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Bell Matrix-Collocation Method to Solve Linear Volterra Integro-Differential Equations

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List of Notations

Notation	Name
$\boldsymbol{\lambda}$	Scalar parameter
$[\mathbf{a}, \mathbf{b}]$	Interval endpoints
$\mathbf{BL}_n(x)$	Bell polynomials of degree n
$\mathbf{S}(n, k)$	Stirling number of the second kind
$\mathbf{S}$	Bell transformation matrix
$\mathbf{X}(x)$	Taylor basis vector
$\boldsymbol{\Omega}$	Computational domain
$\mathbf{E}$	Normed vector space
$\mathbf{F}$	Codomain space
$\mathbf{H}$	Hilbert space
$\\|\mathbf{x}\\|$	Vector norm
$\langle \mathbf{x}, \mathbf{y} \rangle$	Inner product
$\\|\mathbf{A}\\|$	Operator norm
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbf{L}^p$	Lebesgue space
$\mathbf{C}([a, b])$	Space of continuous functions
$\boldsymbol{\phi}, \boldsymbol{\psi}$	Auxiliary functions
$\operatorname{rank}(\mathbf{M})$	Rank of matrix $\mathbf{M}$
$\sum$	Summation sign
$\int$	Integral sign
$k(x, t)$	kernel of the integral equation

Contents

Introduction

In recent years, integral differential equations have emerged as essential mathematical tools for modelling a wide range of phenomena across disciplines such as physics, biology and engineering.
The Volterra integral-differential equation typically arises from the transformation of initial value problems in differential equations. Since it is often difficult or impossible to find exact analytical solutions to these equations, researchers have focused on developing numerical methods to obtain approximate solutions. Many authors have contributed to this field. For example, some have used Chebyshev polynomials, whilst others have used the Lucas summation method, Euler polynomials combined with the method of least squares, or Legendre polynomials.
The linear Volterra integral-differential equation of order $m \geq 1$ as follows:

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = f(x) + \lambda \int_a^x K_v(x,t)y(t)\,dt$$

This equation is subject to the following initial conditions:

$$y^{(k)}(a) = \alpha_k, \quad \text{for } 0 \leq k \leq m-1$$

where the functions $K(x,t)$, $f(x)$ and $P_k(x)$ are known, $y(x)$ is the unknown function to be determined, and a and λ are constants.
In this thesis, we present a new numerical technique based on Bell polynomials, by utilizing the Bell matrix aggregation method, this study provides a robust and effective framework for approximating. This work consists of three chapters covering various aspects of the numerical solution of Volterra differential-integral equations. It is organised as follows:
Chapter 1: presents the basic concepts as well as the definitions and fundamental theories that will be applied in the following chapters (function spaces, Banach and Hilbert spaces, linear operators).
Chapter 2: discusses specific types of integral equations (Fredholm, Volterra, mixed) as well as Volterra, Fredholm and mixed differential-integral equations, introduces Bell polynomials (definition, properties, relation to Stirling numbers)
Chapter 3: Presents numerical methods for solving Volterra integro-differential equations using the collocation method, with illustrative examples

Chapter 1

PRELIMINARIES

1.1 Functional spaces

1.1.1 Normed vector space

Norms

Definition. *Let E be a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we call a norm on the space E any function denoted $\|.\|$, defined on E with values in $\mathbb{R}_+$, such that:*

For all $x, y \in \mathbb{E}$ and $\lambda \in \mathbb{K}$:

1. $\|x\| = 0 \Longleftrightarrow x = 0.$
2. $\|\lambda x\| = |\lambda| \|x\|.$
3. $\|x + y\| \leq \|x\| + \|y\|.$

A vector space E with a norm $(E, \|.\|)$ is called a ***normed space***

Examples:

1. The absolute value is a norm on $\mathbb{R}$.
2. The modulus is a norm on $\mathbb{C}$.
3. The following applications are norms on $\mathbb{R}^n$:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}. \quad \|x\|_\infty = \sup_{1 \leq i \leq n} |x_i|.$$

4. Let $F = (C([a; b]); \mathbb{R})$ be the space of real-valued continuous functions on the interval $([a; b])$for any $g \in F$, the following functions define a norm on F :

$$\|g\|_\infty = \max_{a \leq x \leq b} |g(x)|.$$

1.1.2 Banach space

Cauchy sequences:

let (x_n) be a sequence of elements in a normed space $(E, \|.\|)$, we say that (x_n) is a cauchy sequence if:

$$\forall \epsilon > 0, \exists N_\epsilon, \forall p, q \geq N_\epsilon, \|x_p - x_q\| < \epsilon$$

Lemma. *let (x_n) be a cauchy sequence in a normed space $(E, \|.\|)$, if (x_{nk}) is a subsequence that converges to some $x \in E$; then the whole sequence (x_n) also converges to n.*

Complete spaces:

we say a normed vector space $(E; \|.\|)$ is complet if every cauchy sequence of elements in E converges to an element in E, meaning there is a point $x \in E$, such that the sequence (xn) converges to$x$ in the norm $\|.\|$

$$\lim_{n \to \infty} x_n = x$$

The advantage of complete spaces is that, in such spaces, it is enough to verify that a sequence is Cauchy in order to conclude that it converges. You do not need to explicitly know the limit of the sequence; the completeness of the space guarantees that the sequence will converge to some limit within the space.

Banach spaces:

A normed vector space $(E, \|.\|)$is called a Banach space if it is complete, meaning that every Cauchy sequence (x_n) in E is converges to some limit $x \in E$.
Every finit-dimensional normed vector is automatically complete, meaning it is a Banach space

1.1.3 Hilbert space

An inner product on a real or complex vector space H is a function

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$$

that assigns to each pair $(x, y) \in H \times H$ a complex number $\langle x, y \rangle$, and satisfies certain properties (typically linearity in one argument, conjugate symmetry, and positive-definiteness).

Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space H. Then for all $x, y, z \in H$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$ for real vector spaces), the inner product satisfies:

1. **Linearity in the first argument:**

$$\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$$

2. **Conjugate symmetry:**

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

3. **Positive-definiteness:**

$$\langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \iff x = 0$$

Example

Here is some examples of inner product space which demonstrate that expression : $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm.

1. The inner product for $\mathbb{C}^n$ is given by

$$\langle x, y\rangle = \sum_1^n x_j \bar{y_j}$$

where $\bar{y_i}$ denotes the complex conjugate of y_i.

2. Let extension for infinite vectors, let l_2 be

$$l_2 = \left\{ sequences \{x_j\}_1^\infty : \sum_1^n |x_j|^2 < \infty \right\}$$

3. Let $C([a, b])$ be a space of continuous function on the interval $[a, b] \subset \mathbb{R}$:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx, \quad and\, \|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$$

A Hilbert space is a vector space equipped with an inner product that is complete with respect to the norm induced by that inner product.

1.1.4 Linear operators

Consider two normed vector spaces E and F, and let $A : E \to F$ be a mapping. The operator A is said to be linear if it satisfies the following property:

For any vectors $\varphi_1, \varphi_2 \in E$ and any scalars $\lambda_1, \lambda_2 \in \mathbb{K}$ (where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$), we have:

$$A(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 A(\varphi_1) + \lambda_2 A(\varphi_2)$$

In other words, A preserves vector addition and scalar multiplication.

(Continuous linear operators)

Let E and F be normed spaces, and let A be a linear operator defined on a subset $G \subset E$, mapping into F. The operator A is said to be continuous at a point $x_0 \in G$ if the following condition holds:

For every sequence (x_n) in G that converges to x_0, the sequence $A(x_n)$ converges to $A(x_0)$.

In other words:

$$\lim_{n\to\infty} A(x_n) = A\left(\lim_{n\to\infty} x_n\right) = A(x_0)$$

We can say about the operator A that it is continuous in the set G if it is continuous at every point of G.

1.1.5 Bounded linear operators

Let $A : E \to F$ be a linear operator between normed spaces E and F. The operator A is said to be bounded if there exists a constant $C > 0$ such that:

$$\|A(x)\|_F \leq C\|x\|_E, \quad \forall x \in E$$

In other words, the norm of $A(x)$ in F is controlled by the norm of x in E, uniformly over all $x \in E$.

$$\|A\| = \sup_{x\neq 0} \frac{\|A(x)\|_F}{\|x\|_E} = \sup_{\|x\|=1} \|A(x)\|_F = \sup_{x\in E \; x\neq 0, \|x\|\leq 1} \|A(x)\|_F$$

A is continuous linear operator $\iff$ A is bounded.

1.1.6 Compact linear operators:

Let A be a linear operator from a normed space E into a normed space F. The operator A is said to be a compact linear operator (or completely continuous linear operator) if for every bounded subset $\Omega \subset E$, the image $A(\Omega) \subset F$ has compact closure in F. In other words, the closure $\overline{A}(\Omega)$

Theorem. ***(Finite dimensional domain)***
Let A be a bounded operator defined from E into F with the domain E has a finite dimension, $dimE < \infty$ then the operator A is compact.

Proof. Indeed, the space E has a finite dimension, $dimE < \infty$ implies the finite dimensional range $A(E)$; say

$$dimA(E) \leq dimE,$$

it follows that, A is a compact operator. □

1.1.7 Integral linear operators

Integral operators are key tools in functional analysis, especially useful for transforming complex functions into simpler forms for easier analysis.

An integral operator is any linear operator A defined on a normed space E, taking values in another normed space F, and expressed as:

$$A\varphi(x) = \int_{G_2} k(x, y)\varphi(y)\, dy \quad (x, y) \in G_1$$

Where :

- $k(x, y)$ is a measurable function on the measurable set $G_1 \times G_2$, called the **kernel** of the integral operator A.

- $\varphi(y)$ is a measurable function on G_1.

The **norm** of an integral operator A defined on the space $L^p(G_1)$, where p and q are conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1, with 1 \leq p, q \leq \infty$) is given by:

$$\|A\|_p = \begin{cases} \left(\int_{G_2}\left(\int_{G_2}|k(x,y)|^q dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}, & 1 < p < \infty \\ \int_{G_1} ess\sup_y |k(x,y)|dx, & p = 1 \\ ess\sup_x \int_{G2} |k(x,y)|dy, & p = \infty \end{cases}$$

Let A be an integral operator such that its norm $\|A\|_p$ is finite:

$$\|A\|_p < \infty$$

Then the operator A is a **continuous linear operator** from $L^p(G_1)$ to $L^p(G_2)$. Moreover, for every function $\varphi \in L^p(G_1)$, we have

$$\|A\varphi\|_p \leq \|A\|_p \|\varphi\|_p$$

Chapter 2

Integral Equations

In this chapter, we will discuss certain types of integral equations as well as the Volterra integro-differential equations.

2.1 Classification of integral equations

2.1.1 Integral Equations

Definition. *A functional equation is called a linear integral equation when the unknown function Ψ satisfies an equation of the form:*

$$\Psi(x) = f(x) + \gamma \int_{\alpha(x)}^{\beta(x)} (l(x,\mu)\Psi(\mu))\, d\mu + f(x)$$

where l and f are given functions,$\alpha(x)$ and $\beta(x)$ define the limits of integration, and γ is a nonzero numerical parameter (real or complex).

2.1.2 Fredholm Integral Equations

Fredholm integral equations are characterized by fixed limits of integration ,The unknown function $y(x)$ typically appears under the integral sign;
First kind:
A Fredholm integral equation of the first kind has the form:

$$f(x) = \int_a^b K(x,t)y(t)\, dt$$

Second Kind
the unknown function $y(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)\, dt$$

Example. *examples of the tow kind are given by*

$$\frac{\sin x - x\cos x}{x^2} = \int_0^1 \sin(xt)y(t)\, dt$$

and

$$y(x) = x + \frac{1}{2}\int_{-1}^1 (x-t)y(t)\, dt$$

2.1.3 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the first kind Volterra integral equations, the unknown function $y(x)$ appears only inside integral sign in the form:

$$\int_0^x K(x,t)y(t)\,dt = f(x).$$

However, Volterra integral equations of the second kind, the unknown function$y(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$y(x) = f(x) + \lambda \int_0^x K(x,t)y(t)\,dt.$$

Example. *examples of the Volterra integral equations of the first kind are:*

$$xe^{-x} = \int_0^x e^{t-x}y(t)\,dt$$

and

$$5x^2 + x^3 = \int_0^x (5 + 3x - 3t)y(t)\,dt$$

However, examples of the Volterra integral equations of the second kind are:

$$y(x) = 1 - \int_0^x y(t)\,dt,$$

and

$$y(x) = x + \int_0^x (x-t)y(t)\,dt.$$

2.1.4 Volterra-Fredholm Integral Equations

The combination of Volterra and Fredholm integrals (taken separately) is known as a **VolterraFredholm integral equation**:
•An equation of the form:

$$y(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)y(t)\,dt + \lambda_2 \int_a^b K_2(x,t)y(t)\,dt, \quad x \in [a,b]$$

is called a **VolterraFredholm integral equation**.
where the functions$K_1(x,t)$, $K_2(x,t)$ and $f(x)$ are given, $y(x)$ is the unknown function, and λ_1 and λ_2 are non-zero constants.

•An equation of the form

$$y(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,y(\xi,\tau))\, d\xi d\tau, \quad (x,t) \in \Omega \times [0,T].$$

is called a **mixed integral equation**
where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is closed subset of $\mathbb{R}^n, n = 1,2,3$. It is interesting to note that.

Example. *Examples of the two types are given by:*

$$y(x) = 6x + 3x^2 + 2 - \int_0^x xy(t)\, dt \; - \int_0^1 ty(t)\, dt,$$

and

$$y(x,t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)\, d\xi d\tau.$$

2.1.5 Volterra Integro-Differential Equations

A Volterra integro-differential equation is generally expressed as:

$$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = f(x) + \lambda \int_a^x K_v(x,t) y(t) dt,$$

with initial conditions:

$$y^{(k)}(a) = \alpha_k, \quad for \quad 0 \le k \le m-1,$$

where $y(x)$ is the unknown function , $P_k(x)$, $K_v(x,t)$ and $f(x)$ are given functions, λ is a parameter , $y^{(k)}$ is the k^{th} derivative of $y(x)$.

Example.

1. $$\begin{cases} y'(x) + 2y(x) = \cos(x) + \int_0^x e^{x-t} y(t)\, dt, \\ y(0) = 1 \end{cases}$$

   This is a first-order Volterra integro-differential equation.

2. $$\begin{cases} y''(x) - xy'(x) = \sin(x^2) - \int_0^x t\, y(t)\, dt, \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

   This is a second-order Volterra integro-differential equation.

2.1.6 Fredholm Integro-Differential Equations

Fredholm integro-differential equations arise naturally when differential equations are transformed into integral equations. Such equations involve the unknown function $y(x)$ together with one of its derivatives $y^n(x)$ $(with n \geq 1)$,
where the derivative typically appears outside the integral, while the function itself appears inside the integral.
In this type of equation, the limits of integration are fixed, as in standard Fredholm integral equations. The term integro-differential reflects the presence of both differential and integral operators within the same equation.
To determine a particular solution, appropriate initial conditions must be specified.
A general form of a Fredholm integro-differential equation is given by:

$$y^n(x) = f(x) + \lambda \int_a^b K(x,t)y(t)\,dt,$$

Here, $y^n(x)$ denotes the nth derivative of $y(x)$. Lower-order derivatives may also appear alongside $y^n(x)$ on the left-hand side of the equation.

Example.

$$\begin{cases} y^{'}(x) = 1 - \dfrac{1}{3}x + \displaystyle\int_0^1 xy(t)\,dt \\ \qquad y(0) = 0 \end{cases}$$

This is a first-order Fredholm integro-differential equation.

$$\begin{cases} y^{''}(x) + y^{'}(x) = x - \sin x - \displaystyle\int_0^{\frac{\pi}{2}} xty(t)\,dt, \\ y(0) = 0, \quad y^{'}(0) = 1. \end{cases}$$

This is a second-order Fredholm integro-differential equation.

2.1.7 Volterra Fredholm integro-differential equations

VolterraFredholm integro-differential equations are formulated similarly to VolterraFredholm integral equations, but with the addition of one or more ordinary derivatives alongside the integral terms. These equations are commonly presented in two main forms:

$$y^{(n)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)\,y(t)\,dt + \lambda_2 \int_a^b K_2(x,t)\,y(t)\,dt$$

and

$$y^{(n)}(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,y(\xi,\tau))\,d\xi\,d\tau, \quad (x,t) \in \Omega \times [0,T]$$

where $f(x,t)$ and $F(x,t,\xi,\tau,y(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, where Ω is closed subset of $\mathbb{R}^n$ with$n = 1,2,3$.
It is important to observe that the first equation separates the Volterra and Fredholm integral components, whereas the second equation involves mixed integrals, and the lower-order derivatives may also appear in such formulations.
Additionally, the unknown functions $y(x)$ and $y(x,t)$ may occur both inside and outside the integral operators. This feature characterizes equations of the second kind. In contrast, if

the unknown function appears only within the integrals, the equation is classified as being of the first kind.
To uniquely determine a solution, appropriate initial conditions must be specified.

Example.
$$\begin{cases} y'(x) = 24x + x^4 + 3 - \int_0^x (x-t)y(t)\,dt - \int_0^1 t\,y(t)\,dt, \\ \quad y(0) = 0 \end{cases}$$

and

$$\begin{cases} y'(x,t) = 1 + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)\,d\xi\,d\tau, \\ \quad y(0,t) = t^3 \end{cases}$$

2.1.8 Bell Polynomial

Definition. *Let n be a natural number and $S(n,k)$ be the Stirling numbers of second kind, where $k = 0, \ldots, n$ which are:*

$$s(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i} (k-i)^n \tag{1}$$

Then ;

$$BL_n(x) = \sum_{k=1}^{n} s(n,k)\,x^k, \quad BL_0(x) = 1,\ n \geq 1 \tag{2}$$

Is called a Bell polynomial of degree n.

The Stirling numbers satisfy the recurrence:

$$S(n,1) = S(n,n) = 1, \quad S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$$

2.1.9 Fundamental properties of Bell polynomials

Bell polynomials play a fundamental role in combinatorics, analysis, and the theory of differential equations. We will summarize their most important properties.

Exponential Generating Function

The complete Bell polynomials $BL_n(x_1, \ldots, x_n)$ are defined through the exponential generating function:

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k!}\right) = \sum_{n=0}^{\infty} BL_n(x_1, \ldots, x_n)\frac{t^n}{n!} \quad BL_0 = 1$$

This generating function is the starting point for most of their properties.

Recurrence Relation

Bell polynomials satisfy the recurrence relation:

$$BL_{n+1}(x_1, \ldots, x_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} BL_{n-k}(x_1, \ldots, x_{n-k})\,x_{k+1}.$$

Relation with Stirling Numbers

Bell polynomials are closely related to Stirling numbers of the second kind $S(n,k)$:

$$BL_n(x_1,\ldots,x_n) = \sum_{k=1}^{n} S(n,k)\,x^k.$$

In particular, when $x_1 = x_2 = \cdots = x_n = 1$, we obtain the Bell numbers:

$$BL_n(1,1,\ldots,1) = BL_n.$$

Homogeneity Property

If the weight of x_k is defined as k, then every monomial in $BL_n(x_1,\ldots,x_n)$ satisfies:

$$\sum k \cdot (\text{power of } x_k) = n.$$

This shows that Bell polynomials are homogeneous with respect to this weighting.

Partial Bell Polynomials

The complete Bell polynomials can be decomposed into partial Bell polynomials:

$$BL_n(x_1,\ldots,x_n) = \sum_{k=1}^{n} BL_{n,k}(x_1,\ldots,x_{n-k+1}).$$

Derivative Property

Bell polynomials satisfy the derivative identity:

$$\frac{\partial BL_n}{\partial x_k} = \binom{n}{k} BL_{n-k}(x_1,\ldots,x_{n-k}).$$

Chapter 3

Numerical Implementation and Collocation Approach Via Bell Polynomials

Chapter Three presents a simple and effective numerical method to solve linear Volterra integro-differential equations under initial conditions. This approach combines the Collocation Method with Bell Polynomials.

The chapter begins with the mathematical setup of the problem.Then, it shows how to transform the complex equation into a simple matrix system that is easy to solve. Finally, the chapter tests the accuracy of this method using different practical examples, comparing the approximate results directly with the exact solutions.

3.1 Numerical solution for Volterra integro-differential equations

We consider the linear Volterra integro-differential equations :

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = f(x) + \lambda \int_0^1 K(x,t)y(t)dt, \quad 0 \leq x \leq 1 \tag{04}$$

With initial conditions:

$$y^{(k)}(0) = \alpha_k, \quad for \quad 0 \leq k \leq m-1,$$

First, we seek an approximate solution of eq (04) as a finite linear combination of Bell polynomials:

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n BL_n(x) = BL(x)A \tag{05}$$

with ;

$$A = [a_0, a_1, \ldots, a_N]^T$$

$$\mathbf{BL(x)} = X(x).S \tag{03}$$

where

$$\mathbf{BL}(\mathbf{x}) = [BL_0(x), BL_1(x), \ldots, BL_N(x)]\ ;\ \mathbf{X}(\mathbf{x}) = [1, x, x^2, \ldots, x^N]^T\ ;$$

$$\mathbf{S} = \begin{pmatrix} s(0,0) & 0 & \cdots & 0 \\ s(0,1) & s_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s(0,N) & s(1,N) & \cdots & s(N,N) \end{pmatrix}$$

Once can obtain the entries of this matrix by using equation (1).

Example. *In the case of $N = 3$, we get*

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix}$$

So, the first few Bell polynomials are:

$$BL_0(x) = 1, \quad BL_1(x) = x, \quad BL_2(x) = x + x^2, \quad BL_3(x) = x + 3x^2 + x^3.$$

Since matrix S is a lower triangular matrix with nonzero diagonal elements,

Thus, by means the equality(03):

$$y(x) = X(x).S.A \tag{07}$$

The next step is dedicated to the derivatives of the approximate solution:

$$y^{(k)}(x) = \mathbf{BL}^{(k)}(x) \cdot \mathbf{A} \tag{08}$$

And within it, firstly requires us to compute the derivatives of each Bell polynomial:

$$BL_n^{(k)}(x) = \frac{d^k}{dx^k} BL_n(x) = X^{(k)}(x) \cdot S, \quad k = 0, 1, \ldots, m$$

Where;

$$\mathbf{BL_n^{(k)}}(x) = [BL_0^{(k)}(x), BL_1^{(k)}(x), \ldots, BL_N^{(k)}(x)].$$

And $\mathbf{X^{(k)}}(\mathbf{x})$ is the k-th derivative of each component of $\mathbf{X}(x)$. Then the following equality holds:

$$BL^{(k)}(x) = X^{(k)}(x) \cdot S \tag{09}$$

With;

$$X^{(k)}(x) = X(x) \cdot H^k \tag{10}$$

Where

$$H = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Example. *If we set $N = 2$ we get:*

$$X(x) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}^T$$

$$\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2x \end{bmatrix}$$

By Substituting Eq (09),(10) into (08) we obtain the derivatives of the approximate solution as:

$$\begin{aligned} y^{(k)}(x) &= \mathbf{BL}^{(k)}(x) \cdot A \\ y^{(k)}(x) &= \mathbf{X^{(k)}(x)}.S.A \\ y^{(k)}(x) &= \mathbf{X(x).H^{k}} \cdot S \cdot A \end{aligned} \tag{15}$$

3.1.1 Matrix relation of The Volterra integral part:

Furthermore, Eq(04)'s kernel function $k(x,t)$ is built in matrix form ,and its defined for the **Taylor** and **Bell** polynomials as follows:

For Taylor pollynomials:

$$K_(x,t) = \sum_{n=0}^{N} \sum_{m=0}^{N} K_{ty} x^n t^m = X(x) K_{ty} X^T(t) \tag{11}$$

For Bell pollynomials:

$$K(x,t) = BL(x) K_{BL} BL^T(t) \tag{12}$$

Where $\mathbf{k_{ty}} = [k_{ij}] \qquad i,j = 0,1,\ldots,n$

$$k_{ij} = \frac{1}{i!j!} \cdot \frac{\partial^{i+j} K(0,0)}{\partial x^i \partial t^j}$$

According to eqs (09) and (10) , we obtain:

$$BL(x) K_{BL} BL^T(t) = X(x) K_{ty} X^T(t)$$

Making use of eq(03) , we derive:

$$\mathbf{X(x)S}\ K_{BL}\ (\mathbf{X(t)S})^{\mathbf{T}} = X(x) K_{ty} X^T(t)$$

then

$$S K_{BL} S^T = K_{ty}$$

Therefore

$$\mathbf{K_{BL}(x)} = S^{-1}K_{ty}(S^T)^{-1} \tag{13}$$

Substituting Eqs (12), (06) and (03),(13) . The Volterra's Integral can be expressed in the following form:

$$\begin{aligned}
\int_0^x K(x,t)y(t)\,dt &= \int_0^x BL(x)K_{BL}BL_(t)^T BL(t)A \\
&= \int_0^x X(x)S \;\; S^{-1}K_{ty}(S^T)^{-1} \;\; X^T(t)S^T \;\; X(t)SA \\
&= X(x)K_{ty} \;\; \left(\int_0^x X^T(t)X(t)\,dt\right) \;\; S\;A \\
&= X(x)K_{ty}\,\mathbf{R(x)}\,SA
\end{aligned} \tag{14}$$

Where

$$R(x) = \int_0^x X^T(t)X(t)\,dt$$

Now, we have the the original integro-differential equation is expressed as follows:

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = f(x) + \int_0^x K(x,t)y(t)\,dt$$

Making use of Eqs (14) ,(15) ; we derive

$$\sum_{k=0}^{m} P_k(x)\cdot X(x)\cdot H^k\;SA = f(x) + X(x)K_{ty}\;R(x)\;SA$$

Or shortly

$$\left[\sum_{k=0}^{m} P_k(x)\cdot X(x)\cdot H^k\;S - X(x)K_{ty}\;R(x)\;S\right]\cdot A = f(x)$$

3.1.2 Collocation method:

In this Step , we apply the collocation method by selecting $N+1$ distinct collocation points in the interval $[a,b]$
The collocation points x_i are defined by:

$$x_i = a + \frac{b-a}{N}i, \quad i = 0,1,2,\ldots,N$$

In the interval $[0,1]$;

$$x_i = \frac{i}{N}, \quad i = 0,1,2,\ldots,N$$

Next, we require that the residual equation holds exactly at each collocation point, which yields:

$$\left[\sum_{k=0}^{m} P_k(x_i)\cdot X(x_i)\cdot H^k\;S - X(x_i)K_{ty}\;R(x_i)\;S\right]\cdot A = f(x_i) \tag{16}$$

where

$$P_k = \begin{pmatrix} p_k(x_0) & 0 & \cdots & 0 \\ 0 & p_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k(x_n) \end{pmatrix} \qquad A = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \qquad F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix} X = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Next , we construct the linear algebraic system by defining the matrix M of size $(N+1)\times(N+1)$ with entries:

$$\mathbf{G}=[g_{ij}]=\sum_{k=0}^{m} P_k(x_i)\cdot X(x_i)\cdot H^k\cdot\ S - X(x_i).K_{ty}.\ R(x).\ S$$

The fundamental matrix relation (16) can be expressed as follows:

$$G\mathbf{A}=\mathbf{F} \qquad or \qquad [G;F] \tag{17}$$

$$[\mathbf{G};\mathbf{F}]=\begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,n} & ; & f(x_0) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,n} & ; & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ g_{n,0} & g_{n,1} & \cdots & g_{n,n} & ; & f(x_n) \end{pmatrix}$$

By writing the matrix corresponding to the initial conditions (04), we obtain:

$$\begin{aligned} y^k(0) &= \alpha_k\ , \qquad k=0,1,\ldots,m-1 \\ BL^{(k)}(0)\ A &= \alpha_k \\ X^{(k)}(0)\ S\ A &= \alpha_k \\ X(0)\ H^k\ S\ A &= \alpha_k \end{aligned}$$

Thus, we can get:

$$\omega_i A=\alpha_i, \quad or \quad [\omega_i,\alpha_i]=[\omega_{i,0},\omega_{i,1},\omega_{i,2},\ldots,\omega_{i,n}\ ;\ \alpha_i] \quad ,\ i=0,1 \tag{18}$$

Where

$$\omega_i = X(0)\ H^k\ S \qquad ,\ i=0,1$$

Applying the conditions to Eq. (04), we replace the last m rows of matrix (17) by the corresponding rows of matrix (18). Therefore, the new augmented matrix is given by:

$$[\bar{G};\bar{F}]=\left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,n} & f(x_0) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,n} & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{n-m,0} & g_{n-m,1} & \cdots & g_{n-m,n} & f(x_{n-m}) \\ \omega_{0,0} & \omega_{0,1} & \cdots & \omega_{0,n} & \alpha_0 \\ \omega_{1,0} & \omega_{1,1} & \cdots & \omega_{1,n} & \alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega_{m-1,0} & \omega_{m-1,1} & \cdots & \omega_{m-1,n} & \alpha_m - 1 \end{array}\right)$$

If ;

$$rank(\bar{G}) = rank([\bar{G};\bar{F}]) = n+1$$

Then the augmented matrix solution is written as :

$$A=(\bar{G})^{-1}\ \bar{F} \tag{19}$$

Finally, the Bell coefficients are obtained from the solution of matrix (19) and substituted into the series (05).

3.2 Numerical examples

Example 01: consider the linear Volterra Integro-Differential Equation:

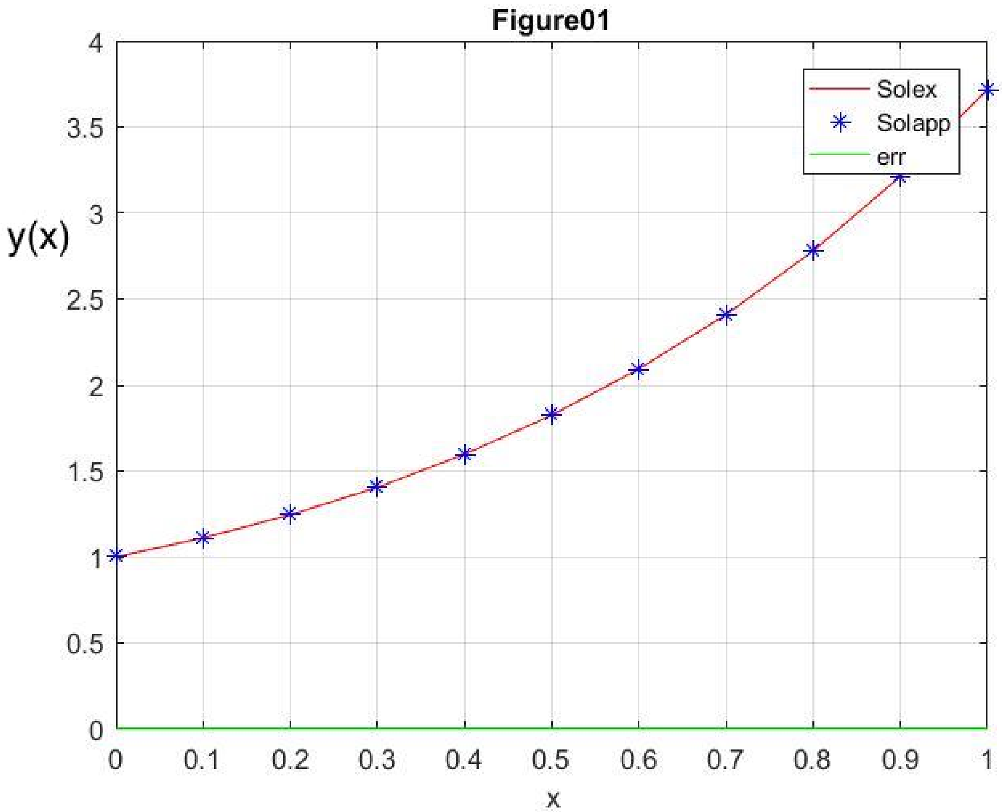
$$\begin{cases} y^{(4)}(x) - y(x) + \int_0^x y(t)\, dt = x(1 + e^x) + 3e^x \\ y(0) = 1 \ , \ y^{'}(0) = 1 \ , \ y^{''}(0) = 2 \ , \ y^{'''}(0) = 3 \end{cases}$$

the exact solution given by :

$$y_{ex}(x) = 1 + xe^x$$

Table 01 we present the approximate solution *Solapp* obtained by the Volterra collocation method, the error is calculated for $N = 10$

x	$Solex$	$Solapp$	err
0	1.0000e+00	1.0000e+00	0
1.0000e-01	1.1105e+00	1.1105e+00	7.7272e-14
2.0000e-01	1.2443e+00	1.2443e+00	1.1553e-12
3.0000e-01	1.4050e+00	1.4050e+00	4.6543e-12
4.0000e-01	1.5967e+00	1.5967e+00	1.1942e-11
5.0000e-01	1.8244e+00	1.8244e+00	2.4393e-11
6.0000e-01	2.0933e+00	2.0933e+00	4.3423e-11
7.0000e-01	2.4096e+00	2.4096e+00	7.0356e-11
8.0000e-01	2.7804e+00	2.7804e+00	1.3270e-10
9.0000e-01	3.2136e+00	3.2136e+00	5.8216e-10
1.0000e+00	3.7183e+00	3.7183e+00	3.6651e-09



Example 02: Consider the integro-differential equation of Volterra:

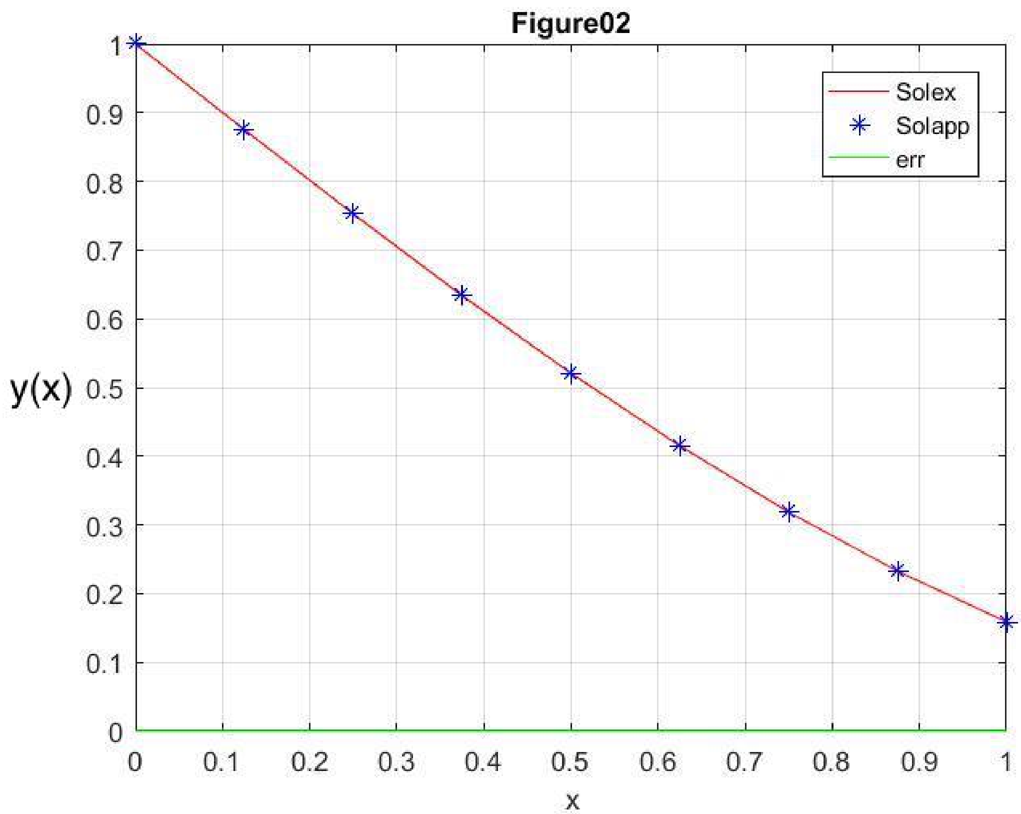
$$\begin{cases} y^{'''}(x) - \int_0^x e^t \, dt = \cos x - e^x - \frac{1}{2} e^x \left[\cos x - \sin x\right] + \frac{3}{2} \\ y(0) = 1 \ , \ y^{'}(0) = -1 \ , \ y^{''}(0) = 0 \end{cases}$$

the exact solution given by :

$$y_{ex}(x) = 1 - \sin x$$

The error is calculated for $N = 8$

x	Solex	Solapp	err
0	1.0000e+00	1.0000e+00	0
1.2500e-01	8.7533e-01	8.7533e-01	1.9338e-11
2.5000e-01	7.5260e-01	7.5260e-01	1.2388e-10
3.7500e-01	6.3373e-01	6.3373e-01	3.0615e-10
5.0000e-01	5.2057e-01	5.2057e-01	5.7346e-10
6.2500e-01	4.1490e-01	4.1490e-01	9.1867e-10
7.5000e-01	3.1836e-01	3.1836e-01	1.2933e-09
8.7500e-01	2.3246e-01	2.3246e-01	1.8677e-09
1.0000e+00	1.5853e-01	1.5853e-01	3.6181e-08

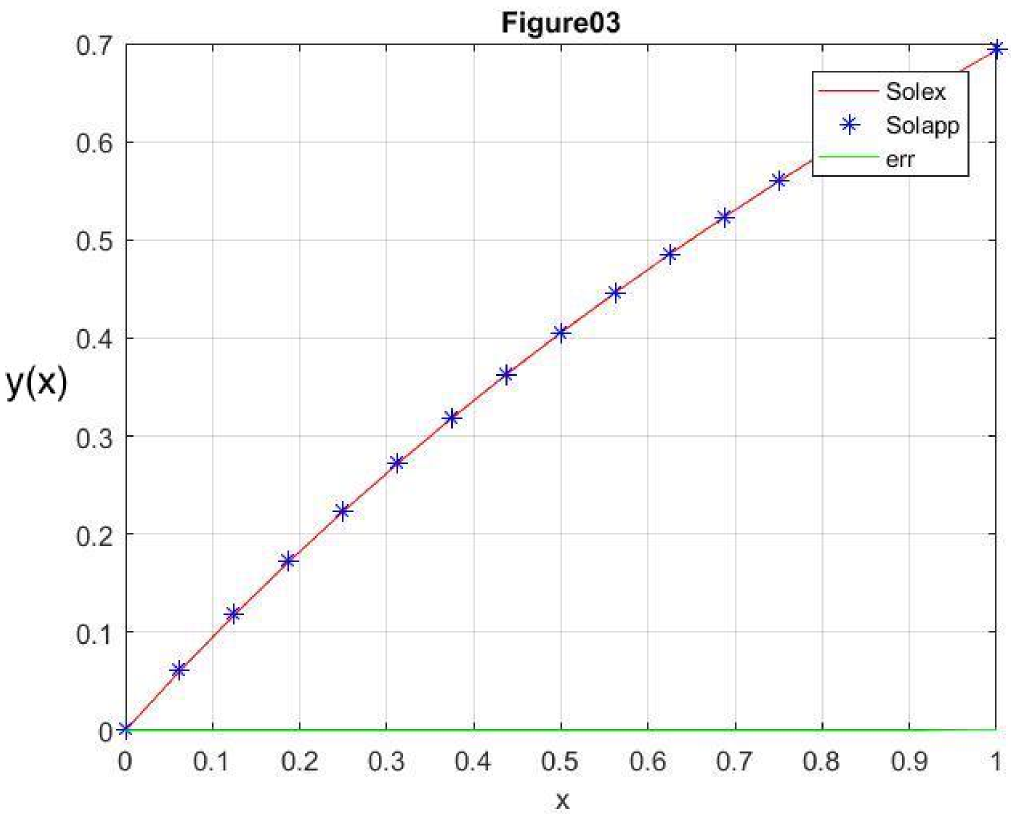


Example 03: consider the linear Volterra Integro-Differential Equation:

$$\begin{cases} y^{'}(x) - y(x) - \int_0^x \frac{x}{t+1} y(t) \ = f(x) \\ y(0) = 0 \\ f(x) = -\ln(1+x) \left[\frac{x}{2} \ln(x+1) + 1\right] + \frac{1}{x+1} \end{cases}$$

The exact solution given by : $\quad y_{ex}(x) = \ln(1+x)$
The error is calculated for $N = 16$

x	Solex	Solapp	err
0	0	3.6082e-07	3.6082e-07
6.2500e-02	6.0625e-02	6.0625e-02	3.8559e-07
1.2500e-01	1.1778e-01	1.1778e-01	4.2703e-07
1.8750e-01	1.7185e-01	1.7185e-01	4.4897e-07
2.5000e-01	2.2314e-01	2.2314e-01	4.3538e-07
3.1250e-01	2.7193e-01	2.7193e-01	4.4196e-07
3.7500e-01	3.1845e-01	3.1845e-01	4.9886e-07
4.3750e-01	3.6291e-01	3.6291e-01	5.2933e-07
5.0000e-01	4.0547e-01	4.0547e-01	6.0033e-07
5.6250e-01	4.4629e-01	4.4629e-01	5.9400e-07
6.2500e-01	4.8551e-01	4.8551e-01	6.6443e-07
6.8750e-01	5.2325e-01	5.2325e-01	1.0723e-06
7.5000e-01	5.5962e-01	5.5962e-01	3.4984e-06
8.1250e-01	5.9471e-01	5.9472e-01	1.5107e-05
8.7500e-01	6.2861e-01	6.2867e-01	6.5785e-05
9.3750e-01	6.6140e-01	6.6166e-01	2.6619e-04
1.0000e+00	6.9315e-01	6.9413e-01	9.8314e-04



Conclusion

In this thesis, we presented a numerical method based on Bell polynomials in combination with collocation method to solve high-order linear Volterra integro-differential equations with initial conditions. The relationship between Bell polynomials and Stirling numbers of the second kind was exploited to represent the approximate solution in matrix form, which facilitates the computation of derivatives and integrals. The properties of Bell polynomials were used to transform these equations into a system of algebraic equations which could be solved very easily after applying the collocation method and initial conditions.

The results obtained from several numerical examples demonstrate the accuracy, and effectiveness of the proposed method.

Abstract

This thesis presents a numerical collocation method based on Bell polynomials to solve high-order linear Volterra integro-differential Equations under the initial conditions. Some examples are presented to illustrate the applicability, efficiency and accuracy of this, scheme in comparison with some other well-known methods.
Keywords:
Bell polynomials, collocation method, high-order linear Volterra integro-differential equations, numerical solution, initial conditions, approximation methods, computational efficiency, error analysis.

ملخـص

تهدف هذه الأطروحة الى إيجاد حل عددي للمعادلات التكاملية التفاضلية الخطية من الرتب العليا من نوع "فولتيرا" تحت شروط ابتدائية محددة بالاعتماد على كثيرات حدود "بيل". كما يتم تقديم بعض الأمثلة لتوضيح مدى قابلية تطبيق هذا المخطط، وكفاءته، ودقته مقارنة ببعض الطرق الأخرى المعروفة.

كلمات مفتاحية:

كثيرات حدود بيل، طريقة التجميع، المعادلات التكاملية التفاضلية الخطية من الرتب العليا من نوع فولتيرا، الحل العددي، الشروط الابتدائية، الطرق التقريبية، الكفاءة الحسابية، تحليل الخطأ.

<u>Résumé</u>

Cette thèse propose une méthode de collocation numérique basée sur les polynômes de Bell pour la résolution des équations intégro-différentielles linéaires de Volterra d'ordre supérieur, sous des conditions initiales. Quelques exemples sont présentés afin d'illustrer l'applicabilité, l'efficacité et la précision de ce schéma, en comparaison avec d'autres méthodes bien connues.

<u>Mots-clés:</u>

Polynômes de Bell , Méthode de collocation, Équations intégro-différentielles linéaires de Volterra d'ordre supérieur, Solution numérique, Conditions initiales, Méthodes d'approximation, Efficacité de calcul, Analyse d'erreur.

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