Then, the unknown function $u(x)$ may appear only inside the integral equation by the form:

$$f(x) = \int_a^b K(x,t)u(t)\,dt. \tag{2.2}$$

This is called Fredholm integral equation of the first kind. But, for Fredholm integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral . The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)\,dt \tag{2.3}$$

2.2.2 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is variable. For the first kind of Volterra integral equation, the unknown function $u(x)$ appears only inside the integral in the form:

$$f(x) = \int_0^x K(x,t)u(t)\,dt. \tag{2.4}$$

However, in Volterra integral equations of the second kind, the unknown function $u(x)$ appears both inside and outside the integral. The second kind is given by the form:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt. \tag{2.5}$$

2.2.3 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely:

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \tag{2.6}$$

and

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau, \quad (x,t) \in \Omega \times [0,T], \tag{2.7}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\mathbb{R}^n$, $n = 1,2,3$.

It is interesting to note that Equation 2.6 contains separate Volterra and Fredholm integral equations, whereas Equation 2.7 contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral, the resulting equations are of the first kind, but will not be examined in this text.

2.2.4 Singular Integral Equations

Volterra integral equations of the first kind is given by:

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)\,dt. \tag{2.8}$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)u(t)\,dt. \tag{2.9}$$

are called singular if one of the limits of integration g(x), h(x) or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration. In this text we will focus our attention on equations of the form:

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)\, dt, \quad 0 < \alpha < 1.$$

or of the second kind:

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t)\, dt, \quad 0 < \alpha < 1.$$

The last two standard forms are called generalized Abel's integral equation and weakly singular integral equations respectively. For $\alpha = \frac{1}{2}$, the equation:

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}}\, dt$$

is called Abel's singular integral equation. It is to be noted that the kernel in each equation becomes infinite at the upper limit $t = x$.

2.3 Existence and Uniqueness of solution of Volterra Integral Equation

[7]

Theorem 2.1 *Let K(x,t) be a continuous function for x,t∈[a,b], such that*

$$|K(x,t)| \leq M \quad , \quad M > 0$$

then the Volterra equation:

$$\varphi(x) - \lambda \int_a^x K(x,t)\varphi(t)\, dt = f(x) \tag{2.10}$$

It admits a unique solution $\varphi(x)$ for every $f(x)$ in $L_2([a,b])$ and every $\lambda \in \mathbb{R}$.

Proof.

For the Volterra integral equation, we consider the operator:

$$T\varphi = f(x) + \lambda A\varphi = f(x) + \lambda \int_a^x K(x,t)\varphi(t)\, dt$$

with

$$A\varphi = \int_a^x K(x,t)\varphi(t)\,dt$$

and we show that the operator T^n is a contraction for $n \in \mathbb{N}$, so T has a fixed point.

$T\varphi = f + \lambda A\varphi$

$T^2\varphi = T(T\varphi) = T(f + \lambda A\varphi) = f + \lambda A(f + \lambda A\varphi) = f + \lambda Af + \lambda^2 A^2\varphi$

.

.

.

$T^n\varphi = f + \lambda Af + \lambda^2 A^2 f + \cdots + \lambda^{n-1}A^{n-1}f + \lambda^n A^n\varphi$

so that :

$$\|T^n(\varphi_1) - T^n(\varphi_2)\| = \|\lambda^n A^n\varphi_1 - \lambda^n A^n\varphi_2\| = |\lambda|^n\|A^n\varphi_1 - A^n\varphi_2\|$$

$$\|T^n(\varphi_1) - T^n(\varphi_2)\| = |\lambda|^n\|\int_a^x K_n(x,t)(\varphi_1(t) - \varphi_2(t)dt)\|.$$

To determine $K_n(x,t)$, we use the relation:

$$\begin{aligned}
A\varphi &= \int_a^x K(x,t)\varphi(t)dt \\
A^2\varphi &= \int_a^x K(x,t)\int_a^t K(t,z)\varphi(z)\,dzdt \\
&= \int_a^x \varphi(z)dz\int_z^x K(x,t)K(t,z)\,dt \\
&= \int_a^x K_2(x,z)\varphi(z)dz,
\end{aligned}$$

Thus:

$$K_2(x,t) = \int_t^x K(x,z)K(z,t)\,dz$$

by reccurence we obtain:

$K_1(x,t) = K(x,t),$

$K_2(x,t) = \int_t^x K(x,z)K(z,t)\,dz,$

$K_3(x,t) = \int_t^x K_2(x,z)K(z,t)\,dz,$

.

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$K_n(x,t) = \int_t^x K_{n-1}(x,z)K(z,t)\,dz$,

since by the hypothesis we have $|K(x,t)| \leq M$, then

$$|K_n(x,t)| \leq \frac{M^n(x-t)^{n-1}}{(n-1)!},\ a \leq t \leq x \leq b \tag{2.11}$$

for n=1 the expression 2.11 is true. we assume that it is true for $m \in \mathbb{N}$,

$$|K_m(x,t)| \leq \frac{M^m(x-t)^{m-1}}{(m-1)!}$$

so :

$$\begin{aligned} |K_{m+1}(x,t)| &= |\int_t^x K_m(x,z)K(z,t),dz| \\ &\leq \int_t^x |K_m(x,z)K(z,t)|,dz \\ &\leq \frac{M^{m+1}}{m-1}\int_t^x (x-z)^{m-1},dz \\ &\leq \frac{M^{m+1}}{m}(x-z)^m \end{aligned}$$

thus

$$\|T^n(\varphi_1) - T^n(\varphi_2)\| \leq \frac{|\lambda|^n M^n}{(n-1)!}\|\varphi_1 - \varphi_1\|$$

for $n \in \mathbb{N}$ large enough ,we obtain :

$$\frac{|\lambda|^n M^n}{(n-1)!} < 1$$

Since T^n is a contraction, T admits a unique fixed point, according to **Theorem(1.3)**

$$T\varphi = \varphi \leftrightarrow \varphi(x) = f(x) + \lambda \int_a^x K(x,t)\varphi(t)dt.$$

this fixed point is the solution of the equation (2.10).

□

2.4 Conversion Between Differential and Integral Equations

[8]

2.4.1 From a differential equation to an integral equation:

Let us consider the following initial value problem:

$$\begin{cases} u''(x) + a(x)u'(x) + b(x)u(x) = f(x) \\ u(0) = \alpha, \quad u(1) = \beta \end{cases}$$

We aim to transform this differential equation into an equivalent integral equation:

suppose that:

$$u'' = \phi(x)$$

So:

$$\begin{aligned} u'(x) - u'(0) &= \int_0^x u''(x)dx \\ &= \int_0^x \phi(x)dx \\ &= \int_0^x \phi(x)dx \end{aligned}$$

$$\begin{aligned} u'(x) &= \int_0^x \phi(t)dt + \beta \\ \int_0^x u'(x)dx &= \int_0^x \int_0^x \phi(t)dt + \int_0^x \beta dt \\ u(x) - u(0) &= \int_0^x (x-t)\phi(t)dt + \beta(x) \\ u(x) &= \int_0^x (x-t)\phi(t)dt + \beta(x) + \alpha \end{aligned}$$

substitute $u''(x), u'(x)$ and $u(x)$ with their values :

$$\phi(x) + a(x)[\int_0^x \phi(t)dt + \beta] + b(x)[\int_0^x (x-t)\phi(t)dt + \beta(x) + \alpha] = f(x)$$

$$\phi(x) + \int_0^x a(x) + b(x)(x-t)\phi(t)dt = f(x) - \beta a(x) - \beta(x)b(x) - \alpha b(x)$$

2.4.2 From an integral equation to a differential equation:

Let us now consider the reverse process, namely, obtaining a differential equation from an integral equation, suppose we are given an integral equation of the form:

$$\phi(x)+\int_a^x K(x,t)\phi(t)dt=f(x)$$
$$\phi(x)+\int_a^x A(x)B(t)\phi(t)dt=f(x)$$
$$\phi(x)+A(x)\int_a^x B(t)\phi(t)dt=f(x)$$

Taking:

$$\begin{aligned} u(x) &= \int_a^x B(t)\phi(t)dt \\ \phi(x) &= f(x)+A(x)u(x) \\ u'(x) &= B(x)\phi(x) \end{aligned}$$

So :

$$u'(x)=B(x)[f(x)+A(x)u(x)]$$

and thermore:

$$\begin{cases} u'(x)-B(x)A(x)u(x)=B(x)f(x) \\ u(a)=0 \end{cases}$$

2.5 Solution Methods for Volterra Integal Equations

[6]

2.5.1 The Adomian Decomposition Method

The Adomian Decomposition Method involves expressing the unknown function $u(x)$ as an infinite series composed of components, each determined through a recursive decomposition process.

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \tag{2.12}$$

Or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \tag{2.13}$$

where the components $u_n(x)$, for $n \geq 0$, are determined recursive manner. The decomposition method concerns itself with finding the components We consider the components $u_0, u_1, u_2, \ldots$ individually. As will be seen through the text, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To establish the recurrence relation, we substitute 2.10 into the Volterra integral equation 2.5 to obtain:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt \tag{2.14}$$

or equivalently,

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_0^x K(x,t) \left[u_0(t) + u_1(t) + \cdots \right] dt. \tag{2.15}$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x),\ j \geq 1$, of the unknown function $u(x)$ are completely determined by setting the recurrence relation:

$$u_0(x) = f(x),$$

$$u_{n+1}(x) = \lambda \int_0^x K(x,t) u_n(t)\, dt, \quad n \geq 0. \tag{2.16}$$

This is equivalent to:

$$\begin{aligned}
u_0(x) &= f(x), \\
u_1(x) &= \lambda \int_0^x K(x,t) u_0(t)\, dt, \\
u_2(x) &= \lambda \int_0^x K(x,t) u_1(t)\, dt, \\
u_3(x) &= \lambda \int_0^x K(x,t) u_2(t)\, dt,
\end{aligned}$$

and so on for other compenents.

In view of2.15 this recurrence relation, the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ are completely determined. As a result, the solution $u(x)$ of the Volterra integral equation 2.5 in a series form is readily obtained by applying the series assumption presented in 2.10.

It is clearly seen that the decomposition method transforms the integral equation into an elegant process of determining computable components. It has been formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converges very rapidly to that solution. The convergence of the decomposition series has been thoroughly studied to confirm its rapid convergence.

However, for concrete problems where a closed-form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we include, the higher the accuracy we can achieve.

Example 2.2 *Solve the following Volterra integral equation:*

$$u(x) = 1 - \int_0^x u(t)\,dt. \tag{2.17}$$

We note that $f(x) = 1$, $\lambda = -1$, and the kernel is $K(x,t) = 1$. Recall that the solution $u(x)$ is assumed to have a series form as given in 2.12. Substituting the decomposition series 2.12 into both sides of equation 2.17 gives:

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt. \tag{2.18}$$

Or equivalently:

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 - \int_0^x [u_0(t) + u_1(t) + u_2(t) + \cdots]\,dt. \tag{2.19}$$

We identify the zeroth component by all terms that are not under the integral sign. Therefore, we obtain the following recurrence relation:

$$u_0(x) = 1, \qquad u_{k+1}(x) = -\int_0^x u_k(t)\,dt, \quad k \geq 0. \tag{2.20}$$

Thus, we get:

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= -\int_0^x u_0(t)\,dt = -\int_0^x 1\,dt = -x, \\ u_2(x) &= -\int_0^x u_1(t)\,dt = -\int_0^x (-t)\,dt = \frac{1}{2!}x^2, \\ u_3(x) &= -\int_0^x u_2(t)\,dt = -\int_0^x \frac{1}{2!}t^2\,dt = -\frac{1}{3!}x^3, \\ u_4(x) &= -\int_0^x u_3(t)\,dt = -\int_0^x \left(-\frac{1}{3!}t^3\right)\,dt = \frac{1}{4!}x^4, \end{aligned} \tag{2.21}$$

and so on. Using2.12 gives the series solution:

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots, \tag{2.22}$$

which converges to the closed-form solution:

$$u(x) = e^{-x}.$$

2.5.2 Successive Approximations Method

The successive approximations method, also known as the Picard iteration method, provides a scheme that can be used to solve initial value problems or integral equations. This method finds successive approximations to the solution by starting with an initial guess, called the zeroth approximation.

As will be seen, the zeroth approximation is any chosen real-valued function that is used in a recurrence relation to determine the subsequent approximations.

Consider the linear Volterra integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)\,dt \tag{2.23}$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel, and λ is a parameter.

The successive approximations method introduces the recurrence relation:

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)\,dt, \quad n \geq 1 \tag{2.24}$$

The zeroth approximation $u_0(x)$ can be any selected real-valued function. We always start with an initial guess for $u_0(x)$; most commonly, we select 0 , 1 ,x , for $u_0(x)$.

and by using the recurrence relation from equation 2.23, several successive approximations $u_k(x),\ k \geq 1$, can be determined as follows:

$$\begin{aligned}
u_1(x) &= f(x) + \lambda \int_0^x K(x,t)u_0(t)\,dt \\
u_2(x) &= f(x) + \lambda \int_0^x K(x,t)u_1(t)\,dt \\
u_3(x) &= f(x) + \lambda \int_0^x K(x,t)u_2(t)\,dt \\
&\vdots \\
u_n(x) &= f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)\,dt
\end{aligned} \tag{2.25}$$

The question of convergence of the sequence $u_n(x)$ is justified by the following theorem.

Theorem 2.3 *If the function $f(x)$ in equation 2.16 is continuous on the interval $0 \leq x \leq a$, and the kernel $K(x,t)$ is also continuous in the triangle $0 \leq t \leq x \leq a$, then the sequence of successive approximations $u_n(x),\ n \geq 0$ converges to the exact solution $u(x)$ of the Volterra integral equation under consideration.*

Example 2.4 *Solve the following Volterra integral equation using the method of successive approximations:*

$$u(x) = 1 - \int_0^x (x-t)u(t)\,dt. \tag{2.26}$$

For the zeroth approximation, we choose:

$$u_0(x) = 1. \tag{2.27}$$

The successive approximation method admits the following iteration formula:

$$u_{n+1}(x) = 1 - \int_0^x (x-t)u_n(t)\,dt, \quad n \geq 0. \tag{2.28}$$

Substituting $u_0(x)$ into the iteration formula gives:

$$u_1(x) = 1 - \int_0^x (x-t)(1)\,dt = 1 - \frac{1}{2!}x^2,$$

$$u_2(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4,$$
$$u_3(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6,$$
$$u_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8,$$

Consequently, we obtain:

$$u_{n+1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

the solution of equation 2.26 :

$$u(x) = \lim_{n \to \infty} u_{n+1}(x) = \cos(x).$$

Chapter 3

Solving Volterra Equations via Galerkin–Hermite Method

In this chapter, we present the Galerkin–Hermite method as an efficient numerical technique for solving Volterra integral equations, particularly those of the second kind. The main idea is to approximate the unknown solution using a finite sum of Hermite polynomials.

3.1 General Approach Using Galerkin-Hermite Method

Consider the general Volterra integral equation of the second kind:

$$u(x) = f(x) + \int_a^x K(x,t)u(t)\,dt$$

where $f(x)$ and $K(x,t)$ are known functions and λ is a given constant.

The Galerkin–Hermite method proceeds as follows:

1. **Approximation:** Assume the approximate solution $u_n(x)$ is a finite sum of Hermite polynomials:

   $$u_n(x) = \sum_{i=0}^{n} C_i H_i(x), \tag{3.1}$$

   where $H_i(x)$ denotes the i-th Hermite polynomial and C_i are unknown coefficients to be determined.

2. **Substitution:** Substitute $u_n(x)$.

   $$\sum_{i=0}^{n} C_i H_i(x) = f(x) + \int_a^x K(x,t) \left(\sum_{i=0}^{n} C_i H_i(t) \right) dt$$

3. **Galerkin Projection:** We apply the Galerkin method using Hermite polynomials as test functions. Multiply both sides by $H_j(x)$ and integrate over $[a, b]$:

$$\int_a^b \left(\sum_{i=0}^n C_i H_i(x) - f(x) - \int_a^x K(x,t) \sum_{i=0}^n C_i H_i(t)\, dt \right) H_j(x)\, dx = 0, \quad j = 0, 1, \ldots, n$$

Interchanging the order of summation and integration:

$$\sum_{i=0}^n C_i \left(\int_a^b H_i(x) H_j(x)\, dx - \int_a^b H_j(x) \left(\int_a^x K(x,t) H_i(t)\, dt \right) dx \right) = \int_a^b f(x) H_j(x)\, dx$$

Define:

$$A_{ji} = \int_a^b H_i(x) H_j(x)\, dx - \int_a^b H_j(x) \left(\int_a^x K(x,t) H_i(t)\, dt \right) dx$$

$$F_j = \int_a^b f(x) H_j(x)\, dx$$

4. **Linear System:** We obtain the linear system:

$$\sum_{i=0}^n A_{ji} C_i = F_j, \quad j = 0, 1, \ldots, n$$

or in matrix form:

$$A \cdot C = F$$

where:

- A is a matrix of size $(n+1) \times (n+1)$:

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0n} \\ A_{10} & A_{11} & A_{12} & \cdots & A_{1n} \\ A_{20} & A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n0} & A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

- $C = [C_0, C_1, \ldots, C_n]^T$ is the vector of unknown coefficients
- $F = [F_0, F_1, \ldots, F_n]^T$ is the right-hand side vector

Once this system is solved for C_i, the approximate solution is:

$$u_n(x) = \sum_{i=0}^n C_i H_i(x)$$

3.2 Some Illustrative Examples

To demonstrate the effectiveness of the Galerkin–Hermite method, let us consider the following example:

Example 3.1 *Consider the following Volterra integral equation of the second kind, for* $n = 2$*:*

$$u(x) = e^x + \int_0^x u(t)\,dt. \qquad 0 \le x \le 1$$

where the exact solution is given by:

$$u_{exact}(x) = (1 + x)e^x.$$

We aim to solve this equation using the Galerkin–Hermite method with $n = 2$*, by approximating the solution as:*

$$u_2(x) = \sum_{i=0}^{2} C_i H_i(x).$$

Substituting this approximation into the original equation leads to:

$$\sum_{i=0}^{2} C_i H_i(x) - \int_0^x \left(\sum_{i=0}^{2} C_i H_i(t) \right) dt = e^x. \tag{3.2}$$

We multiply both sides by each Hermite polynomial $H_i(x)$ for $i = 0, 1, 2$, and integrate over $[0, 1]$. This yields three equations:

For $i = 0$**:**

$$\int_0^1 [C_0 - \int_0^x C_0 dt]dx + \int_0^1 [2xC_1 - \int_0^x 2tC_1 dt]dx + \int_0^1 [(4x^2 - 2)C_2 - \int_0^x (4t^2 - 2)C_2 dt]dx = \int_0^1 e^x dx$$

$$\Leftrightarrow$$

$$\int_0^1 [C_0 - C_0 x]dx + \int_0^1 [2xC_1 - x^2 C_1]dx + \int_0^1 [(4x^2 - 2)C_2 - (\frac{4}{3}x^3 - 2x)C_2]dx = e^1 - 1$$

$$\frac{1}{2}C_0 + \frac{2}{3}C_1 = 1.718$$

For $i = 1$**:**

$$\int_0^1 [C_0 - \int_0^x C_0 dt] 2x dx + \int_0^1 [2xC_1 - \int_0^x 2tC_1 dt] 2x dx + \int_0^1 [(4x^2-2)C_2 - \int_0^x (4t^2-2)C_2 dt] 2x dx = \int_0^1 2xe^x d$$

$\Leftrightarrow$

$$\int_0^1 [C_0 - C_0 x] 2x dx + \int_0^1 [2xC_1 - x^2 C_1] 2x dx + \int_0^1 [(4x^2 - 2)C_2 - (\frac{4}{3}x^3 - 2x)C_2] 2x dx = 2$$

$$\frac{1}{3}C_0 + \frac{5}{6}C_1 + \frac{20}{12}C_2 = 2$$

For $i = 2$**:**

$$\int_0^1 [C_0 - \int_0^x C_0 dt](4x^2-2) dx + \int_0^1 [2xC_1 - \int_0^x 2tC_1 dt](4x^2-2) dx + \int_0^1 [(4x^2-2)C_2 - \int_0^x (4t^2-2)C_2 dt](4x^2-$$

$\Leftrightarrow$

$$\int_0^1 [C_0 - C_0 x](4x^2-2) dx + \int_0^1 [2xC_1 - x^2 C_1](4x^2-2) dx + \int_0^1 [(4x^2-2)C_2 - (\frac{4}{3}x^3 - 2x)C_2](4x^2-2) dx = -0.56$$

$$-\frac{2}{3}C_0 - \frac{2}{15}C_1 + \frac{74}{45}C_2 = -0.563$$

Thus, we obtain the following matrix system:

$$AC = F$$

Where:

$$A = \begin{bmatrix} 0.5 & 0.66 & 0 \\ 0.33 & 0.83 & 0.8 \\ 0.66 & -0.13 & 1.64 \end{bmatrix}$$

$$C = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$$

$$F = \begin{bmatrix} 1.718 \\ 2 \\ -0.563 \end{bmatrix}$$

$$C = \begin{bmatrix} 2.5597 \\ 0.6576 \\ 0.7484 \end{bmatrix}$$

So the matrix form given by:

$$\begin{bmatrix} 0.5 & 0.66 & 0 \\ 0.33 & 0.83 & 0.8 \\ 0.66 & -0.13 & 1.64 \end{bmatrix} \times \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1.718 \\ 2 \\ -0.563 \end{bmatrix}$$

So the approximate solution is given by :

$$\begin{aligned} u_2(x) &= \sum_{n=0}^{2} C_i H_i(x) \\ &= 2.5597 + 1.3152x + 0.7484(4x^2 - 2) \\ u_2(x) &= 1.0629 + 0.2523x + 1.6784x^2 \end{aligned}$$

3.3 Numerical examples

In this section we'll test the proposed method for several examples, we recall the error defined as follows $e = |u_{ext} - u_{app}|$ The table below presents a comparison between the exact and approximate solutions,for example (3.1), along with the corresponding absolute errors, for $n = 4$ and $n = 6$.Results are shown in table(3.1)and figure (3.1).

Xi	**Exact Solution**	**Error (n=4)**	**Error (n=6)**
0.0	1.0000×10^{0}	5.4670×10^{-3}	1.8784×10^{-5}
0.1	1.2157×10^{0}	1.1050×10^{-3}	7.4622×10^{-6}
0.2	1.4657×10^{0}	2.2851×10^{-3}	2.8490×10^{-6}
0.3	1.7548×10^{0}	8.4832×10^{-4}	5.6893×10^{-6}
0.4	2.0886×10^{0}	1.0671×10^{-3}	1.0479×10^{-6}
0.5	2.4731×10^{0}	2.0398×10^{-3}	5.8503×10^{-6}
0.6	2.9154×10^{0}	1.4559×10^{-3}	1.9681×10^{-6}
0.7	3.4234×10^{0}	3.8944×10^{-4}	5.2794×10^{-6}
0.8	4.0060×10^{0}	2.1786×10^{-3}	3.6018×10^{-6}
0.9	4.6732×10^{0}	1.4434×10^{-3}	7.2728×10^{-6}
1.0	5.4366×10^{0}	5.5779×10^{-3}	1.8947×10^{-5}

Table 3.1: Comparison of the exact solution with absolute errors for $n = 4$ and $n = 6$

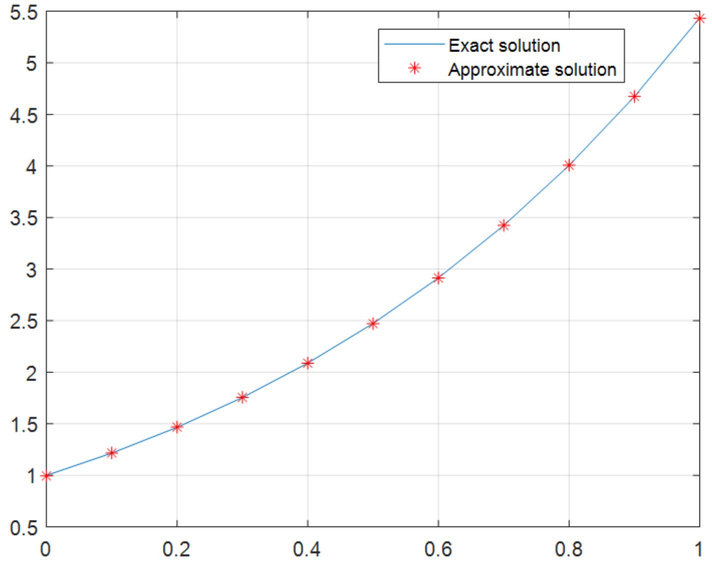

Figure 3.1: comparison between Exact and Approximate solution

The following graph presents the error between the approximate solution and the exact solution

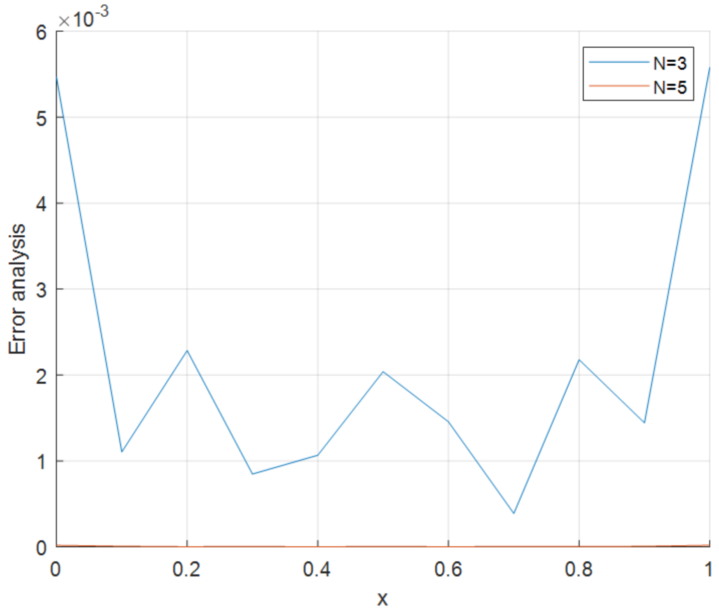

Figure 3.2: The error between the numerical and exact solution

Example 3.2 *Consider the following Volterra integral equation of the second kind:*

$$u(x) = \frac{x^4}{12} - x^4(\frac{x^4 + 1680}{3360}) + 2\int_0^x (x-t)^3\, U(t)\, dt,$$

Assume that the exact solution is known and given by:

$$u_{exact}(x) = 2 + \frac{x^4}{12}.$$

We apply the Galerkin–Hermite method to approximate the solution $u(x)$, using two different values of n, namely $n = 4$ and $n = 6$.

In this comparison, we focus on the exact solution and the corresponding absolute errors for each value of n. The approximate solution is omitted from the table for clarity.

The results are presented in the table (3.2) and figure (3.3):

Xi	Exact Solution	Error (n=4)	Error (n=6)
0.0	2.000000×10^0	1.1901×10^{-3}	5.3428×10^{-12}
0.1	2.000008×10^0	2.7730×10^{-4}	6.6724×10^{-12}
0.2	2.000133×10^0	4.8528×10^{-4}	8.6704×10^{-12}
0.3	2.000675×10^0	1.3383×10^{-4}	1.1533×10^{-11}
0.4	2.002133×10^0	2.7704×10^{-4}	1.5432×10^{-11}
0.5	2.005208×10^0	4.4733×10^{-4}	2.0511×10^{-11}
0.6	2.010800×10^0	2.7704×10^{-4}	2.6888×10^{-11}
0.7	2.020008×10^0	1.3382×10^{-4}	3.4655×10^{-11}
0.8	2.034133×10^0	4.8525×10^{-4}	4.3876×10^{-11}
0.9	2.054675×10^0	2.7726×10^{-4}	5.4591×10^{-11}
1.0	2.083333×10^0	1.1902×10^{-3}	6.6810×10^{-11}

Table 3.2: Comparison of the exact solution and corresponding errors for $n = 4$ and $n = 6$

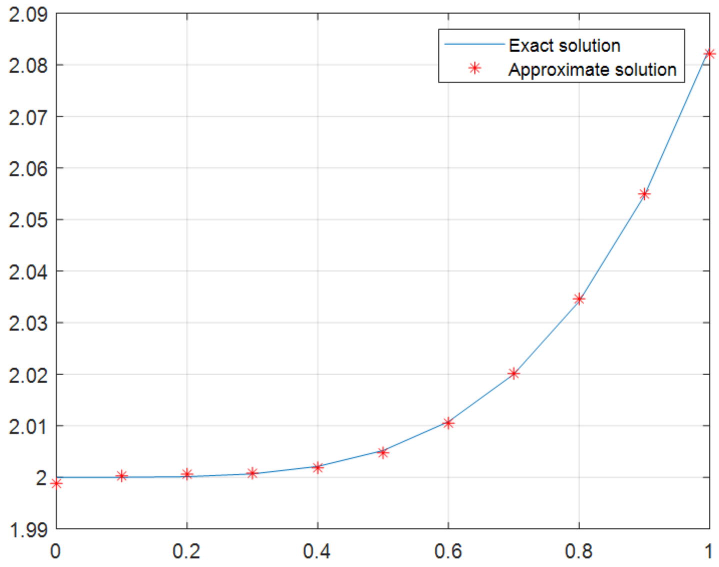

Figure 3.3: comparison between Approximate and Exact solution

This graph highlights the error between the numerical and axact solution

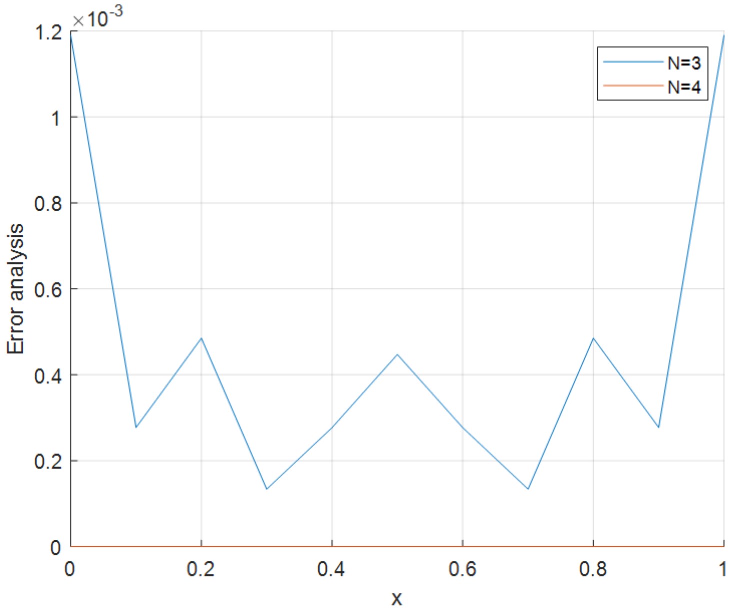

Figure 3.4: The error between the approximate and exact solution

Example 3.3 *Let us consider the following Volterra integral equation of the second kind:*

$$u(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x-t)\,u(t)\,dt,$$

where that the exact solution is given by:

$$u_{exact}(x) = 1 - sinh(x).$$

We apply the Galerkin-Hermite method to approximate the solution. To assess the accuracy of our method, we computed the error for $n = 4$ *and* $n = 6$*, while the error for the TMD method was computed at* $n = 8$*. The TMD results were taken from the article (Hosoya polynomial method for the numerical solution of Volterra integral equations) [9] . Our results at* $n = 4$ *were much better, indicating that we achieved accurate results even with smaller values of* n*. This suggests that increasing* n *will further improve the accuracy. A detailed comparison is presented in the following table.*

Xi	Exact Solution	Error (n=4)	Error (n=6)	Error(TDM)
0.0	1	2.7793×10^{-4}	7.2697×10^{-7}	0
0.1	8.9983×10^{-2}	5.7777×10^{-5}	2.9364×10^{-7}	3.40×10^{-4}
0.2	7.9866×10^{-1}	1.2203×10^{-4}	1.1459×10^{-7}	6.84×10^{-4}
0.3	6.9548×10^{-1}	4.5662×10^{-5}	2.3163×10^{-7}	1.04×10^{-3}
0.4	5.8925×10^{-1}	6.0673×10^{-5}	4.4350×10^{-8}	1.41×10^{-3}
0.5	4.7890×10^{-1}	1.1681×10^{-4}	2.4644×10^{-7}	1.80×10^{-3}
0.6	3.6335×10^{-1}	8.3740×10^{-5}	8.3187×10^{-8}	2.22×10^{-3}
0.7	2.4142×10^{-1}	2.5363×10^{-5}	2.2901×10^{-7}	2.67×10^{-3}
0.8	1.1189×10^{-1}	1.3354×10^{-4}	1.5586×10^{-7}	3.16×10^{-3}
0.9	-2.6517×10^{-2}	8.7867×10^{-5}	3.2654×10^{-7}	3.70×10^{-3}
1.0	-1.7520×10^{-1}	3.5357×10^{-4}	8.5695×10^{-7}	4.29×10^{-3}

Table 3.3: Comparison of the exact solution and errors for $n = 4$ and $n = 6$

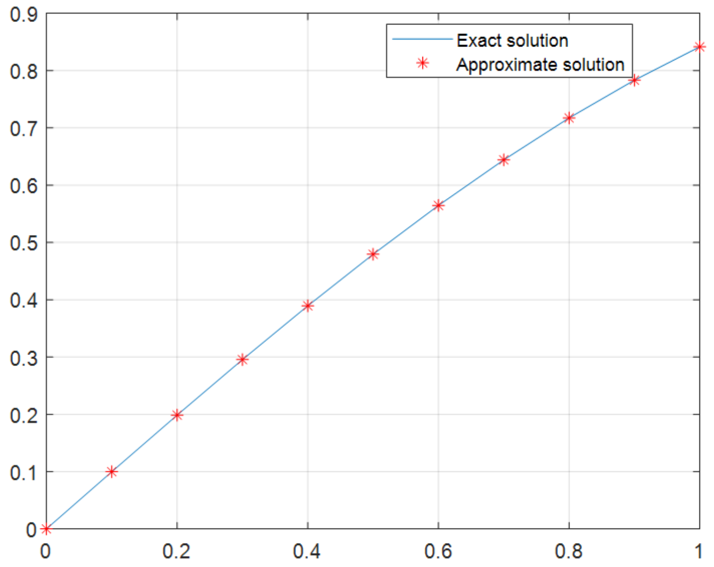

Figure 3.5: comprision between Exact and Approximate solution

Figure (3.6) shows the error between the approximate and exact solutions

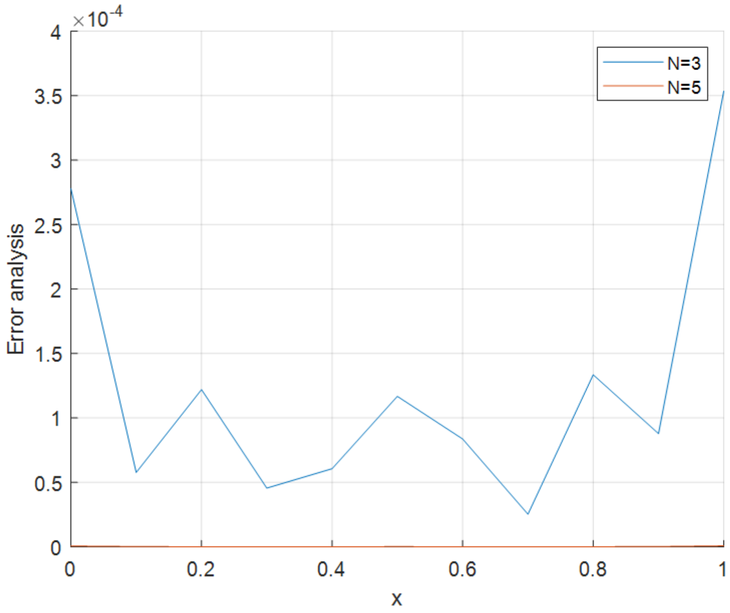

Figure 3.6: The error between numerical and exact solution

Conclusion

In this work, we have successfully studied the numerical solution of linear Volterra integral equations of the second kind using the Galerkin–Hermite method by employing Hermite polynomials as basis functions.

The method was applied to several illustrative examples. A comparison was carried out with the TMD method, using results taken from the referenced article. The errors were computed for different values of n, and it was observed that as n increases, the error decreases significantly. This confirms the efficiency and high accuracy of the Galerkin–Hermite method, even for relatively small values of n.

This study highlights the potential of Hermite-based spectral methods in solving integral equations. Future work could focus on extending this approach to more complex problems, such as nonlinear or multi-dimensional integral equations, or combining it with adaptive techniques to improve convergence and reduce computational cost.

ملخص

يهدف هذا العمل إلى دراسة وحل معادلات فولترّا التكاملية من النوع الثاني باستخدام طريقة جالركن-هيرميت. تعتمد هذه الطريقة على تقريب الحل بكثيرات حدود هيرميت المتعامدة، وتحويل المعادلة الأصلية إلى نظام جبري خطي. تم تقديم الجانب النظري للطريقة، كما تم تطبيقها على عدة أمثلة عددية لمقارنة النتائج مع الحلول الدقيقة. وقد أظهرت النتائج فعالية الطريقة ودقتها في حل هذا النوع من المعادلات التكاملية.

كلمات مفتاحية

معادلات فولترّا التكاملية، طريقة جالركين-هيرميت، كثيرات حدود هيرميت، التقريب العددي.

Abstract

This thesis focuses on the study and numerical solution of second-kind Volterra integral equations using the Galerkin-Hermite method. This method is based on approximating the solution by orthogonal Hermite polynomials, transforming the original equation into a linear algebraic system. The theoretical background of the method is presented, followed by its application to several numerical examples to compare the approximate results with exact solutions. The obtained results demonstrate the efficiency and accuracy of this method in solving such integral equations.

Key words

Volterra integral equations, Galerkin-Hermite method, Hermite polynomials, numerical approximation.

Résumé

Ce mémoire porte sur l'étude et la résolution numérique des équations intégrales de Volterra du deuxième type en utilisant la méthode de Galerkin-Hermite. Cette méthode repose sur l'approximation de la solution par des polynômes de Hermite orthogonaux, transformant l'équation initiale en un système algébrique linéaire. Nous avons présenté l'aspect théorique de la méthode, puis nous l'avons appliquée à plusieurs exemples numériques pour comparer les résultats aux solutions exactes. Les résultats obtenus montrent l'efficacité et la précision de cette méthode pour résoudre ce type d'équations intégrales.

Mot-clés

Équations intégrales de Volterra, méthode de Galerkin-Hermite, polynômes de Hermite, approximation numérique.

Bibliography

[1] Linz, P. (1985). Analytical and numerical methods for Volterra equations. Society for Industrial and Applied Mathematics.

[2] Nadir, M., Rahmoune, A. (2007). Modified method for solving linear Volterra integral equations of the second kind using Simpson's rule. Development, 90,16.

[3] Brunner, H. (2004). Collocation methods for Volterra integral and related functional differential equations(Vol. 15), Cambridge university press.

[4] Atkinson, K. E. (1997). The numerical solution of integral equations of the second kind (Vol. 4). Cambridge university press.

[5] Szeg, G. (1939). Orthogonal polynomials (Vol. 23). American Mathematical Soc.

[6] Wazwaz, A. M. (2011). Linear and nonlinear integral equations (Vol. 639, pp. 35-36). Berlin: Springer.

[7] Bachiri, F. (2017). theoremes du point fixe et applications aux equations integrales (Memoire de Master, Universite Mohamed Boudiaf de M'Sila). Encadre par Amina Khirani.

[8] Nadir, M.,Note: This derivation is based on class notes by Prof. Mostefa Nedir, University of M'Sila

[9] Rouibah, K., Bellour, A., Lima, P., and Rawashdeh, E. (2022). Iterative continuous collocation method for solving nonlinear Volterra integral equations. Kragujevac journal of Mathematics, 46(4),635-648.