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Field: **Material sciences** Branch: **Physics**

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A study of a nonrelativistic energy spectrum produced with an isotropic potential in the framework of extended quantum mechanics symmetries: the case of inversely quadratic Yukawa and inversely quadratic Hellmann potential

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Dedication

To the pure soul of my grandfather, Al-Hajj El-Omari Ziane.To my father and mother. To my wife and my children: Abdelraouf, Baraa, and Taqwa. To all my brothers and siblings

I dedicate this humble work.

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Ziani oussama

To the pure soul of my father — may Allah have mercy on him, To my dear mother — may Allah preserve her, To my wife and my children: Salah and Hadjer, And to all my brothers and sisters,

I dedicate this humble work.

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Rahmani abdelhamid

Acknowledgment

I would like to thank Allah for the Almighty who granted us the strength to reach our successes end of this year. I was fortunate to have been supervised by Pr. MAIRECHE Abdelmadjid, to whom I would like to express My sincerest gratitude for his insightful advice and for proposing this topic, allowing me the chance to work on it, and guiding me through the end. My acknowledgment goes also to Mrs. Salim MEDJBER and Abboud METATLA who will examine and evaluate My work. Last but not least, I would like to thank all those who helped me in any way in my project.

Introduction

It has been a century since the appearance of the Schrödinger equation. It has caused a very great scientific revolution, as it is considered the first bridge that gradually transports us from the macroscopic scale to the microscopic scale. The first signs of her success were clearly evident in her remarkable success in studying the spectrum of the hydrogen atom, which surpassed Bohr's atomic model. The exact solutions of the Schrödinger equation contain all the necessary information for a quantum system. The reason for this is that the eigenfunctions connected to these problems include highly relevant details about the quantum systems. In modern natural science, which includes cosmology, atomic physics, molecular science, materials science, and other fields, quantum mechanics is an essential research tool. The Schrödinger equation is the foundational concept known as quantum mechanics [1]. The Schrödinger equation is defined as a linear differential equation of second order. Researchers currently use several methods to obtain exact or approximate solutions to the Schrödinger equation such as the asymptotic iteration method (AIM), exact-quantization rules, the Nikiforov–Uvarov method, supersymmetric quantum mechanics, Wentzel-Kramers-Brillion Jeffery approximation method, the Nikiforov– Uvarov-functional analysis method [2, 3, 4, 5, 6, 7, 8, 9, 10] and so on. Schrödinger's equation has achieved success in many fields such as particles, light mesons, atomic and nuclear physics, in addition to various problems in chemistry. It is known to researchers and specialists in physics that quantum mechanics, known in the literature, is based on the following postulates [11, 12]:

$$\begin{cases} [x_i, p_j] = \imath\hbar\delta_{ij}, \\ [x_i, x_j] = [p_i, p_j] = 0. \end{cases}$$

Quantum mechanics on noncommutative space was first proposed by Heisenberg in 1930 [13] and then developed by Snyder in late 1947 [14]. The proposal of extended quantum mechanics came as a possible solution to many physical problems that non-relativistic and relativistic quantum mechanics were unable to find solutions to the divergence problem in the standard model, string theory and quantum gravity [15, 16, 17, 18].

We will reserve this study to obtain a master's degree in theoretical physics from Mohammed Boudiaf University in M'sila to study the inversely quadratic Yukawa and inversely quadratic Hellmann potential in the context of nonrelativistic noncommutative quantum mechanics symmetries for the promotion 2024-2025 because noncommutative quantum mechanics includes larger physical symmetries than the quantum mechanics known in the literature.

This master memory is organized as follows. In chapter one, the noncommutative quantum mechanics is represented. In chapter two, the Schrödinger equation is revised under the inversely quadratic Yukawa and inversely quadratic Hellmann potential. In chapter three, we study the effect of phase-space noncommutativity deformation on the inversely quadratic Yukawa and inversely quadratic Hellmann potential.

Chapter 1

The noncommutative phase-space formalism

1.1 Introduction

This chapter will cover the postulates and hypotheses that define the quantum and physical structures of the noncommutative (phase-space/ phase-phase and space-space) and its physical structures. The fundamental ideas that will be covered are as follows:

-A standard quantum structure representation,

-The postulates of the noncommutative (phase-space/ phase-phase and space-space),

-The Star product and its characteristics,

1.2 Review of structure of ordinary quantum mechanics

The beginning of the nineteenth century witnessed a radical in the traditional view of light. The prevailing belief was that light had only wave behavior. The wave nature of light was well established among scientists of that period and those before it, and they demonstrated this through well-known interference and diffraction experiments. The fundamental change in knowledge of other behavior of light began convincingly through experiments on the photoelectric effect. In 1900, Planck quantifies the energy of light:

$$E_\gamma = h\nu$$

which consider the beginning of quantum physics, here $h \approx 6,6262.10^{-34}$ js. Currently, ordinary quantum mechanics is formulated on the commutative space of the coordinates of variable and the canonical moment of hermetic operators (x_i, p_i), as follows [19, 20]:

$$\begin{cases} [x_i, p_j] = \imath\hbar\delta_{ij} \\ [x_i, x_j] = 0 \\ [p_i, p_j] = 0 \end{cases} \tag{1.1}$$

Here, $\hbar$ is the reduced Plank constant $\frac{\hbar}{2\pi}$, and the usual Kronecker symbol δ_{ij} is:

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

The above algebra (in Eq. (1.1)) can be generalized to the Dirac and interaction pictures as follows:

$$\begin{cases} \left[x_i^d(t), p_j^d(t)\right] = \left[x_i^I(t), p_j^I(t)\right] = \imath\hbar\delta_{ij} \\ \left[x_i^d(t), x_j^d(t)\right] = \left[x_i^I(t), x_j^I(t)\right] = 0 \\ \left[p_i^d(t), p_j^d(t)\right] = \left[p_i^I(t), p_j^I(t)\right] = 0 \end{cases} \tag{1.2}$$

where the usual canonical coordinates (x_i, p_i) and the corresponding time-dependent $x_i(t)$ and $p_i(t)$ are determined from the projection relations:

$$\begin{cases} x_i^d(t) = \exp\left(\frac{\imath}{\hbar}H(t-t_0)\right) x_i^s \exp\left(-\frac{\imath}{\hbar}H(t-t_0)\right) \\ p_i^d(t) = \exp\left(\frac{\imath}{\hbar}H(t-t_0)\right) p_i^s \exp\left(-\frac{\imath}{\hbar}H(t-t_0)\right) \\ x_i^I(t) = \exp\left(\frac{\imath}{\hbar}H_0(t-t_0)\right) x_i^s \exp\left(-\frac{\imath}{\hbar}H_0(t-t_0)\right) \\ p_i^I(t) = \exp\left(\frac{\imath}{\hbar}H_0(t-t_0)\right) p_i^s \exp\left(-\frac{\imath}{\hbar}H_0(t-t_0)\right) \end{cases} \tag{1.3}$$

The two indices (s, d, I) represent the Shrodinger, Dirac and joint interaction representations, $(\{x_i(t)\}, \{p_i(t)\})$, $\left(\left\{x_i^d(t)\right\}, \left\{p_i^d(t)\right\}\right)$, $\left(\left\{x_i^I(t)\right\}, \left\{p_i^I(t)\right\}\right)$ and H/H_0 are (total/free) Hermitian operators on a Hilbert space of physical states, which, each, satisfy the Heisenberg equation of motions. We get the following:

$$\begin{cases} \frac{dx_i}{dt} = \frac{\imath}{\hbar}[H, x_i(t)] \\ \frac{dp_i}{dt} = \frac{\imath}{\hbar}[H, p_i(t)] \end{cases} \tag{1.4}$$

Both related concepts relating to energy E and impulsion p_i are satisfied by the quantization procedure:

$$\begin{cases} E \to i\hbar\frac{\partial}{\partial t} \\ p_i \to \frac{\hbar}{i}\frac{\partial}{\partial x_i} \end{cases} \tag{1.5}$$

It is well known that the classical energy E of a particle of mass m_0 subjected to the external forces produced by a potential $V(\vec{r}, t)$, in a classical mechanic is given by:

$$E = \frac{\overrightarrow{p}^2}{2m_0} + V(\vec{r}, t) \tag{1.6}$$

The quantization process in Equation (1.5) made it possible to derive the Shrödinger equation, which is well-known in the literature's framework for quantum mechanics:

$$\left(-\frac{\hbar^2}{2m}\Delta+V(\vec{r},t)\right)\psi\left(\overrightarrow{r},t\right)=\imath\hbar\frac{\partial\psi\left(\overrightarrow{r},t\right)}{\partial t} \tag{1.7}$$

Here Δ is the well known Laplacian operator in spherical coordinates $\overrightarrow{r}\left(r,\theta,\varphi\right)$ as follows:

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\varphi^2} \tag{1.8}$$

which can be expressed in Cartesian coordinates $\overrightarrow{r}\left(x,y,z\right)$ as follows:

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{1.9}$$

while $\psi\left(\overrightarrow{r},t\right)$ denoting the complex wave function. The probability of finding the particle at time t in an elementary volumes $\left(d^3r,d^3p\right)$, rounding the point (r,p) as follows:

$$dp=\left\{\begin{array}{c}\left|\psi\left(\overrightarrow{r},t\right)\right|^2d^3r:\ \text{ In the configuration space}\\ \left|\psi\left(\overrightarrow{p},t\right)\right|^2d^3p:\text{ In the momentum space}\end{array}\right. \tag{1.10}$$

where $\left(d^3r,d^3p\right)$ equals $\left(r^2\sin\theta d\theta d\varphi dr,p^2\sin\theta d\theta d\varphi dp\right)$, respectively.

1.3 Noncommutative (phase-space/ phase-phase and space-space):

One of the most essential aspects of quantum physics is dealing with noncommuting operators, specifically the commutation relations between positions x_i and corresponding momenta p_i. Noncommutative quantum mechanics (NCQM) symmetries is a concept that implies that operators do not commute; for example, consider a situation in which the coordinates and moment operators are noncommutative. During 1930, a hot topic was how to solve the infinity problem in the newly found quantum field theory (QFT). Heisenberg was the first to propose that noncommutativity be extended to coordinate systems. Then, the concept of NCQM was extended to generalize the usual conception of space-time, in which the noncommutativity of some normally commutative variables is assumed, leading to the formation of different Lie algebras. Connes in 1980 revived the ideas of noncommutative geometry while Woronowicz and Drinfel'd, were generalized the notion of a differential structure to the noncommutative setting [21, 22, 23, 24]. The simplest commutation relation that described the noncommutativity idea satisfying the following algebra [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]:

$$\left\{\begin{array}{c}\left[\hat{x}_i\left(t\right)\overset{*}{,}\hat{x}_j\left(t\right)\right]=\left[\hat{x}_i\overset{*}{,}\hat{x}_j\right]=\imath\hbar_{eff}\theta_{ij}\\ \left[\hat{p}_i\left(t\right)\overset{*}{,}\hat{p}_j\left(t\right)\right]=\left[\hat{p}_i\overset{*}{,}\hat{p}_j\right]=\imath\overline{\theta}_{ij}\end{array}\right. \tag{1.11}$$

where $\left(\theta_{ij},\bar{\theta}_{ij}\right)=-\left(\theta_{ji},\bar{\theta}_{ji}\right)=\epsilon_{ij}\left(\theta,\bar{\theta}\right)$ are constants anti-symmetric tensors of dimensions $[x]^2$ and $[p]^2$, $(\left(\theta,\bar{\theta}\right)$ are the non-commutative parameters and ϵ_{ij} is just an antisymmetric number ($\epsilon_{ij}=-\epsilon_{ji}=1$ with $i\neq j$ and $\epsilon_{jj}=0$) and $\hbar_{eff}=\hbar\left(1+\frac{\theta\bar{\theta}}{4\hbar^2}\right)$ is the effective constant of Planck. The noncommutative coordinates $(\hat{x}_i,\hat{p}_i)$ take the form:

$$\begin{cases} x_i\rightarrow\hat{x}_i=f\left(x_i,p_i\right) \\ p_i\rightarrow\hat{p}_i=f\left(x_i,p_i\right)\end{cases} \tag{1.12}$$

In this work, we interred by the phase-phase has three dimensions $N=3$, therefore the indices take the values $(i,j=1,3)$. In this particular case, the rules of canonical commutations become:

$$\begin{cases} [\hat{x}_1,\hat{p}_2]=0 \\ [\hat{x}_1,\hat{p}_3]=0 \\ [\hat{x}_2,\hat{p}_3]=0 \\ [\hat{x}_1,\hat{x}_2]=i\theta_{12} \\ [\hat{x}_1,\hat{x}_3]=i\theta_{13} \\ [\hat{x}_2,\hat{x}_3]=i\theta_{23}\end{cases} \tag{1.13}$$

and

$$\begin{cases} [\hat{x}_1,\hat{p}_1]=i\hbar_{eff} \\ [\hat{x}_2,\hat{p}_2]=i\hbar_{eff} \\ [\hat{x}_3,\hat{p}_3]=i\hbar_{eff} \\ [\hat{p}_1,\hat{p}_2]=i\bar{\theta}_{12} \\ [\hat{p}_1,\hat{p}_3]=i\bar{\theta}_{13} \\ [\hat{p}_2,\hat{p}_3]=i\bar{\theta}_{23}\end{cases} \tag{1.14}$$

1.4 Weyl's quantization:

The fundamentals of quantum physics inspired many of the broad principles behind noncommutative geometry. Weyl proposed an elegant formulation for mapping quantum operators to classical functions of phase-space variables within the framework of canonical quantification. This method establishes a systematic approach to modeling noncommutative spaces in general and examining ancient ideas based on them [38]. Weyl quantization is a technique for describing quantum physics using classical mechanics' phase space. It is a rule that allows a quantum operator to be associated with a classical function that is dependent on phase space variables. The Weyl quantification also applies to commutative relations in a general form. Consider a $f\left(x,p\right)$ and $g\left(x,p\right)$ a general two functions, their product in the notion of noncommutative phase-space can be expressed as a new product called the star product or the Weyl-Moyal star product defined on phase space,

$$f(x,p) * g(x,p) = \\ f(x,p)\,g(x,p) + \frac{i}{2}\sum_m \theta^{mn}\frac{\partial}{\partial x^m} f(x,p)\frac{\partial}{\partial x^n} + \\ + \frac{i}{2}\sum_m \bar{\theta}^{mn}\frac{\partial}{\partial p^m} f(x,p)\frac{\partial g(x,p)}{\partial p^n} + O\left(\theta^2,\bar{\theta}^2\right) \tag{1.15}$$

The formalism of the star product initiated by Weyl and Wigner to allow a description of quantum mechanics in terms of phase space, is articulated not around non-commuting operators, as in the operational approach, but around the deformation of the product between the phase space variables. We will see how this formalism can be used in the context of noncommutative quantum mechanics (NCQM) symmetries. In the symmetries of non-commutative space-space symmetry, the above equation will reduced to the form:

$$f(x,p) * g(x,p) = \\ f(x,p)\,g(x,p) + \frac{i}{2}\sum_m \theta^{mn}\frac{\partial}{\partial x^m} f(x,p)\frac{\partial g(x,p)}{\partial x^n} + O\left(\theta^2\right)$$

In the symmetries of non-commutative phase-phase symmetry, the above equation will reduced to the form:

$$f(x,p) * g(x,p) = \\ f(x,p)\,g(x,p) + \frac{i}{2}\sum_m \bar{\theta}^{mn}\frac{\partial}{\partial p^m} f(x,p)\frac{\partial g(x,p)}{\partial p^n} + O\left(\bar{\theta}^2\right)$$

1.5 Properties of the star product

The formalism of the star product was initiated by Weyl and Wigner to allow a description of quantum mechanics in terms of phase space, the properties of the star product are presented as follows: [39, 40, 41, 42, 43, 44]:

-When $\left(\theta,\overline{\theta}\right) = (0,0)$

$$\underset{\left(\theta,\overline{\theta}\right)\to(0,0)}{Lim} \left(f(x) * g(x)\right) = f(x)g(x) \tag{1.16}$$

-The star product between exponential :

$$e^{\imath kx} * e^{\imath qx} = e^{\imath(k+q)x} e^{-\frac{\imath}{2}(k \curlywedge q)} \qquad \text{with } k \curlywedge q = k^{\imath} q^{\jmath}\theta_{ij} \tag{1.17}$$

-Not commutative property:

$$f(x,p) * g(x,p) \neq g(x,p) * f(x,p) \tag{1.18}$$

-Associative property:

$$(f(x,p)*g(x,p))*h(x,p)=f(x,p)*(g(x,p)*h(x,p)) \tag{1.19}$$

-The relation of the complex conjugate property:

$$(f(x,p)*g(x,p))^*=g(x,p)^* * f(x,p) \tag{1.20}$$

-The integral relation property:

$$\int d^Dx\,(f*g)=\int d^Dx\,(g*f)(x,p)=\int d^Dx f(x,p)\,g(x,p) \tag{1.21}$$

-Cyclic permutation property:

$$\int d^Dx\,(f*g*h)=\int d^D\,(g*f*h)=\int d^Dx\,(g*f*h) \tag{1.22}$$

-Satisfies the Leibniz's rule property:

$$\frac{\partial(f*g)}{\partial x^\alpha}=\left(\frac{\partial f}{\partial x^\alpha}\right)*g+f*\left(\frac{\partial g}{\partial x^\alpha}\right) \tag{1.23}$$

1.6 Boop's shift method

In his study, physicist Fritz Bopp was the first to examine pseudo-differential operators derived from a symbol using quantization methods:

$$\begin{cases} x\to x+\frac{1}{2}\imath\hbar\frac{\partial}{\partial p} \\ p\to p-\frac{1}{2}\imath\hbar\frac{\partial}{\partial x}\end{cases} \tag{1.24}$$

Instead of the usual correspondence $\left(x\to x,\ p\to -\frac{1}{2}\imath\hbar\frac{\partial}{\partial x}\right)$, the operators $x\to x+\frac{1}{2}\imath\hbar\frac{\partial}{\partial p}$ and $p\to p-\frac{1}{2}\imath\hbar\frac{\partial}{\partial x}$ are known as Bopp's shifts, and this quantization procedure is known as the Bopp quantization procedure. This quantization leads us to obtain the following:

$$\begin{cases} \widehat{x}^i=x^i-\sum_j^3\left(\frac{\theta^{ij}}{2}p_j\right) \\ \widehat{p}^i=p^i+\sum_j^3\left(\frac{\overline{\theta}^{ij}}{2}x_j\right)\end{cases} \tag{1.25}$$

To write the Schrödinger equation in the noncommutative phase-space, we follow these steps:

1- The ordinary three-dimensional Hamiltonian operators $\hat{H}(p_i,x_i)$ will be replace with new Hamiltonian operator $\hat{H}(\hat{p}_i,\hat{x}_i)$.

2- The ordinary complex wave function $\psi\left(\overrightarrow{r}\right)$ become a new complex wave function $\hat{\psi}(\hat{r})$.

3- The ordinary energie E_{nl} will be replacing with new values E_{nc}^{gqy}.
4- We replace the ordinary product with the star product.
Hence, we get the following Schrödinger equation in the noncommutative space :

$$H\left(\widehat{x}^{i},\widehat{p}^{i}\right)\hat{\psi}\left(\vec{r},t\right)=E_{nc}\hat{\psi}\left(\vec{r},t\right)\Leftrightarrow\hat{H}\left(x,p\right)*\hat{\psi}\left(\vec{r},t\right)=E_{nc}\hat{\psi}\left(\vec{r},t\right) \tag{1.25}$$

The Bopp's shifts method allows to reduce the above deformed Shrodinger equation to the new translated form:

$$H\left(\widehat{x}^{i},\widehat{p}^{i}\right)*\hat{\psi}\left(\vec{r},t\right)=E_{nc}\hat{\psi}\left(\vec{r},t\right)=E_{nc}\psi(\vec{r}) \tag{1.26}$$

So the Hamiltonian operator takes the three varieties forms as follows [44, 45, 46, 47, 48, 49, 50]:

$$\left\{\begin{array}{c}H\left(\hat{p}_{i},\hat{x}_{i}\right)=H\left(\hat{p}_{i}=p_{i}+\sum\limits_{j=1}^{3}\left(\frac{\bar{\theta}_{ij}}{2}x^{j}\right),\quad\hat{x}_{i}=x_{i}-\sum\limits_{j=1}^{3}\left(\frac{\theta_{ij}}{2}p^{j}\right)\right)\\ \text{For noncommutative phase-space}\\ H\left(\hat{p}_{i},\hat{x}_{i}\right)=H\left(\hat{p}_{i}=p_{i},\quad\hat{x}_{i}=x_{i}-\sum\limits_{j=1}^{3}\left(\frac{\theta_{ij}}{2}p^{j}\right)\right)\\ \text{For noncommutative space-space}\\ H\left(\hat{p}_{i},\hat{x}_{i}\right)=H\left(\hat{p}_{i}=p_{i}+\sum\limits_{j=1}^{3}\left(\frac{\bar{\theta}_{ij}}{2}x^{j}\right),\quad\hat{x}_{i}=x_{i}\right)\\ \text{For noncommutative phase-phase}\end{array}\right. \tag{1.27}$$

The first variety corresponds to noncommutative phase-space (NCPS in short) symmetries which correspond to the new Hamiltonian operator $H\left(\hat{p}_{i}=p_{i}+\frac{\bar{\theta}^{ij}}{2}x_{j},\hat{x}_{i}=x_{i}-\frac{\theta^{ij}}{2}p_{i}\right)$ in Eq. (1.27):

$$\left\{\begin{array}{c}x_{i}\rightarrow\quad\hat{x}_{i}=x_{i}-\sum\limits_{j=1}^{3}\left(\frac{\theta_{ij}}{2}p^{j}\right)\\ p_{i}\rightarrow\hat{p}_{i}=p_{i}+\sum\limits_{j=1}^{3}\left(\frac{\bar{\theta}_{ij}}{2}x^{j}\right)\end{array}\right. \tag{1.28}$$

The second variety corresponds to noncommutative space-space (NCSS in short) symmetries which correspond to the new Hamiltonian operator $H\left(\hat{p}_{i}=p_{i},\hat{x}_{i}=x_{i}-\frac{\theta^{ij}}{2}p_{j}\right)$ in Eq. (1.27):

$$\left\{\begin{array}{c}p_{i}\rightarrow\hat{p}_{i}=p_{i}\\ x_{i}\rightarrow\hat{x}_{i}=x_{i}-\frac{\theta^{ij}}{2}p_{j}\end{array}\right. \tag{1.29}$$

The third variety corresponds to noncommutative phase-phase (NCPP in short) symmetries which correspond to the new Hamiltonian operator $H\left(\hat{p}_{i}=p_{i}+\frac{\bar{\theta}^{ij}}{2}x_{j},\hat{x}_{i}=x_{i}\right)$ in Eq. (1.27):

$$\begin{cases} p_i \to \hat{p}_i = p_i - \frac{\bar{\theta}^{ij}}{2} p_j \\ x_j \to \hat{x}_i = x_i \end{cases} \tag{1.30}$$

In our current master memoir, we are interested in applying the following general procedure to NCSS symmetries which correspond to the first variety of Eq. (1.27). The three-generalized coordinates $(\hat{x} = \hat{x}_1, \hat{y} = \hat{x}_2, \hat{z} = \hat{x}_3)$ in the noncommutative phase-space were depended on corresponding three-usual generalized positions (x, y, z) and three momentum coordinates (p_x, p_y, p_z):

$$\begin{cases} i_1 = 1 & \hat{x}_1 = \hat{x}_{p_1} = p_x \\ i_2 = 2 & \hat{x}_2 = \hat{x}_{p_2} = p_y \\ i_3 = 3 & \hat{x}_3 = \hat{x}_{p_3} = p_z \end{cases} \tag{1.31}$$

It is important to notice that the new operators $\hat{x}_i$ and $\hat{p}_i$ in three-dimensional phase-space noncommutativity was depended on ordinary operator x_i and p_i from the projection relations:

$$\begin{cases} \hat{x}_1 = x_1 - \frac{\theta^{12}}{2} p_2 - \frac{\theta^{13}}{2} p_3 \\ \hat{x}_2 = x_2 - \frac{\theta^{21}}{2} p_1 - \frac{\theta^{23}}{2} p_3 \\ \hat{x}_3 = x_3 - \frac{\theta^{31}}{2} p_1 - \frac{\theta^{32}}{2} p_2 \end{cases} \tag{1.32}$$

and

$$\begin{cases} \hat{p}_1 = p_1 \\ \hat{p}_2 = p_2 \\ \hat{p}_3 = p_3 \end{cases} \tag{1.33}$$

The non-vanish 9-commutators in three-dimensional phase-space noncommutativity can be determined as follows:

$$\begin{cases} [\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i \\ [\hat{x}, \hat{y}] = i\theta_{12}, \\ [\hat{x}, \hat{z}] = i\theta_{13}, \\ [\hat{y}, \hat{z}] = i\theta_{23}, \\ [\hat{p}_y, \hat{p}_y] = 0, \\ [\hat{p}_y, \hat{p}_z] = 0, \\ [\hat{p}_x, \hat{p}_z] = 0. \end{cases} \tag{1.34}$$

The square of $(\vec{r}, \vec{p})$ are given by :

$$\begin{cases} \overrightarrow{\hat{r}}^2 = \overrightarrow{\hat{r}}_x^2 + \overrightarrow{\hat{r}}_y^2 + \overrightarrow{\hat{r}}_z^2 \\ \overrightarrow{\hat{p}}^2 = \overrightarrow{\hat{p}}_x^2 + \overrightarrow{\hat{p}}_y^2 + \overrightarrow{\hat{p}}_z^2 \end{cases} \tag{1.35}$$

To get the solution to the noncommutative Schrödinger equation, we added the star product introduced by Bopp's shift method. That is a consequence of the star product between the potential operator $\hat{V}(\hat{x})$ and the complex wave function $\hat{\Psi}\hat{r}$:

$$\left[\frac{\hat{\vec{P}}^{2}}{2m}+\hat{V}(\hat{r})\right]*\hat{\Psi}(\hat{r})=E_{nc}\hat{\psi}(\vec{r},t)$$

$$\Rightarrow$$

$$\left[\frac{\overrightarrow{p}^{2}_{nc}}{2m}+V(\hat{r})\right]\Psi(\overrightarrow{r})=E\Psi(\overrightarrow{r}) \tag{1.36}$$

The two operators $\hat{x}$ and $\hat{p}$, when on a noncommutative three-dimensional space-phase, can be written as follows :

$$\begin{cases} \hat{r}^{2}=r^{2}-\overrightarrow{L}\vec{\Theta} \\ \frac{\overrightarrow{p}^{2}_{nc}}{2m_{0}}=\frac{\overrightarrow{P}^{2}}{2m_{0}} \end{cases} \tag{1.37}$$

Where the two couplings $\overrightarrow{L}\overrightarrow{\Theta}$ is given by the following relations respectively:

$$\overrightarrow{L}\vec{\Theta}=L_{x}\theta_{12}+L_{y}\theta_{23}+L_{z}\theta_{13} \tag{1.38}$$

Chapter 2

Review of the inversely quadratic Yukawa plus inversely quadratic Hellmann potential

2.1 Introduction

The inversely quadratic Yukawa and inversely quadratic Hellmann's potential is an exponential potential. It consists of the sum of two potentials, the first is Yukawa potential $(-\frac{V_0}{r^2}\exp(-2\delta r))$ that proposed by Yukawa itself in 1935 [51] while the second part is the inersely quadratic Hellmann potential $(-\frac{a}{r}+\frac{b}{r^2}\exp(\delta r))$. Thus, the inverse quadratic Yukawa plus inversely quadratic Hellmann potential is as follows [52, 53]:

$$V(r) = -\frac{V_0}{r^2}\exp(-2\delta r) - \frac{a}{r} + \frac{b}{r^2}\exp(\delta r) \tag{2.1}$$

where V_0, a, b and δ are the dissociation energy, the strengths of the Coulomb and Yukawa potentials and the screening parameter. We expend the two exponential functions $\exp(-2\delta r)$ and $\exp(\delta r)$ to obtains:

$$\begin{cases} \exp(-2\delta r) = 1 - 2\delta r + \frac{1}{2!}4\delta^2 r^2 + O\left(\delta^3\right) \\ \exp(\delta r) = 1 + \delta r + \frac{1}{2!}\delta^2 r^2 + O\left(\delta^3\right) \end{cases} \tag{2.2}$$

Substitution of Eqs.(2.2) into Eq. (2.1) yields:

$$V(r) = -\frac{1}{r^2}(b - V_0) - \frac{1}{r}(2V_0 - \alpha - b\delta) - (b - 2V_0)\delta^2 \tag{2.3}$$

Using the expansion approach and the Nikiforov-Uvarov method, Ita [54] has solved the Schrödinger equation for the Hellman potential and derived the energy eigenvalues and their accompanying wave functions. Additionally, Hamzavi and Rajabi [55] obtained relativistic spin and pseudospin symmetries of the Dirac equation with the Hellmann potential and tensor coupling using the parametric Nikiforov-Uvarov approach. Using the asymptotic iteration method, Kocak *et al.* [56] solved the Schrödinger equation with the Hellmann potential and produced the wave functions and energy eigenvalues. Inyang *et al.* [57] (2024) applied the exact quantization rule approach to solve the radial Schrödinger equation analytically using the class of inversely quadratic Yukawa potential and predicted the mass spectra of heavy mesons, including charmonium and bottomonium and investigated the energy spectra of homonuclear diatomic molecules, like nitrogen N_2 and hydrogen H_2. Inyang *et al.* [58] (2025) used the Nikiforov-Uvarov method to solve the Schrödinger equation for the class of inversely quadratic Yukawa potential and deriving both the energy equation and the normalized wave function. Onate (2013) [59] by using the concept of supersymmetric quantum mechanics, obtained the Klein-Gordon and Schrödinger equations solutions for the inversely quadratic Yukawa potential. [Maireche, (2021) [60] present approximate and analytical solutions of the deformed Klein–Gordon containing an interaction of the equal vector and scalar potential newly generalized modified screened Coulomb plus inversely quadratic Yukawa potential. This study is realized in the relativistic noncommutative 3-dimensional real space symmetries. Chang *et al.* [61] (2022) studied the effects of hydrostatic pressure, temperature, Al-concentration and other structural parameters on the optical absorption coefficients in inversely quadratic Yukawa-Herman potential (IQYHP) GaAs/AlηGa1-ηAs spherical quantum dots using the Nikiforov-Uvarov to solve the Schrödinger equation and obtained the energy levels and wave functions. Maireche (2022) [62] determined the bound state solutions of the deformed Dirac equation (DDE) with the improved generalized inversely quadratic Yukawa potential including Coulomb-like tensor interaction under the condition of spin symmetry and pseudospin symmetry in the deformation Dirac theory symmetries for with the arbitrary spin-orbit quantum number k using the parametric Bopp's shift method and standard perturbation theory to obtain the relativistic and nonrelativistic energy eigenvalues. Now we will study the structure resulting from the combination of the two potentials the inversely quadratic Yukawa potential and the inversely quadratic Hellmann potential. Hitler *et al.* (2017) [63] obtained the exact energy spectrum for inversely quadratic Yukawa plus potential plus inversely quadratic Hellmann potential, via the WentzelK ramers-Brillouin (WKB) approach. Ita *et al.* (2013) [64] presented the solutions to the Schrödinger equation with inversely quadratic Yukawa and inversely quadratic Hellmann (IQYIQH) potential for any angular momentum quantum number l using the Nikiforov-Uvarov method and obtained the bound state energy eigenvalues and the corresponding unnormalized eigenfunctions are obtained in terms of the Laguerre polynomials. It is worth noting that there were previous studies, including: modified inversely quadratic Hellmann plus inversely quadratic potential [65] in the context of three-dimensional non-commutative phase-space. The investigation of an extended relativistic shell model of mirror Nuclei ^{17}O and ^{17}F under spin and pseudo-spin symmetries conditions using the modified quadratic Hellmann potential model within Bopp's shift method and standard perturbation theory framework within the framework in 3-dimensional relativistic non-commutative space symmetries[66]. A new look at a nonrelativistic shell model: study of the mirror nuclei ^{17}O and ^{17}F in the symmetries of non-commutative quantum mechanics.

2.2 Reviewing the eigenfunctions and the energy eigenvalues for inversely quadratic Yukawa and inversely quadratic Hellmann potential

Schrödinger equation is a fundamental equation of quantum mechanics which describes the evolution of the wave function of a physical system over time. It is a first-order partial differential equation concerning time and a second-order partial differential equation concerning the coordinates of ordinary space. It takes the following form:

$$H\psi\left(\overrightarrow{r},t\right)=E\psi\left(\overrightarrow{r},t\right) \tag{2.4}$$

here $\psi\left(\overrightarrow{r}\right)$ is the complex wave function that satisfies the stationary Schrödinger equation and E is a nonrelativistic eigenvalue of the Hamiltonian H, which is written in the form :

$$\begin{aligned} H &= \frac{\hat{P}^2}{2m_0}-\frac{V_0}{r^2}\exp\left(-2\delta r\right)-\frac{a}{r}+\frac{b}{r^2}\exp\left(\delta r\right) \\ &\equiv \frac{\hat{P}^2}{2m_0}+\frac{1}{r^2}(b-V_0)+\frac{1}{r}(2V_0-\alpha-b\delta)+(b-2V_0)\delta^2 \end{aligned} \tag{2.5}$$

where P represents the impulse $\overrightarrow{P}=-i\hbar\overrightarrow{\nabla}$, and $\overrightarrow{\nabla}$ represents the operator of partial derivatives (Nabla). In Cartesian coordinates, it is defined by:

$$\overrightarrow{\nabla}=\overrightarrow{i}\frac{\partial}{\partial x}+\overrightarrow{j}\frac{\partial}{\partial y}+\overrightarrow{k}\frac{\partial}{\partial z} \tag{2.6}$$

Hence, Schrödinger's equation becomes:

$$i\hbar\frac{\partial\psi\left(\overrightarrow{r},t\right)}{\partial t}=\left(-\frac{\hbar^2}{2m_0}\Delta+\frac{1}{r^2}(b-V_0)+\frac{1}{r}(2V_0-\alpha-b\delta)+(b-2V_0)\delta^2\right)\psi\left(\overrightarrow{r},t\right) \tag{2.7}$$

Since the inversely quadratic Yukawa and inversely quadratic Hellmann potential does not depend on time, solutions can be written separately as a part that is only position-dependent and an only time-dependent part:

$$\Psi\left(\overrightarrow{r},t\right)=\exp(-iE/\hbar t)\Psi\left(\overrightarrow{r}\right) \tag{2.8}$$

And by substituting into Schrödinger equation, we find:

$$\left(\frac{-\hbar^2}{2m}\Delta+\frac{1}{r^2}(b-V_0)+\frac{1}{r}(2V_0-\alpha-b\delta)+(b-2V_0)\delta^2\right)\Psi\left(\overrightarrow{r}\right)=E\Psi\left(\overrightarrow{r}\right) \tag{2.9}$$

Using the spherical coordinate system $\overrightarrow{r}\,(r,\theta,\varphi)$, the complex wave function $\Psi\left(\overrightarrow{r}\right)$ can be written as:

$$\Psi\left(r,\theta,\varphi\right)=\chi_{nl}\left(r\right)Y_{lm}\left(\theta,\varphi\right) \tag{2.10}$$

where $\chi_{nl}(r)$ is the radial part of the wave function that depends only on radius r, $Y_{l,m}(\theta,\varphi)$ represented the angular part depends on the angles (θ,φ) and n is the principal quantum number, l the orbital quantum number and m the magnetic quantum number $(-l\leq m\leq +l)$. The Schrödinger equation in the spherical coordinate can be expressed as:

$$\left(\frac{-\hbar}{2m_0}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{\hat{L}^2}{r^2}-\frac{1}{r^2}(b-V_0)-\frac{1}{r}(2V_0-\alpha-b\delta)-(b-2V_0)\delta^2\right)\right)\chi_{nl}(r)=E\chi_{nl}\left(r\right) \tag{23.11}$$

In quantum mechanics, the classical momentum obtains the forms $\overrightarrow{L}$ is the orbital angular momentum. The total moment $\overrightarrow{J}$ is given by:

$$\begin{cases}\overrightarrow{J}=\overrightarrow{L}+\overrightarrow{S}\\ \overrightarrow{L}=\overrightarrow{r}\wedge\overrightarrow{p}\end{cases} \tag{2.12}$$

here $\overrightarrow{S}$ is the spin. The components L_x , L_y and L_z of $\overrightarrow{L}$ which are expressed in Cartesian coordinates (x,y,z) as:

$$\begin{cases}L_x=\frac{\hbar}{\imath}\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\\ L_y=\frac{\hbar}{\imath}\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)\\ L_z=\frac{\hbar}{\imath}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\end{cases} \tag{2.13}$$

In the spherical coordinate system $\overrightarrow{r}\,(r,\theta,\varphi)$, he components L_x , L_y and L_z of $\overrightarrow{L}$ are expressed as:

$$\begin{cases}L_x=\frac{\hbar}{\imath}\left(-\sin\varphi\frac{\partial}{\partial\theta}-\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\\ L_y=\frac{\hbar}{\imath}\left(\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\\ L_z=\frac{\hbar}{\imath}\frac{\partial}{\partial\varphi}\end{cases} \tag{2.14}$$

Note that the operators H, L^2 and L_z commute with each other and they have formed a common set of eigenfunctions $\psi\left(r,\theta,\varphi\right)$; however, the three components of the angular momentum (L_x,L_y,L_z) do not commute with each other:

$$\begin{cases}\left[H,\mathbf{L}^2\right]=\left[H,L_z\right]=0\\ \left[L_i,L_j\right]=\imath\hbar\xi_{ijk}L_k\end{cases} \tag{2.15}$$

here $\mathbf{L}^2$ is the square of the angular momentum :

$$L^2 = -\hbar^2 \left(\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) \right) + \hbar^2 \frac{\partial^2}{\partial \varphi^2} \tag{2.16}$$

the eigenvalues of $L^2$and L_z are determined from:

$$\begin{cases} \hat{L}^2 \psi(r,\theta,\varphi) = \hbar^2 l(l+1)\psi(r,\theta,\varphi) \\ \hat{L}_z \psi(r,\theta,\varphi) = m\hbar\psi(r,\theta,\varphi) \end{cases} \tag{2.17}$$

We introducing the wave function: we have:

$$\chi_{n,l}(r) = \frac{R_{n,l}(r)}{r} \tag{2.18}$$

Thus, the new radial part $u_{n,l}(r)$, will be satisfying the following equation:

$$\frac{d^2 u_{n,l}(r)}{dr^2} + \frac{2m_0}{\hbar^2} \left(E_{nl} - \frac{1}{r^2}(b - V_0) - \frac{1}{r}(2V_0 - \alpha - b\delta) - (b - 2V_0)\delta^2 - \frac{l(l+1)}{r^2} \right) R_{n,l}(r) = 0 \tag{2.19}$$

The energy eigenvalues E_{nl} and corresponding eigenfunctions in closed forms were obtained using the parametric Nikiforov-Uvarov approach by the authors of Refs. [55, 56]. They further show that these results are consistent with those obtained previously in other studies using different approaches**:**

$$E_{nl} = -(2V_0 - b)\delta^2 - \frac{\mu(2V_0\delta - a - b\delta)^2/2\hbar^2}{\left(n + (1/2) + \sqrt{(2\mu(b - V_0)/\hbar^2) + (l + (1/2))^2}\right)^2} \tag{2.20}$$

The corresponding radial part $\chi(z)$ of the complex wave function of the inversely quadratic Yukawa and inversely quadratic Hellmann potential [55, 56]:

$$\chi(z) = B_n z^{\left(-1+\sqrt{1+4\gamma}/2\right)} e^{2\sqrt{\alpha z}} \frac{d^n}{dz^n} \left({}_n z^{\left(n+\sqrt{1+4\gamma}\right)} e^{-2\sqrt{\alpha z}} \right) \tag{2.21}$$

Here N_n is the normalization constant and γ is as follows:

$$\gamma = 2\mu(b - V_0) + l(l+1) \tag{2.22}$$

Thus, the complex wave function $\Psi\left(\overrightarrow{r}\right)$ can be written as:

$$\Psi(r,\theta,\varphi,t) = N_{nl} \frac{\chi_{nl}(r)}{r}(r) Y_{lm}(\theta,\varphi) \exp(-iE_{nl}t) \tag{2.23}$$

Allow us to obtained, the complex wave function $\Psi\left(\overrightarrow{r}\right)$ can be written for fundamental state as:

$$\Psi\left(z,\theta,\varphi,t\right)=N_{nl}\frac{r^{z\left(-1+\sqrt{1+4\gamma}/2\right)}e^{2\sqrt{\alpha}z}L_{n}^{\sqrt{1+4\gamma}}\left(2\sqrt{\alpha}r^{2}\right)}{r}Y_{lm}\left(\theta,\varphi\right)\exp(-iE_{nl}t) \tag{2.24}$$

which take the complete expression:

$$\Psi\left(r,\theta,\varphi,t\right)=N_{nl}\frac{r^{2\left(-1+\sqrt{1+4\gamma}/2\right)}\exp\left(2\sqrt{\alpha}r^{2}\right)L_{n}^{\sqrt{1+4\gamma}}\left(2\sqrt{\alpha}r^{2}\right)}{r}Y_{lm}\left(\theta,\varphi\right)\exp(-iE_{nl}t) \tag{2.25}$$

The bound state solutions of the Schrödinger equation have been successfully revised for the combined inversely quadratic Yukawa and inversely quadratic Hellmann potential.

Chapter 3

The effect of space noncommutativity on the energy spectrum produced by the inversely quadratic Yukawa and inversely quadratic Hellmann potential

3.1 Introduction

The purpose of this chapter is to study the modified Schrödinger equation of the inversely quadratic Yukawa and inversely quadratic Hellmann potential in noncommutative three-dimensional space-space (NCSS). Accordingly, we use Bopp's shift method instead of solving the modified Schrödinger equation directly and the perturbation theorem to find the corresponding energy correction instead to apply the star product in the context of NCSS symmetry.

3.2 The Schrödinger equation on a Noncommutative NCSS symmetry:

We simply replace the wave function products (or fields) with the star product or the Moyal product. The Schrödinger equation for a NCSS symmetry has the form:

$$\begin{cases} H(\hat{p}_i,\hat{x}_i)\hat{\Psi}(\overrightarrow{\hat{r}}) = E_{nc}\Psi\left(\overrightarrow{\hat{r}}\right) \\ \Rightarrow \\ \left[\frac{\overrightarrow{\hat{P}^2}}{2m}+V(\hat{r})\right]*\hat{\Psi}\left(\overrightarrow{\hat{r}},\hat{t}\right) = \imath\hbar\frac{\partial}{\partial t}\Psi\left(\overrightarrow{\hat{r}},\hat{t}\right) \end{cases} \tag{3.1}$$

In the symmetries of non-commutative space and according to the method of Bopp's Shift, which we have seen in the first chapter, the above equation can be simplified into the following form:

$$H_{nc-iqyh}\left(\hat{p}_i = p_i, \hat{x}_i = x_i + \frac{\theta_{ij}}{2}p^j\right)\Psi(\overrightarrow{\hat{r}}) = E_{nc}\Psi(\overrightarrow{\hat{r}}) \tag{3.2}$$

where

$$H\left(\hat{p}_i,\hat{x}_i\right) = H\left(\hat{p}_i = p_i, \quad \hat{x}_i = x_i + \frac{\theta^{ij}}{2}p_i\right) = \frac{\overrightarrow{P}^2}{2m_0} + V\left(\hat{r}\right)$$

and

$$\begin{cases} V\left(\hat{r}\right) = \frac{1}{\hat{r}^2}(b-V_0) + \frac{1}{\hat{r}}(2V_0-\alpha-b\delta) + (b-2V_0)\delta^2 \\ \frac{\overrightarrow{P}^2}{2m_0} = \frac{P^2}{2m_0} \end{cases} \tag{3.4}$$

By using Eq.(1.37), we can obtain both $\left(-\frac{(2V_0-\alpha-b\delta)}{\hat{r}}\right)$ and $\left(-\frac{(b-V_0)}{\hat{r}^2}\right)$ as the sum of corresponding values $\left(-\frac{(2V_0-\alpha-b\delta)}{r}\right)$ and $\left(-\frac{(b-V_0)}{r^2}\right)$ in the symmetries of nonrelativistic quantum mechanics plus the induced terms $-(2V_0-\alpha-b\delta)\frac{\overrightarrow{L}\vec{\Theta}}{2r^3}$ and $-(b-V_0)\frac{\overrightarrow{L}\vec{\Theta}}{r^4}$ with the effect of deformed proprieties of space-space, as follows:

$$\begin{cases} \frac{1}{\hat{r}} = \frac{1}{r} + \frac{\overrightarrow{L}\vec{\Theta}}{2r^3} \\ \frac{1}{\hat{r}^2} = \frac{1}{r^2}\frac{\overrightarrow{L}\vec{\Theta}}{r^4} \end{cases} \tag{3.5(1)}$$

Which gives

$$\begin{cases} -\frac{(2V_0-\alpha-b\delta)}{\hat{r}} = -\frac{(2V_0-\alpha-b\delta)}{r} - (2V_0-\alpha-b\delta)\frac{\overrightarrow{L}\vec{\Theta}}{2r^3} \\ -\frac{(b-V_0)}{\hat{r}^2} = -\frac{(b-V_0)}{r^2} - (b-V_0)\frac{\overrightarrow{L}\vec{\Theta}}{r^4} \end{cases} \tag{3.5(2)}$$

Allow us to get the inversely quadratic Yukawa and inversely quadratic Hellmann potential in the noncommutative space-space symmetry as follows:

$$V(\hat{r}) = \frac{1}{r^2}(b-V_0) + \frac{1}{r}(2V_0-\alpha-b\delta) + (b-2V_0)\delta^2 + \left(A_1\frac{1}{r^3} + A_2\frac{1}{r^4}\right)\overrightarrow{L}\vec{\Theta} \tag{3.6}$$

The parameters A_1 and A_2 are given by:

$$\left\{ \begin{array}{c} A_1 = -\frac{(2V_0-\alpha-b\delta)}{2} \\ A_2 = -(b-V_0) \end{array} \right. \tag{3.7}$$

And the coupling $\overrightarrow{L}\vec{\Theta}$ is equal to $L_x\theta_{12} + L_y\theta_{23} + L_z\theta_{13}$ (see Eq. (1.38)). Thus, the modified potential $V(\hat{r})$ for the modified the inversely quadratic Yukawa and inversely quadratic Hellmann potential:

$$V(\hat{r}) = \frac{1}{r^2}(b - V_0) + \frac{1}{r}(2V_0 - \alpha - b\delta) + (b - 2V_0)\delta^2 + V_{per}^{iqyh}(r, \Theta) \tag{3.8}$$

with the supplementary additive part $V_{per}^{iqyh}(r, \Theta)$:

$$V_{per}^{iqyh}(r, \Theta) = \left(A_1 \frac{1}{r^3} + A_2 \frac{1}{r^4} \right) \overrightarrow{L}\vec{\Theta} \tag{3.9}$$

The global Hamiltonian operator $H_{nc}^{iqyh}\left(\hat{p}_i = p_i, \hat{x}_i = x_i + \frac{\theta^{ij}}{2}p_i\right)$ in the noncommutative three-dimensional space-space symmetries can be written in the following form:

$$H_{nc}^{iqyh}\left(\hat{p}_i = p_i, \hat{x}_i = x_i + \frac{\theta^{ij}}{2}p_i\right) =$$

$$\frac{\overrightarrow{p}^2}{2m} + \frac{1}{r^2}(b - V_0) + \frac{1}{r}(2V_0 - \alpha - b\delta) + (b - 2V_0)\delta^2 + \left(A_1 \frac{1}{r^3} + A_2 \frac{1}{r^4} \right) \overrightarrow{L}\vec{\Theta} \tag{3.10}$$

1-The first two terms in the Hamiltonian operator $H_{nc}^{iqyh}\left(\hat{p}_i = p_i, \hat{x}_i = x_i - \frac{\theta^{ij}}{2}p_i\right)$, which corresponds to the inversely quadratic Yukawa and inversely quadratic Hellmann potential in Eq. (2.1) and the Kinetic term or dynamic $\frac{\overrightarrow{p}^2}{2m}$ in ordinary commutative space which formed the usual Hamiltonian operator:

$$H(\hat{p}_i = p_i, \hat{x}_i = x_i) = \frac{\overrightarrow{p}^2}{2m} + \frac{1}{r^2}(b - V_0) + \frac{1}{r}(2V_0 - \alpha - b\delta) + (b - 2V_0)\delta^2 \tag{3.11}$$

2- The effective Hamiltonian operator H_{iqyh}^{eff}, which corresponds to the inversely quadratic Yukawa and inversely quadratic Hellmann potential is the sum of Hamiltonian operator in Eq. (3.11) in ordinary commutative plus the centrifugal term $-\frac{l(l+1)}{r^2}$:

$$H_{iqyh}^{eff}(\hat{p}_i = p_i, \hat{x}_i = x_i) = \frac{\overrightarrow{p}^2}{2m} + \frac{1}{r^2}(b - V_0) + \frac{1}{r}(2V_0 - \alpha - b\delta) + (b - 2V_0)\delta^2 - \frac{l(l+1)}{r^2} \tag{3.12}$$

and the corresponding effective Hamiltonian operator $H_{iqyh}^{nc-eff}\left(\hat{p}_i = p_i, \hat{x}_i = x_i - \frac{\theta^{ij}}{2}p_i\right)$ in the noncommutative three-dimensional space-space symmetries

$$H_{iqyh}^{nc-eff}\left(\hat{p}_i=p_i,\hat{x}_i=x_i-\frac{\theta^{ij}}{2}p_i\right)=H_{nc}^{iqyh}\left(\hat{p}_i=p_i,\hat{x}_i=x_i-\frac{\theta^{ij}}{2}p_i\right)-l\left(l+1\right)\frac{\vec{L}\vec{\Theta}}{r^4} \tag{3.13}$$

3-The perturbed effective new Hamiltonian operator H_{per}^{nc-eff} or the additive created term H_{per}^{iqyh} which is represent the contributions of the noncommutative space-phase:

$$H_{per}^{nc-eff}\left(b,V_0,\alpha,\delta,\Theta\right)=\left(\frac{A_1}{r^3}+\frac{A_3}{r^4}\right)\vec{L}\vec{\Theta} \tag{3.14}$$

with

$$A_3=A_2-l\left(l+1\right)$$

According to the mathematical form of the couplings $\overrightarrow{L}\overrightarrow{\Theta}$ that observed in Eq.(3.14), it is physically possible to replace $\overrightarrow{L}\overrightarrow{\Theta}$ by $\alpha\Theta\overrightarrow{S}\overrightarrow{L}$ as:

$$\overrightarrow{L}\overrightarrow{\Theta}=\Theta\overrightarrow{L}\frac{\overrightarrow{\Theta}}{\Theta}=\Theta\overrightarrow{L}\frac{\overrightarrow{S}}{S}\rightarrow\alpha\Theta\overrightarrow{L}\overrightarrow{S} \tag{3.15}$$

With $\overrightarrow{S}$ denote to the spin of the particle which interacted with inversely quadratic Yukawa and inversely quadratic Hellmann potential and β is a new constant of proportionality. This enables rewriting Eq.(3.14) as follows:

$$H_{per}^{iqyh}=\beta\Theta\left(A_1\frac{1}{r^3}+A_3\frac{1}{r^4}\right)\overrightarrow{L}\overrightarrow{S} \tag{3.16}$$

The parameter Θ is given by:

$$\Theta=\left(\Theta_{12}^2+\Theta_{23}^2+\Theta_{13}^2\right)^{\frac{1}{2}} \tag{3.17}$$

In ordinary quantum mechanics, we have the sets of operators $\hat{\mathbf{A}}$**,** $\hat{\mathbf{B}}$**,** $\hat{\mathbf{C}}$**......** which form a complete set of complete observable are commute (ECOC). We apply this rule to the sets of operators $\left(\overrightarrow{\mathbf{J}}^2,\overrightarrow{\mathbf{S}}^2,\overrightarrow{\mathbf{L}}^2\text{ and }J_z\right)$, i.e.:

$$\begin{cases}\left[\overrightarrow{\mathbf{J}}^2,\overrightarrow{\mathbf{L}}^2\right]=0\\ \left[\overrightarrow{\mathbf{J}}^2,\overrightarrow{\mathbf{S}}^2\right]=0\\ \left[\overrightarrow{\mathbf{J}}^2,J_z\right]=0\end{cases} \tag{3.18}$$

And the corresponding eigenvalues are $j\left(j+1\right)$ and $l(l+1)$, $s(s+1)$ and $m\left(-l\leq m\leq +l\right)$ in the system $\left(c=\hbar=1\right)$, so:

$$\begin{cases}\overrightarrow{\mathbf{J}}^2\Psi_{n,l,m_l}\left(r,\theta,\varphi\right)=j(j+1)\Psi_{n,l,m_l}\left(r,\theta,\varphi\right)\\ \overrightarrow{\mathbf{L}}^2\Psi_{n,l,m_l}\left(r,\theta,\varphi\right)=l(l+1)\Psi_{n,l,m_l}\left(r,\theta,\varphi\right)\\ \overrightarrow{\mathbf{S}}^2\Psi_{n,l,m_l}\left(r,\theta,\varphi\right)=s(s+1)\Psi_{n,l,m_l}\left(r,\theta,\varphi\right)\\ J_z\Psi_{n,l,m_l}\left(r,\theta,\varphi\right)=m\Psi_{n,l,m_l}\left(r,\theta,\varphi\right)\end{cases} \tag{3.19}$$

With $\overrightarrow{J}$ being the geometric sum of the moments $\overrightarrow{L}$ and $\overrightarrow{S}$, this allows us to find the spin-orbit coupling $\overrightarrow{L}\overrightarrow{S}$ as follows:

$$\overrightarrow{L}\overrightarrow{S} = \frac{1}{2}\left[\overrightarrow{\mathbf{J}}^2 - \overrightarrow{\mathbf{S}}^2 - \overrightarrow{\mathbf{L}}^2\right] \tag{3.20}$$

An immediate result is:

$$\overrightarrow{L}\overrightarrow{S}\Psi = \frac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]\Psi \tag{3.21}$$

With $j \in [|l-s|, |l+s|]$, this permuted us to obtain j to be equal the values $(|l-s|, |l-s|+1, ..., |l+s|)$. For the two extreme values of the total angular momentum, we can write for $s = \frac{1}{2}$:

$$\overrightarrow{L}\overrightarrow{S}\Psi = \begin{cases} \frac{1}{2}\left\{(l+s)(l+s+1) - l(l+1) - 3/4\right\}\Psi \equiv k_+\Psi & \text{if} \quad j = |l+1/2| \\ \frac{1}{2}\left\{(l-s)(l-s+l) - l(l+1) - 3/4\right\}\Psi \equiv k_-\Psi & \text{if} \quad j = |l-1/2| \end{cases} \tag{3.22}$$

The inversely quadratic Yukawa and inversely quadratic Hellmann potential Hamiltonian $H(\hat{p}_i = p_i, \hat{x}_i = x_i)$ is extended by including new additive perturbative Hamiltonian $H_{per}^{iqyh}(r, \Theta)$ expressed to the radial terms:

$$\left\{\frac{1}{r^3}\text{and } \frac{1}{r^4}\right\}$$

to become the modified inversely quadratic Yukawa and inversely quadratic Hellmann potential Hamiltonian $H_{nc}^{iqyh}\left(\hat{p}_i = p_i, \hat{x}_i = x_i - \frac{\theta^{ij}}{2}p_i\right)$ in noncommutative three-dimensional space-space symmetries. The generated new Hamiltonian $H_{per}^{iqyh}(r, b, V_0, \alpha, \delta, \Theta)$ is also proportional to the infinitesimal parameter Θ. This allows us to consider the new additive part of the potential $H_{per}^{iqyh}(r, b, V_0, \alpha, \delta, \Theta)$ as perturbation potential compared with the main Hamiltonian $H(\hat{p}_i = p_i, \hat{x}_i = x_i)$. That is all physical justifications for applying the time-independent perturbation theory become satisfied to calculate the expectation values of previous radial terms. This allows us to give a complete prescription for determining the energy level of the generalized $(n, l, s)^{th}$ excited states. The exact spectrum produced by the spin-orbit effect for the generalized inverse quadratic Yukawa potential in the three dimensional noncommutative space-space $\Delta E_{nc}^{iqyh}(n, b, V_0, \alpha, \delta, \Theta)$ is the sum of the energy corresponding to ordinary space $E_{nl}(n, b, V_0, \alpha, \delta)$ and the corrections $E_{per}^{iqyh}(n, b, V_0, \alpha, \delta, \Theta)$:

$$E_{nc\_nl}^{iqyh}(n, b, V_0, \alpha, \delta, \Theta) = E_{nl}(n, b, V_0, \alpha, \delta) + \Delta E_{nc}^{iqyh}(b, V_0, \alpha, \delta, \Theta) \tag{3.23}$$

The perturbation theorem allows to obtain the first-order corrections as follows:

$$\Delta E_{nc}^{iqyh}(n, b, V_0, \alpha, \delta, \Theta) = \langle\Psi^p(\overrightarrow{r})| H_{per}^{iqyh}(r, b, V_0, \alpha, \delta, \Theta) |\Psi^p(\overrightarrow{r})\rangle \tag{3.24}$$

We can write the equation (3.26) in the form:

$$\Delta E_{nc}^{iqyh}(n, b, V_0, \alpha, \delta, \Theta) = \int \Psi^{(p)}(\overrightarrow{r}) H_{per}^{iqyh}(r, b, V_0, \alpha, \delta, \Theta)(r, \Theta, \theta)\Psi^{(p)}(\overrightarrow{r})\, d\tau \tag{3.25}$$

where $d\tau$ represent the volume element in spherical coordinates (r, θ, φ), which is given by:

$$d\tau = r^2 dr d\Omega \tag{3.26}$$

With the solid angle

$$d\Omega = \sin\theta d\theta d\varphi$$

and the nonperturbative complex wave function , the wave function is defined by :

$$\Psi^{(p)}\left(\overrightarrow{r}\right) = R_{n,l}\left(r\right) Y_l^m\left(\theta, \phi\right) \tag{3.27}$$

So, we can write the equation (3.27) in the form:

$$\Delta E_{nc}^{iqyh}\left(n, b, V_0, \alpha, \delta, \Theta\right) =$$
$$\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle \int_0^\infty R_{n,l}^*\left(r\right) H_{per}^{iqyh}\left(s, \Theta, \bar{\theta}\right) R_{n,l}\left(r\right) r^2 dr \int_0^\pi \int_0^{2\pi} Y_l^{*m_l}\left(\theta, \phi\right) Y_l^m\left(\theta, \phi\right) d\Omega \tag{3.28}$$

The normalized wave function $\Psi\left(\overrightarrow{r}\right)$ allows us to write :

$$\int_0^\pi \int_0^{2\pi} Y_l^{*m_l}\left(\theta, \phi\right) Y_l^m\left(\theta, \phi\right) d\Omega = 1 \tag{3.29}$$

This reduces the corrections found by (3.28) to the form:

$$\Delta E_{nc}^{iqyh}\left(n, b, V_0, \alpha, \delta, \Theta\right) = \left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle \int_0^\infty R_{n,l}^*\left(r\right) H_{per}^{iqyh}\left(s, \Theta, \bar{\theta}\right) R_{n,l}\left(r\right) r^2 dr \tag{3.30}$$

We substituted the spin-orbit coupling term $H_{so}^{iqyh}\left(r, b, V_0, \alpha, \delta, \Theta\right)$, and we find:

$$\Delta E_{nc}^{iqyh}\left(b, V_0, \alpha, \delta, \Theta\right) = \beta \left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle$$
$$\left(A_1 \int_0^\infty R_{n,l}^*\left(r\right) \tfrac{1}{r^3} R_{n,l}\left(r\right) r^2 dr + A_3 \int_0^\infty R_{n,l}^*\left(r\right) \tfrac{1}{r^4} R_{n,l}\left(r\right) r^2 dr\right) \tag{3.31}$$

If we replace the radial part $R_{n,l}\left(r\right)$ which is expressed as:

$$R_{n,l}\left(r\right) = N_{nl} \frac{r^{2\left(-2+\sqrt{1+4\gamma}\right)} \exp\left(2\sqrt{\alpha} r^2\right) L_n^{\sqrt{1+4\gamma}}\left(2\sqrt{\alpha} r^2\right)}{r} \tag{3.32}$$

We obtain the corrections in Eq. (3.31) as follows :

$$\Delta E_{nc}^{iqyh}(n,b,V_0,\alpha,\delta,\Theta)=\beta N_{nl}^2\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle$$
$$\left(\begin{array}{c} A_1\int\limits_0^\infty r^{2\left(-2+\sqrt{1+4\gamma}\right)-3}\exp\left(2\sqrt{\alpha}r^2\right)\left[L_n^{\sqrt{1+4\gamma}}\left(2\sqrt{\alpha}r^2\right)\right]^2 dr \\ +A_3\int\limits_0^\infty r^{2\left(-2+\sqrt{1+4\gamma}\right)-4}\exp\left(2\sqrt{\alpha}r^2\right)\left[L_n^{\sqrt{1+4\gamma}}\left(2\sqrt{\alpha}r^2\right)\right]^2 dr \end{array}\right) \tag{3.33}$$

We have replace $2\sqrt{\alpha}$ and r^2 by ω and t , respectively. This allows us to rewritten Eq. (3.33) in the new form:

$$\Delta E_{nc}^{iqyh}(n,b,V_0,\alpha,\delta,\Theta)=\frac{\mu N_{nl}^2}{2}\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle$$
$$\left(\begin{array}{c} A_1\int\limits_0^\infty t^{\left(\sqrt{1+4\gamma}-9/2\right)-1}\exp(\omega t)\left[L_n^{\sqrt{1+4\gamma}}(\omega t)\right]^2 dt \\ +A_3\int\limits_0^\infty t^{\left(\sqrt{1+4\gamma}-11/2\right)-1}\exp(\omega t)\left[L_n^{\sqrt{1+4\gamma}}(\omega t)\right]^2 dt \end{array}\right) \tag{3.34}$$

We have replace dr with corresponding values$\frac{dt}{2t^{1/2}}$. The integrals referred to in Eq. (3.34) can be accomplished by applying the special integral formula[68]:

$$\int\limits_0^{+\infty} t^{\varepsilon-1}\exp(-\omega t)L_m^{\lambda}(\omega t)L_n^{\beta}(\omega t)\,dt=\omega^{-\varepsilon}\frac{\Gamma(\varepsilon)\Gamma(n-\varepsilon+\beta+1)\Gamma(m+\lambda+1)}{m!n!\Gamma(1-\varepsilon+\beta)\Gamma(\lambda+1)}$$
$$\times\ {}_3F_2(-m,\varepsilon,\varepsilon-\beta;-n+\varepsilon-\beta,\lambda+1;1) \tag{3.35}$$

here $\mathrm{Re}(\varepsilon)\rangle 0\ \Lambda\ \mathrm{Re}(\omega)\rangle 0\ \Lambda\ (m,n)\in N^2$ while $\Gamma(\xi)$ denotes the usual Gamma function while ${}_3F_2(-m,\varepsilon,\varepsilon-\beta;-n+\varepsilon-\beta,\lambda+1;1)$ obtained from generalized confluent hypergeometric function $pFq(a_1,...,a_p;b_1,...,bq;x)$:

$${}_3F_2(a,b,c;d,e;1)=\frac{\left(\frac{1}{2}a!\right)(a-b)!\,(a-c)!\left(\frac{1}{2}a-b-c\right)!}{a!\left(\frac{1}{2}a-b\right)!\left(\frac{1}{2}a-c\right)!\,(a-b-c)!} \tag{3.36}$$

After detailed calculation, we arrive at the following results:

$$\Delta E_{nc}^{iqyh}(n,b,V_0,\alpha,\delta,\Theta)=\frac{\beta N_{nl}^2}{2}\left\langle \overrightarrow{L}\overrightarrow{S}\right\rangle$$
$$\left(\begin{array}{c} A_1\omega^{-\left(\sqrt{1+4\gamma}-9/2\right)}\frac{\Gamma\left(\sqrt{1+4\gamma}-9/2\right)\Gamma(n+11/2)\Gamma\left(n+\sqrt{1+4\gamma}+1\right)}{(n!)^2\Gamma(1+9/2)\Gamma\left(\sqrt{1+4\gamma}+1\right)} \\ +A_3\omega^{-\varepsilon}\frac{\Gamma\left(\sqrt{1+4\gamma}-11/2\right)\Gamma(n+13/2)\Gamma\left(n+\sqrt{1+4\gamma}+1\right)}{(n!)^2\Gamma(13/2)\Gamma\left(\sqrt{1+4\gamma}+1\right)} \end{array}\right) \tag{3.37}$$

Thus, the energy correction that produced with the modified inversely quadratic Yukawa and inversely quadratic Hellmann potential for the ground state $n = 0$ reduced to the following simple form:

$$\Delta E_{nc}^{iqyh}(n = 0, b, V_0, \alpha, \delta, \Theta) = \frac{\beta N_{nl}^2}{2} \left\langle \overrightarrow{L}\overrightarrow{S} \right\rangle \left(\begin{array}{c} A_1 \omega^{-\left(\sqrt{1+4\gamma}-9/2\right)} \frac{\Gamma\left(\sqrt{1+4\gamma}-9/2\right)\Gamma(11/2)\Gamma\left(\sqrt{1+4\gamma}+1\right)}{\Gamma(1+9/2)\Gamma\left(\sqrt{1+4\gamma}+1\right)} \\ +A_3 \omega^{-\varepsilon} \frac{\Gamma\left(\sqrt{1+4\gamma}-11/2\right)\Gamma(13/2)\Gamma\left(\sqrt{1+4\gamma}+1\right)}{\Gamma(13/2)\Gamma\left(\sqrt{1+4\gamma}+1\right)} \end{array} \right) \tag{3.38}$$

The global energy $E_{nc\_0l}^{iqyh}(n = 0, b, V_0, \alpha, \delta, \Theta)$ for the ground state $n = 0$ is the energy spectrums:

$$E_{nc\_0l}^{iqyh} = E_{0l}^{iqyh}(n = 0, b, V_0, \alpha, \delta) + \Delta E_{nc}^{iqyh}(n = 0, b, V_0, \alpha, \delta, \Theta) \tag{3.39}$$

Where E_{0l}^{py} is determined from Eq.(2.20) which we have seen in the second chapter:

$$E_{0l}(n = 0, b, V_0, \alpha, \delta) = - (2V_0 - b)\delta^2 - \frac{\mu(2V_0\delta - a - b\delta)^2 / 2\hbar^2}{\left(1/2 + \sqrt{(2\mu(b - V_0)/\hbar^2) + (l + (1/2))^2}\right)^2} \tag{3.40}$$

here $\left\langle \overrightarrow{L}\overrightarrow{S} \right\rangle$ is determined from:

$$\left\langle \overrightarrow{L}\overrightarrow{S} \right\rangle = \begin{cases} \frac{1}{2}\{(l+s)(l+s+1) - l(l+1) - 3/4\} \\ \equiv k_+ \quad \text{if } j = |l + 1/2| \\ \frac{1}{2}\{(l-s)(l-s+l) - l(l+1) - 3/4\} \\ \equiv k_- \quad \text{if } \; j = |l - 1/2| \end{cases} \tag{3.41}$$

It is clear that the following physical limit procedure:

$$\begin{cases} \underset{\Theta\to 0}{Lim} E_{nc\_0l}^{iqyh}(n = 0, b, V_0, \alpha, \delta, \Theta) = E_{0l}(n = 0, b, V_0, \alpha, \delta) \\ \underset{\Theta,\to 0}{Lim} \Delta E_{nc}^{iqyh}(n, b, V_0, \alpha, \delta, \Theta) = 0 \end{cases} \tag{3.42}$$

Gives us all the results of physical treatments which we have seen in the standard references [64, 65].

Conclusion

Through this master's memory in physics, theoretical specialty: Promotion 2024-2025,

The nonrelativistic study of the energy spectrum producing from a central potential in the extended quantum mechanics symmetries: the case of inversely quadratic Yukawa and inversely quadratic Hellmann potential as a model,

This memory aims to study physical systems within the framework of the modified Schrödinger equation with the modified inversely quadratic Yukawa and inversely quadratic Hellmann potential, in three-dimensional non-commutative space-space.

In the first chapter, we have represented the mathematical and physical formalisms of the noncommutative three-dimensional (phase-phase/phase-phase and space-space).

In the second chapter, we reviewed the Shrodinger equation under the inversely quadratic Yukawa and inversely quadratic Hellmann based on many works.

In the third chapter, we studied the effect of the noncommutativity of the three-dimensional space-space, by applying the Bopp shift method and standard perturbation theory at the first order of parameter Θ, the modifications on the energy corresponding to the any excited states and in particularly the ground state are obtained. We can conclude that the application of the noncommutativity in this work on the modified inversely quadratic Yukawa and inversely quadratic Hellmann potential, includes the spin-orbit coupling effect automatically. This is in contrast to what we observe in the framework of quantum mechanics known in the literature, where the spin-orbit interaction appears by external addition and not through spontaneous birth as a result of space deformation.

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Abstract

In our work on this master memory, in theoretical physics (2024/2025): A study of a nonrelativistic energy spectrum produced with an isotropic potential in the framework of extended quantum mechanics symmetries: the case of inversely quadratic Yukawa and inversely quadratic Hellmann potential. We have studied the Schrödinger equation with the inversely quadratic Yukawa and inversely quadratic Hellmann potential in noncommutative three-dimensional spaces, by applying the Boop's Shift method to the first order of the parameters Θ, in addition to the standard perturbation theory, to obtain the spectrum of energy of the system, which is changing radically, and replaced by degenerate new states depending on the discrete atomic quantum numbers (j, n, l, s). This result was produced by the spin-orbit interaction.

Keywords: Schrödinger equation, inversely quadratic Yukawa and inversely quadratic Hellmann potential, noncommutative quantum mechanics, star product, Boop's shift method.