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## *Master memory*

**Field :** Mathematics and computer sciences

**Branch :** Mathematics

**Option :** Functional analysis

## Theme

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**p-summing Bloch mappings on the complex unit disk**

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**In front of the jury composed of :**

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# Dedication

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To my parents, whose unwavering support and encouragement have been my guiding light throughout this journey. Your sacrifices and love have made this achievement possible.

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[Boumalia Djoumana Riham]

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# Contents

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<b>Introduction</b>	<b>1</b>
<b>1 Preliminaries</b>	<b>2</b>
1.1 Linear $p$ -summing operators . . . . .	2
1.2 Ideal property of $p$ -summing operators . . . . .	2
1.3 Pietsch Domination Theorem . . . . .	3
1.4 Pietsch Factorization Theorem . . . . .	4
1.5 Holomorphic Function . . . . .	4
1.6 Bloch Function . . . . .	5
1.7 The Bloch space $B(\mathbb{D}, X)$ . . . . .	7
1.8 The Bloch space $\hat{B}(\mathbb{D}, X)$ . . . . .	8
<b>2 <math>(p, \sigma)</math>-absolutely Lipschitz mappings</b>	<b>9</b>
2.1 $(p, \sigma)$ -absolutely continuous mappings . . . . .	9
2.2 Lipschitz compact and weakly compact operators . . . . .	10
2.3 Lipschitz $p$ -summing operators . . . . .	11
2.4 $(p, \sigma)$ -absolutely Lipschitz mappings . . . . .	12
2.5 Factorization Theorem . . . . .	19
<b>3 <math>p</math>-summing Bloch mapping on the complex unit disc.</b>	<b>24</b>
3.1 $p$ -Summing Bloch Mappings on the Unit Disc . . . . .	24
3.2 Injective Banach Ideal Property . . . . .	26
<b>4 Pietsch domination</b>	<b>31</b>
4.1 Pietsch Domination . . . . .	31
<b>Bibliography</b>	<b>33</b>



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# Notations

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$\mathbb{K}$	The field of real or complex numbers.
$E^*$	The topological dual of $E$ .
$L(E; F)$	The set of all linear operators.
$\mathcal{L}(E; F)$	The sets of all continuous linear operators.
$\mathcal{I}$	The ideal of all linear operator.
$T^*$	The adjoint operator of $T$ .
$\mathcal{K}$	The set of all compact linear operators.
$\mathcal{L}_f$	The set of all finite rank linear operators.
$H(D, X)$	A holomorphic function
$B(\mathbb{D}, X)$	The Bloch space
$p_B$	the Bloch seminorm
$\hat{B}(\mathbb{D}, X)$	The normalized Bloch space
$\Pi_p(E, F)$	The collection of all $p$ -summing operators
$\Pi_{p,\sigma}(E, F)$	The Banach space of all $(p, \sigma)$ -absolutely continuous operators
$I_{p,\infty}$	The formal inclusion map defined between $L_\infty(\mu)$ and $L_p(\mu)$ ( $1 \leq p < \infty$ ).
$i_E$	The isometric embedding, $i_E : E \rightarrow C(E^*)$ given by $i_E(x) := \langle x, \cdot \rangle$ .
$\mathcal{K}$	The set of all compact linear operators.
$Lip_{0K}$	The set of all Lipschitz compact operators.
$\mathcal{W}$	The set of all weakly compact linear operators.
$Lip_{0W}$	The set of all Lipschitz weakly compact operators.
$\mathcal{L}_f$	The set of all finite rank linear operators.
$Lip_{0F}$	The set of Lipschitz finite rank operators.
$\Pi_p^L$	The set of all Lipschitz $p$ -summing operator ( $1 \leq p < \infty$ ).
$\Pi_p^B$	The Banach space of all $p$ -summing Bloch mappings

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# Introduction

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The study of operator ideals has played a pivotal role in understanding linear and nonlinear mappings within functional analysis. Among these, the class of  $p$ -summing operators has emerged as a central concept due to its deep connections with Banach space theory, and integral operator theory. These operators, which generalize absolutely summing operators, offer a powerful framework for analyzing bounded linear maps between Banach spaces, especially those with geometric .

In recent years, attention has turned to extending these ideas beyond the linear setting, particularly toward nonlinear operators and holomorphic function spaces, where classical linear tools require significant refinement. In this context, Bloch-type mappings have drawn interest due to their intrinsic connection to complex analysis, operator theory, and the geometry of function spaces defined on the unit disk. The Bloch space, consisting of holomorphic functions on the unit disk with bounded hyperbolic derivative, provides a natural ground for studying nonlinear analogs of operator ideals.

This memory investigates  $p$ -summing Bloch mappings, a novel concept recently introduced by Antonio Jiménez-Vargas, that blends the analytic structure of Bloch functions with the summability conditions characteristic of  $p$ -summing operators. Our aim is to explore the geometry and analytic properties of this new class of operators, develop appropriate domination and factorization theorems, and examine how classical results such as Pietsch's Domination Theorem and Maurey's Extrapolation Theorem can be extended or adapted in this setting. Our work is composed of four chapters as follows: first Chapter contains some fundamental definitions and theorems concerning the ideal of  $p$ -summing operators. Second Chapter is devoted to studying the ideal of  $p$ -summing linear and Lipschitz operators. We present the domination theorems related to each category. In the Third Chapter we study the ideal of  $(p, \sigma)$ -absolutely Lipschitz mappings . The last Chapter studies  $p$ -summing Bloch mappings, which were recently introduced by Antonio Jiménez-Vargas

# Preliminaries

In this Chapter we give some concepts we need it in the sequel of this memory and we fixed some notations

## 1.1 Linear $p$ -summing operators.

**Definition 1.1** *Let  $1 \leq p < \infty$ . A linear operator  $T : E \rightarrow F$  is said to be  $p$ -summing if there exists a constant  $C \geq 0$  such that for all finite sequence  $(x_i)_{1 \leq i \leq n}$  in  $E$*

$$\left( \sum_{i=1}^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*} \leq 1} \left( \sum_{i=1}^n |\xi(x_i)|^p \right)^{\frac{1}{p}}. \quad (1.1)$$

The infimum of all such constants  $C \geq 0$  is denoted by  $\pi_p(T)$ . The collection of all  $p$ -summing operators between  $E$  and  $F$  is denoted by  $\Pi_p(E, F)$ . We mention that  $(\Pi_p^L, \pi_p^L)$  is an injective Banach Lipschitz operator ideal.

**Remark 1.1** *For all  $T \in \pi_p(E; F)$ , the inequality (1.1) is equivalent to:*

$$\left( \sum_{i=1}^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq \pi_p(T) \sup_{\|\xi\|_{E^*} \leq 1} \left( \sum_{i=1}^n |\xi(x_i)|^p \right)^{\frac{1}{p}}. \quad \text{for all } (x_i)_{i=1}^n \in E.$$

**Theorem 1.1** *If  $1 \leq p \leq q < \infty$ , then  $\Pi_p(E, F) \subset \Pi_q(E, F)$ . Moreover,  $\pi_q(T) \leq \pi_p(T)$  for every  $u \in \Pi_p(E, F)$ .*

## 1.2 Ideal property of $p$ -summing operators

Let  $1 \leq p < \infty$  and let  $v \in \Pi_p(X, Y)$ . then the composition of  $v$  with any bounded linear operator is  $p$ -summing. More specifically, if  $X_0$  and  $Y_0$  are Banach space then, regardless

of how we choose  $u \in \mathcal{L}(X, Y)$  and  $w \in \mathcal{L}(X_0, X)$ , we always have  $uvw \in \Pi_p(X_0, Y_0)$  with  $\pi_p(uvw) \leq \|u\| \cdot \pi_p(v) \cdot \|w\|$ .

. The operators  $u, v, w$  give rise to canonical operators  $\hat{u} : \ell_p(Y) \rightarrow \ell_p(Y_0)$ ,  $\hat{v} : \ell_p^{weak}(X) \rightarrow \ell_p^{strong}(Y)$  and  $\hat{w} : \ell_p^{weak}(X_0) \rightarrow \ell_p^{weak}(X)$  with  $\|u\| = \|\hat{u}\|$ ,  $\pi_p(v) = \|\hat{v}\|$  and  $\|w\| = \|\hat{w}\|$ . the operator  $u\hat{v}w$  corresponding to  $uvw : X_0 \rightarrow Y_0$  coincides with  $\hat{u}\hat{v}\hat{w}$  and thus induces a map  $\ell_p^{weak}(X_0) \rightarrow \ell_p^{strong}(Y_0)$ . Hence  $uvw$  is  $p$ -summing, with

$$\pi_p(uvw) = \|u\hat{v}w\| \leq \|\hat{u}\| \cdot \|\hat{v}\| \cdot \|\hat{w}\| = \|u\| \cdot \pi_p(v) \cdot \|w\|$$

The following basic result about  $p$ -summing operators is due to A. Pietsch, and it characterizes the  $p$ -summability by means of a domination theorem.

### 1.3 Pietsch Domination Theorem

**Theorem 1.2** *Let  $1 \leq p < \infty$  and  $T \in \mathcal{L}(E, F)$ . Then  $T$  is  $p$ -summing if and only if there exist a constant  $C$  and a regular Borel probability measure  $\mu$  on  $B_{E^*}$  (with the weak star topology) so that*

$$\|T(x)\| \leq C \int_{B_{E^*}} |\langle x, x^* \rangle|^p d\mu(x^*), \quad x \in E. \quad (1.2)$$

*In this case,  $\pi_p(T)$  is the least of all the constants  $C$  such that (1.2) holds.*

In order to adapt the previous result into a factorization theorem, we present basic examples of  $p$ -summing linear operators.

**Example 1.1** (1) *Let  $K$  be a compact Hausdorff space, let  $\mu$  be a positive regular Borel measure on  $K$ , and let  $1 \leq p < \infty$ . The canonical inclusion*

$$J_p : C(K) \longrightarrow L_p(\mu),$$

*is  $p$ -summing with  $\pi_p(J_p) = \|J_p\| = \mu(K)^{\frac{1}{p}}$ .*

(2) *Let  $(\Omega, \Sigma, \mu)$  be a finite measure space and let  $1 \leq p < \infty$ . The formal inclusion map*

$$I_{\infty, p} : L_{\infty}(\mu) \longrightarrow L_p(\mu),$$

is  $p$ -summing, with  $\pi_p(I_{\infty,p}) = \mu(\Omega)^{\frac{1}{p}}$ .

We denote by  $i_E$  the isometric embedding  $E \longrightarrow C(B_{E^*})$  given by  $i_E(x) = \langle x, \cdot \rangle$ .

## 1.4 Pietsch Factorization Theorem

**Theorem 1.3** *Let  $1 \leq p < \infty$  and  $T \in \mathcal{L}(E, F)$ . The following are equivalent*

(i)  $T$  is  $p$ -summing.

(ii) *There exist a regular Borel probability measure  $\mu$  on  $B_{E^*}$  (with the weak star topology), a closed subspace  $E_p$  of  $L_p(\mu)$  and a linear continuous operator  $\tilde{u} : E_p \longrightarrow F$  such that  $J_p \circ i_E(E) \subset E_p$  and  $\tilde{u} \circ J_p \circ i_E(x) = T(x)$  for all  $x \in E$ .*

*In other words, if  $\bar{J}_p$  is the map  $i_E(E) \longrightarrow E_p$  induced by  $J_p$ , then the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ i_E \downarrow & & \uparrow \tilde{T} \\ i_E(E) & \xrightarrow{\bar{J}_p} & E_p \\ \cap & & \cap \\ C(B_{E^*}) & \xrightarrow{J_p} & L_p(\mu). \end{array}$$

*In addition, we may choose  $\mu$  and  $\tilde{T}$  so that  $\|\tilde{T}\| = \pi_p(T)$ .*

## 1.5 Holomorphic Function

**Definition 1.2** : *A holomorphic function is a complex function that is differentiable at every point in its domain. Formally, a function  $f : D \rightarrow \mathbb{C}$ , where  $D \subseteq \mathbb{C}$  is an open set, is said to be holomorphic if, for every point  $z_0 \in D$ , the derivative of  $f$  exists in a neighborhood of  $z_0$  in  $D$ .*

*In other words,  $f$  is holomorphic in  $D$  if it is complex differentiable at every point in  $D$ .*

*The formal condition for differentiability is:*

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

*exists for each  $z_0 \in D$ .*

**Example 1.2** A simple example of a holomorphic function is the exponential function  $f(z) = e^z$ .  
 - For any  $z \in \mathbb{C}$ , the derivative of  $f(z) = e^z$  is given by:

$$f'(z) = e^z$$

which is also a holomorphic function.

Thus,  $f(z) = e^z$  is holomorphic in  $\mathbb{C}$ , since its derivative exists everywhere in the complex plane.

## 1.6 Bloch Function

**Definition 1.3** : A **Bloch function** is a specific type of analytic function defined in the complex plane. It is named after the physicist Felix Bloch. A Bloch function satisfies the following conditions:

- The function  $f$  is holomorphic on an open domain  $D \subset \mathbb{C}$ .
- There exists a constant  $C$  such that for all  $z \in D$ , the function satisfies the condition

$$\sup_{z \in D} |f'(z)| \cdot (1 - |z|^2) < C$$

where  $|z|$  is the modulus of  $z$ . This condition ensures that the function does not grow too rapidly as  $z$  approaches the boundary of the domain.

**Example 1.3** Consider the function

$$f(z) = \frac{z}{1-z} \quad \text{for } |z| < 1.$$

This function is analytic in the open unit disk and satisfies the Bloch condition. Specifically, as  $|z| \rightarrow 1$ , the function's growth rate is controlled by the factor  $(1 - |z|)$ , ensuring that  $f(z)$  does not grow unboundedly.

- For this function, the modulus is given by

$$|f(z)| = \left| \frac{z}{1-z} \right|.$$

As  $|z|$  approaches 1, this function becomes large, but it remains bounded according to the Bloch condition.

**Definition 1.4** : The Bloch space  $B(\mathbb{D}, X)$  is defined as the set of all functions in  $H(D, X)$  such that

$$p_B(f) := \sup_{z \in D} (1 - |z|^2) \|f'(z)\| < \infty.$$

Under the norm

$$\|f\|_{B(X)} = \sup_{z \in D} (1 - |z|^2) \|f'(z)\|,$$

it becomes a Banach space.

**Proposition 1.1**  $B(\mathbb{D}, X)$  is vector space

**Proof** Closure under Addition: If  $f, g \in B(X)$ . Then  $f, g$  are holomorphic and

$$(f + g)'(z) = f'(z) + g'(z)$$

Hence,

$$\|f + g\|_{B(X)} = \sup_{z \in \mathbb{D}} (1 - |z|^2) \|f'(z) + g'(z)\|$$

Using the triangle inequality:

$$\|f + g\|_{B(X)} \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) (\|f'(z)\| + \|g'(z)\|) \leq \|f\|_{B(X)} + \|g\|_{B(X)} < \infty$$

Thus  $f + g \in B(X)$ .

Closure under Scalar Multiplication; Let  $\lambda \in \mathbb{C}$  and  $f \in B(X)$ . Then:

$$(\lambda f)'(z) = \lambda f'(z)$$

and hence:

$$\|\lambda f\|_{B(X)} = \sup_{z \in \mathbb{D}} (1 - |z|^2) \|\lambda f'(z)\| = |\lambda| \sup_{z \in \mathbb{D}} (1 - |z|^2) \|f'(z)\| = |\lambda| \|f\|_{B(X)} < \infty$$

So  $\lambda f \in B(X)$ .

## 1.7 The Bloch space $B(\mathbb{D}, X)$

The Bloch space  $B(\mathbb{D}, X)$  is the linear space of all those mappings  $f \in H(\mathbb{D}, X)$  such that

$$p_B(f) := \sup (1 - |z|^2) \|f'(z)\| : z \in \mathbb{D} < \infty.$$

equipped with the Bloch seminorm  $p_B$

To prove that  $p_B(f) := \sup (1 - |z|^2) \|f'(z)\| < \infty$  is a seminorm, we need to show that it satisfies the properties of a seminorm. A seminorm  $p$  on a vector space must satisfy the following conditions for all  $f, g \in D$  and scalar  $\alpha \in \mathbb{C}$ :

1. **Non-negativity:**  $p_B(f) \geq 0$  for all  $f \in D$ .
2. **Homogeneity:**  $p_B(\alpha f) = |\alpha| p_B(f)$  for all  $f \in D$  and  $\alpha \in \mathbb{C}$ .
3. **Subadditivity:**  $p_B(f + g) \leq p_B(f) + p_B(g)$  for all  $f, g \in D$ .

Here, we are working with functions defined on the unit disk  $D = \{z \in \mathbb{C} : |z| < 1\}$ , and  $f$  is assumed to be a function defined on  $D$ .

1. **Non-negativity:** We need to show that  $p_B(f) \geq 0$  for all functions  $f$ .

We have:

$$p_B(f) = \sup_{z \in D} ((1 - |z|^2) \|f'(z)\|).$$

- The term  $(1 - |z|^2)$  is non-negative for all  $z \in D$ , because  $|z|^2 < 1$  for all  $z \in D$ , so  $1 - |z|^2 > 0$ .

- and the norm  $\|f'(z)\|$  is also non-negative for any function  $f$ , the product  $(1 - |z|^2) \|f'(z)\|$  is non-negative for all  $z \in D$ .

Therefore, the supremum of a non-negative quantity is also non-negative, so:

$$p_B(f) \geq 0 \quad \text{for all } f.$$

2. **Homogeneity:** We need to show that  $p_B(\alpha f) = |\alpha| p_B(f)$  for any scalar  $\alpha$  and function  $f$ .

We have:

$$p_B(\alpha f) = \sup_{z \in D} ((1 - |z|^2) \|(\alpha f)'(z)\|) = \sup_{z \in D} ((1 - |z|^2) |\alpha| \|f'(z)\|).$$

Since  $|\alpha|$  is a scalar, it can be factored out of the supremum:

$$p_B(\alpha f) = |\alpha| \sup_{z \in D} ((1 - |z|^2) \|f'(z)\|) = |\alpha| p_B(f).$$

This satisfies the homogeneity condition.

3. **Sub-additivity (Triangle inequality):** We need to show that  $p_B(f + g) \leq p_B(f) + p_B(g)$  for all functions  $f, g \in D$ . We have:

$$p_B(f + g) = \sup_{z \in D} (1 - |z|^2) \|(f + g)'(z)\|.$$

By the triangle inequality for derivatives, we have:

$$\|(f + g)'(z)\| = \|f'(z) + g'(z)\| \leq \|f'(z)\| + \|g'(z)\|.$$

Therefore:

$$p_B(f + g) \leq \sup_{z \in D} (1 - |z|^2) (\|f'(z)\| + \|g'(z)\|) \leq \sup_{z \in D} (1 - |z|^2) \|f'(z)\| + \sup_{z \in D} (1 - |z|^2) \|g'(z)\|.$$

This implies:

$$p_B(f + g) \leq p_B(f) + p_B(g),$$

which shows that the seminorm is subadditive.

**Conclusion:** Since we have shown that  $p_B(f)$  satisfies non-negativity, homogeneity, and subadditivity, we conclude that  $p_B(f)$  is a seminorm.

## 1.8 The Bloch space $\hat{B}(\mathbb{D}, X)$

**Definition 1.5** The normalized Bloch space  $\hat{B}(\mathbb{D}, X)$  is the Banach space of all mappings  $f \in H(\mathbb{D}, X)$  with  $f(0) = 0$  such that

$$p_B(f) := \sup (1 - |z|^2) \|f'(z)\| < \infty,$$

endowed with the Bloch norm  $p_B$ .

# \$(p, \sigma)\$-absolutely Lipschitz mappings

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## 2.1 \$(p, \sigma)\$-absolutely continuous mappings

**Definition 2.1** [20, Definition 3.1]

Let \$1 \leq p < \infty\$ and \$0 \leq \sigma < 1\$. Recall that a linear operator \$T \in \mathcal{L}(E, F)\$ between two Banach spaces \$E\$ and \$F\$ is called \$(p, \sigma)\$-absolutely continuous if there exist a Banach space \$G\$ and a linear operator \$S \in \Pi\_p(E, G)\$ such that

$$\|T(x)\| \leq \|x\|^\sigma \|S(x)\|^{1-\sigma}, \quad x \in E. \tag{2.1}$$

Let \$\pi\_{p,\sigma}(T) = \inf \pi\_p(S)^{1-\sigma}\$, where the infimum is taken over all Banach spaces \$G\$ and \$S \in \Pi\_p(E, G)\$ such that (2.1) holds. By \$\Pi\_{p,\sigma}(E, F)\$ we denote the Banach space of all \$(p, \sigma)\$-absolutely continuous operators between \$E\$ and \$F\$.

Note that if \$T \in \Pi\_{p,\sigma}(E, F)\$ then \$\|T\| \leq \pi\_{p,\sigma}(T)\$. The class \$(\Pi\_{p,\sigma}, \pi\_{p,\sigma}(\cdot))\$ is an injective Banach operator ideal (see [20, Theorem 3.2]).

**Proposition 2.1** [8, Proposition 1] Let \$1 \leq p \leq q < \infty\$ and \$0 \leq \sigma < 1\$. Then the following holds

$$\Pi_{p,\sigma}(E, F) \subset \Pi_{q,\sigma}(E, F)$$

for all Banach spaces \$E\$ and \$F\$.

The Pietsch Domination Theorem concerning the \$(p, \sigma)\$-absolutely continuous linear operators is proved by Matter in [20]. As usual, the unit ball \$B\_{E^\*}\$, of \$E^\*\$ is considered as a compact space with respect to the weak star topology.

**Theorem 2.1** [20, Theorem 4.1]

For a linear operator \$T \in \mathcal{L}(E, F)\$ the following statements are equivalent.

(i)  $T \in \Pi_{p,\sigma}(E, F)$

(ii) There are a constant  $C > 0$  and a regular Borel probability measure  $\mu$  on  $B_{E^*}$ , such that

$$\|T(x)\| \leq C \left( \int_{B_{E^*}} (|\langle x, x^* \rangle|^{1-\sigma} \|x\|^\sigma)^{\frac{p}{1-\sigma}} d\mu(x^*) \right)^{\frac{1-\sigma}{p}}, \quad x \in E. \quad (2.2)$$

(iii) There is a constant  $C \geq 0$  such that for all  $(x_i)_{i=1}^n$  in  $E$  we have

$$\left( \sum_{i=1}^n \|T(x_i)\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \leq C \sup_{x^* \in B_{E^*}} \left( \sum_{i=1}^n (|\langle x^*, x_i \rangle|^{1-\sigma} \|x_i\|^\sigma)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}$$

In addition,  $\pi_{p,\sigma}(T)$  is the smallest number  $C$  for which (ii) and (iii) hold.

## 2.2 Lipschitz compact and weakly compact operators

A Lipschitz map  $T \in Lip_0(X, E)$  is Lipschitz compact (Lipschitz weakly compact) if the set

$$\left\{ \frac{T(x) - T(x')}{d(x, x')}, x, x' \in X, x \neq x' \right\}$$

is relatively compact (respectively, relatively weakly compact) in  $E$ . Denote by  $Lip_{0\mathcal{K}}(X, E)$  and  $Lip_{0\mathcal{W}}(X, E)$  the sets of Lipschitz compact operators and Lipschitz weakly compact operators from  $X$  to  $E$ , respectively. The relationship between the compactness (weakly compactness) of a Lipschitz operator  $T \in Lip_0(X, E)$  and the compactness (respectively, weakly compactness) of its linearization  $T_L \in \mathcal{L}(\mathcal{A}(X), E)$  has been established:

**Proposition 2.2** Let  $T \in Lip_0(X, E)$ . The following statements are equivalent:

- (1)  $T$  is compact Lipschitz.
- (2)  $T_L$  is compact from  $\mathcal{A}(X)$  to  $E$ .

The Lipschitz versions of Schauder's theorem (respectively, Gantmacher's theorem) on the compactness (respectively, weak compactness) of the adjoint of a compact (respectively, weakly compact) linear operators are also studied in terms of the Lipschitz transpose map of a Lipschitz operator.

**Proposition 2.3** (Gantmacher's theorem) Let  $T \in Lip_0(X, E)$ . The following statements are equivalent:

- (1)  $T$  is weakly compact Lipschitz.

(2)  $T^t$  is weakly compact from  $E^*$  to  $X^\#$ .

**Proposition 2.4** [?, Proposition 3.5](Schauder's theorem) Let  $T \in \text{Lip}_0(X, E)$ . The following statements are equivalent:

(1)  $T$  is Lipschitz compact .

(2)  $T^t$  is compact from  $E^*$  to  $X^\#$ .

## 2.3 Lipschitz $p$ -summing operators

The Lipschitz version of  $p$ -summing operators ( $1 \leq p < \infty$ ) was introduced by J.D. Farmer and W.B. Johnson in [14]. A mapping  $T \in \text{Lip}_0(X, Y)$  is Lipschitz  $p$ -summing if there exists a constant  $C \geq 0$  such that for all  $(x_i)_{i \leq n}, (x'_i)_{i \leq n}$  in  $X$  and all  $(a_i)_{i \leq n} \subset \mathbb{R}^+$

$$\sum_{i=1}^n a_i d(T(x_i), T(x'_i))^p \leq C^p \sup_{f \in B_{X^\#}} \sum_{i=1}^n a_i |f(x_i) - f(x'_i)|^p.$$

The infimum of all such constants  $C \geq 0$  is denoted by  $\pi_p^L(T)$ . This class of mappings is denoted by  $\Pi_p^L(X, Y)$ . Thanks to an argument detailed in [14] the scalars  $a_1, \dots, a_n$  can be removed from the definition.

The Theorem characterizing the Lipschitz  $p$ -summing operators is the following.

**Theorem 2.2** [14, Theorem 1] Let  $1 \leq p < \infty$ . The following properties are equivalent for a mapping  $T \in \text{Lip}_0(X, Y)$  and a positive constant  $C$ .

1.  $\pi_p^L(T) \leq C$ .

2. There is a probability  $\mu$  on  $B_{X^\#}$  such that

$$d(T(x), T(y)) \leq C \left( \int_{B_{X^\#}} |f(x) - f(y)|^p d\mu(f) \right)^{\frac{1}{p}}.$$

(Pietsch domination theorem).

3. For some (or any) isometric embedding  $J$  of  $Y$  into a 1-injective space  $Z$ , there is a factorization

$$\begin{array}{ccc} L_\infty(\mu) & \xrightarrow{I_{\infty,p}} & L_p(\mu) \\ A \uparrow & & \downarrow B \\ X & \xrightarrow{T} & Y \xrightarrow{J} Z \end{array}$$

with  $\mu$  a probability and  $\text{Lip}(A) \cdot \text{Lip}(B) \leq C$ .

(Pietsch factorization theorem).

The domination theorem immediately implies the monotonicity of the Lipschitz  $p$ -summing norm. Then we have the following theorem.

**Theorem 2.3** *If  $1 \leq p \leq q < \infty$ , then  $\Pi_p^L(X, Y) \subset \Pi_q^L(X, Y)$ . Moreover,  $\pi_p^L(T) \leq \pi_q^L(T)$ .*

For a linear operator  $T \in \mathcal{L}(E, F)$  it is clear that  $\pi_p^L(T) \leq \pi_p(T)$ . J.D. Farmer and W.B. Johnson proved that the reverse inequality is true. This justifies that the notion of Lipschitz  $p$ -summing operator is really a generalization of the concept of linear  $p$ -summing operator.

**Theorem 2.4** [14, Theorem 2] *Let  $T$  be a bounded linear operator from  $E$  into  $F$  and  $1 \leq p < \infty$ . Then  $\pi_p^L(T) = \pi_p(T)$*

## 2.4 $(p, \sigma)$ -absolutely Lipschitz mappings

This section based on the article of Yahi ae all. (see [28]).

**Definition 2.2** *Let  $1 \leq p < \infty$ , and  $0 \leq \sigma < 1$ . Let  $X$  be a pointed metric space and  $E$  be a Banach space. A base point preserving mapping  $T \in \text{Lip}_0(X, E)$  is called  $(p, \sigma)$ -absolutely Lipschitz if there exist a Banach space  $F$  and a Lipschitz operator  $S \in \Pi_p^L(X, F)$  such that*

$$\|T(x) - T(x')\| \leq \|S(x) - S(x')\|^{1-\sigma} d(x, x')^\sigma$$

for all  $x, x' \in X$ . Let  $\pi_{p,\sigma}^L(T)$  denote the infimum of all  $\pi_p^L(S)^{1-\sigma}$ , when  $S$  varies over all Lipschitz  $p$ -summing operators defined on  $X$  that fulfill the above condition. The space of all  $(p, \sigma)$ -absolutely Lipschitz mappings between  $X$  and  $E$  is denoted by  $\Pi_{p,\sigma}^L(X, E)$ .

An easy calculation shows that

$$\Pi_p^L \subset \Pi_{p,\sigma}^L \subset \text{Lip}_0 \quad (2.3)$$

and  $\text{Lip} \leq \pi_{p,\sigma}^L \leq \pi_p^L$  for every  $0 < \sigma < 1$ .

**Proposition 2.5** *Let  $X$  be a pointed metric space and  $E$  be a Banach space, then for  $1 \leq p < \infty$  and  $0 \leq \sigma < 1$  the space  $\Pi_{p,\sigma}^L(X, E)$  is a Banach space.*

**Proof :** *We prove the triangle inequality. Consider  $T_1, T_2 \in \Pi_{p,\sigma}^L(X, E)$ ,  $F_1, F_2$  Banach spaces and  $S_i \in \Pi_p^L(X, F_i)$ ,  $i = 1, 2$ , such that*

$$\|T_i(x) - T_i(x')\| \leq \|S_i(x) - S_i(x')\|^{1-\sigma} d(x, x')^\sigma \text{ for all } x, x' \in X.$$

Let  $F$  be the  $\ell_1$ -sum of  $F_1$  and  $F_2$  and let  $I_i : F_i \rightarrow F$  be the canonical injections. The map

$$S := \pi_p^L(S_1)^{-\sigma} I_1 \circ S_1 + \pi_p^L(S_2)^{-\sigma} I_2 \circ S_2$$

belongs to  $\Pi_p^L(X, F)$  and

$$\pi_p^L(S) \leq \pi_p^L(S_1)^{1-\sigma} + \pi_p^L(S_2)^{1-\sigma}.$$

Using Hölder's inequality we get

$$\begin{aligned} & \| (T_1 + T_2)(x) - (T_1 + T_2)(x') \| \\ & \leq \| T_1(x) - T_1(x') \| + \| T_2(x) - T_2(x') \| \\ & \leq \sum_{i=1}^2 \left\| \pi_p^L(S_i)^{-\sigma} (S_i(x) - S_i(x')) \right\|_{F_i}^{1-\sigma} \left( \pi_p^L(S_i)^{1-\sigma} \right)^\sigma d(x, x')^\sigma \\ & \leq \left( \sum_{i=1}^2 \left\| \pi_p^L(S_i)^{-\sigma} (S_i(x) - S_i(x')) \right\|_{F_i} \right)^{1-\sigma} \left( \sum_{i=1}^2 \pi_p^L(S_i)^{1-\sigma} \right)^\sigma d(x, x')^\sigma \\ & = \left( \pi_p^L(S_1)^{1-\sigma} + \pi_p^L(S_2)^{1-\sigma} \right)^\sigma \| S(x) - S(x') \|_F^{1-\sigma} d(x, x')^\sigma \end{aligned}$$

for all  $x, x' \in X$ . Thus  $T_1 + T_2 \in \Pi_{p,\sigma}^L(X, E)$  and

$$\begin{aligned} \pi_{p,\sigma}^L(T_1 + T_2) & \leq \left( \pi_p^L(S_1)^{1-\sigma} + \pi_p^L(S_2)^{1-\sigma} \right)^\sigma \pi_p^L(S)^{1-\sigma} \\ & \leq \pi_p^L(S_1)^{1-\sigma} + \pi_p^L(S_2)^{1-\sigma}. \end{aligned}$$

Taking the infimum we finally get that  $\pi_{p,\sigma}^L(T_1 + T_2) \leq \pi_{p,\sigma}^L(T_1) + \pi_{p,\sigma}^L(T_2)$ .

To prove the completeness of the space, take a sequence  $(T_n)_n$  in  $\Pi_{p,\sigma}^L(X, E)$  such that  $\sum_{n=1}^{\infty} \pi_{p,\sigma}^L(T_n) < \infty$ . Since  $\text{Lip} \leq \pi_{p,\sigma}^L$  and  $(\text{Lip}_0(X, E), \text{Lip})$  is a Banach space, there exists  $T := \sum_{n=1}^{\infty} T_n \in \text{Lip}_0(X, E)$ . We prove that  $\sum_{n=1}^{\infty} T_n = T$  for  $\pi_{p,\sigma}^L$ . Let  $\epsilon > 0$  and, for each  $n \in \mathbb{N}$ , let  $S_n \in \Pi_p^L(X, F_n)$  be such that

$$\|T_n(x) - T_n(x')\| \leq \|S_n(x) - S_n(x')\|^{1-\sigma} d(x, x')^\sigma$$

for all  $x, x' \in X$  and  $\pi_p^L(S_n)^{1-\sigma} \leq \pi_{p,\sigma}^L(T_n) + \epsilon/2^n$ . Then

$$\left( \sum_{n=1}^{\infty} \pi_p^L(S_n) \right)^{1-\sigma} \leq \sum_{n=1}^{\infty} \pi_p^L(S_n)^{1-\sigma} \leq \sum_{n=1}^{\infty} \pi_{p,\sigma}^L(T_n) + \epsilon < \infty.$$

Let  $S = \sum_{n=1}^{\infty} \pi_p^L(S_n)^{-\sigma} (I_n \circ S_n) \in \Pi_p^L(X, F)$ , where  $F$  is the  $\ell_1$ -sum of all  $F_n$  and  $I_n : F_n \rightarrow F$  is the natural inclusion. Hence,

$$\begin{aligned} \|T(x) - T(x')\| &\leq \sum_{n=1}^{\infty} \|T_n(x) - T_n(x')\| \\ &\leq \sum_{n=1}^{\infty} \|S_n(x) - S_n(x')\|_{F_n}^{1-\sigma} d(x, x')^\sigma \\ &\leq \|S(x) - S(x')\|_F^{1-\sigma} \left( \sum_{n=1}^{\infty} \pi_p^L(S_n)^{1-\sigma} \right)^\sigma d(x, x')^\sigma. \end{aligned}$$

This implies  $T \in \Pi_{p,\sigma}^L(X, E)$  and

$$\pi_{p,\sigma}^L(T) \leq \sum_{n=1}^{\infty} \pi_p^L(S_n)^{1-\sigma} \leq \sum_{n=1}^{\infty} \pi_{p,\sigma}^L(T_n) + \epsilon.$$

We have

$$\pi_{p,\sigma}^L\left(T - \sum_{k=1}^n T_k\right) = \pi_{p,\sigma}^L\left(\sum_{k=n+1}^{\infty} T_k\right) \leq \sum_{k=n+1}^{\infty} \pi_p^L(S_k)^{1-\sigma}.$$

Thus,  $\sum_{n=1}^{\infty} T_n = T$  for  $\pi_{p,\sigma}^L$ .

Note that  $\Pi_{p,0}^L = \Pi_p^L$  and  $\pi_{p,0}^L = \pi_p^L$ . Therefore, the class  $\Pi_{p,\sigma}^L$  for  $0 \leq \sigma < 1$  can be considered as an interpolating class between  $\Pi_p^L$  and  $\text{Lip}_0$ .

The next result is an extension of Domination Theorem 3.2, for  $(p, \sigma)$ -absolutely Lipschitz map-

pings. To prove the domination part, we will use a alternative technique: the unified abstract version of the Pietsch domination theorem given in [11, Theorem 2.2] (see also [22]).

**Theorem 2.5** Let  $1 \leq p < \infty$ ,  $0 \leq \sigma < 1$  and  $T \in \text{Lip}_0(X, E)$ . The following statements are equivalent.

1.  $T \in \Pi_{p, \sigma}^L(X, E)$ .
2. There is a constant  $C \geq 0$  and a regular Borel probability measure  $\mu$  on  $B_{X^\#}$  such that

$$\|T(x) - T(x')\| \leq C \left( \int_{B_{X^\#}} (|f(x) - f(x')|^{1-\sigma} d(x, x')^\sigma)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}}$$

for all  $x, x' \in X$ .

3. There is a constant  $C \geq 0$  such that for all  $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$  in  $X$  and all  $(a_i)_{i=1}^n \subset \mathbb{R}^+$  we have

$$\begin{aligned} & \left( \sum_{i=1}^n a_i \|T(x_i) - T(x'_i)\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\ & \leq C \sup_{f \in B_{X^\#}} \left( \sum_{i=1}^n a_i (|f(x_i) - f(x'_i)|^{1-\sigma} d(x_i, x'_i)^\sigma)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}. \end{aligned}$$

Furthermore, the infimum of the constants  $C \geq 0$  in (2) and (3) is  $\pi_{p, \sigma}^L(T)$ .

**Proof :** (1) $\Rightarrow$ (2) If  $T \in \Pi_{p, \sigma}^L(X, E)$ , then there exist a Banach space  $F$  and  $S \in \Pi_p^L(X, F)$  such that

$$\|T(x) - T(x')\| \leq \|S(x) - S(x')\|^{1-\sigma} d(x, x')^\sigma$$

for all  $x, x' \in X$ . By [14, Theorem 1], since  $S$  is Lipschitz  $p$ -summing then there exists a regular Borel probability measure  $\mu$  on  $B_{X^\#}$  such that

$$\begin{aligned} \|T(x) - T(x')\| & \leq \|S(x) - S(x')\|^{1-\sigma} d(x, x')^\sigma \\ & \leq \pi_p^L(S)^{1-\sigma} \left( \int_{B_{X^\#}} |f(x) - f(x')|^p d\mu(f) \right)^{\frac{1-\sigma}{p}} d(x, x')^\sigma \\ & = \pi_p^L(S)^{1-\sigma} \left( \int_{B_{X^\#}} (|f(x) - f(x')|^{1-\sigma} d(x, x')^\sigma)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}} \end{aligned}$$

for all  $x, x' \in X$ .

(2) $\Rightarrow$ (1) Let  $A$  be the natural isometric embedding from  $X$  into  $C(B_{X^\#})$  composed with the formal identity from  $C(B_{X^\#})$  into  $L_\infty(\mu)$  given by  $A(x)(f) = f(x)$ ,  $x \in X, f \in B_{X^\#}$ . Let  $I_{\infty,p} : L_\infty(\mu) \rightarrow L_p(\mu)$  be the canonical mapping  $I_{\infty,p}(g) = g$ . Note that  $\pi_p^L(I_{\infty,p}) = 1$ . Therefore by (2)

$$\begin{aligned} \|T(x) - T(x')\| &\leq C \left( \int_{B_{X^\#}} (|f(x) - f(x')|^{1-\sigma} d(x, x')^\sigma)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}} \\ &= \left( \int_{B_{X^\#}} |C^{\frac{1}{1-\sigma}} I_{\infty,p} A(x)(f) - C^{\frac{1}{1-\sigma}} I_{\infty,p} A(x')(f)|^p d\mu(f) \right)^{\frac{1-\sigma}{p}} d(x, x')^\sigma. \end{aligned}$$

Consequently there is a Banach space  $F = L_p(\mu)$  and a Lipschitz  $p$ -summing operator  $S = C^{\frac{1}{1-\sigma}} I_{\infty,p} A$  such that

$$\|T(x) - T(x')\| \leq \|S(x) - S(x')\|^{1-\sigma} d(x, x')^\sigma,$$

as required.

(2) $\Rightarrow$ (3) If

$$\|T(x) - T(x')\| \leq C \left( \int_{B_{X^\#}} (|f(x) - f(x')|^{1-\sigma} d(x, x')^\sigma)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}},$$

for all  $x, x' \in X$  then, for  $n \in \mathbb{N}$ ,  $a_1, \dots, a_n \in \mathbb{R}^+$  and  $x_1, \dots, x_n, x'_1, \dots, x'_n \in X$ , we have

$$\begin{aligned} &\sum_{i=1}^n a_i \|T(x_i) - T(x'_i)\|^{\frac{p}{1-\sigma}} \\ &\leq C^{\frac{p}{1-\sigma}} \sum_{i=1}^n \int_{B_{X^\#}} a_i (|f(x_i) - f(x'_i)|^{1-\sigma} d(x_i, x'_i)^\sigma)^{\frac{p}{1-\sigma}} d\mu(f) \\ &= C^{\frac{p}{1-\sigma}} \int_{B_{X^\#}} \sum_{i=1}^n a_i (|f(x_i) - f(x'_i)|^{1-\sigma} d(x_i, x'_i)^\sigma)^{\frac{p}{1-\sigma}} d\mu(f) \\ &\leq C^{\frac{p}{1-\sigma}} \sup_{f \in B_{X^\#}} \sum_{i=1}^n a_i (|f(x_i) - f(x'_i)|^{1-\sigma} d(x_i, x'_i)^\sigma)^{\frac{p}{1-\sigma}}. \end{aligned}$$

(3) $\Rightarrow$ (2) We will use the unified abstract version of the Pietsch domination theorem given in [11, Theorem 2.2].

Let  $R : B_{X^\#} \times (X \times X \times \mathbb{R}) \times \mathbb{R} \rightarrow [0, \infty[$  be given by

$$R(f, (x, x', a), \lambda) = |a|^{\frac{1-\sigma}{p}} |f(x) - f(x')|^{1-\sigma} d(x, x')^\sigma |\lambda|$$

and let  $S : \text{Lip}_0(X, E) \times (X \times X \times \mathbb{R}) \times \mathbb{R} \rightarrow [0, \infty[$  be given by

$$S(T, (x, x', a), \lambda) = |a|^{\frac{1-\sigma}{p}} \|T(x) - T(x')\| |\lambda|.$$

Then  $T$  is  $R$ - $S$ -abstract  $p/(1-\sigma)$ -summing (see [11, Definition 2.1]):

$$\begin{aligned} & \left( \sum_{i=1}^n S(T, (x_i, x'_i, a_i), \lambda_i)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\ &= \left( \sum_{i=1}^n |a_i| |\lambda_i|^{\frac{p}{1-\sigma}} \|T(x_i) - T(x'_i)\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\ &\leq C \sup_{f \in B_{X^\#}} \left( \sum_{i=1}^n |a_i| |\lambda_i|^{\frac{p}{1-\sigma}} (|f(x_i) - f(x'_i)|^{1-\sigma} d(x_i, x'_i)^\sigma)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\ &= C \sup_{f \in B_{X^\#}} \left( \sum_{i=1}^n R(f, (x_i, x'_i, a_i), \lambda_i)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}. \end{aligned}$$

Then, by [11, Theorem 2.2] there is a constant  $C \geq 0$  and a regular Borel probability measure  $\mu$  on  $B_{X^\#}$  such that

$$S(T, (x, x', a), \lambda) \leq C \left( \int_{B_{X^\#}} R(f, (x, x', a), \lambda)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}}$$

for all  $(x, x', a) \in X \times X \times \mathbb{R}$  and  $\lambda \in \mathbb{R}$ . In particular, we have

$$\|T(x) - T(x')\| \leq C \left( \int_{B_{X^\#}} (|f(x) - f(x')|^{1-\sigma} d(x, x')^\sigma)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}}$$

for all  $x, x' \in X$ .

**Remark 2.1** The notion of  $(p, \sigma)$ -absolutely Lipschitz mapping can be defined for Lipschitz mappings between pointed metric spaces. Given pointed metric spaces  $X$  and  $Y$ , a map  $T \in \text{Lip}_0(X, Y)$  is called  $(p, \sigma)$ -absolutely Lipschitz, if there exist a constant  $k \geq 0$ , a pointed metric space  $G$  and

a Lipschitz operator  $S \in \Pi_p^L(X, G)$  such that

$$d(T(x), T(x')) \leq kd(S(x), S(x'))^{1-\sigma} d(x, x')^\sigma$$

for all  $x, x' \in X$ . In this case  $\pi_{p,\sigma}^L(T)$  denotes the infimum of all  $k\pi_p^L(S)^{1-\sigma}$ . Theorem 2.5 can be easily adapted whenever  $T \in \text{Lip}_0(X, Y)$  and  $X$  and  $Y$  are pointed metric spaces.

**Proposition 2.6 (Ideal property)** Let  $X, Y, X_0, Y_0$  be pointed metric spaces. If  $v : X_0 \rightarrow X$ ,  $w : Y \rightarrow Y_0$  are Lipschitz mappings and  $T : X \rightarrow Y$  is  $(p, \sigma)$ -absolutely Lipschitz then,  $wTv$  is  $(p, \sigma)$ -absolutely Lipschitz and

$$\pi_{p,\sigma}^L(wTv) \leq \text{Lip}(w)\text{Lip}(v)\pi_{p,\sigma}^L(T).$$

**Proof:** Since  $T$  is  $(p, \sigma)$ -absolutely Lipschitz then there exist a constant  $k \geq 0$ , a pointed metric space  $G$  and a Lipschitz operator  $S \in \Pi_p^L(X, G)$  such that

$$d(T(x), T(x')) \leq kd(S(x), S(x'))^{1-\sigma} d(x, x')^\sigma \text{ for all } x, x' \in X.$$

Let  $x_0, x'_0 \in X_0$ , then

$$\begin{aligned} d(wTv(x_0), wTv(x'_0)) &\leq \text{Lip}(w)d(Tv(x_0), Tv(x'_0)) \\ &\leq k\text{Lip}(w)d(S \circ v(x_0), S \circ v(x'_0))^{1-\sigma} d(v(x_0), v(x'_0))^\sigma \\ &\leq k\text{Lip}(w)\text{Lip}(v)^\sigma d(S \circ v(x_0), S \circ v(x'_0))^{1-\sigma} d(x_0, x'_0)^\sigma. \end{aligned}$$

Since  $S \circ v \in \Pi_p^L(X_0, G)$  it follows that  $w \circ T \circ v \in \Pi_{p,\sigma}^L(X_0, Y_0)$  and

$$\begin{aligned} \pi_{p,\sigma}^L(wTv) &\leq k\text{Lip}(w)\text{Lip}(v)^\sigma \pi_p^L(S \circ v)^{1-\sigma} \\ &\leq k\text{Lip}(w)\text{Lip}(v)\pi_p^L(S)^{1-\sigma}. \end{aligned}$$

Taking infimum we get

$$\pi_{p,\sigma}^L(wTv) \leq \text{Lip}(w)\text{Lip}(v)\pi_{p,\sigma}^L(T).$$

**Remark 2.2** Using (2.3), Proposition 2.5 and Proposition 2.6 it can be shown that all  $(p, \sigma)$ -absolutely Lipschitz mappings form a Lipschitz operator ideal .

## 2.5 Factorization Theorem

Let  $\mu$  be a Borel probability measure on  $B_{X^\#}$ . Consider the canonical inclusion  $i : \mathcal{A}(X) \longrightarrow C(B_{X^\#})$ , given by  $i(\sum_{j=1}^n \lambda_j m_{x_j, x'_j}) := \sum_{j=1}^n \lambda_j \langle m_{x_j, x'_j}, \cdot \rangle$ . On  $i(\mathcal{A}(X))$ , we define the semi-norm

$$\|i(m)\|_{p, \sigma} := \inf \left\{ \sum_{j=1}^n |\lambda_j| d(x_j, x'_j)^\sigma \left( \int_{B_{X^\#}} |f(x_j) - f(x'_j)|^p d\mu(f) \right)^{\frac{1-\sigma}{p}} \right\}$$

where the infimum is taken over all representations of  $m$  of the form  $m = \sum_{j=1}^n \lambda_j m_{x_j, x'_j}$ . Consider on  $i \circ \delta_X(X)$  the pseudo-metric induced by  $\|\cdot\|_{p, \sigma}$  :

$$d_{p, \sigma}(i \circ \delta_X(x), i \circ \delta_X(x')) := \|i \circ \delta_X(x) - i \circ \delta_X(x')\|_{p, \sigma}$$

and the relation of equivalence  $\mathcal{R}$  given by

$$i \circ \delta_X(x) \mathcal{R} i \circ \delta_X(x') \Leftrightarrow d_{p, \sigma}(i \circ \delta_X(x), i \circ \delta_X(x')) = 0.$$

We set  $X_{p, \sigma}^\mu := \frac{i \circ \delta_X(X)}{\mathcal{R}}$  and let  $q : i \circ \delta_X(X) \longrightarrow X_{p, \sigma}^\mu$  be the projection.

Note that, if we consider the canonical map  $j_p : C(B_{X^\#}) \longrightarrow L_p(\mu)$ , then  $i \circ \delta_X(x) \mathcal{R} i \circ \delta_X(x')$  if and only if  $j_p(i \circ \delta_X(x)) = j_p(i \circ \delta_X(x'))$ . Hence  $X_{p, \sigma}^\mu$  can be seen as a subset of  $L_p(\mu)$  via the formal identity  $I$ .

**Theorem 2.6** Let  $1 \leq p < \infty$  and  $0 \leq \sigma < 1$ . Let  $X$  and  $Y$  be pointed metric spaces and  $T \in \text{Lip}_0(X, Y)$ . The following statements are equivalent.

1.  $T \in \Pi_{p, \sigma}^L(X, Y)$ .
2. There exist a regular Borel probability measure  $\mu$  on  $B_{X^\#}$  and a Lipschitz operator  $v : X_{p, \sigma}^\mu \rightarrow$

$Y$  such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow \delta_X & & \uparrow v \\ \delta_X(X) & \xrightarrow{q \circ i} & X_{p,\sigma}^\mu \end{array}$$

(1) $\Rightarrow$ (2) Assume first that  $T \in \Pi_{p,\sigma}^L(X, Y)$ . By Theorem 2.5 and Remark 2.1 there is a regular Borel probability measure  $\mu$  on  $B_{X^\#}$  such that

$$d(T(x), T(x')) \leq \pi_{p,\sigma}^L(T) \left( \int_{B_{X^\#}} (|f(x) - f(x')|^{1-\sigma} d(x, x')^\sigma)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}}$$

for all  $x, x' \in X$ . Define  $v(q \circ i \circ \delta_X(x)) := T(x)$ ,  $x \in X$ . If  $x, x' \in X$  are so that  $q(i \circ \delta_X(x)) = q(i \circ \delta_X(x'))$  then  $0 = d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x')) = \|\langle m_{xx'}, \cdot \rangle\|_{p,\sigma}$ . Therefore, given  $\epsilon > 0$ , there exists a representation of  $m_{xx'}$ ,  $m_{xx'} = \sum_{j=1}^n \lambda_j m_{x_j x'_j}$  such that

$$\sum_{j=1}^n |\lambda_j| d(x_j, x'_j)^\sigma \left( \int_{B_{X^\#}} |f(x_j) - f(x'_j)|^p d\mu \right)^{\frac{1-\sigma}{p}} < \epsilon.$$

Let  $g \in B_{Y^\#}$ . Then,

$$\begin{aligned} |g(T(x)) - g(T(x'))| &= |\langle m_{xx'}, g \circ T \rangle| \\ &\leq \sum_{j=1}^n |\lambda_j| |\langle m_{x_j x'_j}, g \circ T \rangle| \leq \sum_{j=1}^n |\lambda_j| d(T(x_j), T(x'_j)) \\ &\leq \pi_{p,\sigma}^L(T) \sum_{j=1}^n |\lambda_j| d(x_j, x'_j)^\sigma \left( \int_{B_{X^\#}} |f(x_j) - f(x'_j)|^p d\mu \right)^{\frac{1-\sigma}{p}} < \epsilon \pi_{p,\sigma}^L(T). \end{aligned}$$

Letting  $\epsilon \rightarrow 0$  it follows that  $g(T(x)) - g(T(x')) = 0$  for all  $g \in B_{Y^\#}$ . Hence  $T(x) = T(x')$ .

This proves that  $v$  is well defined.

We now show that  $v$  is Lipschitz. Take  $g \in B_{Y^\#}$  and let  $m_{xx'} = \sum_{j=1}^n \lambda_j m_{x_j x'_j}$ . Then, by Proposition 2.6

$$\begin{aligned} &|g \circ v(q \circ i \circ \delta_X(x)) - g \circ v(q \circ i \circ \delta_X(x'))| \\ &= |g \circ T(x) - g \circ T(x')| = |\langle m_{xx'}, g \circ T \rangle| \end{aligned}$$

$$\leq \pi_{p,\sigma}^L(T) \sum_{j=1}^n |\lambda_j| d(x_j, x'_j)^\sigma \left( \int_{B_{X^\#}} |f(x_j) - f(x'_j)|^p d\mu \right)^{\frac{1-\sigma}{p}}.$$

Taking the infimum over all representations of  $m_{xx'}$ , we get

$$|g \circ v(q \circ i \circ \delta_X(x)) - g \circ v(q \circ i \circ \delta_X(x'))| \leq \pi_{p,\sigma}^L(T) d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x'))$$

for all  $g \in B_{Y^\#}$ . We conclude now that

$$d(v(q \circ i \circ \delta_X(x)), v(q \circ i \circ \delta_X(x'))) \leq \pi_{p,\sigma}^L(T) d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x')).$$

(2) $\Rightarrow$ (1) Assume that  $T$  factors as in (2). By Proposition 2.6 it suffices to prove that  $q \circ i : \delta_X(X) \rightarrow X_{p,\sigma}^\mu$  is  $(p, \sigma)$ -absolutely Lipschitz, but this is clear as

$$d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x')) = \|i(m_{xx'})\|_{p,\sigma} \leq \|m_{xx'}\|^\sigma \left( \int_{B_{X^\#}} |f(x) - f(x')|^p d\mu \right)^{\frac{1-\sigma}{p}}.$$

Farmer and Johnson [14, Theorem 1] proved that  $\pi_p^L(T) \leq C$  if and only if for some (or any) isometric embedding  $J$  of  $Y$  into a 1-injective space  $Z$ , there is a factorization

$$\begin{array}{ccc} L_\infty(\mu) & \xrightarrow{I_{\infty,p}} & L_p(\mu) \\ A \uparrow & & \downarrow B \\ X & \xrightarrow{T} & Y \xrightarrow{J} Z \end{array}$$

with  $\mu$  a probability and  $\text{Lip}(A) \cdot \text{Lip}(B) \leq C$ .

Letting  $\sigma = 0$  in Theorem 2.6, we obtain a factorization theorem for Lipschitz absolutely  $p$ -summing operators which is equivalent to the above. In that case,  $X_{p,0}^\mu = j_p \circ i \circ \delta_X(X)$ , where  $j_p : C(B_{X^\#}) \rightarrow L_p(\mu)$  is the canonical mapping, and the induced metric  $d_{p,0}$  generates the  $L_p$ -norm on  $X_{p,0}^\mu$ . So, Theorem 2.6 is a generalization of the Farmer and Johnson factorization.

**Theorem 2.7** Let  $1 \leq p < \infty$ . Let  $X$  and  $Y$  be pointed metric spaces. The following statements are equivalent for a mapping  $T \in \text{Lip}_0(X, Y)$  and a positive constant  $C$ .

1.  $T \in \Pi_p^L(X, Y)$  and  $\pi_p^L(T) \leq C$ .

2. There exist a regular Borel probability measure  $\mu$  on  $B_{X^\#}$  such that

$$d(T(x), T(x')) \leq C \left( \int_{B_{X^\#}} |\langle \delta_X(x) - \delta_X(x'), f \rangle|^p d\mu(f) \right)^{1/p}$$

for all  $x, x' \in X$ .

3. There exist a regular Borel probability measure  $\mu$  on  $B_{X^\#}$  and a Lipschitz operator  $v : X_{p,0}^\mu \rightarrow Y$  such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow i \circ \delta_X & & \uparrow v \\ i \circ \delta_X(X) & \xrightarrow{j_p} & X_{p,0}^\mu \end{array}$$

Furthermore, the infimum of the constants  $C \geq 0$  in (1) and (2) is  $\pi_p^L(T)$ .

Let us end showing the duality for  $(p, \sigma)$ -absolutely Lipschitz operators.

Let  $1 \leq p, r < \infty$  and  $0 \leq \sigma < 1$  such that  $r' = \frac{p'}{1-\sigma}$ , where  $p'$  is the conjugate of  $p$ , that is,  $\frac{1}{p} + \frac{1}{p'} = 1$ . For  $x_1, \dots, x_n, x'_1, \dots, x'_n$  in  $X$  and scalars  $\lambda_1, \dots, \lambda_n$ , we define

$$\delta_{p,\sigma}^{Lip} \left( (\lambda_j, x_j, x'_j)_{j=1}^n \right) := \sup_{f \in B_{X^\#}} \left( \sum_{j=1}^n \left( |\lambda_j| |f(x_j) - f(x'_j)|^{1-\sigma} d(x_j, x'_j)^\sigma \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.$$

If we denote

$$w_{\frac{p}{1-\sigma}}^{Lip} \left( (\lambda_j, x_j, x'_j)_{j=1}^n \right) := \sup_{f \in B_{X^\#}} \left( \sum_{j=1}^n \left( |\lambda_j| |f(x_j) - f(x'_j)| \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.$$

then, we have

$$w_{\frac{p}{1-\sigma}}^{Lip} \left( (\lambda_j, x_j, x'_j)_{j=1}^n \right) \leq \delta_{p,\sigma}^{Lip} \left( (\lambda_j, x_j, x'_j)_{j=1}^n \right).$$

As a remark, the above inequality shows that  $\Pi_{p/(1-\sigma)}^L(X, Y) \subset \Pi_{p,\sigma}^L(X, Y)$ .

For a molecule  $m \in \mathcal{M}(X, E)$  we define its  $(p, \sigma)$ -Chevet-Saphar norm by

$$cs_{p,\sigma}(m) = \inf \left\{ \left\| (\lambda_j \|v_j\|)_{j=1}^n \right\|_{r}^{\delta_{p',\sigma}^{Lip}} \left( (\lambda_j^{-1}, x_j, x'_j)_{j=1}^n \right) : m = \sum_{j=1}^n v_j m_{x_j x'_j}, \lambda_j > 0 \right\}.$$

We denote by  $CS_{p,\sigma}(X, E)$  the space  $\mathcal{M}(X, E)$  endowed with the norm  $cs_{p,\sigma}$ .

# p-summing Bloch mapping on the complex unit disc.

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## 3.1 p-Summing Bloch Mappings on the Unit Disc

**Definition 3.1** For any  $1 \leq p \leq \infty$ , we say that a mapping  $f \in H(\mathbb{D}, X)$  is  $p$ -summing Bloch if there is a constant  $c \geq 0$  such that for any  $n \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  and  $z_1, \dots, z_n \in \mathbb{D}$ , we have:

$$\left( \sum_{i=1}^n |\lambda_i|^p \|f'(z_i)\|^p \right)^{1/p} \leq c \sup_{g \in \mathcal{B}_{\mathbb{B}(\mathbb{D})}} \left( \sum_{i=1}^n |\lambda_i|^p \|g'(z_i)\|^p \right)^{1/p} \quad \text{if } 1 \leq p < \infty,$$

$$\max_{1 \leq i \leq n} |\lambda_i| \|f'(z_i)\| \leq c \sup_{g \in \mathcal{B}_{\mathbb{B}(\mathbb{D})}} \max_{1 \leq i \leq n} |\lambda_i| \|g'(z_i)\| \quad \text{if } p = \infty.$$

This section gathers the most important properties of  $p$ -summing Bloch mappings on  $\mathbb{D}$ . From now on, unless otherwise stated,  $X$  will denote a complex Banach space.

**Proposition 3.1** Let  $1 \leq p < q \leq \infty$ . Then

$$\pi_p^{\mathcal{B}}(\mathbb{D}, X) \subseteq \pi_q^{\mathcal{B}}(\mathbb{D}, X) \text{ with } \pi_q^{\mathcal{B}}(f) \leq \pi_p^{\mathcal{B}}(f) \text{ for all } f \in \pi_p^{\mathcal{B}}(\mathbb{D}, X).$$

Moreover,  $\pi_{\infty}^{\mathcal{B}}(\mathbb{D}, X) = B(\mathbb{D}, X)$  with  $\pi_{\infty}^{\mathcal{B}}(f) = p_{\mathcal{B}}(f)$  for all  $f \in \pi_{\infty}^{\mathcal{B}}(\mathbb{D}, X)$ .

**Proof :** Let  $n \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ , and  $z_1, \dots, z_n \in \mathbb{D}$ . We will first prove the second assertion. Let  $f \in \pi_\infty^{\mathcal{B}}(\mathbb{D}, X)$ . For all  $z \in \mathbb{D}$ , we have

$$(1 - |z|^2) \|f'(z)\| \leq \pi_\infty^{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathbb{B}}}(\mathbb{D})} (1 - |z|^2) |g'(z)| = \pi_\infty^{\mathcal{B}}(f),$$

hence  $f \in \mathcal{B}(\mathbb{D}, X)$  with  $p_{\mathcal{B}}(f) \leq \pi_\infty^{\mathcal{B}}(f)$ . Conversely, let  $f \in \mathcal{B}(\mathbb{D}, X)$ . For  $i = 1, \dots, n$ , we have

$$|\lambda_i| \|f'(z_i)\| \leq \frac{|\lambda_i|}{1 - |z_i|^2} p_{\mathcal{B}}(f) = |\lambda_i| |f'_{z_i}(z_i)| p_{\mathcal{B}}(f) \leq p_{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathbb{B}}}(\mathbb{D})} |\lambda_i| |g'(z_i)|,$$

this implies that

$$\max_{1 \leq i \leq n} |\lambda_i| \|f'(z_i)\| \leq p_{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathbb{B}}}(\mathbb{D})} \max_{1 \leq i \leq n} |\lambda_i| |g'(z_i)|,$$

and thus  $f \in \Pi_\infty^{\mathcal{B}}(\mathbb{D}, X)$  with  $\pi_\infty^{\mathcal{B}}(f) \leq p_{\mathcal{B}}(f)$ .

To prove the first assertion, let  $f \in \Pi_p^{\mathcal{B}}(\mathbb{D}, X)$ . Assume  $q < \infty$ . Taking  $\beta_i = |\lambda_i|^{q/p} \|f'(z_i)\|^{(q/p)-1}$  for  $i = 1, \dots, n$ , we have

$$\left( \sum_{i=1}^n |\lambda_i|^q \|f'(z_i)\|^q \right)^{1/p} = \left( \sum_{i=1}^n |\beta_i|^p \|f'(z_i)\|^p \right)^{1/p} \leq \pi_p^{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathbb{B}}}(\mathbb{D})} \left( \sum_{i=1}^n |\beta_i|^p |g'(z_i)|^p \right)^{1/p}.$$

Since  $q/p > 1$  and  $(q/p)^* = q/(q-p)$ , Hölder Inequality yields

$$\begin{aligned} \sup_{g \in \mathcal{B}_{\hat{\mathbb{B}}}(\mathbb{D})} \left( \sum_{i=1}^n |\beta_i|^p |g'(z_i)|^p \right)^{\frac{1}{p}} &= \sup_{g \in \mathcal{B}_{\hat{\mathbb{B}}}(\mathbb{D})} \left( \sum_{i=1}^n (|\lambda_i| \|f'(z_i)\|)^{q-p} (|\lambda_i| |g'(z_i)|)^p \right)^{\frac{1}{p}} \\ &\leq \left( \sum_{i=1}^n |\lambda_i|^q \|f'(z_i)\|^q \right)^{\frac{1}{p} - \frac{1}{q}} \sup_{g \in \mathcal{B}_{\hat{\mathbb{B}}}(\mathbb{D})} \left( \sum_{i=1}^n |\lambda_i|^q |g'(z_i)|^q \right)^{1/q}, \end{aligned}$$

and thus we obtain

$$\left( \sum_{i=1}^n |\lambda_i|^q \|f'(z_i)\|^q \right)^{1/q} \leq \pi_p^{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathbb{B}}}(\mathbb{D})} \left( \sum_{i=1}^n |\lambda_i|^q |g'(z_i)|^q \right)^{1/q}.$$

This shows that  $f \in \Pi_q^{\mathcal{B}}(\mathbb{D}, X)$  with  $\pi_q^{\mathcal{B}}(f) \leq \pi_p^{\mathcal{B}}(f)$  if  $q < \infty$ . For the case  $q = \infty$ , note that

$$(1 - |z|^2)\|f'(z)\| \leq \pi_p^{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\mathcal{B}}(\mathbb{D})} (1 - |z|^2)|g'(z)| = \pi_p^{\mathcal{B}}(f)$$

for all  $z \in \mathbb{D}$ , and thus  $f \in \mathcal{B}(\mathbb{D}, X) = \Pi_{\infty}^{\mathcal{B}}(\mathbb{D}, X)$  with  $\pi_{\infty}^{\mathcal{B}}(f) = p_{\mathcal{B}}(f) \leq \pi_p^{\mathcal{B}}(f)$ .

## 3.2 Injective Banach Ideal Property

**Definition 3.2** Let us recall (see [1, Definition 5.11]) that a normalized Bloch ideal is a subclass  $\mathcal{I}^{\hat{\mathcal{B}}}$  of the class of all normalized Bloch mappings  $\hat{\mathcal{B}}$  such that for every complex Banach space  $X$ , the components

$$\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X) := \mathcal{I}^{\hat{\mathcal{B}}} \cap \hat{\mathcal{B}}(\mathbb{D}, X),$$

satisfy the following properties:

- (I1)  $\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  is a linear subspace of  $\hat{\mathcal{B}}(\mathbb{D}, X)$ ,
- (I2) For every  $g \in \hat{\mathcal{B}}(\mathbb{D})$  and  $x \in X$ , the mapping  $g \cdot x : z \mapsto g(z)x$  from  $\mathbb{D}$  to  $X$  is in  $\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ ,
- (I3) The ideal property: If  $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ ,  $h : \mathbb{D} \rightarrow \mathbb{D}$  is a holomorphic function with  $h(0) = 0$ , and  $T \in L(X, Y)$  where  $Y$  is a complex Banach space, then  $T \circ f \circ h$  belongs to  $\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, Y)$ .

A normalized Bloch ideal  $\mathcal{I}^{\hat{\mathcal{B}}}$  is said to be normed (Banach) if there is a function  $\|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}} : \mathcal{I}^{\hat{\mathcal{B}}} \rightarrow \mathbb{R}_0^+$  such that for every complex Banach space  $X$ , the following three conditions are satisfied:

- (N1)  $(\mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X), \|\cdot\|_{\mathcal{I}^{\hat{\mathcal{B}}}})$  is a normed (Banach) space with  $p_{\mathcal{B}}(f) \leq \|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$  for all  $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ ,
- (N2)  $\|g \cdot x\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = p_{\mathcal{B}}(g)\|x\|$  for all  $g \in \hat{\mathcal{B}}(\mathbb{D})$  and  $x \in X$ ,
- (N3) If  $h : \mathbb{D} \rightarrow \mathbb{D}$  is a holomorphic function with  $h(0) = 0$ ,  $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  and  $T \in L(X, Y)$  where  $Y$  is a complex Banach space, then

$$\|T \circ f \circ h\|_{\mathcal{I}^{\hat{\mathcal{B}}}} \leq \|T\| \|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}.$$

A normed normalized Bloch ideal  $[\mathcal{I}_{\hat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}_{\hat{\mathcal{B}}}}]$  is said to be:

(I) Injective if for any mapping  $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$ , any complex Banach space  $Y$  and any isometric linear embedding  $\iota : X \rightarrow Y$ , we have that  $f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  with  $\|f\|_{\mathcal{I}^{\hat{\mathcal{B}}}} = \|\iota \circ f\|_{\mathcal{I}^{\hat{\mathcal{B}}}}$  whenever  $\iota \circ f \in \mathcal{I}^{\hat{\mathcal{B}}}(\mathbb{D}, Y)$ .

We are now ready to establish the following result which can be compared to Satz 2 [2, 1-4]:

**Proposition 3.2**  $[\Pi_p^{\hat{\mathcal{B}}}, \pi_p^{\hat{\mathcal{B}}}]$  is an injective Banach normalized Bloch ideal for any  $1 \leq p \leq \infty$ .

**Proof** : Note that  $\Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X) \subseteq \hat{\mathcal{B}}(\mathbb{D}, X)$  with  $p_{\mathcal{B}}(f) \leq \pi_p^{\hat{\mathcal{B}}}(f)$  for all  $f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  by Proposition 3.1.

We will only prove the case  $1 < p < \infty$ . The cases  $p = 1$  and  $p = \infty$  follow similarly.

Let  $n \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ , and  $z_1, \dots, z_n \in \mathbb{D}$ .

(N1) If  $f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  and  $\pi_p^{\hat{\mathcal{B}}}(f) = 0$ , then  $p_{\mathcal{B}}(f) = 0$ , and so  $f = 0$ .

Given  $f_1, f_2 \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ , we have

$$\begin{aligned} \left( \sum_{i=1}^n |\lambda_i|^p \|(f_1 + f_2)'(z_i)\| \right)^{\frac{1}{p}} &\leq \left( \sum_{i=1}^n |\lambda_i|^p (\|f_1'(z_i)\|^p + \|f_2'(z_i)\|^p) \right)^{\frac{1}{p}} \\ &\leq \left( \sum_{i=1}^n |\lambda_i|^p \|f_1'(z_i)\|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |\lambda_i|^p \|f_2'(z_i)\|^p \right)^{\frac{1}{p}} \\ &\leq \left( \pi_p^{\hat{\mathcal{B}}}(f_1) + \pi_p^{\hat{\mathcal{B}}}(f_2) \right) \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} \left( \sum_{i=1}^n |\lambda_i|^p |g'(z_i)|^p \right)^{\frac{1}{p}} \end{aligned}$$

and therefore,  $f_1 + f_2 \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  with  $\pi_p^{\hat{\mathcal{B}}}(f_1 + f_2) \leq \pi_p^{\hat{\mathcal{B}}}(f_1) + \pi_p^{\hat{\mathcal{B}}}(f_2)$ .

Let  $\lambda \in \mathbb{C}$  and  $f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ . We have

$$\left( \sum_{i=1}^n |\lambda_i|^p \|(\lambda f)'(z_i)\|^p \right)^{\frac{1}{p}} = |\lambda| \left( \sum_{i=1}^n |\lambda_i|^p \|f'(z_i)\|^p \right)^{\frac{1}{p}} \leq |\lambda| \pi_p^{\hat{\mathcal{B}}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} \left( \sum_{i=1}^n |\lambda_i|^p |g'(z_i)|^p \right)^{\frac{1}{p}},$$

and thus,  $\lambda f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  with  $\pi_p^{\hat{\mathcal{B}}}(\lambda f) \leq |\lambda| \pi_p^{\hat{\mathcal{B}}}(f)$ .

This implies that  $\pi_p^{\hat{\mathcal{B}}}(\lambda f) = 0 = |\lambda| \pi_p^{\hat{\mathcal{B}}}(f)$  if  $\lambda = 0$ . For  $\lambda \neq 0$ , we have  $\pi_p^{\hat{\mathcal{B}}}(f) = \pi_p^{\hat{\mathcal{B}}}(\lambda^{-1}(\lambda f)) \leq |\lambda|^{-1} \pi_p^{\hat{\mathcal{B}}}(\lambda f)$ , hence

$$|\lambda| \pi_p^{\hat{\mathcal{B}}}(f) \leq \pi_p^{\hat{\mathcal{B}}}(\lambda f),$$

and so  $\pi_p^{\mathcal{B}}(\lambda f) = |\lambda| \pi_p^{\mathcal{B}}(f)$ .

Thus, we have proved that  $(\Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X), \pi_p^{\mathcal{B}})$  is a normed space.

To show that it is a Banach space, it is enough to see that every absolutely convergent series is convergent. So let  $(f_n)_{n \geq 1}$  be a sequence in  $\Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  such that  $\sum \pi_p^{\mathcal{B}}(f_n)$  converges. Since  $p_{\mathcal{B}}(f_n) \leq \pi_p^{\mathcal{B}}(f_n)$  for all  $n \in \mathbb{N}$  and  $(\hat{\mathcal{B}}(\mathbb{D}, X), p_{\mathcal{B}})$  is a Banach space, then  $\sum f_n$  converges in  $(\hat{\mathcal{B}}(\mathbb{D}, X), p_{\mathcal{B}})$  to a function  $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$ .

Given  $m \in \mathbb{N}$ ,  $z_1, \dots, z_m \in \mathbb{D}$ , and  $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ , we have

$$\begin{aligned} \left( \sum_{k=1}^m |\lambda_k|^p \left\| \sum_{i=1}^n f'_i(z_k) \right\|^p \right)^{1/p} &\leq \pi_p^{\mathcal{B}} \left( \sum_{i=1}^n f_i \right) \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}}(\mathbb{D})} \left( \sum_{k=1}^m |\lambda_k|^p |g'(z_k)|^p \right)^{\frac{1}{p}}. \\ &\leq \sum_{i=1}^n \pi_p^{\mathcal{B}}(f_i) \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}}(\mathbb{D})} \left( \sum_{k=1}^m |\lambda_k|^p |g'(z_k)|^p \right)^{\frac{1}{p}}. \end{aligned}$$

For all  $n \in \mathbb{N}$ , and by taking limits with  $n \rightarrow \infty$  yields

$$\left( \sum_{k=1}^m |\lambda_k|^p \left\| \sum_{i=1}^{\infty} f'_i(z_k) \right\|^p \right)^{\frac{1}{p}} \leq \sum_{i=1}^{\infty} \pi_p^{\mathcal{B}}(f_i) \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}}(\mathbb{D})} \left( \sum_{k=1}^m |\lambda_k|^p |g'(z_k)|^p \right)^{\frac{1}{p}}$$

Hence,  $f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  with  $\pi_p^{\mathcal{B}}(f) \leq \sum_{n=1}^{\infty} \pi_p^{\mathcal{B}}(f_n)$ . Moreover, we have:

$$\pi_p^{\mathcal{B}} \left( f - \sum_{i=1}^n f_i \right) = \pi_p^{\mathcal{B}} \left( \sum_{i=n+1}^{\infty} f_i \right) \leq \sum_{i=n+1}^{\infty} \pi_p^{\mathcal{B}}(f_i)$$

for all  $n \in \mathbb{N}$ , and thus  $f$  is the  $\pi_p^{\mathcal{B}}$ -limit of the series  $\sum f_n$ .

**(N2)** Let  $g \in \hat{\mathcal{B}}(\mathbb{D})$  and  $x \in X$ . Note that  $g \cdot x \in \hat{\mathcal{B}}(\mathbb{D}, X)$  with  $p_{\mathcal{B}}(g \cdot x) = p_{\mathcal{B}}(g) \|x\|$  by [1, Proposition 5.13]. If  $g = 0$ , there is nothing to prove. Assume  $g \neq 0$ . We have:

$$\begin{aligned} \left( \sum_{i=1}^n |\lambda_i|^p \|(g \cdot x)'(z_i)\|^p \right)^{\frac{1}{p}} &= \|x\| p_{\mathcal{B}}(g) \left( \sum_{i=1}^n |\lambda_i|^p \left| \left( \frac{g}{p_{\mathcal{B}}(g)} \right)'(z_i) \right|^p \right)^{\frac{1}{p}} \\ &\leq \|x\| p_{\mathcal{B}}(g) \sup_{h \in \mathcal{B}_{\hat{\mathcal{B}}}(\mathbb{D})} \left( \sum_{i=1}^n |\lambda_i|^p |h'(z_i)|^p \right)^{\frac{1}{p}} \end{aligned}$$

and Thus,  $g \cdot x \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  with  $\pi_p^{\mathcal{B}}(g \cdot x) \leq p_{\mathcal{B}}(g)\|x\|$ . Conversely, we have:

$$p_{\mathcal{B}}(g)\|x\| = p_{\mathcal{B}}(g \cdot x) \leq \pi_p^{\mathcal{B}}(g \cdot x)$$

**(N3)** Let  $h : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic function with  $h(0) = 0$ ,  $f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$ , and  $T \in L(X, Y)$  where  $Y$  is a complex Banach space. Note that  $T \circ f \circ h \in \hat{\mathcal{B}}(\mathbb{D}, Y)$  by [1, Proposition 5.13]. We have:

$$\begin{aligned} \left( \sum_{i=1}^n |\lambda_i|^p \|(T \circ f \circ h)'(z_i)\|^p \right)^{\frac{1}{p}} &= \left( \sum_{i=1}^n |\lambda_i|^p \|T(f'(h(z_i))h'(z_i))\|^p \right)^{\frac{1}{p}} \\ &\leq \|T\| \left( \sum_{i=1}^n |\lambda_i|^p |h'(z_i)|^p \|f'(h(z_i))\|^p \right)^{\frac{1}{p}} \\ &\leq \|T\| \pi_p^{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}}(\mathbb{D})} \left( \sum_{i=1}^n |\lambda_i|^p |h'(z_i)|^p |g'(h(z_i))|^p \right)^{\frac{1}{p}} \\ &= \|T\| \pi_p^{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}}(\mathbb{D})} \left( \sum_{i=1}^n |\lambda_i|^p |(g \circ h)'(z_i)|^p \right)^{\frac{1}{p}} \\ &\leq \|T\| \pi_p^{\mathcal{B}}(f) \sup_{K \in \mathcal{B}_{\hat{\mathcal{B}}}(\mathbb{D})} \left( \sum_{i=1}^n |\lambda_i|^p |K'(z_i)|^p \right)^{\frac{1}{p}}, \end{aligned}$$

where We have used that  $p_{\mathcal{B}}(g \circ h) \leq p_{\mathcal{B}}(g)$  by [1, Proposition 3.6]. Therefore,  $T \circ f \circ h \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, Y)$  with  $\pi_p^{\mathcal{B}}(T \circ f \circ h) \leq \|T\| \pi_p^{\mathcal{B}}(f)$ .

**(I)** Let  $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$  and let  $\iota : X \rightarrow Y$  be a linear (not necessarily surjective) isometry. Assume that  $\iota \circ f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, Y)$ . We have

$$\begin{aligned} \left( \sum_{i=1}^n |\lambda_i|^p \|f'(z_i)\|^p \right)^{\frac{1}{p}} &= \left( \sum_{i=1}^n |\lambda_i|^p \|\iota(f'(z_i))\|^p \right)^{\frac{1}{p}} \\ &= \left( \sum_{i=1}^n |\lambda_i|^p \|(\iota \circ f)'(z_i)\|^p \right)^{\frac{1}{p}} \end{aligned}$$

$$\leq \pi_p^{\mathcal{B}}(\iota \circ f) \sup_{g \in \mathcal{B}_{\hat{\mathbb{D}}}(\mathbb{D})} \left( \sum_{i=1}^n |\lambda_i|^p |g'(z_i)|^p \right)^{\frac{1}{p}}.$$

and Thus,  $f \in \Pi_p^{\mathcal{B}}(\mathbb{D}, X)$  with  $\pi_p^{\mathcal{B}}(f) \leq \pi_p^{\mathcal{B}}(\iota \circ f)$ . The reverse inequality follows from (N3).

### 1.3 Möbius Invariance

The Möbius group of  $\mathbb{D}$ , denoted  $\text{Aut}(\mathbb{D})$ , is formed by all biholomorphic bijections  $\phi : \mathbb{D} \rightarrow \mathbb{D}$ . Each  $\phi \in \text{Aut}(\mathbb{D})$  has the form  $\phi = \lambda \phi_a$  with  $\lambda \in \mathbb{T}$  and  $a \in \mathbb{D}$ , where

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \quad (z \in \mathbb{D}).$$

Given a complex Banach space  $X$ , let us recall (see [3]) that a linear space  $A(\mathbb{D}, X)$  of holomorphic mappings from  $\mathbb{D}$  into  $X$ , endowed with a seminorm  $p_A$ , is Möbius-invariant if it holds:

- (i)  $A(\mathbb{D}, X) \subseteq \mathcal{B}(\mathbb{D}, X)$  and there exists  $c > 0$  such that  $p_B(f) \leq cp_A(f)$  for all  $f \in A(\mathbb{D}, X)$ ,
- (ii)  $f \circ \phi \in A(\mathbb{D}, X)$  with  $p_A(f \circ \phi) = p_A(f)$  for all  $\phi \in \text{Aut}(\mathbb{D})$  and  $f \in A(\mathbb{D}, X)$ .

By Proposition 3.1, each  $p$ -summing Bloch mapping  $f : \mathbb{D} \rightarrow X$  is Bloch with  $p_B(f) \leq \pi_p^{\mathcal{B}}(f)$ . Moreover, following the argument of the proof of (N3) in Proposition 3.2, it is easy to prove that if  $f : \mathbb{D} \rightarrow X$  is  $p$ -summing Bloch and  $\phi \in \text{Aut}(\mathbb{D})$ , then  $f \circ \phi$  is  $p$ -summing with  $\pi_p^{\mathcal{B}}(f \circ \phi) \leq \pi_p^{\mathcal{B}}(f)$ , and using this fact, we also

We deduce that

$$\pi_p^{\mathcal{B}}(f) = \pi_p^{\mathcal{B}}((f \circ \phi) \circ \phi^{-1}) \leq \pi_p^{\mathcal{B}}(f \circ \phi).$$

In this way, we have proved the following.

**Proposition 3.3**  $(\Pi_p^{\mathcal{B}}(\mathbb{D}, X), \pi_p^{\mathcal{B}})$  is a Möbius-invariant space for  $1 \leq p \leq \infty$ .

# Pietsch domination

## 4.1 Pietsch Domination

We establish a version for  $p$ -summing Bloch mappings on  $\mathbb{D}$  of the known Pietsch domination Theorem for  $p$ -summing linear operators between Banach spaces [2, Theorem 2].

Let us recall that  $\hat{\mathcal{B}}(\mathbb{D})$  is a dual Banach space (see [4]) and therefore we can consider this space equipped with its weak\* topology. Let  $P(\mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})})$  denote the set of all Borel regular probability measures  $\mu$  on  $(\mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}, w^*)$ .

**Theorem 4.1** *Let  $1 \leq p < \infty$  and  $f \in \hat{\mathcal{B}}(\mathbb{D}, X)$ . The following statements are equivalent:*

(i)  $f$  is  $p$ -summing Bloch.

(ii) (Pietsch domination). There is a constant  $c \geq 0$  and a Borel regular probability measure  $\mu$  on  $(\mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}, w^*)$  such that

$$\|f'(z)\| \leq c \left( \int_{\mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} |g'(z)|^p d\mu(g) \right)^{\frac{1}{p}}$$

for all  $z \in \mathbb{D}$ .

In this case,  $\pi_p^{\mathcal{B}}(f)$  is the infimum of all constants  $c \geq 0$  satisfying the preceding inequality, and this infimum is attained.

**Proof (i)  $\Rightarrow$  (ii):** We will apply an unified abstract version of the Pietsch domination Theorem (see [5, 6]). For it, consider the functions

$$S : \hat{\mathcal{B}}(\mathbb{D}, X) \times \mathbb{D} \times \mathbb{C} \rightarrow [0, \infty[, \quad S(f, z, \lambda) = |\lambda| \|f'(z)\|,$$

and

$$R : \mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})} \times \mathbb{D} \times \mathbb{C} \rightarrow [0, \infty[, \quad R(g, z, \lambda) = |\lambda| |g'(z)|.$$

Note first that for any  $z \in \mathbb{D}$  and  $\lambda \in \mathbb{C}$ , the function  $R_{z,\lambda} : \mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})} \rightarrow [0, \infty[$ , given by

$$R_{z,\lambda}(g) = R(g, z, \lambda),$$

is continuous. For every  $n \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  and  $z_1, \dots, z_n \in \mathbb{D}$ , we have

$$\begin{aligned} \left( \sum_{i=1}^n S(f, z_i, \lambda_i)^p \right)^{\frac{1}{p}} &= \left( \sum_{i=1}^n |\lambda_i|^p \|f'(z_i)\|^p \right)^{\frac{1}{p}} \\ &\leq \pi_p^{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} \left( \sum_{i=1}^n |\lambda_i|^p |g'(z_i)|^p \right)^{1/p} \\ &= \pi_p^{\mathcal{B}}(f) \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} \left( \sum_{i=1}^n R(g, z_i, \lambda_i)^p \right)^{1/p}, \end{aligned}$$

and therefore  $f$  is  $R$ - $S$ -abstract  $p$ -summing. Hence, by applying [6, Theorem 3.1], there is a constant  $c \geq 0$  and a measure  $\mu \in P(\mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})})$  such that

$$S(f, z, \lambda) \leq c \left( \int_{\mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} R(g, z, \lambda)^p d\mu(g) \right)^{1/p}$$

for all  $z \in \mathbb{D}$  and  $\lambda \in \mathbb{C}$ , and therefore

$$\|f'(z)\| \leq c \left( \int_{\mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} |g'(z)|^p d\mu(g) \right)^{1/p}$$

for all  $z \in \mathbb{D}$ . Furthermore, we have

$$\|f'(z)\| = \left( \sum_{i=1}^n |\lambda_i|^p \|f'(z_i)\|^p \right)^{1/p} \leq \pi_p^{\mathcal{B}}(f) \left( \int_{\mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} |g'(z)|^p d\mu(g) \right)^{1/p}$$

for every  $z \in \mathbb{D}$ , by taking, for example,  $n \in \mathbb{N}$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = \dots = \lambda_n = 0$ , and  $z_1 = \dots = z_n = z$ .

(ii)  $\Rightarrow$  (i): Given  $n \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ , and  $z_1, \dots, z_n \in \mathbb{D}$ , we have

$$\left( \sum_{i=1}^n |\lambda_i|^p \|f'(z_i)\|^p \right)^{1/p} \leq c \sum_{i=1}^n \left( \int_{\mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} |\lambda_i|^p |g'(z_i)|^p d\mu(g) \right)^{1/p} \leq c \sup_{g \in \mathcal{B}_{\hat{\mathcal{B}}(\mathbb{D})}} \left( \sum_{i=1}^n |\lambda_i|^p |g'(z_i)|^p \right)^{1/p}.$$

Hence,  $f \in \Pi_p^{\hat{\mathcal{B}}}(\mathbb{D}, X)$  with  $\pi_p^{\mathcal{B}}(f) \leq c$ .

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# Abstract

## ملخص

في هذه الأطروحة، تناولنا واحدة من أهم مواضيع التحليل الدالي، وهي التحويلات  $p$ -المجمعة من نوع Bloch على قرص الوحدة العقدي  $D$ . حاولنا قدر الإمكان، بجهد متواضع جدا، أن نتعرف على التعريفات الأكثر أهمية، والنظريات والخصائص في جميع فصول هذا البحث. تتناول هذه الرسالة دراسة نظرية المؤثرات  $p$ -المجمعة وامتداداتها في سياقات رياضية مختلفة، وتركز بشكل خاص على هندسة المؤثرات غير الخطية. في الفصل الأول، نستعرض التعاريف الأساسية والنظريات المتعلقة بالمؤثرات  $p$ -المجمعة. أما في الفصل الثاني، ندرس المؤثرات الخطية  $(p, \sigma)$  المستمرة بشكل مطلق، مع التركيز على المؤثرات الخطية وليبتشيزية من النوع  $p$ -المجموع، مع عرض نظريات الهيمنة المناسبة لكل صنف. يقدم الفصل الثالث المؤثرات  $p$ -المجمعة من نوع Bloch، وهو مفهوم حديث تم تطويره بواسطة أنطونيو خيمينيز فارغاس. أما الفصل الرابع، فيخصص لدراسة شكل من نوع Bloch لنتائج كلاسيكية، مثل مبرهنة بيتش للهيمنة والتفكيك، ومبرهنة الاستقراء لماوري. تسهم النتائج المقدمة فهذه الرسالة الى تعميق الفهم لنماذج المؤثرات ودورها في التحليل الدالي والتحليل غير الخطي.

## Abstract

*This thesis investigates the theory of  $p$ -summing operators and its extensions in various mathematical and more specifically in the geometry of nonlinear operators settings. Our work is divided on four Chapters: In Chapter One, we present fundamental definitions and theorems concerning the ideal of  $p$ -summing operators. Chapter Two treats Ideal of  $(p, \sigma)$ -absolutely continuous linear operators, focuses on both linear and Lipschitz  $p$ -summing operators, highlighting relevant domination theorems for each class. In Chapter Three we introduce the ideal of  $p$ -summing Bloch mappings, a recent development by Antonio Jiménez-Vargas. Finally, Chapter Four is dedicated to Bloch-type versions of classical results, including Pietsch's domination and factorization theorem and Maurey's extrapolation theorem. The results presented contribute to the understanding of operator ideals and their role in functional and nonlinear analysis.*

## Résumé

*Cette thèse explore la théorie des opérateurs  $p$ -sommants et ses extensions dans divers contextes mathématiques, et plus particulièrement dans la géométrie des opérateurs non linéaires.*

*Notre travail est divisé en quatre chapitres :*

*Le Chapitre Un présente les définitions fondamentales et les théorèmes concernant l'idéal des opérateurs  $p$ -sommants.*

*Le Chapitre Deux traite l'idéal des opérateurs linéaires  $(p, \sigma)$ -absolument continus, en se concentrant à la fois sur les opérateurs linéaires et les opérateurs de Lipschitz  $p$ -sommants, en mettant en évidence les théorèmes de domination pertinents pour chaque classe.*

*Dans le Chapitre Trois, nous introduisons l'idéal des applications de Bloch  $p$ -sommantes, et lequel est un développement récent dû à Antonio Jiménez-Vargas.*

*Enfin, le Chapitre Quatre est consacré aux versions de type Bloch de résultats classiques, incluant le théorème de domination et de factorisation de Pietsch ainsi que le théorème d'extrapolation de Maurey.*

*Les résultats présentés contribuent à une meilleure compréhension des idéaux d'opérateurs et de leur rôle dans l'analyse fonctionnelle et non linéaire.*