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Boundedness of singular integral operators on Besov and Triebel-Lizorkin type spaces with applications

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DEDICATE

I cannot begin and finish my work without thanking the greatest and the most powerful ***"Allah"****.*

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List of publications

1. Z. ZEGHAD AND D. DRIHEM, *Variable Besov-type spaces*, Acta Mathematica Sinica, English Series, https://doi.org/10.1007/s10114-022-1223-2 http://www.ActaMath.com
2. H. BRAHIM BOULARES, D.DRIHEM AND Z. ZEGHAD, *Weak type estimate of commutators on variable Herz type Hardy spaces*, preprint.
3. D. DRIHEM, W. HEBBACHE AND Z. ZEGHAD, *Boundedness of some commutators on variable Herz-type Hardy spaces*, preprint.

NOTATION

- $\mathbb{R}^n$ is the n-dimensional real Euclidean space and

$$\mathbb{R}^{n+1}_{+} = \{(x,t) : x \in \mathbb{R}^n, \quad 0 < t < \infty\}.$$

- $\mathbb{N}$ is the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- $\mathbb{Z}$ is the set of all integer numbers.
- For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, and $x = (x_1, ..., x_n) \in \mathbb{R}^n$ we write $|\alpha| = \alpha_1 + ... + \alpha_n, \quad x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$ and $\partial^\alpha = \left(\frac{\partial}{x_1}\right)^{\alpha_1} ... \left(\frac{\partial}{x_n}\right)^{\alpha_n}$.
- The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by $x \cdot y = x_1 y_1 + ... + x_n y_n$.
- $c_{\alpha,\beta,...}$ is a positive constant depending on the indicated parameters $\alpha, \beta, ...$.
- The expression $f \lesssim g$ means that $f \leqslant cg$ for some independent constant c (and non-negative functions f and g).
- $f \approx g$ means $f \lesssim g \lesssim f$ for non-negative functions f and g.
- For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the largest integer smaller than or equal to x.
- $\mathrm{supp} f$ is the support of the function f, i.e., the closure of its non-zero set.
- Let $E \subset \mathbb{R}^n$ be a measurable set. $|E|$ stands for the (Lebesgue) measure of E.
- χ_E denotes the characteristic function for the set $E \subset \mathbb{R}^n$.
- $\mathscr{S}(\mathbb{R}^n)$ is the set of all Schwartz functions on $\mathbb{R}^n$.

- $\mathscr{S}'(\mathbb{R}^n)$ the set of all tempered distributions on $\mathbb{R}^n$.
- $L^1_{loc}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$.
- For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x.
- Let $f \in L^1_{loc}(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator is defined by

$$\mathscr{M}(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad \forall x \in \mathbb{R}^n.$$

- The Fourier transform of a function $f \in \mathscr{S}(\mathbb{R}^n)$ is defined by

$$\mathscr{F}\left(f\right)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx, \quad x \in \mathbb{R}^n.$$

  Its inverse is denoted by $\mathscr{F}^{-1}\left(f\right)$ or $\check{f}$.

- The convolution $f * g$ is is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f\left(x-y\right) g\left(y\right) dy, \quad f, g \in L^1(\mathbb{R}^n).$$

- $Q_{v,m}$ be the dyadic cube in $\mathbb{R}^n$ is defined by

$$Q_{v,m} = \{(x_1, ..., x_n) : m_i \leqslant 2^v x_i < m_i + 1, i = 1, 2, ..., n\} \quad v \in \mathbb{Z}, m = (m_1, ..., m_n) \in \mathbb{Z}^n,$$

  for the collection of all such cubes we use

$$\mathscr{Q} = \{Q_{v,m} : v \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

  For each cube Q, we denote its center by c_Q, its lower left-corner by $x_{Q_{v,m}} = 2^{-v} m$ of $Q = Q_{v,m}$ and its side length by $l(Q)$.

- We denote by rQ the cube concentric with Q having the side length $r l(Q)$.
- "i.e." stands simply for "in other words".
- "a.e." stands simply for "almost everywhere".

- Let f and g tow measurable functions an operator T is called a sublinear operator if, for $\lambda \in \mathbb{C}$, we have

  $$\left|T\left(f+g\right)\right| \leqslant \left|T\left(f\right)\right| + \left|T\left(g\right)\right|$$

  and

  $$\left|T\left(\lambda f\right)\right| = |\lambda| \left|T\left(f\right)\right|.$$

- We say a quasi-Banach space A_1 is continuously embedded in another quasi-Banach space A_2, $A_1 \hookrightarrow A_2$, if $A_1 \subset A_2$ and there is a $c > 0$ such that $\left\|f\right\|_{A_2} \leqslant c \left\|f\right\|_{A_1}$ for all $f \in A_1$.

INTRODUCTION

It is well known that Herz spaces and Besov spaces play an important role in harmonic analysis and partial differential equations, and have been systematically studied and developed see, for instance, [31, 44, 46] for classical Herz spaces, [41, 43, 45] for Herz-type Hardy spaces, [24, 53, 54] for Besov spaces, and [68] for Besov-type spaces.

Function spaces with variable exponents have been intensively studied in recent years by a large number of authors. These function spaces are applied in partial differential equations, fluid dynamics and image processing, see for example, [48]. Some examples of these spaces can be mentioned as variable Lebesgue spaces, variable Herz spaces, variable Herz-type Hardy spaces, variable Besov and Triebel-Lizorkin spaces, see [1, 5, 6, 8, 7, 9, 10, 16, 32, 33, 34, 37, 49, 63]. Meanwhile, variable Besov–Morrey spaces, variable Besov and Triebel-Lizorkin-type spaces and their generalizations have been developed, see [3, 14, 15, 17, 65, 69, 66, 67].

Herz spaces introduced by Herz [31] in the study of absolute convergent Fourier transform. Izuki [32, 33] introduced Herz space with one variable exponent $K^{\alpha}_{p(\cdot),q}$ and $\dot{K}^{\alpha}_{p(\cdot),q}$. These spaces with two variable exponents $K^{\alpha(\cdot)}_{p(\cdot),q}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ were studied in [2]. They gave the boundedness of a wide class of classical operators on these spaces. The spaces $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, were first introduced by Izuki and Noi in [34]. Drihem and Seghiri [16, Proposition 1] gave a new equivalent norms for $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. This result is very useful to consider the boundedness of some operators on these function spaces.

Herz-type Hardy spaces were first considered by Chen and Lau [4]. They introduced

Hardy spaces associated with the Beurling algebras A^q on the real line with $1 < q \leqslant 2$. In higher-dimensional and all $1 < q < \infty$ by García-Cuerva [26]. Independently García-Cuerva and Herrero [27] and Lu and Yang [41, 43] generalized the Hardy space theory for Herz spaces. See [22] where another class of Herz-type Hardy spaces are given.

Herz-type Hardy spaces with variable exponent $HK^{\alpha}_{p(\cdot),q}$ and $H\dot{K}^{\alpha}_{p(\cdot),q}$ have recently been investigated in [59]. In [16], the authors introduced the variable Herz-type Hardy spaces $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ where all parameters defining the space are variable and established their characterization in terms of atom. Moreover, applying the atomic decomposition , they obtained the boundedness of some singular integral operators on these spaces.

Besov spaces of variable smoothness and integrability, $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$, initially appeared in the paper of Almeida and Hästö [1]. Several basic properties were established, such as the Fourier analytical characterization and Sobolev embeddings. When p, q, s are constants they coincide with the usual function spaces $B^s_{p,q}$. Later, [12] characterized these spaces by local means and established the atomic characterization. Afterwards, Kempka and Vybíral [36] characterized these spaces by the ball means of differences and also by local means, see [35] for the duality of $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ spaces.

Variable Besov-type spaces have been introduced in [14] and [15], where their basic properties are given, such as the Sobolev type embeddings and that under some conditions these spaces are just the variable Besov spaces. For constant exponents, these spaces unify and generalize many classical function spaces including Besov spaces, Besov-Morrey spaces (see, for example, [68, Corollary 3.3]). Independently, D. Yang, C. Zhuo and W. Yuan, [66] studied these function spaces where several properties are obtained such as atomic decomposition and the boundedness of trace operator.

Tyulenev [56], [57] has studied a new function spaces of variable smoothness. Triebel-Lizorkin spaces with variable smoothness and integrability $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ were introduced in [8]. They proved a discretization by the so called φ-transform. Also atomic and molecular decompositions of these function spaces are obtained and used it to derive trace results. Subsequently, Vybiral [58] established Sobolev-Jawerth embeddings of these spaces. In [67], Triebel-Lizorkin type spaces of variable smoothness and integrability were introduced and studied, their function spaces generalize classical Triebel-Lizorkin-type spaces and Triebel-Lizorkin spaces with variable smoothness and integrability.

The main target of this thesis is to study the boundedness of a class of pseudo-differential operators and singular integral operators of convolution type on variable Herz-type Hardy spaces. In addition, we present another Besov-type spaces with variable smoothness and integrability which covers Besov-type spaces with fixed exponents. Our thesis consists of five chapters. In the first chapter, we give some basic properties of variable Lebesgue spaces and the mixed variable Lebesgue's-sequence spaces. We also give some key technical lemmas that we will use later. In the second chapter, we introduce the local variable Herz-type Hardy spaces where we give their atomic decomposition and we present the boundedness of a class of pseudo-differential operators on such spaces. In the third chapter, we establish the boundedness singular integral operators of convolution type on inhomogeneous Herz-type Hardy spaces $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$. Our result is based on the molecular decompositions of such spaces. In the forth chapter, we present another Besov-type spaces with variable smoothness and integrability which covers Besov-type spaces with fixed exponents. We establish their φ-transform characterization in the sense of Frazier and Jawerth. We also give some basic properties and Sobolev-type embeddings. In the last chapter, we give the atomic decomposition of variable Besov-type spaces $\mathfrak{B}_{p(\cdot),q(\cdot)}^{s(\cdot),\tau(\cdot)}$.

CHAPTER 1

THE MIXED LEBESGUE-SEQUNCE SPACES

In this chapter, we expose the concepts and results used throughout this thesis. We recall some fundamental proprieties on modular space, variable Lebesgue spaces and mixed Lebesgue-sequence spaces. We also give some key technical lemmas that we will use later.

1.1 Definition and basic properties of modular space

We refer to the monographs [10] and [47] for an exposition on semi-modular spaces. We start by recalling about the semi-modular functional space.

Definition 1.1. *Let X be a (real or complex) vector space. A function $\varrho : X \longrightarrow [0,+\infty]$ is called a semi-modular on X if it satisfies the following conditions:*

1. $\varrho(0)=0$.
2. $\varrho(\lambda x)=\varrho(x)$ *for all* $x\in X$, *and all* λ *with* $|\lambda|=1$.
3. $\varrho(\lambda x)=0$ *for all* $\lambda>0$ *implies* $x=0$.
4. ϱ *is left-continuous i.e.* $\lim\limits_{\underset{<}{\lambda\longrightarrow 1}}\varrho(\lambda x)=\varrho(x)$ *for all* $x\in X$.

The function ϱ is called a modular if, in addition,

5. $\varrho(x)=0$ *implies* $x=0$.

If the mapping $\lambda\longrightarrow\varrho(\lambda x)$ is continuous on $[0,+\infty)$ for all $x\in X$, we say that ϱ is continuous. A semi-modular ϱ is said to be quasi-convex if there exists $L\geqslant 1$ such that $\varrho\left(\delta x+(1-\delta)\,y\right)\leqslant L\left(\delta\varrho\,(x)+(1-\delta)\,\varrho\left(y\right)\right)$ for all $x,y\in X$ and $0\leqslant\delta\leqslant 1$, if $L=1$ it's said to be convex.

Now we give examples of modular functions :

Example 1.2. *Let* Ω *be a Lebesgue measurable subset of* $\mathbb{R}^n$.

(a) *If* $1 \leqslant p < \infty$, *then*

$$\varrho_p(g) = \int_\Omega |g(x)|^p \, dx,$$

defines a continuous modular on the space of all measurable functions on Ω.

(b) *If* $1 \leqslant p < \infty$, *then*

$$\varrho_p((x_j)) = \sum_{j=0}^{\infty} |x_j|^p,$$

defines a continuous modular on $\mathbb{R}^n$.

(c) *Let* $\varphi_\infty(t) = \infty \cdot \chi_{(1,\infty)}(t)$ *for* $t \geqslant 0$, *i.e.* $\varphi_\infty(t) = 0$ *if* $t \in [0,1]$ *and* $\varphi_\infty(t) = \infty$ *for* $t \in (1,\infty)$. *Then*

$$\varrho_\infty(f) = \int_\Omega \varphi_\infty\big(|f(x)|\big) \, dx,$$

defines a semi-modular on the space of all measurable functions on Ω *which is not continuous.*

(d) *Let* $\omega \in L^1_{loc}(\mathbb{R}^n)$ *with* $\omega > 0$ *almost everywhere and* $1 \leqslant p < \infty$. *Then*

$$\varrho(f) = \int_\Omega |f(x)| \omega(x) \, dx,$$

defines a continuous modular on the space of all measurable functions on Ω.

Definition 1.3. *If* ϱ *be a semi-modular or modular on* X, *then*

$$X_\varrho = \{x \in X : \lim_{\lambda \longrightarrow 0} \varrho(\lambda x) = 0\},$$

is called a semi-modular space or modular space, respectively, where the limit $\lambda \longrightarrow 0$ *takes place in* $\mathbb{k}$.

We will give other definition of modular function.

Let ϱ be a semimodular or modular on X, then by convexity and non-negative of ϱ and $\varrho(0) = 0$ it follows that $\varrho(\lambda x)$ is non-decreasing on $[0,\infty)$ for every $x \in X$. Also,

$$\begin{aligned} \varrho(\lambda x) &= \varrho(|\lambda|\, x) \leqslant |\lambda|\, \varrho(x) \quad \text{for all } |\lambda| \leqslant 1, \\ \varrho(\lambda x) &= \varrho(|\lambda|\, x) \geqslant |\lambda|\, \varrho(x) \quad \text{for all } |\lambda| \geqslant 1. \end{aligned} \tag{1.1}$$

From (1.1) we can alternatively define X_ϱ by

$$X_\varrho = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

Theorem 1.4. *Let ϱ be a semi-modular. Then X_ϱ is a quasi-normed $\Bbbk$-vector space. The quasi-norm, called the Luxemburg quasi-norm, is defined by*

$$\|x\|_\varrho = \inf\left\{\lambda > 0 : \varrho(\frac{x}{\lambda}) \leqslant 1\right\}.$$

Example 1.5. *The Lebesgue space $L^p(\Omega)$ where $1 \leqslant p < \infty$, is normed space with Luxemburg norm defined by*

$$\|x\|_\varrho = \inf\left\{\lambda > 0 : \varrho(\frac{x}{\lambda}) \leqslant 1\right\},$$

with modular function

$$\varrho_p(f) = \int_\Omega |f(x)|^p \, dx,$$

on the space of all measurable functions on Ω.

The following result is very useful from the technical point of view since it allows one to skip working with the complicated structure of the quasi-norm directly, in many situations of interest.

Lemma 1.6. *Let ϱ be a semi-modular on X. Then*
(a) *$\|x\|_\varrho \leqslant 1$ and $\varrho(x) \leqslant 1$ are equivalent.*
(b) *If ϱ is left-continuous then $\|x\|_\varrho < 1$ and $\varrho(x) < 1$ are equivalent.*
(c) *If ϱ is left-continuous then $\|x\|_\varrho = 1$ and $\varrho(x) = 1$ are equivalent.*

1.2 Variable Lebesgue spaces

In this section, we recall the definition of Lebesgue spaces with variable exponents $L^{p(\cdot)}(\mathbb{R}^n)$, we also mention that main results on the basic properties on $L^{p(\cdot)}(\mathbb{R}^n)$. The spaces $L^{p(\cdot)}(\mathbb{R}^n)$ fit into the framework of Musielak-Orlicz spaces and are therefore also semimodular spaces.

1.2.1 Variable exponent

The kind of variable exponents that we are interested in must be introduced in order to define the variable Lebesgue spaces.

Definition 1.7. *The variable exponents that we consider are always measurable functions on* $\mathbb{R}^n$ *with range in* $[c,\infty[$ *for some* $c>0$. *We denote the set of such functions by* $\mathscr{P}_0(\mathbb{R}^n)$. *The subset of variable exponents with range* $[1,\infty)$ *is denoted by* $\mathscr{P}(\mathbb{R}^n)$.

We define some notation to describe the range of exponent functions. We use the standard notation

$$p^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x), \quad p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x).$$

In the following example, we will mention the functions of the exponent, see [10, Definition 2.1].

Example 1.8. *Some examples of exponent functions on* $\mathbb{R}$ *include* $p(x)=p$ *for some constant* p, *or* $p(x)=2+\sin(x)$.

Definition 1.9. *The* variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ *is the class of all measurable functions* f *on* $\mathbb{R}^n$ *such that the modular*

$$\varrho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx,$$

is finite for some $\lambda>0$. *This is a quasi-Banach space equipped with the quasi-norm*

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda>0 : \varrho_{p(\cdot)}\left(\frac{1}{\lambda}f\right) \leqslant 1\right\}.$$

This function define a norm when $p^- \geqslant 1$.

It is easy to see that $\|\cdot\|_{p(\cdot)}$ has the following property :

$$\left\||f|^{\sigma}\right\|_{p(\cdot)} = \|f\|^{\sigma}_{\sigma p(\cdot)},$$

for any $p\in\mathscr{P}(\mathbb{R}^n)$ with $p^+<\infty$ and $\sigma\geqslant\frac{1}{p^-}$, (see [10, Lemma 3.2.6]). Furthermore, if $p(x)\equiv p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ is the classical Lebesgue space.

In the development of the variable exponent function spaces, the concept of log-Hölder continuity is the cornerstone, which was introduced in [7, 5].

1.2.2 Logarithmic Hölder continuity

Definition 1.10. *We say that a function* $g : \mathbb{R}^n \to \mathbb{R}$ *is* locally log-Hölder continuous*, abbreviated* $g \in C^{\log}_{loc}$*, if there exists* $c_{\log}(g) > 0$ *such that*

$$\left|g(x)-g(y)\right| \leqslant \frac{c_{\log}(g)}{\log(e+1/|x-y|)}, \tag{1.2}$$

for all $x, y \in \mathbb{R}^n$*. We say that* g *satisfies the* log-Hölder continuous at the origin *(or has a* log decay at the origin*), if there exist a constant* $c_{\log} > 0$ *such that*

$$|g(x)-g(0)| \leqslant \frac{c_{\log}}{\log(e+1/|x|)},$$

for all $x \in \mathbb{R}^n$*. We say that* g *satisfies the* log-Hölder continuous at infinity *(or has a* log *decay at infinity), if there exists* $g_\infty \in \mathbb{R}$ *and a constant* $c_{\log} > 0$ *such that*

$$\left|g(x)-g_\infty\right| \leqslant \frac{c_{\log}}{\log(e+|x|)},$$

for all $x \in \mathbb{R}^n$*. We say that* g *is* globally-log -Hölder continuous*, abbreviated* $g \in C^{\log}$*, if it is* locally log-Hölder continuous *and satisfies the* log-Hölder decay *condition. The constants* $c_{\log}(g)$ *and* $c_{\log}$ *are called the* locally log-Hölder *constant and the* log-Hölder decay *constant, respectively.*

Remark 1.11. *We note that all functions* $g \in C^{\log}_{loc}$ *always belong to* L^∞*.*

Notation 1.12. *By* $\tilde{\mathscr{P}}^{\log}_0(\mathbb{R}^n)$ *and* $\tilde{\mathscr{P}}^{\log}_\infty(\mathbb{R}^n)$ *we denote the class of all exponents* $p \in \mathscr{P}(\mathbb{R}^n)$ *which have a log decay at the origin and at infinity, respectively. The notation* $\tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$ *is used for all those exponents* $p \in \mathscr{P}(\mathbb{R}^n)$ *which are locally* log*-Hölder continuous and have a* log *decay at infinity, with* $p_\infty = \lim_{|x|\to\infty} p(x)$*. Obviously we have* $\tilde{\mathscr{P}}^{\log}(\mathbb{R}^n) \subset \tilde{\mathscr{P}}^{\log}_0(\mathbb{R}^n) \cap \tilde{\mathscr{P}}^{\log}_\infty(\mathbb{R}^n)$*. Note that* $p \in \tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$ *if and only if* $p' \in \tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$*, and since* $(p')_\infty = (p_\infty)'$ *we write only* p'_∞ *for any of these quantities. We define the following class of variable exponents*

$$\mathscr{P}^{\log}(\mathbb{R}^n) = \left\{p \in \mathscr{P}(\mathbb{R}^n) : \frac{1}{p} \in C^{\log}\right\},$$

were introduced in [9, Section 2]. The class $\mathscr{P}^{\log}_0(\mathbb{R}^n)$ *is defined analogously. We define* $\frac{1}{p_\infty} = \lim_{|x|\to\infty} \frac{1}{p(x)}$ *and we use the convention* $\frac{1}{\infty} = 0$*. Note that although* $\frac{1}{p}$ *is bounded,* *the variable exponent p itself can be unbounded.*

1.2.3 Some important inequalities

It was shown in [10], Theorem 4.3.8 that $\mathscr{M} : L^{p(\cdot)}(\mathbb{R}^n) \longrightarrow L^{p(\cdot)}(\mathbb{R}^n)$ is bounded if $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$ and $p^- > 1$, see also [9], Theorem 1.2. Also if $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$, then the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$:

$$\|\varphi * f\|_{p(\cdot)} \leqslant c\|\varphi\|_1 \|f\|_{p(\cdot)}.$$

We also refer to the papers [6] and [7], where various results on maximal function in variable Lebesgue spaces were obtained.

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various operators in Harmonic Analysis. In classical L^p spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for $p \in \mathscr{P}^{\log}$ we have

$$\|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \approx |B|. \tag{1.3}$$

Also,

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B, \tag{1.4}$$

for small balls $B \subset \mathbb{R}^n$ ($|B| \leqslant 2^n$), and

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p_\infty}}, \tag{1.5}$$

for large balls ($|B| \geqslant 1$), with constants only depending on the log-Hölder constant of p (see, for example, [10, Section 4.5]).

In variable Lebesgue spaces there are some important lemmas as follows :

Lemma 1.13 (norm-modular unit ball property)**.** *If* $p \in \mathscr{P}(\mathbb{R}^n)$*, then* $\|f\|_{p(\cdot)} \leqslant 1$ *and* $\varrho_{p(\cdot)}(f) \leqslant 1$ *are equivalent. For* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *we have*
(i) *If* $\|f\|_{p(\cdot)} \leqslant 1$*, then* $\varrho_{p(\cdot)}(f) \leqslant \|f\|_{p(\cdot)}$.
(ii) *If* $1 < \|f\|_{p(\cdot)}$*, then* $\|f\|_{p(\cdot)} \leqslant \varrho_{p(\cdot)}(f)$.

For the proof of this Lemma, see [10].

Lemma 1.14 (generalized Hölder's inequality)**.** *Let $p, q, s \in \mathscr{P}(\mathbb{R}^n)$ such that*

$$\frac{1}{s(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$ then $fg \in L^{s(\cdot)}(\mathbb{R}^n)$ and

$$\|fg\|_{s(\cdot)} \leqslant c\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

For the proof of this Lemma, see [38].

1.3 The space $\ell^{q(\cdot)}(L^{p(\cdot)})$

The targets of this section are twofold. The first one is to recall the concept of variable mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ originally introduced by Almeida and Hästö in [1]. The second one is to recall some fundamental properties related to $\ell^{q(\cdot)}(L^{p(\cdot)})$.

Definition 1.15. *Let $p, q \in \mathscr{P}_0^{\log}(\mathbb{R}^n)$. The* mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ *is defined on sequences of $L^{p(\cdot)}$ -functions by the modular*

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf\Big\{\lambda_v > 0 : \varrho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \leqslant 1\Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\big\|(f_v)_v\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\Big\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}(f_v)_v\Big) \leqslant 1\Big\}. \tag{1.6}$$

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$.

Remark 1.16. *If $q^+ < \infty$, then*

$$\inf\Big\{\lambda > 0 : \varrho_{p(\cdot)}\Big(\frac{f_v}{\lambda^{1/q(\cdot)}}\Big) \leqslant 1\Big\} = \big\||f|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}},$$

since the right-hand side expression is much simpler, we can replace (1.6) by the simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \big\||f_v|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Observed in [1, Proposition 3.5] that $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a modular if $p^+ < \infty$ and it is continuous if $p^+, q^+ < \infty$. Also in [1, Theorem 3.6] it was shown that (1.6) defines a norm in $\ell^{q(\cdot)}(L^{p(\cdot)})$ if $q(\cdot) \geqslant 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $p(\cdot) \geqslant 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leqslant 1$ a.e. More recently, it was observed in [37, Theorem 1] that it also becomes a norm if $1 \leqslant q(x) \leqslant p(x) \leqslant \infty$ a.e. on $\mathbb{R}^n$. It is not difficult to verify that $\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty$ implies $(f_v)_v \in \ell^{q(\cdot)}(L^{p(\cdot)})$, which in turn implies $f_v \in L^{p(\cdot)}$ for any $v \in \mathbb{N}_0$. Simple calculations show that

$$\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \left\|(|f_v|^r)_v\right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}, \ r > 0.$$

Remark 1.17. *Let $p, q \in \mathscr{P}_0^{\log}(\mathbb{R}^n)$.*

(i) *The values of q have no influence on $\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ when we restrict ourselves to sequences having just one non-zero entry. In fact, as in the constant exponent case, there holds $\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \left\|f\right\|_{p(\cdot)}$ when $f_{v_0} = f$ for some fixed $v_0 \in N_0$ and $f_v = 0$ for all $v \neq v_0$.*

(ii) *Note that the left-continuity of the semi-modular ensures the useful equivalence*

$$\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant 1 \quad \text{if and only if} \quad \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) \leqslant 1 \text{ (unit ball property).}$$

The following statement are from [36, Lemma 9].

Lemma 1.18. *Let $p, q \in \mathscr{P}^{\log}$ and $\delta > 0$. Let $\{f_k\}_{k\in\mathbb{Z}}$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$ and define*

$$F_v(x) = \sum_{k\in\mathbb{Z}} 2^{-|v-k|\delta} f_k(x), \ x \in \mathbb{R}^n, \ v \in \mathbb{Z}.$$

Then there exist a constant c depending on p, q and δ such that

$$\|F_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c \left\|f_k\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

We know that the Hardy-Littlewood maximal operator is bounded in variable exponent Lebesgue spaces if $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$ and $p^- > 1$. This fact is very important, to develop harmonic analysis in $L^{p(\cdot)}$. It's different and complicated when we think of $\ell^{q(\cdot)}(L^{p(\cdot)})$. Almeida and Hästö proved that the maximal operator is in general not bounded on $\ell^{q(\cdot)}(L^{p(\cdot)})$ when

q is non-constant see [1, Example 4.1]. Therefore, the space $\ell^{q(\cdot)}(L^{p(\cdot)})$ loses an important feature of the iterated space, namely the inheritance of properties from the starting space $L^{p(\cdot)}$. The difficulty described above was successfully overcome, in [1], through the study of convolutions involving nice kernels, namely the η-functions defined by

$$\eta_{v,m}(x) = \frac{2^{nv}}{(1+2^v|x|)^m}, \ v \in \mathbb{N}_0, \quad m > 0.$$

Note that $\eta_{v,m} \in L^1$ when $m > n$ and that $\left\|\eta_{v,m}\right\|_1 = c_m$ is independent of v, where this type of function was introduced in [30] and [10]. Convolution inequalities with kernels given by the functions above have been heavily used as a replacement of the boundedness of the maximal operator in the mixed Lebesgue-sequence spaces. The following inequality was proved in [1, Lemma 4.7].

Lemma 1.19. *Let $p, q \in \mathscr{P}^{\log}(\mathbb{R}^n)$. For any $m > n$, there exists $c > 0$ such that*

$$\left\|(\eta_{v,m} * f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c \left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

holds for all sequence $(f_v)_v \in \ell^{q(\cdot)}(L^{p(\cdot)})$.

CHAPTER 2

BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS ON LOCAL VARIABLE HERZ-TYPE HARDY SPACES

The local Herz-type Hardy spaces $h\dot{K}^{\alpha}_{p(\cdot),q}$ and $hK^{\alpha}_{p(\cdot),q}$ with one variable exponent p were introduced by H. Wang and Z. Liu, [60]. The authors gave their atomic decomposition characterizations also proved the boundedness of a pseudo-differential operators of order zero on these spaces.

Motivated by the mentioned works, the aim of the chapter is twofold. Firstly, we introduce local Herz-type Hardy spaces with variable exponent, where all three parameters are variable and then establish their atomic decomposition of these function spaces. Secondly, we study the boundedness of a class of pseudo-differential operators on such spaces.

2.1 Technical lemmas

In this section, we need to recall some results that will be used in the proofs of our main results of the present chapter. The proof of the following results are given in [2], where its a generalization of (1.3), (1.4) and (1.5) to the case of dyadic annuli.

Lemma 2.1. *Let* $\alpha \in L^{\infty}(\mathbb{R}^n)$ *and* $r_1 > 0$. *If* α *is* log*-Hölder continuous both at the origin and at infinity, then*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} \left(\frac{r_1}{r_2}\right)^{\alpha^+} & \textit{if} \quad 0 < r_2 \leqslant \frac{r_1}{2}\,; \\ 1 & \textit{if} \quad \frac{r_1}{2} < r_2 \leqslant 2r_1\,; \\ \left(\frac{r_1}{r_2}\right)^{\alpha^-} & \textit{if} \quad r_2 > 2r_1\,, \end{cases}$$

for any $x \in B(0, r_1) \setminus B(0, \frac{r_1}{2})$ *and* $y \in B(0, r_2) \setminus B(0, \frac{r_2}{2})$*, with the implicit constant not depending on* x, y, r_1 *and* r_2*.*

Lemma 2.2. *Let* $p \in \tilde{\mathscr{P}}_\infty^{\log}(\mathbb{R}^n)$ *and let* $R = B(0, r) \setminus B(0, \frac{r}{2})$*. If* $|R| \geqslant 2^{-n}$*, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}},$$

with the implicit constants independent of r *and* $x \in R$*.*

The left-hand side equivalence remains true for every $|R| > 0$ *if we assume, additionally,* $p \in \tilde{\mathscr{P}}_0^{\log}(\mathbb{R}^n) \cap \tilde{\mathscr{P}}_\infty^{\log}(\mathbb{R}^n)$*.*

The next lemma is a Hardy-type inequality, see [16, Lemma 2].

Lemma 2.3. *Let* $0 < a < 1$ *and* $0 < q \leqslant \infty$*. Let* $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *be a sequence of positive real numbers, such that*

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\leqslant k} a^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ *and* $\left\{\eta_k : \eta_k = \sum_{j\geqslant k} a^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ *belong to* ℓ^q *and*

$$\left\|\{\delta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} + \left\|\{\eta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = cI,$$

with $c > 0$ *only depending on* a *and* q*.*

The following lemma is from [14, Lemma 2.11], see also [39, Lemma 2.6].

Lemma 2.4. *Let* $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$*. For any cubes (balls)* P *and* Q*, such that* $P \subset Q$ *we have*

$$C\left(\frac{|Q|}{|P|}\right)^{1/p^+} \leqslant \frac{\|\chi_Q\|_{p(\cdot)}}{\|\chi_P\|_{p(\cdot)}} \leqslant c\left(\frac{|Q|}{|P|}\right)^{1/p^-},$$

with $c, C > 0$ *are independent of* $|Q|$ *and* $|P|$*.*

2.2 Variable Herz-type Hardy spaces

The targets of this section are twofold. The first one is to recall the concept of variable exponent Herz spaces and variable Herz-type Hardy spaces. The second one is to recall some important results related to these function spaces. Most of the results of this section was proved by Drihem and Seghiri in [16].

First, we present the concept of variable exponent Herz spaces. For convenience, we set

$$B_k = B(0,2^k), \quad R_k = B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

Definition 2.5. *Let $p, q \in \mathscr{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The* inhomogeneous Herz space $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$\big\|f\big\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \big\|f\,\chi_{B_0}\big\|_{p(\cdot)} + \big\|\big(2^{k\alpha(\cdot)} f\,\chi_k\big)_{k\geqslant 1}\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{2.1}$$

Similarly, the homogeneous Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ such that*

$$\big\|f\big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \big\|\big(2^{k\alpha(\cdot)} f\,\chi_k\big)_{k\in\mathbb{Z}}\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{2.2}$$

If α and p, q are constant, then $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = K^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ are the classical Herz spaces.

Let us denote

$$\|\{g_k\}\|_{\ell^q_>(L^{p(\cdot)})} = \left(\sum_{k=0}^{\infty} \|g_k\|^q_{p(\cdot)}\right)^{1/q} \quad \text{and} \quad \|\{g_k\}\|_{\ell^q_<(L^{p(\cdot)})} = \left(\sum_{k=-\infty}^{-1} \|g_k\|^q_{p(\cdot)}\right)^{1/q}$$

for sequences $\{g_k\}_{k\in\mathbb{Z}}$ of measurable functions (with the usual modification if $q = \infty$).

The following statement plays a crucial role in our work, see [16, Proposition 1].

Proposition 2.6. *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathscr{P}_0(\mathbb{R}^n)$. If α and q are* log*-Hölder continuous at infinity, then*

$$K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = K^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log *decay at the origin, then*

$$\big\|f\big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \big\|\{2^{k\alpha(0)} f\,\chi_k\}\big\|_{\ell^{q(0)}_<(L^{p(\cdot)})} + \big\|\{2^{k\alpha_\infty} f\,\chi_k\}\big\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}. \tag{2.3}$$

The following lemmas are proved in [14, Theorems 1 and 2].

Lemma 2.7. *Let* $q \in \tilde{\mathscr{P}}_0^{\log}(\mathbb{R}^n)$, $p \in \tilde{\mathscr{P}}_\infty^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \leqslant p^+ < \infty$, *and let* α *and* q *be log-Hölder continuous, at infinity, with* $\alpha \in L^\infty(\mathbb{R}^n)$ *and*

$$-\frac{n}{p_\infty} < \alpha_\infty < \frac{n}{p'_\infty}.$$

If a sublinear operator T *satisfies*

$$\left|Tf(x)\right| \lesssim \int_{\mathbb{R}^n} \frac{\left|f(y)\right|}{\left|x-y\right|^n} dy, \quad x \notin suppf, \tag{2.4}$$

for any integrable and compactly supported functions f, *and* T *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$, *then* T *is bounded on* $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Lemma 2.8. *Let* $q \in \mathscr{P}_0(\mathbb{R}^n)$, $p \in \mathscr{P}_0(\mathbb{R}^n)$ *with* $1 < p^- \leqslant p^+ < \infty$, *and let* α *and* q *are* log-*Hölder continuous, both at the origin and at infinity, such that* $\alpha \in L^\infty(\mathbb{R}^n)$ *and*

$$-\frac{n}{p^+} < \alpha^- \leqslant \alpha^+ < n\left(1 - \frac{1}{p^-}\right).$$

Let T *be as in Lemma 2.7. Then* T *is bounded on* $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Now, we recall the definition of variable Herz-type Hardy spaces. Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) := \sup_{\varphi \in \mathscr{A}_N} |\varphi_N^*(f)(x)|,$$

where $\mathscr{A}_N := \{\varphi \in \mathscr{S}(\mathbb{R}^n) : \sup_{|\alpha| \leqslant N, |\beta| \leqslant N} |x^\alpha \partial^\beta \varphi(x)| \leqslant 1\}$ and

$$\varphi_N^*(f)(x) := \sup_{t>0} |\varphi_t * f(x)|,$$

with $\varphi_t := t^{-n}\varphi(\frac{\cdot}{t})$.

Definition 2.9. *Let* $p, q \in \mathscr{P}_0(\mathbb{R}^n)$ *and* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha \in L^\infty(\mathbb{R}^n)$ *and* $N > n+1$. *The* inhomogeneous Herz-type Hardy space $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ *consists of all* $f \in \mathscr{S}'(\mathbb{R}^n)$ *such that*

$G_N f \in K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and we define*

$$\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|G_N f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Similarly, the homogeneous Herz-type Hardy space $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all* $f \in \mathscr{S}'(\mathbb{R}^n)$ *such that* $G_N f \in \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and we define*

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|G_N f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Definition 2.10. *Let* $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathscr{P}(\mathbb{R}^n)$, $q \in \mathscr{P}_0(\mathbb{R}^n)$ *and* $s \in \mathbb{N}_0$. *A function* a *is said to be a central* $(\alpha(\cdot), p(\cdot))$*-atom, if*
(i) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leqslant r\}, \quad r > 0,$
(ii) $\|a\|_{p(\cdot)} \leqslant |\overline{B(0,r)}|^{-\alpha(0)/n}, \quad 0 < r < 1,$
(iii) $\|a\|_{p(\cdot)} \leqslant |\overline{B(0,r)}|^{-\alpha_\infty/n}, \quad r \geqslant 1,$
(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \quad |\beta| \leqslant s.$

A function a *on* $\mathbb{R}^n$ *is said to be a central* $(\alpha(\cdot), p(\cdot))$*-atom of restricted type, if it satisfies the conditions* (iii), (iv) *above and* $\operatorname{supp} a \subset B(0,r)$, $r \geqslant 1$

The following results was proved in [16], see Theorems 3 and 4.

Theorem 2.11. *Let* α *and* q *are* log*-Hölder continuous at infinity and* $p \in \tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \leqslant p^+ < \infty$. *For any* $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, *we have*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \tag{2.5}$$

where the series converges in the sense of distributions, $\lambda_k \geqslant 0$, *each* a_k *is a central* $(\alpha(\cdot), p(\cdot))$*-atom of restricted type with* $\operatorname{supp} a_k \subset B_k$ *and*

$$\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \leqslant c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha_\infty \geqslant n(1-\frac{1}{p_\infty})$ *and* $s \geqslant \left\lfloor \alpha_\infty + n(\frac{1}{p_\infty} - 1)\right\rfloor$, *and if* (2.5) *holds, then* $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, *and*

$$\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\left\{\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f as above.

Theorem 2.12. *Let α and q are* log-*Hölder continuous, both at the origin and at infinity and $p \in \tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. For any $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \tag{2.6}$$

where the series converges in the sense of distributions, $\lambda_k \geqslant 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with supp$a_k \subset B_k$ *and*

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \leqslant c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha(\cdot) \geqslant n(1-\frac{1}{p^-})$ and $s \geqslant \left\lfloor \alpha^+ + n(\frac{1}{p^-}-1)\right\rfloor$, and if (2.6) holds, then $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\left\{\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f as above.

2.3 Some characterizations for variable Herz-type Hardy spaces

In this section, we establish some real-variable characterizations of variable Herz-type Hardy spaces. To begin with, present the concepts of various maximal functions as follows.

Definition 2.13. *Let $f \in \mathscr{S}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. For $t > 0$, set $\varphi_t = t^{-n}\varphi(\cdot/t)$. For any $x \in \mathbb{R}^n$ we define the maximal functions $\varphi^*_+(f)$, $\varphi^*_\nabla(f)$, $\varphi^*_{\nabla,N}(f)$ and $\varphi^{**}_M(f)$ by*

$$\varphi^*_+(f)(x) = \sup_{t>0}\left|(f * \varphi_t)(x)\right|,$$

$$\varphi^*_\nabla(f)(x) = \sup_{|x-y|<t}\left|(f * \varphi_t)(y)\right|,$$

$$\varphi^*_{\nabla,N}(f)(x) = \sup_{t>0}\sup_{|x-y|<Nt}\left|(f * \varphi_t)(y)\right|, \quad N > 1,$$

and

$$\varphi_M^{**}(f)(x)=\sup_{(y,t)\in\mathbb{R}_+^{n+1}}\left|(f*\varphi_t)(y)\right|\left(\frac{t}{|x-y|+t}\right)^M,\quad M\in\mathbb{N}.$$

Next we give the following characterization.

Theorem 2.14. *Let $q\in\mathscr{P}_0(\mathbb{R}^n)$ and $p\in\tilde{\mathscr{P}}_\infty^{\log}(\mathbb{R}^n)$ with $1<p^-\leqslant p^+<\infty$, and let α and q be log-Hölder continuous, at infinity such that $\alpha\in L^\infty(\mathbb{R}^n)$ and $\alpha^->0$. For $\varphi\in\mathscr{S}(\mathbb{R}^n)$, the following statements are equivalent:*
(a) $f\in HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.
(b) *For some* $N>1$, $\varphi_{\nabla,N}^*(f)\in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.
(c) $\varphi_\nabla^*(f)\in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.
(d) $\varphi_+^*(f)\in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Moreover,

$$\|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\|\varphi_{\nabla,N}^*(f)\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\|\varphi_\nabla^*(f)\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\|\varphi_+^*(f)\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Theorem 2.15. *Let $q\in\mathscr{P}_0(\mathbb{R}^n)$, $p\in\tilde{\mathscr{P}}_0^{\log}(\mathbb{R}^n)\cap\tilde{\mathscr{P}}_\infty^{\log}(\mathbb{R}^n)$ with $1<p^-\leqslant p^+<\infty$, and let α and q be log-Hölder continuous, both at the origin and at infinity such that $\alpha\in L^\infty(\mathbb{R}^n)$ and $\alpha^->0$. For $\varphi\in\mathscr{S}(\mathbb{R}^n)$, the following statements are equivalent:*
(a) $f\in H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.
(b) *For some* $N>1$, $\varphi_{\nabla,N}^*(f)\in\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.
(c) $\varphi_\nabla^*(f)\in\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.
(d) $\varphi_+^*(f)\in\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Moreover,

$$\|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\|\varphi_{\nabla,N}^*(f)\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\|\varphi_\nabla^*(f)\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\|\varphi_+^*(f)\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Proof. By similarity we only consider the homogeneous case. (b)$\Longrightarrow$(c)$\Longrightarrow$(d): This follows from

$$\varphi_+^*(f)(x)\leqslant\varphi_\nabla^*(f)(x)\leqslant\varphi_{\nabla,N}^*(f)(x).$$

Note that for any $N>n+1$,

$$\varphi_\nabla^*(f)(x)\lesssim G_N(f)(x).\tag{2.7}$$

(a)$\Longrightarrow$(c): This is a consequence of (2.7). To complete the proof, it suffices to show that (d)$\Longrightarrow$(b) and (d)$\Longrightarrow$(a). We first prove (d)$\Longrightarrow$(b). For $l,N\in\mathbb{N}$, define

$$g^{*}_{\varepsilon,l,N}(x)=\sup_{|x-y|<Nt<1/\varepsilon}\left|(f*\varphi_t)(y)\right|\Big(\frac{Nt}{Nt+\varepsilon}\Big)\big(1+\varepsilon N|y|\big)^{-l}.$$

So by permutation between series and integration, we only need to prove that

$$\left\|g^{*}_{\varepsilon,l,N}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim N^{n/r}\left\|\varphi^{*}_{+}(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},\quad 0<r<1.$$

Now, we further define

$$h^{*}_{\varepsilon,l,N}(x)=\sup_{|x-y|<Nt<1/\varepsilon}t\left|\nabla_y(f*\varphi_t)(y)\right|\Big(\frac{Nt}{Nt+\varepsilon}\Big)\big(1+\varepsilon N|y|\big)^{-l}.$$

As in [43] if l is sufficiently large, then we have

$$h^{*}_{\varepsilon,l,N}(x)\leqslant c\left(\mathcal{M}\left(g^{*}_{\varepsilon,l,N}\right)^{\delta}(x)\right)^{1/\delta},\quad 0<\delta<1,$$

where the positive constant c is independent of ε,N and f. Let

$$A_\varepsilon=\left\{x:h^{*}_{\varepsilon,l,N}(x)\leqslant\tilde{c}\,g^{*}_{\varepsilon,l,N}(x)\right\},\quad\text{and}\quad A^{c}_{\varepsilon}=\mathbb{R}^n\backslash A_\varepsilon$$

where $\tilde{c}$ will be chosen later. Take $0<\delta<1$ such that $0<\alpha^{-}\leqslant\alpha^{+}<n\big(\frac{1}{\delta}-\frac{1}{p^{-}}\big)$, then by Lemma 2.8, we obtain that

$$\begin{aligned}\left\|g^{*}_{\varepsilon,l,N}\chi_{A^{c}_{\varepsilon}}\right\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}&\leqslant\frac{1}{\tilde{c}}\left\|h^{*}_{\varepsilon,l,N}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\\&\leqslant\frac{c}{\tilde{c}}\left\|\mathcal{M}\left(\left(g^{*}_{\varepsilon,l,N}\right)^{\delta}\right)\right\|^{1/\delta}_{\dot{K}^{\delta\alpha(\cdot)}_{p(\cdot)/\delta,q(\cdot)/\delta}}\\&\leqslant\frac{c}{\tilde{c}}\left\|\left(g^{*}_{\varepsilon,l,N}\right)^{\delta}\right\|^{1/\delta}_{\dot{K}^{\delta\alpha(\cdot)}_{p(\cdot)/\delta,q(\cdot)/\delta}}=\frac{c}{\tilde{c}}\left\|g^{*}_{\varepsilon,l,N}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|g^{*}_{\varepsilon,l,N}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}&\leqslant\left\|g^{*}_{\varepsilon,l,N}\chi_{A_\varepsilon}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}+\left\|g^{*}_{\varepsilon,l,N}\chi_{A^{c}_{\varepsilon}}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\\&\leqslant\left\|g^{*}_{\varepsilon,l,N}\chi_{A_\varepsilon}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}+\frac{c}{\tilde{c}}\left\|g^{*}_{\varepsilon,l,N}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\\&\leqslant 2\left\|g^{*}_{\varepsilon,l,N}\chi_{A_\varepsilon}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},\end{aligned}$$

provided that $\tilde{c}$ is sufficiently large. Thus the proof that (d) $\Longrightarrow$ (b) can be reformulated as showing that

$$\left\|g^*_{\varepsilon,l,N}\chi_{A_\varepsilon}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim N^{n/r}\left\|\varphi^*_+(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \qquad 0<r<1. \tag{2.8}$$

It will know that, see [43]

$$g^*_{\varepsilon,l,N}(x) \leqslant cN^{n/r}\left(\mathcal{M}\left(\varphi^*_+(f)\right)^r(x)\right)^{1/r}, \quad \text{for any } 0<r<1 \quad \text{and } x\in A_\varepsilon.$$

Now choosing r such that $0<\alpha^- \leqslant \alpha^+ < n\left(\frac{1}{r}-\frac{1}{p^-}\right)$, then by Lemma 2.8, and the last inequality we get

$$\begin{aligned}
\left\|g^*_{\varepsilon,l,N}\chi_{A_\varepsilon}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &\lesssim N^{n/r}\left\|\left(\mathcal{M}\left(\varphi^*_+(f)\right)^r\right)^{1/r}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\\
&\lesssim N^{n/r}\left\|\mathcal{M}\left(\varphi^*_+(f)\right)^r\right\|^{1/r}_{\dot{K}^{r\alpha(\cdot)}_{p(\cdot)/r,q(\cdot)/r}}\\
&\lesssim N^{n/r}\left\|\varphi^*_+(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.
\end{aligned}$$

This finishes the proof of (d) $\Longrightarrow$ (b). Moreover,

$$\left\|\varphi^*_{\nabla,N}(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim N^{n/r}\left\|\varphi^*_+(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{2.9}$$

Now we consider (d) $\Longrightarrow$ (a). It is easy to verify that

$$\varphi^{**}_M(f)(x) \lesssim \varphi^*_\nabla(f)(x) + \sum_{k=0}^{\infty} 2^{-kM}\varphi^*_{\nabla,2^{k+1}}(f)(x), \tag{2.10}$$

and since

$$G_N(f)(x) \lesssim \varphi^{**}_M(f)(x), \tag{2.11}$$

for all $N>M+n+1$ see [23]. From (2.10), (2.11) and (2.9), we have gives that

$$\begin{aligned}
\left\|G_N(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &\lesssim \left\|\varphi^*_\nabla(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} + \sum_{k=0}^{\infty} 2^{-kM}\left\|\varphi^*_{\nabla,2^{k+1}}(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\\
&\lesssim \left\|\varphi^*_+(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \sum_{k=0}^{\infty} 2^{-k(M-n/r)}\\
&\lesssim \left\|\varphi^*_+(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},
\end{aligned}$$

for any N large enough, where $M > n/r$. Hence the proof of Theorem is completed. $\square$

Remark 2.16. *If α and q are constants, then the statements corresponding to Theorems 2.14 and 2.15 can be found in [61, Theorem3.3].*

2.4 Local variable Herz-type Hardy spaces

In this section, we give the definition of local variable Herz-type Hardy spaces and the atomic decomposition of these function spaces.

Let $\tilde{G}_N f$ be the maximal function f defined by

$$\tilde{G}_N f(x) = \sup_{\varphi\in\mathscr{A}_N} \sup_{0<t<1, |x-y|<Nt} |(f*\varphi_t)(y)|,$$

where $\mathscr{A}_N = \{\varphi \in \mathscr{S}(\mathbb{R}^n) : \sup_{|\alpha|\leqslant N, |\beta|\leqslant N} |x^\alpha \partial^\beta \varphi(x)| \leqslant 1\}$ and $\varphi_t = t^{-n}\varphi(\frac{\cdot}{t})$. Then we introduce localized variable Herz-type Hardy spaces as follows.

Definition 2.17. *Let $q \in \mathscr{P}_0(\mathbb{R}^n)$, $\alpha \in L^\infty(\mathbb{R}^n)$ and $p \in \tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$.*

(i) *If α and q be log-Hölder continuous, both at the origin and at infinity and let $\alpha(\cdot) \geqslant n\big(1-\frac{1}{p^-}\big)$. Then a function $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ is said to be in the space $h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if $\tilde{G}_N f$ belongs to the space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ for any $N > \frac{\alpha^+ p^-}{p^- -1} + 1$. Moreover, we define that*

$$\|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|\tilde{G}_N(f)\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

(ii) *If α and q be log-Hölder continuous, at infinity and $\alpha_\infty \geqslant n\big(1-\frac{1}{p_\infty}\big)$. Then a function $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ is said to be in the space $hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if $\tilde{G}_N f$ belongs to the space $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ for any $N > \frac{\alpha_\infty p_\infty}{p_\infty - 1} + 1$. Moreover, we define that*

$$\|f\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|\tilde{G}_N(f)\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Now, we establish the maximal function characterizations of the local variable Herz-type Hardy spaces.

Theorem 2.18. *Let α and q are* log*-Hölder continuous at infinity and $p \in \tilde{\mathscr{P}}^{\log}_\infty(\mathbb{R}^n)$ such that $1 < p^- \leqslant p^+ < \infty$, and $\alpha_\infty \geqslant n\big(1-\frac{1}{p_\infty}\big)$. For $f \in \mathscr{S}'(\mathbb{R}^n)$, the following statements are* *equivalent:*

(a) $f \in hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(b) $\tilde{\varphi}^*_{\nabla}(f) = \sup_{0<t<1} \sup_{|x-y|<t} \left|(f * \varphi_t)(y)\right| \in K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(c) $\tilde{\varphi}^*_{+}(f) = \sup_{0<t<1} \left|(f * \varphi_t)(y)\right| \in K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Moreover,

$$\|f\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|\tilde{\varphi}^*_{\nabla}(f)\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|\tilde{\varphi}^*_{+}(f)\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Theorem 2.19. *Let α and q are* log*-Hölder continuous both at the origin and at infinity and $p \in \tilde{\mathscr{P}}^{\log}_0(\mathbb{R}^n) \cap \tilde{\mathscr{P}}^{\log}_\infty(\mathbb{R}^n)$ such that $1 < p^- \leqslant p^+ < \infty$, and $\alpha(\cdot) \geqslant n\big(1-\frac{1}{p^-}\big)$. For $f \in \mathscr{S}'(\mathbb{R}^n)$, the following statements are equivalent:*

(a) $f \in h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(b) $\tilde{\varphi}^*_{\nabla}(f) = \sup_{0<t<1} \sup_{|x-y|<t} \left|(f * \varphi_t)(y)\right| \in \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(c) $\tilde{\varphi}^*_{+}(f) = \sup_{0<t<1} \left|(f * \varphi_t)(y)\right| \in \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Moreover,

$$\|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|\tilde{\varphi}^*_{\nabla}(f)\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|\tilde{\varphi}^*_{+}(f)\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

We omit the proofs of Theorems 2.18 and 2.19 since they are essentially is similar to the proof of Theorem 2.15.

On the other hand, we obtain the following relation between the local variable Herz-type Hardy spaces and the variable Herz-type Hardy spaces as follows.

Theorem 2.20. *Let α and q are* log*-Hölder continuous both at infinity and $p \in \tilde{\mathscr{P}}^{\log}_\infty(\mathbb{R}^n)$ such that $1 < p^- \leqslant p^+ < \infty$, and $\alpha_\infty \geqslant n\big(1-\frac{1}{p_\infty}\big)$. Suppose that $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$, and*

$$\int_{\mathbb{R}^n} x^\beta \varphi(x)\,dx = 0, \quad \textit{for all } \beta \in \mathbb{N}^n, |\beta| \leqslant N.$$

Then we have

$$\|f - \varphi * f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Moreover, if $f \in hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then $f - \varphi * f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Theorem 2.21. *Let α and q are* log*-Hölder continuous both at the origin and at infinity, and $p \in \tilde{\mathscr{P}}^{\log}_0(\mathbb{R}^n) \cap \tilde{\mathscr{P}}^{\log}_\infty(\mathbb{R}^n)$ such that $1 < p^- \leqslant p^+ < \infty$, and $\alpha(\cdot) \geqslant n\big(1-\frac{1}{p^-}\big)$. Suppose that*

$\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$, and

$$\int_{\mathbb{R}^n} x^\beta \varphi(x)\,dx = 0, \quad \textit{for all } \beta \in \mathbb{N}, |\beta| \leqslant N.$$

Then we have

$$\|f - \varphi * f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Moreover, if $f \in h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then $f - \varphi * f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Proof. By similarity we only consider the homogeneous case. Take $\psi \in \mathscr{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)\,dx = 1$. Using Theorem 2.15, we have

$$\begin{aligned}
\|f - \varphi * f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &\approx \Big\|\sup_{t>0} \big|\psi_t * (f - \varphi * f)\big|\Big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \\
&\lesssim \Big\|\sup_{0<t<1} \big|\psi_t * f\big|\Big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} + \Big\|\sup_{0<t<1} \big|\psi_t * \varphi * f\big|\Big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \\
&\quad + \Big\|\sup_{t\geqslant 1} \big|\varphi_t * (f - \varphi * f)\big|\Big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.
\end{aligned}$$

It will know that, see [43]

$$\sup_{0<t<1} \big|\psi_t * f\big| \lesssim \tilde{G}_N f(x), \tag{2.12}$$

$$\sup_{0<t<1} \big|\psi_t * \varphi * f\big| \lesssim \tilde{G}_N f(x), \tag{2.13}$$

and

$$\sup_{t\geqslant 1} \big|\psi_t * (f - \varphi * f)\big| \lesssim \tilde{G}_N f(x). \tag{2.14}$$

From (2.12), (2.13) and (2.14), we see that

$$\|f - \varphi * f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|\tilde{G}_N f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

This finishes the proof of Theorem 2.21. □

Now we give the notion of block.

Definition 2.22. *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and , $p \in \mathscr{P}(\mathbb{R}^n)$. A function a is said to be a central $(\alpha(\cdot), p(\cdot))$-block, if*

(i) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leqslant r\}, \quad r > 0,$

(ii) $\|a\|_{p(\cdot)} \lesssim r^{-\alpha(0)}, \quad 0 < r < 1,$
(iii) $\|a\|_{p(\cdot)} \lesssim r^{-\alpha_\infty}, \quad r \geqslant 1.$

A function a is said to be a central $(\alpha(\cdot), p(\cdot))$-block of restricted type, if it satisfies the condition (iii)*, above and* $\operatorname{supp} a \subset B(0, r)$, $r \geqslant 1$.

Next we come to the atomic decomposition theorem for $hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Which plays an important role in this chapter.

Theorem 2.23. *Let α and q are* log*-Hölder continuous at infinity and $p \in \tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. For any $f \in hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \tag{2.15}$$

where the series converges in the sense of distributions, $\lambda_k \geqslant 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-block of restricted type with $\operatorname{supp} a_k \subset B_k$ *and*

$$\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \leqslant c\|f\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha_\infty \geqslant n(1 - \frac{1}{p_\infty})$, and if (2.15) *holds, then $f \in hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and*

$$\|f\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\left\{\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f as above.

Theorem 2.24. *Let α and q are* log*-Hölder continuous, both at the origin and at infinity and $p \in \tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$. For any $f \in h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \quad \textit{in the sense of } \mathscr{S}'(\mathbb{R}^n), \tag{2.16}$$

where for $k \leqslant 0$, a_k is a central $(\alpha(\cdot), p(\cdot))$-atom, while for $k > 0$, a_k is a central $(\alpha(\cdot), p(\cdot))$-block, with $\operatorname{supp} a_k \subset B_k$ *and*

$$\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)}+\left(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\right)^{1/q_\infty}\leqslant c\|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha(\cdot)\geqslant n(1-\frac{1}{p^-})$ *and* $s\geqslant\left\lfloor\alpha^++n(\frac{1}{p^-}-1)\right\rfloor$, *and if* (2.16) *holds, then* $f\in h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, *and*

$$\|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\approx\inf\left\{\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)}+\left(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f *as above.*

Proof. By similarity we only consider the homogeneous case. We follows the idea of [16] and [60]. We divide the proof into tow steps.

Step 1. To prove the necessity part of Theorem 2.24. Let $f\in h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\varphi\in\mathscr{S}(\mathbb{R}^n)$ be as in Theorem 2.21. Then $g=f-\varphi*f\in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, which implies that

$$f=g+\varphi*f,\quad \text{in the sense of } \mathscr{S}'(\mathbb{R}^n).$$

For g we have the central $(\alpha(\cdot),p(\cdot))$-atom decomposition by Theorem 2.12. So it suffices to decompose $\varphi*f$. Let ψ be a radial smooth function such that

$$\operatorname{supp}\psi\subset\left\{x:\frac{1}{2}-\varepsilon\leqslant|x|\leqslant1+\varepsilon\right\},\ 0<\varepsilon<\frac{1}{4},\ \text{and } \psi(x)=1 \text{ if } \frac{1}{2}\leqslant|x|\leqslant1.$$

Let $\tilde{R}_{k,\varepsilon}=\left\{x:2^{k-1}-2^k\varepsilon\leqslant|x|\leqslant2^k+2^k\varepsilon\right\}$ and $\psi_k=\varphi\left(2^{-k}\cdot\right)$. It is easy to see that $\operatorname{supp}\psi_k\subset\tilde{R}_{k,\varepsilon}$, $\psi_k(x)=1$ if $x\in R_k$, and $1\leqslant\sum_{k=-\infty}^{\infty}\psi_k(x)\leqslant2$, $x\neq0$. Let

$$\Phi_k(x)=\begin{cases}\frac{\psi_k(x)}{\sum_{j=-\infty}^{\infty}\psi_j(x)}, & x\neq0;\\ 0, & x=0,\end{cases}$$

then $\sum_{k=-\infty}^{\infty}\Phi_k(x)=1$ for $x\neq0$. We denote by $\mathscr{P}_m$ the class of all real polynomials of degree no more than m. Let $P_k(x)=P_{\tilde{R}_{k,\varepsilon}}\big((f*\varphi)\Phi_k\big)(x)\chi_{\tilde{R}_{k,\varepsilon}}(x)\in\mathscr{P}_m(\mathbb{R}^n)$ be the unique polynomial satisfying

$$\int_{\tilde{R}_{k,\varepsilon}}\big((f*\varphi)\Phi_k(x)-P_k(x)\big)x^\beta\,dx=0,\quad \left|\beta\right|\leqslant[\alpha^++n(\frac{1}{p^-}-1)]=m.$$

Observe that

$$\begin{aligned} f*\varphi(x) &= \sum_{k=-\infty}^{\infty}\left(\left(f*\varphi\right)(x)\Phi_k(x)-P_k(x)\right)+\sum_{k=-\infty}^{\infty}P_k(x) \\ &= \mathrm{I}_k(x)+\mathrm{II}_k(x). \end{aligned}$$

For $\mathrm{I}_k(x)$, let $F_k(x)=\left(f*\varphi\right)(x)\Phi_k(x)-P_k(x)$ and $a_k=\frac{F_k}{\lambda_k}$, where

$$\lambda_k=b\sum_{l=k-1}^{k+1}\left\||B_{k+1}|^{\alpha(\cdot)/n}\left(\tilde{G}_N f\right)\chi_l\right\|_{p(\cdot)}$$

and b is a constant which will be chosen later. Obviously, $\operatorname{supp}F_k\subset\tilde{R}_{k,\varepsilon}\subset B_{k+1}$ and

$$\mathrm{I}_k(x)=\sum_{k=-\infty}^{\infty}\lambda_k a_k(x).$$

Let us prove that a_k is a central $(\alpha(\cdot),p(\cdot))$-atom. Set $\left\{\varphi_d^k:|d|\leqslant m\right\}$ be the orthogonal polynomials restricted to $\tilde{R}_{k,\varepsilon}$ with respect to the weight $\frac{1}{|\tilde{R}_{k,\varepsilon}|}$, which are obtained from $\left\{x^\beta:\left|\beta\right|\leqslant m\right\}$ by the Gram-Schmidt method, that is

$$\left\langle\varphi_d^k,\varphi_v^k\right\rangle=\frac{1}{|\tilde{R}_{k,\varepsilon}|}\int_{\tilde{R}_{k,\varepsilon}}\varphi_d^k(x)\varphi_v^k(x)\,dx=\delta_{dv}.$$

Therefore,

$$P_k(x)=\sum_{|d|\leqslant m}\left\langle\left(f*\varphi\right)\Phi_k,\varphi_d^k\right\rangle\varphi_d^k(x)\quad\text{for } x\in\tilde{R}_{k,\varepsilon}.$$

Observe that $\varphi_v^k(x)=\varphi_v^1\left(2^{k-1}x\right)$ a.e $x\in\tilde{R}_{k,\varepsilon}$, by Hölder's inequality, we have

$$\begin{aligned} |B_{k+1}|^{\alpha(x)/n}|P_k(x)| &\leqslant \frac{c}{|\tilde{R}_{k,\varepsilon}|}\int_{\tilde{R}_{k,\varepsilon}}|B_{k+1}|^{\alpha(x)/n}\left|\left(f*\varphi\right)\left(y\right)\Phi_k\left(y\right)\right|dy \\ &\leqslant \frac{c\left\||B_{k+1}|^{\alpha(x)/n}\left(f*\varphi\right)\Phi_k\right\|_{p(\cdot)}\left\|\chi_{\tilde{R}_{k,\varepsilon}}\right\|_{p'(\cdot)}}{|\tilde{R}_{k,\varepsilon}|}. \end{aligned}$$

Using Lemma 2.1 to obtain $|B_{k+1}|^{\alpha(x)/n}\approx 2^{k\alpha(x)}\approx 2^{k\alpha(y)}$ for any $x,y\in\tilde{R}_{k,\varepsilon}$. Hence

$$\begin{aligned} &\big\||B_{k+1}|^{\alpha(x)/n}F_k\big\|_{p(\cdot)}\\ \leqslant\ & c\big\||B_{k+1}|^{\alpha(x)/n}(f*\varphi)\Phi_k\big\|_{p(\cdot)}+\frac{c}{|\tilde{R}_{k,\varepsilon}|}\big\||B_{k+1}|^{\alpha(x)/n}(f*\varphi)\Phi_k\big\|_{p(\cdot)}\big\|\chi_{\tilde{R}_{k,\varepsilon}}\big\|_{p(\cdot)}\big\|\chi_{\tilde{R}_{k,\varepsilon}}\big\|_{p'(\cdot)}\\ \leqslant\ & c\big\||B_{k+1}|^{\alpha(x)/n}(f*\varphi)\Phi_k\big\|_{p(\cdot)}\leqslant C\sum_{l=k-1}^{k+1}\big\||B_{k+1}|^{\alpha(\cdot)/n}(\tilde{G}_N f)\chi_l\big\|_{p(\cdot)}, \end{aligned}$$

where we wsud the fact that $\big\|\chi_{\tilde{R}_{k,\varepsilon}}\big\|_{p(\cdot)}\big\|\chi_{\tilde{R}_{k,\varepsilon}}\big\|_{p'(\cdot)}\approx|\tilde{R}_{k,\varepsilon}|$. Choose $b=C$, we obtain

$$\big\||B_{k+1}|^{\alpha(\cdot)/n}a_k\big\|_{p(\cdot)}\leqslant c.$$

This relation is equivalent to the inequalities (ii) and (iii) in Definition 2.10 and hence each a_k is a central $(\alpha(\cdot),p(\cdot))$-atom with supp$a_k\subset B_{k+1}$. Furthermore, since $|B_{k+1}|^{\alpha(x)/n}\approx 2^{k\alpha(x)}\approx 2^{l\alpha(x)}\approx 2^{k\alpha(0)}$ for any $x\in R_l$ with $k\leqslant -1$ and $k-1\leqslant l\leqslant k+1$,

$$\begin{aligned} \sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)} &\lesssim \sum_{k=-\infty}^{-1}\Big(\sum_{l=k-1}^{k+1}\big\||B_{k+1}|^{\alpha(\cdot)/n}(\tilde{G}_N f)\chi_l\big\|_{p(\cdot)}\Big)^{q(0)}\\ &\lesssim \sum_{k=-\infty}^{-1}\Big(\sum_{l=k-1}^{k+1}\big\|2^{l\alpha(0)}(\tilde{G}_N f)\chi_l\big\|_{p(\cdot)}\Big)^{q(0)}\lesssim\big\|\tilde{G}_N f\big\|^{q(0)}_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \end{aligned}$$

Similarly, we have

$$\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\lesssim\big\|\tilde{G}_N f\big\|^{q_\infty}_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Here we use the fact that $|B_{k+1}|^{\alpha(x)/n}\approx 2^{k\alpha(x)}\approx 2^{l\alpha(x)}\approx 2^{k\alpha_\infty}$ for any $x\in R_l$ with $k\geqslant 0$ and $k-1\leqslant l\leqslant k+1$. It remains to estimate $\mathrm{II}_k(x)$. Let $\{\psi^k_d:|d|\leqslant m\}$ be the dual basis of $\{x^\beta:|\beta|\leqslant m\}$ with respected to the weight $\frac{1}{|\tilde{R}_{k,\varepsilon}|}$, on $\tilde{R}_{k,\varepsilon}$ that is

$$\langle\psi^k_d,x^\beta\rangle=\frac{1}{|\tilde{R}_{k,\varepsilon}|}\int_{\tilde{R}_{k,\varepsilon}}\psi^k_d(x)x^\beta\,dx=\delta_{\beta,d}.$$

We can prove that if $\varphi^k_d(x)=\sum_{|v|\leqslant m}\beta^k_{d,v}x^v$, then $\psi^k_d(x)=\sum_{|v|\leqslant m}\beta^k_{d,v}\varphi^k_v(x)$. In fact, let $\psi^k_d(x)=\sum_{|v|\leqslant m}\tau^k_{dv}\varphi^k_v(x)$, then

$$\tau_{dv}^{k}=\left\langle \psi_d^k,\varphi_v^k\right\rangle=\left\langle \psi_d^k,\sum_{|\gamma|\leqslant m}\beta_{v\gamma}^{k}x^{\gamma}\right\rangle=\sum_{|\gamma|\leqslant m}\beta_{v\gamma}^{k}\left\langle \psi_d^k,x^{\gamma}\right\rangle=\beta_{v,d}^{k}.$$

Thus, for $x\in\tilde{R}_{k,\varepsilon}$,

$$P_k(x)=\sum_{|d|\leqslant m}\left\langle \left(f*\varphi\right)\Phi_k,y^d\right\rangle\psi_d^k(x).$$

It is easy to prove that if $x\in\tilde{R}_{k,\varepsilon}$, then $\left|\psi_d^k(x)\right|\leqslant C2^{-k|d|}$ see [42]. It follows that

$$\begin{aligned}
\mathrm{II}_k(x) &= \sum_{k=-\infty}^{\infty}\sum_{|d|\leqslant m}\left\langle \left(f*\varphi\right)\Phi_k,y^d\right\rangle\psi_d^k(x)\chi_{\tilde{R}_{k,\varepsilon}}(x)\\
&= \sum_{|d|\leqslant m}\sum_{k=-\infty}^{\infty}\left(\int_{\mathbb{R}^n}\left(f*\varphi\right)\Phi_k y^d\,dy\right)\frac{\psi_d^k(x)\chi_{\tilde{R}_{k,\varepsilon}}(x)}{|\tilde{R}_{k,\varepsilon}|}\\
&= \sum_{|d|\leqslant m}\sum_{k=-\infty}^{0}h_{k,d}^{1}(x)+\sum_{|d|\leqslant m}\left(h_{1,d}^{2}(x)+\sum_{k=2}^{\infty}h_{k,d}^{2}(x)\right)=\mathrm{II}_k^1(x)+\mathrm{II}_k^2(x),
\end{aligned}$$

where

$$h_{k,d}^{1}(x)=\left(\sum_{l=-\infty}^{k}\int_{\mathbb{R}^n}\left(f*\varphi\right)\Phi_l(x)y^d\,dy\right)\left(\frac{\psi_d^k(x)\chi_{\tilde{R}_{k,\varepsilon}}(x)}{|\tilde{R}_{k,\varepsilon}|}-\frac{\psi_d^{k+1}(x)\chi_{\tilde{R}_{k+1,\varepsilon}}(x)}{|\tilde{R}_{k+1,\varepsilon}|}\right),$$

$$h_{1,d}^{2}(x)=\left(\sum_{l=-\infty}^{1}\int_{\mathbb{R}^n}\left(f*\varphi\right)\Phi_l(x)y^d\,dy\right)\frac{\psi_d^k(x)\chi_{\tilde{R}_{k,\varepsilon}}(x)}{|\tilde{R}_{k,\varepsilon}|},$$

and

$$h_{k,d}^{2}(x)=\left(\int_{\mathbb{R}^n}\left(f*\varphi\right)\Phi_k(x)y^d\,dy\right)\frac{\psi_d^k(x)\chi_{\tilde{R}_{k,\varepsilon}}(x)}{|\tilde{R}_{k,\varepsilon}|}.$$

For $\mathrm{II}_k^1(x)$, let

$$\mathrm{II}_k^1(x)=\sum_{|d|\leqslant m}\sum_{k=-\infty}^{0}\sigma_{k,d}^{1}\frac{h_{k,d}^{1}}{\sigma_{k,d}^{1}}=\sum_{|d|\leqslant m}\sum_{k=-\infty}^{0}\sigma_{k,d}^{1}a_{k,d}^{1}$$

where

$$\sigma_{k,d}^{1}=\tilde{b}\sum_{l=k-1}^{k+2}\left\||B_{k+1}|^{\alpha(\cdot)/n}\left(\tilde{G}_N f\right)\chi_l\right\|_{p(\cdot)}$$

and $\tilde{b}>0$ is a constant which will be chosen later. Observe that

$$\left|\sum_{l=-\infty}^{k}\int_{\mathbb{R}^n}\big(f*\varphi\big)\Phi_l(x)y^d\,dy\right|\leqslant c2^{k(n+|d|)}\tilde{G}_N\big(f\big)(x),\quad x\in B_{k+2},$$

and

$$\left|\frac{\psi_d^k(x)\chi_{\tilde{R}_{k,\varepsilon}}(x)}{|\tilde{R}_{k,\varepsilon}|}-\frac{\psi_d^{k+1}(x)\chi_{\tilde{R}_{k+1,\varepsilon}}(x)}{|\tilde{R}_{k+1,\varepsilon}|}\right|\leqslant c2^{-k(n+|d|)}\sum_{l=k-1}^{k+1}\chi_l(x),$$

then

$$\big\||B_{k+2}|^{\alpha(\cdot)/n}h_{k,d}^1\big\|_{p(\cdot)}\leqslant C\sum_{l=k-1}^{k+2}\big\||B_{k+2}|^{\alpha(\cdot)/n}\big(\tilde{G}_Nf\big)\chi_l\big\|_{p(\cdot)}.$$

Thus if we take $\tilde{b}=C$, then $a_{k,d}^1$ is a central $(\alpha(\cdot),p(\cdot))$-atom, with supp$a_k\subset B_{k+2}$. Furthermore, since $|B_{k+1}|^{\alpha(x)/n}\approx 2^{k\alpha(x)}\approx 2^{l\alpha(x)}\approx 2^{k\alpha(0)}$ for any $x\in R_l$ with $k\leqslant -1$ and $k-1\leqslant l\leqslant k+2$,

$$\sum_{k=-\infty}^{-1}|a_{k,d}^1|^{q(0)}\leqslant C\sum_{k=-\infty}^{-1}\Big(\sum_{l=k-1}^{k+2}\big\|2^{l\alpha(0)}\big(\tilde{G}_Nf\big)\chi_l\big\|_{p(\cdot)}\Big)^{q(0)}\leqslant c\big\|\tilde{G}_Nf\big\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{q(0)}.$$

Now, we come to decompose $\mathrm{II}_k^2(x)$. A trivail computation as above gives us that

$$\left|\sum_{l=-\infty}^{1}\int_{\mathbb{R}^n}\big(f*\varphi\big)\varphi_l(x)y^d\,dy\right|\leqslant c\tilde{G}_N\big(f\big)(x),\quad x\in B_2,$$

then

$$\big\|h_{1,d}^2\big\|_{p(\cdot)}\lesssim\sum_{l=0}^{2}\big\|\big(\tilde{G}_Nf\big)\chi_l\big\|_{p(\cdot)}.$$

Set

$$\sigma_{1,d}^2=2^{\alpha_\infty}\big\|h_{1,d}^2\big\|_{p(\cdot)}.$$

Then $a_{1,d}^2(x)=\frac{h_{1,d}^2(x)}{\sigma_{1,d}^2}$ is a central $(\alpha(\cdot),p(\cdot))$-block supported on B_2 and $h_{1,d}^2(x)=\sigma_{1,d}^2a_{1,d}^2(x)$. For $k\geqslant 2$, let

$$a_{k,d}^2(x)=\frac{h_{k,d}^2(x)}{\sigma_{k,d}^2},$$

and

$$\sigma_{k,d}^2=\tilde{C}\sum_{l=k-1}^{k+1}\big\||B_{k+1}|^{\alpha(\cdot)/n}\big(\tilde{G}_Nf\big)\chi_l\big\|_{p(\cdot)},$$

where $\tilde{C}>0$ is a constant which will be chosen later. by Hölder's inequality and the the fact that $\left\|\chi_{\tilde{R}_{k,\varepsilon}}\right\|_{p(\cdot)}\left\|\chi_{\tilde{R}_{k,\varepsilon}}\right\|_{p'(\cdot)}\approx|\tilde{R}_{k,\varepsilon}|$, we have

$$\begin{aligned}\left\||B_{k+1}|^{\alpha(\cdot)/n}h_{k,d}^{2}\right\|_{p(\cdot)} &\leqslant c\left\||B_{k+1}|^{\alpha(\cdot)/n}\tilde{G}_N\left(f\right)\chi_{\tilde{R}_{k,\varepsilon}}\right\|_{p(\cdot)}\\ &\leqslant C\sum_{l=k-1}^{k+1}\left\||B_{k+1}|^{\alpha(\cdot)/n}\tilde{G}_N\left(f\right)\chi_l\right\|_{p(\cdot)}.\end{aligned}$$

Thus if we take $\tilde{C}=C$ then $a_{k,d}^{2}$ is a central $(\alpha(\cdot),p(\cdot))$-block which is supported on B_{k+1}. It then follows that

$$\mathrm{II}_k^2(x)=\sum_{|d|\leqslant m}\sum_{k=1}^{\infty}\sigma_{k,d}^{2}a_{k,d}^{2}(x).$$

Furthermore, since $|B_{k+1}|^{\alpha(x)/n}\approx 2^{k\alpha_\infty}\approx 2^{l\alpha_\infty}$ for any $x\in R_l$ with $k\geqslant 2$ and $k-1\leqslant l\leqslant k+1$, we see that

$$\sum_{k=1}^{\infty}|\sigma_{k,d}^{2}|^{q_\infty}\leqslant c\sum_{k=0}^{\infty}\left(\sum_{l=k-1}^{k+1}\left\|2^{l\alpha_\infty}\left(\tilde{G}_Nf\right)\chi_l\right\|_{p(\cdot)}\right)^{q_\infty}\leqslant c\left\|\tilde{G}_Nf\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}^{q_\infty}.$$

Using the Banach-Alaoglu Theorem and the usual method, we obtain a subsequence $\left\{j_v\right\}$ of $\mathbb{N}$ such that for each $k\in\mathbb{Z}$, $\lim_{v\to\infty}a_k^{(j_v)}=a_k$ in $\mathscr{S}'(\mathbb{R}^n)$, which for $k\leqslant 0, a_k$ is a central $(\alpha(\cdot),p(\cdot))$-atom, while for $k>0, a_k$ is a central $(\alpha(\cdot),p(\cdot))$-block, supported on B_{k+2}. Now it remains to prove that

$$f=\sum_{k=-\infty}^{\infty}\lambda_k a_k,\quad\text{in the sense of }\mathscr{S}'(\mathbb{R}^n).\tag{2.17}$$

For each $\varphi\in\mathscr{S}(\mathbb{R}^n)$, note that $\mathrm{supp}a_k^{(j_v)}\subset\cup_{i=k-1}^{k+1}R_i=\tilde{R}_k$ and

$$\left\langle f,\varphi\right\rangle=\lim_{v\to\infty}\sum_{k=-\infty}^{\infty}\lambda_k\int a_k^{(j_v)}(x)\varphi(x)dx.$$

It will know that, see [59]

$$\left|\int a_k^{(j_v)}(x)\varphi(x)dx\right|\leqslant c2^{k(m+1)}\left\|a_k^{(j_v)}\chi_{\tilde{R}_k}\right\|_1.$$

This term is bounded by

$$\begin{aligned}
& c2^{k(m+1-\alpha(0))}\left\|2^{k\alpha(\cdot)}a_k^{(j_\nu)}\chi_{\tilde{R}_k}\right\|_1 \\
\leqslant\ & c2^{k(m+1-\alpha(0))}\left\|2^{k\alpha(\cdot)/n}a_k^{(j_\nu)}\chi_{\tilde{R}_k}\right\|_{p(\cdot)}\left\|\chi_{\tilde{R}_k}\right\|_{p'(\cdot)} \\
\leqslant\ & c2^{k\left(m+1-\alpha(0)+n-\frac{n}{p^-}\right)}\left\|2^{k\alpha(\cdot)/n}a_k^{(j_\nu)}\chi_{\tilde{R}_k}\right\|_{p(\cdot)}, k\leqslant 0 \\
\leqslant\ & c2^{k\left(m+1-\alpha(0)+n-\frac{n}{p^-}\right)},
\end{aligned}$$

where we have used successively $2^{k\alpha(\cdot)}\approx 2^{k\alpha(0)}$, $k\leqslant 0$, Hölder's inequality and Lemma 2.2. If $k>0$, let $k_0\in\mathbb{N}$ such that $k_0+\alpha_\infty-n+\frac{n}{p_\infty}>0$, then by the fact that $2^{k\alpha(\cdot)}\approx 2^{k\alpha_\infty}$, Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned}
\left|\int a_k^{(j_\nu)}(x)\varphi(x)\,dx\right| &\leqslant c\int_{\tilde{R}_k}\left|a_k^{(j_\nu)}(x)\right||x|^{-k_0}\,dx \\
&\leqslant c2^{-k(k_0+\alpha_\infty)}\left\|2^{k\alpha(\cdot)/n}a_k^{(j_\nu)}\chi_{\tilde{R}_k}\right\|_{p(\cdot)}\left\|\chi_{\tilde{R}_k}\right\|_{p'(\cdot)} \\
&\leqslant c2^{-k\left(k_0+\alpha_\infty-n-\frac{n}{p_\infty}\right)},
\end{aligned}$$

where $c>0$ is independent of k. If we set

$$b_k=\begin{cases}\lambda_k 2^{k\left(m+1-\alpha(0)+n-\frac{n}{p^-}\right)} & \text{if}\quad k\leqslant 0;\\ \lambda_k 2^{-k\left(k_0+\alpha_\infty-n-\frac{n}{p_\infty}\right)} & \text{if}\quad k>0,\end{cases}$$

then

$$\sum_{k=-\infty}^{\infty}|b_k|\leqslant c\Big(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Big)^{1/q(0)}+c\Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\leqslant c\left\|\tilde{G}_N f\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}<\infty.$$

Therefore,

$$\langle f,\varphi\rangle=\sum_{k=-\infty}^{\infty}\lambda_k\int a_k(x)\varphi(x)\,dx.$$

This means that (2.17) holds in the sense of distribution.

Step 2. We prove the sufficiency part of the Theorem 2.24. Let $f=\sum_{l=-\infty}^{\infty}\lambda_l a_l$ where if $l\leqslant 0$, a_l is a central $(\alpha(\cdot),p(\cdot))$-atom, supported on B_l and if $l>0$, a_l is a central $(\alpha(\cdot),p(\cdot))$-block. Take $\varphi\in\mathscr{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$, with $\operatorname{supp}\varphi\subset B_0$. According to Theorem 2.19 it suffices to prove

$$\left\|\tilde{\varphi}^*_+(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} < \infty.$$

By Proposition 2.6, we have

$$\begin{aligned}\left\|\tilde{\varphi}^*_+(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &\approx \left\|\{2^{k\alpha(0)}\tilde{\varphi}^*_+(f)\chi_k\}\right\|_{\ell^{q(0)}_{<}(L^{p(\cdot)})} + \left\|\{2^{k\alpha_\infty}\tilde{\varphi}^*_+(f)\chi_k\}\right\|_{\ell^{q_\infty}_{>}(L^{p(\cdot)})}\\ &\lesssim E_1+E_2+E_3+E_4+E_5,\end{aligned}$$

where

$$E_1 = \left\{\sum_{k=-\infty}^{-1}\left(2^{k\alpha(0)}\sum_{l=-\infty}^{k-2}|\lambda_l|\left\|\tilde{\varphi}^*_+(a_l)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)},$$

$$E_2 = \left\{\sum_{k=-\infty}^{-1}\left(2^{k\alpha(0)}\sum_{l=k-1}^{-1}|\lambda_l|\left\|\tilde{\varphi}^*_+(a_l)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)},$$

$$E_3 = \left\{\sum_{k=-\infty}^{-1}\left(2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|\left\|\tilde{\varphi}^*_+(a_l)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)},$$

$$E_4 = \left\{\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=-\infty}^{-1}|\lambda_l|\left\|\tilde{\varphi}^*_+(a_l)\chi_k\right\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty},$$

and

$$E_5 = \left\{\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=0}^{\infty}|\lambda_l|\left\|\tilde{\varphi}^*_+(a_l)\chi_k\right\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty}.$$

To estimate E_1. Let $m \in \mathbb{N}$ such that $m > \alpha^+ + n(\frac{1}{p^-}-1)-1,$. Denote by P_m the m-order Taylor expansion for φ at x/t. For $x \in R_k$ with $k \geqslant l+2$, by m-order vanishing moment of a_l, we have

$$\begin{aligned}\left|a_l * \varphi_t(x)\right| &= t^{-n}\left|\int_{\mathbb{R}^n} a_l(y)\varphi\left(\frac{x-y}{t}\right)dy\right|\\ &= t^{-n}\left|\int_{\mathbb{R}^n} a_l(y)\left(\varphi\left(\frac{x-y}{t}\right)-P_m\left(\frac{y}{t}\right)\right)dy\right|\\ &\lesssim \int_{\mathbb{R}^n}\left|a_l(y)\right|\left|y\right|^{m+1}\left(t+\left|x-\theta y\right|\right)^{-(n+m+1)}dy\\ &\lesssim 2^{l(m+1)}|x|^{-(n+m+1)}\left\|a_l\right\|_1,\end{aligned}$$

where $0<\theta<1$. This implies that

$$\begin{aligned}2^{k\alpha(0)}\tilde{\varphi}^*_+(a_l)(x) &\lesssim 2^{l(m+1)+k\alpha(0)}|x|^{-(n+m+1)}\left\|a_l\right\|_1\\ &\lesssim 2^{(k-l)\alpha^+}2^{l(m+1)-(n+m+1)}\left\|2^{l\alpha(\cdot)}a_l\right\|_1.\end{aligned}$$

Here we used that fact $2^{l\alpha(0)}\approx 2^{l\alpha(y)}$, $y\in B_l$ and $l\leqslant 0$, since

$$-l\,|\alpha(x)-\alpha(0)|\lesssim \frac{-c\,l}{\log\left(e+\frac{1}{|x|}\right)}\lesssim c,\quad x\in B_l,\ l\leqslant 0.$$

Applying Hölder's inequality and the fact that a_l is a dyadic central $(\alpha(\cdot),p(\cdot))$-atom, we obtain

$$\begin{aligned}2^{k\alpha(0)}\left\|\tilde{\varphi}^*_+(a_l)\chi_k\right\|_{p(\cdot)} &\lesssim 2^{(k-l)\alpha^+}2^{l(m+1)-k(n+m+1)}\left\|\chi_{B_l}\right\|_{p'(\cdot)}\left\|\chi_{B_k}\right\|_{p(\cdot)}\left\|2^{l\alpha(\cdot)}a_l\right\|_{p(\cdot)}\\ &\lesssim 2^{(k-l)\alpha^+}2^{l(m+1)-k(n+m+1)}\left\|\chi_{B_l}\right\|_{p'(\cdot)}\left\|\chi_{B_k}\right\|_{p(\cdot)}\\ &\lesssim 2^{(l-k)\left(m+1+n-\alpha^+-\frac{n}{p^-}\right)},\end{aligned}$$

where in the last estimate we have used Lemma 2.4. Since $m+1+n-\alpha^+-\frac{n}{p^-}>0$, by Lemma 2.3 we obtain that

$$\begin{aligned}E_1 &\lesssim \left\{\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\,2^{(l-k)\left(m+1+n-\alpha^+-\frac{n}{p^-}\right)}\right)^{q(0)}\right\}^{1/q(0)}\\ &\lesssim \left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\right)^{1/q(0)}.\end{aligned}$$

For E_2 and E_3. We use the fact that $\tilde{\varphi}^*_+(a_l)(x)\lesssim \mathcal{M}(a_l)(x)$ and the $L^{p(\cdot)}$ boundedness of $\mathcal{M}$, we get

$$2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|\left\|\tilde{\varphi}^*_+(a_l)\right\|_{p(\cdot)}\lesssim 2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|\left\|a_l\right\|_{p(\cdot)}.$$

This term is bounded by

$$c\,2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|2^{-l\alpha^-}\left\|2^{l\alpha(\cdot)}a_l\right\|_{p(\cdot)}\lesssim 2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|2^{-l\alpha^-}\lesssim 2^{k\alpha(0)}\Big(\sum_{l=0}^{\infty}|\lambda_l|^{q_\infty}\Big)^{1/q_\infty}.$$

This implies that

$$E_3 \lesssim \Big(\sum_{l=0}^{\infty} |\lambda_l|^{q_\infty} \Big)^{1/q_\infty}.$$

Again, be $\tilde{\varphi}_+^*(a_l)(x) \lesssim \mathcal{M}(a_l)(x)$, the $L^{p(\cdot)}$ boundedness of $\mathcal{M}$ and the fact that fact $2^{l\alpha(0)} \approx 2^{l\alpha(y)}$, ($y \in B_l$ and $l \leqslant 0$), it follows that

$$2^{k\alpha(0)} \sum_{l=k-1}^{-1} |\lambda_l| \left\| \tilde{\varphi}_+^*(a_l)\chi_k \right\|_{p(\cdot)} \lesssim \sum_{l=k-1}^{-1} |\lambda_l| 2^{(k-l)\alpha(0)}.$$

So by Lemma 2.3, we get

$$E_2 \lesssim \Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \Big)^{1/q(0)}.$$

For E_4 . By the same argument as E_1, we have

$$2^{k\alpha_\infty} \left\| \tilde{\varphi}_+^*(a_l)\chi_k \right\|_{p(\cdot)} \lesssim 2^{(l-k)\left(m+1+n-\alpha^+-\frac{n}{p^-}\right)}, \quad k \geqslant 0 > l.$$

Hence

$$\begin{aligned} 2^{k\alpha_\infty} \sum_{l=-\infty}^{-1} |\lambda_l| \left\| \tilde{\varphi}_+^*(a_l)\chi_k \right\|_{p(\cdot)} &\lesssim 2^{-k\left(m+1+n-\alpha^+-\frac{n}{p^-}\right)} \sum_{l=-\infty}^{-1} |\lambda_l| 2^{l\left(m+1+n-\alpha^+-\frac{n}{p^-}\right)} \\ &\lesssim 2^{-k\left(m+1+n-\alpha^+-\frac{n}{p^-}\right)} \Big(\sum_{l=-\infty}^{-1} |\lambda_l|^{q(0)} \Big)^{1/q(0)}. \end{aligned}$$

Thus we get

$$E_4 \lesssim \Big(\sum_{l=-\infty}^{-1} |\lambda_l|^{q(0)} \Big)^{1/q(0)}.$$

For E_5. We use the fact that $\tilde{\varphi}_+^*(a_l)(x) \lesssim \mathcal{M}(a_l)(x)$ and the $L^{p(\cdot)}$ boundedness of $\mathcal{M}$, and $\operatorname{supp}\tilde{\varphi}_+^*(a_l) \subset B_{l+1}$, we have

$$\begin{aligned} \sum_{l=0}^{\infty} |\lambda_l| 2^{k\alpha_\infty} \left\| \tilde{\varphi}_+^*(a_l)\chi_k \right\|_{p(\cdot)} &= \sum_{l=k-1}^{\infty} |\lambda_l| 2^{k\alpha_\infty} \left\| \tilde{\varphi}_+^*(a_l)\chi_k \right\|_{p(\cdot)} \\ &\lesssim \sum_{l=k-1}^{\infty} |\lambda_l| 2^{k\alpha_\infty} \left\| a_l \right\|_{p(\cdot)} \lesssim \sum_{l=k-1}^{\infty} |\lambda_l| 2^{(k-l)\alpha_\infty}. \end{aligned}$$

Thus, from Lemma 2.3, it follows that

$$E_5 \lesssim \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty}.$$

This finishes the proof of Theorem 2.24. □

Remark 2.25. *In the necessity part of the Theorems 2.23 and 2.24, the atoms in the decompositions (2.15) and (2.16) can be taken to be supported in dyadic annuli.*

2.5 Pseudo-differential operators on local variable Herz-type Hardy spaces

Recently, the boundedness of pseudo-differential operators on variable function spaces was studied by many authors (see [55, 60, 18]). In this section is to show the boundedness of pseudo-differential operators of order zero on local variable Herz-type Hardy spaces $hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Theorem 2.26. *Let $q \in \mathscr{P}_0(\mathbb{R}^n)$ and $p \in \tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$, and let α and q are* log*-Hölder continuous, at infinity such that $\alpha \in L^\infty(\mathbb{R}^n)$ and $\alpha_\infty \geqslant n\big(1-\frac{1}{p_\infty}\big)$. If*

$$Tf(x) = \int_{\mathbb{R}^n} \hat{f}(x)\sigma(x,\xi)e^{2\pi i x\xi}d\xi,$$

with $\sigma \in S^0$, that is $\sigma \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\big|\partial_x^\gamma \partial_\xi^\beta \sigma(x,\xi)\big| \leqslant c_{\gamma,\beta}(1+|\xi|)^{-|\beta|}$, then

$$\big\|T(f)\big\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c\,\big\|f\big\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Theorem 2.27. *Let $q \in \mathscr{P}_0(\mathbb{R}^n)$ and $p \in \tilde{\mathscr{P}}^{\log}(\mathbb{R}^n)$, and let α and q are* log*-Hölder continuous, both at the origin and at infinity such that $\alpha \in L^\infty(\mathbb{R}^n)$ and $\alpha(\cdot) \geqslant n\big(1-\frac{1}{p^-}\big)$. If T be as in theorem 2.26, then*

$$\big\|T(f)\big\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c\,\big\|f\big\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Proof. By similarity we only consider the homogeneous case. Let $f \in h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, by Theorem 2.24, we have

$$f = \sum_{l=-\infty}^{\infty} \lambda_l a_l, \tag{2.18}$$

in the sense of distributions, where for $l \leqslant 0$, a_l is a central $(\alpha(\cdot), p(\cdot))$-atom, while for $l > 0$, a_l is a central $(\alpha(\cdot), p(\cdot))$-block, with $\operatorname{supp} a_l \subset B_l$ and

$$\|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\left\{\left(\sum_{l=-\infty}^{-1} |\lambda_l|^{q(0)}\right)^{1/q(0)} + \left(\sum_{l=0}^{\infty} |\lambda_l|^{q_\infty}\right)^{1/q_\infty}\right\}.$$

In view of Proposition 2.6, it suffices to estimate

$$\left\|\left\{2^{k\alpha(0)} \tilde{\varphi}_+^*\left(T\left(f\right)\right)\chi_k\right\}\right\|_{\ell_<^{q(0)}(L^{p(\cdot)})} \quad \text{and} \quad \left\|\left\{2^{k\alpha_\infty} \tilde{\varphi}_+^*\left(T\left(f\right)\right)\chi_k\right\}\right\|_{\ell_>^{q_\infty}(L^{p(\cdot)})}.$$

For $k < 0$, we have

$$\left\|\left\{2^{k\alpha(0)} \tilde{\varphi}_+^*\left(T\left(f\right)\right)\chi_k\right\}\right\|_{\ell_<^{q(0)}(L^{p(\cdot)})} \lesssim F_1 + F_2 + F_3,$$

where

$$F_1 = \left(\sum_{k=-\infty}^{-1}\left(2^{k\alpha(0)} \sum_{l=-\infty}^{k-2} |\lambda_l| \left\|\tilde{\varphi}_+^*(Ta_l)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$F_2 = \left(\sum_{k=-\infty}^{-1}\left(2^{k\alpha(0)} \sum_{l=k-1}^{-1} |\lambda_l| \left\|\tilde{\varphi}_+^*(Ta_l)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right)^{\frac{1}{q(0)}},$$

and

$$F_3 = \left(\sum_{k=-\infty}^{-1}\left(2^{k\alpha(0)} \sum_{l=0}^{\infty} |\lambda_l| \left\|\tilde{\varphi}_+^*(Ta_l)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right)^{\frac{1}{q(0)}}.$$

For F_3, by the $L^{p(\cdot)}$-boundedness of $\mathcal{M}$, we get

$$\begin{aligned} 2^{k\alpha(0)} \sum_{l=0}^{\infty} |\lambda_l| \left\|\tilde{\varphi}_+^*(Ta_l)\chi_k\right\|_{p(\cdot)} &\lesssim \sum_{l=0}^{\infty} |\lambda_l| 2^{k\alpha(0)} \left\|\mathcal{M}(Ta_l)\right\|_{p(\cdot)} \\ &\lesssim \sum_{l=0}^{\infty} |\lambda_l| 2^{k\alpha(0)} \left\|Ta_l\right\|_{p(\cdot)} \\ &\lesssim 2^{k\alpha(0)} \sum_{l=0}^{\infty} |\lambda_l| 2^{-l\alpha^-} \left\|2^{l\alpha(\cdot)} a_l\right\|_{p(\cdot)} \\ &\lesssim 2^{k\alpha(0)} \Big(\sum_{l=0}^{\infty} |\lambda_l|^{q_\infty}\Big)^{1/q_\infty}. \end{aligned}$$

Therefore, we conclude that

$$F_3 \lesssim \Big(\sum_{l=0}^{\infty}|\lambda_l|^{q_\infty}\Big)^{1/q_\infty}.$$

For F_2, similar to the estimate of F_3 and applying Lemma 2.3, we obtain

$$F_2 \lesssim \left(\sum_{k=-\infty}^{-1}\left(\sum_{l=k-1}^{-1}|\lambda_l|2^{(k-l)\alpha(0)}\right)^{q(0)}\right)^{\frac{1}{q(0)}} \lesssim \Big(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Big)^{1/q(0)}.$$

For F_1. By Theorem 4 in [29], we can write

$$\varphi_t*(Ta_l)(x)=\int_{\mathbb{R}^n}K_t\big(x;x-y\big)a_l\big(y\big)dy,$$

then we expand $K_t\big(x;x-y\big)$ in a Taylor series about $y=0$. By the vanishing moment of a_l, we get that

$$\varphi_t*(Ta_l)(x)=\sum_{|\beta|=N+1}\int_{\mathbb{R}^n}\partial_y^{\beta}K_t\big(x;x-\theta y\big)y^{\beta}a_l\big(y\big)dy.$$

where $0<\theta<1$ and $N\in\mathbb{N}$ satisfying that $N+1>\alpha^++n(\frac{1}{p^-}-1)$. Noting that $x\in R_k$ with $k\geqslant l+2$, by Theorem 4 in [29], we have

$$\begin{aligned}\big|\varphi_t*(Ta_l)(x)\big| &\lesssim |x|^{-(n+N+1)}\int_{\mathbb{R}^n}\big|y\big|^{N+1}a_l\big(y\big)dy\\ &\lesssim 2^{l(N+1)}2^{-k(n+N+1)}\|a_l\|_1\\ &\lesssim 2^{(k-l)\alpha^+}2^{l(N+1)}2^{-k(n+N+1)}\big\|2^{l\alpha(\cdot)}a_l\big\|_1.\end{aligned}$$

Applying Hölder's inequality and the fact that a_l is a central $(\alpha(\cdot),p(\cdot))$-atom, we obtain

$$\begin{aligned}\big\|2^{k\alpha(0)}\tilde{\varphi}_+^*(Ta_l)\chi_k\big\|_{p(\cdot)} &\lesssim 2^{(k-l)\alpha^+}2^{l(N+1)-k(n+N+1)}\big\|\chi_{B_l}\big\|_{p'(\cdot)}\big\|\chi_{B_k}\big\|_{p(\cdot)}\\ &\lesssim 2^{(k-l)\alpha^+}2^{l(N+1)-k(N+n+1)}\big\|\chi_{B_l}\big\|_{p'(\cdot)}\big\|\chi_{B_k}\big\|_{p(\cdot)}\\ &\lesssim 2^{(l-k)\left(N+1+n-\alpha^+-\frac{n}{p^-}\right)},\end{aligned}$$

where in the last estimate we have used Lemma 2.4. Then, we obtain

$$2^{k\alpha(0)}\sum_{l=-\infty}^{k-2}|\lambda_l|\left\|\tilde{\varphi}_+^*(Ta_l)\chi_k\right\|_{p(\cdot)}\lesssim\sum_{l=-\infty}^{k-2}|\lambda_l|2^{(l-k)\left(N+1+n-\alpha^+-\frac{n}{p^-}\right)}.$$

Therefore, by Lemma 2.3, we have

$$F_1\lesssim\Big(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Big)^{1/q(0)}.$$

For $k\geqslant 0$, we choose a radial smooth function η such that $\operatorname{supp}\eta\subset B(0,1)$ and $\eta=1$ near the origin. We split $T=T_1+T_2$ by decomposing $K(x;y)=K_1(x;y)+K_2(x;y)=\eta K(x;y)+(1-\eta)K(x;y)$. Then T_1 and T_2 are of order zero. We have

$$\begin{aligned}\left\|\left\{2^{k\alpha_\infty}\tilde{\varphi}_+^*(T(f))\chi_k\right\}\right\|_{\ell_>^{q_\infty}(L^{p(\cdot)})}&\lesssim\left(\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=-\infty}^{\infty}|\lambda_l|\left\|\tilde{\varphi}_+^*(T_1a_l)\chi_k\right\|_{p(\cdot)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}}\\&+\left(\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=-\infty}^{-1}|\lambda_l|\left\|\tilde{\varphi}_+^*(T_2a_l)\chi_k\right\|_{p(\cdot)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}}\\&+\left(\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=0}^{k-2}|\lambda_l|\left\|\tilde{\varphi}_+^*(T_2a_l)\chi_k\right\|_{p(\cdot)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}}\\&+\left(\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=k-1}^{\infty}|\lambda_l|\left\|\tilde{\varphi}_+^*(T_2a_l)\chi_k\right\|_{p(\cdot)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}}\\&=H_1+H_2+H_3+H_4.\end{aligned}$$

To estimate H_1, since $\operatorname{supp}\tilde{\varphi}_+^*(T_1a_l)\subset B_{l+1}$ and $L^{p(\cdot)}$ boundedness of $\mathcal{M}$, we deduce that

$$\begin{aligned}\sum_{l=-\infty}^{\infty}|\lambda_l|2^{k\alpha_\infty}\left\|\tilde{\varphi}_+^*(T_1a_l)\chi_k\right\|_{p(\cdot)}&\lesssim\sum_{l=k-1}^{\infty}|\lambda_l|2^{k\alpha_\infty}\left\|\|\mathcal{M}(T_1a_l)\right\|_{p(\cdot)}\\&\lesssim\sum_{l=k-1}^{\infty}|\lambda_l|2^{k\alpha_\infty}\left\|\|T_1a_l\right\|_{p(\cdot)}.\end{aligned}$$

The last term is bounded by

$$c\sum_{l=k-1}^{\infty}|\lambda_l|2^{k\alpha_\infty}\left\|\|a_l\right\|_{p(\cdot)}\lesssim\sum_{l=k-1}^{\infty}|\lambda_l|2^{(k-l)\alpha_\infty}.$$

Therefore, by Lemma 2.3, we obtain

$$H_1 \lesssim \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty}.$$

For H_4, it is obvious that

$$\sum_{l=k-1}^{\infty} |\lambda_l| \left\|\tilde{\varphi}_+^*(T_2 a_l)\chi_k\right\|_{p(\cdot)} \lesssim \sum_{l=k-1}^{\infty} |\lambda_l| 2^{(k-l)\alpha_\infty}.$$

Again by Lemma 2.3, we get

$$H_4 \lesssim \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty}.$$

For H_2, it will know that

$$\left|(K_2)_t\left(x,y\right)\right| \lesssim c_m \left(1+\left|y\right|\right)^{-m}, \tag{2.19}$$

for any $m \geqslant n$ see [23, Theorem 4] . If $x \in R_k$ and $l < 0 < k$, by (2.10) we obtain that

$$\begin{aligned}
\varphi_t * \left(T_2 a_j\right)(x) &= \left|\int_{\mathbb{R}^n} (K_2)_t\left(x, x-y\right) a_l\left(y\right) dy\right| \\
&\lesssim \int_{\mathbb{R}^n} \frac{1}{\left(1+\left|x-y\right|\right)^m} \left|a_l\left(y\right)\right| dy \\
&\lesssim 2^{l(N+1)} |x|^{-(N+n+1)} \int_{\mathbb{R}^n} \left|a_l\left(y\right)\right| dy \\
&\lesssim 2^{l(N+1)} 2^{-k(N+n+1)} \|a_l\|_1 .
\end{aligned}$$

So by the fact that $2^{l\alpha(0)} \approx 2^{l\alpha(y)}$ $\left(y \in B_l \text{ and } l < 0\right)$, we get

$$\begin{aligned}
2^{k\alpha_\infty} \varphi_t * (T_2 a_l)(x) &\lesssim 2^{l(N+1)+k\alpha_\infty} |x|^{-(n+N+1)} \left\|a_l\right\|_1 \\
&\lesssim 2^{(k-l)\alpha^+} 2^{l(N+1)-k(n+N+1)} \left\|2^{l\alpha(\cdot)} a_l\right\|_1, \quad k \geqslant 0 > l.
\end{aligned}$$

Applying Hölder's inequality and the fact that a_l is a dyadic central $(\alpha(\cdot), p(\cdot))$-atom, we obtain the term $\left\|2^{k\alpha_\infty} \varphi_t * (T_2 a_l)\chi_k\right\|_{p(\cdot)}$ is bounded by

$$c 2^{(k-l)\alpha^+} 2^{l(N+1)-k(n+N+1)} \left\|\chi_{B_l}\right\|_{p'(\cdot)} \left\|\chi_{B_k}\right\|_{p(\cdot)} \lesssim 2^{(l-k)\left(N+1+n-\alpha^+-\frac{n}{p^-}\right)},$$

where in the last estimate we have used Lemma 2.4. We take $N \in \mathbb{N}$ satisfying that $N+1 > \alpha^+ + n(\frac{1}{p^-}-1)$, then

$$2^{k\alpha_\infty}\sum_{l=-\infty}^{-1}|\lambda_l|\left\|\tilde{\varphi}_+^*(T_2a_l)\chi_k\right\|_{p(\cdot)} \lesssim c2^{-k\left(N+1+n-\alpha^+-\frac{n}{p^-}\right)}\sum_{l=-\infty}^{-1}|\lambda_l|2^{l\left(N+1+n-\alpha^+-\frac{n}{p^-}\right)}.$$

The last term is bounded by

$$c2^{-k\left(N+1+n-\alpha^+-\frac{n}{p^-}\right)}\Big(\sum_{l=-\infty}^{-1}|\lambda_l|^{q(0)}\Big)^{1/q(0)}.$$

Therefore,

$$H_2 \lesssim \Big(\sum_{l=-\infty}^{-1}|\lambda_l|^{q(0)}\Big)^{1/q(0)}.$$

For H_3, by similar argument in the estimate H_2, we obtain

$$2^{k\alpha_\infty}\sum_{l=0}^{k-2}|\lambda_l|\left\|\tilde{\varphi}_+^*(T_2a_l)\chi_k\right\|_{p(\cdot)} \lesssim \sum_{l=0}^{k-2}2^{(l-k)\left(N+1+n-\alpha^+-\frac{n}{p^-}\right)}.$$

Again by Lemma 2.3, we get

$$H_3 \lesssim \Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}.$$

The proof is complete. □

Remark 2.28. *Corresponding statements to Theorems 2.26 and 2.27, with α, p and q constants, can be found in [41], while with α and q constants Theorems 2.26 and 2.27 are proved in [60], under the assumption that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Here we are requiring the log-Hölder continuity at two points only (zero and infinity).*

CHAPTER 3

BOUNDEDNESS OF SINGULAR INTEGRAL OPERATOR OF CONVOLUTION TYPE ON VARIABLE HERZ-TYPE HARDY SPACES

In recent years, it turned out that atomic and molecular decomposition of some function spaces are extremely useful in many aspects. This concerns, for instance, the investigation of (compact) embeddings between function spaces. But this applies equally to questions of mapping properties of some operators, such as Calderón-Zygmund operators, the commutator of Calderón-Zygmund operator with a BMO function and to trace problems, where arguments can be equivalently transferred to the sequence space, which is often more convenient to handle. In this chapter, we shall give the boundedness of singular integral operators of convolution type on $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. To do this, we establish the molecular decompositions of inhomogeneous Herz-type Hardy spaces.

3.1 Molecular decompositions of $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

The main goal of this section is to prove an molecular decomposition result for variable inhomogeneous Herz-type Hardy spaces $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. First we give the notation of molecule.

Definition 3.1. *Let* $p \in \mathscr{P}(\mathbb{R}^n)$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha \in L^\infty(\mathbb{R}^n)$ *and* $\alpha \geqslant n\big(1 - \frac{1}{p_\infty}\big)$. *Let* $l \in \mathbb{N}_0$,

$$s \geqslant \left\lfloor \alpha_\infty + n\left(\frac{1}{p_\infty} - 1\right) \right\rfloor.$$

Let

$$\varepsilon > \max\Big(\frac{s}{n}, \frac{\alpha_\infty}{n} + \frac{1}{p_\infty} - 1\Big), \quad b_\infty = 1 - \frac{1}{p_\infty} + \varepsilon,$$

and

$$a_\infty = 1 - \frac{1}{p_\infty} - \frac{\alpha_\infty}{n} + \varepsilon.$$

A function $M_l \in L^{p(\cdot)}(\mathbb{R}^n)$ *is said to be a dyadic central* $\big(\alpha(\cdot), p(\cdot); s, \varepsilon\big)_l$*-molecule of restricted type if it satisfies*

(i) $\|M_l\|_{p(\cdot)} \leqslant 1$,

(ii) $\mathscr{R}_{p(\cdot)}(M_l) = \|M_l\|_{p(\cdot)}^{a_\infty/b_\infty} \big\||\cdot|^{nb_\infty} M_l\big\|_{p(\cdot)}^{1-a_\infty/b_\infty} < \infty$,

(iii) $\int_{\mathbb{R}^n} M_l(x) x^\beta dx = 0$ *for any* β *with* $|\beta| \leqslant s$.

In the next statement we shows that the molecular is a generalization of atom.

Lemma 3.2. *Let* $\alpha, p, s, \varepsilon, a_\infty, b_\infty$ *be as in Definition 3.1. Let* $l \in \mathbb{N}_0$. *If* M_l *is central* $\big(\alpha(\cdot), p(\cdot)\big)$*-atom of restricted type supported on* $B(0, 2^l)$*, then* M_l *is a central* $\big(\alpha(\cdot), p(\cdot); s, \varepsilon\big)_l$*-molecule of restricted type.*

Proof. We only need to verify

$$\mathscr{R}_{p(\cdot)}(M_l) = \|M_l\|_{p(\cdot)}^{a_\infty/b_\infty} \big\||\cdot|^{nb_\infty} M_l\big\|_{p(\cdot)}^{1-a_\infty/b_\infty} < \infty.$$

We have

$$\mathscr{R}_{p(\cdot)}(M_l) \leqslant c2^{nb(1-a_\infty/b_\infty)l} \|M_l\|_{p(\cdot)} \leqslant c2^{l(\alpha_\infty-\alpha_\infty)} = c,$$

where the positive constant c is independent of l. □

We immediately arrive at the following result.

Lemma 3.3. *Let* $\alpha, s, \varepsilon, a_\infty, b_\infty, l \in \mathbb{N}_0$ *be as in Definition 3.1, and* $p \in \tilde{\mathscr{P}}^{\log}_\infty(\mathbb{R}^n)$ *with* $1 < p^- \leqslant p^+ < \infty$. *Let* α *and* q *be log-Hölder continuous, both at the origin and at infinity. Then there exists a constant* C *such that for any dyadic central* $\big(\alpha(\cdot), p(\cdot); s, \varepsilon\big)_l$*-molecule of restricted type* M_l,

$$\|M_l\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant C,$$

where $C > 0$ *is independent of* l.

Proof. We follows the idea of [45] and [64]. Let M_l be a dyadic central $\big(\alpha(\cdot), p(\cdot); s, \varepsilon\big)_l$-molecule. Assume that $2^{v_l} < \|M_l\|_{p(\cdot)}^{-\frac{1}{\alpha_\infty}} \leqslant 2^{v_l+1}$, $v_l \geqslant 0$. Recall that

$$B_{v_l} = \left\{x \in \mathbb{R}^n : |x| < 2^{v_l-1}\right\} \quad \text{and} \quad R_{k+v_l} = \left\{x \in \mathbb{R}^n : 2^{k+v_l-1} \leqslant |x| < 2^{k+v_l}\right\}$$

for any $k \in \mathbb{N}_0$. Therefore

$$M_l(x) = \sum_{k=0}^{\infty} M_l(x)\chi_{k+v_l}(x), \quad x \in \mathbb{R}^n,$$

where χ_{k+v_l} and χ_{v_l} are respectively the characteristic function of R_{k+v_l} and B_{v_l}, $k > 0$. Let $M_{l,k} = M_l\chi_{k+v_l}$. We denote by $\mathscr{P}_s$ the class of all real polynomials of degree less than s. Let $P_{R_k}M_{l,k} \in \mathscr{P}_s$ be the unique polynomial satisfying

$$\int_{R_k} \big(M_{l,k}(x) - P_{R_k}M_{l,k}(x)\big)x^\beta = 0, \quad \left|\beta\right| \leqslant s.$$

Let $\mathrm{Q}_{l,k} = \big(P_{R_k}M_{l,k}\big)\chi_{k+v_l}$, $k > 0$.

Step 1. In this step we prove that there is a positive constant C and sequences of numbers $\left\{\lambda_{l,k}\right\}_{k\in\mathbb{N}_0}$ such that

$$\Big(\sum_{k=0}^{\infty}\left|\lambda_{l,k}\right|^{q_\infty}\Big)^{\frac{1}{q_\infty}} < C, \tag{3.1}$$

and

$$M_{l,k} = \lambda_{l,k}a_{l,k} + \mathrm{Q}_{l,k},$$

where each $a_{l,k}$ is a $\big(\alpha(\cdot), p(\cdot)\big)$-atom and the constant C is independent of l. Without loss of generality, assume that $\mathscr{R}_{p(\cdot)}(M_l) = 1$, which leads to

$$\left\| |\cdot|^{nb_\infty} M_l \right\|_{p(\cdot)} = \left\|M_l\right\|_{p(\cdot)}^{-\frac{a_\infty}{b_\infty - a_\infty}} \leqslant c2^{v_l a_\infty n},$$

where $c > 0$ is independent of l. Let $\{\varphi_j^{l,k} : \left|j\right| \leqslant s\} \in \mathscr{P}_s(\mathbb{R}^n)$ be such that

$$\left\langle\varphi_\mu^{l,k}, \varphi_v^{l,k}\right\rangle_{R_{k+v_l}} = \frac{1}{|R_{k+v_l}|}\int_{R_{k+v_l}} \varphi_\mu^{l,k}(x)\varphi_v^{l,k}(x)dx = \delta_{\mu,\nu}.$$

Then

$$Q_{l,k}(x)=\sum_{|\beta|\leqslant s}\langle M_l,\varphi_\beta^{l,k}\rangle_{R_{k+v_l}}\varphi_\beta^{l,k}(x),\quad x\in R_k. \tag{3.2}$$

In addition by Hölder's inequality

$$\begin{aligned}\left|Q_{l,k}(x)\right| &\lesssim \frac{1}{|R_{k+v_l}|}\int_{R_{k+v_l}}\left|M_{l,k}(x)\right|dx\\ &\lesssim \frac{\left\|\chi_{R_{k+v_l}}\right\|_{p'(\cdot)}}{|R_{k+v_l}|}\left\|M_{l,k}\right\|_{p(\cdot)}\\ &\lesssim \frac{\left\|M_{l,k}\right\|_{p(\cdot)}}{\left\|\chi_{R_{k+v_l}}\right\|_{p(\cdot)}},\end{aligned}$$

for any $x\in R_k$, where the last estimate follows by Lemma 2.2. Therefore

$$\begin{aligned}\left\|M_{l,k}-Q_{l,k}\right\|_{p(\cdot)} &\lesssim \left\|M_{l,k}\right\|_{p(\cdot)}\\ &\lesssim \left\||\cdot|^{nb_\infty}M_{l,k}\right\|_{p(\cdot)}2^{-(k+v_l)nb_\infty}\\ &\lesssim 2^{v_l a_\infty n}2^{-(k+v_l)nb_\infty}\\ &=c2^{-nka_\infty}\left|B_{k+v_l}\right|^{-\alpha_\infty/n}.\end{aligned}$$

Consequently $M_{l,k}-Q_{l,k}=\gamma_k a_{l,k}$, where $\gamma_k=c2^{-nka_\infty}$ and $a_{l,k}$ is a central $(\alpha(\cdot),p(\cdot))$-atom of support contained in B_{k+v_l}, $k\geqslant 0$. The constant c is independent of k and l. Let

$$\lambda_k=\begin{cases}\gamma_k, & \text{if} \quad k\geqslant 0,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore we obtain the estimate (3.1).

Step 2. In this step we prove that $\sum_{k=0}^{\infty}Q_{l,k}$ has a $(\alpha(\cdot),p(\cdot))$-atom decomposition. Let $\{\varphi_j^{l,k}:|j|\leqslant s\}\in\mathscr{P}_s(\mathbb{R}^n)$ be the dual basis of $\{x^\gamma:|\gamma|\leqslant s\}$ with respect to the weight $\frac{1}{|R_{k+v_l}|}$ on R_{k+v_l}, that is

$$\langle\varphi_j^{l,k},x^\gamma\rangle=\frac{1}{|R_{k+v_l}|}\int_{R_{k+v_l}}\varphi_j^{l,k}(x)x^\gamma dx=\delta_{j,\gamma}.$$

If set $\varphi_j^{l,k}(x)=\sum_{|v|\leqslant s}\beta_{vj}^{l,k}x^v$ and $\psi_j^{l,k}(x)=\sum_{|v|\leqslant s}\tau_{vj}^{l,k}\varphi_v^{l,k}(x)$, then we have

$$\tau^{l,k}_{vj} = \left\langle \psi^{l,k}_j, \varphi^{l,k}_v \right\rangle = \sum_{|\gamma|\leqslant s} \beta^{l,k}_{v\gamma} \left\langle \varphi^{l,k}_j, x^\gamma \right\rangle = \sum_{|\gamma|\leqslant s} \beta^{l,k}_{v\gamma}\delta_{j\gamma} = \beta^{l,k}_{vj}.$$

So $\varphi^{l,k}_j(x) = \sum\limits_{|v|\leqslant s} \beta^{l,k}_{vj}\varphi^{l,k}_v(x)$. For $x\in R_{k+v_l}$, we have

$$\begin{aligned}\left\langle M_{l,k}, \varphi^{l,k}_j\right\rangle_{R_{k+v_l}} \varphi^{l,k}_j(x) &= \left\langle M_{l,k}, \sum_{|v|\leqslant s}\beta^{l,k}_{vj} x^v\right\rangle_{R_{k+v_l}}\varphi^{l,k}_j(x) \\ &= \sum_{|v|\leqslant s}\left\langle M_{l,k},x^v\right\rangle_{R_{k+v_l}}\beta^{l,k}_{vj}\varphi^{l,k}_j(x),\end{aligned}$$

which together with (3.2) implies that

$$\mathrm{Q}_{l,k}(x) = \sum_{|j|\leqslant s}\left\langle M_{l,k},x^j\right\rangle_{R_{k+v_l}}\varphi^{l,k}_j(x), \quad \text{if} \quad x\in R_{k+v_l}. \tag{3.3}$$

We set $E=\{x\in\mathbb{R}^n : 1\leqslant |x| \leqslant 2\}$, $F=\{x\in\mathbb{R}^n : |x|\leqslant 1\}$, $\left\{e_j : \left|j\right| \leqslant s\right\}\in\mathscr{P}_s(\mathbb{R}^n)$ satisfying

$$\frac{1}{|E|}\int_E e_j(x)x^\gamma dx = \delta_{j,\gamma},$$

and $\left\{\tilde{e}_j : \left|j\right| \leqslant s\right\}\in\mathscr{P}_s(\mathbb{R}^n)$ satisfying

$$\frac{1}{|F|}\int_F \tilde{e}_j(x)x^\gamma dx = \delta_{j,\gamma}.$$

Noting that

$$\delta_{j,\gamma} = \frac{1}{\left|R_{k+v_l}\right|}\int_{R_{k+v_l}} \psi^{l,k}_j(x)x^\gamma dx = \frac{1}{|E|}\int_E 2^{(k+v_l)|\gamma|}\psi^{l,k}_j\left(2^{k+v_l}y\right)y^\gamma dy,$$

we get $e_j\left(y\right) = 2^{k|\gamma|}\psi^{l,k}_j\left(2^k y\right)$. Thus in turn leads to that for $k>0$,

$$\psi^{l,k}_j(x) = 2^{-(k+v_l)|j|}e_j\left(2^{-(k+v_l)}x\right),\ x\in R_{k+v_l}.$$

Similarly, we have

$$\psi_j^{l,k}(y) = 2^{-v_l|j|}\tilde{e}_j\left(2^{-v_l-1}x\right),\ x \in F.$$

Let $C = \sup\limits_{j:|j|\leqslant s}\{\|e_j\|_\infty, \|\tilde{e}_j\|_\infty\}$. Then we have

$$|\psi_j^{l,k}(x)| \leqslant C2^{-(k+v_l)|j|}, \quad \text{for } k \geqslant 0, \tag{3.4}$$

where the constant C is independent of l. Let

$$N_j^{l,k} = \sum_{i=k}^{\infty}|\tilde{R}_{i+v_l}|\langle M_{l,i}, x^j\rangle_{\tilde{R}_{i+v_l}}, \quad k \geqslant 0.$$

First observe that

$$N_j^{l,l} = \sum_{i=0}^{\infty}|\tilde{R}_{i+v_l}|\langle M_{l,i}, x^j\rangle_{\tilde{R}_i} = \sum_{i=0}^{\infty}\int_{\tilde{R}_{i+v_l}} M_l(x)x^j\,dx = \int_{\mathbb{R}^n} M_l(x)x^j\,dx = 0.$$

Now by Hölder's inequality and Lemma 2.2 it follows

$$|N_j^{l,k}| \leqslant \sum_{i=k}^{\infty}\int_{R_{i+v_l}}\left|M_{l,i}(x)x^j\right|dx \leqslant \sum_{i=k}^{\infty}\left\||\cdot|^j M_{l,i}\right\|_{p(.)}\left\|\chi_{R_{i+v_l}}\right\|_{p'(\cdot)}.$$

By Lemma 2.2 we get

$$\left\|\chi_{R_{i+v_l}}\right\|_{p'(\cdot)} \approx |R_{i+v_l}|^{1-\frac{1}{p_\infty}},$$

which yields that

$$\begin{aligned}
|N_j^{l,k}| &\leqslant c2^{v_l(|j|-nb_\infty-\frac{n}{p_\infty}+n)}\sum_{i=k}^{\infty}2^{i(|j|-nb_\infty-\frac{n}{p_\infty}+n)}\left\||\cdot|^{nb_\infty}M_l\right\|_{p(\cdot)} \\
&\lesssim 2^{v_l(|j|-nb_\infty-\frac{n}{p_\infty}+n+a_\infty n)}\sum_{i=k}^{\infty}2^{i(|j|-n\varepsilon)} \\
&\lesssim 2^{v_l(|j|-nb_\infty-\frac{n}{p_\infty}+n+a_\infty n)}2^{k(|j|-n\varepsilon)},
\end{aligned}$$

with the implicit constant not depending on i and l, which yields that

$$|N_j^{l,k}| \lesssim 2^{v_l(|j|-nb_\infty-\frac{n}{p_\infty}+n)}2^{k(|j|-n\varepsilon)}.$$

Consequently, with the help of (3.4),

$$\frac{\left|N_j^{l,k}\psi_j^{l,k}(x)\chi_k(x)\right|}{\left|R_{k+v_l}\right|}\lesssim 2^{v_l(-nb_\infty-\frac{n}{p_\infty}+a_\infty n)}2^{-kn(\varepsilon+1)}, \tag{3.5}$$

which tends to zero if k tends to infinity. Using Abel's transform and (3.5) we obtain

$$\sum_{k=0}^{\infty}\mathrm{Q}_{l,k}(x),$$

can be rewritten as

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{|j|\leqslant s}\langle M_k,x^j\rangle_{\tilde{R}_{k+v_l}}\psi_j^{l,k}(x)\\&=\sum_{|j|\leqslant s}\sum_{k=0}^{\infty}\Big(\sum_{i=0}^{k}|\tilde{R}_{k+v_l}|\langle M_{l,i},x^j\rangle_{\tilde{R}_{k+v_l}}\Big)\Big(\frac{\psi_j^{l,k}(x)\chi_{k+v_l}(x)}{|\tilde{R}_{k+v_l}|}-\frac{\psi_j^{l,k+1}(x)\chi_{k+v_l+1}(x)}{|\tilde{R}_{k+v_l+1}|}\Big)\\&=-\sum_{|j|\leqslant s}\sum_{k=0}^{\infty}N_j^{l,k+1}\Big(\frac{\psi_j^{l,k}(x)\chi_{k+v_l}(x)}{|\tilde{R}_{k+v_l}|}-\frac{\psi_j^{l,k+1}(x)\chi_{k+v_l+1}(x)}{|\tilde{R}_{k+v_l+1}|}\Big).\end{aligned}$$

On the other hand, we have

$$\Big|N_j^{l,k+1}\Big(\frac{\psi_j^{l,k}(x)\chi_{k+v_l}(x)}{|\tilde{R}_{k+v_l}|}-\frac{\psi_j^{l,k+1}(x)\chi_{k+v_l+1}(x)}{|\tilde{R}_{k+v_l+1}|}\Big)\Big|,$$

can be estimated by

$$\begin{aligned}&C|N_j^{l,k+1}||\psi_j^{l,k+1}(x)||\tilde{R}_{k+v_l}|^{-1}\\&\leqslant C2^{v_l(-nb_\infty-\frac{n}{p_\infty}+a_\infty n)}2^{-k(n\varepsilon+n)}\\&\leqslant C2^{-kna_\infty}|\tilde{R}_{k+v_l}|^{-\frac{1}{p_\infty}-\frac{a_\infty}{n}},\end{aligned}$$

for some positive constant c independent of l and k. Let

$$\lambda_{l,k}=\begin{cases}C2^{-kna_\infty}, & \text{if} \quad k\geqslant 0,\\ 0, & \text{otherwise}\end{cases} \tag{3.6}$$

and

$$a^{(j)}_{l,k} = \begin{cases} \frac{(-N^{l,k+1}_j)}{\lambda_{l,k}}\Big(\frac{\psi^{l,k}_j(x)\chi_{k+v_l}(x)}{|\tilde{R}_{k+v_l}|} - \frac{\psi^{l,k+1}_j(x)\chi_{k+v_l+1}(x)}{|\tilde{R}_{k+v_l+1}|}\Big), & \text{if} \quad k \geqslant 0 \text{ and } \left|j\right| \leqslant s, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{k=0}^{\infty} \mathrm{Q}_{l,k}(x) = \sum_{\left|j\right| \leqslant s} \sum_{k=0}^{\infty} \lambda_{l,k} a^{(j)}_{l,k},$$

where $a^{(j)}_{l,k}$ is an $\big(\alpha(\cdot), p(\cdot)\big)$-atom supported on B_{k+v_l}. In addition we obtain the estimate (3.1). □

Now we come to the molecular decomposition theorems.

Theorem 3.4. *Let $\alpha, p, q, s, \varepsilon, a_\infty, b_\infty$ be as in Lemma 3.3. Then $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if f can be represented as*

$$f = \sum_{k=0}^{\infty} \lambda_k M_k,$$

where the series converges in the sense of distributions, $\lambda_k \geqslant 0$, each M_k is a dyadic central $\big(\alpha(.), p(.); s, \varepsilon\big)$-molecule of restricted type, and

$$\Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}} \lesssim \left\|f\right\|_{HK^{\alpha(.)}_{p(.),q(.)}}.$$

Moreover,

$$\left\|f\right\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \approx \inf\Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

where the infimum is taken over all the decompositions of f as above.

Remark 3.5. *Corresponding statement to Theorem 3.4 were proved by Liu and Wang [59], with α and q constants, under the assumption that the maximal operator $\mathscr{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Also Here we are requiring the log-Hölder continuity at infinity.*

3.2 Main result

The target of this section is to show the boundedness of a convolution type singular integral operator on $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. The main theorem of this chapter is the following.

Theorem 3.6. *Let α and q are* log*-Hölder continuous at infinity and $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$ and $0 < q^- \leqslant q^+ < \infty$. Let*

$$Tf(x) = \int_{\mathbb{R}^n} K\left(x; y\right) f\left(y\right) dy, \quad x \notin suppf, \tag{3.8}$$

for any central $(\alpha(\cdot), p(\cdot))$-atom f with the kernel K satisfies

$$\left|k\left(x; y\right) - k\left(x; 0\right)\right| \leqslant c \frac{\left|y\right|^{\delta}}{|x|^{n+\delta}}, \quad |x| \geqslant 2\left|y\right|,$$

where c is a positive constant and $0 < \delta \leqslant 1$ such that

$$n\left(1 - \frac{1}{p_\infty}\right) \leqslant \alpha_\infty < n\left(1 - \frac{1}{p_\infty}\right) + \delta.$$

If T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and satisfying $\int_{\mathbb{R}^n} Tf(x)dx = 0$. Then T is bounded on $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Proof. Let $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, by Theorem 2.11, we have

$$f = \sum_{l=0}^{\infty} \lambda_l a_l,$$

in the sense of distributions, where each a_l is a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type with supp$a_l \subset B_l$ and

$$\left(\sum_{l=0}^{\infty} |\lambda_l|^{q_\infty}\right)^{1/q_\infty} \leqslant c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

By Theorem it suffices to show Tf is a central $\left(\alpha(\cdot), p(\cdot); 0, \varepsilon\right)_l$-molecule of restricted type where $\left(\frac{\alpha_\infty}{n} - 1 + \frac{1}{p_\infty}\right) \leqslant \varepsilon < \frac{\delta}{n} + \frac{1}{p_\infty}$, to be determined later, $a_\infty = 1 - \frac{1}{p_\infty} - \frac{\alpha_\infty}{n} + \varepsilon$, $\quad b_\infty = 1 - \frac{1}{p_\infty} + \varepsilon$. Obviously, we only need to verify the size condition for molecules, that is

$$\mathscr{R}_{p(\cdot)}\left(T\left(f\right)\right) = \left\|T\left(f\right)\right\|_{p(\cdot)}^{a_\infty/b_\infty} \left\||\cdot|^{nb_\infty} T\left(f\right)\right\|_{p(\cdot)}^{1-a_\infty/b_\infty} < \infty.$$

We first estimate $\left\| |.|^{nb} \, T\left(f\right)(.) \right\|_{L^{p(.)}}$. In fact, we have

$$\begin{aligned}\left\| |\cdot|^{nb_\infty} \, T\left(f\right) \right\|_{L^{p(.)}(B(0,2r))} &\lesssim r^{nb_\infty} \left\| T\left(f\right) \right\|_{p(.)} \\ &\lesssim r^{nb_\infty} \left\| f \right\|_{p(.)} \\ &\lesssim r^{nb_\infty - \alpha_\infty}.\end{aligned}$$

On the other hand, for any x with $|x| > 2r$ the vanishing moment of f and the regularity of K, we have

$$\begin{aligned}\left|T\left(f\right)(x)\right| &= \left| \int_{\mathbb{R}^n} K\left(x;y\right) f\left(y\right) dy \right| \\ &= \left| \int_{\mathbb{R}^n} \left(k\left(x;y\right) - k\left(x;0\right)\right) f\left(y\right) dy \right| \\ &\lesssim \int_{\mathbb{R}^n} \frac{\left|y\right|^\delta}{\left|x-y\right|^{n+\delta}} \left|f\left(y\right)\right| dy \\ &\lesssim r^{n+\delta} |x|^{-(n+\delta)} \mathscr{M} f\left(x\right)\end{aligned}$$

and since $nb_\infty - n - \delta < 0$, we have

$$\begin{aligned}\left\| |\cdot|^{nb_\infty} \, T\left(f\right) \right\|_{L^{p(.)}(\mathbb{R}^n/B(0,2r))} &\lesssim r^{n+\delta} \left\| |\cdot|^{nb_\infty - (n+\delta)} \mathscr{M}\left(f\right) \right\|_{L^{p(.)}(\mathbb{R}^n/B(0,2r))} \\ &\lesssim r^{nb_\infty} \left\| \mathscr{M}\left(f\right) \right\|_{p(\cdot)} \\ &\lesssim r^{nb_\infty} \left\| f \right\|_{p(\cdot)} \lesssim r^{nb_\infty - \alpha_\infty}.\end{aligned}$$

Thus, we get

$$\mathscr{R}_{p(\cdot)}\left(T\left(f\right)\right) \lesssim r^{-\alpha_\infty a_\infty / b_\infty} \, r^{(nb_\infty - \alpha_\infty)(1 - a_\infty / b_\infty)} \lesssim 1.$$

This finishes the proof. □

Remark 3.7. *Corresponding statement to Theorem 3.6 , with α, p and q constants, can be found in [45, Theorem 6.2.3], while with α and q constants Theorem 3.6 is proved in [59, Theorem 3.3].*

CHAPTER 4

BESOV-TYPE SPACES WITH VARIABLE SMOOTHNESS AND INTEGRABILITY

In this chapter, we introduce Besov-type spaces with variable smoothness and integrability. We show that these spaces are characterized by the φ-transforms in appropriate sequence spaces. Moreover the Sobolev embeddings for these function spaces are obtained.

4.1 Some properties of Besov-type spaces of fixed exponents

The $B^{s,\tau}_{p,q}$ spaces have been studied extensively in recent years. When $\tau = 0$ they coincide with the usual function $B^{s}_{p,q}$, studied in detail by Triebel in [53]. When $s \in \mathbb{R}, 0 \leqslant \tau < \infty$ and $1 \leqslant p, q \leqslant \infty$ the $B^{s,\tau}_{p,q}$ spaces were first introduced by El Baraka in 2002, see [20, 21]. In these papers, El Baraka investigated embeddings as well as Littlewood-Paley characterization of Campanato spaces. Later Drihem gave in [17] a characterization for $B^{s,\tau}_{p,q}$ spaces by local means and maximal functions. For a complete treatment of $B^{s,\tau}_{p,q}$ spaces we refer the reader the work of Yuan et al. [68].

In this section, we first recall the concept of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ given in [68] and we present some fundamental properties related to these spaces.

Select a pair of Schwartz functions Φ and φ such that

$$\operatorname{supp}\mathscr{F}\Phi \subset \overline{B(0,2)} \quad \text{and} \quad |\mathscr{F}\Phi(\xi)| \geqslant c \quad \text{if} \quad |\xi| \leqslant \frac{5}{3}, \tag{4.1}$$

and

$$\operatorname{supp}\mathscr{F}\varphi \subset \overline{B(0,2)}\backslash B(0,1/2) \quad \text{and} \quad |\mathscr{F}\varphi(\xi)| \geqslant c \quad \text{if} \quad \frac{3}{5} \leqslant |\xi| \leqslant \frac{5}{3}, \tag{4.2}$$

where $c > 0$. We put $\varphi_v = 2^{vn}\varphi(2^v\cdot)$, $v \in \mathbb{N}$.

To begin with, we recall the following definitions of Besov spaces.

Definition 4.1. *Let* $s \in \mathbb{R}, 0 < p, q \leqslant \infty$, *and let* Φ *and* φ *satisfy* (4.1) *and* (4.2), *respectively. The Besov space* $B^s_{p,q}(\mathbb{R}^n)$ *is the collection of all* $f \in \mathscr{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{B^s_{p,q}} = \left(\sum_{v=0}^{\infty} 2^{vsq}\left\|\varphi_v * f\right\|_p^q\right)^{1/q} < \infty,$$

where φ_0 *is replaced by* Φ.

The spaces $B^s_{p,q}(\mathbb{R}^n)$ include many classical spaces such as Lebesgue spaces, Hölder and Sobolev spaces.

The following results, can be found in [53].

Proposition 4.2. *Let* $s \in \mathbb{R}$ *and* $0 < p, q \leqslant \infty$.

(i) *The space* $B^s_{p,q}$ *is quasi-Banach space.*

(ii) *The space* $B^s_{p,q}$ *is monotone with respect to* q, *namely if* $q_0 \leqslant q_1$, *then* $B^s_{p,q_0} \hookrightarrow B^s_{p,q_1}$.

(iii) $\mathscr{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q} \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$.

Definition 4.3. *Let* $s \in \mathbb{R}, 0 \leqslant \tau < \infty$ *and* $0 < p, q \leqslant \infty$. *The Besov-type space* $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ *is the collection of all* $f \in S'(\mathbb{R}^n)$ *such that*

$$\|f\|_{B^{s,\tau}_{p,q}} = \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\left(\sum_{v=v_P^+}^{\infty} 2^{vsq}\left(\int_P \left|\varphi_v * f(x)\right|^p dx\right)^{q/p}\right)^{\frac{1}{q}} < \infty,$$

where φ_0 *is replaced by* Φ.

It is well-known that these spaces do not depend on the choice of the system $(\varphi_v)_{v\in\mathbb{N}_0}$ (up to equivalence of quasi-norms) see,[68, Corollary 2.1] . If we replace dyadic cubes P in Definition 4.3 by arbitrary cubes P, we then obtain equivalent quasi-norms. In particular if $\tau = 0$, the spaces $B^{s,0}_{p,q}$ are just the classical Besov spaces.

We collect a few basic facts.

Proposition 4.4. *Let* $s \in \mathbb{R}, 0 \leqslant \tau < \infty$ *and* $0 < p, q \leqslant \infty$.

(i) *The spaces* $B^{s,\tau}_{p,q}$ *are quasi-Banach spaces.*

(ii) *The spaces* $B^{s,\tau}_{p,q}$ *is monotone with respect to* q, *namely if* $q_0 \leqslant q_1$, *then* $B^{s,\tau}_{p,q_0} \hookrightarrow B^{s,\tau}_{p,q_1}$.

(iii) $\mathscr{S}(\mathbb{R}^n) \hookrightarrow B^{s,\tau}_{p,q} \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$.

These results can be found in [68]. In particular, (i) can be found in Lemma 2.1, (ii) is proved in Proposition 2.1 and (iii) can be found in Proposition 2.1.

The following proposition is given [68, Lemma 2.2].

Proposition 4.5. *Let $s \in \mathbb{R}, 0 < p, q \leqslant \infty$ and $\frac{1}{p} \leqslant \tau < \infty$. A tempered distribution f belongs to $B^{s,\tau}_{p,q}$ if and only if,*

$$\left\| f \right\|^{\#}_{B^{s,\tau}_{p,q}} = \sup_{P \in \mathcal{Q}, |P| \leqslant 1} \frac{1}{|P|^{\tau}} \left(\sum_{v=v_P^+}^{\infty} 2^{vsq} \left(\int_P \left| \varphi_v * f(x) \right|^p dx \right)^{q/p} \right)^{\frac{1}{q}} < \infty.$$

Furthermore, the quasi-norms $\left\| f \right\|_{B^{s,\tau}_{p,q}}$ and $\left\| f \right\|^{\#}_{B^{s,\tau}_{p,q}}$ are equivalent.

The following Theorem implies under some suitable conditions the Besov-type spaces $\mathfrak{B}^{s,\tau}_{p,q}$ are just the Besov spaces $B^{s+n(\tau-\frac{1}{p})}_{\infty,\infty}$, the proof can be found in [65, Theorem 2].

Theorem 4.6. *Let $s \in \mathbb{R}, 0 < p \leqslant \infty$. If $\tau \geqslant \frac{1}{p} > 0, 0 < q \leqslant \infty$ or $\tau = \frac{1}{p}$ and $q = \infty$, then*

$$B^{s,\tau}_{p,q} = B^{s+n(\tau-\frac{1}{p})}_{\infty,\infty}$$

with equivalent quasi-norms.

The following Proposition implies under certain conditions the definition of Besov-type spaces $B^{s,\tau}_{p,q}$ can be simplified, see [51, Proposition 3.1]

Proposition 4.7. *Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < q \leqslant \infty$. If $0 \leqslant \tau < \frac{1}{p}$ or $0 \leqslant \tau \leqslant \frac{1}{p}$ and $q = \infty$, then*

$$\left\| f \right\|^{\blacktriangle}_{B^{s,\tau}_{p,q}} = \sup_{P \in \mathcal{Q}} \frac{1}{|P|^{\tau}} \left(\sum_{v=0}^{\infty} 2^{vsq} \left(\int_P \left| \varphi_v * f(x) \right|^p dx \right)^{q/p} \right)^{\frac{1}{q}} < \infty.$$

is an equivalent quasi-norm in $B^{s,\tau}_{p,q}$.

Further details on the classical theory of these spaces can be found in [11], [65] and [68], see also [13] for recent developments.

In the following section, some important lemmas are provided in order to study the variable Besov-type spaces.

4.2 Technical lemmas

The following lemma is from [36, Lemma 19], see also [8, Lemma 6.1].

Lemma 4.8. *Let $s \in C^{\log}_{\mathrm{loc}}$ and let $R \geqslant c_{\log}(s)$, where $c_{\log}(s)$ is the constant from (1.2) for s. Then*

$$2^{\upsilon s(x)}\eta_{\upsilon,m+R}(x-y) \leqslant c\ 2^{\upsilon s(y)}\eta_{\upsilon,m}(x-y)$$

with $c>0$ independent of $x,y\in\mathbb{R}^n$ and $\upsilon,m\in\mathbb{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$2^{\upsilon s(x)}\eta_{\upsilon,m+R} * f(x) \leqslant c\ \eta_{\upsilon,m} * (2^{\upsilon s(\cdot)}f)(x), \quad x\in\mathbb{R}^n,$$

where $c>0$ is independent of υ and m.

Lemma 4.9. *Let $r,R,N>0$, $m>n$ and $\theta,\omega\in\mathscr{S}(\mathbb{R}^n)$ with $\operatorname{supp}\mathscr{F}\omega\subset\overline{B(0,1)}$. Then there exists $c=c(r,m,n)>0$ such that for all $g\in\mathscr{S}'(\mathbb{R}^n)$, we have*

$$\left|\theta_R*\omega_N*g\,(x)\right| \leqslant c\ A(\eta_{N,m}*\left|\omega_N*g\right|^r(x))^{1/r}, \quad x\in\mathbb{R}^n, \tag{4.3}$$

where $\theta_R=R^n\theta(R\cdot)$, $\omega_N=N^n\omega(N\cdot)$, $\eta_{N,m}=N^n(1+N\,|\cdot|)^{-m}$ and

$$A=\max\left(1,\left(NR^{-1}\right)^m\right).$$

The proof of this lemma is given in [15, Lemma 2.2].

We will make use of the following statement, see [9], Lemma 3.3.

Lemma 4.10. *Let $p\in\mathscr{P}^{\log}$. Then for every $m>0$ there exists $\beta\in(0,1)$ only depending on m and $c_{\log}(p)$ such that*

$$\begin{aligned}&\Big(\frac{\beta}{|Q|}\int_Q\left|f(y)\right|dy\Big)^{p(x)}\\ &\leqslant\frac{1}{|Q|}\int_Q\left|f(y)\right|^{p(y)}dy\\ &+\min(|Q|^m,1)\Big(\frac{1}{|Q|}\int_Q\Big((e+|x|)^{-m}+\left(e+\left|y\right|\right)^{-m}\Big)dy\Big),\end{aligned}$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, all $x \in Q \subset \mathbb{R}^n$ and all $f \in L^{p(\cdot)} + L^\infty$ such that $\left\| f \right\|_{L^{p(\cdot)}+L^\infty} \leqslant 1$.

Let $p \in \mathscr{P}^{\log}$. In the proof of this lemma we need only that

$$\int_Q \left| f(y) \right|^{p(y)} dy \leqslant 1,$$

and/or $\left\| f \right\|_\infty \leqslant 1$. We set

$$\left\| \left(f_v \right)_v \right\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})} = \sup_{P \in \mathcal{Q}} \left\| \left(\frac{f_v}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

where, $v_P = -\log_2 l(P)$ and $v_P^+ = \max(v_P, 0)$.

Lemma 4.11. *Let $0 < a < 1$, $J \in \mathbb{Z}$ and $0 < q \leqslant \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ be a sequence of positive real numbers denote $\delta_k = \sum_{j=J^+}^{k} a^{k-j} \varepsilon_j$, $k \geqslant J^+$. Then there exists constant $c > 0$, depending only on a and q such that*

$$\left(\sum_{k=J^+}^{\infty} \delta_k^q \right)^{1/q} \leqslant c \left(\sum_{k=J^+}^{\infty} \varepsilon_k^q \right)^{1/q}.$$

Lemma 4.12. *Let $s, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- \geqslant 0$ and $p, q \in \mathscr{P}^{\log}_0$ with $0 < q^- \leqslant q^+ < \infty$. Let $(f_k)_{k \in \mathbb{N}_0}$ be a sequence of measurable functions on $\mathbb{R}^n$. For all $v \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, let*

$$g_v(x) = \sum_{k=0}^{\infty} 2^{-|k-v|\delta} f_k(x).$$

Then there exists a positive constant c, independent of $(f_k)_{k \in \mathbb{N}_0}$ such that

$$\left\| (g_v)_v \right\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})} \leqslant c \left\| (f_v)_v \right\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})}, \quad \delta > 0.$$

Proof. We follow the details given in the proof of [15, Lemma 2.10] and [36, Lemma 8] : let $\left(f_v \right)_v \in \ell^{q(\cdot)}\left(L^{\tau(\cdot)}_{p(\cdot)} \right)$, the problem can be reduced to the case when $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a normed space. Then we have

$$\begin{aligned} \left\| \left(\frac{g_v}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} &\leqslant \left\| \left(\sum_{k=0}^{v_P^+} \frac{2^{-|k-v|\delta} f_k}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\quad + \left\| \left(\sum_{k=v_P^+}^{v} \ldots \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} + \left\| \left(\sum_{k=v}^{\infty} \ldots \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \end{aligned} \tag{4.4}$$

The first norm is bounded by

$$\sum_{k=0}^{v_P^+} 2^{\left(k-v_P^+\right)\delta} \left\| \Big(\frac{2^{\left(v_P^+-v\right)\delta} f_k}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Let $Q_{k;h}$ be a dyadic cube such that $P \subset Q_{k;h}$. Obviously $v_{Q_{k;h}}^+ = k$ and since $\frac{|Q_{k;h}|^{\tau(\cdot)}}{|P|^{\tau(\cdot)}} \lesssim 2^{\left(v_P^+-k\right)n\tau^+}$. Then the last sum is bounded by

$$\begin{aligned} &\sum_{k=0}^{v_P^+} 2^{\left(k-v_P^+\right)(\delta-n\tau^+)} \left\| \Big(\frac{f_j}{|Q_{k;h}|^{\tau(\cdot)}} \chi_{Q_{k;h}} \Big)_{j \geqslant v_{Q_{k;h}}^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\leqslant \sum_{k=0}^{v_P^+} 2^{\left(k-v_P^+\right)(\delta-n\tau^+)} \left\| \left(f_v\right)_v \right\|_{\ell^{q(\cdot)}\left(L_{p(\cdot)}^{\tau(\cdot)}\right)} \lesssim \left\| \left(f_v\right)_v \right\|_{\ell^{q(\cdot)}\left(L_{p(\cdot)}^{\tau(\cdot)}\right)}, \end{aligned}$$

Let $\sigma > \max\left(q^+, \frac{q^+}{p^-}\right)$ and $\left\| \left(f_v\right)_v \right\|_{\ell^{q(\cdot)}\left(L_{p(\cdot)}^{\tau(\cdot)}\right)} = 1$. Then

$$\begin{aligned} &\sum_{k=v_P^+}^{\infty} \left\| \left| \frac{\sum_{k=v_P^+}^{v} 2^{(k-v)\delta} f_v}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} = \sum_{k=v_P^+}^{\infty} \left\| \left| \frac{\sum_{k=v_P^+}^{v} 2^{(k-v)\delta} f_v}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)/\sigma} \chi_P \right\|_{\frac{\sigma p(\cdot)}{q(\cdot)}}^{\sigma} \\ &\leqslant \sum_{k=v_P^+}^{\infty} \left(\sum_{k=v_P^+}^{v} 2^{\frac{(k-v)\delta q^-}{\sigma}} \left\| \left| \frac{f_v}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)/\sigma} \chi_P \right\|_{\frac{\sigma p(\cdot)}{q(\cdot)}} \right)^{\sigma} \\ &\lesssim \sum_{k=v_P^+}^{\infty} \left\| \left| \frac{f_v}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1, \end{aligned}$$

by Lemma 4.11 The desired estimate is completed by the scaling argument. Now the last norm in (4.4) is bounded by

$$\begin{aligned} &\left\| \Big(\sum_{i=0}^{\infty} \frac{2^{-i\delta} f_{i+v}}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\leqslant \sum_{i=0}^{\infty} 2^{-i\delta} \left\| \Big(\frac{f_k}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v \geqslant v_P^+ + i} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\leqslant \sum_{i=0}^{\infty} 2^{-i\delta} \left\| \Big(\frac{f_k}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \left\| \left(f_v\right)_v \right\|_{\ell^{q(\cdot)}\left(L_{p(\cdot)}^{\tau(\cdot)}\right)}. \end{aligned}$$

This finishes the proof of Lemma 4.12 $\square$

Let $\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}$ be the collection of functions $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} = \sup \left\| \frac{f\chi_P}{|P|^{\tau(\cdot)}} \right\|_{p(\cdot)} < \infty, \quad p \in \mathscr{P}, \quad \tau : \mathbb{R}^n \to \mathbb{R}^+,$$

where the supremum is taken over all dyadic cubes P with $|P| \geqslant 1$. Notice that

$$\|f\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} \leqslant 1 \Leftrightarrow \sup_{P\in\mathscr{Q},|P|\geqslant 1} \left\| \left| \frac{f}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{p(\cdot)/q(\cdot)} \leqslant 1. \tag{4.5}$$

Recall that $\theta_v = 2^{vn}\theta(2^v\cdot)$, $v \in \mathbb{Z}$.

Lemma 4.13. *Let $v \in \mathbb{Z}$, $\tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p \in \mathscr{P}^{\log}_0$ and $\theta, \omega \in \mathscr{S}(\mathbb{R}^n)$ with* $\operatorname{supp}\mathscr{F}\omega \subset \overline{B(0,1)}$. *For any $f \in \mathscr{S}'(\mathbb{R}^n)$ and any dyadic cube P with $|P| \geqslant 1$, we have*

$$\left\| \frac{\theta_v * \omega_v * f}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leqslant c \|\omega_v * f\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}},$$

such that the right-hand side is finite, where $c > 0$ is independent of v and $l(P)$.

Proof. We claim that

$$2^{-v\frac{n}{r}} |\omega_v * f(x)| \lesssim \|\omega_v * f\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}}, \tag{4.6}$$

for any $x \in \mathbb{R}^n$, any $0 < r < p^-$ and any $v \in \mathbb{N}_0$. Indeed, by Lemma 4.9, we have

$$|\omega_v * f(x)| \leqslant c\ (\eta_{v,m} * |\omega_v * f|^r(x))^{1/r},$$

for any $x \in \mathbb{R}^n$, any $m > n$, $0 < r < p^-$ and any $v \in \mathbb{N}_0$. We write

$$\eta_{v,m} * |\omega_v * f|^r(x) \lesssim \sum_{i=0}^{\infty} 2^{-i(m-n)} M_{B(x,2^{i-v})}(|\omega_v * f|^r),$$

where the implicit constant independent of x and v. Hölder's inequality leads to

$$\begin{aligned} M_{B(x,2^{i-v})}(\left|\omega_v * f\right|^r) &\lesssim 2^{(v-i)n} \|(\omega_v * f)\chi_{B(x,2^{i-v})}\|^r_{p(\cdot)} \|\chi_{B(x,2^{i-v})}\|^r_{h(\cdot)} \\ &\lesssim 2^{(v-i)n+inr\tau^+} \|\omega_v * f\|^r_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} \|\chi_{B(x,2^i)}\|^r_{h(\cdot)}, \end{aligned}$$

where $\frac{1}{r} = \frac{1}{p(\cdot)} + \frac{1}{h(\cdot)}$. Making m large enough (4.6) follows. Let P be any dyadic cube. We use again Lemma 4.9, in the form

$$\left|\theta_v * \omega_v * f(x)\right| \leqslant c\,(\eta_{v,m} * |\omega_v * f|^r(x))^{1/r},$$

where $0 < r < \min(p^-, \frac{(p\tau)^-}{\tau^+})$, $m > n$ large enough and $x \in P$. By the scaling argument, we see that it suffices to prove that

$$\left\|\frac{\theta_v * \omega_v * f}{|P|^{\tau(\cdot)}}\chi_P\right\|_{p(\cdot)} \lesssim 1,$$

for any dyadic cube P, with $l(P) \geqslant 1$, whenever $\left\|\omega_v * f\right\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} \leqslant 1$. Let $Q_v \subset P$ be a cube, with $l(Q_v) = 2^{-v}$ and $x \in Q_v \subset P$. As in Lemma 5.1,

$$\eta_{v,m} * |\omega_v * f|^r(x) \leqslant J^1_v(|\omega_v * f|^r \chi_{3Q_v})(x) + \sum_{k\in\mathbb{Z}^n, \|k\|_\infty \geqslant 2} J^2_{v,k}(|\omega_v * f|^r \chi_{Q^k_v})(x).$$

Thus we obtain

$$\begin{aligned}\left\|\frac{\theta_v * \omega_v * f}{|P|^{\tau(\cdot)}}\chi_P\right\|^r_{p(\cdot)} \lesssim & \left\|\frac{J^1_v(|\omega_v * f|^r \chi_{3Q_v})}{|P|^{r\tau(\cdot)}}\chi_P\right\|_{\frac{p(\cdot)}{r}} \\ & + \sum_{k\in\mathbb{Z}^n, \|k\|_\infty \geqslant 2}\left\|\frac{J^2_{v,k}(|\omega_v * f|^r \chi_{Q^k_v})}{|P|^{r\tau(\cdot)}}\chi_P\right\|_{\frac{p(\cdot)}{r}}. \end{aligned} \tag{4.7}$$

Let us prove that the first norm on the right-hand side is bounded. We have

$$J^1_v(\left|\omega_v * f\right|^r \chi_{3Q_v})(x) \lesssim M_{3Q_v}\left(\left|\omega_v * f\right|^r\right).$$

Let $d > 0$ be such that $\tau^+ < d < \frac{(p\tau)^-}{r}$. We have

$$\frac{M_{3Q_v}\left(\left|\omega_v * f\right|^r\right)}{|P|^{r\tau(\cdot)}} = \left(2^{v\frac{nd}{\tau(\cdot)}}\frac{\left(M_{3Q_v}\left(2^{-vn}|\omega_v * f|^r\right)\right)^{\frac{d}{\tau(\cdot)}}}{|P|^{dr}}\right)^{\frac{\tau(\cdot)}{d}}.$$

By (4.6), Lemma 4.10 and the fact that $2^{-\frac{vnd}{\tau(x)}} \approx 2^{-\frac{vnd}{\tau(y)}}$, $x, y \in 3Q_v$,

$$2^{v\frac{nd}{\tau(x)}}\left(M_{3Q_v}\left(2^{-vn}\left|\omega_v * f\right|^r\right)\right)^{\frac{d}{\tau(x)}} \lesssim M_{3Q_v}\left(\left|\omega_v * f\right|^{\frac{rd}{\tau(\cdot)}}\right) + 2^{\frac{vnd}{\tau^-}}2^{-snv}h(x),$$

for any $s > 0$ large enough and any $x \in Q_v$, where the implicit constant is independent of x and v. Hence

$$\Big\|\Big(\frac{J_v^1(\left|\omega_v * f\right|^r \chi_{3Q_v})}{|P|^{r\tau(\cdot)}}\chi_P\Big)^{\frac{d}{\tau(\cdot)}}\Big\|_{\frac{p(\cdot)\tau(\cdot)}{dr}} \lesssim \Big\|\mathcal{M}\Big(\frac{\left|\omega_v * f\right|^{\frac{dr}{\tau(\cdot)}} \chi_{3Q_v}}{|P|^{rd}}\Big)\Big\|_{\frac{p(\cdot)\tau(\cdot)}{dr}} + c$$
$$\lesssim \Big\|\frac{\left|\omega_v * f\right|^{\frac{dr}{\tau(\cdot)}}}{|P|^{rd}}\chi_{3P}\Big\|_{\frac{p(\cdot)\tau(\cdot)}{dr}} + c,$$

after using the fact that $\mathcal{M} : L^{\frac{p(\cdot)\tau(\cdot)}{rd}} \to L^{\frac{p(\cdot)\tau(\cdot)}{rd}}$ is bounded. The last norm is bounded by 1 if and only if

$$\Big\|\frac{\omega_v * f}{|P|^{\tau(\cdot)}}\chi_{3P}\Big\|_{p(\cdot)} \lesssim 1.$$

Notice that $3P = \cup_{h=1}^{3^n} P_h$, where $\{P_h\}_{h=1}^{3^n}$ are disjoint dyadic cubes with side length $l(P_h) = l(P)$. Therefore $\chi_{3P} = \sum_{h=1}^{3^n} \chi_{P_h}$ and

$$\Big\|\frac{\omega_v * f}{|P|^{\tau(\cdot)}}\chi_{3P}\Big\|_{p(\cdot)} \leqslant c\sum_{h=1}^{3^n}\Big\|\frac{\omega_v * f}{|P_h|^{\tau(\cdot)}}\chi_{P_h}\Big\|_{p(\cdot)}$$
$$\lesssim \big\|\omega_v * f\big\|_{\widetilde{L^{p(\cdot)}_{\tau(\cdot)}}}$$
$$\lesssim 1.$$

Using a combination of the arguments used in the corresponding case of the proof of Lemma 5.1 and those used in the estimate of J_v^1 above, we arrive at the desired estimate.

□

4.3 The spaces $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$

In this section, we present the Fourier analytical definition of Besov-type spaces with variable smoothness and integrability and we prove their basic properties in analogy to the Besov-type spaces with fixed exponents.

Definition 4.14. *Let $s : \mathbb{R}^n \to \mathbb{R}$, $\tau : \mathbb{R}^n \to \mathbb{R}^+$ and $p, q \in \mathcal{P}_0$. Let Φ and φ satisfy (4.1) and (4.2), respectively. The Besov-type space $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{P\in\mathcal{Q}} \Big\| \Big(\frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v\geqslant v_P^+} \Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty, \tag{4.8}$$

where φ_0 *is replaced by* Φ.

One recognizes immediately that if s, τ, p and q are constants, then

$$\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} = B^{s,\tau}_{p,q},$$

in the sense of equivalent quasi-norms. When $q=\infty$ the Besov-type space $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),\infty}$ consist of all distributions $f\in\mathscr{S}'(\mathbb{R}^n)$ such that

$$\sup_{P\in\mathcal{Q},v\geqslant v_P^+} \Big\| \frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big\|_{p(\cdot)} < \infty.$$

Let B_J be any ball of $\mathbb{R}^n$ with radius 2^{-J}, $J\in\mathbb{Z}$. In the definition of the spaces $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ if we replace the dyadic cubes P by the balls B_J, then we obtain equivalent quasi-norms. From these if we replace dyadic cubes P in Definition 4.14 by arbitrary cubes P, we then obtain equivalent quasi-norms.

The Besov space of variable smoothness and integrability $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f\in\mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \Big\| \big(2^{vs(\cdot)}\varphi_v * f\big)_{v\in\mathbb{N}_0} \Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty,$$

which introduced and investigated in [1], see [36] for further results. Taking $s\in\mathbb{R}$ and $q\in(0,\infty)$ as constants we derive the spaces $B^{s}_{p(\cdot),q}$ studied by Xu in [63]. Obviously,

$$\mathfrak{B}^{s,0}_{p(\cdot),q} = B^{s}_{p(\cdot),q},$$

in the sense of equivalent quasi-norms. We refer the reader to the recent paper [69] for further details, historical remarks and more references on embeddings of Besov-type spaces with fixed exponents. We mention that the variable Triebel-Lizorkin version of our spaces introduced on this paper is given in [19]. Variable Besov-Morrey spaces are given in [3], see [28] and [62] for the variable 2-microlocal Besov-Triebel-Lizorkin-type spaces.

Sometimes it is of great service if one can restrict $\sup_{P\in\mathcal{Q}}$ in the definition of $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ to a supremum taken with respect to dyadic cubes with side length $\leqslant 1$.

Lemma 4.15. *Let* $s,\tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- \geqslant 0$ *and* $p,q \in \mathscr{P}^{\log}_0$ *with* $\left(\tau p - 1\right)^- \geqslant 0$ *and* $0 < q^+ < \infty$. *A tempered distribution* f *belongs to* $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ *if and only if,*

$$\left\|f\right\|^{\#}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{P\in\mathscr{Q},|P|\leqslant 1} \left\|\Big(\frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\geqslant v_P}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Furthermore, the quasi-norms $\left\|f\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ *and* $\left\|f\right\|^{\#}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ *are equivalent.*

Proof. We follow the scheme of the proof of [14, Lemma 3.6]. Let P be a dyadic cube such that $|P| = 2^{-Jn}$, for some $-J \in \mathbb{N}$. Let $\left\{Q_m : m = 1,...,2^{-Jn}\right\}$ be the collection of all dyadic cubes with volume 1 and such that $P = \cup_{m=1}^{2^{-Jn}} Q_m$. In view of the definition of $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, it suffices to show that $\|f\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant \|f\|^{\#}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$. By the scaling argument, it suffices to consider the case $\|f\|^{\#}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$ and show that the modular of f on the left-hand side is bounded. In particular, we will show that

$$\sum_{v=v_P^+}^{\infty} \left\| \left|\frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

for any dyadic cube P, with $|P| > 1$. We set

$$\max_{m=1,...,2^{-Jn}} \left\| \left|\frac{2^{vs(\cdot)}\varphi_v * f}{|Q_m|^{\tau(\cdot)}}\right| \chi_{Q_m} \right\|_{\frac{p(\cdot)}{q(\cdot)}} = \max_{m=1,...,2^{-Jn}} \left\| \left|\frac{2^{vs(\cdot)}\varphi_v * f}{|Q_{m_0}|^{\tau(\cdot)}}\right|^{q(\cdot)} \chi_{Q_{m_0}} \right\|_{\frac{p(\cdot)}{q(\cdot)}} = \delta.$$

Thus it remains only to prove that

$$\left\| \left|\frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \delta.$$

This claim can be reformulated as showing that

$$\left\| \delta^{-\frac{1}{q(\cdot)}} \frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leqslant 1,$$

which is equivalent to

$$\int_P \delta^{-\frac{p(x)}{q(x)}} \frac{\left|2^{vs(x)}\varphi_v * f\right|^{p(x)}}{|P|^{\tau(x)p(x)}} dx \leqslant 1.$$

This integral can be rewritten us

$$\begin{aligned} &\sum_{m=1}^{2^{-Jn}} \int_{Q_m} \delta^{-\frac{p(x)}{q(x)}} \frac{\left|2^{vs(x)}\varphi_v * f\right|^{p(x)}}{|P|^{\tau(x)p(x)}} dx \\ &\lesssim \sum_{m=1}^{2^{-Jn}} 2^{Jn} \int_{Q_m} \delta^{-\frac{p(x)}{q(x)}} \frac{\left|2^{vs(x)}\varphi_v * f\right|^{p(x)}}{|Q_m|^{\tau(x)p(x)}} dx \lesssim 1. \end{aligned}$$

where the second inequality is obtained from the estimate $|P|^{\tau(\cdot)p(\cdot)} \geqslant 2^{-Jn}$, the last inequality follows from the fact that

$$\left\| \delta^{-\frac{1}{q(\cdot)}} \frac{2^{vs(\cdot)}\varphi_v * f}{|Q_m|^{\tau(x)}} \chi_{Q_m} \right\|_{p(\cdot)} \leqslant 1,$$

for any $m = 1, ..., 2^{-Jn}$, which follows immediately from the definition of δ. (combined with, if $\|g\|_{p(\cdot)} \leqslant 1$, then $\varrho_{p(\cdot)}(g) \leqslant \|g\|^t_{p(\cdot)}$, where $t \leqslant p^-$, which completes the proof. □

Remark 4.16. *We like to point out that this result with fixed exponents is given in [68, Lemma 2.2].*

The following conclusion implies under some suitable conditions the variable Besov-type spaces $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ are just the Besov spaces $B^{s(\cdot)+n(\tau(\cdot)-\frac{1}{p(\cdot)})}_{\infty,\infty}$, whose proof is similar to that of [14, Theorem 3.8], the details being omitted.

Theorem 4.17. *Let $s, \tau \in C^{\log}_{\text{loc}}$, $\tau^- \geqslant 0$ and $p, q \in \mathscr{P}^{\log}_0$ with $p^+, q^+ < \infty$. If $\left(\tau p - 1\right)^- > 0$ or $\left(\tau p - 1\right)^- \geqslant 0$ and $q = \infty$, then*

$$\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} = B^{s(\cdot)+n(\tau(\cdot)-\frac{1}{p(\cdot)})}_{\infty,\infty},$$

with equivalent quasi-norms.

Proof. We consider only $\left(\tau p - 1\right)^- > 0$. The case $\left(\tau p - 1\right)^- \geqslant 0$ and $q \equiv \infty$ can be proved analogously with the necessary modifications. Since $\left(\tau p - 1\right)^- > 0$, then we use the equivalent norm given in the previous lemma. First let us prove the following estimate

$$\|f\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{B^{s(\cdot),n\left(\tau(\cdot)-1/p(\cdot)\right)}_{\infty,\infty}}$$

for any $f \in B^{s(\cdot)+n(\tau(\cdot)-\frac{1}{p(\cdot)})}_{\infty,\infty}$. Let P be a dyadic cube such that $|P| = 2^{-nv_P}, v_P \in \mathbb{N}_0$. We obtain that

$$\begin{aligned}
&\frac{2^{vs(x)}\left|\varphi_v * f(x)\right|}{|P|^{\tau(x)}} \\
&\leqslant c2^{v\left(s(x)+n\left(\tau(x)-1/p(x)\right)\right)+n(v_P-v)\left(\tau(x)-1/p(x)\right)+nv_P/p(x)}\left|\varphi_v * f(x)\right| \\
&\leqslant c2^{n(v_P-v)\left(\tau(x)-1/p(x)\right)+nv_P/p(x)}\|f\|_{B^{s(\cdot),n\left(\tau(\cdot)-1/p(\cdot)\right)}_{\infty,\infty}}
\end{aligned}$$

for any $x \in P$ and any $v \geqslant v_P$. Then

$$\begin{aligned}
\left\|\left|\frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\right|^{q(\cdot)} \chi_P\right\|_{\frac{p(\cdot)}{q(\cdot)}} &\lesssim \left\|\left|2^{n(v_P-v)\left(\tau(\cdot)-1/p(\cdot)\right)+nv_P/p(\cdot)}\right|^{q(\cdot)} \chi_P\right\|_{\frac{p(\cdot)}{q(\cdot)}} \\
&\lesssim 2^{n(v_P-v)\left(\tau-1/p\right)^{-}q^{-}} \left\|\left|2^{nv_P/p(\cdot)}\right|^{q(\cdot)} \chi_P\right\|_{\frac{p(\cdot)}{q(\cdot)}}.
\end{aligned}$$

The norm on the right-hand side is bounded by 1. To show that, we investigate the corresponding modular:

$$\varrho_{\frac{p(\cdot)}{q(\cdot)}}\left(\left|2^{nv_P/p(\cdot)}\right|^{q(\cdot)} \chi_P\right) = \int_P \left|2^{nv_P/p(x)}\right|^{p(x)} dx = 2^{nv_P}\int_P dx = 1.$$

Hence

$$\sum_{v=v_P^+}^{\infty} \left\|\left|\frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\right|^{q(\cdot)} \chi_P\right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim \sum_{v=v_P^+}^{\infty} 2^{n(v_P-v)\left(\tau-1/p\right)^{-}q^{-}} \lesssim 1,$$

since $\left(\tau p-1\right)^{-} > 0$. We are now going to prove the converse inequality. Let $f \in \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ with $\|f\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$. By Lemma 4.9 we have for any $x \in \mathbb{R}^n, m > n, \delta < p^-$

$$\begin{aligned}
&2^{v\left(s(x)+n\left(\tau(x)-1/p(x)\right)\right)}\left|\varphi_v * f(x)\right| \\
&\lesssim 2^{v\left(s(x)+n\left(\tau(x)-1/p(x)\right)\right)}\left(\eta_{v,m} * \left|\varphi_v * f(x)\right|^{\delta}(x)\right)^{1/\delta} \\
&\lesssim \left\|2^{v(s(x)+n\tau(x))}\varphi_v * f(\cdot)(1+2^v|x-\cdot|)^{-m/2\delta}\right\|_{p(\cdot)} \left\|2^{vn/t(x)}(1+2^v|x-\cdot|)^{-m/2\delta}\right\|_{t(\cdot)},
\end{aligned}$$

by Hölder's inequality, with $\frac{1}{\delta} = \frac{1}{p(\cdot)} + \frac{1}{t(\cdot)}$. The second quasi-norm on the right-hand side is bounded if $m > n\frac{2\delta}{t^-} + c_{\log}(1/t)$ (this is possible since m can be taken large enough). To show that, we investigate the corresponding modular:

$$\begin{aligned}\varrho_{t(\cdot)}\left(2^{vn/t(x)}(1+2^v|x-\cdot|)^{-m/2\delta}\right) &= \int_{\mathbb{R}^n} 2^{vnt(y)/t(x)}\left(1+2^v\left|x-y\right|\right)^{-mt(y)/2\delta} dy \\ &\leqslant 2^{nv}\int_{\mathbb{R}^n}\left(1+2^v\left|x-y\right|\right)^{-\left(m-c_{\log}(1/t)\right)t^-/2\delta} dy \\ &< \infty,\end{aligned}$$

where we used Lemma 4.8. Again by the same lemma the first norm is bounded by

$$\left\|2^{v(s(\cdot)+n\tau(\cdot))}\varphi_v * f(\cdot)(1+2^v|x-\cdot|)^{-h}\right\|_{p(\cdot)},$$

where $h = \frac{m}{2\delta} - c_{\log}(s+n\tau)$. Let now prove that this expression is bounded. We investigate the corresponding modular:

$$\begin{aligned}&\varrho_{p(\cdot)}\left(2^{v(s(\cdot)+n\tau(\cdot))}\varphi_v * f(\cdot)(1+2^v|x-\cdot|)^{-h}\right) \\ &= \int_{\mathbb{R}^n} 2^{v(s(y)+n\tau(y))p(y)}\left|\varphi_v * f(\cdot)\right|^{p(y)}\left(1+2^v\left|x-y\right|\right)^{-hp(y)} dy \\ &= \int_{\left|x-y\right|<2^{-v}}(\ldots)\,dy + \sum_{i=0}^{\infty}\int_{2^{i-v}\leqslant\left|x-y\right|<2^{i-v+1}}(\ldots)\,dy \\ &\leqslant \sum_{i=0}^{\infty} 2^{-ihp^-}\int_{\left|x-y\right|<2^{i-v+1}} 2^{v(s(y)+n\tau(y))p(y)}\left|\varphi_v * f(\cdot)\right|^{p(y)}.\end{aligned} \tag{4.9}$$

Then the right hand side of (4.9) is bounded by

$$\begin{aligned}&\sum_{i=0}^{\infty} 2^{\left(n\tau^+p^+-hp^-\right)i}\int_{B(x,2^{i-v+1})}\frac{2^{vs(y)p(y)}\left|\varphi_v * f(\cdot)\right|^{p(y)}}{|B(x,2^{i-v+1})|^{p(y)\tau(y)}} \\ &\leqslant \sum_{i=0}^{\infty} 2^{\left(n\tau^+p^+-hp^-\right)i}\times\left(\sum_{j=(v-i-1)^+}^{\infty}\left(\int_{B(x,2^{i-v+1})}\frac{2^{js(y)p(y)}\left|\varphi_j * f(\cdot)\right|^{p(y)}}{|B(x,2^{i-v+1})|^{p(y)\tau(y)}}\,dy\right)^{1/(p/q)^-}\right)^{(p/q)^-},\end{aligned}$$

and since $\|f\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$, we obtain

$$\sum_{j=J^+}^{\infty}\left\|\left|\frac{2^{js(\cdot)}\varphi_j*f}{|B_J|^{\tau(\cdot)}}\right|^{q(\cdot)}\chi_{B_J}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\leqslant 1,$$

and

$$\int_{B_J}2^{js(y)p(y)}\frac{\left|\varphi_j*f\right|^{p(y)}}{|B_J|^{\tau(y)p(y)}}dy\leqslant\left\|\left|\frac{2^{js(\cdot)}\varphi_j*f}{|B_J|^{\tau(\cdot)}}\right|^{q(\cdot)}\chi_{B_J}\right\|_{\frac{p(\cdot)}{q(\cdot)}}^{(p/q)^-},$$

for any ball B_J and $j\geqslant J^+$. Therefore,

$$\varrho_{p(\cdot)}\left(2^{v(s(\cdot)+n\tau(\cdot))}\varphi_v*f(\cdot)(1+2^v|x-\cdot|)^{-h}\right)\leqslant c\sum_{i=0}^{\infty}2^{\left(n\tau^+p^+-hp^-\right)i}<\infty,$$

for any $h>n\tau^+p^+/p^-$. The proof is completed by the scaling argument. □

Remark 4.18. From this theorem we obtain

$$2^{v(s(x)+n(\tau(x)-\frac{1}{p(x)}))}|\varphi_v*f(x)|\leqslant c\left\|f\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}\tag{4.10}$$

for any $f\in\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, $x\in\mathbb{R}^n$, $s,\tau\in C^{\log}_{\text{loc}}$, $\tau^-\geqslant 0$ and $p,q\in\mathscr{P}^{\log}_0$, where $c>0$ is independent of v and x.

In the following theorem we have the possibility to define these spaces by replacing $v\geqslant v_P^+$ by $v\in\mathbb{N}_0$ in Definition 4.14, where the main arguments used in its proof rely on [14, Theorem 3.12] and when $\tau=0$, was obtained by Sickel [51].

Theorem 4.19. *Let $s,\tau\in C^{\log}_{\text{loc}}$, $\tau^-\geqslant 0$ and $p,q\in\mathscr{P}^{\log}_0$ with $p^+,q^+<\infty$. If $(\tau p-1)^+<0$ or $(\tau p-1)^+\leqslant 0$ and $q=\infty$, then*

$$\left\|f\right\|^{\blacktriangle}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}=\sup_{P\in\mathcal{Q}}\left\|\left(\frac{2^{s(\cdot)}\varphi_v*f}{|P|^{\tau(\cdot)}}\chi_P\right)_{v\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

is an equivalent quasi-norm in $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. Clerly, it suffices to prove that

$$\left\|f\right\|^{\blacktriangle}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim\left\|f\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.$$

In view of the proof of previous theorem, we have

$$2^{v\left(s(x)+n\left(\tau(\cdot)-1/p(x)\right)\right)}\left|\varphi_v * f(x)\right| \leqslant c,$$

for any $x \in \mathbb{R}^n$. Then for any $0 \leqslant v \leqslant v_p$

$$\begin{aligned}\left\|\left|\frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\right|^{q(\cdot)}\chi_P\right\|_{\frac{p(\cdot)}{q(\cdot)}} &\lesssim \left\|\left|2^{(v_P-v)\left(\tau(\cdot)-1/p(\cdot)\right)+nv_P/p(\cdot)}\right|^{q(\cdot)}\chi_P\right\|_{\frac{p(\cdot)}{q(\cdot)}}\\ &\lesssim 2^{(v_P-v)\left(\tau-1/p\right)^+q^-}\left\|\left|2^{nv_P/p(\cdot)}\right|^{q(\cdot)}\chi_P\right\|_{\frac{p(\cdot)}{q(\cdot)}}\\ &\lesssim 2^{(v_P-v)\left(\tau-1/p\right)^+q^-}.\end{aligned}$$

Therefore,

$$\sum_{v=0}^{v_p}\left\|\left|\frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\right|^{q(\cdot)}\chi_P\right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1.$$

The proof is completed by the scaling argument. The case $q \equiv \infty$ can be easily proved. □

For any $\gamma \in \mathbb{Z}$, we put

$$\left\|f\right\|^*_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{P\in\mathcal{Q}}\left\|\left(\frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\right)_{v\geqslant v_P^+-\gamma}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty,$$

where $\varphi_{-\gamma}$ is replaced by $\Phi_{-\gamma}$.

Lemma 4.20. *Let* $s,\tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p,q \in \mathscr{P}^{\log}_0$ *and* $0 < q^+ < \infty$. *The quasi-norms* $\left\|f\right\|^*_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ *and* $\left\|f\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ *are equivalent with equivalent constants depending on* γ.

Proof. The proof is a straightforward adaptation of [15, Lemma 3.9] and [68, Lemma 2.6]. We will present the proof into two steps. *Step 1.* In this step we prove that

$$\left\|f\right\|^*_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \left\|f\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.$$

We need only to consider the case $\gamma > 0$. By the scaling argument, it suffices to consider the case

$$\left\|f\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant 1, \tag{4.11}$$

and show that the modular of f on the left-hand side is bounded. In particular, we will show that

$$\sum_{v=v_P^+-\gamma}^{\infty} \Big\| \Big| \frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant c,$$

for any dyadic cube P. As in [68, Lemma 2.6], it suffices to prove that for all dyadic cube P with $l(P) \geqslant 1$,

$$I_P = \sum_{v=-\gamma}^{0} \Big\| \Big| \frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant c,$$

and for all dyadic cube P with $l(P) < 1$,

$$J_P = \sum_{v=v_p-\gamma}^{v_p-1} \Big\| \Big| \frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant c.$$

The estimate of I_P, clearly follows from the inequality

$$\Big\| \Big| \frac{\varphi_v * f}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant c,$$

for any $v = -\gamma, ..., 0$ and any dyadic cube P with $l(P) \geqslant 1$. This claim can be reformulated as showing that

$$\Big\| \frac{\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big\|_{p(\cdot)} \leqslant c. \tag{4.12}$$

From (4.1) and (4.2), we find $\omega_v \in \mathcal{S}(\mathbb{R}^n)$, $v = -\gamma, ..., -1$ and $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_v = \omega_v * \Phi, \quad v = -\gamma, ..., -1 \quad \text{and} \quad \varphi = \varphi_0 = \eta_1 * \Phi + \eta_2 * \varphi_1.$$

Therefore

$$\varphi_v * f = \omega_v * \Phi * f \quad \text{for} \quad v = -\gamma, ..., -1,$$

and

$$\varphi_0 * f = \eta_1 * \Phi * f + \eta_2 * \varphi_1 * f.$$

Using Lemma 4.13, (4.5) and (4.11) to estimate the left-hand side of (4.12) by

$$C \big\| \Phi * f \big\|_{\widetilde{L^{p(\cdot)}_{\tau(\cdot)}}} + C \big\| \varphi_1 * f \big\|_{\widetilde{L^{p(\cdot)}_{\tau(\cdot)}}} \leqslant c.$$

To estimate J_P, denote by $P(\gamma)$ the dyadic cube containing P with $l(P(\gamma)) = 2^{\gamma} l(P)$. If $v_P \geqslant \gamma + 1$, applying the fact that $v_{P(\gamma)} = v_P - \gamma$, and $P \subset P(\gamma)$, we then have

$$J_P \leqslant \sum_{v=v_{P(\gamma)}}^{v_p-1} \Big\| \Big| \frac{2^{vs(\cdot)}\varphi_v * f}{|P(\gamma)|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_{P(\gamma)} \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant c.$$

If $1 \leqslant v_P \leqslant \gamma$, we write

$$J_P = J_P^1 + J_P^2,$$

where

$$J_P^1 = \sum_{v=v_P-\gamma}^{-1} \Big\| \Big| \frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}},$$

and

$$J_P^2 = \sum_{v=0}^{v_P-1} \Big\| \Big| \frac{2^{vs(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Let $P(2^{v_P})$ the dyadic cube containing P with $l(P(2^{v_P})) = 2^{v_P} l(P) = 1$. By the fact that

$$\frac{|P(2^{v_P})|^{\tau(\cdot)}}{|P|^{\tau(\cdot)}} \lesssim 2^{n v_P \tau^+} \lesssim c(\gamma),$$

we have

$$J_P^2 \lesssim \sum_{v=v_{P(2^{v_P})}}^{v_p-1} \Big\| \Big| \frac{2^{vs(\cdot)}\varphi_v * f}{|P(2^{v_P})|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_{P(2^{v_P})} \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant c.$$

By the arguments similar to that used in the estimate for I_P , we obtain $J_P^1 \leqslant c$. *Step 2.* We will prove that

$$\|f\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|^{*}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.$$

It suffices to show that

$$\Big\| \Big| \frac{\Phi * f}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant c,$$

for all $P \in \mathcal{Q}$ with $l(P) \geqslant 1$ and all $f \in \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ with

$$\|f\|^{*}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant 1. \tag{4.13}$$

This claim can be reformulated as showing that

$$\Big\| \frac{\Phi * f}{|P|^{\tau(\cdot)}} \chi_P \Big\|_{p(\cdot)} \leqslant c.$$

There exist $\varrho_v \in \mathcal{S}(\mathbb{R}^n)$, $v = -\gamma, ..., 1$, such that

$$\Phi * f = \varrho_{-\gamma} * \Phi_{-\gamma} * f + \sum_{v=1-\gamma}^{1} \varrho_v * \varphi_v * f,$$

see [25, p. 130]. Using Lemma 4.13 we get

$$\big\| \varrho_{-\gamma} * \Phi_{-\gamma} * f \big\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} \lesssim \big\| \Phi_{-\gamma} * f \big\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} \leqslant c,$$

and

$$\big\| \varrho_v * \varphi_v * f \big\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} \lesssim \big\| \varphi_v * f \big\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} \leqslant c, \quad v = 1-\gamma, ..., 1,$$

by using (4.5) and (4.13). The proof is complete. □

4.4 The φ-transform for $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$

Let Φ and φ satisfy, respectively (4.1) and (4.2). By [25, pp. 130–131], there exist functions $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (4.1) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (4.2) such that for all $\xi \in \mathbb{R}^n$

$$\mathcal{F}\widetilde{\Phi}(\xi)\mathcal{F}\Psi(\xi) + \sum_{j=1}^{\infty} \mathcal{F}\widetilde{\varphi}(2^{-j}\xi)\mathcal{F}\psi(2^{-j}\xi) = 1, \tag{4.14}$$

where $\widetilde{\Phi} = \overline{\Phi(-\cdot)}$ and $\widetilde{\varphi} = \overline{\varphi(-\cdot)}$. Furthermore, we have the following identity for all $f \in \mathcal{S}'(\mathbb{R}^n)$; see [25, (12.4)]

$$\begin{aligned} f &= \Psi * \widetilde{\Phi} * f + \sum_{v=1}^{\infty} \psi_v * \widetilde{\varphi}_v * f \\ &= \sum_{m \in \mathbb{Z}^n} \widetilde{\Phi} * f(m)\Psi(\cdot - m) + \sum_{v=1}^{\infty} 2^{-vn} \sum_{m \in \mathbb{Z}^n} \widetilde{\varphi}_v * f(2^{-v}m)\psi_v(\cdot - 2^{-v}m). \end{aligned}$$

Recall that the φ-transform S_φ is defined by setting $(S_\varphi)_{0,m} = \langle f, \Phi_m \rangle$ where $\Phi_m(x) = \Phi(x-m)$ and $(S_\varphi)_{v,m} = \langle f, \varphi_{v,m} \rangle$ where

$\varphi_{v,m}(x) = 2^{v\frac{n}{2}}\varphi(2^v x - m)$ and $v \in \mathbb{N}$. The inverse φ-transform T_ψ is defined by

$$T_{\psi}\lambda = \sum_{m\in\mathbb{Z}^n} \lambda_{0,m}\Psi_m + \sum_{v=1}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\psi_{v,m},$$

where $\lambda = \{\lambda_{v,m}\}_{v\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C}$, see [25].

Definition 4.21. *Let $p, q \in \mathscr{P}_0$, $\tau : \mathbb{R}^n \to \mathbb{R}^+$ and let $s : \mathbb{R}^n \to \mathbb{R}$. Then we define*

$$\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} = \Big\{\lambda = \{\lambda_{v,m}\}_{v\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C} : \big\|\lambda\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} < \infty\Big\},$$

where

$$\big\|\lambda\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{P\in\mathscr{Q}} \Big\|\Big(\frac{\sum_{m\in\mathbb{Z}^n} 2^{v(s(\cdot)+\frac{n}{2})}\lambda_{v,m}\chi_{v,m}}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\geqslant v_P^+}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

If we replace dyadic cubes P by arbitrary balls B_J of $\mathbb{R}^n$ with $J \in \mathbb{Z}$, we then obtain equivalent quasi-norms, where the supremum is taken over all $J \in \mathbb{Z}$ and all balls B_J of $\mathbb{R}^n$.

Lemma 4.22. *Let $s, \tau \in C^{\log}_{\text{loc}}$, $\tau^- \geqslant 0$, $p, q \in \mathscr{P}^{\log}_0$, $0 < q^+ < \infty$, $v \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $x \in Q_{v,m}$ and $\lambda \in \mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$. Then there exists $c > 0$ independent of x, v and m such that*

$$|\lambda_{v,m}| \leqslant c\ 2^{-v(s(x)+\frac{n}{2})}|Q_{v,m}|^{\tau(x)}\big\|\lambda\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}\big\|\chi_{v,m}\big\|^{-1}_{p(\cdot)}.$$

Proof. Let $\lambda \in \mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, $v \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $x \in Q_{v,m}$, with $Q_{v,m} \in \mathscr{Q}$. Then

$$|\lambda_{v,m}|^{p^-} = |Q_{v,m}|^{-1}\int_{Q_{v,m}} |\lambda_{v,m}|^{p^-}\chi_{v,m}(y)dy.$$

Using the fact that $2^{v(s(x)-s(y))} \leqslant c$ and $2^{v(\tau(x)-\tau(y))} \leqslant c$ for any $x, y \in Q_{v,m}$, we obtain

$$\frac{2^{v(s(x)+\frac{n}{2})p^-}}{|Q_{v,m}|^{p^-\tau(x)}}|\lambda_{v,m}|^{p^-} \lesssim |Q_{v,m}|^{-1}\int_{Q_{v,m}} \frac{2^{v(s(y)+\frac{n}{2})p^-}}{|Q_{v,m}|^{p^-\tau(y)}}|\lambda_{v,m}|^{p^-}\chi_{v,m}(y)dy.$$

Applying Hölder's inequality to estimate this expression by

$$\begin{aligned} &c|Q_{v,m}|^{-1}\Big\|\frac{2^{v(s(\cdot)+\frac{n}{2})p^-}}{|Q_{v,m}|^{p^-\tau(\cdot)}}|\lambda_{v,m}|^{p^-}\chi_{v,m}\Big\|_{\frac{p}{p^-}}\big\|\chi_{v,m}\big\|_{(\frac{p}{p^-})'} \\ &\lesssim \big\|\lambda\big\|^{p^-}_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}\big\|\chi_{v,m}\big\|^{-1}_{\frac{p}{p^-}}, \end{aligned}$$

where we have used (1.3). Therefore for any $x \in Q_{v,m}$

$$|\lambda_{v,m}| \lesssim 2^{-v(s(x)+\frac{n}{2})}|Q_{v,m}|^{\tau(x)}\big\|\lambda\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}\big\|\chi_{v,m}\big\|^{-1}_{p(\cdot)},$$

which completes the proof. □

As in [15], and using Lemma 4.22 we obtain the following statement.

Lemma 4.23. *Let $s,\tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- \geqslant 0$, $p,q \in \mathscr{P}^{\log}_0$ and Ψ, $\psi \in \mathscr{S}(\mathbb{R}^n)$ satisfy, respectively, (4.1) and (4.2). Then for all $\lambda \in \mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$*

$$T_\psi\lambda = \sum_{m\in\mathbb{Z}^n}\lambda_{0,m}\Psi_m + \sum_{v=1}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{v,m}\psi_{v,m},$$

converges in $\mathscr{S}'(\mathbb{R}^n)$; moreover, $T_\psi : \mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \to \mathscr{S}'(\mathbb{R}^n)$ is continuous.

For a sequence $\lambda = \{\lambda_{v,m}\}_{v\in\mathbb{N}_0,m\in\mathbb{Z}^n} \subset \mathbb{C}, 0 < r \leqslant \infty$ and a fixed $d > 0$, set

$$\lambda^*_{v,m,r,d} = \Big(\sum_{h\in\mathbb{Z}^n}\frac{|\lambda_{v,h}|^r}{(1+2^v|2^{-v}h-2^{-v}m|)^d}\Big)^{\frac{1}{r}},$$

and $\lambda^*_{r,d} = \{\lambda^*_{v,m,r,d}\}_{v\in\mathbb{N}_0,m\in\mathbb{Z}^n} \subset \mathbb{C}$.

Lemma 4.24. *Let $s,\tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p,q \in \mathscr{P}^{\log}_0$, $0 < q^+ < \infty$ and $0 < r < \frac{(\tau p)^-}{\tau^+}$. Then for d large enough*

$$\big\|\lambda^*_{r,d}\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \approx \big\|\lambda\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Proof. Obviously,

$$\big\|\lambda\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant \big\|\lambda^*_{r,d}\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.$$

We will prove that

$$\big\|\lambda^*_{r,d}\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \big\|\lambda\big\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.$$

For each $k \in \mathbb{N}_0$ define

$$\Omega_k = \{h \in \mathbb{Z}^n : 2^{k-1} < 2^v|2^{-v}h-2^{-v}m| \leqslant 2^k\},$$

and

$$\Omega_0 = \{h \in \mathbb{Z}^n : 2^v|2^{-v}h-2^{-v}m| \leqslant 1\}.$$

Then for any $x \in Q_{v,m} \cap P$,

$$\sum_{h\in\mathbb{Z}^n} \frac{2^{vr(s(x)+\frac{n}{2})}|\lambda_{v,h}|^r}{(1+2^v|2^{-v}h-2^{-v}m|)^d}, \tag{4.15}$$

can be rewritten as

$$\begin{aligned} &\sum_{k=0}^{\infty}\sum_{h\in\Omega_k} \frac{2^{vr(s(x)+\frac{n}{2})}\left|\lambda_{v,h}\right|^r}{(1+2^v\,|2^{-v}h-2^{-v}m|)^d} \\ \lesssim\ & \sum_{k=0}^{\infty} 2^{-dk}\sum_{h\in\Omega_k} 2^{vr(s(x)+\frac{n}{2})}\left|\lambda_{v,h}\right|^r \\ =\ & \sum_{k=0}^{\infty} 2^{(n-d)k+(v-k)n+vr(s(x)+\frac{n}{2})} \int\limits_{\cup_{z\in\Omega_k} Q_{v,z}} \sum_{h\in\Omega_k}\left|\lambda_{v,h}\right|^r \chi_{v,h}(y)dy. \end{aligned} \tag{4.16}$$

Let $x \in Q_{v,m}\cap P$ and $y \in \cup_{z\in\Omega_k} Q_{v,z}$. Then $y \in Q_{v,z}$ for some $z \in \Omega_k$ and $2^{k-1} < 2^v\,|2^{-v}z - 2^{-v}m| \leqslant 2^k$. From this it follows that y is located in the cube $Q(x, 2^{k-v+3})$. Therefore, (4.16) does not exceed

$$\begin{aligned} & c\sum_{k=0}^{\infty} 2^{(n-d+a)k+(v-k)n} \int\limits_{Q(x,2^{k-v+3})} 2^{v(s(y)+\frac{n}{2})r} \sum_{h\in\Omega_k}\left|\lambda_{v,h}\right|^r \chi_{v,h}(y)dy \\ =\ & c\sum_{k=0}^{\infty} 2^{(n-d+a)k} M_{Q(x,2^{k-v+3})}\left(g_v\right), \end{aligned}$$

for some positive constant c independent of v and k and

$$g_v = 2^{v(s(\cdot)+\frac{n}{2})r}\sum_{h\in\mathbb{Z}^n}\left|\lambda_{v,h}\right|^r \chi_{v,h}, \quad v \geqslant v_P^+.$$

Observe that

$$M_{Q(x,2^{k-v+3})}\left(g_v\right) \lesssim 2^{kL}\eta_{v,L} * g_v(x),$$

for any $x \in Q_{v,m}\cap P$ and any $L > n$ large enough, where the implicit constant is independent of x, k and v. Therefore (4.15) is bounded by

$$c\ \eta_{v,L} * g_v(x), \quad x \in Q_{v,m}\cap P.$$

Thanks to Lemma 5.1, we have

$$\left\|\lambda^*_{r,d}\right\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \left\|(\eta_{v,L} * g_v)_v\right\|^{\frac{1}{r}}_{\ell^{\frac{q(\cdot)}{r}}(L^{r\tau(\cdot)}_{\frac{p(\cdot)}{r}})} \lesssim \left\|(g_v)_v\right\|^{\frac{1}{r}}_{\ell^{\frac{q(\cdot)}{r}}(L^{r\tau(\cdot)}_{\frac{p(\cdot)}{r}})} \lesssim \left\|\lambda\right\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}},$$

provided that d is sufficiently large such that $d > n + a + L$. The proof of the lemma is thus complete. $\square$

By this result, Lemma 4.20 and by the same arguments given in [15, Theorem 3.14] we obtain the following statement.

Theorem 4.25. *Let $s, \tau \in C^{\log}_{\text{loc}}$, $\tau^- > 0$, $p, q \in \mathscr{P}^{\log}_0$ and $0 < q^+ < \infty$. Suppose that $\Phi, \Psi \in \mathscr{S}(\mathbb{R}^n)$ satisfying (4.1) and $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$ satisfy (4.2) such that (4.14) holds. The operators $S_\varphi : \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \to \mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ and $T_\psi : \mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \to \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.*

From Theorem 4.25, we obtain the next important property of spaces $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Corollary 4.26. *Let $s, \tau \in C^{\log}_{\text{loc}}$, $\tau^- > 0$, $p, q \in \mathscr{P}^{\log}_0$ and $0 < q^+ < \infty$, The definition of the spaces $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ is independent of the choices of Φ and φ.*

4.5 Embeddings

For the spaces $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ introduced above we want to show some embedding theorems. We begin with the following elementary embeddings.

Theorem 4.27. *Let $s, \tau \in C^{\log}_{\text{loc}}$, $\tau^- > 0$ and $p, q, q_0, q_1 \in \mathscr{P}^{\log}_0$.*
(i) If $q_0 \leqslant q_1$, then

$$\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

(ii) If $(s_0 - s_1)^- > 0$, then

$$\mathfrak{B}^{s_0(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow \mathfrak{B}^{s_1(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

The proof can be obtained by using the same method as in [1, Theorem 6.1]. We next consider embeddings of Sobolev-type. It is well-known that

$$B^{s_0,\tau}_{p_0,q} \hookrightarrow B^{s_1,\tau}_{p_1,q},$$

if $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$, where $0 < p_0 < p_1 \leqslant \infty, 0 \leqslant \tau < \infty$ and $0 < q \leqslant \infty$ (see e.g. [68, Corollary 2.2]). In the following theorem we generalize these embeddings to variable exponent case.

Theorem 4.28. *Let* $s_0, s_1, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$ *and* $p_0, p_1, q \in \mathscr{P}^{\log}_0$ *with* $q^+ < \infty$. *If* $s_0(\cdot) > s_1(\cdot)$ *and* $s_0(\cdot) - \frac{n}{p_0(\cdot)} = s_1(\cdot) - \frac{n}{p_1(\cdot)}$ *with* $\left(\frac{p_0}{p_1}\right)^+ < 1$, *then*

$$\mathfrak{B}^{s_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{s_1(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Proof. Let $f \in \mathfrak{B}^{s_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)}$ with $\left\|f\right\|_{\mathfrak{B}^{s_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)}} \leqslant 1$ and P be any dyadic cube of $\mathbb{R}^n$. *Case 1.* $l(P) > 1$. Let $Q_v \subset P$ be a cube, with $\ell(Q_v) = 2^{-v}$ and $x \in Q_v \subset P$. By Lemma 4.9 we have for any $m > n$, $d > 0$

$$|\varphi_v * f(x)| \leqslant c(\eta_{v,m} * |\varphi_v * f|^d(x))^{\frac{1}{d}}.$$

We have

$$\begin{aligned}
&\eta_{v,m} * |\varphi_v * f|^d(x) \\
&= 2^{vn} \int_{\mathbb{R}^n} \frac{\left|\varphi_v * f(z)\right|^d}{(1 + 2^v |x - z|)^m} dz \\
&= \int_{3Q_v} \frac{\left|\varphi_v * f(z)\right|^d}{(1 + 2^v |x - z|)^m} dz + \sum_{k \in \mathbb{Z}^n, \|k\|_\infty \geqslant 2} \int_{Q_v^k} \frac{\left|\varphi_v * f(z)\right|^d}{(1 + 2^v |x - z|)^m} dz,
\end{aligned}$$

where $Q_v^k = Q_v + kl(Q_v)$. Let $z \in Q_v^k$ with $k \in \mathbb{Z}^n$ and $|k| > 4\sqrt{n}$. Then $|x - z| \geqslant |k| 2^{-v-1}$ and the second integral is bounded by

$$c \ |k|^{-m} M_{Q_v^k}\left(|\varphi_v * f|^d\right),$$

where the positive constant c independent of x, k and v. Fix $r, d > 0$ such that

$$0 < r < \min\left(1, (p_0\tau)^-\right) \quad \text{and} \quad 0 < d < \min\left(\frac{p_1^-}{2r}, \frac{q^-}{2r}, \frac{r}{\tau^+}\right).$$

Then

$$0 < r < \frac{1}{2} \min\left(\frac{p_1^-}{d}, \frac{q^-}{d}, 2\right).$$

We have

$$\Big\|\Big(\frac{2^{v s_1(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}^{rd},$$

can be estimated by

$$\lesssim \Big\|\Big(\frac{2^{v s_1(\cdot)}}{|P|^{\tau(\cdot)}}(M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}}\chi_P\Big)_{v\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}^{rd} \tag{4.17}$$

$$+ \sum_{k\in\mathbb{Z}^n, \|k\|_\infty\geqslant 2} |k|^{\sigma rd}$$

$$\times \Big\|\Big(\frac{2^{v s_1(\cdot)}|k|^{b(\cdot)}}{|P|^{\tau(\cdot)}}(M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}}\chi_P\Big)_{v\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}^{rd}, \tag{4.18}$$

where

$$b(\cdot) = -\frac{\theta n\tau(\cdot)}{dr} - 2c_{\log}(\frac{1}{q\tau})\tau(\cdot) - c_{\log}\big(s_1 - \frac{n}{p_1}\big) - \frac{n\tau(\cdot)}{rt^-} - \kappa, \quad \kappa > 0,$$

and $\sigma = (-b)^+ - \frac{m}{d}$ with $\frac{r}{p_0(\cdot)\tau(\cdot)} + \frac{1}{t(\cdot)} = 1$, where θ will be chosen later. *Estimate of* (4.17). We will prove that (4.17), with power $\frac{1}{rd}$, is bounded by

$$c\Big\|\Big(\frac{2^{v s_1(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_{3P}\Big)_{v\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \lesssim \|f\|_{\mathfrak{B}_{p_0(\cdot),q(\cdot)}^{s_0(\cdot),\tau(\cdot)}}. \tag{4.19}$$

By the scaling argument, we see that it suffices to consider the case when the left-hand side of (4.19) is less than or equal 1. Therefore we will prove that

$$\sum_{v=0}^{\infty}\Big\|\Big|\frac{c\ 2^{v s_1(\cdot)}}{|P|^{\tau(\cdot)}}(M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}}\chi_P\Big|^{q(\cdot)}\Big\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leqslant 1,$$

for some positive constant $c > 0$. This clearly follows from the inequality

$$\Big\|\Big|\frac{c\ 2^{v s_1(\cdot)}}{|P|^{\tau(\cdot)}}(M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}}\chi_P\Big|^{q(\cdot)}\Big\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leqslant \Big\|\Big|\frac{2^{v s_0(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\chi_{3P}\Big\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-v} = \delta.$$

This claim can be reformulated as showing that

$$\Big\|\Big|\frac{c\ \delta^{-\frac{1}{q(\cdot)}}2^{v s_1(\cdot)}}{|P|^{\tau(\cdot)}}(M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}}\Big|^{q(\cdot)}\chi_P\Big\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leqslant 1,$$

which is equivalent to

$$\int_P \frac{\delta^{-\frac{p_1(x)}{q(x)}} 2^{\nu s_1(x) p_1(x)}}{|P|^{\tau(x) p_1(x)}} \big(M_{3Q_\nu}(|\varphi_\nu * f|^d)\big)^{\frac{p_1(x)}{d}} \, dx \lesssim 1. \tag{4.20}$$

Since s_1 and p_1 are log-Hölder continuous, we can move $2^{\nu(s_1(x)-\frac{n}{p_1(x)})}$ inside the integral by Lemma 4.8:

$$\delta^{-\frac{1}{q(x)}} \frac{2^{\nu(s_1(x)-\frac{n}{p_1(x)})}}{|P|^{\tau(x)}} (M_{3Q_\nu}(|\varphi_\nu * f|^d))^{\frac{1}{d}} \lesssim \frac{\delta^{-\frac{1}{q(x)}}}{|P|^{\tau(x)}} \Big(M_{3Q_\nu}\big(2^{\nu(s_1(\cdot)-\frac{n}{p_1(\cdot)})d} |\varphi_\nu * f|^d\big)\Big)^{\frac{1}{d}}, \tag{4.21}$$

for any $x \in Q_\nu \subset P$. Observe that $0 < d < \frac{r}{\tau+}$. The right-hand side of (4.21) can be rewritten us

$$\Big(\frac{1}{|P|^r} \Big(\delta^{-\frac{d}{q(x)}} M_{3Q_\nu}\big(2^{\nu(s_1(\cdot)-\frac{n}{p_1(\cdot)})d} |\varphi_\nu * f|^d\big)\Big)^{\frac{r}{d\tau(x)}}\Big)^{\frac{\tau(x)}{r}}. \tag{4.22}$$

By Lemma 4.10, Remark 4.18 and since $\frac{1}{q}$ and τ are log-Hölder continuous,

$$\delta^{-\frac{r}{q(x)\tau(x)}} \Big(\frac{1}{|3Q_\nu|} \int_{3Q_\nu} 2^{\nu(s_1(y)-\frac{n}{p_1(y)})d} |\varphi_\nu * f(y)|^d \, dy\Big)^{\frac{r}{d\tau(x)}},$$

can be estimated by

$$\begin{aligned} &\frac{c}{|3Q_\nu|} \int_{3Q_\nu} \delta^{-\frac{r}{q(y)\tau(y)}} 2^{\frac{\nu(s_1(y)-\frac{n}{p_1(y)})r}{\tau(y)}} |\varphi_\nu * f(y)|^{\frac{r}{\tau(y)}} \, dy + |Q_\nu|^\theta g(x) \\ &\lesssim \int_{3Q_\nu} 2^{\nu n} \delta^{-\frac{r}{q(y)\tau(y)}} 2^{\frac{\nu(s_1(y)-\frac{n}{p_1(y)})r}{\tau(y)}} |\varphi_\nu * f(y)|^{\frac{r}{\tau(y)}} \, dy + h(x) \end{aligned}$$

for any $\theta > 0$ large enough, where

$$g(x) = (e + |x|)^{-\theta} + M_{3Q_\nu}((e + |\cdot|)^{-\theta}), \quad x \in \mathbb{R}^n, \theta > 0,$$

and

$$h(x) = (e + |x|)^{-\theta} + \mathcal{M}((e + |\cdot|)^{-\theta})(x), \quad x \in \mathbb{R}^n, \theta > 0.$$

These two functions will be used throughout in the squeal. Therefore (4.22), with power

$\frac{1}{\tau(x)}$, is bounded by

$$c\left\|\frac{\delta^{-\frac{r}{q(\cdot)\tau(\cdot)}}2^{v\frac{s_0(\cdot)r}{\tau(\cdot)}}|\varphi_v*f|^{\frac{r}{\tau(\cdot)}}}{|P|^r}\chi_{3P}\right\|_{\frac{p_0(\cdot)\tau(\cdot)}{r}}^{\frac{1}{r}}\left\|2^{v\frac{n}{t(\cdot)}}\chi_{3Q_v}\right\|_{t(\cdot)}^{\frac{1}{r}}+c,$$

by Hölder's inequality, with $\frac{r}{p_0(\cdot)\tau(\cdot)}+\frac{1}{t(\cdot)}=1$. The second norm is bounded and the first norm is bounded if and only if

$$\left\|\frac{\delta^{-\frac{1}{q(\cdot)}}2^{vs_0(\cdot)}\left|\varphi_v*f\right|\chi_{3P}}{|P|^{\tau(\cdot)}}\right\|_{p_0(\cdot)}\lesssim 1,$$

which follows immediately from the definition of δ. Now, we find that the left-hand side of (4.20) can be rewritten as

$$\begin{aligned}&\int_P\Big(\frac{\delta^{-\frac{1}{q(x)}}2^{v(s_1(x)-\frac{n}{p_1(x)})}}{|P|^{\tau(x)}}(M_{3Q_v}(|\varphi_v*f|^d))^{\frac{1}{d}}\Big)^{p_1(x)-p_0(x)}\\&\times\Big(\frac{\delta^{-\frac{1}{q(x)}}2^{v(s_1(x)+\frac{n}{p_0(x)}-\frac{n}{p_1(x)})}}{|P|^{\tau(x)}}(M_{3Q_v}(|\varphi_v*f|^d))^{\frac{1}{d}}\Big)^{p_0(x)}dx\\&\lesssim\int_P\frac{1}{|P|^{\tau(x)p_0(x)}}\Big(\delta^{-\frac{d}{q(x)}}M_{3Q_v}(2^{vs_0(\cdot)d}|\varphi_v*f|^d)\Big)^{\frac{p_0(x)}{d}}dx.\end{aligned}$$

The last expression is bounded if and only if

$$\left\|\frac{1}{|P|^r}\Big(\delta^{-\frac{d}{q(\cdot)}}M_{3Q_v}(2^{vs_0(\cdot)d}|\varphi_v*f|^d)\chi_P\Big)^{\frac{r}{d\tau(\cdot)}}\right\|_{\frac{p_0(\cdot)\tau(\cdot)}{r}}\lesssim 1.$$

This norm is bounded by

$$\left\|\mathscr{M}\Big(\frac{\delta^{-\frac{r}{q(\cdot)\tau(\cdot)}}2^{\frac{vs_0(\cdot)r}{\tau(\cdot)}}|\varphi_v*f|^{\frac{r}{\tau(\cdot)}}}{|P|^r}\chi_{3P}\Big)\right\|_{\frac{p_0(\cdot)\tau(\cdot)}{r}}+c,$$

where we have used again Lemma 4.10 and Remark 4.18. Since the maximal function is bounded in $L^{p(\cdot)}$ when $p\in\mathscr{P}^{\log}$ and $p^->1$, this expression is bounded by

$$\left\|\frac{\delta^{-\frac{1}{q(\cdot)\tau(\cdot)}}2^{\frac{vs_0(\cdot)}{\tau(\cdot)}}|\varphi_v*f|^{\frac{1}{\tau(\cdot)}}\chi_{3P}}{|P|}\right\|_{p_0(\cdot)\tau(\cdot)}^r+c.$$

The last quasi-norm is bounded if and only if

$$\left\| \frac{\delta^{-\frac{1}{q(\cdot)}} 2^{v s_0(\cdot)} |\varphi_v * f| \chi_{3P}}{|P|^{\tau(\cdot)}} \right\|_{p_0(\cdot)} \lesssim 1.$$

due to the choice of δ. *Estimate of* (4.18). The summation in (4.18) can be rewritten as

$$\sum_{k \in \mathbb{Z}^n, |k| \leqslant 4\sqrt{n}} \cdots + \sum_{k \in \mathbb{Z}^n, |k| > 4\sqrt{n}} \cdots.$$

The estimation of the first sum follows in the same manner as before. Let us prove that

$$\left\| \left(\frac{2^{v s_1(\cdot)} |k|^{b(\cdot)} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}}}{|\tilde{Q}^k|^{\tau(\cdot)}} \chi_P \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \lesssim 1, \tag{4.23}$$

for any $k \in \mathbb{Z}^n$ with $|k| > 4\sqrt{n}$, where $\tilde{Q}^k = Q(c_P, 2|k|\, l(P))$. Therefore we will prove that

$$\sum_{v=0}^{\infty} \left\| \left| \frac{2^{v s_1(\cdot)} |k|^{b(\cdot) - n\tau(\cdot)}}{|P|^{\tau(\cdot)}} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}} \chi_P \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \lesssim 1.$$

This clearly follows from the inequality

$$\begin{aligned} &\left\| \left| \frac{c\ 2^{v s_1(\cdot)} |k|^{b(\cdot) - n\tau(\cdot)}}{|P|^{\tau(\cdot)}} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}} \chi_P \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \\ &\leqslant \left\| \left| \frac{2^{v s_0(\cdot)} \varphi_v * f}{|\tilde{Q}^k|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{\tilde{Q}^k} \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-v} \\ &= \delta, \end{aligned}$$

for some positive constant c. This claim can be reformulated as showing that

$$\int_P \frac{\delta^{-\frac{p_1(x)}{q(x)}} 2^{v s_1(x) p_1(x)} |k|^{(b(x) - n\tau(x)) p_1(x)}}{|P|^{\tau(x) p_1(x)}} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{p_1(x)}{d}} \, dx \lesssim 1.$$

Since, again, s_1 and p_1 are log-Hölder continuous, we can move $2^{v(s_1(x) - \frac{n}{p_1(x)})}$ inside the integral by Lemma 4.8:

$$\begin{aligned}
&\frac{|k|^{-c_{\log}(s_1-\frac{n}{p_1})-\frac{n}{d}}\,2^{v(s_1(x)-\frac{n}{p_1(x)})}}{|P|^{\tau(x)}}(M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}}\\
&\lesssim \frac{1}{|P|^{\tau(x)}}\Big(M_{Q_v^k}\big(|k|^{-n}\,2^{v(s_1(\cdot)-\frac{n}{p_1(\cdot)})d}|\varphi_v * f|^d\big)\Big)^{\frac{1}{d}},
\end{aligned}$$

where the implicit constant is independent of x, v and k. We have

$$\begin{aligned}
&\frac{|k|^{b(\cdot)-n\tau(\cdot)}\,\delta^{-\frac{1}{q(\cdot)}}}{|P|^{\tau(\cdot)}}\Big(M_{Q_v^k}\big(\frac{|\varphi_v * f|^d}{|k|^n\,2^{-v(s_1(\cdot)-\frac{n}{p_1(\cdot)})d}}\big)\Big)^{\frac{1}{d}}\\
&=\Big(\frac{|k|^{(b(\cdot)-n\tau(\cdot))\frac{r}{\tau(\cdot)}}\,\delta^{-\frac{r}{q(\cdot)\tau(\cdot)}}}{|P|^r}\big(M_{Q_v^k}\big(\frac{|\varphi_v * f|^d}{|k|^n\,2^{-v(s_1(\cdot)-\frac{n}{p_1(\cdot)})d}}\big)\big)^{\frac{r}{d\tau(\cdot)}}\Big)^{\frac{\tau(\cdot)}{r}}.
\end{aligned}\tag{4.24}$$

As before, let us prove that this expression, with power $\frac{1}{\tau(\cdot)}$ is bounded. Observe that $Q_v^k \subset Q(x,|k|2^{1-v}) = \tilde{Q}_v^k$. We have

$$\delta^{-\frac{1}{q(x)\tau(x)}} = (2^v\delta)^{-\frac{1}{q(x)\tau(x)}+\frac{1}{q(y)\tau(y)}}(2^v\delta)^{-\frac{1}{q(y)\tau(y)}}\,2^{v\frac{1}{q(x)\tau(x)}}, \quad x \in Q_v \subset P, y \in \tilde{Q}_v^k.$$

From Lemma 4.8 it follows that

$$2^{v\frac{1}{q(x)\tau(x)}} \lesssim |k|^{c_{\log}(\frac{1}{q\tau})}2^{v\frac{1}{q(y)\tau(y)}},$$

and

$$(2^v\delta)^{-\frac{1}{q(x)\tau(x)}+\frac{1}{q(y)\tau(y)}} \lesssim |k|^{c_{\log}(\frac{1}{q\tau})},$$

for any $x \in Q_v$, $y \in \tilde{Q}_v^k$, where the implicit constant is independent of x, y, k and v. Again by Lemma 4.10 combined with Remark 4.18 and since $\frac{1}{q}$ and τ are log-Hölder continuous,

$$\begin{aligned}
&|k|^{\left(-\frac{\theta n\tau(x)}{dr}-2c_{\log}(\frac{1}{q\tau})\tau(x)-\kappa\right)\frac{r}{\tau(x)}-\frac{n}{t^-}}\,\delta^{-\frac{r}{q(x)\tau(x)}}\\
&\times\Big(\frac{1}{|\tilde{Q}_v^k|}\int_{\tilde{Q}_v^k}2^{v(s_1(y)-\frac{n}{p_1(y)})d}|\varphi_v*f(y)|^d\,dy\Big)^{\frac{r}{d\tau(x)}}\\
&\lesssim\frac{|k|^{-\frac{n}{t^-}}}{|\tilde{Q}_v^k|}\int_{\tilde{Q}_v^k}\delta^{-\frac{r}{q(y)\tau(y)}}2^{\frac{vr(s_1(y)-\frac{n}{p_1(y)})}{\tau(y)}}|\varphi_v*f(y)|^{\frac{r}{\tau(y)}}\,dy\\
&+(e+|x|)^{-\theta}+\frac{1}{|\tilde{Q}_v^k|}\int_{\tilde{Q}_v^k}\left(e+\left|y\right|\right)^{-\theta}dy\\
&\lesssim\frac{|k|^{-\frac{n}{t^-}}}{|\tilde{Q}_v^k|}\int_{\tilde{Q}_v^k}\delta^{-\frac{r}{q(y)\tau(y)}}2^{\frac{v(s_1(y)-\frac{n}{p_1(y)})r}{\tau(y)}}|\varphi_v*f(y)|^{\frac{r}{\tau(y)}}\,dy+h(x),
\end{aligned}$$

for any θ large enough with $\frac{r}{p_0(\cdot)\tau(\cdot)}+\frac{1}{t(\cdot)}=1$. Therefore the left-hand side of (4.24), with power $\frac{1}{\tau(\cdot)}$, is bounded by

$$c\left\|\frac{|k|^{-nr}\,\delta^{-\frac{r}{q(\cdot)\tau(\cdot)}}2^{\frac{vs_0(\cdot)r}{\tau(\cdot)}}|\varphi_v*f|^{\frac{r}{\tau(\cdot)}}}{|P|^r}\chi_{\tilde{Q}^k}\right\|^{\frac{1}{r}}_{\frac{p_0(\cdot)\tau(\cdot)}{r}}\left\|2^{v\frac{n}{t(\cdot)}}\chi_{\tilde{Q}_v^k}\right\|^{\frac{1}{r}}_{t(\cdot)}|k|^{-\frac{n}{rt^-}}+c,$$

by Hölder's inequality. As before the second term is bounded and the first norm is bounded if and only if

$$\int_{\tilde{Q}^k}\frac{\delta^{-\frac{p_0(y)}{q(y)}}2^{vs_0(y)p_0(y)}|\varphi_v*f\left(y\right)|^{p_0(y)}}{|\tilde{Q}^k|^{p_0(y)\tau(y)}}dy\lesssim 1,$$

which follows immediately from the definition of δ. The desired estimate, follows using similar arguments as above, with suitable choice of κ, and by taking m large enough. *Case 2.* $l(P)\leqslant 1$. Since τ is log-Hölder continuous, we have

$$|P|^{-\tau(x)}\leqslant c\,|P|^{-\tau(y)}(1+2^{v_P}\left|x-y\right|)^{c_{\log}(\tau)}\leqslant c\,|P|^{-\tau(y)}(1+2^{v}\left|x-y\right|)^{c_{\log}(\tau)},$$

for any $x,y\in\mathbb{R}^n$ and any $v\geqslant v_P$. Therefore,

$$\frac{1}{|P|^{\tau(\cdot)d}}\eta_{v,m}*(|\varphi_v*f|^d\chi_{3Q_v})\lesssim\eta_{v,m-c_{\log}(\tau)}*\Big(\frac{|\varphi_v*f|^d\chi_{3Q_v}}{|P|^{\tau(\cdot)d}}\Big),$$

and

$$\frac{1}{|P|^{\tau(\cdot)d}}\eta_{v,m}*(|\varphi_v*f|^d\chi_{Q_v^k}) \lesssim \eta_{v,m-c_{\log}(\tau)}*\Big(\frac{|\varphi_v*f|^d\chi_{Q_v^k}}{|P|^{\tau(\cdot)d}}\Big).$$

The arguments here are quite similar to those used in the case $l(P)>1$, where we did not need to use Lemma 4.10, which could be used only to move $|P|^{\tau(\cdot)}$ inside the convolution and hence the proof is complete. □

Remark 4.29. We would like to mention that similar arguments give

$$\mathfrak{B}^{s_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{s_0(\cdot)-\frac{n}{p_0(\cdot)},\tau(\cdot)}_{\infty,q(\cdot)},$$

if $s_0,\tau\in C^{\log}_{\mathrm{loc}}$, $\tau^->0$ and $p_0,q,\tau\in\mathscr{P}^{\log}_0$, *with* $q^+<\infty$.

Let $s,\tau\in C^{\log}_{\mathrm{loc}}$, $\tau^->0$, $p,q\in\mathscr{P}^{\log}$. From (4.10), we obtain

$$\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow B^{s(\cdot)+n\tau(\cdot)-\frac{n}{p(\cdot)}}_{\infty,\infty} \hookrightarrow \mathscr{S}'(\mathbb{R}^n).$$

Similar arguments of [68, Proposition 2.3] can be used to prove that

$$\mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}.$$

Therefore, we obtain the following statement.

Theorem 4.30. *Let* $s,\tau\in C^{\log}_{\mathrm{loc}}$, $\tau^->0$ *and* $p,q\in\mathscr{P}^{\log}_0$ *with* $q^+<\infty$. *Then*

$$\mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathscr{S}'(\mathbb{R}^n).$$

Now we establish some further embedding of the spaces $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Theorem 4.31. *Let* $s,\tau\in C^{\log}_{\mathrm{loc}}$, $\tau^->0$ *and* $p,q\in\mathscr{P}^{\log}_0$ *with* $q^+<\infty$. *If* $(p_2-p_1)^+\leqslant 0$, *then*

$$\mathfrak{B}^{s(\cdot)+n\tau(\cdot)+\frac{n}{p_2(\cdot)}-\frac{n}{p_1(\cdot)},0}_{p_2(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Proof. Recall that

$$\mathfrak{B}^{s(\cdot),0}_{p(\cdot),q(\cdot)} = B^{s(\cdot)}_{p(\cdot),q(\cdot)},$$

in the sense of equivalent quasi-norms. Using the Sobolev embeddings

$$B^{s(\cdot)+n\tau(\cdot)+\frac{n}{p_2(\cdot)}-\frac{n}{p_1(\cdot)}}_{p_2(\cdot),q(\cdot)} \hookrightarrow B^{s(\cdot)+n\tau(\cdot)}_{p_1(\cdot),q(\cdot)},$$

see [1, Theorem 6.4] it is sufficient to prove that $B^{s(\cdot)+n\tau(\cdot)}_{p_1(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}$. We have

$$\sup_{P\in\mathcal{Q},|P|>1} \Big\| \Big(\frac{2^{v s(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v\geqslant v_P^+} \Big\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \leqslant \Big\| \big(2^{v s(\cdot)} \varphi_v * f \big)_{v\in\mathbb{N}_0} \Big\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}.$$

In view of the definition of $B^{s(\cdot)}_{p_1(\cdot),q(\cdot)}$ spaces the last expression is bounded by

$$\big\| f \big\|_{B^{s(\cdot)}_{p_1(\cdot),q(\cdot)}} \leqslant \big\| f \big\|_{B^{s(\cdot)+n\tau(\cdot)}_{p_1(\cdot),q(\cdot)}}.$$

Now we have the estimates

$$\begin{aligned}
&\sup_{P\in\mathcal{Q},|P|\leqslant 1} \Big\| \Big(\frac{2^{v s(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v\geqslant v_P^+} \Big\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \\
&\leqslant \sup_{P\in\mathcal{Q},|P|\leqslant 1} \Big\| \big(2^{v(s(\cdot)+n\tau(\cdot))+n\tau(\cdot)(v_P-v)} \varphi_v * f \big)_{v\geqslant v_P} \Big\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \\
&\leqslant \sup_{P\in\mathcal{Q},|P|\leqslant 1} \Big\| \big(2^{v(s(\cdot)+n\tau(\cdot))} \varphi_v * f \big)_{v\in\mathbb{N}_0} \Big\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \\
&\leqslant \big\| f \big\|_{B^{s(\cdot)+n\tau(\cdot)}_{p_1(\cdot),q(\cdot)}},
\end{aligned}$$

which completes the proof. □

CHAPTER 5

ATOMIC DECOMPOSITION FOR VARIABLE BESOV-TYPE SPACES

The idea of atomic decompositions leads back to M. Frazier and B. Jawerth in their series of papers [24], [25]. The main goal of this chapter is to prove an atomic decomposition result for $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

5.1 Maximal function characterization

In this section, we establish the Peetre maximal function characterization of $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$. Recall that the Peetre maximal function of $f \in \mathscr{S}'(\mathbb{R}^n)$ is defined by setting, for all $a > 0$, $s : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$

$$\varphi_v^{*,a} 2^{vs(\cdot)} f(x) = \sup_{y\in\mathbb{R}^n} \frac{2^{vs(y)}\left|\varphi_v * f(y)\right|}{\left(1+2^v\left|x-y\right|\right)^a}, \qquad v \in \mathbb{N}_0.$$

where φ_0 is replaced by Φ.

The following lemma plays an important role in this chapter and it is the $\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})$-version of Lemma 1.19 (we use it, since the maximal operator is in general not bounded on $\ell^{q(\cdot)}(L^{p(\cdot)})$, see [1, Example 4.1]).

Lemma 5.1. *Let* $\tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p \in \mathscr{P}^{\log}$, $q \in \mathscr{P}^{\log}_0$ *with* $0 < q^- \leqslant q^+ < \infty$ *and* $\tau^+ < \left(\tau p\right)^-$. *For any* m *large enough, there exists* $c > 0$ *such that*

$$\left\|(\eta_{v,m} * f_v)_v\right\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})} \leqslant c\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})},$$

for any $(f_v)_v \in \ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})$.

Proof. By the scaling argument, we see that it suffices to consider when

$$\big\| (f_v)_v \big\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})} \leqslant 1, \tag{5.1}$$

and show that for any dyadic cube P

$$\sum_{v=v_P^+}^{\infty} \Big\| \Big| \frac{c\ \eta_{v,m} * |f_v|}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

for some constant $c > 0$. We distinguish two cases: **Case 1.** $l(P) > 1$. Let $Q_v \subset P$ be a cube, with $\ell(Q_v) = 2^{-v}$ and $x \in Q_v \subset P$. We have

$$\begin{aligned}
&\eta_{v,m} * |f_v|(x) \\
&= 2^{vn} \int_{\mathbb{R}^n} \frac{|f_v(z)|}{(1+2^v|x-z|)^m} dz \\
&= \int_{3Q_v} \frac{|f_v(z)|}{(1+2^v|x-z|)^m} dz + \sum_{k\in\mathbb{Z}^n, \|k\|_\infty \geqslant 2} \int_{Q_v^k} \frac{|f_v(z)|}{(1+2^v|x-z|)^m} dz \\
&= J_v^1(f_v\chi_{3Q_v})(x) + \sum_{k\in\mathbb{Z}^n, \|k\|_\infty \geqslant 2} J_{v,k}^2(f_v\chi_{Q_v^k})(x),
\end{aligned}$$

where $Q_v^k = Q_v + kl(Q_v)$. Let $0 < r < \frac{1}{2}\min(p^-, q^-, 2)$ and define $\tilde{p} = \frac{p}{r}$ and $\tilde{q} = \frac{q}{r}$. Then clearly, $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \leqslant 1$. Thus we obtain

$$\begin{aligned}
&\Big\| \Big(\frac{\eta_{v,m} * |f_v|}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v\in\mathbb{N}_0} \Big\|^r_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&\leqslant \Big\| \Big(\frac{J_v^1(f_v\chi_{3Q_v})}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v\in\mathbb{N}_0} \Big\|^r_{\ell^{q(\cdot)}(L^{p(\cdot)})} && (5.2) \\
&+ \sum_{k\in\mathbb{Z}^n, \|k\|_\infty \geqslant 2} \Big\| \Big(\frac{J_{v,k}^2(f_v\chi_{Q_v^k})}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v\in\mathbb{N}_0} \Big\|^r_{\ell^{q(\cdot)}(L^{p(\cdot)})}. && (5.3)
\end{aligned}$$

Estimate of (5.2). We will prove that (5.2) is bounded by a constant independent of P. Clearly, we need to show that

$$\Big\| \Big| \frac{c\ J_v^1(f_v\chi_{3Q_v})}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant \Big\| \Big| \frac{f_v}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_{3P} \Big\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-v} = \delta,$$

for some positive constant c. This claim can be reformulated as showing that

$$\Big\|\delta^{-\frac{1}{q(\cdot)}}\frac{c\,J_v^1(f_v\chi_{3Q_v})}{|P|^{\tau(\cdot)}}\chi_P\Big\|_{p(\cdot)}\leqslant 1. \tag{5.4}$$

Let $d>0$ be such that $\tau^+<d<\big(\tau p\big)^-$. We have

$$\frac{M_{3Q_v}\big(f_v\big)}{|P|^{\tau(\cdot)}}=\Big(\frac{\big(M_{3Q_v}\big(f_v\big)\big)^{\frac{d}{\tau(\cdot)}}}{|P|^d}\Big)^{\frac{\tau(\cdot)}{d}}.$$

Hence, we will prove that

$$\Big\|\frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}}\big(M_{3Q_v}\big(f_v\big)\big)^{\frac{d}{\tau(\cdot)}}}{|P|^d}\chi_P\Big\|_{\frac{p(\cdot)\tau(\cdot)}{d}}\lesssim 1.$$

By Hölder's inequality,

$$|Q_v|M_{3Q_v}\big(\big|f_v\big|^{\frac{d}{\tau(\cdot)}}\big)\lesssim\Big\|\frac{\big|f_v\big|^{\frac{1}{\tau(\cdot)}}}{|3Q_v|}\chi_{3Q_v}\Big\|_{p(\cdot)\tau(\cdot)}^d\big\||3Q_v|\chi_{3Q_v}\big\|_{t(\cdot)}^d,$$

where $\frac{1}{d}=\frac{1}{p(\cdot)\tau(\cdot)}+\frac{1}{t(\cdot)}$. The second quasi-norm is bounded, while the first is bounded if and only if

$$\Big\|\frac{f_v}{|Q_v|^{\tau(\cdot)}}\chi_{3Q_v}\Big\|_{p(\cdot)}\lesssim 1.$$

Notice that $3Q_v\subset\cup_{h=1}^{3^n}\check{Q}_v^h$, where $\{\check{Q}_v^h\}_{h=1}^{3^n}$ are disjoint dyadic cubes with side length $l(\check{Q}_v^h)=l(Q_v)$. Therefore $\chi_{3Q_v}\leqslant\sum_{h=1}^{3^n}\chi_{\check{Q}_v^h}$ and

$$\Big\|\frac{f_v}{|Q_v|^{\tau(\cdot)}}\chi_{3Q_v}\Big\|_{p(\cdot)}\leqslant c\sum_{h=1}^{3^n}\Big\|\frac{f_v}{|\check{Q}_v^h|^{\tau(\cdot)}}\chi_{\check{Q}_v^h}\Big\|_{p(\cdot)}\lesssim 1,$$

where we used (5.1). We can use Lemma 4.10 to obtain that

$$\big(M_{3Q_v}\big(f_v\big)\big)^{\frac{d}{\tau(x)}},$$

can be estimated by

$$M_{3Q_v}\big(\big|f_v\big|^{\frac{d}{\tau(\cdot)}}\big)+|3Q_v|^{\theta}\,g(x),$$

for any $x \in Q_v$ and any $\theta > 0$, where g is the same function as in Theorem 4.28. Taking into account that $\frac{1}{q}$ and τ are log-Hölder continuous, $\delta \in [2^{-v}, 1+2^{-v}]$, by Lemma 4.8;

$$\delta^{-\frac{d}{q(x)\tau(x)}} \big(M_{3Q_v}\big(f_v\big)\big)^{\frac{d}{\tau(x)}},$$

does not exceed

$$c\,M_{3Q_v}\Big(\big|\delta^{-\frac{1}{q(\cdot)}} f_v\big|^{\frac{d}{\tau(\cdot)}}\Big) + c\,2^{\frac{vd}{q(x)\tau(x)}} \,|Q_v|^{\theta}\, g(x) \lesssim M_{3Q_v}\Big(\big|\delta^{-\frac{1}{q(\cdot)}} f_v\big|^{\frac{d}{\tau(\cdot)}}\Big) + h(x),$$

where the positive constant c is independent of v and x and we used the fact that

$$\max_{x \in Q_v} 2^{vd/q(x)\tau(x)} |Q_v|^{\theta} \leqslant 1,$$

since $\theta > 0$ can be taken large enough. Here h is the same function as in Theorem 4.28. Therefore,

$$\begin{aligned}\Bigg\| \frac{\Big(\delta^{-\frac{1}{q(\cdot)}} M_{3Q_v}\big(f_v\big)\Big)^{\frac{d}{\tau(\cdot)}}}{|P|^d} \Bigg\|_{\frac{p(\cdot)\tau(\cdot)}{d}} &\lesssim \Bigg\| \mathscr{M}\Big(\frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} \big|f_v\big|^{\frac{d}{\tau(\cdot)}}}{|P|^d} \chi_{3Q_v}\Big)\Bigg\|_{\frac{p(\cdot)\tau(\cdot)}{d}} + c \\ &\lesssim \Bigg\| \frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} \big|f_v\big|^{\frac{d}{\tau(\cdot)}}}{|P|^d} \chi_{3P} \Bigg\|_{\frac{p(\cdot)\tau(\cdot)}{d}} + c,\end{aligned}$$

since $\frac{p\tau}{d} \in \tilde{\mathscr{P}}^{\log}$, $(\frac{p\tau}{d})^{-} > 1$ and $\mathscr{M} : L^{\frac{p(\cdot)\tau(\cdot)}{d}} \to L^{\frac{p(\cdot)\tau(\cdot)}{d}}$ is bounded. The last norm is bounded if and only if

$$\Bigg\| \frac{\delta^{-\frac{1}{q(\cdot)}} \big|f_v\big| \chi_{3P}}{|P|^{\tau(\cdot)}} \Bigg\|_{p(\cdot)} \lesssim 1,$$

which follows immediately from the definition of δ. *Estimate of* (5.3). We will prove that (5.3) is bounded by a constant independent of P. The summation in (5.3) can be rewritten as

$$\sum_{k \in \mathbb{Z}^n, |k| \leqslant 4\sqrt{n}} \cdots + \sum_{k \in \mathbb{Z}^n, |k| > 4\sqrt{n}} \cdots. \tag{5.5}$$

The estimation of the first sum follows in the same manner as in the estimate of $J_v^1(f_v \chi_{3Q_v})$, so we need only to estimate the second sum. Let now prove that

$$\Big\|\Big(|k|^{b(\cdot)}\frac{J_{v,k}^2(f_v\chi_{Q_v^k})}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}\lesssim\Big\|\Big(\frac{f_v}{|\tilde{Q}^k|^{\tau(\cdot)}}\chi_{\tilde{Q}^k}\Big)_{v\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

where $\tilde{Q}^k=Q(c_P,2|k|\,l(P))$ and

$$b(\cdot)=m-n(1+\frac{1}{t^-})\tau^+-2\frac{c_{\log}\big(\frac{d}{q\tau}\big)\tau(\cdot)}{d}-\frac{\theta\tau(\cdot)}{d},$$

and θ will be chosen later. Again, by the scaling argument, we see that it suffices to consider when the last norm is less than or equal 1 and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that for any dyadic cube P

$$\sum_{v=0}^{\infty}\Big\|\Big|\frac{c\;|k|^{b(\cdot)}J_{v,k}^2(f_v\chi_{Q_v^k})}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\chi_P\Big\|_{\frac{p(\cdot)}{q(\cdot)}}\leqslant 1,$$

for some positive constant c. This estimate follows from the inequality

$$\begin{aligned}\Big\|\Big|\frac{c\;|k|^{b(\cdot)}J_{v,k}^2(f_v\chi_{Q_v^k})}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\chi_P\Big\|_{\frac{p(\cdot)}{q(\cdot)}}&\leqslant\Big\|\Big|\frac{f_v}{|\tilde{Q}^k|^{\tau(\cdot)}}\Big|^{q(\cdot)}\chi_{\tilde{Q}^k}\Big\|_{\frac{p(\cdot)}{q(\cdot)}}+2^{-v}\\&=\delta,\end{aligned}$$

for any $v\in\mathbb{N}_0$. This claim can be reformulated as showing that

$$\Big\|\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}}\frac{\big(|k|^{b(\cdot)}J_{v,k}^2(f_v\chi_{Q_v^k})\big)^{\frac{d}{\tau(\cdot)}}}{|P|^d}\chi_P\Big\|_{\frac{p(\cdot)\tau(\cdot)}{d}}\lesssim 1. \tag{5.6}$$

Let $z\in Q_v^k$, $x\in Q_v$ with $k\in\mathbb{Z}^n$ and $|k|>4\sqrt{n}$. Then $z=h+k2^{-v}$ for some $h\in Q_v$, $|x-z|\geqslant|k|2^{-v-1}$. Hence

$$\begin{aligned}\delta^{-\frac{1}{q(x)}}|k|^{b(x)}J_{v,k}^2(f_v\chi_{Q_v^k})(x)&\lesssim\delta^{-\frac{1}{q(x)}}|k|^{b(x)-m}M_{Q_v^k}\big(f_v\big)\\&\lesssim\delta^{-\frac{1}{q(x)}}|k|^{b(x)-m+n(1+\frac{1}{t^-})\tau^+}M_{Q_v^k}(|k|^{-n(1+\frac{1}{t(\cdot)})\tau(\cdot)}f_v)\\&\lesssim\delta^{-\frac{1}{q(x)}}|k|^{-(2c_{\log}(\frac{d}{q\tau})+\theta)\frac{\tau(x)}{d}}M_{Q_v^k}(|k|^{-n(1+\frac{1}{t(\cdot)})\tau(\cdot)}f_v),\end{aligned}$$

for any $x\in Q_v$ and any $v\in\mathbb{N}_0$, where $\frac{1}{d}=\frac{1}{p(\cdot)\tau(\cdot)}+\frac{1}{t(\cdot)}$. Observe that $Q_v^k\subset Q(x,|k|2^{1-v})=\tilde{Q}_v^k$.

By Hölder's inequality,

$$|\tilde{Q}_v^k| M_{\tilde{Q}_v^k}\big(|k|^{-n(1+\frac{1}{t(\cdot)})d} \big|f_v\big|^{\frac{d}{\tau(\cdot)}}\big) \lesssim \Big\| \frac{\big|f_v\big|^{\frac{1}{\tau(\cdot)}}}{|\tilde{Q}_v^k|} \chi_{\tilde{Q}_v^k} \Big\|_{p(\cdot)\tau(\cdot)}^d \Big\| |\tilde{Q}_v^k| |k|^{-n(1+\frac{1}{t(\cdot)})} \chi_{\tilde{Q}_v^k} \Big\|_{t(\cdot)}^d.$$

The second quasi-norm is bounded, while the first is bounded if and only if

$$\Big\| \frac{f_v}{|\tilde{Q}_v^k|^{\tau(\cdot)}} \chi_{\tilde{Q}_v^k} \Big\|_{p(\cdot)} \lesssim 1, \quad v \in \mathbb{N}_0,$$

which follows by (5.1). Again by Lemma 4.10,

$$\Big(M_{\tilde{Q}_v^k}\big(|k|^{-n(1+\frac{1}{t(\cdot)})\tau(\cdot)} f_v\big)\Big)^{\frac{d}{\tau(x)}},$$

does not exceed

$$c\, M_{\tilde{Q}_v^k}\Big(\Big| |k|^{-n(1+\frac{1}{t(\cdot)})\tau(\cdot)} f_v \Big|^{\frac{d}{\tau(\cdot)}}\Big) + c \min(1, |k|^{n\theta} 2^{(1-v)n\theta}) \big((e+|x|)^{-\theta} + M_{\tilde{Q}_v^k}((e+|\cdot|)^{-\theta})\big),$$

for any $\theta > 0$ large enough. Hence,

$$\delta^{-\frac{d}{q(x)\tau(x)}} \Big(M_{\tilde{Q}_v^k}\big(|k|^{-n(1+\frac{1}{t(\cdot)})\tau(\cdot)} f_v\big)\Big)^{\frac{d}{\tau(x)}},$$

is bounded by

$$\begin{aligned} & c\, |k|^{2c_{\log}(\frac{d}{q\tau})} M_{\tilde{Q}_v^k}\big(\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} |k|^{-n(1+\frac{1}{t(\cdot)})d} \big|f_v\big|^{\frac{d}{\tau(\cdot)}}\big) + 2^{\frac{vd}{(q\tau)^-}} \min(1, |k|^{n\theta} 2^{(1-v)n\theta}) h(x) \\ & \lesssim |k|^{2c_{\log}(\frac{d}{q\tau})} M_{\tilde{Q}_v^k}\big(\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} |k|^{-n(1+\frac{1}{t(\cdot)})d} \big|f_v\big|^{\frac{d}{\tau(\cdot)}}\big) + |k|^{n\theta} h(x), \end{aligned}$$

where $\theta > 0$ large enough such that $\theta > \frac{d}{n(q\tau)^-}$. Therefore, the left-hand side of (5.6) is bounded by

$$\begin{aligned} & c\, \Big\| \mathscr{M}\Big(|k|^{-nd} \frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} \big|f_v\big|^{\frac{d}{\tau(\cdot)}} \chi_{\tilde{Q}^k}}{|P|^d}\Big) \Big\|_{\frac{p(\cdot)\tau(\cdot)}{d}} + C \\ & \lesssim \Big\| \frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} \big|f_v\big|^{\frac{d}{\tau(\cdot)}} \chi_{\tilde{Q}^k}}{|\tilde{Q}^k|^d} \Big\|_{\frac{p(\cdot)\tau(\cdot)}{d}} + C, \end{aligned}$$

after using the fact that $\mathcal{M} : L^{\frac{p(\cdot)\tau(\cdot)}{d}} \to L^{\frac{p(\cdot)\tau(\cdot)}{d}}$ is bounded. The last norm is bounded if and only if

$$\left\| \frac{\delta^{-\frac{1}{q(\cdot)}} f_v \chi_{\tilde{Q}^k}}{|\tilde{Q}^k|^{\tau(\cdot)}} \right\|_{p(\cdot)} \leqslant 1,$$

which follows immediately from the definition of δ. Since m can be taken large enough, then the second sum in (5.5) is bounded by

$$\sum_{k\in\mathbb{Z}^n, |k|>4\sqrt{n}} |k|^{-b^- r} \left\| \left(\frac{f_v}{|\tilde{Q}^k|^{\tau(\cdot)}} \chi_{\tilde{Q}^k} \right)_{v\in\mathbb{N}_0} \right\|^r_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant \sum_{k\in\mathbb{Z}^n, |k|>4\sqrt{n}} |k|^{-b^- r} \left\| (f_v)_v \right\|^r_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})}$$
$$\lesssim 1.$$

Case 2. $l(P) \leqslant 1$. As before,

$$\eta_{v,m} * |f_v|(x) \lesssim J_v^1(f_v \chi_{3P})(x) + \sum_{k\in\mathbb{Z}^n, \|k\|_\infty \geqslant 2} J^2_{v,k}(f_v \chi_{P+kl(P)})(x).$$

We see that

$$J_v^1(f_v\chi_{3P})(x) = \eta_{v,m} * (|f_v| \chi_{3P})(x), \quad x \in P,$$

and since τ is log-Hölder continuous, we have

$$|P|^{-\tau(x)} \leqslant c\,|P|^{-\tau(y)} (1 + 2^{v_P} |x-y|)^{c_{\log}(\tau)} \leqslant c\,|P|^{-\tau(y)},$$

for any $x \in P$ any $y \in 3P$ and any $v \geqslant v_P$. Hence

$$|P|^{-\tau(x)} J_v^1(f_v\chi_{3P})(x) \lesssim \eta_{v,m-c_{\log}(\tau)} * \left(|P|^{-\tau(\cdot)} |f_v| \chi_{3P} \right)(x), \quad x \in P.$$

Also, we have

$$|P|^{-\tau(x)} J^2_{v,k}(f_v\chi_{P+kl(P)})(x) \lesssim \eta_{v,m-c_{\log}(\tau)} * \left(|P|^{-\tau(\cdot)} |f_v| \chi_{P+kl(P)} \right)(x).$$

As before, we obtain

$$\sum_{v=v_P}^{\infty} \left\| \left| \frac{c\,\eta_{v,m} * f_v}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leqslant 1,$$

where we did not need to use Lemma 4.10, which could be used only to move $|P|^{\tau(\cdot)}$ inside the convolution. The proof is complete.

$\square$

We now present a fundamental characterization of spaces under consideration.

Theorem 5.2. *Let* $\tau, s \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$ *and* $p, q \in \mathscr{P}^{\log}_0$. *Let* m *be as in Lemma 5.1,* $a > \frac{m\tau^+}{(\tau p)^-}$ *and* Φ *and* φ *satisfy (4.1) and (4.2), respectively. Then*

$$\|f\|^{\blacktriangledown}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{P\in\mathcal{Q}} \left\| \left(\frac{\varphi_v^{*,a} 2^{vs(\cdot)} f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}, \tag{5.7}$$

is an equivalent quasi-norm in $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. We divide the proof in two steps. *Step* 1. It is easy to see that for any $f \in \mathscr{S}'(\mathbb{R}^n)$ with $\|f\|^{\blacktriangledown}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} < \infty$ and any $x \in \mathbb{R}^n$ we have

$$2^{vs(x)} \left|\varphi_v * f(x)\right| \leqslant \varphi_v^{*,a} 2^{vs(\cdot)} f(x).$$

This shows that the right-hand side in (4.8) is less than or equal (5.7).

Step 2. We will prove in this step that there is a constant $C > 0$ such that for every $f \in \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$

$$\|f\|^{\blacktriangledown}_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant C \|f\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{5.8}$$

We choose $t > 0$ such that $a > \frac{m}{t} > \frac{m}{p^-}$. By Lemmas 4.9 and 4.8 the estimates

$$\begin{aligned} 2^{vs(y)} \left|\varphi_v * f(y)\right| &\leqslant C_1\, 2^{vs(y)} (\eta_{v,w} * |\varphi_v * f|^t(y))^{\frac{1}{t}} \\ &\leqslant C_2\, (\eta_{v,w-c_{\log}(s)} * (2^{vs(\cdot)} |\varphi_v * f|)^t(y))^{\frac{1}{t}}, \end{aligned} \tag{5.9}$$

are true for any $y \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and any $w > n$. Now divide both sides of (5.9) by $\left(1 + 2^v \left|x - y\right|\right)^a$, in the right-hand side we use the inequality

$$\left(1 + 2^v \left|x - y\right|\right)^{-a} \leqslant (1 + 2^v |x - z|)^{-a} \left(1 + 2^v \left|y - z\right|\right)^a, \quad x, y, z \in \mathbb{R}^n,$$

in the left-hand side take the supremum over $y \in \mathbb{R}^n$ and get for all $f \in \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, any $x \in P$

any $v \geqslant v_P^+$ and any $w > \max(n, at + c_{\log}(s))$

$$\left(\varphi_v^{*,a} 2^{vs(\cdot)} f(x)\right)^t \leqslant C_2\, \eta_{v,at} * (2^{vs(\cdot)} |\varphi_v * f|)^t(x),$$

where $C_2 > 0$ is independent of x, v and f. An application of Lemma 5.1 gives that the left hand side of (5.8) is bounded by

$$\begin{aligned} & C \sup_{P \in \mathcal{Q}} \left\| \left(\frac{\eta_{v,at} * (2^{vs(\cdot)} |\varphi_v * f|)^t}{|P|^{\tau(\cdot)t}} \chi_P \right)_{v \geqslant v_P^+} \right\|_{\ell^{\frac{q(\cdot)}{t}}(L^{\frac{p(\cdot)}{t}})}^{\frac{1}{t}} \\ & \lesssim \left\| (2^{vs(\cdot)} \varphi_v * f)_v \right\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})} \\ & = C \left\| f \right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}. \end{aligned}$$

The proof is complete. □

5.2 Atomic characterization

In this section we shall show that under certain restrictions on the parameters our spaces $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ allow characterizations smooth atoms.

Atoms are the building blocks for the atomic decomposition.

Definition 5.3. *Let $K \in \mathbb{N}_0, L+1 \in \mathbb{N}_0$ and let $\gamma > 1$. A K-times continuous differentiable function $a \in C^K(\mathbb{R}^n)$ is called $[K, L]$-atom centered at $Q_{v,m}$, $v \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if*

$$\operatorname{supp} a \subseteq \gamma Q_{v,m} \tag{5.10}$$

$$|\partial^\beta a(x)| \leqslant 2^{v(|\beta| + \frac{1}{2})}, \quad \textit{for} \quad 0 \leqslant |\beta| \leqslant K, x \in \mathbb{R}^n, \tag{5.11}$$

and if

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0, \quad \textit{for} \quad 0 \leqslant |\beta| \leqslant L \textit{ and } v \geqslant 1. \tag{5.12}$$

If the atom a located at $Q_{v,m}$, that means if it fulfills (5.10), then we will denote it by $a_{v,m}$. For $v = 0$ or $L = -1$ there are no moment conditions (5.12) required.

For proving the decomposition by atoms we need the following lemma, see Frazier and Jawerth [24, Lemma 3.3].

Lemma 5.4. *Let Φ and φ satisfy, respectively, (4.1) and (4.2) and let $\varrho_{\nu,m}$ be an $[K,L]$-atom. Then*

$$\left|\varphi_j * \varrho_{\nu,m}(x)\right| \leqslant c\, 2^{(\nu-j)K+\nu\frac{n}{2}}\left(1+2^{\nu}|x-x_{Q_{\nu,m}}|\right)^{-M},$$

if $\nu \leqslant j$, and

$$\left|\varphi_j * \varrho_{\nu,m}(x)\right| \leqslant c\, 2^{(j-\nu)(L+n+1)+\nu\frac{n}{2}}\left(1+2^{j}|x-x_{Q_{\nu,m}}|\right)^{-M},$$

if $\nu \geqslant j$, where M is sufficiently large, $\varphi_j = 2^{jn}\varphi(2^j\cdot)$ and φ_0 is replaced by Φ.

Now we come to the atomic decomposition theorem.

Theorem 5.5. *Let $s,\tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$ and $p,q \in \mathscr{P}^{\log}_0$ with $0 < q^- \leqslant q^+ < \infty$. Let $0 < p^- \leqslant p^+ < \infty$ and let $K, L+1 \in \mathbb{N}_0$ such that*

$$K \geqslant ([s^+ + n\tau^+]+1)^+, \tag{5.13}$$

and

$$L \geqslant \max\Big(-1, [n(\frac{1}{\min(1,\frac{(\tau p)^-}{\tau^+})}-1)-s^-]\Big). \tag{5.14}$$

Then $f \in \mathscr{S}'(\mathbb{R}^n)$ belongs to $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, if and only if it can be represented as

$$f = \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}\varrho_{\nu,m}, \quad \text{converging in } \mathscr{S}'(\mathbb{R}^n), \tag{5.15}$$

where $\varrho_{\nu,m}$ are $[K,L]$-atoms and $\lambda = \{\lambda_{\nu,m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$. Moreover, $\inf\left\|\lambda\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$, where the infimum is taken over admissible representations (5.15), is an equivalent quasi-norm in $\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. The proof follows the ideas in [24, Theorem 6] and [15].

Step 1. Assume that $f \in \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ and let Φ and φ satisfy, respectively (4.1) and (4.2). There exist functions $\Psi \in \mathscr{S}(\mathbb{R}^n)$ satisfying (4.1) and $\psi \in \mathscr{S}(\mathbb{R}^n)$ satisfying (4.2) such that

$$f = \Psi * \widetilde{\Phi} * f + \sum_{\nu=1}^{\infty}\psi_\nu * \widetilde{\varphi}_\nu * f,$$

see Section 4.2. Using the definition of the cubes $Q_{\nu,m}$ we obtain for all $x \in \mathbb{R}^n$

$$f(x)=\sum_{m\in\mathbb{Z}^n}\int_{Q_{0,m}}\widetilde{\Phi}(x-y)\Psi*f(y)dy+\sum_{v=1}^{\infty}2^{vn}\sum_{m\in\mathbb{Z}^n}\int_{Q_{v,m}}\widetilde{\varphi}(2^v(x-y))\psi_v*f(y)dy,$$

with convergence in $\mathscr{S}'(\mathbb{R}^n)$. We define for every $v\in\mathbb{N}$ and all $m\in\mathbb{Z}^n$

$$\lambda_{v,m}=C_{\widetilde{\varphi}}\sup_{y\in Q_{v,m}}\left|\psi_v*f(y)\right|, \tag{5.16}$$

where

$$C_{\widetilde{\varphi}}=\max\{\sup_{|y|\leqslant 1}\left|D^s\widetilde{\varphi}(y)\right|:|s|\leqslant K\}.$$

Define also

$$\varrho_{v,m}(x)=\begin{cases}\frac{1}{\lambda_{v,m}}2^{vn}\int_{Q_{v,m}}\widetilde{\varphi}_v(2^v(x-y))\psi_v*f(y)dy & \text{if}\quad \lambda_{v,m}\neq 0\\ 0 & \text{if}\quad \lambda_{v,m}=0\end{cases}. \tag{5.17}$$

Similarly we define for every $m\in\mathbb{Z}^n$ the numbers $\lambda_{0,m}$ and the functions $\varrho_{0,m}$ taking in (5.16) and (5.17) $v=0$ and replacing ψ_v and $\widetilde{\varphi}$ by Ψ and $\widetilde{\Phi}$, respectively. Let us now check that such $\varrho_{v,m}$ are atoms in the sense of Definition 5.3. Note that the support and moment conditions are clear by (4.1) and (4.2), respectively. It thus remains to check (5.11) in Definition 5.3. We have

$$\begin{aligned}\left|D^\beta\varrho_{v,m}(x)\right| &\leqslant \frac{2^{v(n+|\beta|)}}{C_{\widetilde{\varphi}}}\int_{Q_{v,m}}\left|(D^\beta\widetilde{\varphi})(2^v(x-y))\right|\left|\psi_v*f(y)\right|dy\Big(\sup_{y\in Q_{v,m}}\left|\psi_v*f(y)\right|\Big)^{-1}\\ &\leqslant \frac{2^{v(n+|\beta|)}}{C_{\widetilde{\varphi}}}\int_{Q_{v,m}}\left|(D^\beta\widetilde{\varphi})(2^v(x-y))\right|dy\\ &\leqslant 2^{v(n+|\beta|)}\left|Q_{v,m}\right|\\ &\leqslant 2^{v|\beta|}.\end{aligned}$$

The modifications for the terms with $v=0$ are obvious.

Step 2. Next we show that there is a constant $c>0$ such that $\left\|\lambda\right\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}\leqslant c\left\|f\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$. For that reason we exploit the equivalent quasi-norms given in Theorem 5.2 involving Peetre's maximal function. Let $v\in\mathbb{N}$. Taking into account that $\left|x-y\right|\leqslant c\,2^{-v}$ for $x,y\in Q_{v,m}$ we obtain

$$2^{v(s(x)-s(y))} \leqslant \frac{c_{\log}(s)v}{\log(e+\frac{1}{|x-y|})} \leqslant \frac{c_{\log}(s)v}{\log(e+\frac{2^v}{c})} \leqslant c,$$

if $v \geqslant [\log_2 c]+2$. If $0 < v < [\log_2 c]+2$, then $2^{v(s(x)-s(y))} \leqslant 2^{v(s^+-s^-)} \leqslant c$. Therefore,

$$2^{vs(x)}\left|\psi_v * f(y)\right| \leqslant c\, 2^{vs(y)}\left|\psi_v * f(y)\right|,$$

for any $x, y \in Q_{v,m}$ and any $v \in \mathbb{N}$. Hence,

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} \lambda_{v,m} 2^{vs(x)} \chi_{v,m}(x) &= C_{\tilde{\varphi}} \sum_{m\in\mathbb{Z}^n} 2^{vs(x)} \sup_{y\in Q_{v,m}} \left|\psi_v * f(y)\right| \chi_{v,m}(x)\\ &\leqslant c \sum_{m\in\mathbb{Z}^n} \sup_{|z|\leqslant c\, 2^{-v}} \frac{2^{vs(x-z)}\left|\psi_v * f(x-z)\right|}{(1+2^v|z|)^a}(1+2^v|z|)^a \chi_{v,m}(x)\\ &\leqslant c\, \psi_v^{*,a} 2^{vs(\cdot)} f(x) \sum_{m\in\mathbb{Z}^n} \chi_{v,m}(x)\\ &= c\, \psi_v^{*,a} 2^{vs(\cdot)} f(x),\end{aligned}$$

where $a > \frac{m\tau^+}{(\tau p)^-}$ and we have used $\sum_{m\in\mathbb{Z}^n} \chi_{v,m}(x) = 1$. This estimate and its counterpart for $v = 0$ (which can be obtained by a similar calculation) give

$$\begin{aligned}\left\|\lambda\right\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} &\leqslant c\left\|\left(\psi_v^{*,a} 2^{vs(\cdot)} f\right)_v\right\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})}\\ &\leqslant c\left\|f\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}},\end{aligned}$$

by Theorem 5.2.

Step 3. Assume that f can be represented by (5.15), with K and L satisfying (5.13) and (5.14), respectively. Similar arguments of [15, Theorem 4.3], by using Lemmas 5.1, 4.12, show that $f \in \mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ and that for some $c > 0$,

$$\left\|f\right\|_{\mathfrak{B}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c\left\|\lambda\right\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Step 4. The convergence of (5.15). Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Using (5.10)-(5.11)-(5.12) and the Taylor expansion of φ up to order L with respect to the off-points $x_{Q_{v,m}}$, we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\lambda_{v,m}\varrho_{v,m}(y)\varphi(y)dy\\&=\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\lambda_{v,m}\varrho_{v,m}(y)\Big(\varphi(y)-\sum_{|\beta|\leqslant L}(y-x_{Q_{v,m}})^{\beta}\frac{\partial^{\beta}\varphi(x_{Q_{v,m}})}{\beta!}\Big)dy,\end{aligned}$$

for fixed v. The last factor in the integral can be uniformly estimated from the above by

$$c\,2^{-v(L+1)}(1+\left|y\right|^2)^{-\frac{M}{2}}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^{\frac{M}{2}}\sum_{|\beta|\leqslant L+1}\left|\partial^{\beta}\varphi(x)\right|,$$

where $M>0$ is at our disposal and the positive constant c is independent of x and v. From (5.14) we obtain

$$L+1>-s(\cdot)+n\Big(\frac{1}{\min(1,\frac{(\tau p)^-}{\tau^+})}-1\Big).$$

Let

$$\max\Big(0,\big(p(\cdot)(1-\frac{1}{\min(1,\frac{(\tau p)^-}{\tau^+})})\big)^{+}+1\Big)<t<1,$$

and $s(\cdot)=s(\cdot)+\frac{n}{p(\cdot)}(t-1)$. Here if $\frac{(\tau p)^-}{\tau^+}\geqslant 1$ then we take $t=1$. Then $L+1>-s(\cdot)$. Since $\varrho_{v,m}$ are $[K,L]$-atoms, then for every $S>0$, we have

$$\left|\varrho_{v,m}(y)\right|\leqslant c2^{v\frac{n}{2}}\left(1+2^{v}\left|y-x_{Q_{v,m}}\right|\right)^{-S}.$$

Therefore,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\lambda_{v,m}\varrho_{v,m}(y)\varphi(y)dy\Big| &&(5.18)\\&\leqslant c\,2^{-v(L+1)}\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}2^{v\frac{n}{2}}\left|\lambda_{v,m}\right|\frac{(1+\left|y\right|^2)^{-\frac{M}{2}}}{\left(1+2^{v}\left|y-x_{Q_{v,m}}\right|\right)^{S}}dy\\&=c\,2^{-v(L+1)}\sum_{h\in\mathbb{Z}^n}\int_{Q_{0,h}}\sum_{m\in\mathbb{Z}^n}2^{v\frac{n}{2}}\left|\lambda_{v,m}\right|\frac{(1+\left|y\right|^2)^{-\frac{M}{2}}}{\left(1+2^{v}\left|y-x_{Q_{v,m}}\right|\right)^{S}}dy.\end{aligned}$$

Applying [8, Lemma A.2] to obtain

$$\sum_{m\in\mathbb{Z}^n}\left|\lambda_{v,m}\right|\left(1+2^v\left|y-x_{Q_{v,m}}\right|\right)^{-S}\lesssim\sum_{m\in\mathbb{Z}^n}\left|\lambda_{v,m}\right|\eta_{v,S}*\chi_{v,m}(y).$$

Since we have in addition the factor $(1+\left|y\right|^2)^{-\frac{M}{2}}$, Hölder's inequality, the fact that $\left\|\chi_{Q_{0,h}}\right\|_{(\frac{p(\cdot)}{t})'}\approx 1$, and $(1+\left|y\right|^2)^{-\frac{M}{2}}\lesssim(1+|h|^2)^{-\frac{M}{2}}$ give that the term (5.18) can be estimated by

$$\begin{aligned}&c\;2^{-v(L+1)}\sum_{h\in\mathbb{Z}^n}(1+|h|^2)^{-\frac{M}{2}}\left\|\frac{1}{|Q_{0,h}|^{\tau(\cdot)}}\eta_{v,S}*[\sum_{m\in\mathbb{Z}^n}2^{v\frac{n}{2}}\left|\lambda_{v,m}\right|\chi_{v,m}]\chi_{Q_{0,h}}\right\|_{\frac{p(\cdot)}{t}}\\ &\lesssim\sup_{P\in\mathcal{Q},j\geqslant j_P^+}\left\|\frac{1}{|P|^{\tau(\cdot)}}2^{(s(\cdot)+\frac{n}{2})j}\sum_{m\in\mathbb{Z}^n}\left|\lambda_{j,m}\right|\chi_{j,m}\chi_P\right\|_{\frac{p(\cdot)}{t}}\lesssim\left\|\lambda\right\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{\frac{p(\cdot)}{t},\infty}},\end{aligned}$$

where the first inequality follows by Lemma 5.1, $L+1+s(\cdot)>0$ and by taking M large enough. We claim that

$$\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}\hookrightarrow\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),\infty}\hookrightarrow\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{\frac{p(\cdot)}{t},\infty}.$$

Then the convergence of (5.15) is now clear. We prove our claim. By the scaling argument, we see that it suffices to consider $\lambda\in\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),\infty}$ with $\left\|\lambda\right\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{p(\cdot),\infty}}\leqslant 1$. From Lemma 4.22

$$|\lambda_{v,m}|\leqslant c\;2^{-v(s(x)-\frac{n}{p(x)}+\frac{n}{2})}|Q_{v,m}|^{\tau(x)},$$

for any $x\in Q_{v,m}$, $v\in\mathbb{N}_0$, $m\in\mathbb{Z}^n$. Let P be a dyadic cube and $v\geqslant v_P^+$. We have

$$\begin{aligned}&\int_P\Big(\frac{1}{|P|^{\tau(x)}}\sum_{m\in\mathbb{Z}^n}2^{v(s(x)+\frac{n}{2})}|\lambda_{v,m}|\chi_{v,m}(x)\Big)^{\frac{p(x)}{t}}dx\\ &=\int_P\Big(\frac{1}{|P|^{\tau(x)}}\sum_{m\in\mathbb{Z}^n}2^{v(s(x)-\frac{nt}{p(x)}+\frac{n}{2})}|\lambda_{v,m}|\chi_{v,m}(x)\Big)^{\frac{p(x)}{t}-p(x)}\\ &\times\Big(\frac{1}{|P|^{\tau(x)}}\sum_{m\in\mathbb{Z}^n}2^{v(s(x)+\frac{n}{2})}|\lambda_{v,m}|\chi_{v,m}(x)\Big)^{p(x)}dx\\ &\lesssim\int_P\Big(\frac{1}{|P|^{\tau(x)}}\sum_{m\in\mathbb{Z}^n}2^{v(s(x)+\frac{n}{2})}|\lambda_{v,m}|\chi_{v,m}(x)\Big)^{p(x)}dx\lesssim 1.\end{aligned}$$

Therefore

$$\left\|\lambda\right\|_{\mathfrak{b}^{s(\cdot),\tau(\cdot)}_{\frac{p(\cdot)}{t},\infty}}\lesssim 1.$$

The proof is completed. ☐

Remark 5.6. *If* p, q, τ *and* s *are constants, then the restriction* (5.13)*, and their counterparts, in the atomic decomposition theorem are* $K \geqslant ([s+n\tau]+1)^{+}$ *and* $L \geqslant \max\big(-1,[n(\frac{1}{\min(1,p)}-1)-s]\big)$*, which are essentially the restrictions from the works of [13, Theorem 3.12].*

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الملخص: في هذه الأطروحة قمنا بدراسة إستمرارية المؤثرات الشبه تفاضلية على فضاءات معينة من نوع هارز–هاردي ذات الأدلة المتغيرة، و درسنا إستمرارية المؤثرات التكاملية الشاذة على فضاءات هارز–هاردي الغير متجانسة و أيضا قدمنا فضاءات من نوع بيزوف ذات الأدلة المتغيرة و تطرقنا إلى توصيفات φ–تحويل و التفكيك الذري لهذه الفضاءات. كما تحصلنا على إحتواءات سوبولاف لهذه الفضاءات الدالية.

الكلمات المفتاحية : فضاءات هارز، فضاءات هارز–هاردي ،فضاءات بيزوف، فضاءات من نوع بيزوف، الدالة الأعظمية، الأدلة المتغيرة، مؤثرات شبه تفاضلية، مؤثرات تكاملية شاذة.

Résumé: Dans cette thèse, nous étudions la continuité des opérateurs pseudo-différentiels sur certains espaces de Herz-Hardy avec des exposants variables et nous prouvions la continuté des opérateurs intégraux singuliers sur les espaces de Herz-Hardy non-homogènes avec des exposants variables. Nous avons également introduits des espaces de type de Besov où nous avons montré que ces espaces sont caractérisés par les φ-transforme dans les espaces de séquence appropriés et nous avons donné la décomposition atomique de ces espaces. De plus, les inclusions de type de Sobolev pour ces espaces fonctionnels sont obtenus.

Mots-clés: Espaces de Herz, Espaces de Herz-Hardy, Espaces de Besov, Espace de type-Besov, Exposants variable, Fonction maximale, Atome, Opérateurs pseudo-différentiels, Opérateurs intégraux singuliers
.

Abstract: In this thesis, we present the boundedness of a class of pseudo-differential operators on local variable Herz-type Hardy spaces and we establish the boundedness of singular integral operators on inhomogeneous variable Herz-type Hardy spaces. We also introduce Besov-type spaces with variable smoothness and integrability. We show that these spaces are characterized by the φ-transforms in appropriate sequence spaces and we obtain atomic decompositions for these spaces. Moreover the Sobolev embeddings for these function spaces are obtained.

Keywords: Herz space, Herz-type Hardy spaces, Maximal function, Besov spaces, Besov-type spaces, Variable exponent, Pseudo-differential operators, Singular integral operators.