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On some new classes of Lipschitz summing operators

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Didications

To my parents defunts Said and Messaouda Medjir.

To my virtuous wife

To my sons

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I would like to thank my advisor, Dr. Abdelmoumen TIAIBA for the interesting subject than it proposed to me. I am also grateful to him for confidence that it my granted. It is impossible for me to express to him all my gratitude in only some lines.

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Abstract: The context of this thesis falls within the framework of functional analysis and more precisely in nonlinear geometry. The ideas of this work were inspired from the article of J.D. Farmer and W.B. Johnson entitled "Lipschitz p -summing operators" and from the paper of R. Khalil and W. Deeb entitled " ϕ -summing operators in Banach spaces"; also that of K. Saadi entitled "On the composition ideals of Lipschitz mappings".

In the first part, our contribution consisted in the following idea: from a linear ϕ -summing operator between Banach spaces, where ϕ is a modulus function, we have introduced the notion of Lipschitz ϕ -summing operator between metric spaces giving a nonlinear version of Pietsch domination theorem for these operators and some properties concerning this class.

In the second part, from the notion of Lipschitz strictly p -summing operators we have given new characterizations of this class, where we have adapted this definition for constructing other class of Lipschitz mappings which are called M -strictly Lipschitz p -summing operators and strictly Lipschitz p -nuclear operators and strictly Lipschitz (p, r, s) -summing operators, in addition to all that, we obtained also some interesting results for these new classes.

In the end of this thesis, we present a short section collecting some interesting open problems.

Key-words: ϕ -summing linear, Lipschitz p -summing operator, Lipschitz (p, r, s) -summing operator, Lipschitz tensor norm, Pietsch Domination theorem, strictly Lipschitz p -summing operator, tensor norm.

ملخص

محتوى هذه الأطروحة يندرج في إطار التحليل الدالي، وبعبارة أدق ضمن الهندسة غير الخطية.

إن أفكار هذا العمل مستوحاة من مقال W.B.Johnson و J.D.Farmer المعنون بـ

(Lipschitz p -summing operators)

ومقال رشدي خليل ووسيم ديب بعنوان (ϕ -summing operators in Banach spaces) وكذلك مقال خليل سعدي المعنون بـ (On the composition ideals of Lipschitz mappings).

في الجزء الأول، تمثلت مساهمتنا في الفكرة الآتية: انطلاقاً من مؤثر خطي ϕ -جمعي بين فضاءي بناخ حيث ϕ هي دالة modulus أدخلنا مفهوم مؤثر ليبتشيز ϕ -جمعي بين فضاءين مترين معطينا طرحاً غير خطي لنظرية الهيمنة لـ (Pietsch) الخاصة بهذه المؤثرات وبعض الخواص المتعلقة بهذا الصنف.

في الجزء الثاني، انطلاقاً من مفهوم مؤثر ليبتشيزي تماماً p -جمعي ومؤثر ليبتشيز (p,r,s) -جمعي أدخلنا تميزات جديدة لهذه الأصناف وحيث كيفنا هذه التعاريف لإنشاء أصناف أخرى من المؤثرات الليبتشيزية والمسماة M -ليبتشيز تماماً p -جمعية وكذلك مؤثرات ليبتشيز p -نووية وأيضاً مؤثرات ليبتشيز تماماً (p,r,s) -جمعية بالإضافة إلى ذلك بعض الخواص الهامة وتميزات متعلقة بهذه الأصناف.

في نهاية هذه الأطروحة قدمنا بعض الإشكاليات المفتوحة.

كلمات مفتاحية:

مؤثر ليبتشيزي p -جمعي، مؤثر خطي p -جمعي، مؤثر ليبتشيز تماماً p -جمعي، ناظم موتري، ناظم ليبتشيز موتري، نظرية الهيمنة لـ Pietsch .

Résumé: Le contenu de cette thèse s'inscrit dans le cadre de l'analyse fonctionnelle et plus précisément dans la géométrie non linéaire. Les idées de ce travail nous ont été inspirées de l'article de J.D. Farmer et W.B. Johnson intitulé "les opérateurs Lipschitz p -sommants" et de l'article de Roshdi Khalil et W. Deeb intitulé "les opérateurs ϕ -sommants dans les espaces de Banach", ainsi du papier de Khalil saadi intitulé "Sur la composition des idéaux d'opérateurs lipschitziens".

Dans la première partie, notre contribution se consiste dans l'idée suivante, à partir d'un opérateur linéaire ϕ -sommant entre espaces de Banach où ϕ est une fonction modulus, on introduit la notion d'opérateur Lipschitz ϕ -sommant entre espaces métriques en donnant une version non linéaire du théorème de domination de Pietsch pour ces opérateurs et quelques propriétés concernant cette classe.

Dans la deuxième partie, à partir de la définition d'opérateur Lipschitzien strictement p -sommant et lipschitzien (p, r, s) -sommant, nous avons donné des nouvelles caractérisations de ces classes, où nous adaptons ces définitions pour construire d'autres classes d'opérateurs lipschitziens qui sont appelés opérateurs M -strictement Lipschitziens p -sommants, et de la définition d'un opérateur p -nucléaire dans le cas linéaire, on a introduit la notion d'opérateurs strictement lipschitziens p -nucléaires et aussi opérateurs strictement lipschitzien (p, r, s) -sommant, en plus nous avons obtenu encore quelques importantes propriétés et factorisations pour ces nouvelles classes.

A la fin de cette thèse, nous présentons une brève section contenant certains problèmes ouverts.

Mots-clés: opérateur lipschitzien p -sommant, opérateur linéaire p -sommant, opérateur lipschitzien strictement p -sommant, norme tensorielle, norme lipschitzienne tensorielle, théorème de domination de Pietsch.

Notation

\mathbb{K}	The field of real or complex numbers
\mathbb{R}_+	The set of non negative real numbers
p^*	The conjugate of the number p ($1 \leq p \leq \infty$), that is $\frac{1}{p} + \frac{1}{p^*} = 1$
E, F	Banach spaces
X, Y	Metric spaces
E^*	The topological dual of E
B_E	The closed unit ball of E
$L(E, F)$	The set of all linear operators
$\mathcal{L}(E, F)$	The set of all bonded linear operators
w	The weak topology
w^*	The weak * topology
$\text{Lip}_0(X, Y)$	The set of all Lipschitz operators between X and Y that vanish at 0
$X^\# = \text{Lip}_0(X, \mathbb{R})$	The Lipschitz dual of the pointed metric space X
$\mathcal{M}(X)$	The linear space of all molecules on the metric space X
$m_{xx'}$	The molecule defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$ for $x, x' \in X$, where χ_A is the characteristic function of the set A
Π_p^{SL}	The space of all strictly Lipschitz p -summing operators
$\mathcal{A}\mathcal{E}(X)$	The Arens-Eells space of X
$\mathcal{F}(X)$	The Lipschitz free space of X
T^*	The adjoint of linear operator T
$T^\#$	The Lipschitz adjoint of Lipschitz operator T
T_L	The linearization of the Lipschitz operator T
$T^t = T^\# \setminus E^*$	The Lipschitz transpose of Lipschitz map $T : X \rightarrow E$
$J_p(1 \leq p < \infty)$	The canonical inclusion map defined from $C(K)$ to $L_p(\mu)$
$I_{\infty, p}$	The formal inclusion map defined from $L_\infty(\mu)$ to $L_p(\mu)$ for $(1 \leq p < \infty)$

i_E	The isometric embedding, $i_E : E \longrightarrow C(E^*)$ given by $i_E(x) := \langle x, \cdot \rangle$
Π_p	The set of all linear p -summing operators ($1 \leq p < \infty$)
$\cdot \Pi_p^L$	The set of all Lipschitz p -summing operator ($1 \leq p < \infty$)
$\Pi_{p,r,s}$	The set of all (p, r, s) -summing operators ($1 \leq p, r, s \leq \infty$)
$\Pi_{p,r,s}^L$	The set of all Lipschitz (p, r, s) -summing operators ($1 \leq p, r, s \leq \infty$)
\mathcal{D}_p	The set of all strongly linear p -summing operators ($1 < p \leq \infty$)
$\mathcal{D}_{st,p}^L$	The set of all strongly Lipschitz p -summing operators ($1 < p \leq \infty$)
\mathcal{N}_p	The set of all p -nuclear linear operators ($1 \leq p < \infty$)
\mathcal{N}_p^L	The set of all Lipschitz Cohen p -nuclear operators ($1 \leq p < \infty$)
Π_p^{SL}	The set of all strictly Lipschitz p -summing operator ($1 \leq p < \infty$)
$E \otimes F$	The tensor product between two Banach spaces
$E \otimes_\alpha F$	The tensor product $E \otimes F$ endowed with the tensor norm α
$E \widehat{\otimes}_\alpha F$	The completion of $E \otimes_\alpha F$
α^t	The transpose of the tensor norm α
α' and α^*	The dual and the conjugate of the tensor norm α
d_p and g_p	Chavet-Saphar tensor norms
ε and π	The injective and projective tensor norms
$X \boxtimes Y$	Lipschitz tensor product between X and Y
$X \widehat{\boxtimes}_\varepsilon E$	The injective Lipschitz tensor product of X and E
$X \widehat{\boxtimes}_\pi E$	The projectiv tensor product of X and E .

List of publications

1. MAATOUGUI BELAALA, Lipschitz ϕ -summing operators. *Analele Universităţii Oradea Fasc. Matematica*, Tom XXIV (2017), Issue No. 2, 175(180).
2. MAATOUGUI BELAALA AND KHALIL SAADI , Further results on strictly Lipschitz summing operators (submitted).

Introduction

The beginning point of the theory of linear summing operators is the fundamental work of A.Grothendieck [17]. His work was done in the middle of the last century, just in 1955, he had introduced the notion of p -summing operators (for $p = 1$), but also in 1967 and 1968, the classical works of Pietsch [26] and Lindenstrauss-Pełczyński [23] clarified Grothendieck's precious ideas and contributed clearly to the vigorous development of this notion, for ($p > 0$), especially, the book [27] of A.Pietsch which constitutes a culminating point in the progress of this theory. After that a lot of authors have interested to study this notion by other ways and giving results in the linear case like Cohen Strongly and p -nuclear operators, also in the multilinear case like p -dominated and Cohen Strongly p -summing operators. The same procedure happened in the polynomial and sublinear cases.

A linear operator T between two Banach spaces E and F is called p -summing ($1 \leq p < \infty$) if there exists a positive constant C such that for all x_1, \dots, x_n in E , we have

$$\left(\sum_{i=1}^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}}.$$

The infimum of such constants is denoted by $\pi_p(T)$ and the space of all p -summing operators denoted by $\Pi_p(E, F)$ which is a Banach space under the norm $\pi_p(T)$.

In 1987 Roshdi Khalil and Wassim Deeb have given in [20] the notion of ϕ -summing operators. A linear operator $T \in \mathcal{L}(E, F)$ is called ϕ -summing if there exists a constant $C \geq 0$ such that for all $(x_i)_{i=1}^n$ in E

$$\sum_{i=1}^n \phi \|T(x_i)\| \leq C \sup_{x^* \in B_{E^*}} \sum_{i=1}^n \phi |\langle x^*, x_i \rangle|$$

where ϕ is a modulus function.

The smallest constant $C \geq 0$ such this inequality holds is called the metric of T , and denoted by $\pi_\phi(T)$. This class of mappings is denoted by $\Pi_\phi(E, F)$. A nonlinear generalization

of p -summing operators, was introduced by Farmer and Johnson in [14]. Recall that a Lipschitz mapping $T : X \rightarrow Y$ between two metric spaces is *Lipschitz p -summing* ($1 \leq p < \infty$) if there exists a constant $C \geq 0$ such that for all $(x_i)_{i \leq n}, (x'_i)_{i \leq n}$ in X and all $(a_i)_{i \leq n} \subset \mathbb{R}^+$

$$\left(\sum_{i=1}^n a_i d(T(x_i), T(x'_i))^p \right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n a_i |f(x_i) - f(x'_i)|^p \right)^{\frac{1}{p}}.$$

This notion had already played a role in earlier works of several others, so the Lipschitz p -summing operators are expected to play a similar important role for applications in the nonlinear case as the linear p -summing operators for the linear theory.

J.D.Farmer and W.B.Johnson proved that this is true extension of the linear concept and obtained a nonlinear counterpart of interesting fundamental characterization theorem with this theorem, they proved that the *Lipschitz p -summing* norm of a linear operator is the same as its p -summing norm.

In 2017, Khalil Saadi has introduced in [4] the notion of strictly Lipschitz p -summing, Let $1 \leq p \leq \infty$, a Lipschitz operator $T : X \rightarrow E$ is said to be *strictly Lipschitz p*

-summing if there exists a positive constant C such that for every $x_k, y_k \in X$ and $s_k^* \in E^*$ ($1 \leq k \leq l$) we have

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C d_p^L(u),$$

where $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^*$. We denote by $\Pi_p^{SL}(X, E)$ the Banach space of all strictly Lipschitz p -summing operators from X into E which its norm $\pi_p^{SL}(T)$ is the smallest constant C verifying the above inequality.

The aim of this thesis is the generalization and the extending of some results and properties from linear ϕ -summing operators to new notion of Lipschitz ϕ -summing operators and Lipschitz strictly p -summing operators to M -strictly Lipschitz p -summing operators, and from Lipschitz p -nuclear and strictly Lipschitz (p, r, s) -summing .

J.D.Farmer and W.B.Johnson in their paper [14], titled(The Lipschitz p -summing operators)have put six open problems, in problem six they have posed the following question. What results about p -summing operators have analogues for Lipschitz p -summing operators? some have answered this question by generalizing some notions from linear case to the lipschitz case, for example: p -integral, p -nuclear, Lipschitz strongly p -summig operator, compact operator and weakly compact operator.

This work is divided in two important parts, in the first we based on our paper [3] which is titled by " Lipschitz ϕ -summing operators, using this notion, our aim is to show some properties of these operators and to prove the domination theorem of such class.

In the next part of this thesis, we have based on the second paper [4] (submitted) presented by M. Belaala and K. Saadi which is called "Further results on strictly Lipschitz summing operators". The main focus of this article is giving some new characterizations of strictly Lipschitz p -summing operators which have been introduced in order to improve the Lipschitz p -summing operators. Therefore, we adapt this definition for constructing other classes of Lipschitz mappings which are called strictly p -nuclear and strictly Lipschitz (p, r, s) -summing operators. Some properties and factorizations results are obtained for these new classes (see [4]).

This thesis is organized as follows:

In chapter 1, we remained some basic definitions and properties which will be used throughout this paper: sequence Banach spaces, linear operator ideal, linear p -summing operator, Lipschitz maps, Lipschitz spaces, the Lipschitz dual of metric spaces, Arens-Eells space, injective operator ideal, molecule space, adjoint of Lipschitz mappings, ideal property, Lipschitz operator ideal, Lipschitz p -summing operator and linear ϕ -summing operator.

The second chapter is one of answers to Farmer's open problem 6 posed in [14].

The basic idea here is to extend as many notions as possible from the linear ϕ -summing operators to the Lipschitz ϕ -summing operators. Therefore, we start by recalling the definition of a modulus function and its properties and a linear ϕ -summing operators defined by R. Khalil and W.Deeb in [20] and their Domination Theorem see also [3].

In this chapter, we introduce a new notion which is called Lipschitz ϕ -summing operator between metric spaces, which is the nonlinear version of linear ϕ -summing operators between Banach spaces. We prove a nonlinear Pietsch domination theorem for such operators and it is shown that this class of operators is a Lipschitz operator ideal in the sense of [1]. Showing that the class of Lipschitz p -summing operators ($0 < p < 1$) given by M.A.S. Saleh [31] is a particular case of this class. In the end, we prove some properties, giving also some results concerning the same class.

In the chapter 3, first we recall the notions of Lipschitz tensor product and strictly Lipschitz p -summing operators and some characterizations theorems. In the second, the notion of M -strictly Lipschitz p -summing operator is introduced and also some characterizations

theorems, in the third we have introduced the notion of strictly Lipschitz p -nuclear operators, giving some characterizations theorem.

In the last chapter, we recall the notions of the linear (p, r, s) -summing operators, Lipschitz (p, r, s) -summing operators and we have also introduced a new notion which is the strictly Lipschitz (p, r, s) -summing operators and some characterizations theorems for this class.

In the end of this thesis, we present a short section collecting some interesting open problems.

Chapter 1

Preliminaries

This chapter is dedicated to recalling some previous concepts that accompany us during the completion of this work to exploit and use them as a means of clarifying the new notions.

Let E and F be Banach spaces. We denote by $\mathcal{L}(E, F)$ the Banach space of all bounded linear operators from E into F with the usual norm. As usual, E^* stands for the dual space of E and B_E for the closed unit ball of E .

For $x \in X$, we shall write $\langle x, x^* \rangle$ (or $\langle x^*, x \rangle$) for the action of the functional x^* on x . The norm of $x^* \in X^*$ is given by

$$\|x^*\| = \sup \{ |\langle x, x^* \rangle| : x \in B_X \}.$$

A linear operator $u : X \rightarrow Y$ between two normed spaces X and Y is an *isomorphism* if u is a continuous bijection whose inverse u^{-1} is also continuous. In such case the spaces X and Y are said to be *isomorphic*. Such a mapping u is an *isometric isomorphism* when $\|u(x)\| = \|x\|$ for all $x \in X$.

A linear operator u is an *embedding of X into Y* if u is an isomorphism onto its image $u(X)$. In this case we say that X *embeds in Y* . If $u : X \rightarrow Y$ is an embedding such that $\|u(x)\| = \|x\|$ for all $x \in X$, then u is said to be an *isometric embedding*.

Given the continuous linear operator $u : X \rightarrow Y$, the continuous linear operator $u^* : Y^* \rightarrow X^*$ defined as

$$u^*(y^*)(x) = y^*(u(x)),$$

for every $y^* \in Y^*$ and $x \in X$ is called the *adjoint of u* and has the property that $\|u^*\| = \|u\|$.

Some sequences Banach spaces Consider $1 \leq p \leq \infty$ and $n \in \mathbb{N}^*$. We denote by $l_p^n(E)$ the Banach space of all sequences $(e_i)_{i=1}^n$ in E with the norm

$$\|(e_i)_i\|_{l_p^n(E)} = (\sum_{i=1}^n \|e_i\|^p)^{\frac{1}{p}},$$

and by $l_p^{n,w}(E)$ the Banach space of all weakly sequences $(e_i)_{i=1}^n$ in E with the norm

$$\|(e_i)_i\|_{l_p^{n,w}(E)} = \sup_{e^* \in B_{E^*}} (\sum_{i=1}^n |\langle e_i, e^* \rangle|^p)^{\frac{1}{p}}.$$

If $E = \mathbb{K}$, we simply write l_p^n .

The $C(K)$ space. If K is a topological space, then by $C(K)$ we mean the space of all scalar valued (i.e. real or complex valued), bounded, continuous functions on K . This is a Banach space with the norm

$$\|f\|_\infty = \sup_{x \in K} |f(x)|.$$

Clearly, if K is a compact space then $C(K)$ consists of all continuous, scalar valued functions. The dual of the space $C(K)$ equals the space $M(K)$ of all regular Borel measures (scalar valued, but obviously not necessarily positive) on K . The duality is defined as

$$\langle f, \mu \rangle = \mu(f) = \int_K f d\mu, \quad f \in C(K), \quad \mu \in M(K).$$

A Banach space Z is injective if whenever Y_0 is a subspace of a Banach space Y , any $u \in \mathcal{L}(Y_0, Z)$ has an extension $\tilde{u} \in \mathcal{L}(Y, Z)$ with $\|u\| = \|\tilde{u}\|$

1.1 Linear operator ideals

Recall that, a linear operator $T \in \mathcal{L}(E, F)$ is said to have finite rank if $T(E)$ is a finite dimensional subspace of F . The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(E, F)$. An operator has rank one if and only if has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle y$$

i.e. if $u \in \mathcal{L}_f(E, F)$ we have

$$u = \sum_{i=1}^n x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$ (see [27, Page 25]).

Definition 1.1. An operator ideal \mathcal{I} is a subclass of the class \mathcal{L} of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components $\mathcal{I}(E, F) := \mathcal{L}(E, F) \cap \mathcal{I}$ satisfy:

(i) $\mathcal{I}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$ which contains the finite rank operators.

(ii) The ideal property: if $v \in \mathcal{L}(G, E)$, $u \in \mathcal{I}(E, F)$ and $w \in \mathcal{L}(F, H)$, then the composition $w \circ u \circ v$ is in $\mathcal{I}(G, H)$.

If $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \rightarrow \mathbb{R}^+$ satisfies:

(i') $(\mathcal{I}(E, F), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F ,

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') if $v \in \mathcal{L}(G, E)$, $u \in \mathcal{I}(E, F)$ and $w \in \mathcal{L}(F, H)$, we have

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \|v\|_{\mathcal{I}} \|u\|,$$

then $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal, for all Banach spaces G, H .

The ideal \mathcal{L}_f of finite rank linear operators is the smallest operator ideal and \mathcal{L} the largest one [27, Theorem 1.2.2].

The operator ideal \mathcal{I} is said to be *closed* if each $\mathcal{I}(E, F)$ is a closed subspace of $\mathcal{L}(E, F)$ for the sup norm.

Definition 1.2. (*injective operator ideal*)

A normed operator ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is said to be *injective* if for every metric injection $i : F \hookrightarrow G$ and every $u \in \mathcal{L}(E, F)$ it follows from $i \circ u \in \mathcal{I}(E, G)$ that $u \in \mathcal{I}(E, F)$. Moreover

$$\|i \circ u\|_{\mathcal{I}} = \|u\|_{\mathcal{I}}.$$

1.2 Ideal of p -summing linear operators.

The theory of p -summing operators is based on a crucial criterion due to Pietsch [27]. We mention that the linear p -summing operators are the starting point in the study of Lipschitz p -summing mappings.

Let $1 \leq p < \infty$. A linear operator $T : E \rightarrow F$ is said to be p -summing if there exists a constant $C \geq 0$ such that for all finite sequence $(x_i)_{1 \leq i \leq n}$ in E

$$\left(\sum_{i=1}^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*} \leq 1} \left(\sum_{i=1}^n |\xi(x_i)|^p \right)^{\frac{1}{p}}. \quad (1.1)$$

The infimum of all such constants C is denoted by $\pi_p(T)$. The collection of all p -summing operators between E and F is denoted by $\Pi_p(E, F)$. We mention that (Π_p, π_p) is an injective Banach operator ideal.

Theorem 1.3. [13, Page 39] If $1 \leq p \leq q < \infty$, then $\Pi_p(E, F) \subset \Pi_q(E, F)$. Moreover, for $T \in \Pi_p(E, F)$ we have $\pi_q(T) \leq \pi_p(T)$.

The following basic result about p -summing operators is due to A. Pietsch, and it characterizes the p -summability by means of a domination theorem.

Theorem 1.4. (Pietsch Domination Theorem) [13, page 44]

Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E, F)$, then T is p -summing if and only if there exist a constant C and a regular Borel probability measure μ on B_{E^*} (with the weak $*$ -topology) so that

$$\|T(x)\| \leq C \left(\int_{B_{E^*}} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{\frac{1}{p}}, \quad x \in E. \quad (1.2)$$

In this case, $\pi_p(T)$ is the least of all constants C such that (1.2) holds.

1.3 Lipschitz spaces

A pointed metric space X is a metric space with a base point denoted by 0. If X is a normed space, its base point will be its origin. If there is no place to confusion, we will denote by d the distance in any metric space.

The natural morphism between metric spaces are Lipschitz functions like linear operators between Banach spaces.

Definition 1.5. Let X and Y be pointed metric spaces. Let us recall that a map $T: X \rightarrow Y$ is Lipschitz if there exists a real constant $C \geq 0$ such that

$$d(T(x), T(y)) \leq Cd(x, y) \quad (1.3)$$

for all $x, y \in X$.

For a Lipschitz map $T: X \rightarrow Y$ we define its Lipschitz constant by

$$Lip(T) = \sup \left\{ \frac{d(T(x), T(y))}{d(x, y)}, x \neq y \right\}.$$

i.e. the smallest constant C such that the inequality (1.3) holds. By $Lip_0(X, Y)$ the set of all Lipschitz maps f from X into Y for which $T(0) = 0$. If Y is a Banach space, $Lip_0(X, Y)$ is a Banach space with the Lipschitz norm $Lip(\cdot)$. It is known that $Lip_0(X, \mathbb{R})$ denoted frequently as $X^\#$, It is a dual Banach space and $B_{X^\#}$ is a compact Hausdorff space endowed with the pointwise topology. We denote by $C(B_{X^\#})$ the Banach space of all bounded continuous real-valued functions with the supremum norm.

Proposition 1.6. *Let X, Y and Z be metric spaces and let $f : (X, d_X) \longrightarrow (Y, d_Y)$, $g : (Y, d_Y) \longrightarrow (Z, d_Z)$ be Lipschitz maps. Then $g \circ f : (X, d_X) \longrightarrow (Z, d_Z)$ is Lipschitz and $\text{Lip}(g \circ f) \leq \text{Lip}(g) \text{Lip}(f)$.*

Proof. For x, y in X , we have

$$\begin{aligned} d_Z(g \circ f(x), g \circ f(y)) &\leq \text{Lip}(g) d_Y(f(x), f(y)) \\ &\leq \text{Lip}(g) \text{Lip}(f) d_X(x, y) \end{aligned}$$

and this shows the proposition. \square

Now, we are going to present some concepts about the space of molecules, the reader can see [35] or [2] for more details.

Definition 1.7. *Let X be a metric space. A molecule on X is a scalar valued function m on X with finite support that satisfies $\sum_{x \in X} m(x) = 0$. We denote by $\mathcal{M}(X)$ the linear space of all molecules on X .*

For $x, x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A . For $m \in \mathcal{M}(X)$ we can write

$$m = \sum_{j=1}^n \lambda_j m_{x_j x'_j}$$

for some suitable scalars λ_j , and we write

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^n |\lambda_j| d(x_j, x'_j), m = \sum_{j=1}^n \lambda_j m_{x_j x'_j} \right\},$$

where the infimum is taken over all representations of the molecule m . Denote by $\mathcal{A}(X)$ the completion of the normed space $(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells [2] in 1956. The terminology of *Arens-Eells space* $\mathcal{A}(X)$ is due to Weaver [35]. A different notations were used in [16] by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X)$ that is the closed linear span of the linear forms $\delta_{(x,y)}$ of $\text{Lip}_0(X)^*$ such that

$$\delta_{(x,y)}(f) = f(x) - f(y), \text{ for every } f \in \text{Lip}_0(X),$$

i.e.,

$$\mathcal{F}(X) = \overline{\text{span} \{ \delta_{(x,y)} : x, y \in X \}}^{\text{Lip}_0(X)^*}$$

We have $X^\# = \mathcal{F}(X)^*$ holds isometrically through the mapping

$$Q_X(f)(m) = m(f), \text{ for every } f \in X^\# \text{ and } m \in \mathcal{F}(X).$$

We can see $\mathcal{F}(X)$ as the completion of the set of all measures μ of finite support under the norm

$$\|\mu\| = \sup \{ \int f d\mu : \text{Lip}(f) \leq 1 \}.$$

Let X be a pointed metric space

1. The map $\delta_X : X \longrightarrow \mathcal{A}(X)$ defined by

$$\delta_X(x) = m_{x0}$$

is an isometric embedding of X into $\mathcal{A}(X)$ (see [35, Theorem 2.2.4]).

2. The map $Q_X : X^\# \longrightarrow \mathcal{A}(X)^*$ defined by

$$Q_X(f) = f_L, \text{ where } f_L(m) = \sum_{x \in X} f(x)m(x),$$

establishes an isometric isomorphism between $X^\#$ and $\mathcal{A}(X)^*$. On a bounded subsets of $X^\#$ its pointwise topology (see [35, Theorem 2.2.2]).

3. $\|\cdot\|_{\mathcal{A}}$ is a norm on the space of molecules and $\|m_{xy}\|_{\mathcal{A}} = d(x, y)$ for all $x, y \in X$.

4. For any molecule m and by Hahn-Banach theorem we have $\|m\|_{\mathcal{A}} = \sup_{f \in B_{X^\#}} |\langle m, f \rangle|$

The Banach space $\mathcal{A}(X)$ has some remarkable properties. We mention the following ones.

Theorem 1.8. ([35, Theorem 2.2.4])

Let $T : X \longrightarrow E$ be a Lipschitz map which preserves the base point; that is $T(0) = 0$. Then there is a unique bounded linear map $T_L : \mathcal{A}(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$, i.e. the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & E, \\ & \searrow \delta_X & \nearrow T_L \\ & \mathcal{A}(X), & \end{array}$$

commutes. Furthermore $\|T_L\| = \text{Lip}(T)$

The operator T_L is referred as the linearisation of T . The correspondence

$$T \longleftrightarrow T_L$$

establishes an isomorphism isometric between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}(\mathcal{A}(X), E)$.

Theorem 1.9. ([16, Lemma 3.1])

Let $T : X \rightarrow Y$ be a Lipschitz map which preserves the base point. Then there is a unique bounded linear map $\hat{T} : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)$ such that $\hat{T}\delta_X = \delta_Y T$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow \delta_X & & \downarrow \delta_Y \\ \mathcal{A}(X) & \xrightarrow{\hat{T}} & \mathcal{A}(Y) \end{array}$$

commutes. Furthermore $\|\hat{T}\| = Lip(T)$.

If X is a Banach space and $T : X \rightarrow E$ is a linear operator, then the corresponding linear operator \hat{T} is given by

$$\hat{T} = T \circ \beta_X,$$

where $\beta_X : \mathcal{F}(X) \rightarrow X$ is linear quotient map which verifies $\beta_X \circ \delta_X = id_X$ and $\|\beta_X\| \leq 1$, see [16, p 124] for more details about the operator β_X .

Theorem 1.10. Let X, Y be two metric spaces. Let $T : X \rightarrow Y$ be a Lipschitz operator, then there is a unique linear operator \tilde{T} such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow \delta_X & & \downarrow \delta_Y \\ \mathcal{F}(X) & \xrightarrow{\tilde{T}} & \mathcal{F}(Y) \end{array}$$

i.e., $\delta_Y \circ T = \tilde{T} \circ \delta_X$.

1.4 Adjoint of Lipschitz mapping

Let X, Y be pointed metric spaces. Sawashima in [33] defined the Lipschitz adjoint (or dual) $T^\# : Lip_0(Y) \rightarrow Lip_0(X)$ of a Lipschitz map $T \in Lip_0(X, Y)$ by the formula

$$T^\#(g) = g \circ T, \quad g \in Lip_0(Y)$$

He showed that $T^\#$ is a continuous linear operator and

$$\|T^\#\| = \text{Lip}(T) = \|T^\#|_{Y^*}\| \text{ (if } Y \text{ is a Banach space).}$$

If $Y = E$ is a Banach space, the restriction of $T^\#$ on E^* is called the Lipschitz transpose map of T and is denoted here by T^t .

$$T^\# = Q_X^{-1} \circ \tilde{T}^* \circ Q_Y \text{ and } T^t = Q_X^{-1} \circ \hat{T}^*. \quad (1.4)$$

The correspondence

$$T \longleftrightarrow T^t$$

establishes an isomorphism between the vector spaces $\text{Lip}_0(X, E)$ and $\mathcal{L}((E^*, w^*), (X^\#, w^*))$, where w^* denote the weak*-topology (see [18, Theorem 3.1]).

1.5 Lipschitz p -summing operators

The nonlinear version of p -summing operators ($1 \leq p < \infty$) was introduced by J.D. Farmer and W.B. Johnson in [14]. Let X, Y be metric spaces. In the sequel, we consider (X, d_X) a pointed metric space. A mapping $T : X \rightarrow Y$ is *Lipschitz p -summing* if there exists a constant $C \geq 0$ such that for all $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n}$ in X and all $(a_i)_{1 \leq i \leq n} \subset \mathbb{R}^+$

$$\sum_{i=1}^n a_i d(T(x_i), T(x'_i))^p \leq C^p \sup_{f \in B_{X^\#}} \sum_{i=1}^n a_i |f(x_i) - f(x'_i)|^p.$$

The infimum of all such constants $C \geq 0$ is denoted by $\pi_p^L(T)$. The class of all this mappings is denoted by $\Pi_p^L(X, Y)$.

Notice that the definition is the same if we restrict to $a_i = 1$, we can find it implicitly in [14].

Proposition 1.11. (*Ideal property*)

Let X, Y be metric spaces and E, F be Banach spaces. Let $v : E \rightarrow X, w : Y \rightarrow F$ be Lipschitz functions and $T : X \rightarrow Y$ be a Lipschitz p -summing operator. Then wTv is Lipschitz p -summing operator and $\pi_p^L(wTv) \leq \text{Lip}(w)\pi_p^L(T)\text{Lip}(v)$.

Proof. According to the definition of Lipschitz function and the definition of Lipschitz p -summing operator, we can find

$$\begin{aligned}
\sum_{i=1}^n d_F(wTv(x_i), wTv(y_i))^p &\leq Lip(w)^p \sum_{i=1}^n d_Y(Tv(x_i), Tv(y_i))^p \\
&\leq Lip(w)^p \pi_p^L(T)^p \sup_{f \in B_{X^\#}} \sum_{i=1}^n |f(v(x_i)) - f(v(y_i))|^p \\
&\leq Lip(w)^p \pi_p^L(T)^p Lip(v)^p \sup_{f \in B_{X^\#}} \sum_{i=1}^n \left| \frac{f \circ v}{Lip(v)}(x_i) - \frac{f \circ v}{Lip(v)}(y_i) \right|^p \\
&\leq Lip(w)^p \pi_p^L(T)^p Lip(v)^p \sup_{g \in B_{E^\#}} \sum_{i=1}^n |g(x_i) - g(y_i)|^p.
\end{aligned}$$

□

The following theorem is a characterization of the Lipschitz p -summing operators.

Theorem 1.12. [14, Theorem 1] *Let $1 \leq p < \infty$. The following properties are equivalent, for a mapping $T \in Lip_0(X, Y)$ and a positive constant C .*

1. $\pi_p^L(T) \leq C$.
2. *There is a probability μ on $B_{X^\#}$ such that*

$$d(T(x), T(y)) \leq C \left(\int_{B_{X^\#}} |f(x) - f(y)|^p d\mu(f) \right)^{\frac{1}{p}}.$$

(Pietsch domination theorem).

The domination theorem immediately implies the monotonicity of the Lipschitz p -summing norm.

Theorem 1.13. *If $1 \leq p \leq q < \infty$, then $\Pi_p^L(X, Y) \subset \Pi_q^L(X, Y)$. Moreover, $\pi_q^L(T) \leq \pi_p^L(T)$.*

For a linear operator $T \in \mathcal{L}(E, F)$, it is clear that $\pi_p^L(T) \leq \pi_p(T)$. J.D. Farmer and W.B. Johnson proved that the reverse inequality is true. This justifies that the notion of Lipschitz p -summing operator is really a generalization of the concept of linear p -summing operator.

Theorem 1.14. [14, Theorem 2] *Let T be a bounded linear operator from E into F and $1 \leq p < \infty$. Then $\pi_p^L(T) = \pi_p(T)$.*

1.6 Lipschitz operator ideals

Definition 1.15. [1] A Lipschitz operator ideal \mathcal{I}_{Lip} is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components

$$\mathcal{I}_{Lip}(X, E) := Lip_0(X, E) \cap \mathcal{I}_{Lip}$$

satisfy:

(i) $\mathcal{I}_{Lip}(X, E)$ is a linear subspace of $Lip_0(X, E)$.

(ii) $vg \in \mathcal{I}_{Lip}(X, E)$ for $v \in E$ and $g \in X^\#$.

(iii) The ideal property: if $S \in Lip_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $w \in \mathcal{L}(E, F)$, then the composition wTS is in $\mathcal{I}_{Lip}(Y, F)$.

A Lipschitz operator ideal \mathcal{I}_{Lip} is a normed (Banach) Lipschitz operator ideal if there is $\|\cdot\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \rightarrow [0, +\infty[$ that satisfies

(i') For every pointed metric space X and every Banach space E , the pair $(\mathcal{I}_{Lip}(X, E), \|\cdot\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X, E)$.

(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \rightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.

(iii') If $S \in Lip_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $w \in \mathcal{L}(E, F)$, then

$$\|wTS\|_{\mathcal{I}_{Lip}} \leq Lip(S) \|T\|_{\mathcal{I}_{Lip}} \|w\|.$$

1.7 Some definitions

In what follows we recall some important definitions which we need in the sequel. We refer to [12, 13, 22] for more details about the following notions. Let $1 \leq p \leq \infty$ and p^* its conjugate, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. Let E, F be two Banach spaces and $R : E \rightarrow F$ be a linear operator. Then,

- The linear operator R is Cohen strongly p -summing if there exists a constant $C > 0$ such that, for any $x_1, \dots, x_n \in E$, and any $y_1^*, \dots, y_n^* \in F^*$, we have

$$\sum_{i=1}^n |(R(x_i), y_i^*)| \leq C \| (x_i) \|_{l_p^n(E)} \| (y_i^*) \|_{l_{p^*}^{n,w}}.$$

The class of Cohen strongly p -summing operators from E into F , which is denoted by $\mathcal{D}_p(E, F)$, is a Banach space for the norm $d_p(R)$, i.e., the smallest constant C such that the

above inequality holds. If $1 \leq p < \infty$, we have by [12, Theorem 2.2.2]

$$R \in \Pi_p(E; F) \Leftrightarrow R^* \in \mathcal{D}_{p^*}(F^*; E^*).$$

- The linear operator R is p -nuclear if there exists a constant $C > 0$ such that, for any $x_1, \dots, x_n \in E$, and any $y_1^*, \dots, y_n^* \in F^*$, we have

$$\sum_{i=1}^n |\langle R(x_i), y_i^* \rangle| \leq C \| (x_i) \|_{l_p^{n,w}(E)} \| (y_i^*) \|_{l_{p^*}^{n,w}}.$$

The class of Cohen p -nuclear linear operators from E into F , which is denoted by $\mathcal{N}_p(E, F)$, is a Banach space for the norm $\mathcal{N}_p(\cdot)$, i.e., the smallest constant C such that the above inequality holds. We have by [12, 21]: R is p -nuclear if and only if, $R = R_1 \circ R_2$ where R_1 is Cohen strongly p -summing and R_2 is p summing.

Let $1 \leq p \leq \infty$. Lipschitz p -nuclear operators from metric spaces to Banach spaces are introduced in [9]. A map $T \in Lip_0(X, E)$ is *Lipschitz p -nuclear* if there are two Lipschitz mappings $A : \ell_p \rightarrow E$ and $B : X \rightarrow \ell_\infty$ and $\lambda = (\lambda_n)_n \in \ell_p$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ \downarrow B & & \uparrow A \\ \ell_\infty & \xrightarrow{M_\lambda} & \ell_p \end{array}$$

where $M_\lambda((x_n)_n) := (\lambda_n x_n)_n$, for all $(x_n)_n \in \ell_\infty$. Let $\nu_p^L(T)$ denote the infimum of $Lip(A) \|M_\lambda\| Lip(B)$ over all factorizations as above; note that $\|M_\lambda\| = \|\lambda\|_{\ell_p}$. The set of all such T is denoted by $\mathcal{N}_p^L(X, E)$.

Chapter 2

Lipschitz ϕ -summing operators

The results obtained in this chapter have been published in Annals of Oradea University - Mathematics Fascicola [3]. In this chapter we contribute to the theory of absolutely p -summing Lipschitz mappings by studying the class of Lipschitz ϕ -summing mappings. we have shown that this class of operators is a Lipschitz operator ideal in the sense of [1]. After giving the domination Theorem of this class, we have studied the relationships between a Lipschitz operator T and its linearization \hat{T} .

2.1 Introduction

Our purpose in this chapter is to extend this concept to a more general setting introducing the notion of Lipschitz ϕ -summing operator between pointed metric spaces. Our notion generalizes that of Lipschitz p -summing operator given by M. A. S. Saleh in [31] for $0 < p < 1$. The study of Lipschitz p -summing operators with $p \geq 1$ between metric spaces was initiated by J. D. Farmer and W. B. Johnson in [14]. They stated a nonlinear version for such operators of the celebrated Pietsch factorization-domination theorem of p -summing linear operators [27]. In [5], G. Botelho, D. Pellegrino and P. Rueda proved an abstract version of the Pietsch domination theorem (improved in [24] by D. Pellegrino and J. Santos) which gives a unified proof for the Pietsch domination theorem, for the result of J. D. Farmer and W. B. Johnson and for other known Pietsch domination theorems in classes of mappings that generalize the ideal of p -summing linear operators. We study in this chapter the basic properties of Lipschitz ϕ -summing operators and establish a nonlinear version of Pietsch domination theorem for such operators which is not covered by the general Pietsch domination theorem stated in [24].

2.2 Modulus function

Definition 2.1. *A function $\phi: [0, \infty[\rightarrow [0, \infty[$ is called a modulus function if*

1. ϕ is a continuous function on $[0, \infty[$
2. $\phi(0) = 0$.
3. $\phi(x + y) \leq \phi(x) + \phi(y), \forall x, y \in [0, \infty[$ (subadditive)
4. strictly increasing.

Properties :

From the definition of modulus function, we can find:

1. If ϕ a modulus function then
 - ϕ is bijective
 - ϕ^{-1} is not modulus function

2. The composition of two modulus functions is a modulus function and is not commutative. $\forall n \in \mathbb{N}^* : \phi^n = \underbrace{\phi \circ \dots \circ \phi}_{n \text{ times}}$ is a modulus function whenever ϕ is a modulus function and $\phi^n(\alpha x) \leq (n + \alpha)\phi^n(x), \alpha > 0$.
3. The composition of modulus functions is associative
4. There is a modulus function e such that for every modulus function $\phi : \forall x \in [0, +\infty[$:
 $\phi \circ e(x) = \phi(x)$ and $e \circ \phi(x) = \phi(x)$ where $e(x) = x$

Example 2.2. *The functions $\phi(t) = t^p$, $\phi(t) = \max\{t, t^p\}$ and $\phi(t) = \ln(1 + t)$, $\phi(t) = t/(1 + t)$ ($t \geq 0$) for $t \geq 0$ when $0 < p \leq 1$ are modulus. We prove that $\phi(t) = t/(1 + t)$ is a modulus function.*

- ϕ is continuous on $[0, +\infty[$.
- $\phi(0) = 0$.
- ϕ is strictly increasing on $[0, +\infty[$.

$$\begin{aligned} \phi(x + y) &= \frac{x+y}{1+x+y}, x, y \in [0, +\infty[\\ &= \frac{x}{1+x+y} + \frac{y}{1+x+y} \\ &\leq \frac{x}{1+x} + \frac{y}{1+y} \end{aligned}$$

i.e $\phi(x + y) \leq \phi(x) + \phi(y)$

Remark 2.3. *From the property 2 and the above example, we note for example that $\phi(t) = \ln(1 + t^p)$ is modulus function, also if ϕ a modulus function then $\frac{\phi}{\phi+1}$ is a modulus function.*

Through this Chapter, we will use the important inequality

$$\phi(at) \leq (1 + a)\phi(t)$$

for all $t \geq 0$ when $a > 0$, which can be deduced of the properties of ϕ .

2.3 Linear ϕ -summing operators

For Banach spaces X and Y , a bounded linear operator $T: X \rightarrow Y$ is called ϕ -summing if there exists a constant $C \geq 0$ such that

$$\sum_{i=1}^n \phi(\|T(x_i)\|) \leq C \sup_{x^* \in B_{X^*}} \sum_{i=1}^n \phi(|x^*(x_i)|)$$

for all finite set $(x_i)_{i=1}^n$ in X . The infimum of such constants C is denoted by $\pi_\phi(T)$ and the set of such maps T by $\Pi_\phi(X; Y)$.

This concept was first considered with the preceding definition by B. Rosenberger [28] and R. Khalil and W. Deeb [20], and more generally by N. Tita [34].

The following results can be found in [28].

Theorem 2.4. (*ideal property*)

Let $S \in \mathcal{L}(Y, X)$, $T \in \Pi_\phi(X, E)$, and $w \in \mathcal{L}(E, F)$, then wTS is linear ϕ -summing and $\pi_\phi(wTS) \leq (1 + \|w\|)\pi_\phi(T)(1 + \text{Lip}(S))$.

Theorem 2.5. *The followings are equivalent for a bounded linear operator $T : E \rightarrow F$ and a positive constant C .*

(i) $\pi_\phi(T) \leq C$.

(ii) *There is a probability μ on B_{E^*} such that*

$$\phi(\|T(x)\|) \leq C \int_{B_{E^*}} \phi|\xi(x)| d\mu(\xi).$$

Theorem 2.6. *Let X a Banach space with the metric approximation property and ϕ a modulus function, then $\Pi_\phi(X, X) = \Pi_p(X, X)$, for $0 < p \leq 1$.*

2.4 Lipschitz ϕ -summing operators

Definition 2.7. *Let ϕ be a modulus function and X, Y be pointed metric spaces.*

A Lipschitz map $T \in \text{Lip}(X, Y)$ is called Lipschitz ϕ -summing if there exists a constant $C \geq 0$ such that regardless of the choice of points $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X and the choice of numbers $(a_i)_{i=1}^n$ in \mathbb{R}^+ , we have the inequality

$$\sum_{i=1}^n a_i \phi(d(T(x_i), T(x'_i))) \leq C \sup_{f \in B_{X^\#}} \sum_{i=1}^n a_i \phi(|f(x_i) - f(x'_i)|).$$

The infimum of such constants C is denoted by $\pi_\phi^L(T)$, and the set of such maps T by $\Pi_\phi^L(X, Y)$.

Remark 2.8. *Definition 2.7 is the same if we restrict to $a_i = 1$ (see [14]).*

Taking the modulus function $\phi(t) = t^p$ ($0 < p \leq 1$) for all $t \geq 0$, Definition 2.7 covers the notions of Lipschitz p -summing operators with $p \in (0, 1)$ introduced by M. A. S. Saleh [31], and of Lipschitz 1-summing operators given by J. D. Farmer and W. B. Johnson [14].

Lemma 2.9. *If (X, d) is a pointed metric space and ϕ a modulus function then $d_\phi = \phi \circ d$ is a distance on X .*

Proof. For all $x, y \in X$ we have

$$\begin{aligned} d_\phi(x, y) = 0 &\Leftrightarrow \phi \circ d(x, y) = 0 \\ &\Leftrightarrow d(x, y) = 0 \\ &\Leftrightarrow x = y. \end{aligned}$$

And

$$\begin{aligned} d_\phi(x, y) &= \phi \circ d(x, y) \\ &= \phi \circ d(y, x) \\ &= d_\phi(y, x) \end{aligned}$$

Finally for all $x, y, z \in X$, we have

$$d(x, z) \leq d(x, y) + d(y, z) \implies \phi \circ d(x, z) \leq \phi \circ d(x, y) + \phi \circ d(y, z)$$

i.e. $d_\phi(x, z) \leq d_\phi(x, y) + d_\phi(y, z)$. □

From this lemma we can define $\text{Lip}_\phi(X, Y) = \text{Lip}((X, \phi \circ d), (Y, \phi \circ d))$ and

$$\text{Lip}_\phi(T) = \sup \left\{ \frac{\phi(d(T(x), T(y)))}{\phi(d(x, y))} : x, y \in X, x \neq y \right\}$$

for all $T \in \text{Lip}_\phi(X, Y)$.

Note that when $\phi(x) = x$, we find $\text{Lip}_\phi(T) = \text{Lip}(T)$.

The following assertions can be checked easily.

Proposition 2.10. *Let ϕ be a modulus function and X, Y are pointed metric spaces, then*

1. $\text{Lip}(X, Y) \subset \text{Lip}_\phi(X, Y)$ and $\text{Lip}_\phi(T) \leq 1 + \text{Lip}(T)$ for all $T \in \text{Lip}(X, Y)$.
2. $\Pi_\phi^L(X, Y) \subset \text{Lip}_\phi(X, Y)$ and $\text{Lip}_\phi(T) \leq \pi_\phi^L(T)$ for all $T \in \Pi_\phi^L(X, Y)$.

Proof. (1) If $T \in \text{Lip}(X, Y)$, then $d(T(x), T(y)) \leq \text{Lip}(T)d(x, y)$ for all $x, y \in X$, and thus

$$\phi(d(T(x), T(y))) \leq \phi(\text{Lip}(T)d(x, y)) \leq (1 + \text{Lip}(T))\phi(d(x, y))$$

for all $x, y \in X$, and this proves (1).

(2) If $T \in \Pi_\phi^L(X, Y)$, we have

$$\sum_{i=1}^n \phi(d(T(x_i), T(x'_i))) \leq \pi_\phi^L(T) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|)$$

for all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X$. In particular,

$$\phi(d(T(x), T(y))) \leq \pi_\phi^L(T) \sup_{f \in B_{X^\#}} \phi(|f(x) - f(y)|) = \pi_\phi^L(T) \phi(d(x, y))$$

for all $x, y \in X$, and thus we obtain (2). For the last equality above, notice that

$$\phi(d(x, y)) = \phi(|h_y(x) - h_y(y)|) \leq \sup_{f \in B_{X^\#}} \phi(|f(x) - f(y)|) \leq \phi(d(x, y)),$$

where h is the function in $B_{X^\#}$ defined by $h_y(z) = d(z, y) - d(0, y)$ for all $z \in X$ and we have $h_y(x) - h_y(y) = d(x, y)$. \square

We now study the relation between a Lipschitz ϕ -summing operator and its linear version. Using the definition of ϕ -summing operator, the following proposition can be checked easily.

Proposition 2.11. *Let ϕ be a modulus function, X, Y are Banach spaces and $T: X \rightarrow Y$ be a ϕ -summing bounded linear operator. Then T is Lipschitz ϕ -summing and $\pi_\phi^L(T) \leq \pi_\phi(T)$.*

Proof. Let $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X$. We have

$$\begin{aligned} \sum_{i=1}^n \phi(\|T(x_i) - T(x'_i)\|) &= \sum_{i=1}^n \phi(\|T(x_i - x'_i)\|) \\ &\leq \pi_\phi(T) \sup_{x^* \in B_{X^*}} \sum_{i=1}^n \phi(|x^*(x_i - x'_i)|) \\ &= \pi_\phi(T) \sup_{x^* \in B_{X^*}} \sum_{i=1}^n \phi(|x^*(x_i) - x^*(x'_i)|) \\ &\leq \pi_\phi(T) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|), \end{aligned}$$

and the proposition follows. \square

We next show that $\Pi_\phi^L(X, Y)$ has the ideal property.

Proposition 2.12. *Let ϕ be a modulus function, X, Y, Z, W be pointed metric spaces, $B \in \text{Lip}(X, Y)$, $T \in \Pi_\phi^L(Y, Z)$ and $A \in \text{Lip}_\phi(Z, W)$. Then $ATB \in \Pi_\phi^L(X, W)$ and $\pi_\phi^L(ATB) \leq \text{Lip}_\phi(A) \pi_\phi^L(T) (1 + \text{Lip}(B))$.*

Proof. For all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X , using the definition of Lipschitz ϕ -summing operator, We have

$$\begin{aligned} \sum_{i=1}^n \phi(d(ATB(x_i), ATB(x'_i))) &\leq \text{Lip}_\phi(A) \sum_{i=1}^n \phi(d(TB(x_i), TB(x'_i))) \\ &\leq \text{Lip}_\phi(A) \pi_\phi^L(T) \sup_{g \in B_{Y^\#}} \sum_{i=1}^n \phi(|g(B(x_i)) - g(B(x'_i))|), \end{aligned}$$

we use the inequality $\phi(at) \leq (1+a)\phi(t)$, we find

$$\begin{aligned} \phi(|g(B(x_i)) - g(B(x'_i))|) &= \phi(\text{Lip}(B) \left| \frac{g \circ B}{\text{Lip}(B)}(x_i) - \frac{g \circ B}{\text{Lip}(B)}(x'_i) \right|) \\ &\leq (1 + \text{Lip}(B)) \phi\left(\left| \frac{g \circ B}{\text{Lip}(B)}(x_i) - \frac{g \circ B}{\text{Lip}(B)}(x'_i) \right| \right). \end{aligned}$$

Then, the above inequality can be estimated as

$$\sum_{i=1}^n \phi(d(ATB(x_i), ATB(x'_i))) \leq \text{Lip}_\phi(A) \pi_\phi^L(T) (1 + \text{Lip}(B)) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|).$$

Then $ATB \in \Pi_\phi^L(X, W)$ and $\pi_\phi^L(ATB) \leq \text{Lip}_\phi(A) \pi_\phi^L(T) (1 + \text{Lip}(B))$. \square

It is clear that $\Pi_\phi^L(X, Y)$ also enjoys the injectivity property. More specifically, we have:

Proposition 2.13. *Let ϕ be a modulus function and let X, Y, Z be pointed metric spaces. If $\iota: Y \rightarrow Z$ is an isometry, then $T \in \Pi_\phi^L(X, Y)$ if and only if $\iota T \in \Pi_\phi^L(X, Z)$. In this case, $\pi_\phi^L(\iota T) = \pi_\phi^L(T)$.*

Proof. It suffices to observe that

$$\sum_{i=1}^n \phi(d(\iota T(x_i), \iota T(x'_i))) = \sum_{i=1}^n \phi(d(T(x_i), T(x'_i)))$$

for all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X$. \square

Proposition 2.14. *Let X be a pointed metric space and E be a Banach space. The space $\Pi_\phi^L(X, E)$ is a Lipschitz operator ideal.*

Proof. (i) Let $T \in \Pi_\phi^L(X, E)$, $\alpha \in \mathbb{K}$. Let ϕ be a modulus function. Then given $x_1, \dots, x_n \in X$; $x'_1, \dots, x'_n \in X$ we obtain

$$\begin{aligned} \sum_{i=1}^n \phi(\|\alpha T(x_i) - \alpha T(x'_i)\|) &= \sum_{i=1}^n \phi(|\alpha| \|T(x_i) - T(x'_i)\|) \\ &\leq (1 + |\alpha|) \sum_{i=1}^n \phi(\|T(x_i) - T(x'_i)\|) \\ &\leq (1 + |\alpha|) C \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi |f(x_i) - f(x'_i)|. \end{aligned}$$

so, αT is Lipschitz ϕ -summing and $\pi_\phi^L(\alpha T) \leq (1 + |\alpha|)\pi_\phi^L(T)$.

Let $T_1, T_2 \in \Pi_\phi^L(X, E)$, ϕ be a modulus function. Then given $x_1, \dots, x_n \in X$; $x'_1, \dots, x'_n \in X$ and from the definition of Lipschitz ϕ -summing operators T_1, T_2 , we obtain

$$\begin{aligned} &\sum_{i=1}^n \phi(\|(T_1 + T_2)(x_i) - (T_1 + T_2)(x'_i)\|) \\ &= \sum_{i=1}^n \phi(\|(T_1(x_i) - T_1(x'_i)) + (T_2(x_i) - T_2(x'_i))\|) \\ &\leq \sum_{i=1}^n \phi(\|T_1(x_i) - T_1(x'_i)\|) + \sum_{i=1}^n \phi(\|T_2(x_i) - T_2(x'_i)\|) \\ &\leq \pi_\phi^L(T_1) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi |f(x_i) - f(x'_i)| + \pi_\phi^L(T_2) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi |f(x_i) - f(x'_i)|. \\ &\leq (\pi_\phi^L(T_1) + \pi_\phi^L(T_2)) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi |f(x_i) - f(x'_i)| \end{aligned}$$

so, $T_1 + T_2$ is Lipschitz ϕ -summing and $\pi_\phi^L(T_1 + T_2) \leq \pi_\phi^L(T_1) + \pi_\phi^L(T_2)$.

(ii) Now let $0 \neq g \in X^\#$, and $v \in E$. For all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X we have

$$\begin{aligned} \sum_{i=1}^n \phi(\|vg(x_i) - vg(x'_i)\|) &= \sum_{i=1}^n \phi(\|v\| |g(x_i) - g(x'_i)|) \\ &\leq (1 + \|v\|) Lip(g) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|) \end{aligned}$$

then vg is Lipschitz ϕ -summing and $\pi_\phi^L(vg) \leq (1 + \|v\|) Lip(g)$.

(iii) Now we prove Π_ϕ^L having the ideal property. Let $S \in Lip_0(Y, X)$, $T \in \Pi_\phi^L(X, E)$, and

$w \in \mathcal{L}(E, F)$. For all $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X we have

$$\begin{aligned} \sum_{i=1}^n \phi(\|wTS(x_i) - wTS(x'_i)\|) &\leq (1 + \|w\|) \sum_{i=1}^n \phi(\|TS(x_i) - TS(x'_i)\|) \\ &\leq (1 + \|w\|) \pi_\phi^L(T) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(\|f \circ S(x_i) - f \circ S(x'_i)\|) \\ &= (1 + \|w\|) \pi_\phi^L(T) (1 + \text{Lip}(S)) \sup_{g \in B_{Y^\#}} \sum_{i=1}^n \phi(|g(x_i) - g(x'_i)|) \end{aligned}$$

which means that wTS is Lipschitz ϕ -summing and $\pi_\phi^L(wTS) \leq (1 + \|w\|) \pi_\phi^L(T) (1 + \text{Lip}(S))$. Finally, $\Pi_\phi^L(X, E)$ is a Lipschitz operator ideal. \square

We next state a version of Pietsch domination theorem for Lipschitz ϕ -summing operators, which is inspired by [14, Theorem 1].

Theorem 2.15. *Let ϕ be a modulus function, let X, Y be pointed metric spaces and let $T \in \text{Lip}(X, Y)$. Then T is Lipschitz ϕ -summing if and only if there exist a constant C and a regular Borel probability measure μ on $B_{X^\#}$ such that*

$$\phi(d(T(x), T(x'))) \leq C \int_{B_{X^\#}} \phi(|f(x) - f(x')|) d\mu(f)$$

for all $x, x' \in X$. In such a case, $\pi_\phi^L(T)$ is the minimum of all the constants C for which such a measure exists.

Proof. Assume that $T \in \Pi_\phi^L(X, Y)$. For any $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X and $(\lambda_i)_{i=1}^n$ in \mathbb{R}^+ , define the function $\varphi_{(\lambda_i, x_i, x'_i)_{i=1}^n} : B_{X^\#} \rightarrow \mathbb{R}$ by

$$\varphi_{(\lambda_i, x_i, x'_i)_{i=1}^n}(f) = \sum_{i=1}^n \lambda_i \phi(d(T(x_i), T(x'_i))) - \pi_\phi^L(T) \sum_{i=1}^n \lambda_i \phi(|f(x_i) - f(x'_i)|).$$

Let A be the set consisting of all such functions.

A is a convex subset of the vector space $C(B_{X^\#})$ indeed let $\varphi_{(x_i, y_i, \alpha_i)_{1 \leq i \leq n}}$ and $\varphi_{(x'_j, y'_j, \beta_j)_{1 \leq j \leq m}} \in A$ such that $\alpha_i > 0, x_i, x'_i \in X$ ($1 \leq i \leq n$) $\beta_j > 0, y_j, y'_j \in X$ ($1 \leq j \leq m$). We have

$$\begin{aligned} \varphi_{(x_i, x'_i, \alpha_i)_{1 \leq i \leq n}}(f) &= \sum_{i=1}^n \alpha_i C \phi(|f(x_i) - f(x'_i)|) - \sum_{i=1}^n \alpha_i \phi(\|T(x_i) - T(x'_i)\|). \\ \varphi_{(y_j, y'_j, \beta_j)_{1 \leq j \leq m}}(f) &= \sum_{j=1}^m \beta_j C \phi(|f(y_j) - f(y'_j)|) - \sum_{j=1}^m \beta_j \phi(\|T(y_j) - T(y'_j)\|). \end{aligned}$$

so,

$$\begin{aligned} & \varphi_{(x_i, x'_i, \alpha_i)_{1 \leq i \leq n}}(f) + \varphi_{(y_j, y'_j, \beta_j)_{1 \leq j \leq m}}(f) = \\ & \sum_{k=1}^{n+m} \gamma_k C\phi(|f(z_k) - f(z'_k)|) - \sum_{k=1}^{n+m} \gamma_k \phi(\|T(z_k) - T(z'_k)\|) \in A, \gamma_k > 0, z_k, z'_k \in X, \end{aligned}$$

which

$$\begin{aligned} T(z_k) &= \begin{cases} T(x_k), & 1 \leq k \leq n \\ T(y_{k-n}), & 1+n \leq k \leq n+m \end{cases}, \\ T(z'_k) &= \begin{cases} T(x'_k), & 1 \leq k \leq n \\ T(y'_{k-n}), & 1+n \leq k \leq n+m \end{cases}, \\ \gamma_k &= \begin{cases} \alpha_k, & 1 \leq k \leq n \\ \beta_{k-n}, & 1+n \leq k \leq n+m \end{cases}. \end{aligned}$$

Let $\lambda > 0$, $\lambda \varphi_{(x_i, x'_i, \alpha_i)_{1 \leq i \leq n}}(f) = \sum_{i=1}^n \lambda \alpha_i C\phi(|f(x_i) - f(x'_i)|) - \sum_{i=1}^n \lambda \alpha_i \phi(\|T(x_i) - T(x'_i)\|) \in A$.

Consider now the set

$$B = \{\varphi \in C(B_{X^\#}): \varphi(f) > 0, \forall f \in B_{X^\#}\}.$$

Clearly, B is a convex open subset of the Banach space $C(B_{X^\#})$. Moreover, A and B are disjoint. Indeed, for otherwise, we could find $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ in X and $(\lambda_i)_{i=1}^n$ in \mathbb{R}^+ such that $\varphi_{(\lambda_i, x_i, x'_i)_{i=1}^n}(f) > 0$ for all $f \in B_{X^\#}$, that is,

$$\pi_\phi^L(T) \sum_{i=1}^n \lambda_i \phi(|f(x_i) - f(x'_i)|) < \sum_{i=1}^n \lambda_i \phi(d(T(x_i), T(x'_i)))$$

for all $f \in B_{X^\#}$. But since $B_{X^\#}$ is weak* compact and $f \mapsto \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|)$ is a continuous function from $(B_{X^\#}, w^*)$ to \mathbb{R} , we would have

$$\pi_\phi^L(T) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \lambda_i \phi(|f(x_i) - f(x'_i)|) < \sum_{i=1}^n \lambda_i \phi(d(T(x_i), T(x'_i))),$$

and this is impossible because T is in $\Pi_\phi^L(X, Y)$.

Then, by the Hahn–Banach separation theorem and the Riesz representation theorem, there exists a regular Borel probability measure μ on $B_{X^\#}$ and a scalar $c \in \mathbb{R}$ such that

$$\int_{B_{X^\#}} \varphi_{(\lambda_i, x_i, x'_i)_{i=1}^n} d\mu \leq c < \int_{B_{X^\#}} \varphi d\mu$$

for all $\varphi_{(\lambda_i, x_i, x'_i)_{i=1}^n} \in A$ and all $\varphi \in B$. Observe that $c = 0$. Indeed, $0 \leq c$ since the zero function $\varphi_{(\lambda_i, x_i, x'_i)_{i=1}^n}$ is in A ; and $c \leq 0$ since all positive constant functions are in B . It follows that

$$\int_{B_{X^\#}} \varphi_{(1, x, x')}(f) d\mu(f) \leq 0$$

for all $x, x' \in X$, and since μ is a probability measure, we conclude that

$$\phi(d(T(x), T(x'))) \leq \pi_\phi^L(T) \int_{B_{X^\#}} \phi(|f(x) - f(x')|) d\mu(f).$$

for all $x, x' \in X$.

Conversely, assume that the second statement of the theorem holds and let $(x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X$. We have

$$\phi(d(T(x_i), T(x'_i))) \leq C \int_{B_{X^\#}} \phi(|f(x_i) - f(x'_i)|) d\mu(f)$$

for $i = 1, \dots, n$. It follows that

$$\begin{aligned} \sum_{i=1}^n \phi(d(T(x_i), T(x'_i))) &\leq C \int_{B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|) d\mu(f) \\ &\leq C \int_{B_{X^\#}} \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|) d\mu(f) \\ &= C \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(x'_i)|). \end{aligned}$$

Hence $T \in \Pi_\phi^L(X, Y)$ and $\pi_\phi^L(T) \leq C$.

Finally, it is clear from the proof that $\pi_\phi^L(T)$ is the minimum value of C . \square

Remark 2.16. *Theorem 2.15 extends a nonlinear version of Pietsch domination theorem given in [31, Corollary 6.14] for Lipschitz p -summing operators with $0 < p < 1$.*

The Lipschitz-free Banach space $\mathcal{F}(X)$ over a pointed metric space X is the closed linear span in $(X^\#)^*$ of the evaluation functionals $\delta_x: X^\# \rightarrow \mathbb{R}$ with $x \in X$, where $\delta_x(f) = f(x)$ for all $f \in X^\#$. It is known that the Dirac map $\delta_X: X \rightarrow \mathcal{F}(X)$, given by $\delta_X(x) = \delta_x$, is a (nonlinear) isometry, and $\mathcal{F}(X)^*$ is isometrically isomorphic to $X^\#$ via the evaluation map $Q_X: X^\# \rightarrow \mathcal{F}(X)^*$ given by $Q_X(f)(\gamma) = \gamma(f)$ for all $f \in X^\#$ and $\gamma \in \mathcal{F}(X)$.

Let us recall (see [19, Lemma 3.1]) that if X and Y are pointed metric spaces and $T \in \text{Lip}(X, Y)$, then there exists a unique operator $\widehat{T} \in \mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$ such that $\widehat{T}\delta_X = \delta_Y T$. Furthermore, $\|\widehat{T}\| = \text{Lip}(T)$.

We can now study the relationship of a Lipschitz ϕ -summing operator T in $\text{Lip}(X, Y)$ and its linearization \widehat{T} in $\mathcal{L}(\mathcal{F}(X), \mathcal{F}(Y))$.

Proposition 2.17. *Let ϕ be a modulus function, let X, Y be pointed metric spaces and let $T \in \text{Lip}(X, Y)$. If \widehat{T} is ϕ -summing, then T is Lipschitz ϕ -summing and $\pi_\phi^L(T) \leq \pi_\phi(\widehat{T})$.*

Proof. Notice first that

$$d(T(x), T(y)) = \|\delta_{T(x)} - \delta_{T(y)}\| = \|\widehat{T}(\delta_x) - \widehat{T}(\delta_y)\| = \|\widehat{T}(\delta_x - \delta_y)\|$$

for all $x, y \in X$. If \widehat{T} is ϕ -summing, we have

$$\sum_{i=1}^n \phi(\|\widehat{T}(\gamma_i)\|) \leq \pi_\phi(\widehat{T}) \sup_{F \in B_{\mathcal{F}(X)^*}} \sum_{i=1}^n \phi(|F(\gamma_i)|)$$

for any set $(\gamma_i)_{i=1}^n \subset \mathcal{F}(X)$. Then

$$\begin{aligned} \sum_{i=1}^n \phi(d(T(x_i), T(y_i))) &\leq \pi_\phi(\widehat{T}) \sup_{F \in B_{\mathcal{F}(X)^*}} \sum_{i=1}^n \phi(|F(\delta_{x_i} - \delta_{y_i})|) \\ &= \pi_\phi(\widehat{T}) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|Q_X(f)(\delta_{x_i} - \delta_{y_i})|) \\ &= \pi_\phi(\widehat{T}) \sup_{f \in B_{X^\#}} \sum_{i=1}^n \phi(|f(x_i) - f(y_i)|). \end{aligned}$$

Consequently, T is Lipschitz ϕ -summing and $\pi_\phi^L(T) \leq \pi_\phi(\widehat{T})$. □

Chapter 3

Characterization of strictly Lipschitz p -summing operators.

The results obtained in this chapter have been submitted for publication (see [4]). we give some new characterizations of strictly Lipschitz p -summing operators ($1 \leq p < \infty$), which have been introduced in order to improve the Lipschitz p -summing operators. Therefore, using the notion of p -nuclear in the linear case, we adapt this definition for constructing other classes of Lipschitz mappings which are called strictly Lipschitz p -nuclear operators, and introduce a new strong notion which is M -strictly Lipschitz p -summing giving its characterization.

3.1 Introduction and preliminaries

In the last years, many concepts of the theory of p -summing operators have been developed in several ways, namely the multilinear and Lipschitz settings. The fundamental purpose of nonlinear theory of Lipschitz mappings is an attempt to borrow linear properties in order to make an analog in the nonlinear case. Let X be a pointed metric space and E be a Banach space. It is well known that each Lipschitz operator $T : X \rightarrow E$ can be factorized through a Lipschitz map and a linear operator. Let \mathcal{I} be a linear operator ideal, when we want to form an analogous class of Lipschitz operators, it is natural to think how to preserve the connection between their linearization operators, which appear in the factorization, and the original ideal \mathcal{I} . If this property holds for a given Lipschitz class, then it can be represented by [1] as follows

$$\mathcal{I} \circ Lip_0(X; E) = \mathcal{I}(\mathcal{F}(X); E).$$

Then, the above representation, that we consider interesting, expresses good relation between Lipschitz operators and their linearizations. To make the relation above attainable, it has been improved in [30] the definition of Lipschitz p -summing by introducing the strictly Lipschitz p -summing operators whose original ideal is Π_p , the Banach space of p -summing operators, and admits a similar representation of the same relation. The goal of this chapter is to explore more properties of the class of strictly Lipschitz p -summing by showing some characterizations of these operators by means of fundamental inequalities and the Pietsch domination theorem. We give a strong version of this concept, called M -strictly Lipschitz p -summing, when we consider the class of Lipschitz operators defined on metric spaces. Furthermore, we will introduce the class of strictly Lipschitz p -nuclear by generalizing the linear definition introduced by Cohen [12].

This chapter is organized as follows.

First, we recall some standard notations which will be used throughout this chapter. In section 2, we study some characterizations of strictly Lipschitz p -summing operators which are defined from a metric space into a Banach sapce. We will give the definition for a general case, called M -strictly Lipschitz p -summing, where X and Y are metric spaces for which there is a beautiful equivalence between T and its corresponding linear operators for the concept of p -summing. In Section 3, the definition of strictly Lipschitz p -nuclear operators is given.

This class has surprising properties namely their connections with linearization operators.

A representation through a Lipschitz tensor product is given for this class. We end this section by investigating certain relations with other classes.

3.2 Lipschitz tensor product

The elements of this subsection are derived from an important new article [7] of Cabrera-Padilla et al. which help us to treat the rest of paragraphs of this chapter. Let X be a pointed metric space and let E be a Banach space. The Lipschitz tensor product of X, E is denoted by $X \boxtimes E$ and defined as the vector subspace of $\text{Lip}_0(X, E^*)'$ (the algebraic dual of $\text{Lip}_0(X, E^*)$) which is spanned by the set

$$\{\delta_{(x,y)} \boxtimes e : (x, y) \in X^2, e \in E\}$$

where

$$(\delta_{(x,y)} \boxtimes e)(f) = \langle f(x) - f(y), e \rangle, \forall f \in \text{Lip}_0(X, E^*).$$

We say that, $\delta_{(x,y)} \boxtimes e$ is an elementary Lipschitz tensor. We have $u \in X \boxtimes E$ if, and only if, $u = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \boxtimes e_i$ and this representation is not unique. Note that each element $u \in X \boxtimes E$ can be represented as $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ since $\lambda \delta_{(x,y)} \boxtimes e = \delta_{(x,y)} \boxtimes \lambda e$. In other words, for all $f \in \text{Lip}_0(X, E^*)$ we have

$$u(f) = \sum_{i=1}^n \langle f(x_i) - f(y_i), e_i \rangle$$

Denote by $\mathcal{F}((X^\#, \tau_p); E)$ the space of all finite rank continuous linear operators from $(X^\#, \tau_p)$ to E , where τ_p denotes the topology of pointwise convergence of $X^\#$.

Theorem 3.1. [7] *The map $J : X \boxtimes E \longrightarrow \mathcal{F}((X^\#, \tau_p); E)$, given by*

$$J(u)(f) = \sum_{i=1}^n (f(x_i) - f(y_i)) e_i$$

for $u \in X \boxtimes E, u = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \boxtimes e_i$ and $f \in X^\#$, is a linear isomorphism.

Lipschitz injective norm

Let E be a Banach space and X be a metric space. In what follows we present the definition of Lipschitz injective norm on $X \boxtimes E$.

Definition 3.2. [7] For each $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E$, we define the injective norm of u by

$$\varepsilon(u) = \sup \left\{ \left| \sum_{i=1}^n (f(x_i) - f(y_i)) \langle \varphi, e_i \rangle \right| : f \in \mathcal{B}_{X^\#}, \varphi \in B_{E^*} \right\}.$$

Notice that the supremum on the right side in the previous definition exists since

$$\begin{aligned} \left| \sum_{i=1}^n (f(x_i) - f(y_i)) \langle \varphi, e_i \rangle \right| &\leq \sum_{i=1}^n |(f(x_i) - f(y_i)) \langle \varphi, e_i \rangle| \\ &\leq \sum_{i=1}^n \text{Lip}(f) d(x_i, y_i) \|\varphi\| \|e_i\| \\ &\leq \sum_{i=1}^n d(x_i, y_i) \|e_i\| \end{aligned}$$

for all $f \in \mathcal{B}_{X^\#}$ and $\varphi \in B_{E^*}$. Note that

$$\sum_{i=1}^n |(f(x_i) - f(y_i)) \langle \varphi, e_i \rangle| = (f \boxtimes \varphi) \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right)$$

and, consequently, $\varepsilon(u)$ does not depend on the representation of u by [7, Lemma2.2], so ε defines a map from $X \boxtimes E$ to \mathbb{R} .

ε is a dualizable Lipschitz cross-norm on $X \boxtimes E$.

Cross-norm, i.e., $\varepsilon(\delta_{(x, y)} \boxtimes e) = d(x, y) \|e\|$, $\forall x, y \in X, \forall e \in E$.

Dualizable, i.e.,

$$\left| \sum_{i=1}^n (f(x_i) - f(y_i)) \langle \varphi, e_i \rangle \right| \leq \text{Lip}(f) \|\varphi\| \varepsilon \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right), \forall f \in X^\#, \forall \varphi \in E^*,$$

ε is the least dualizable Lipschitz cross-norm on $X \boxtimes E$.

The completion $X \widehat{\boxtimes}_\varepsilon E$ of $X \boxtimes_\varepsilon E$ is called the injective Lipschitz tensor product of X and E . We can identify $X \widehat{\boxtimes}_\varepsilon E$ with the space of approximable bounded linear operators of $(X^\#, \tau_p)$ ($= (X^\#, \sigma(X^\#, X))$).

Proposition 3.3. [7] *The map $J : X \boxtimes_\varepsilon E \rightarrow \mathcal{F}((X^\#, \tau_p); E)$, defined by*

$$J(u)(f) = \sum_{i=1}^n (f(x_i) - f(y_i)) e_i$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ and $f \in X^\#$ is an isometric isomorphism. As a consequence, $X \widehat{\boxtimes}_\varepsilon E$ is isometrically isomorphic to the closure in the operator norm topology of $\mathcal{F}((X^\#, \tau_p); E)$.

Proof. By Theorem 3.1, J is a linear bijection. If $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$, we have

$$\begin{aligned} \|J(u)\| &= \sup\{\|J(u)(f)\| : f \in \mathcal{B}_{X^\#}\} \\ &= \sup\left\{\left|\varphi\left(\sum_{i=1}^n (f(x_i) - f(y_i)) e_i\right)\right| : f \in \mathcal{B}_{X^\#}, \varphi \in B_{E^*}\right\} \\ &= \sup\left\{\left|\sum_{i=1}^n (f(x_i) - f(y_i)) \langle \varphi, e_i \rangle\right| : f \in \mathcal{B}_{X^\#}, \varphi \in B_{E^*}\right\} \\ &= \varepsilon(u). \end{aligned}$$

The consequence is immediate. □

Let $\mathcal{F}(X)$ be the Lipschitz-free Banach space over a pointed metric space X . Let us recall that $\mathcal{F}(X)$ is the closed linear subspace of $(X^\#)^*$ spanned by the set $\{\delta_x : x \in X\}$, where for each $x \in X$, δ_x is the evaluation functional at the point x defined on $X^\#$. It was proved that if X is a pointed metric space, then $X \widehat{\boxtimes}_\varepsilon \mathbb{K}$ is isometrically isomorphic to $\mathcal{F}(X)$; in fact, much more is true. We show below that the space $X \widehat{\boxtimes}_\varepsilon E$ can be identified with the injective tensor product $\mathcal{F}(X) \widehat{\otimes}_\varepsilon E$.

Proposition 3.4. [7] *The map $I : X \boxtimes_\varepsilon E \rightarrow \mathcal{F}(X) \otimes_\varepsilon E$, defined by*

$$I(u) = \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$$

for $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ is a linear isometry. As a consequence, $X \widehat{\boxtimes}_\varepsilon E$ is isometrically isomorphic to $\mathcal{F}(X) \widehat{\otimes}_\varepsilon E$.

Proof. Let $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$. Since $\mathcal{F}(X)^* = X^\#$, note that the norm of $\sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$ in $\mathcal{F}(X) \otimes_\varepsilon E$ is given by

$$\sup\left\{\left|\sum_{i=1}^n \langle f, \delta_{x_i} - \delta_{y_i} \rangle \langle \varphi, e_i \rangle\right| : f \in \mathcal{B}_{X^\#}, \varphi \in B_{E^*}\right\}.$$

Since $\langle f, \delta_{x_i} - \delta_{y_i} \rangle$ is precisely $f(x_i) - f(y_i)$, [7, Proposition 1.7] shows that I is well defined (and thus linear) and moreover a quick glance at Definition 3.2 shows that I is an isometry. \square

Recall that the linear span of $\{\delta_x\}_{x \in X}$ is dense in $\mathcal{F}(X)$, hence the tensors of the form $\sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i$, with $x_i, y_i \in X$ and $e_i \in E$ are dense in $\mathcal{F}(X) \widehat{\otimes}_\varepsilon E$. This shows that the map I has dense range, and thus $X \widehat{\otimes}_\varepsilon E$ is isometrically isomorphic to $\mathcal{F}(X) \widehat{\otimes}_\varepsilon E$.

Lipschitz projective norm

We introduce the Lipschitz projective norm on $X \boxtimes E$.

Definition 3.5. [7] For each $u \in X \boxtimes E$, we define the projective norm of u by

$$\pi(u) = \inf \left\{ \sum_{i=1}^n d(x_i, y_i) \|e_i\| : u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right\}$$

the infimum being taken over all representations of u .

The completion $X \widehat{\otimes}_\pi E$ of $X \boxtimes_\pi E$ is called the projective Lipschitz tensor product of X and E .

Theorem 3.6. [7] The Lipschitz projective norm π is the greatest Lipschitz cross-norm on $X \boxtimes E$.

Proof. We have seen in the precedent Theorem that π is a Lipschitz cross-norm on $X \boxtimes E$. Now, let α be a Lipschitz cross-norm on $X \boxtimes E$ and let $u \in X \boxtimes E$. If $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i$ is a representation of u , we have

$$\alpha(u) = \alpha \left(\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \right) \leq \sum_{i=1}^n \alpha(\delta_{(x_i, y_i)} \boxtimes e_i) = \sum_{i=1}^n d(x_i, y_i) \|e_i\|.$$

Now the definition of π gives $\alpha(u) \leq \pi(u)$. \square

3.3 Characterization of strictly Lipschitz p -summing operators

Let X be a pointed metric space and E be a Banach space. In order to establish a relation between a Lipschitz operator $T : X \rightarrow E$ and its linearisation $\widehat{T} : \mathcal{F}(X) \rightarrow E$ for the concept

of p -summing, K. Saadi has introduced in [30] the notion of strictly Lipschitz p -summing operators. Indeed, both operators are related in the sense that: T is strictly Lipschitz p -summing if and only if, \widehat{T} is p -summing. In fact, that report was not true for the class of Lipschitz p -summing operators which is introduced by J. D. Farmer and W. B. Johnson [14], see [29, Remark 3.3]. In this section, we give some characterizations of the class of strictly Lipschitz p -summing operators and we adapt its definition to the general case where the spaces are metric for which we obtain a good relation between the Lipschitz operator T and their corresponding linear operators \widetilde{T} and $T^\#$. Now, we start by the definition of d_p^L , the corresponding Lipschitz cross-norm of the tensor norm d_p , see [30] for more detail about this norm. Recall the definition of the norms of Chevet-Saphar d_p and g_p [10, 32] defined on Banach spaces,

$$d_p(u) = \inf \left\{ \|(x_i)_i\|_{l_p^{n,w}(E)} \|(y_i)_i\|_{l_p^n(F)} \right\},$$

and

$$g_p(u) = \inf \left\{ \|(x_i)_i\|_{l_p^n(E)} \|(y_i)_i\|_{l_p^{n,w}(F)} \right\}.$$

where the infimum is taking over all representations of u of the form $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$.

We have $g_p = d_p^t$, where d_p^t is defined as follows: $d_p^t(u; E \otimes F) = d_p(u^t; F \otimes E)$ and the transpose u^t of $\sum_{i=1}^n x_i \otimes y_i$ is given by $u^t = \sum_{i=1}^n y_i \otimes x_i$

For every $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k \in X \boxtimes E$, we put

$$A_u = \left\{ m = \sum_{i=1}^n m_i \otimes e_i \in \mathcal{F}(X) \otimes E : u = \sum_{k=1}^l \delta_{(x_k, y_k)} \otimes s_k \right\}. \quad (3.1)$$

Since the linearization \widehat{T} can be seen as a linear form on $\mathcal{F}(X) \otimes E^*$, then for every $m \in A_u$ we have

$$\begin{aligned} \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle &= \widehat{T}(\sum_{k=1}^l \delta_{(x_k, y_k)} \otimes s_k^*) = \widehat{T}(m) \\ &= \sum_{i=1}^{n_1} \langle \widehat{T}(m_i), e_i^* \rangle \\ &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \langle \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \rangle \end{aligned} \quad (3.2)$$

where

$$m_i = \sum_{j=1}^{k_i} \lambda_i^j \delta_{(x_i^j, y_i^j)} = \sum_{j=1}^{n_2} \lambda_i^j \delta_{(x_i^j, y_i^j)},$$

with $n_2 = \max k_i (1 \leq i \leq n)$ and the terms between k_i and n_2 are zero. Now, Let α be a tensor norm defined on two Banach spaces, by [30, Theorem 3.1], there is a Lipschitz

cross-norm α^L which is defined on Lipschitz tensor product $X \boxtimes E$ by

$$\alpha^L(u) = \alpha^L\left(\sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k\right) = \alpha\left(\sum_{k=1}^l \delta_{(x_k, y_k)} \otimes s_k\right). \quad (3.3)$$

where $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \otimes s_k \in \mathcal{F}(X) \otimes E$. So, we have

$$d_p^L(u) = \inf_{m \in A_u} \left\{ \|m_i\|_{l_p^{n, w}(\mathcal{F}(X))} \| (e_i)_i \|_{l_{p^*}^n(E)} \right\}.$$

Lemma 3.7. [8] *When X is a finite metric space and E a Banach space then the spaces $\Pi_p^L(X, E)^{**}$ and $\Pi_p^L(X, E^{**})$ are isometrically isomorphic. Moreover if E is a reflexive then the spaces $\Pi_p^L(X, E)^{**}$ and $\Pi_p^L(X, E)$ are isometrically isomorphic.*

Definition 3.8. [30]. *Let $1 \leq p \leq \infty$. A Lipschitz operator $T : X \rightarrow E$ is said to be strictly Lipschitz p -summing if there exists a positive constant C such that for every $x_k, y_k \in X$ and $s_k^* \in E^*$ ($1 \leq k \leq l$) we have*

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C d_p^L(u), \quad (3.4)$$

where $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^*$. We denote by $\Pi_p^{SL}(X, E)$ the Banach space of all strictly Lipschitz p -summing operators from X into E which its norm $\pi_p^{SL}(T)$ is the smallest constant C verifying (3.4).

If we consider linear operators defined on Banach spaces, Khalil Saadi has shown in [30, Proposition 3.8] that the notions of p -summing, Lipschitz p -summing and strictly Lipschitz p -summing are coincide. The following characterization is the main result of this section.

Theorem 3.9. *Let $1 \leq p \leq \infty$. Let X be a pointed metric space and E be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following properties are equivalent.*

- 1) T is strictly Lipschitz p -summing.
- 2) \widehat{T} is p -summing.
- 3) There exists a constant $C > 0$ and a Radon probability μ on $B_{X^\#}$ such that for all $(x^j)_{j=1}^n, (y^j)_{j=1}^n \subset X$ and $(\lambda^j)_{j=1}^n \subset \mathbb{K}; (n \in \mathbb{N}^*)$, we have

$$\left\| \sum_{j=1}^n \lambda^j (T(x^j) - T(y^j)) \right\| \leq C \left(\int_{B_{X^\#}} \left| \sum_{j=1}^n \lambda^j (f(x^j) - f(y^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}}. \quad (3.5)$$

4) There is a constant $C > 0$ such that for every $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1}$ in X and $(\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K}; (1 \leq j \leq n_2)$ and $n_1, n_2 \in \mathbb{N}^*$, we have

$$\sum_{i=1}^{n_1} \left(\left\| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{f \in X^\#} \left(\sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right\|^p \right)^{\frac{1}{p}}. \quad (3.6)$$

Proof. (1) \implies (2) : Theorem 3.5 in [30].

(2) \implies (3) : We apply Pietsch Domination Theorem for p -summing linear operators [13, Theorem 2.12], there is a Radon probability μ on $B_{X^\#}$ such that for any $m \in \mathcal{F}(X)$ we have

$$\left\| \widehat{T}(m) \right\| \leq C \left(\int_{B_{X^\#}} |f(m)|^p d\mu(f) \right)^{\frac{1}{p}}.$$

Now, let $(x^j)_{j=1}^n, (y^j)_{j=1}^n \subset X$ and $(\lambda^j)_{j=1}^n \subset \mathbb{K}$, we put

$$m = \sum_{j=1}^n \lambda^j \delta_{(x^j, y^j)} \in \mathcal{F}(X),$$

Then

$$\left\| \widehat{T} \left(\sum_{j=1}^n \lambda^j \delta_{(x^j, y^j)} \right) \right\| \leq C \left(\int_{B_{X^\#}} \left| f \left(\sum_{j=1}^n \lambda^j \delta_{(x^j, y^j)} \right) \right|^p d\mu(f) \right)^{\frac{1}{p}}$$

thus

$$\left\| \sum_{j=1}^n \lambda^j (T(x^j) - T(y^j)) \right\| \leq C \left(\int_{B_{X^\#}} \left| \sum_{j=1}^n \lambda^j (f(x^j) - f(y^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}}.$$

(3) \implies (4) : Let $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1}$ in X and $(\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K} (1 \leq j \leq n_2)$, by (3.5) we have for every $1 \leq i \leq n_1$

$$\left\| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)) \right\| \leq C \left(\int_{B_{X^\#}} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}}$$

Therefore,

$$\begin{aligned} & \sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)) \right\|^p \\ & \leq C \int_{B_{X^\#}} \sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p d\mu(f) \\ & \leq \sup_{f \in B_{X^\#}} \sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p. \end{aligned}$$

Finally, we have

$$\left(\sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{f \in X^\#} \left(\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p \right)^{\frac{1}{p}}$$

(4) \implies (1) : Let $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^* \in X \boxtimes E^*$. Let A_u the set as defined in (3.1). Let $m = \sum_{i=1}^{n_1} m_i \otimes e_i^* \in A_u$ ($m_i = \sum_{j=1}^{n_2} \lambda_i^j \delta_{(x_i^j, y_i^j)}$), by (3.2)

$$\begin{aligned} & \left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \\ &= \left| \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \langle \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \rangle \right| \quad (\text{by Hölder}) \\ &\leq \left(\sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)) \right\|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n_1} \|e_i^*\|^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq C \cdot \sup_{f \in X^\#} \left(\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n_1} \|e_i^*\|^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq C \| (m_i) \|_{l_p^{n_1, w}(\mathcal{F}(X))} \| (e_i^*) \|_{l_{p^*}^{n_1}(E)}. \end{aligned}$$

By taking the infimum over all representations of $m \in A_u$, we obtain

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C d_p^L(u).$$

Then, T is strictly Lipschitz p -summing. □

If we put $n_2 = 1$ in the formula (3.6), we obtain exactly the definition of Lipschitz p -summing operators. Now, let X and Y be two metric spaces, J. D. Farmer and W. B. Johnson [14] have introduced the general case of Lipschitz p -summing between metric spaces. Inspired by the definition of strictly Lipschitz p -summing, we give the following general setting.

Definition 3.10. . Let $1 \leq p \leq \infty$. Let X and Y be two metric spaces. A Lipschitz operator $T : X \rightarrow Y$ is said to be M -strictly Lipschitz p -summing, if there exists a positive constant C such that for every $x_k, y_k \in X$ and $g_k \in Y^\#$ ($1 \leq k \leq l$) we have

$$\left| \sum_{k=1}^l g_k (T(x_k)) - g_k (T(y_k)) \right| \leq C d_p^L(u),$$

where $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes g_k \in X \boxtimes Y^\#$.

Note that if Y is a Banach space, for every $y^* \in Y^*$ we have $Lip(y^*) = \|y^*\|$, then

$$\| (y_i^*) \|_{l_{p^*}^{n_1}(Y^\#)} = \| (y_i^*) \|_{l_{p^*}^{n_1}(Y^*)}.$$

Therefore, the above definition leads to that of strictly Lipschitz p -summing. As an interesting characterization of M -strictly Lipschitz p -summing operators, we have the following result.

For the proof of the following proposition, we need the following lemma.

Lemma 3.11. . Let X be a pointed metric space and E, F be two Banach spaces. Suppose that E and F are isometrically isomorphic via the application Q . Let $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k \in X \boxtimes E$, then

$$d_p^L(u) = d_p^L(w),$$

where $w = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes Q(s_k) \in X \boxtimes F$.

Proof. The identification $\mathcal{F}(X) \otimes E = \mathcal{F}(X) \otimes F$ holds via the transformation

$$\sum_{i=1}^n m_i \otimes e_i \mapsto \sum_{i=1}^n m_i \otimes Q(e_i).$$

Let $m = \sum_{i=1}^n m_i \otimes e_i \in A_u$, then $\sum_{i=1}^n m_i \otimes Q(e_i) \in A_w$. We have

$$\begin{aligned} d_p^L(w) &\leq \|m_i\|_{l_p^{n,w}(\mathcal{F}(X))} \|(Q(e_i))_i\|_{l_{p^*}^n(F)} \\ &\leq \|m_i\|_{l_p^{n,w}(\mathcal{F}(X))} \|(e_i)_i\|_{l_{p^*}^n(E)}, \end{aligned}$$

by taking the infimum on A_u , we find $d_p^L(w) \leq d_p^L(u)$. With the same argument, the inverse inequality is immediate. \square

Proposition 3.12. Let $1 \leq p \leq \infty$. Let X and Y be two metric spaces. The following properties are equivalent.

- 1) $T : X \rightarrow Y$ is M -strictly Lipschitz p -summing.
- 2) The linearization operator $\tilde{T} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ is p -summing.
- 3) The Lipschitz adjoint $T^\# : Y^\# \rightarrow X^\#$ is Cohen strongly p -summing.

Proof. (1) \Rightarrow (2) : Let $(x_k)_{k=1}^l \subset X$ and $(f_k^*)_{k=1}^l \subset \mathcal{F}(Y)^*$. We will show that $\delta_Y \circ T$ is strictly Lipschitz p -summing. So, there is $(g_k)_{k=1}^l \subset Y^\#$ such that $f_k^* = Q_Y(g_k)$ for every $1 \leq k \leq l$. We have

$$\begin{aligned} \left| \sum_{k=1}^l \langle \delta_Y \circ T(x_k) - \delta_Y \circ T(y_k), f_k^* \rangle \right| &= \left| \sum_{k=1}^l \langle \delta_{(T(x_k), T(y_k))}, Q_Y(g_k) \rangle \right| \\ &= \left| \sum_{k=1}^l g_k(T(x_k)) - g_k(T(y_k)) \right| \\ &\leq C d_p^L(u) = C d_p^L(w), \end{aligned}$$

where $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes g_k \in X \boxtimes Y^\#$ and $w = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes f_k^* \in X \boxtimes \mathcal{F}(Y)^*$. Then, $\delta_Y \circ T$ is strictly Lipschitz p -summing. Now, by the factorization (??) we have

$$\delta_Y \circ T = \tilde{T} \circ \delta_X,$$

this shows that the linearization of $\delta_Y \circ T$ is \tilde{T} . Then \tilde{T} is p -summing.

(2) \Rightarrow (3) : We have by (1.4)

$$T^\# = Q_X^{-1} \circ \tilde{T}^* \circ Q_Y,$$

hence, by the ideal property, $T^\#$ is Cohen strongly p^* -summing.

(3) \Rightarrow (1) : Suppose that $T^\#$ is Cohen strongly p^* -summing. Let $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes g_k \in X \boxtimes Y^\#$ then

$$\begin{aligned} & \left| \sum_{k=1}^l g_k(T(x_k)) - g_k(T(y_k)) \right| \\ &= \left| \sum_{k=1}^l \langle T^\#(g_k), \delta_{(x_k, y_k)} \rangle \right| \\ &\leq C g_p(\sum_{k=1}^l g_k \otimes \delta_{(x_k, y_k)}) \quad (\text{we know that } g_p = d_p^t) \\ &\leq C d_p(\sum_{k=1}^l \delta_{(x_k, y_k)} \otimes g_k) \\ &= C d_p^L(u). \end{aligned}$$

Therefore, T is M-strictly Lipschitz p -summing. \square

The definition of Lipschitz dual of a given operator ideal \mathcal{I} is given in [1] as follows

$$\mathcal{I}^{Lip_0\text{-dual}}(X, E) = \{T \in Lip_0(X, E) : T^t \in \mathcal{I}(E^*, X^\#)\}.$$

Corollary 3.13. . Let \mathcal{D}_p be the linear operator ideal of Cohen strongly p -summing operators. Then, we have

$$\mathcal{D}_p^{Lip_0\text{-dual}}(X, E) = \Pi_{p^*}^{SL}(X, E).$$

Proof. Let $T \in \Pi_{p^*}^{SL}(X, E)$, by Theorem 3.9 its linearization \hat{T} is p^* -summing. By (3.1) and the ideal property, T^t is Cohen strongly p -summing, then $T \in \mathcal{D}_p^{Lip_0\text{-dual}}(X, E)$. Conversely, let $T \in \mathcal{D}_p^{Lip_0\text{-dual}}(X, E)$, then $T^t : E^* \rightarrow X^\#$ is Cohen strongly p -summing. We have

$$\hat{T}^* = Q_X \circ T^t,$$

then \hat{T} is p^* -summing by (1.4) and the result follows by Theorem 3.9 \blacksquare

3.4 Strictly Lipschitz p -nuclear operators

In this section, we adopt the same procedure of the previous section for defining the strictly Lipschitz p -nuclear operators. Then, we obtain a nice class that has many interesting properties with good relations with other classes of Lipschitz mappings where the linearisation

operators play the key of all obtained results. Among the results of this section, we obtain an analog of Pietsch domination theorem and then the same factorization result to Cohen-Kwapień which is given for p -nuclear linear operators. Let $p \in [1, \infty]$ and E, F be two Banach spaces. The definition of tensor norm w_p on $E \otimes F$ is given in [12] by

$$w_p(u) = \inf \left\{ \|(x_i)\|_{l_p^{n,w}(E)} \|(y_i)\|_{l_{p^*}^{n,w}(F)} \right\},$$

where the infimum is taken over all representations of u of the form $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$. Then by [12, Lemma 2.5.1], we have the following identification

$$\mathcal{N}_p(E, F) = (E \widehat{\otimes}_{w_p} F^*)^*. \quad (3.7)$$

Next, we will give an analog approach for the class of strictly Lipschitz p -nuclear operators. Let X be a pointed metric space and E be a Banach space.

Let $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^* \in X \boxtimes E^*$ and A_u be the set as in (3.1). We consider

$$w_p^{SL}(u) = \inf_{m \in A_u} \left\{ \|(m_i)_i\|_{l_p^{n,w}(\mathcal{F}(X))} \|(e_i^*)\|_{l_{p^*}^{n,w}(E^*)} \right\}.$$

The following Proposition shows that w_p^{SL} is nothing else than the corresponding Lipschitz cross-norm to the tensor norm w_p .

Proposition 3.14. . *Let X be a pointed metric space and E be a Banach space. For every $u \in X \boxtimes E$ we have*

$$w_p^{SL}(u) = w_p^L(u),$$

where w_p^L is the Lipschitz cross-norm corresponding to the tensor norm w_p .

Proof. Let $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k \in X \boxtimes E$, we have

$$\begin{aligned} w_p^{SL}(u) &= \inf_{m \in A_u} \left\{ \|(m_i)\|_{l_p^{n,w}(\mathcal{F}(X))} \|(e_i)\|_{l_{p^*}^{n,w}(E)} \right\} \\ &= w_p(\sum_{k=1}^l \delta_{(x_k, y_k)} \otimes s_k) \\ &= w_p^L(u). \end{aligned}$$

□

It is not difficult to prove the following results.

Proposition 3.15. *Let X be a pointed metric space and E be a Banach space.*

- (1) *For every $u \in X \boxtimes E$ we have $w_p^{SL}(u) \leq d_p^L(u)$.*
- (2) *If $p = 1$, we have $w_1^{SL}(u) = d_1^L(u)$.*

Now, we give the following definition of strictly Lipschitz p -nuclear operators for which we use the Lipschitz cross-norms.

Definition 3.16. *Let $1 \leq p \leq \infty$. The Lipschitz operator $T : X \rightarrow E$ is strictly Lipschitz p -nuclear if for every $x_k, y_k \in X$ and $s_k^* \in E^*$ ($1 \leq k \leq l$) we have*

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C w_p^{SL}(u), \quad (3.8)$$

where $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^*$.

The class of all strictly Lipschitz p -nuclear operators from X into E is denoted by $\mathcal{N}_p^{SL}(X, E)$, which is a Banach space with the norm $\mathcal{N}_p^{SL}(T)$ which is the smallest constant C such that the inequality (3.8) holds.

As an immediate consequence of the Proposition 3.15, the following assertions can be checked easily.

Proposition 3.17. *Let X be a pointed metric space and E be a Banach space.*

- (1) *For every $1 < p \leq \infty$ we have*

$$\mathcal{N}_p^{SL}(X, E) \subset \Pi_p^{SL}(X, E).$$

- (2) *If $p = 1$, then*

$$\mathcal{N}_1^{SL}(X, E) = \Pi_1^{SL}(X, E).$$

In the following result, we connect between a Lipschitz operator and its linearization for the concept of p -nuclear. So, this connection helps in considering the space of these operators as a Lipschitz ideal generated by the composition method from the linear ideal \mathcal{N}_p .

Theorem 3.18. *Let $1 \leq p \leq \infty$. Let X be a pointed metric space and E be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following properties are equivalent.*

- 1) *T is strictly Lipschitz p -nuclear.*
- 2) *The linearization operator \widehat{T} is p -nuclear.*

On the other hand

$$\mathcal{N}_p^{SL}(X, E) = \mathcal{N}_p(\mathcal{F}(X), E) \text{ holds isometrically.}$$

Proof. Suppose that \widehat{T} is p -nuclear. Let x_k, y_k in X and $e_k^* \in E^*$ ($1 \leq k \leq l$), we put $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^*$. Let A_u the set as defined in (3.1). For every $m \in A_u$, by (3.2) we have

$$\sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle = \sum_{i=1}^{n_1} \langle \widehat{T}(m_i), e_i^* \rangle.$$

Now, let us prove that T is strictly Lipschitz p -nuclear. Let $m = \sum_{i=1}^{n_1} m_i \otimes e_i^* \in A_u$, by (??) we have

$$\begin{aligned} \left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| &= \left| \sum_{i=1}^{n_1} \langle \widehat{T}(m_i), e_i^* \rangle \right| \\ &\leq N_p(\widehat{T}) \| (m_i) \|_{l_p^{n_1, w}(\mathcal{F}(X))} \| (e_i^*) \|_{l_{p^*}^{n_1, w}(E)}. \end{aligned}$$

By taking the infimum over all representations of $m \in A_u$, we obtain

$$\begin{aligned} \left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| &\leq N_p(\widehat{T}) w_p \left(\sum_{k=1}^l \delta_{(x_k, y_k)} \otimes s_k^* \right) \\ &\leq N_p(\widehat{T}) w_p^{SL}(u). \end{aligned}$$

Then, T is strictly Lipschitz p -nuclear and

$$N_p^{SL}(T) \leq N_p(\widehat{T}).$$

Conversely, let $m_i \in \mathcal{F}(X)$ and $e_i^* \in E^*$ ($1 \leq i \leq n_1$). Then

$$\begin{aligned} &\sum_{i=1}^{n_1} \left| \langle \widehat{T}(m_i), e_i^* \rangle \right| \\ &= \sup_{(\xi_i)_{i \in B_{l_\infty}^{n_1}}} \left| \sum_{i=1}^{n_1} \xi_i \langle \widehat{T}(m_i), e_i^* \rangle \right| \\ &= \sup_{(\xi_i)_{i \in B_{l_\infty}^{n_1}}} \left| \sum_{i=1}^{n_1} \xi_i \left\langle \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle \right| \\ &= \sup_{(\xi_i)_{i \in B_{l_\infty}^{n_1}}} \left| \left\langle \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} T(x_i^j) - T(y_i^j), \lambda_i^j \xi_i e_i^* \right\rangle \right| \\ &\leq \sup_{(\xi_i)_{i \in B_{l_\infty}^{n_1}}} N_p^{SL}(T) w_p^{SL}(u), \end{aligned}$$

where

$$\begin{aligned} u &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \delta_{(x_i^j, y_i^j)} \boxtimes \lambda_i^j \xi_i e_i^* \\ &= \sum_{i=1}^{n_1} \xi_i \sum_{j=1}^{n_2} \lambda_i^j \delta_{(x_i^j, y_i^j)} \boxtimes e_i^* \\ &= \sum_{i=1}^{n_1} \xi_i m_i \boxtimes e_i^* \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{n_1} \left| \left\langle \widehat{T}(m_i), e_i^* \right\rangle \right| &\leq \sup_{(\xi_i)_{i \in B_{l_\infty}^{n_1}}} N_p^{SL}(T) \|(\xi_i m_i)\|_{l_{p^*}^{n_1, w}(\mathcal{F}(X))} \| (e_i^*) \|_{l_{p^*}^{n_1, w}(E^*)} \\ &\leq N_p^{SL}(T) \| (m_i) \|_{l_{p^*}^{n_1, w}(\mathcal{F}(X))} \| (e_i^*) \|_{l_{p^*}^{n_1, w}(E^*)}. \end{aligned}$$

Then \widehat{T} is p -nuclear and

$$N_p(\widehat{T}) \leq N_p^{SL}(T).$$

□

The Pietsch domination theorem is one of the interesting characterizations which is verified by the class of strictly Lipschitz p -nuclear. It can be proved by means of the linearization operators. Consequently, there is an equivalent definition in which we use only fundamental inequalities.

Theorem 3.19. *Let X be a pointed metric space and E be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following properties are equivalent.*

1) T is strictly Lipschitz p -nuclear.

2) There exist a constant $C > 0$, a Radon probability μ on $B_{X^\#}$ and $\eta \in B_{E^{**}}$ such that for every $(x^j)_{j=1}^n, (y^j)_{j=1}^n \subset X$, $(\lambda^j)_{j=1}^n \subset \mathbb{K}$ and $e^* \in E^*$, we have

$$\left| \left\langle \sum_{j=1}^n \lambda^j (T(x^j) - T(y^j)), e^* \right\rangle \right| \leq C \left(\int_{B_{X^\#}} \left| \sum_{j=1}^n \lambda^j (f(x^j) - f(y^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}} \|e^*\|_{L_{p^*}(\eta)}. \quad (3.9)$$

3) For any $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1} \subset X_j$, $(\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K}$, $(j = 1, \dots, n_2)$ and any $(e_i^*)_{i=1}^{n_1} \subset E^*$, we have

$$\sum_{i=1}^{n_1} \left| \left\langle \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle \right| \leq C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p \right)^{\frac{1}{p}} \| (e_i^*) \|_{l_{p^*}^{n_1, w}}. \quad (3.10)$$

In this case, we have

$$\begin{aligned} N_p^{SL}(T) &= \inf \{ C : \text{verifying (3.9)} \} \\ &= \inf \{ C : \text{verifying (3.10)} \}. \end{aligned}$$

Proof. (1) \Rightarrow (2) : Let $(x^j)_{j=1}^n, (y^j)_{j=1}^n \subset X$, $(\lambda^j)_{j=1}^n \subset \mathbb{K}$ and $e^* \in E^*$. We put

$$m = \sum_{j=1}^n \lambda^j \delta_{(x^j, y^j)},$$

□

By the previous theorem, the linearization operator \widehat{T} is p -nuclear. Then, by [21, Proposition 2], there exist a Radon probability μ on $B_{X\#}$ and $\eta \in B_{E^{**}}$ such that

$$\begin{aligned} & \left| \left\langle \sum_{j=1}^n \lambda^j (T(x^j) - T(y^j)), e^* \right\rangle \right| = \left| \left\langle \widehat{T}(m), e^* \right\rangle \right| \\ & \leq \left(\int_{B_{X\#}} |f(m)|^p d\mu(f) \right)^{\frac{1}{p}} \|e^*\|_{L_{p^*}(\eta)} \\ & \leq C \left(\int_{B_{X\#}} \left| \sum_{j=1}^n \lambda^j (f(x^j) - f(y^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}} \|e^*\|_{L_{p^*}(\eta)}. \end{aligned}$$

(2) \Rightarrow (3) : Let $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1} \subset X, (\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K} (1 \leq j \leq n_2)$ and $e_1^*, \dots, e_{n_1}^* \in E^*$.

By (3.10) we have for every $1 \leq i \leq n_1$

$$\left| \left\langle \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle \right| \leq C \left(\int_{B_{X\#}} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}} \|e_i^*\|_{L_{p^*}(\eta)}.$$

So, we have

$$\begin{aligned} & \sum_{i=1}^{n_1} \left| \left\langle \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle \right| \\ & \leq C \sum_{i=1}^{n_1} \left(\int_{B_{X\#}} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}} \|e_i^*\|_{L_{p^*}(\eta)} \text{ by Hölder} \\ & \leq C \left(\int_{B_{X\#}} \sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}} \left(\int_{B_{E^{**}}} \sum_{i=1}^{n_1} |e^{**}(e_i^*)|^{p^*} d\eta(e^{**}) \right)^{\frac{1}{p^*}} \\ & \leq C \sup_{f \in B_{X\#}} \left(\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p \right)^{\frac{1}{p}} \|(e_i^*)\|_{l_p^{n_1, w}(E^*)}. \end{aligned}$$

(3) \Rightarrow (1) : Let x_k, y_k in X and $e_k^* \in E^* (1 \leq k \leq l)$, we put $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^*$. Let $m = \sum_{i=1}^{n_1} m_i \otimes e_i^* \in A_u$. By (3.6)

$$\begin{aligned} \left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| &= \left| \sum_{i=1}^{n_1} \left\langle \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle \right| \\ &\leq C \sup_{f \in B_{X\#}} \left(\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p \right)^{\frac{1}{p}} \|(e_i^*)\|_{l_p^{n_1, w}(E^*)} \\ &\leq C \|(m_i)\|_{l_p^{n_1, w}(\mathcal{F}(X))} \|(e_i^*)\|_{l_p^{n_1, w}(E^*)}. \end{aligned}$$

By taking the infimum over all representations of $m \in A_u$, we obtain

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C w_p^{SL}(u).$$

Then, T is strictly Lipschitz p -nuclear.

In the following proposition, we give new examples of strictly Lipschitz p -nuclear operators.

Proposition 3.20. *Let $1 \leq p \leq \infty$. Let X be a pointed metric space, E and F be Banach spaces. Let $v \in \mathcal{L}(E, F)$ and $T \in Lip_0(X, E)$.*

(1) If T is strictly Lipschitz p -nuclear, then $v \circ T$ is strictly Lipschitz p -nuclear. We have

$$\mathcal{N}_p^{SL}(v \circ T) \leq \|v\| \mathcal{N}_p^{SL}(T).$$

(2) If $L \in \Pi_p^{SL}(X, E)$ and $v \in \mathcal{D}_p(E, F)$, then $v \circ L$ is strictly Lipschitz p -nuclear and

$$\mathcal{N}_p^{SL}(v \circ L) \leq d_p(v) \pi_p^{SL}(L).$$

Theorem 3.21. *Let X be a pointed metric space and E be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following properties are equivalent.*

- 1) T is strictly Lipschitz p -nuclear.
- 2) There exist a Banach space G , a strictly Lipschitz p -summing operator $R : X \rightarrow G$ and a Cohen strongly p -summing linear operator $S : G \rightarrow E$ such that $T = S \circ R$.

Proof. The second implication is immediate by Proposition 3.20. For the first, let T be a strictly Lipschitz p -nuclear operator. Since \widehat{T} is p -nuclear, it factors as follows

$$\begin{array}{ccc} \widehat{T} : \mathcal{F}(X) & \longrightarrow & E \\ & L \searrow & \nearrow S \\ & & G \end{array}$$

where L is p -summing and S is Cohen strongly p -summing linear operators. Consequently, T admits the following factorization

$$\begin{array}{ccc} T : X & \longrightarrow & E \\ & \delta_X \downarrow & \uparrow S \\ & \mathcal{F}(X) & \xrightarrow{L} G \end{array}$$

On the other hand, $T = S \circ R$ where $R = L \circ \delta_X$. So, as $\widehat{R} = L$ which is p -summing, the mapping R is strictly Lipschitz p -summing by Theorem 3.9. \square

According to the last results and by applying [30, Corollary 3.2], we can identify the space of strictly Lipschitz p -nuclear operators with the dual of the space $X \boxtimes E$ endowing with the norm w_p^{SL} . This identification views as a Lipschitz analog of (3.2).

Definition 3.22. [18] *Let X be a pointed metric space and E a Banach space. The mapping $f \in Lip_0(X, E)$ is said to be Lipschitz compact (Lipschitz weakly compact) if its Lipschitz image is relatively compact (respectively, relatively, weakly compact) in E .*

Corollary 3.23. . *Let X be a pointed metric space and E be a Banach space. Then*

$$\mathcal{N}_p^{SL}(X, E) = (X \widehat{\boxtimes}_{w_p^{SL}} E^*)^* = (\mathcal{F}(X) \widehat{\otimes}_{w_p} E^*)^*.$$

Now, we survey some relations between certain classes of Lipschitz operators basing on linear results and by using the linearisation operators. First, in the linear case we know that a p -nuclear operator is compact whenever its domain or range is reflexive. In [18], there is an equivalence between a Lipschitz map and its linearisation, so we have the following fact.

Corollary 3.24. *Let $1 < p \leq \infty$. Let X be a pointed metric space and E be a reflexive Banach space. Then, every strictly Lipschitz p -nuclear operator is Lipschitz compact.*

Definition 3.25. [23] *Let be $1 < p \leq \infty$ and $\lambda > 1$. A Banach space X is said to be \mathcal{L}_p -space if for every finite dimension subspace $E \subset X$ included in finite dimension subspace $F \subset X$, there exist an isomorphism $u : F \rightarrow \ell_p^{\dim F}$ verifies $\|u\| \|u^{-1}\| < \lambda$.*

Definition 3.26. [6] *Let X a pointed metric space and let E a Banach space. A Lipschitz operator $f \in Lip_0(X, E)$ is called a Lipschitz Grothendiek-integral (G -integral for short) operator if there exists a finite measure space (Ω, Σ, μ) a bounded linear operator $A \in \mathcal{L}(L_1(\mu), E^{**})$ and Lipschitz operator $b \in Lip_0(X, L_\infty(\mu))$ such that the following diagram Commutes*

$$\begin{array}{ccccc} X & \xrightarrow{f} & E & \xrightarrow{k_E} & E^{**} \\ B \downarrow & & & & \uparrow A \\ L_\infty(\mu) & \longrightarrow & \xrightarrow{I_{\infty,1}} & \longrightarrow & L_1(\mu) \end{array}$$

where $k_E : E \longrightarrow E^{**}$ is the canonical isometric embedding, $I_{\infty,1} : L_\infty(\mu) \longrightarrow L_1(\mu)$ is the formal inclusion operator and $Lip_{GI}(f) = \inf \|A\| Lip(B)\mu(\Omega)$.

Corollary 3.27. *Let $1 < p \leq \infty$. Let X be a pointed metric space and E be a Banach space. Then*

- (1) $\mathcal{N}_p^{SL}(X, E) \subset \mathcal{D}_p^L(X, E)$.
- (2) $\mathcal{N}_p^{SL}(X, E) \subset \Pi_p^{SL}(X, E) \subset \Pi_p^L(X, E)$.

Next, we have the same linear inclusion and a good relation with Lipschitz G -integral operators which introduced in [6]. By [6, Proposition 2.4], [12, Theorem 2.5.2] and Theorem 3.11, we have the following inclusion.

Corollary 3.28. *Let $1 < p \leq \infty$. Let X be a pointed metric space and E be a Banach space. Then*

$$\text{Lip}_{0GI}(X, E) \subset \mathcal{N}_p^{SL}(X, E).$$

If E is an \mathcal{L}_{p^*} -space, we obtain by [12, Theorem 3.3.3] the following coincidence.

Corollary 3.29. *Let $1 < p \leq \infty$. Let X be a pointed metric space and E is an \mathcal{L}_{p^*} -space. Then, the class of Lipschitz G -integral operators and the class of strictly Lipschitz p -nuclear operators are coincide.*

As a particular case, every Hilbert space H is an \mathcal{L}_2 -space, we have

$$\mathcal{N}_2^{SL}(X, H) = \text{Lip}_{0GI}(X, H).$$

Corollary 3.30. *Let X, E be two Banach spaces and \mathcal{N}_p the linear operator ideal of p -nuclear operators. Then we have,*

$$\mathcal{N}_p^{\text{Lip}_0\text{-dual}}(X, E) = \mathcal{N}_{p^*}^{SL}(X, E).$$

Proof. Let $T \in \mathcal{N}_{p^*}^{SL}(X, E)$, then by Theorem 3.19, its linearisation \widehat{T} is p^* -nuclear. By the ideal property and (3.1), T^t is p -nuclear, then

$$\mathcal{N}_{p^*}^{SL}(X, E) \subset \mathcal{N}_p^{\text{Lip}_0\text{-dual}}(X, E).$$

Let $T \in \mathcal{N}_p^{\text{Lip}_0\text{-dual}}(X, E)$, then $T^t : E^* \rightarrow X^\#$ is p -nuclear, then

$$T^t = v_1 \circ v_2$$

where v_1 is Cohen strongly p -summing and v_2 is p -summing. Therefore, $Q_X^{-1} \circ \widehat{T}^* = v_1 \circ v_2$, and then

$$\widehat{T}^* = Q_X \circ v_1 \circ v_2,$$

which is p -nuclear by the ideal property. So, \widehat{T} is p^* -nuclear by [12, Theorem 2.2.4]. Finally, T is strictly Lipschitz p^* -nuclear. \square

Chapter 4

Strictly Lipschitz (p, r, s) -summing operators

The results obtained in this chapter have been submitted in [4]. In this chapter we contribute to the theory of Lipschitz (p, r, s) -summing operators by studying the class of strictly Lipschitz (p, r, s) -summing operators, some properties and factorization results are obtained for these classes.

4.1 Introduction and Preliminaries

In the same circle of ideas, we study the strong version of Lipschitz (p, r, s) -summing linear operators. The linear class has been stated by Lapreste in [22] and generalized to Lipschitz case by Chávez-Domínguez [8]. The results of this section are analogous of the setting of strictly Lipschitz p -nuclear operators.

Next, we will naturally define the class of strictly Lipschitz (p, r, s) -summing. In analogy with the definition of strictly Lipschitz p -nuclear, some interesting results are obtained for the class of strictly Lipschitz (p, r, s) -summing.

Now, we recall the following definition as stated in [22].

- The linear operator R is (p, r, s) -summing if there exists a constant $C > 0$ such that, for any $x_1, \dots, x_n \in X$, and any $y_1^*, \dots, y_n^* \in F^*$, we have

$$\|(\langle R(x_i), y_i^* \rangle)\|_{l_p^n} \leq C \| (x_i) \|_{l_r^{n,w}(E)} \| (y_i^*) \|_{l_s^{n,w}}. \quad (4.1)$$

The class of (p, r, s) -summing linear operators from E into F , which is denoted by $\Pi_{p,r,s}(E, F)$, is a Banach space for the norm $\pi_{p,r,s}(R)$, i.e., the smallest constant C such that the inequality (4.1) holds. We have by [22]: R is (p, r, s) -summing if and only if, $R = R_1 \circ R_2$ where R_1 is Cohen strongly s^* -summing and R_2 is r -summing. In a particular case when $p = 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, we have

$$\mathcal{N}_r(E, F) = \Pi_{1,r,s}(E, F).$$

Definition 4.1. For a molecule $m \in \mathcal{M}(X, E)$,

$$\mu_{p,r,s}(u) = \inf \left\{ \left\| (\lambda_j)_j \right\|_p \omega_r^{Lip}((k_j^{-1}, \lambda_j^{-1}, x_j, x'_j)_j) : m = \sum_j V_j m_{x_j x'_j} \lambda_j, k_j > 0 \right\}$$

Recall that for $0 < \beta \leq 1$, a non-negative positively homogenous functional μ defined on a vector space U is called a β -seminorm if $(u_1 + u_2)^\beta \leq \mu(u_1)^\beta + \mu(u_2)^\beta$ for all $u_1, u_2 \in U$. In addition, if μ vanishes only at 0, it is called a β -norm. Suppose $\frac{1}{\beta} := \frac{1}{p} + \frac{1}{r} + \frac{1}{s} \geq 1$, then $\mu_{p,r,s}$ is β -norm.

Definition 4.2. Let X be a pointed metric space and E be Banach space. Let $T : X \rightarrow E$ be a Lipschitz map. T is Lipschitz (p, r, s) -summing if there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $(x_i)_i, (y_i)_i$ in X ; $(e_i^*)_i$ in E^* and $(\lambda_i)_i, (k_i)_i$ in \mathbb{R}_+^* ($1 \leq i \leq n$), we have

$$\|(\lambda_i \langle T(x_i) - T(y_i), e_i^* \rangle)_i\|_{l_p^n} \leq C w_r^{Lip}((\lambda_i k_i^{-1}, x_i, y_i)_i) \| (k_i e_i^*)_i \|_{l_s^{n,w}(E^*)}, \quad (4.2)$$

where $w_r^{Lip}((\lambda_i k_i^{-1}, x_i, e_i)_{i=1}^n)$ is the weak Lipschitz p -norm defined by

$$\begin{aligned} w_r^{Lip}((\lambda_i, x_i, y_i)_{i=1}^n) &= \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n |\lambda_i (f(x_i) - f(y_i))|^r \right)^{\frac{1}{r}} \\ &= \left\| (\lambda_i \delta_{(x_i, y_i)}) \right\|_{l_r^{n, w}(\mathcal{F}(X))}. \end{aligned}$$

We denote by $\Pi_{p,r,s}^L(X, E)$ the Banach space of all Lipschitz (p, r, s) -summing operators with the norm $\pi_{p,r,s}^L(T)$ which is the smallest constant C such that the inequality (4.2) holds.

Theorem 4.3. ([8, Theorem 5.2]) Suppose $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$ and let $T \in Lip_0(X, E^*)$, $C > 0$.

The following are equivalent:

(a) $|\langle T, m \rangle| \leq C \mu_{p,r,s}(m)$ for all $m \in \mathcal{M}(X, E)$.

(b) There exists regular Borel probability measures μ and ν on the unit balls $B_{X^\#}, B_{E^*}$ with weak*-topology such that for all $x, x' \in X$ and $v \in E$,

$$|\langle T(x) - T(x'), v \rangle| \leq C \left(\int_{B_{X^\#}} (|f(x) - f(x')|)^r d\mu(f) \right)^{\frac{1}{r}} \left(\int_{B_{E^*}} |(v^*(v))^s| d\nu(v^*) \right)^{\frac{1}{s}}.$$

(c) There exists a Banach space Z , a Lipschitz r -summing operator $R : X \rightarrow Z^*$ and a linear s -summing operator $S : E \rightarrow Z$ such that $\pi_r^L(R) \cdot \pi_s(S) \leq C$ and

$$\langle T(x), v \rangle = \langle R(x), S(v) \rangle \text{ for all } x \in X, v \in E;$$

and

that is the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & E^* \\ & \searrow R & \nearrow S^* \\ & & Z^* \end{array}$$

4.2 Strictly Lipschitz (p, r, s) -summing operators

In this subsection, we give a new definition of Strictly Lipschitz (p, r, s) -summing operators, giving in the end some characterizations of these operators and presenting also some remarks concerning this class.

Definition 4.4. Let $0 < p, r, s < \infty$ such that $\frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}$. Let X be a pointed metric space and E be a Banach space. The Lipschitz operator $T : X \rightarrow E$ is strictly Lipschitz (p, r, s) -summing if there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1} \subset X$, $(\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K} (j = 1, \dots, n_2)$ and any $e_1^*, \dots, e_{n_1}^* \in E^*$, we have

$$\left(\sum_{i=1}^{n_1} \left| \left\langle \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle \right|^p \right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^r \right)^{\frac{1}{r}} \|(e_i^*)\|_{l_s^{n_1 w}}. \quad (4.3)$$

The class of all strictly Lipschitz (p, r, s) -summing operators from X into E is denoted by $\Pi_{p,r,s}^{SL}(X, E)$, which is a Banach space with the norm $\pi_{p,r,s}^{SL}(T)$ which is the smallest constant C such that the inequality (4.3) holds.

Remark 4.5. (1) If we put $n_2 = 1$ and $\lambda_i^j = 1$, we obtain

$$\left(\sum_{i=1}^{n_1} |\langle T(x_i) - T(y_i), e_i^* \rangle|^p \right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n_1} |(f(x_i) - f(y_i))|^r \right)^{\frac{1}{r}} \|(e_i^*)\|_{l_s^{n_1 w}}$$

i.e.,

$$\Pi_{p,r,s}^{SL}(X, E) \subset \Pi_{p,r,s}^L(X, E).$$

(2) If $p = 1$, r and s verify $\frac{1}{r} + \frac{1}{s} = 1$, the definition coincides with the definition of strictly Lipschitz r -nuclear. i.e., $\Pi_{1,r,s}^{SL}(X, E) = \mathcal{N}_r^{SL}(X, E)$

The following theorem characterizes the strictly Lipschitz (p, r, s) -summing operators by their adjoints (p, r, s) -summing operators.

Theorem 4.6. Suppose that $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Let X be a pointed metric space and E be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following properties are equivalent.

1) T is strictly Lipschitz (p, r, s) -summing.

2) \widehat{T} is (p, r, s) -summing.

In this case we have $\pi_{p,r,s}(\widehat{T}) = \pi_{p,r,s}^{SL}(T)$.

Proof. (1) \Rightarrow (2) : Let T be a strictly Lipschitz (p, r, s) -summing operator. Let $m_i \in \mathcal{F}(X)$ of the form $m_i = \sum_{j=1}^{n_2} \lambda_i^j \delta_{(x_i^j, y_i^j)}$ and $e_1^*, \dots, e_{n_1}^* \in E^*$. Then

$$\begin{aligned} \left(\sum_{i=1}^{n_1} \left| \left\langle \widehat{T}(m_i), e_i^* \right\rangle \right|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{n_1} \left| \left\langle \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &\leq \pi_{p,r,s}^{SL}(T) \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n_1} |f(m_i)|^r \right)^{\frac{1}{r}} \|(e_i^*)\|_{l_s^{n_1 w}} \\ &\leq \pi_{p,r,s}^{SL}(T) \|(m_i)\|_{l_r^{n_1, w}(\mathcal{F}(X))} \|(e_i^*)\|_{l_s^{n_1 w}}. \end{aligned}$$

Then, \widehat{T} is (p, r, s) -summing and we have

$$\pi_{p,r,s}(\widehat{T}) \leq \pi_{p,r,s}^{SL}(T).$$

(2) \Rightarrow (1) : Let $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1} \in X, (\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K}(j = 1, \dots, n_2)$ and any $e_1^*, \dots, e_{n_1}^* \in E^*$

$$\begin{aligned} & \left(\sum_{i=1}^{n_1} \left| \left\langle \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)), e_i^* \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^{n_1} \left| \left\langle \widehat{T}(m_i), e_i^* \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &\leq \pi_{p,r,s}(\widehat{T}) \| (m_i) \|_{l_r^{n_1, w}(\mathcal{F}(X))} \| (e_i^*) \|_{l_s^{n_1 w}} \\ &\leq \pi_{p,r,s}(\widehat{T}) \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^r \right)^{\frac{1}{r}} \| (e_i^*) \|_{l_s^{n_1 w}}. \end{aligned}$$

Then, T is strictly Lipschitz (p, r, s) -summing and we have

$$\pi_{p,r,s}^{SL}(T) \leq \pi_{p,r,s}(\widehat{T}).$$

□

The following theorem gives an integral characterization of the strictly Lipschitz (p, r, s) -summing operators. The proof is an adaptation of the one in the last section, for this we will omit it.

Theorem 4.7. *Suppose that $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Let X be a pointed metric space and E be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following properties are equivalent.*

- 1) T is strictly Lipschitz (p, r, s) -summing.
- 2) There exist a constant $C > 0$ and a Radon probability μ on $B_{X^\#}$ and η on E such that for all $x^j, y^j \in X, \lambda^j \in \mathbb{K}(1 \leq j \leq n)$ and $e^* \in E^*$, we have

$$\left| \left\langle \sum_{j=1}^n \lambda^j (T(x^j) - T(y^j)), e^* \right\rangle \right| \leq C \left(\int_{B_{X^\#}} \left| \sum_{j=1}^n \lambda^j (f(x^j) - f(y^j)) \right|^r d\mu(f) \right)^{\frac{1}{r}} \|e^*\|_{L_s(\eta)}.$$

Corollary 4.8. *Let $p \in [0, \infty]$ and $r, s \geq 1$. Suppose that $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Let X be a pointed metric space and E be a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following properties are equivalent.*

- 1) T is strictly Lipschitz (p, r, s) -summing.
- 2) There exists a Banach space G , a strictly Lipschitz s^* -summing operator $R : X \rightarrow G$ and a Cohen strongly r -summing linear operator $S : G \rightarrow E$ such that

$$T = S \circ R.$$

Let $1 \leq p, r, s < \infty$ such that $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$. Let E, F be two Banach spaces, Lapreste [22] has defined the norm $\mu_{p,r,s}$ as follows

$$\mu_{p,r,s}(u) = \inf \left\{ \|(\lambda_i)\|_{l_p^n} \| (e_i) \|_{l_r^{n,w}(E)} \| (y_i) \|_{l_s^{n,w}(F)} \right\},$$

where the infimum is taken over all representations of $u = \sum_{i=1}^n \lambda_i e_i \otimes y_i \in E \otimes F$. For the Lipschitz case, Chávez-Domínguez in [8], has defined the corresponding Lipschitz norm $\mu_{p,r,s}$ (with the same notation of Lapreste) on the space of molecules $\mathcal{F}(X; E)$ and shown that the class $\Pi_{p^*,r,s}^L(X, E^*)$ coincides with the dual of $(\mathcal{F}(X; E^*), \mu_{p,r,s})$. Note that the space $\mathcal{F}(X; E)$ plays the same role of Lipschitz tensor product $X \boxtimes E$. Let $u = \sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i \in X \boxtimes E^*$, the norm $\mu_{p,r,s}$ is defined as follows

$$\mu_{p,r,s}(u) = \inf \left\{ \|(\lambda_i)\|_{l_p^n} \| (\lambda_i^{-1} \kappa_i^{-1} \delta_{(x_i, y_i)}) \|_{l_r^{n,w}(\mathcal{F}(X))} \| (\kappa_i e_i^*) \|_{l_s^{n,w}(E^*)} \right\},$$

where the infimum is taken over all representations of u in $X \boxtimes E^*$ and $\lambda_i, \kappa_i > 0$. Let X be a pointed metric space and E be a Banach space. Let $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^* \in X \boxtimes E^*$ and

$$A_u = \left\{ m = \sum_{i=1}^n \lambda_i m_i \otimes e_i^* \in \mathcal{F}(X) \otimes E^* : \mu = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^* \right\}.$$

We consider

$$\mu_{p,r,s}^{SL}(u) = \inf_{m \in A_u} \left\{ \|(\lambda_i)\|_{l_p^{n_1}} \| (m_i) \|_{l_r^{n_1,w}(\mathcal{F}(X))} \| (e_i^*) \|_{l_s^{n_1,w}(E^*)} \right\}.$$

In fact, in the Lipschitz definition of the norm $\mu_{p,r,s}$ we have just used elements in $X \boxtimes E^*$ of the form $\sum_{i=1}^n \delta_{(x_i, y_i)} \boxtimes e_i^*$ which equals to u and $\lambda_i, \kappa_i > 0$, but in the definition of $\mu_{p,r,s}^{SL}$, we have to consider all elements of the set A_u . Therefore, the infimum in $\mu_{p,r,s}^{SL}$ will in general be smaller.

The following proposition shows that the Lipschitz norm and the strictly Lipschitz norm of an element u from $X \boxtimes E$ are equal.

Proposition 4.9. . *Let X be a pointed metric space and E be a Banach space. For every $u \in X \boxtimes E$ we have*

$$\mu_{p,r,s}^{SL}(u) = \mu_{p,r,s}^L(u),$$

where $\mu_{p,r,s}^L$ is the Lipschitz cross-norms corresponding to the Lapreste tensor norm $\mu_{p,r,s}$.

Proof. Let $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^* \in X \boxtimes E^*$. Let $m = \sum_i^n \lambda_i m_i \otimes e_i^* \in A_u$, we have

$$\begin{aligned} \mu_{p,r,s}^{SL}(u) &= \inf_{m \in A_u} \left\{ \|(\lambda_i)\|_{l_p^{n_1}} \| (m_i) \|_{l_r^{n_1, w}(\mathcal{F}(X))} \| (e_i^*) \|_{l_s^{n_1, w}(Y^*)} \right\} \\ &= \mu_{p,r,s}(\sum_{k=1}^l \delta_{(x_k, y_k)} \otimes s_k^*) \\ &= \mu_{p,r,s}^L(u). \end{aligned}$$

□

Corollary 4.10. *Let $1 \leq p \leq \infty$. The following properties are equivalent.*

- 1) *The Lipschitz operator $T : X \rightarrow E$ strictly Lipschitz (p, r, s) -summing.*
- 2) *There exists a positive constant C such that for every $x_k, y_k \in X$ and $s_k^* \in E^*$ ($1 \leq k \leq l$) we have*

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C \mu_{p,r,s}^{SL}(u),$$

where $u = \sum_{k=1}^l \delta_{(x_k, y_k)} \boxtimes s_k^*$.

Corollary 4.11. [30] *For every pointed metric space X and Banach space E , we have*

$$X \widehat{\boxtimes}_{\alpha^L} E = \mathcal{F}(X) \widehat{\otimes}_{\alpha} E.$$

holds isometrically.

Then, by this corollary we get the following coincidence result.

Corollary 4.12. *Let X be a pointed metric space and E be a Banach space. Then*

$$\Pi_{p^*, r, s}^{SL}(X, E^*) = \left(X \widehat{\boxtimes}_{\mu_{p^*, r, s}^{SL}} E^* \right)^* = \left(\mathcal{F}(X) \widehat{\otimes}_{\mu_{p, r, s}} E^* \right)^*.$$

Proof. $\mu_{p,r,s}^{SL}(u) = \inf_{m \in A_u} \left\{ \|(\lambda_i)\|_{l_p^{n_1}} \| (m_i) \|_{l_r^{n_1, w}(\mathcal{F}(X))} \| (e_i^*) \|_{l_s^{n_1, w}(Y^*)} \right\}$
 $= \mu_{p,r,s}(\sum_{k=1}^l \delta_{(x_k, y_k)} \otimes s_k^*)$
 $= \mu_{p,r,s}^L(u).$

□

Perspectives

In this research, we were able to achieve the following aims as mentioned in the general introduction which are, first the extension of results about linear ϕ -summing operators to Lipschitz ϕ -summing operators, second the extension of Lipschitz p -summing operators

($1 \leq p < \infty$) to Lipschitz Strictly p -summing operators and to Lipschitz p -nuclear operators and Lipschitz M -Strictly p -summing operators. Third, the extension of Lipschitz (p,r,s) -summing to Strictly Lipschitz (p,r,s) -summing operators. During our study and research, we have met a number of questions that are worth mentioning here, in the form open problems similar to those raised by the authors J.D.Farmer and W.B.johnson in their paper [14], especially problem 6.

We expose in the following several special cases of these problems:

Problem 1 Is there a composition formula for Lipschitz ϕ -summing operators?

Problem 2 We have in the linear case $\prod_{\phi}(X; Y) = \prod_p(X; Y)$. Is it true that $\prod_{\phi}^L(X; Y) = \prod_p^L(X; Y)$? $0 < p < 1$

Problem 3 Is there a Pietsch factorization theorem for the class $\prod_{\phi}^L(X; Y)$?

Problem 4 Is there a composition formula for strictly Lipschitz p -nuclear operators?

Problem 5 Is there a composition formula for M -strictly Lipschitz p -summing operators?

Problem 6 Is there a Pietsch factorization theorem for the space $\prod_{(p,r,s)}^L(X; E)$?

Problem 7 In the linear case R. Khalil and W. Deeb in [20] have shown that $\prod_{\phi}(E, E) =$

$\Pi_p(E, E)$ for $0 < p < 1$ and E is a Banach space with the metric approximation property, is it true that $\Pi_\phi^L(E, E) = \Pi_p^L(E, E)$?

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