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Université Mohamed Boudiaf- M'sila
كلية الرياضيات و الاعلام الالي
Faculté des Mathématiques et de l'Informatique
Conseil scientifique de la faculté
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وافق المجلس العلمي للكلية على اعتماد هذه المطبوعة كمرجع لدروس السنة الثانية هندسة مدنية.

رئيس المجلس العلمي

جامعة محمد بوضياف بالمسيلة
رئيس المجلس العلمي
رئيس المجلس العلمي لكلية الرياضيات والإعلام الآلي
عشور دحمان

Mohamed Boudiaf University of M'sila

Faculty of Technology

Department of Sciences and Technology




# Applied Mathematics

Course and Exercises

Second-Year Engineering Students
Civil Engineering – Mechanical Engineering – Electrotechnics

**Prepared by:**

Dr. Khaled Hamidi

Academic Year 2025–2026

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# Introduction

The document I present to you is a comprehensive and detailed resource on the main concepts of Applied Mathematics, focusing on several fundamental topics in mathematical analysis and differential equation theory.

It begins with an in-depth exploration of simple and multiple integrals, offering an exhaustive understanding of integration methods, applications, and techniques used to solve various types of problems. Improper integrals are also covered in detail, highlighting cases where the limits of integration are infinite or where functions exhibit discontinuities.

A substantial part of the book is devoted to differential equations, covering various aspects, including first-order linear differential equations. Analytical methods for solving these equations are explained in detail, providing a deep understanding of solutions and their geometric interpretation. Furthermore, second-order linear differential equations are studied in depth, exploring particular solutions, boundary value problems, and practical applications of these equations.

This manuscript also includes a section dedicated to numerical series and power series, presenting their properties, convergence, and applications in various mathematical fields.

It offers a thorough analysis of Fourier series, exploring their construction, convergence, and use in solving practical problems in engineering, physics, and applied mathematics.

Additionally, an in-depth study of the Fourier transform is proposed, describing its definition, properties, and use in signal analysis and periodic functions.

The Laplace transform is also addressed in detail, presenting its applications in solving differential equations and other mathematical and scientific problems. Finally, each chapter is enriched with a collection of solved exercises, offering readers the opportunity to consolidate their understanding of the presented concepts and refine their problem-solving skills. These varied and comprehensive exercises allow for active learning and better assimilation of the knowledge presented in the book.

I ask Almighty God that this work be sincere for Him, and to bless this work so as to make it not only pure, but also enriching and beneficial for all who will read it. May the knowledge it contains enlighten the minds of students, specialists, and anyone interested in this field. May each page be imbued with clarity and wisdom, and may reading it inspire and uplift those who delve into it. May this book be a source of light and guidance for all who read it, and may it contribute to the advancement of knowledge and the well-being of humanity.

## General Objective

Our program aims to provide students with a deep and complete understanding of the main concepts of Applied Mathematics, with a particular emphasis on mathematical analysis and differential equation theory. We wish to develop in students a solid expertise in these fundamental fields, essential for advanced understanding of mathematical and applied sciences.

Within the framework of our pedagogical approach, we will implement interactive and varied teaching methods, such as dynamic lectures, tutorial sessions, and practical exercises in small groups. These methods aim to foster active student participation and encourage collaborative learning.

Furthermore, we will pay particular attention to the development of transversal skills, such as problem-solving, critical thinking, and effective mathematical communication. These skills are essential for students to not only master theoretical concepts but also apply them creatively and innovatively in their future academic or professional careers.

In summary, our objective is to offer students an enriching and stimulating learning experience, which not only strengthens their understanding of advanced mathematics but also develops their ability to tackle complex mathematical challenges and contribute significantly to research and development in the field of mathematical and applied sciences.

# Chapter 1

# Simple and Multiple Integrals

## 1.1 Riemann Integral

### Subdivision

Let $[a, b]$ be a compact interval of $\mathbb{R}$.

**Definition 1.1** (Subdivision)**.** Any finite set of points of $[a, b]$ containing $a$ and $b$ is called a subdivision. It is written uniquely as:

$$d = \{x_0, x_1, ..., x_n\},$$

where $a = x_0 \leq x_1 \leq ... \leq x_n = b$ and $n$ is any natural number.

*Remark* 1.1. If $d$ is a subdivision of $[a, b]$ and $E$ is a finite subset of $[a, b]$, then $d \cup E$ is again a subdivision of $[a, b]$.

**Definition 1.2** (Finer Subdivision)**.** Let $d, d'$ be two subdivisions of $[a, b]$. We say that $d$ is fine than $d'$ if $d' \subset d$.

**Definition 1.3** (Mesh of a Subdivision)**.** Let $d = \{x_0, x_1, ..., x_n\}$ be a subdivision of $[a, b]$. The strictly positive real number $\delta(d)$, where

$$\delta(d) = \max_{1 \leq i \leq n} (x_i - x_{i-1})$$

is called the mesh of the subdivision.

*Remark* 1.2 (Equidistant Subdivision). If the mesh is uniform, i.e.,

$$x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1},$$

the subdivision is said to be equidistant.

### Darboux Sums

Let $f : [a, b] \to \mathbb{R}$.

**Definition 1.4** (Bounded Function)**.** The function $f$ defined on $[a,b]$ is bounded if

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)| < +\infty.$$

Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$ and $d = \{x_0, x_1, ..., x_n\}$ a subdivision of $[a,b]$. Let

$$m_i(f,d) = m_i = \inf_{x_{i-1}\le x\le x_i} f(x), \quad M_i(f,d) = M_i = \sup_{x_{i-1}\le x\le x_i} f(x).$$

Let $S_{a,b}$ be the set of all subdivisions of $[a,b]$.

**Definition 1.5** (Darboux Sums)**.** The lower (resp. upper) Darboux sum of $f$ relative to the subdivision $d$ is the number $s$ (resp. $S$) given by:

$$s = s(f,d) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad S = S(f,d) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Let $s_a^b := \sup_{d\in S_{a,b}} s(f,d)$ and $S_a^b := \inf_{d\in S_{a,b}} S(f,d)$.

**Definition 1.6** (Riemann-Integrable Function)**.** The function $f$ is Riemann-integrable on $[a,b]$ if and only if $s_a^b$ coincides with $S_a^b$. This number is then called the Riemann integral of $f$ on $[a,b]$, and is denoted:

$$\int_a^b f(x)dx.$$

*Remark* 1.3. The integration variable $x$ in $\int_a^b f(x)dx$ is a dummy variable; it can be replaced by any other variable.

**Theorem 1.1** (Riemann Integrability Criterion)**.** *A function $f$ is Riemann-integrable on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists a subdivision $X \in S_{a,b}$ such that*

$$S(f,d) - s(f,d) < \varepsilon.$$

**Theorem 1.2.** *Every monotonic or continuous function on an interval $[a,b]$ is Riemann-integrable.*

### Riemann Sums

Darboux sums are not very useful for the effective calculation of an integral (e.g., with a computer) because it is generally difficult to find the inf and sup on the subintervals. Instead, consider:

$$s_n(f) = \sum_{i=1}^{n} (x_i - x_{i-1}) f(x_{i-1}), \quad S_n(f) = \sum_{i=1}^{n} (x_i - x_{i-1}) f(x_i).$$

More generally, if $\xi = (\xi_1, ..., \xi_n)$ satisfies $\forall i \in \{1, ..., n\}, \xi_i \in [x_{i-1}, x_i]$, we call $(X, \xi)$ a tagged subdivision and $S(f, X, \xi)$ the associated Riemann sum:

$$S(f, X, \xi) = \sum_{i=1}^{n} (x_i - x_{i-1}) f(\xi_i).$$

**Theorem 1.3.** *If $f$ is Riemann-integrable, then the Riemann sums $S(f, X, \xi)$ tend towards $\int f(x)dx$, independently of the choice of $\xi_i$, as the subdivision becomes finer and finer.*

### Properties of the Riemann Integral

Let $f, g$ be two Riemann-integrable functions. Then for any subdivision $X$ of $[a, b]$:

1. $s(f, X) \leq \int_a^b f(x)dx \leq S(f, X)$. In particular, $(b-a) \inf f([a, b]) \leq \int_a^b f(x)dx \leq (b-a) \sup f([a, b])$.
2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (Chasles relation).
3. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
4. $\int_a^a f(x)dx = 0$.
5. $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$, for all $\alpha, \beta \in \mathbb{R}$.
6. $f \geq 0 \implies \int_a^b f(x)dx \geq 0$ (The converse is false).
7. $f \leq g \implies \int_a^b f(x)dx \leq \int_a^b g(x)dx$.
8. If $|f|$ is Riemann-integrable, then $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

**Theorem 1.4** (Mean Value Theorem)**.** *Let $f \in C([a, b])$. Then*

$$\exists c \in [a, b] : \frac{1}{b-a} \int_a^b f(x)dx = f(c).$$

### 1.1.1 Primitive of a Continuous Function

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a numerical function defined on $D$.

**Definition 1.7** (Primitive)**.** A function $F : D \to \mathbb{R}$ is a primitive of $f$ in $D$ if and only if:

1. $F$ is differentiable on $D$.
2. $F' = f$ in $D$.

**Proposition 1.1.** *If $F$ and $G$ are two primitives of $f$, then $F - G$ is constant on every interval $I \subset D$.*

#### Existence of a Primitive

**Theorem 1.5.** *Every continuous function $f : [a, b] \to \mathbb{R}$ has a primitive. We write:*

$$F(x) = \int_a^x f(t)dt.$$

*And*

$$\int_a^b f(x)dx = F(b) - F(a).$$

*Remark* 1.4. There exist continuous functions whose primitives do not necessarily have an explicit closed-form expression.

#### Primitives of Common Functions

$$\int x^\alpha \, dx = \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1} + C, & \text{if } \alpha \in \mathbb{R} \setminus \{-1\}, \\ \ln|x| + C, & \text{if } \alpha = -1. \end{cases}$$

$$\int e^x \, dx = e^x + C$$

$$\int \sin x \, dx = -\cos x + C, \qquad \int \cos x \, dx = \sin x + C$$

$$\int \cosh x \, dx = \sinh x + C, \qquad \int \sinh x \, dx = \cosh x + C$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C, \quad x \in (-1, 1)$$

$$\int \frac{1}{\sqrt{1+x^2}} \, dx = \operatorname{arcsinh}(x) + C = \ln\left(x + \sqrt{1+x^2}\right) + C$$

**Example**

Compute the following integral:

$$I = \int (3x^2 + e^{-x} - \cos 2x)\, dx.$$

We have:

$$I = \int 3x^2\, dx + \int e^{-x}\, dx - \int \cos 2x\, dx = 3\int x^2\, dx - \int e^{-x}\, dx - \tfrac{1}{2}\int 2\cos 2x\, dx.$$

Thus:

$$I = x^3 - e^{-x} - \tfrac{1}{2}\sin 2x + C.$$

—

### 1.1.2 Integration by Parts

**Proposition 1.2.** *For all $f, g \in C^1(I, \mathbb{R})$, we have:*

$$\int f'(x)g(x)\, dx = f(x)g(x) - \int f(x)g'(x)\, dx.$$

*If the integral is definite on $[a, b]$, then:*

$$\int_a^b f'(x)g(x)\, dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)\, dx.$$

**Example**

Compute

$$I = \int \ln(x+1)\, dx.$$

Let $u = \ln(x+1)$, hence $u' = \frac{1}{x+1}$, and let $v' = 1 \Rightarrow v = x + 1$.
Then, by integration by parts:

$$I = (x+1)\ln(x+1) - \int dx = (x+1)\ln(x+1) - x + C.$$

### 1.1.3 Change of Variable

**Proposition 1.3.** *Let $f : I \to \mathbb{R}$ be continuous and let $\varphi : J \to I$ be a diffeomorphism. Then*

$$\int_{\varphi(a)}^{\varphi(b)} f(x)\, dx = \int_a^b f(\varphi(t))\, \varphi'(t)\, dt.$$

**Example**

Compute

$$I = \int x\sqrt{1+x^2}\,dx.$$

Let $u = 1 + x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$.
Then

$$I = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(1+x^2)^{3/2} + C.$$

## 1.2 Double Integrals and Applications

Multiple integrals constitute the generalization of simple integrals (integrals of a function of a single real variable). Here we focus on the generalization to functions with a larger number of variables (two or three).

### 1.2.1 Definition of the Double Integral

Let $D$ be a domain in the $xOy$-plane bounded by a closed curve. Consider in the same domain the function $z = f(x, y)$ of the two independent variables $x$ and $y$.

We divide the domain $D$ into $n$ parts (subdomains)

$$\Delta s_1,\ \Delta s_2,\ \ldots,\ \Delta s_n,$$

and in each subdomain $\Delta s_k$, choose an arbitrary point $P_k$, whose image by the mapping $f$ is $f(P_k)$.

We then consider the sum of the products

$$V_n = \sum_{k=1}^{n} f(P_k)\,\Delta s_k,$$

which is called the *integral sum* of the function $f(x, y)$ over the domain $D$.

Now, consider the sequence $(V_n)_{n\in\mathbb{N}}$ of all such integral sums corresponding to all possible partitions of $D$, such that the largest subdomain $\Delta s_k$ tends to 0 as $n \to +\infty$.

**Theorem 1.6.** *If $f(x, y)$ is continuous on $D$, then the sequence $(V_n)_{n\in\mathbb{N}}$ of integral sums has a limit when the largest subdomain $\Delta s_k$ tends to 0 and $n \to +\infty$. This limit does not depend on the way $D$ is subdivided nor on the choice of the points $P_k$ in each $\Delta s_k$.*

**Definition 1.8.** The *double integral* of the function $f(x, y)$ over the domain $D$ is defined as the limit of the sequence of integral sums $(V_n)$ when the largest subdomain $\Delta s_k \to 0$ and $n \to +\infty$.
It is denoted by

$$\lim_{\substack{\max \Delta s_k \to 0 \\ n\to+\infty}} \sum_{k=1}^{n} f(P_k)\,\Delta s_k = \iint_D f(x, y)\,dy\,dx.$$

### 1.2.2 Properties of Double Integrals

- $\iint_D [f(x,y) + g(x,y)]\,dx\,dy = \iint_D f(x,y)\,dx\,dy + \iint_D g(x,y)\,dx\,dy$
- If $D$ is split into $D_1$ and $D_2$, then:

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D_1} f(x,y)\,dx\,dy + \iint\limits_{D_2} f(x,y)\,dx\,dy$$

- If $a$ is a constant, then:

$$\iint_D af(x,y)\,dx\,dy = a\iint_D f(x,y)\,dx\,dy$$

- **Linearity:**

$$\iint_D (\lambda f + \mu g)(x,y)\,dx\,dy = \lambda\iint_D f(x,y)\,dx\,dy + \mu\iint_D g(x,y)\,dx\,dy$$

- If $f(x,y) \geq 0$ at every point of $D$, then $\iint_D f(x,y)\,dx\,dy > 0$.
- If $\forall (x,y) \in D, f(x,y) \leq g(x,y)$, then $\iint_D f(x,y)\,dx\,dy \leq \iint_D g(x,y)\,dx\,dy$.
- $\left|\iint_D f(x,y)\,dx\,dy\right| \leq \iint_D |f(x,y)|\,dx\,dy$

### 1.2.3 Successive Simple Integrals — Fubini's Theorem

Let $R = [a,b] \times [c,d]$ with $a < b$ and $c < d$ be a closed rectangle in the plane $\mathbb{R}^2$, and let $f : R \to \mathbb{R}$ be a continuous function.

For each fixed $x \in [a,b]$, the function $f(x,\cdot) : [c,d] \to \mathbb{R}$, defined by $f(x,\cdot)(y) = f(x,y)$, is integrable on $[c,d]$.

The number

$$\int_c^d f(x,\cdot)(t)\,dt$$

depends on $x$. Hence we have a function $A : [a,b] \to \mathbb{R}$ defined by

$$A(x) = \int_c^d f(x,t)\,dt.$$

Integrating $A$ over the interval $[a,b]$, we obtain the formula

$$(*) \quad \int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx.$$

**Definition 1.9.** In the expression $\int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$ from formula $(*)$, we say that we first integrate with respect to $y$, and then with respect to $x$.
Similarly, in the expression $\int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$, we say that we first integrate with respect to $x$, and then with respect to $y$.

**Example**

Consider the rectangle $R = [1, 2] \times [0, 3] \subset \mathbb{R}^2$ with $1 \leq x \leq 2$ and $0 \leq y \leq 3$, and let the function $f$ be defined on $R$ by

$$f(x, y) = xy + y^2 + 1.$$

Here, $a = 1, b = 2, c = 0, d = 3$.

- First, we integrate with respect to $y$:

$$\int_0^3 f(x, y)\, dy = \int_0^3 (xy + y^2 + 1)\, dy = \left[\tfrac{1}{2}xy^2 + \tfrac{1}{3}y^3 + y\right]_0^3 = \tfrac{9}{2}x + 12.$$

- Then, integrating the result with respect to $x$:

$$\int_1^2 \left(\int_0^3 (xy + y^2 + 1)\, dy\right) dx = \int_1^2 \left(\tfrac{9}{2}x + 12\right) dx = \left[\tfrac{9}{4}x^2 + 12x\right]_1^2 = \tfrac{75}{4}.$$

Now, let us integrate first with respect to $x$, and then with respect to $y$:

- First,

$$\int_1^2 f(x, y)\, dx = \int_1^2 (xy + y^2 + 1)\, dx = \left[\tfrac{1}{2}x^2y + (y^2 + 1)x\right]_1^2 = y^2 + \tfrac{3}{2}y + 1.$$

- Then,

$$\int_0^3 \left(\int_1^2 (xy + y^2 + 1)\, dx\right) dy = \int_0^3 \left(y^2 + \tfrac{3}{2}y + 1\right) dy = \left[\tfrac{1}{3}y^3 + \tfrac{3}{4}y^2 + y\right]_0^3 = \tfrac{75}{4}.$$

In Example 3.8, we observe that the two successive integrations give the same result. This is not a coincidence, but a consequence of the following theorem.

**Theorem 1.7** (Fubini's Theorem for Closed Rectangles)**.** *Let $R = [a, b] \times [c, d]$ with $a < b$, $c < d$ be a closed rectangle in the plane $\mathbb{R}^2$, and let $f : R \to \mathbb{R}$ be a continuous function. Then $f$ is integrable on $R$, and*

$$\iint_R f(x, y)\, dx\, dy = \int_a^b \left(\int_c^d f(x, y)\, dy\right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx\right) dy.$$

*Remark* 1.5. If $f(x, y) = f_1(x)f_2(y)$ we have

$$\iint_R f(x, y)\, dx\, dy = \int_a^b f_1(x)dx \int_c^d f_2(y)\, dy = \int_c^d f_2(y)\, dy \int_a^b f_1(x)dx.$$

**Example**

Let $R = [1, 2] \times [0, 2] \subset \mathbb{R}^2$ and $f : R \to \mathbb{R}$ be defined by $f(x, y) = ye^{xy}$. We compute $I = \iint_R f(x, y)\, dx\, dy$. By Fubini's Theorem, we have

$$I = \int_0^2 \left(\int_1^2 ye^{xy}\, dx\right) dy.$$

Now,

$$\int_1^2 ye^{xy}\,dx = [e^{xy}]_1^2 = e^{2y} - e^y.$$

Hence,

$$I = \int_0^2 (e^{2y} - e^y)\,dy = \left[\tfrac{1}{2}e^{2y} - e^y\right]_0^2 = \tfrac{1}{2}e^4 - e^2 + \tfrac{1}{2}.$$

*Remark* 1.6. Integrating first with respect to $x$ required only two lines of computation. If we start instead by integrating with respect to $y$, the calculation becomes more involved, since $\int_0^2 ye^{xy}\,dy$ requires integration by parts, leading to

$$e^{xy}\left(\tfrac{y}{x} - \tfrac{1}{x^2}\right)\Big|_{y=0}^{y=2} = e^{2x}\left(\tfrac{2}{x} - \tfrac{1}{x^2}\right) + \tfrac{1}{x^2}.$$

The subsequent integration of this expression with respect to $x$ again demands another integration by parts:

$$\int_1^2 \left[e^{2x}\left(\tfrac{2}{x} - \tfrac{1}{x^2}\right) + \tfrac{1}{x^2}\right] dx = \tfrac{1}{2}e^4 - e^2 + \tfrac{1}{2}.$$

**Example**

1- Let $D = [1,2] \times [0,2] \subset \mathbb{R}^2$, $f(x,y) = ye^{xy}$. Compute $I = \iint_D f(x,y)\,dx\,dy$. we have, $I = \int_1^2 \int_0^2 ye^{xy}\,dy\,dx$.

First, integrate with respect to $y$:

$$\int_0^2 ye^{xy}\,dy = \left[\frac{ye^{xy}}{x} - \frac{e^{xy}}{x^2}\right]_0^2 = \frac{2e^{2x}}{x} - \frac{e^{2x}}{x^2} + \frac{1}{x^2}.$$

Hence,

$$I = \int_1^2 \left(\frac{2e^{2x}}{x} - \frac{e^{2x}}{x^2} + \frac{1}{x^2}\right) dx.$$

Notice that

$$\frac{d}{dx}\left(\frac{e^{2x}}{x}\right) = \frac{2e^{2x}}{x} - \frac{e^{2x}}{x^2}.$$

Therefore,

$$I = \int_1^2 \frac{d}{dx}\left(\frac{e^{2x}}{x}\right) dx + \int_1^2 \frac{1}{x^2}\,dx = \left[\frac{e^{2x}}{x}\right]_1^2 + \left[-\frac{1}{x}\right]_1^2.$$

Simplify:

$$I = \left(\frac{e^4}{2} - e^2\right) + \frac{1}{2} = \frac{e^4}{2} - e^2 + \frac{1}{2}.$$

2- Compute the integral $\iint_D (x^2+y)\,dy\,dx$, where $D = \{(x,y) : 1 \le x \le 2,\ 0 \le y \le 2\}$.

We have

$$\int_0^2 (x^2 + y)\,dy = \left[x^2 y + \frac{1}{2}y^2\right]_0^2 = 2x^2 + 2,$$

so

$$\iint_D (x^2 + y)\,dy\,dx = \int_1^2 (2x^2 + 2)\,dx = 2\int_1^2 x^2\,dx + 2\int_1^2 1\,dx.$$

Thus

$$2\left[\frac{x^3}{3}\right]_1^2 + 2[x]_1^2 = 2\left(\frac{8-1}{3}\right) + 2(2-1) = \frac{14}{3} + 2 = \frac{20}{3}.$$

$$\iint_D (x^2 + y)\, dy\, dx = \frac{20}{3}.$$

### 1.2.4 Double Integrals over Non-Rectangular Domains

Let $D$ be a bounded domain in the real plane $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$. We wish to compute (if it exists) the integral

$$\iint_D f(x, y)\, dx\, dy.$$

If the domain $D$ under consideration is not a rectangle but is bounded, we can include it in a rectangle $R$, consider a partition (grid) of $R$, and define a double Riemann sum over those subrectangles of the grid that are entirely contained in $D$.

If the double Riemann sum tends to a limit $I \in \mathbb{R}$ as the mesh of the partition tends to 0, then the function $f$ is said to be integrable over $D$, and we write

$$\iint_D f(x, y)\, dx\, dy = I.$$

#### Integrals over a Domain Bounded by Two Graphs and Two Vertical Lines

**Theorem 1.8.** *Let $[a, b]$ $(a < b)$ be a closed and bounded interval in $\mathbb{R}$, and let $u$ and $v$ be two continuous functions on $[a, b]$ such that $\forall x \in [a, b], \quad u(x) \leq v(x)$. Let $D$ be the domain in $\mathbb{R}^2$ defined by*

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ u(x) \leq y \leq v(x)\}.$$

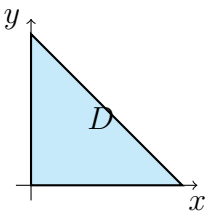
*If $f : D \to \mathbb{R}$ is a continuous function, then $f$ is integrable over $D$, and*

$$\iint_D f(x, y)\, dx\, dy = \int_a^b \left( \int_{u(x)}^{v(x)} f(x, y)\, dy \right) dx.$$

#### Example

1- Let $f(x, y) = x^2 y$, and let $D$ be the triangle with vertices $B_1(0, 0)$, $B_2(1, 0)$, $B_3(0, 1)$.Compute $\iint_D f(x, y)\, dx\, dy$.

The region $D$ is bounded by the lines $x = 0, \quad y = 0, \quad$ and $\quad x + y = 1$.



Hence, for a fixed $x \in [0,1]$, $y$ varies from 0 to $1-x$.
Thus,

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_0^{1-x} x^2 y\,dy\,dx.$$

Compute the inner integral:

$$\int_0^{1-x} x^2 y\,dy = x^2\left[\frac{y^2}{2}\right]_0^{1-x} = \frac{x^2}{2}(1-x)^2.$$

Now integrate with respect to $x$:

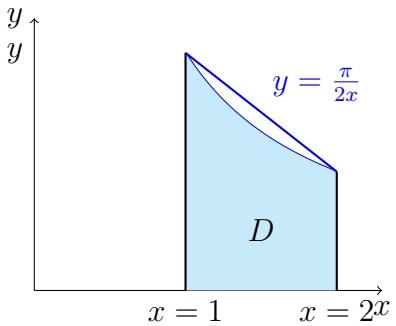
$$\int_0^1 \frac{x^2}{2}(1-x)^2\,dx = \frac{1}{2}\int_0^1 (x^2 - 2x^3 + x^4)\,dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1.$$

Simplify:

$$\frac{1}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{2}\left(\frac{10-15+6}{30}\right) = \frac{1}{2}\cdot\frac{1}{30} = \frac{1}{60}.$$

2 - Let $f(x,y) = \cos(xy)$, $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 2,\ 0 \le xy \le \frac{\pi}{2}$. Compute $I = \iint_D f(x,y)\,dx\,dy$.

Since $0 \le xy \le \frac{\pi}{2}$, we have $0 \le y \le \frac{\pi}{2x}$. Hence $D$ is the region bounded by the curves $x = 1,\ x = 2,\ y = 0,\ y = \frac{\pi}{2x}$. Then the Graph of the region $D$:




$$I = \int_1^2 \int_0^{\pi/(2x)} \cos(xy)\,dy\,dx.$$

Compute the inner integral:

$$\int_0^{\pi/(2x)} \cos(xy)\,dy = \left[\frac{\sin(xy)}{x}\right]_0^{\pi/(2x)} = \frac{1}{x}\sin\left(\frac{\pi}{2}\right) = \frac{1}{x}.$$

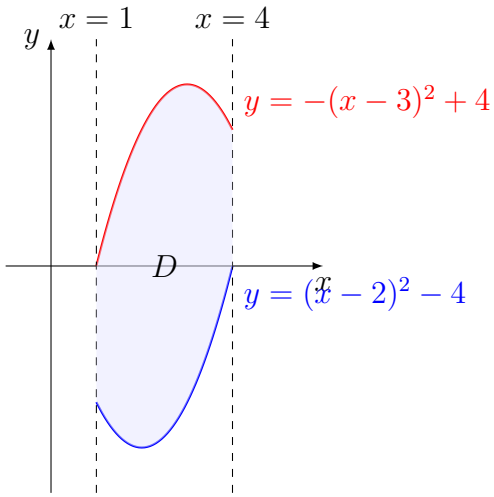
Thus,

$$I = \int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2 = \ln 2.$$

3 - Let $D$ be the domain of $\mathbb{R}^2$ bounded by the vertical lines $x = 1$ and $x = 4$, and the two parabolas $y = (x-2)^2 - 4$, $y = -(x-3)^2 + 4$.

Consider the function $f(x,y) = 3x - 2y + 1$ defined on $D$. We wish to compute $I = \iint_D f(x,y)\,dx\,dy$.

Set $u(x) = (x-2)^2 - 4, v(x) = -(x-3)^2 + 4$. It is easy to see that for all $x \in [1,4]$, we have $u(x) \leq v(x)$. Thus $D = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4,\ u(x) \leq y \leq v(x)\}$.




Applying Theorem 3.11, we obtain:

$$\iint_D f(x,y)\,dx\,dy = \int_1^4 \left( \int_{u(x)}^{v(x)} (3x - 2y + 1)\,dy \right) dx = \int_1^4 \Big[ (3x+1)y - y^2 \Big]_{y=u(x)}^{y=v(x)} dx.$$

Hence

$$I = \int_1^4 \Big( (3x+1)(v(x) - u(x)) - \tfrac{1}{2}\big(v(x)^2 - u(x)^2\big) \Big)\,dx.$$

After simplification, we find

$$I = \int_1^4 \big( -2x^3 - 2x^2 + 55x - 30 \big)\,dx = \Big[ -\tfrac{1}{2}x^4 - \tfrac{2}{3}x^3 + \tfrac{55}{2}x^2 - 30x \Big]_1^4 = \tfrac{153}{2}.$$

### Integrals over a Domain Bounded by Two Graphs and Two Horizontal Lines

The results of this section are obtained from those of the previous one by interchanging the roles of $x$ and $y$.

Let $[c,d]$ $(c < d)$ be a closed and bounded interval in $\mathbb{R}$, and let $u$ and $v$ be two continuous functions defined on $[c,d]$ such that

$$\forall y \in [c,d], \quad u(y) \leq v(y).$$

Let $D$ be the domain of the plane bounded by the graphs of $u$ and $v$, and by the horizontal lines $y = c$ and $y = d$. Formally,

$$D = \{(x,y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ u(y) \leq x \leq v(y)\}.$$

**Theorem 1.9.** *If $f : D \to \mathbb{R}$ is a continuous function, then $f$ is integrable over $D$, and*

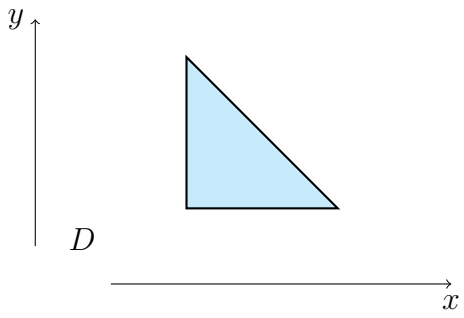
$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left( \int_{u(y)}^{v(y)} f(x,y)\,dx \right) dy.$$

**Example**

Let $f(x,y) = \dfrac{1}{(x+y)^3}$, $D = \{(x,y) \in \mathbb{R}^2 : 1 \le y \le 3,\ 2 \le x,\ x+y \le 5\}$. Compute $I = \iint_D f(x,y)\,dx\,dy$.

Set $v(y) = 2, u(y) = 5-y$. It is easy to see that for all $y \in [1,3]$, we have $u(x) \le v(x)$. Thus $D = \{(x,y) \in \mathbb{R}^2 \mid 1 \le y \le 3,\ 2 \le x \le 5-y\}$. We want to compute $I = \iint_D f(x,y)\,dx\,dy$.

Thus the region is bounded by the points $A(2,1)$, $B(4,1)$, and $C(2,3)$.




To compute the integral: $I = \displaystyle\int_{y=1}^{3}\int_{x=2}^{5-y} \frac{1}{(x+y)^3}\,dx\,dy$.

Compute the inner integral:

$$\int_{x=2}^{5-y} \frac{1}{(x+y)^3}\,dx = \left[-\frac{1}{2(x+y)^2}\right]_{x=2}^{x=5-y} = \frac{1}{2(2+y)^2} - \frac{1}{2(5)^2} = \frac{1}{2}\left(\frac{1}{(2+y)^2} - \frac{1}{25}\right).$$

Hence,

$$I = \frac{1}{2}\int_1^3 \left(\frac{1}{(2+y)^2} - \frac{1}{25}\right) dy = \frac{1}{2}\left(\int_1^3 \frac{dy}{(2+y)^2} - \frac{2}{25}\right).$$

Now,

$$\int \frac{dy}{(2+y)^2} = -\frac{1}{2+y}.$$

So

$$\int_1^3 \frac{dy}{(2+y)^2} = -\frac{1}{5} + \frac{1}{3} = \frac{2}{15}.$$

Therefore,

$$I = \frac{1}{2}\left(\frac{2}{15} - \frac{2}{25}\right) = \frac{1}{2}\cdot\frac{10-6}{75} = \frac{1}{2}\cdot\frac{4}{75} = \frac{2}{75}.$$

### 1.2.5 Double Integrals over a General Domain $D$

If $D \subset \mathbb{R}^2$ is not a rectangle and cannot be defined by the graphs of two functions and two vertical (or horizontal) lines, we can, if possible, decompose $D$ into "elementary" domains of the types previously studied. Then we use the following property to perform the computation.

**Property 1.1.** *Let $D$ be a closed and bounded domain of $\mathbb{R}^2$. Suppose that $D$ is the union of two closed domains $D_1$ and $D_2$ such that*

$$D = D_1 \cup D_2, \qquad \text{and} \qquad int(D_1) \cap int(D_2) = \varnothing.$$

*If $f : D \to \mathbb{R}$ is continuous on $D$, then $f$ is integrable over $D$, and*

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x,y)\,dx\,dy + \iint_{D_2} f(x,y)\,dx\,dy.$$

**Example**

Let

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -2 \le x \le 2, \\ -\frac{1}{4}y^2 + 3 \ge y \ge 2x + 1 \end{array} \right\}$$

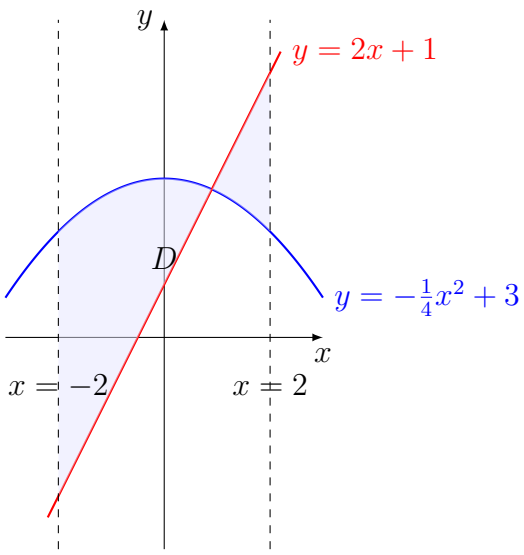
Let $f : D \to \mathbb{R}$ be a continuous function, and consider $I = \iint_D f(x,y)\,dx\,dy$. Set

$$D_1 = \{(x,y) \in D \mid y \ge 2\}, \qquad D_2 = \{(x,y) \in D \mid y \le 2\}.$$

The domains $D_1$ and $D_2$ are bounded by vertical or horizontal lines and by graphs of functions as in the previous sections. Their intersection is a line segment.

Thus,

$$D_1 = \{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 2,\ 2 \le y \le -\tfrac{1}{4}x^2 + 3\},$$
$$D_2 = \{(x,y) \in \mathbb{R}^2 \mid -2 \le y \le 2,\ -2 \le x \le \tfrac{1}{4}y^2 + 1\}.$$




Hence,

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x,y)\,dx\,dy + \iint_{D_2} f(x,y)\,dx\,dy.$$

Compute the double integral $I = \iint_D f(x,y)\,dx\,dy$ for the function $f(x,y) = x - y$.
We have

$$I_1 = \iint_{D_1} f(x,y)\,dx\,dy = \int_{-2}^{2} \left( \int_{2}^{-\frac{1}{4}x^2+3} (x-y)\,dy \right) dx,$$

and

$$I_2 = \iint_{D_2} f(x,y)\,dx\,dy = \int_{-2}^{2} \left( \int_{-2}^{\frac{1}{4}y^2+1} (x-y)\,dx \right) dy.$$

A direct computation gives

$$I_1 = \frac{1}{32}\int_{-2}^{2}(-x^4 - 8x^3 + 24x^2 + 32x - 80)\,dx = -\tfrac{32}{5},$$

and

$$I_2 = \frac{1}{32}\int_{-2}^{2}(y^4 - 8y^3 + 8y^2 - 96y - 48)\,dy = -\tfrac{64}{15}.$$

Therefore,

$$I = I_1 + I_2 = -\tfrac{32}{3}.$$

### 1.2.6 Change of Variables in a Double Integral

**General Case**

Suppose we perform the change of variables:

$$\begin{cases} x = \varphi(u,v) \\ y = \psi(u,v) \end{cases}$$

When $(x,y)$ varies in the domain $D$, $(u,v)$ varies in a domain $D_1$. Consider the determinant of the Jacobian matrix:

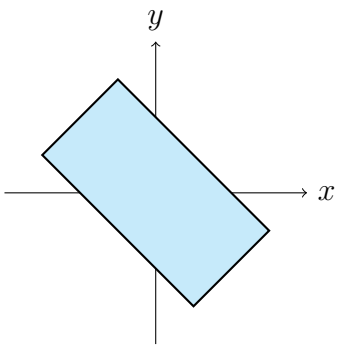
$$J = \begin{vmatrix} \dfrac{\partial \varphi}{\partial u} & \dfrac{\partial \varphi}{\partial v} \\ \dfrac{\partial \psi}{\partial u} & \dfrac{\partial \psi}{\partial v} \end{vmatrix}$$

Then:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} F(u,v)\cdot |J|\,du\,dv$$

**Example**

Calculate $I = \iint_D (x-1)^2\,dx\,dy$, where the domain is $D = \{(x,y) \in \mathbb{R}^2, -1 \le x+y \le 1, -2 \le x-y \le 2\}$



Perform the change of variables:

$$\begin{cases} u = x + y \\ v = x - y \end{cases}$$

Then: $D_1 = \{(u, v) \in \mathbb{R}^2, -1 \leq u \leq 1 \text{ and } -2 \leq v \leq 2\}$
We also have:

$$x = \frac{u+v}{2}; \quad y = \frac{u-v}{2}$$

The Jacobian of this change of variables is:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Thus, we have:

$$\begin{aligned} I &= \frac{1}{2} \int_{-2}^{2} \int_{-1}^{1} \left( \frac{u+v}{2} - 1 \right)^2 du\, dv \\ &= \frac{1}{8} \int_{-2}^{2} \left[ \int_{-1}^{1} \left( (u+v)^2 - 2(u+v) + 4 \right) du \right] dv \\ I &= \frac{17}{3} \end{aligned}$$

**Change to Polar Coordinates**

In this case $u = \theta$, $v = r$:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

The Jacobian of the transformation from Cartesian coordinates $x$ and $y$ to polar coordinates $\theta$ and $r$ is given by:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

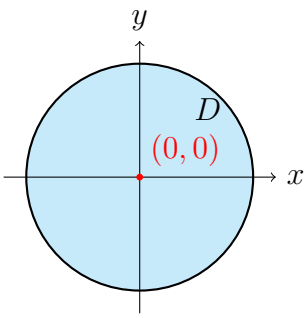
Therefore:

$$\iint_D f(x, y)\, dx\, dy = \iint_{D_1} F(r, \theta) \cdot r\, dr\, d\theta$$

**Example**

1 - Calculate $I = \iint_D \frac{1}{1 + x^2 + y^2}\, dx\, dy$, where $D = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 1\}$

By performing a change of variables to polar coordinates, we have: $D_1 = \{(r, \theta), 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi\}$

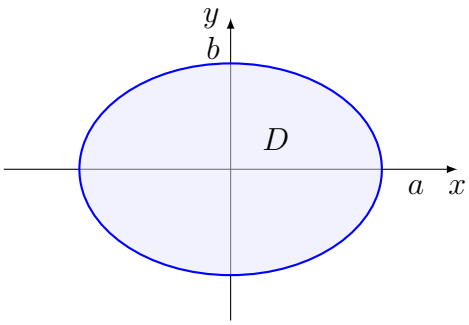



$$I = \iint_D \frac{1}{1+x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 \frac{r}{1+r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\frac{1}{2}\ln(1+r^2)\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{2}\ln 2\,d\theta = \pi\ln 2$$

2 - Compute the integral

$$I = \iint_D \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dy\,dx, \qquad D = \left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\},$$

where $a$ and $b$ are nonzero real numbers.




Change to elliptical polar coordinates Use the change of variables

$$x = ar\cos\theta, \qquad y = br\sin\theta, \qquad 0\le r\le 1, \quad 0\le\theta<2\pi.$$

Then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = r^2.$$

The Jacobian is

$$J = \left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det\begin{pmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta\end{pmatrix}\right| = |ab|\,r.$$

Integral in new variables Thus

$$I = \int_0^{2\pi}\int_0^1 \left(r^2\right)\cdot\left(|ab|\,r\right)dr\,d\theta = |ab|\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta.$$

Compute the radial integral:

$$\int_0^1 r^3\,dr = \frac{1}{4},$$

and the angular integral gives a factor $2\pi$. Therefore

$$I = |ab| \cdot 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}\,|ab|.$$

If $a > 0$ and $b > 0$ you may drop the absolute value and write $I = \frac{\pi}{2}ab$.

### 1.2.7 Some applications of double integrals

#### Area of a Domain

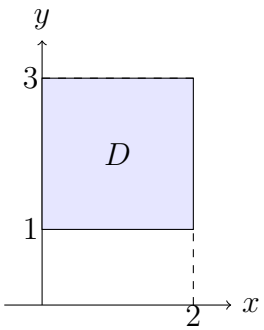
The area of a planar region $D$ is given by

$$A = \iint_D dA = \iint_D dy\,dx.$$

#### Example

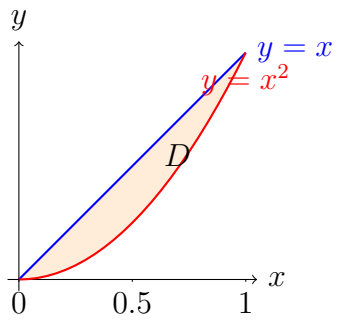
1 - Rectangle $D = [0, 2] \times [1, 3]$
Compute the area:

$$A = \iint_D dA = (2 - 0)(3 - 1) = 4.$$




2 - Region between curves $D = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1,\ x^2 \le y \le x\}$
Set up the integral:

$$A = \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 (x - x^2)\,dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{6}.$$




3 - For Region between curves $D = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 0, x^2 + 2x + 1 \le y \le (x^3 + 1)$ The area of $D$ is given by

$$A = \int_{-1}^{0} \left[(x^3 + 1) - (x^2 + 2x + 1)\right] dx.$$

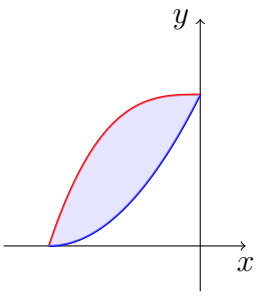
Simplifying the integrand:

$$A = \int_{-1}^{0} (x^3 - x^2 - 2x)\, dx.$$

Now compute the integral:

$$\begin{aligned} A &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \\ &= \Big(0\Big) - \left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2\right) \\ &= -\left(\frac{1}{4} + \frac{1}{3} - 1\right) \\ &= -\left(\frac{7}{12} - 1\right) = \frac{5}{12}. \end{aligned}$$

Thus, the area of the region $D$ is




**Surface Area**

The area of the surface $z = f(x, y)$ over a planar region $D$ is

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy.$$

**Example**

1- For the Plane $z = x + 2y$ on the unit square $D = [0, 1] \times [0, 1]$

Compute partial derivatives:

$$z_x = 1, \quad z_y = 2, \qquad \sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 1 + 4} = \sqrt{6}.$$

Therefore the surface area is

$$A = \iint_D \sqrt{6}\, dx\, dy = \sqrt{6} \cdot \text{Area}(D) = \sqrt{6} \cdot 1 = \sqrt{6}.$$

2- For the Plane $z = x^2 + y^2$ over the unit disk $D = \{x^2 + y^2 \leq 1\}$

We have $z_x = 2x,\ z_y = 2y$. In polar coordinates $x = r\cos\theta,\ y = r\sin\theta$,

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 4r^2}, \qquad dA = r\,dr\,d\theta.$$

Thus

$$A = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\; r\,dr\,d\theta = 2\pi\int_0^1 r\sqrt{1+4r^2}\,dr.$$

Set $u = 1 + 4r^2$, so $du = 8r\,dr$. Then

$$A = 2\pi\int_1^5 \frac{\sqrt{u}}{8}\,du = \frac{\pi}{4}\int_1^5 u^{1/2}\,du = \frac{\pi}{6}\big(5^{3/2} - 1\big).$$

**Mass of a Lamina**

Let $r(x, y)$ be the surface density of matter in a planar domain $D$. The total mass of the lamina that occupies $D$ is

$$m = \iint_D r(x, y)\,dx\,dy.$$

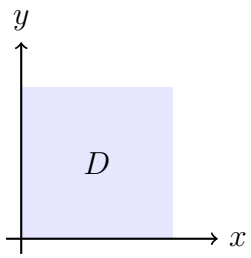
*Remark* 1.7.

- If $r(x, y) = 1$, the mass equals the area of $D$.
- If the region $D$ is described by iterated bounds, evaluate the double integral as an iterated integral.

**Example**

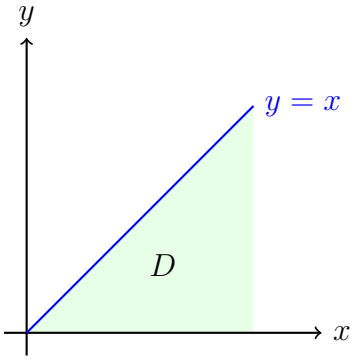
1- Unit square, $r(x, y) = x + y$ Let $D = [0, 1] \times [0, 1]$ and $r(x, y) = x + y$. Then

$$m = \int_0^1\int_0^1 (x+y)\,dy\,dx = \int_0^1 \Big[xy + \tfrac{1}{2}y^2\Big]_{y=0}^1 dx = \int_0^1 \Big(x + \tfrac{1}{2}\Big)\,dx = \Big[\tfrac{1}{2}x^2 + \tfrac{1}{2}x\Big]_0^1 = 1.$$




2- Triangular region, $r(x, y) = 1$ Let $D = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$ and $r \equiv 1$. Then the mass equals the area:

$$m = \iint_D 1\,dx\,dy = \int_0^1\int_0^x dy\,dx = \int_0^1 x\,dx = \frac{1}{2}.$$



### Moment of Inertia of a Plane Figure

The moment of inertia of a plane figure $D$ with surface density $r(x, y)$ about the $x$-axis is defined by

$$I_{ox} = \iint_D y^2\, r(x, y)\, dx\, dy.$$

similarly, The moment of inertia of a plane figure $D$ with surface density $r(x, y)$ about the $y$-axis is defined by

$$I_{oy} = \iint_D x^2\, r(x, y)\, dx\, dy.$$

and the moment of inertia of a plane figure $D$ with surface density $r(x, y)$ about the origin is

$$I_o = \iint_D (x^2 + y^2)\, r(x, y)\, dx\, dy.$$

### Example

1- Rectangle $D = [0, 2] \times [0, 1]$, $r = 1$

Compute

$$I_{ox} = \iint_D y^2\, dx\, dy = \int_0^2 \int_0^1 y^2\, dy\, dx = \int_0^2 \left[\frac{y^3}{3}\right]_0^1 dx = \int_0^2 \frac{1}{3}\, dx = \frac{2}{3}.$$

2- Rectangle $D = [0, 2] \times [0, 1]$, $r \equiv 1$

Compute

$$I_{oy} = \iint_D x^2\, dx\, dy = \int_{y=0}^1 \int_{x=0}^2 x^2\, dx\, dy = \int_0^1 \left[\frac{x^3}{3}\right]_0^2 dy = \int_0^1 \frac{8}{3}\, dy = \frac{8}{3}.$$

3- Rectangle $D = [0, 2] \times [0, 1]$, $r \equiv 1$

Compute

$$\begin{aligned} I_o &= \iint_D (x^2 + y^2)\, dx\, dy = \int_0^2 \int_0^1 (x^2 + y^2)\, dy\, dx \\ &= \int_0^2 \left(x^2 + \tfrac{1}{3}\right) dx = \left[\tfrac{x^3}{3}\right]_0^2 + \tfrac{2}{3} = \tfrac{10}{3}. \end{aligned}$$

### Center of Gravity

### Static Moment about the $x$ and $y$-axis

The static moment of a plane figure $D$ with surface density $r(x, y)$ about the $x$-axis is

$$M_x = \iint_D y\, r(x, y)\, dx\, dy.$$

Similarly, The static moment of a plane figure $D$ with surface density $r(x, y)$ about the $y$-axis is

$$M_y = \iint_D x\, r(x, y)\, dx\, dy.$$

**Example**

1- - Rectangle $D = [0,2] \times [0,1]$, $r \equiv 1$ Compute

$$M_x = \iint_D y\,dx\,dy = \int_0^2 \int_0^1 y\,dy\,dx = \int_0^2 \frac{1}{2}\,dx = 1.$$

2- Triangle $D = \{0 \le x \le 1,\ 0 \le y \le x\}$, $r \equiv 1$ Compute

$$M_x = \int_0^1 \int_0^x y\,dy\,dx = \int_0^1 \frac{1}{2}x^2\,dx = \frac{1}{6}.$$

3- Rectangle $D = [0,2] \times [0,1]$, $r \equiv 1$- Compute

$$M_y = \iint_D x\,dx\,dy = \int_0^1 \int_0^2 x\,dx\,dy = \int_0^1 \frac{x^2}{2}\Big|_0^2 dy = \int_0^1 2\,dy = 2.$$

4- Triangle $D = \{0 \le x \le 1,\ 0 \le y \le x\}$, $r \equiv 1$ Compute

$$M_y = \int_0^1 \int_0^x x\,dy\,dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$

## Coordinates of the Center of Gravity

The coordinates of the center of gravity of a plane figure are given by the formulas:

$$x_c = \frac{M_y}{m} \quad \text{and} \quad y_c = \frac{M_x}{m},$$

where

$$m = \iint_D r(x,y)\,dx\,dy, \qquad M_x = \iint_D y\,r(x,y)\,dx\,dy, \qquad M_y = \iint_D x\,r(x,y)\,dx\,dy.$$

**Example**

Let the density be uniform, $r(x,y) = 1$, and let $D$ be the rectangle

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 0 \le y \le 4\}.$$

Then:

$$m = \iint_D 1\,dx\,dy = (2-0)(4-0) = 8,$$

$$M_x = \iint_D y\,dx\,dy = \int_0^2 \int_0^4 y\,dy\,dx = 2 \times \frac{4^2}{2} = 16,$$

$$M_y = \iint_D x\,dx\,dy = \int_0^4 \int_0^2 x\,dx\,dy = 4 \times \frac{2^2}{2} = 8.$$

Thus, the coordinates of the center of gravity are:

$$x_c = \frac{M_y}{m} = \frac{8}{8} = 1, \qquad y_c = \frac{M_x}{m} = \frac{16}{8} = 2.$$

Hence, the center of gravity is at the point $G(1,2)$.

# 1.3 Triple Integrals and Applications

## 1.3.1 Definition of the Triple Integral

Let $E$ be a bounded solid region of the three–dimensional space $xOyOz$, enclosed by a closed surface. Consider in this region the function $f(x, y, z)$ of the three independent variables $x$, $y$, and $z$.

We divide the region $E$ into $n$ smaller parts (subregions)

$$\Delta v_1,\ \Delta v_2,\ \ldots,\ \Delta v_n,$$

and in each subregion $\Delta v_k$, choose an arbitrary point $M_k(x_k, y_k, z_k)$, whose image by the mapping $f$ is $f(M_k) = f(x_k, y_k, z_k)$.

We then consider the sum of the products

$$V_n = \sum_{k=1}^{n} f(M_k)\,\Delta v_k,$$

which is called the *integral sum* of the function $f(x, y, z)$ over the region $E$.

Now, consider the sequence $(V_n)_{n\in\mathbb{N}}$ of all such integral sums corresponding to all possible partitions of $E$, such that the largest subregion $\Delta v_k$ tends to 0 as $n \to +\infty$.

**Theorem 1.10.** *If $f(x, y, z)$ is continuous on $E$, then the sequence $(V_n)_{n\in\mathbb{N}}$ of integral sums has a limit when the largest subregion $\Delta v_k$ tends to 0 and $n \to +\infty$. This limit does not depend on the way $E$ is subdivided nor on the choice of the points $M_k$ in each $\Delta v_k$.*

**Definition 1.10.** The *triple integral* of the function $f(x, y, z)$ over the region $E$ is defined as the limit of the sequence of integral sums $(V_n)$ when the largest subregion $\Delta v_k \to 0$ and $n \to +\infty$.
It is denoted by

$$\lim_{\substack{\max \Delta v_k \to 0 \\ n\to+\infty}} \sum_{k=1}^{n} f(M_k)\,\Delta v_k = \iiint_E f(x, y, z)\,dx\,dy\,dz.$$

## 1.3.2 Properties of the Triple Integral

Let $f, g : E \to \mathbb{R}$ be integrable on $E$, and $\lambda, \mu \in \mathbb{R}$. Then:

- $\iiint_E (f + g)\,dx\,dy\,dz = \iiint_E f\,dx\,dy\,dz + \iiint_E g\,dx\,dy\,dz.$
- $\iiint_E \lambda f\,dx\,dy\,dz = \lambda \iiint_E f\,dx\,dy\,dz.$
- If $f \le g$ on $E$, then $\iiint_E f\,dx\,dy\,dz \le \iiint_E g\,dx\,dy\,dz.$
- If $E = E_1 \cup E_2$ with disjoint interiors, then

$$\iiint_E f\,dx\,dy\,dz = \iiint_{E_1} f\,dx\,dy\,dz + \iiint_{E_2} f\,dx\,dy\,dz.$$

### 1.3.3 Successive Simple Integrals — Fubini's Theorem for Triple Integrals

Let $E = [a, b] \times [c, d] \times [p, q]$ with $a < b$, $c < d$, and $p < q$ be a closed rectangular box in the space $\mathbb{R}^3$, and let $f : E \to \mathbb{R}$ be a continuous function.

For each fixed pair $(x, y) \in [a, b] \times [c, d]$, the function $f(x, y, \cdot) : [p, q] \to \mathbb{R}$, defined by

$$f(x, y, \cdot)(z) = f(x, y, z),$$

is integrable on $[p, q]$.

The number $\int_p^q f(x, y, \cdot)(t)\, dt$ depends on $x$ and $y$. Hence, we define a function $A : [a, b] \times [c, d] \to \mathbb{R}$ by

$$A(x, y) = \int_p^q f(x, y, t)\, dt.$$

Since $A$ is continuous on the rectangle $[a, b] \times [c, d]$, we can integrate it successively with respect to $y$ and $x$, obtaining

$$(*) \quad \int_a^b \int_c^d A(x, y)\, dy\, dx = \int_a^b \int_c^d \left( \int_p^q f(x, y, z)\, dz \right) dy\, dx.$$

In the expression

$$\int_a^b \int_c^d \left( \int_p^q f(x, y, z)\, dz \right) dy\, dx$$

from formula $(*)$, we say that we *first integrate with respect to* $z$, then with respect to $y$, and finally with respect to $x$.

Similarly, we may reverse the order of integration, e.g.,

$$\int_p^q \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz,$$

which corresponds to integrating successively with respect to $x$, then $y$, and finally $z$.

**Example**

Consider the rectangular box $E = [0, 1] \times [0, 2] \times [0, 3] \subset \mathbb{R}^3$, and let the function $f$ be defined on $E$ by $f(x, y, z) = x + y + z$.

Here, $a = 0, b = 1, c = 0, d = 2, p = 0, q = 3$.

- First, we integrate with respect to $z$:

$$\int_0^3 f(x, y, z)\, dz = \int_0^3 (x + y + z)\, dz = \left[ xz + yz + \tfrac{1}{2} z^2 \right]_0^3 = 3x + 3y + \tfrac{9}{2}.$$

- Then, we integrate with respect to $y$:

$$\int_0^2 (3x + 3y + \tfrac{9}{2})\, dy = \left[ 3xy + \tfrac{3}{2} y^2 + \tfrac{9}{2} y \right]_0^2 = 6x + 6 + 9 = 6x + 15.$$

- Finally, we integrate with respect to $x$:

$$\int_0^1 (6x + 15)\, dx = \left[ 3x^2 + 15x \right]_0^1 = 18.$$

Thus,

$$\iiint_E (x+y+z)\,dx\,dy\,dz = 18.$$

**Theorem 1.11** (Fubini's Theorem for Rectangular Boxes)**.** *Let $E = [a,b]\times[c,d]\times[p,q] \subset \mathbb{R}^3$, and let $f : E \to \mathbb{R}$ be a continuous function. Then $f$ is integrable on $E$, and all successive triple integrals are equal, that is,*

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \int_a^b\int_c^d\int_p^q f(x,y,z)\,dz\,dy\,dx = \int_p^q\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz = \cdots$$

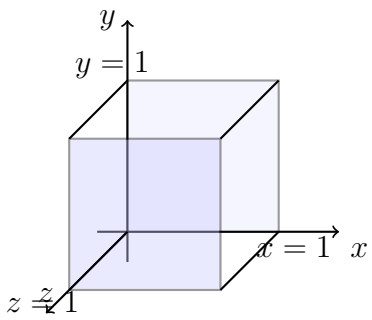
*The result is independent of the order of integration.*

*Remark* 1.8. If $f(x,y,z) = f_1(x)\,f_2(y)\,f_3(z)$, then by separation of variables we have

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \left(\int_a^b f_1(x)\,dx\right)\left(\int_c^d f_2(y)\,dy\right)\left(\int_p^q f_3(z)\,dz\right).$$

**Example**

1- Let $E = [0,1]\times[0,1]\times[0,1] \subset \mathbb{R}^3$ and $f(x,y,z) = xyz$.




Then

$$I = \iiint_E f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 xyz\,dz\,dy\,dx.$$

Integrating successively, we get

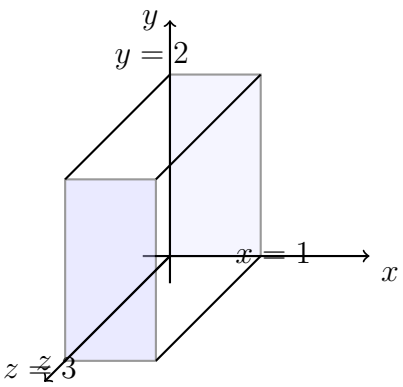
$$\int_0^1 xyz\,dz = \tfrac{1}{2}xy, \quad \int_0^1 \tfrac{1}{2}xy\,dy = \tfrac{1}{4}x, \quad \int_0^1 \tfrac{1}{4}x\,dx = \tfrac{1}{8}.$$

Thus,

$$I = \tfrac{1}{8}.$$

2- Consider the rectangular box $E = [0,1]\times[0,2]\times[0,3] \subset \mathbb{R}^3$, and let $f(x,y,z) = x^2y + z$. Compute

$$I = \iiint_E f(x,y,z)\,dx\,dy\,dz.$$




We apply Fubini's theorem by integrating successively with respect to $x$, $y$, and $z$.

- Integrate first with respect to $x$:

$$\int_0^1 f(x,y,z)\,dx = \int_0^1 (x^2y + z)\,dx = \left[\tfrac{1}{3}x^3y + xz\right]_0^1 = \tfrac{1}{3}y + z.$$

- Integrate the result with respect to $y$:

$$\int_0^2 \left(\tfrac{1}{3}y + z\right)\,dy = \left[\tfrac{1}{6}y^2 + yz\right]_0^2 = \tfrac{2}{3} + 2z.$$

- Finally, integrate with respect to $z$:

$$I = \int_0^3 \left(\tfrac{2}{3} + 2z\right)\,dz = \left[\tfrac{2}{3}z + z^2\right]_0^3 = 2 + 9 = 11.$$

Thus, the value of the triple integral is $I = 11$.

### 1.3.4 Triple Integrals over Non-Rectangular Regions

Let $E$ be a bounded region in the three-dimensional space $\mathbb{R}^3$, and let $f : E \to \mathbb{R}$. We wish to compute (if it exists) the integral

$$\iiint_E f(x,y,z)\,dx\,dy\,dz.$$

If the region $E$ under consideration is not a rectangular box but is bounded, we can include it in a rectangular solid $R$, consider a partition (grid) of $R$, and define a triple Riemann sum over those subregions of the grid that are entirely contained in $E$.

If the triple Riemann sum tends to a limit $I \in \mathbb{R}$ as the mesh of the partition tends to 0, then the function $f$ is said to be integrable over $E$, and we write

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = I.$$

#### Integrals over a Region Bounded by Two Surfaces and Two Planes Parallel to the $xOy$-Plane

Let $[a,b]$ $(a < b)$ and $[c,d]$ $(c < d)$ be closed and bounded intervals in $\mathbb{R}$. Let $u(x,y)$ and $v(x,y)$ be two continuous functions defined on the rectangle $[a,b] \times [c,d]$ such that

$$\forall (x,y) \in [a,b] \times [c,d], \quad u(x,y) \le v(x,y).$$

Let $E$ be the region in $\mathbb{R}^3$ defined by

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b,\ c \le y \le d,\ u(x, y) \le z \le v(x, y)\}.$$

Then $E$ is the solid bounded above and below by the surfaces $z = v(x, y)$ and $z = u(x, y)$, and laterally by the planes $x = a, x = b, y = c, y = d$.

**Theorem 1.12.** *If $f : E \to \mathbb{R}$ is continuous on $E$, then $f$ is integrable over $E$, and*

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \int_a^b \int_c^d \left( \int_{u(x,y)}^{v(x,y)} f(x, y, z)\, dz \right) dy\, dx.$$

**Example**

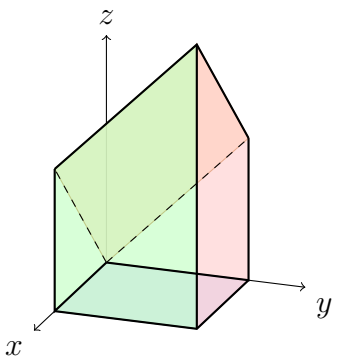
Let

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le x + y\},$$

and let $f(x, y, z) = z$. Compute

$$I = \iiint_E f(x, y, z)\, dx\, dy\, dz.$$

For each fixed $(x, y) \in [0, 1] \times [0, 1]$, the variable $z$ varies from 0 to $x + y$. Thus, $E$ is bounded by the planes $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1$, and the oblique plane $z = x + y$.




The integral can be written as

$$I = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{x+y} z\, dz\, dy\, dx.$$

Compute the inner integral:

$$\int_0^{x+y} z\, dz = \frac{1}{2}(x + y)^2.$$

Hence,

$$I = \frac{1}{2} \int_0^1 \int_0^1 (x + y)^2\, dy\, dx.$$

Expand:

$$I = \frac{1}{2} \int_0^1 \int_0^1 (x^2 + 2xy + y^2)\, dy\, dx = \frac{1}{2} \int_0^1 \left[ x^2 y + xy^2 + \frac{y^3}{3} \right]_0^1 dx.$$

Simplify:

$$I = \frac{1}{2}\int_0^1 \left(x^2 + x + \frac{1}{3}\right) dx = \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{3}\right]_0^1.$$

Thus,

$$I = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{3}\right) = \frac{1}{2} \times \frac{7}{6} = \frac{7}{12}.$$

**Integrals over a Region Bounded by Two Surfaces and Two Planes Parallel to the $xOz$-Plane**

**Theorem 1.13.** *Let $[a, b]$ $(a < b)$ be a closed and bounded interval in $\mathbb{R}$, and let $u$ and $v$ be two continuous functions on $[a, b]$ such that*

$$\forall x \in [a, b], \quad u(x) \leq v(x).$$

*Let $E$ be the region in $\mathbb{R}^3$ defined by*

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b,\ u_1(x, z) \leq y \leq v_1(x, z),\ c \leq z \leq d\}.$$

*If $f : E \to \mathbb{R}$ is a continuous function, then $f$ is integrable over $E$, and*

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \int_a^b \int_c^d \left(\int_{u_1(x,z)}^{v_1(x,z)} f(x, y, z)\, dy\right) dz\, dx.$$

**Example**

Let

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1,\ 0 \leq z \leq 1,\ 0 \leq y \leq x + z\},$$

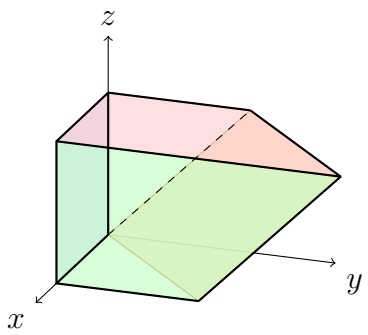
and let $f(x, y, z) = y$. Compute

$$I = \iiint_E f(x, y, z)\, dx\, dy\, dz.$$

The region $E$ is bounded by:

$$x = 0, \quad x = 1, \quad z = 0, \quad z = 1, \quad y = 0, \quad \text{and} \quad y = x + z.$$

It lies below the plane $y = x + z$ in the first octant.



We have for fixed $x$ and $z$: $0 \le y \le x+z$. Thus,

$$I = \int_{x=0}^{1}\int_{z=0}^{1}\int_{y=0}^{x+z} y\,dy\,dz\,dx.$$

Compute the inner integral:

$$\int_{y=0}^{x+z} y\,dy = \left[\tfrac{1}{2}y^2\right]_0^{x+z} = \tfrac{1}{2}(x+z)^2.$$

Then,

$$I = \tfrac{1}{2}\int_{x=0}^{1}\int_{z=0}^{1}(x+z)^2\,dz\,dx = \tfrac{1}{2}\int_{x=0}^{1}\int_{z=0}^{1}(x^2+2xz+z^2)\,dz\,dx.$$

Compute the integral with respect to $z$:

$$\int_{z=0}^{1}(x^2+2xz+z^2)\,dz = x^2z + xz^2 + \tfrac{1}{3}z^3\Big|_0^1 = x^2+x+\tfrac{1}{3}.$$

Hence,

$$I = \tfrac{1}{2}\int_0^1 \left(x^2+x+\tfrac{1}{3}\right)dx = \tfrac{1}{2}\left[\tfrac{x^3}{3}+\tfrac{x^2}{2}+\tfrac{x}{3}\right]_0^1 = \tfrac{1}{2}\left(\tfrac{1}{3}+\tfrac{1}{2}+\tfrac{1}{3}\right) = \tfrac{1}{2}\cdot\tfrac{7}{6} = \tfrac{7}{12}.$$

### Integrals over Regions Bounded by Two Surfaces and Two Planes Parallel to the $yOz$-Plane

The results of this section are obtained from those of the previous one by interchanging the roles of $x$ and $y$.

Let $[c,d]$ $(c<d)$ be a closed and bounded interval in $\mathbb{R}$, and let $u$ and $v$ be two continuous functions defined on $[c,d]$ such that

$$\forall y \in [c,d], \quad u(y) \le v(y).$$

Let $E$ be the region of the space bounded by the graphs of $u$ and $v$, and by the planes $y=c$ and $y=d$. Formally,

$$E = \{(x,y,z)\in\mathbb{R}^3 \mid c \le y \le d,\ u(y,z) \le x \le v(y,z),\ p \le z \le q\}.$$

**Theorem 1.14.** *If $f : E \to \mathbb{R}$ is a continuous function, then $f$ is integrable over $E$, and*

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \int_c^d\int_p^q\left(\int_{u(y,z)}^{v(y,z)} f(x,y,z)\,dx\right)dz\,dy.$$

### Example

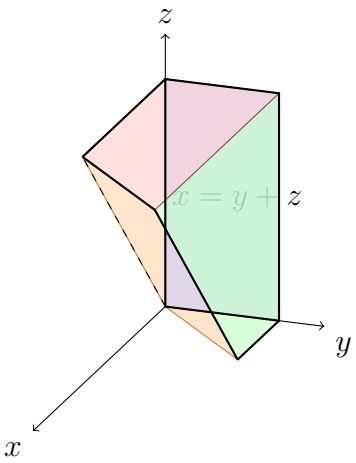
Let

$$E = \{(x,y,z)\in\mathbb{R}^3 \mid 0\le y\le 1,\ 0\le z\le 2,\ 0\le x\le y+z\},$$

and let $f(x, y, z) = x$. Compute

$$I = \iiint_E f(x, y, z)\, dx\, dy\, dz.$$

For each fixed $y$ and $z$, the variable $x$ varies from 0 to $y + z$. Thus, the region $E$ is bounded by the coordinate planes $x = 0,\ y = 0,\ z = 0$, the plane $y = 1$, the plane $z = 2$, and the surface $x = y + z$.




We can write

$$I = \int_{y=0}^{1} \int_{z=0}^{2} \int_{x=0}^{y+z} x\, dx\, dz\, dy.$$

Compute the inner integral:

$$\int_{x=0}^{y+z} x\, dx = \left[\tfrac{1}{2}x^2\right]_0^{y+z} = \tfrac{1}{2}(y + z)^2.$$

Then,

$$I = \tfrac{1}{2} \int_{y=0}^{1} \int_{z=0}^{2} (y + z)^2\, dz\, dy = \tfrac{1}{2} \int_{y=0}^{1} \int_{z=0}^{2} (y^2 + 2yz + z^2)\, dz\, dy.$$

Integrate with respect to $z$:

$$\int_0^2 (y^2 + 2yz + z^2)\, dz = y^2 z + yz^2 + \tfrac{1}{3}z^3 \Big|_0^2 = 2y^2 + 4y + \tfrac{8}{3}.$$

Hence,

$$I = \tfrac{1}{2} \int_0^1 \left(2y^2 + 4y + \tfrac{8}{3}\right) dy = \tfrac{1}{2} \left[\tfrac{2y^3}{3} + 2y^2 + \tfrac{8y}{3}\right]_0^1.$$

Simplify:

$$I = \tfrac{1}{2} \left(\tfrac{2}{3} + 2 + \tfrac{8}{3}\right) = \tfrac{1}{2} \cdot \tfrac{16}{3} = \tfrac{8}{3}.$$

### 1.3.5 Triple Integrals over a General Region $E$

If $E \subset \mathbb{R}^3$ is not a rectangular box and cannot be defined by the graphs of two functions and two planes, we can, if possible, decompose $E$ into "elementary" regions of the types previously studied.

Then we use the following property to perform the computation.

**Property 1.2.** *Let $E$ be a closed and bounded region of $\mathbb{R}^3$. Suppose that $E$ is the union of two closed regions $E_1$ and $E_2$ such that*

$$E = E_1 \cup E_2, \qquad \text{and} \qquad \text{int}(E_1) \cap \text{int}(E_2) = \varnothing.$$

*If $f : E \to \mathbb{R}$ is continuous on $E$, then $f$ is integrable over $E$, and*

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iiint_{E_1} f(x,y,z)\,dx\,dy\,dz + \iiint_{E_2} f(x,y,z)\,dx\,dy\,dz.$$

**Example**

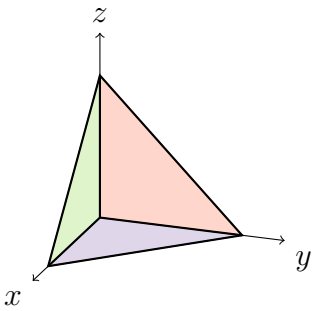
Let $f(x,y,z) = xyz$, and let $E$ be the solid bounded by the coordinate planes and the plane $x + y + z = 1$.

We wish to compute

$$I = \iiint_E f(x,y,z)\,dx\,dy\,dz.$$

The region $E$ is defined by

$$E = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + z \leq 1\}.$$




For a fixed $x$ and $y$, $z$ varies from 0 to $1 - x - y$. Hence,

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\,dz\,dy\,dx.$$

Compute the inner integral:

$$\int_0^{1-x-y} xyz\,dz = xy\left[\tfrac{1}{2}z^2\right]_0^{1-x-y} = \tfrac{1}{2}xy(1-x-y)^2.$$

Next, integrate with respect to $y$:

$$\int_0^{1-x} \tfrac{1}{2}xy(1-x-y)^2\,dy = \tfrac{x}{2}\int_0^{1-x} y(1-x-y)^2\,dy.$$

Let $t = 1 - x - y \Rightarrow dt = -dy$. When $y = 0$, $t = 1 - x$; when $y = 1 - x$, $t = 0$. Then

$$\tfrac{x}{2}\int_{1-x}^0 (1-x-t)t^2(-dt) = \tfrac{x}{2}\int_0^{1-x} (1-x-t)t^2\,dt.$$

Expand:

$$\tfrac{x}{2}\int_0^{1-x} \left((1-x)t^2 - t^3\right)\,dt = \tfrac{x}{2}\left[(1-x)\tfrac{t^3}{3} - \tfrac{t^4}{4}\right]_0^{1-x}.$$

Thus,

$$I = \tfrac{x}{2}\left[(1-x)\tfrac{(1-x)^3}{3} - \tfrac{(1-x)^4}{4}\right] = \tfrac{x(1-x)^4}{24}.$$

Finally, integrate with respect to $x$:

$$I = \int_0^1 \tfrac{x(1-x)^4}{24}\,dx = \tfrac{1}{24}\int_0^1 (x - 4x^2 + 6x^3 - 4x^4 + x^5)\,dx.$$

Compute:

$$I = \tfrac{1}{24}\left[\tfrac{x^2}{2} - \tfrac{4x^3}{3} + \tfrac{3x^4}{2} - \tfrac{4x^5}{5} + \tfrac{x^6}{6}\right]_0^1 = \tfrac{1}{24}\left(\tfrac{1}{2} - \tfrac{4}{3} + \tfrac{3}{2} - \tfrac{4}{5} + \tfrac{1}{6}\right) = \tfrac{1}{720}.$$

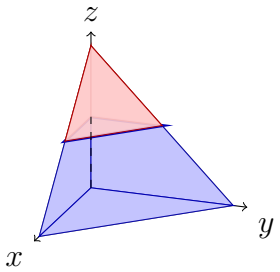
**Example**

Let $f(x,y,z) = x + y + z$, and let $E$ be the solid bounded by the coordinate planes $x = 0, y = 0, z = 0$, and the plane $x + y + z = 1$.

We divide $E$ into two simpler solids:

$$E_1 = \{(x,y,z) \in \mathbb{R}^3 : 0 \le z \le \tfrac{1}{2},\ x,y \ge 0,\ x+y+z \le 1\},$$

$$E_2 = \{(x,y,z) \in \mathbb{R}^3 : \tfrac{1}{2} \le z \le 1,\ x,y \ge 0,\ x+y+z \le 1\}.$$




Thus $E = E_1 \cup E_2$ and $\operatorname{int}(E_1) \cap \operatorname{int}(E_2) = \varnothing$.

We wish to compute

$$I = \iiint_E f(x,y,z)\,dx\,dy\,dz.$$

- Integration over $E_1$:

$$I_1 = \int_{z=0}^{1/2}\int_{x=0}^{1-z}\int_{y=0}^{1-z-x} (x+y+z)\,dy\,dx\,dz.$$

Compute the inner integral:

$$\int_0^{1-z-x} (x+y+z)\,dy = (x+z)(1-z-x) + \tfrac{1}{2}(1-z-x)^2.$$

Then integrate with respect to $x$:

$$I_1 = \int_0^{1/2}\int_0^{1-z}\left[(x+z)(1-z-x) + \tfrac{1}{2}(1-z-x)^2\right]dx\,dz.$$

A direct computation gives

$$I_1 = \frac{13}{96}.$$

- Integration over $E_2$:

$$I_2 = \int_{z=1/2}^{1} \int_{x=0}^{1-z} \int_{y=0}^{1-z-x} (x+y+z)\, dy\, dx\, dz.$$

Repeating the same steps, we find

$$I_2 = \frac{5}{96}.$$

Additivity:

$$I = I_1 + I_2 = \frac{13}{96} + \frac{5}{96} = \frac{18}{96} = \frac{3}{16}.$$

### 1.3.6 Change of Variables in a Triple Integral

#### General Case

Suppose we perform the change of variables:

$$\begin{cases} x = \varphi(u,v,w) \\ y = \psi(u,v,w) \\ z = \chi(u,v,w) \end{cases}$$

When $(x,y,z)$ varies in the domain $E$, $(u,v,w)$ varies in a domain $E_1$. The Jacobian of the transformation is given by:

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Then:

$$\iiint_E f(x,y,z)\, dx\, dy\, dz = \iiint_{E_1} F(u,v,w)\, |J|\, du\, dv\, dw$$

#### Example

Evaluate

$$I = \iiint_E (x+y+z)\, dx\, dy\, dz,$$

where $E = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x+y+z \le 1,\ 0 \le x-y \le 1,\ 0 \le y-z \le 1\}$.

Perform the change of variables:

$$\begin{cases} u = x+y+z, \\ v = x-y, \\ w = y-z. \end{cases}$$

Then the domain becomes $E_1 = \{(u,v,w) : 0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le w \le 1\}$.

The inverse transformation gives:

$$\begin{cases} x = \dfrac{u+2v+w}{3}, \\ y = \dfrac{u-v+w}{3}, \\ z = \dfrac{u-v-2w}{3}. \end{cases}$$

The Jacobian is:

$$J = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \frac{1}{3}.$$

Thus:

$$I = \frac{1}{3} \iiint_{E_1} u \, du \, dv \, dw = \frac{1}{3} \int_0^1 \int_0^1 \int_0^1 u \, du \, dv \, dw = \frac{1}{3} \cdot \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{6}.$$

**Change to Cylindrical Coordinates**

We define:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = z, \end{cases}$$

where $r \geq 0$, $0 \leq \theta < 2\pi$, and $z$ is free. The Jacobian is:

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Hence:

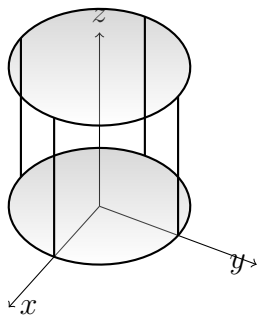
$$\iiint_E f(x,y,z) \, dx \, dy \, dz = \iiint_{E_1} F(r,\theta,z) \, r \, dr \, d\theta \, dz.$$

**Example**

Compute

$$I = \iiint_E r^2 \, dx \, dy \, dz,$$

where $E$ is the cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 2$.



Then:

$$I = \int_0^{2\pi} \int_0^1 \int_0^2 r^2 \cdot r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 2r^3 \, dr \, d\theta = 2\pi \cdot \frac{1}{2} = \pi.$$

**Change to Spherical Coordinates**

We define:

$$\begin{cases} x = r \sin\phi \cos\theta, \\ y = r \sin\phi \sin\theta, \\ z = r \cos\phi, \end{cases}$$

where $0 \leq r < \infty$, $0 \leq \phi \leq \pi$, and $0 \leq \theta < 2\pi$. The Jacobian is: $J = r^2 \sin\phi$. Thus:

$$\iiint_E f(x,y,z) \, dx \, dy \, dz = \iiint_{E_1} F(r,\phi,\theta) \, r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

**Example**

Compute

$$I = \iiint_E (x^2 + y^2 + z^2) \, dx \, dy \, dz,$$

where $E$ is the ball $x^2 + y^2 + z^2 \leq R^2$.

In spherical coordinates:

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^R r^2 \cdot r^2 \sin\phi \, dr \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_0^R r^4 \sin\phi \, dr \, d\phi \, d\theta.$$

Then:

$$I = \left(\frac{R^5}{5}\right)(2)\,(2\pi) = \frac{4\pi R^5}{5}.$$

### 1.3.7 Some Applications of Triple Integrals

**Volume of a Solid**

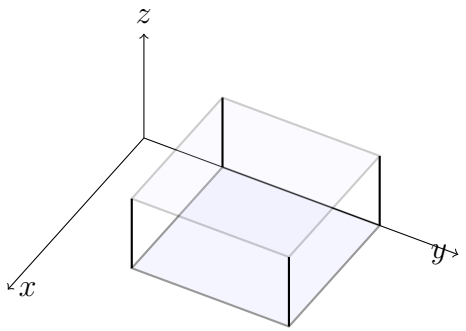
The volume of a solid $E$ in $\mathbb{R}^3$ is

$$V = \iiint_E dx \, dy \, dz.$$

**Example**

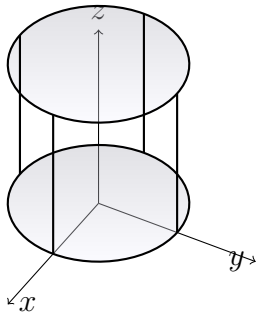
1- Rectangular

$$E = [0,2] \times [1,3] \times [0,1], \quad V = \iiint_E dx \, dy \, dz = (2-0)(3-1)(1-0) = 4.$$



2- Cylinder $E = \{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq 2\}$,

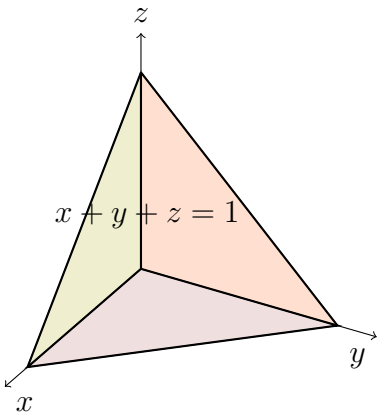
$$V = \int_0^2 \iint_{x^2+y^2\leq 1} dx\, dy\, dz = 2\pi.$$




3- Compute the volume of the solid bounded by the coordinate planes and the plane $x + y + z = 1$.

The region $V$ lies in the first octant where $x, y, z \geq 0$, and is bounded above by the plane $x + y + z = 1$. We can describe $V$ as follows:

$$E = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + z \leq 1\}.$$




The volume is given by

$$V = \iiint_E dx\, dy\, dz.$$

We first integrate with respect to $z$:

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx.$$

Compute the inner integral:

$$\int_0^{1-x-y} dz = 1 - x - y.$$

Substitute back:

$$V = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx.$$

Integrate with respect to $y$:

$$\int_0^{1-x} (1 - x - y)\, dy = (1 - x)(1 - x) - \tfrac{1}{2}(1 - x)^2 = \tfrac{1}{2}(1 - x)^2.$$

Finally, integrate with respect to $x$:

$$V = \int_0^1 \tfrac{1}{2}(1 - x)^2\, dx = \tfrac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

Thus, the volume of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$ is $\frac{1}{6}$.

**Mass of a Solid**

If $r(x, y, z)$ is the density of $E$, then

$$m = \iiint_E r(x, y, z)\, dx\, dy\, dz.$$

**Example**

1- Unit Cube with $r = x + y + z$ and $E = [0, 1]^3$,

$$m = \int_0^1 \int_0^1 \int_0^1 (x + y + z)\, dz\, dy\, dx = \frac{3}{2}.$$

2- Cylinder with $r = 1$ and $E = \{x^2 + y^2 \leq 1, 0 \leq z \leq 2\}$,

$$\begin{aligned} m &= \iiint_E 1\, dx\, dy\, dz &&= \int_0^1 \int_0^2 \int_0^{2\pi} r\ d\theta\, dz dr \\ &= \int_0^1 r\ dr. \int_0^2 dz. \int_0^{2\pi}\ d\theta &&= \left[\frac{1}{2}r^2\right]_0^1 [z]_0^2\, [\theta]_0^{2\pi} = 2\pi. \end{aligned}$$

3- Find the mass of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, if the density is given by $\rho(x, y, z) = x + y + z$.

The region $E$ is

$$E = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ x + y + z \leq 1\}.$$

Thus,

$$m = \iiint_E (x + y + z)\, dx\, dy\, dz.$$

We first integrate with respect to $z$:

$$m = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z)\, dz\, dy\, dx.$$

Compute the inner integral:

$$\int_0^{1-x-y}(x+y+z)\,dz = (x+y)(1-x-y)+\tfrac{1}{2}(1-x-y)^2.$$

Simplify:

$$(x+y)(1-x-y)+\tfrac{1}{2}(1-x-y)^2 = \tfrac{1}{2}(1-x-y)(2x+2y+1-x-y) = \tfrac{1}{2}(1-x-y)(x+y+1).$$

Then

$$m = \tfrac{1}{2}\int_0^1\int_0^{1-x}(1-x-y)(x+y+1)\,dy\,dx.$$

We expand and integrate with respect to $y$:

$$(1-x-y)(x+y+1) = (1-x)(x+1) - y(2x+1) - y^2.$$

Thus,

$$\int_0^{1-x}(1-x-y)(x+y+1)\,dy = (1-x)^2 - \tfrac{1}{2}(1-x)^2(2x+1) - \tfrac{1}{3}(1-x)^3.$$

After simplification, the result is $m = \frac{1}{8}$.

Hence, the total mass of the tetrahedron is $\frac{1}{8}$ (in suitable mass units).

**Moment of Inertia of a Solid Region**

**Moment of Inertia of a Plane Figure with Respect to the $x$, $y$ and $z$-axis and Origin $O$**

The moment of inertia of a solid region $E \subset \mathbb{R}^3$ with respect to the $x$-axis, when the density of the material at each point is given by $\rho(x,y,z)$, is defined by

$$I_{Ox} = \iiint_E (y^2+z^2)\,\rho(x,y,z)\,dx\,dy\,dz.$$

Similarly, the moments of inertia with respect to the $y$- and $z$-axes are defined as

$$I_{Oy} = \iiint_E (x^2+z^2)\,\rho(x,y,z)\,dx\,dy\,dz, \qquad I_{Oz} = \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dx\,dy\,dz.$$

**Example**

1- Find the moment of inertia of the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$ with respect to the $x$-axis, assuming the density is constant, $\rho(x,y,z)=1$.

The region $E$ is

$$E = \{(x,y,z)\in\mathbb{R}^3 : x,y,z \geq 0,\ x+y+z \leq 1\}.$$

Then

$$I_{Ox} = \iiint_E (y^2+z^2)\,dx\,dy\,dz.$$

We integrate with respect to $z$ first:

$$I_{Ox} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (y^2 + z^2)\, dz\, dy\, dx.$$

Compute the inner integral:

$$\int_0^{1-x-y} (y^2 + z^2)\, dz = y^2(1-x-y) + \tfrac{1}{3}(1-x-y)^3.$$

Thus,

$$I_{Ox} = \int_0^1 \int_0^{1-x} \left[y^2(1-x-y) + \tfrac{1}{3}(1-x-y)^3\right] dy\, dx.$$

Compute the first term:

$$\int_0^{1-x} y^2(1-x-y)\, dy = (1-x)\frac{(1-x)^3}{3} - \frac{(1-x)^4}{4} = \frac{(1-x)^4}{12}.$$

Compute the second term:

$$\int_0^{1-x} (1-x-y)^3\, dy = \frac{(1-x)^4}{4}.$$

So,

$$I_{Ox} = \int_0^1 \left[\frac{(1-x)^4}{12} + \frac{1}{3}\cdot\frac{(1-x)^4}{4}\right] dx = \int_0^1 \frac{(1-x)^4}{6}\, dx.$$

Compute:

$$I_{Ox} = \cdot\frac{1}{6}\cdot\frac{1}{5} = \frac{1}{30}.$$

2- Find the moments of inertia of the right triangle bounded by the coordinate axes and the line $x + y = a$ with respect to the $x$-axis and $y$-axis, assuming constant surface density $\rho(x, y) = 1$.

The region $E$ is defined by

$$E = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x + y \leq a\}.$$

The required integrals are:

$$I_{Ox} = \iint_E y^2\, dx\, dy, \qquad I_{Oy} = \iint_E x^2\, dx\, dy.$$

We compute $I_{Ox}$ first. For each $x \in [0, a]$, $y$ varies from 0 to $a - x$:

$$I_{Ox} = \int_0^a \int_0^{a-x} y^2\, dy\, dx = \int_0^a \frac{(a-x)^3}{3}\, dx.$$

Integrate with respect to $x$:

$$I_{Ox} = \frac{1}{3}\int_0^a (a-x)^3\, dx = \frac{1}{3}\cdot\frac{a^4}{4} = \frac{a^4}{12}.$$

Similarly,

$$I_{Oy} = \int_0^a \int_0^{a-y} x^2\, dx\, dy = \int_0^a \frac{(a-y)^3}{3}\, dy = \frac{a^4}{12}.$$

Thus, the moments of inertia of the triangular lamina about the coordinate axes are equal.

3- Find the moment of inertia of the right triangle bounded by the coordinate axes and the line $x + y = a$ with respect to the $z$-axis, assuming a constant surface density $\rho(x, y) = \rho_0$.

The region of the lamina is

$$E = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x + y \leq a\}.$$

Then,

$$I_{Oz} = \iint_E (x^2 + y^2)\, dx\, dy.$$

We first compute:

$$I_{Oz} = \left[\iint_D x^2\, dx\, dy + \iint_D y^2\, dx\, dy\right] = I_x + I_y,$$

Because of the symmetry of the triangle with respect to the line $x = y$, we have $I_x = I_y$. Hence,

$$I_{Oz} = 2I_x.$$

Now compute $I_x$:

$$I_x = \int_0^a \int_0^{a-x} x^2\, dy\, dx = \int_0^a x^2(a - x)\, dx.$$

Integrate:

$$I_x = a\int_0^a x^2\, dx - \int_0^a x^3\, dx = a\frac{a^3}{3} - \frac{a^4}{4} = \frac{a^4}{12}.$$

Therefore,

$$I_{Oz} = 2 \cdot \frac{a^4}{12} = \frac{a^4}{6}.$$

**Moment of Inertia of a Plane Figure with Respect to the $Oyz$,$Oxz$, and $Oxy$-Plane and Origin $O$**

Let a plane figure (or a solid region) occupy a volume $E \subset \mathbb{R}^3$, and let $\rho(x, y, z)$ denote the density of the material at each point. The moment of inertia of this figure with respect to the coordinate plane $Oyz$ is defined by the triple integral

$$I_{yz} = \iiint_E x^2\, \rho(x, y, z)\, dx\, dy\, dz.$$

Similarly, the moments of inertia with respect to the the $Oxz$ and $Oxy$ -Plane and Origin $O$ are defined as

$$I_{xy} = \iiint_E z^2\, \rho(x, y, z)\, dx\, dy\, dz. \quad I_{xz} = \iiint_E y^2\, \rho(x, y, z)\, dx\, dy\, dz. \quad I_O = \iiint_E (x^2+y^2+z^2)\, \rho(x, y, z)\, dx\, d$$

**Example**

1- Compute the moment of inertia of the right triangle bounded by the coordinate axes and the line $x + y = a$ with respect to the $Oyz$-plane, assuming constant surface density $\rho(x, y) = 1$.

The lamina occupies the region

$$E = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x + y \leq a\}.$$

Then,

$$I_{yz} = \iint_E x^2 \, dx \, dy.$$

We express the domain using the limits:

$$I_{yz} = \int_0^a \int_0^{a-x} x^2 \, dy \, dx.$$

Integrate with respect to $y$:

$$I_{yz} = \int_0^a x^2(a - x) \, dx = \left( a \int_0^a x^2 \, dx - \int_0^a x^3 \, dx \right).$$

Compute the integrals:

$$\int_0^a x^2 \, dx = \frac{a^3}{3}, \qquad \int_0^a x^3 \, dx = \frac{a^4}{4}.$$

Thus,

$$I_{yz} = a \cdot \frac{a^3}{3} - \frac{a^4}{4} = \frac{a^4}{12}.$$

2- Find the moment of inertia with respect to the $Oxy$-plane of a homogeneous cube of side $a$, whose edges are parallel to the coordinate axes, occupying the region

$$E = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq a,\ 0 \leq y \leq a,\ 0 \leq z \leq a\},$$

and having constant density $\rho(x, y, z) = 1$.

$$I_{xy} = \iiint_E z^2, dx \, dy \, dz = \int_0^a \int_0^a \int_0^a z^2 \, dz \, dy \, dx.$$

Compute the innermost integral:

$$\int_0^a z^2 \, dz = \frac{a^3}{3}.$$

Hence,

$$I_{xy} = \int_0^a \int_0^a \frac{a^3}{3} \, dy \, dx = \frac{a^3}{3} \cdot a^2 = \frac{a^5}{3}.$$

3- Compute the moment of inertia with respect to the $Oxz$-plane of a homogeneous cube of side $a$, whose edges are parallel to the coordinate axes and occupies the region

$$E = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq a,\ 0 \leq y \leq a,\ 0 \leq z \leq a\},$$

with constant density $\rho(x, y, z) = 1$.

$$I_{xz} = \iiint_E y^2 \, dx \, dy \, dz = \int_0^a \int_0^a \int_0^a y^2 \, dy \, dz \, dx.$$

Compute the innermost integral:

$$\int_0^a y^2 \, dy = \frac{a^3}{3}.$$

Thus,

$$I_{xz} = \int_0^a \int_0^a \frac{a^3}{3} \, dz \, dx = \frac{a^3}{3} \cdot a^2 = \frac{a^5}{3}.$$

4- Find the moment of inertia with respect to the origin $O$ of a homogeneous cube of side $a$, whose edges are parallel to the coordinate axes and occupies the region

$$E = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq a,\ 0 \leq y \leq a,\ 0 \leq z \leq a\},$$

with constant density $\rho(x, y, z) = 1$.

$$I_O = \iiint_E (x^2 + y^2 + z^2) \, dx \, dy \, dz = \iiint_E (x^2 + y^2 + z^2) \, dx \, dy \, dz.$$

Since the region is a rectangular box and the integrand is separable, we can write

$$I_O = \left( \iiint_E x^2 \, dx \, dy \, dz + \iiint_E y^2 \, dx \, dy \, dz + \iiint_E z^2 \, dx \, dy \, dz \right).$$

Each of these three integrals is the same by symmetry:

$$\iiint_E x^2 \, dx \, dy \, dz = \int_0^a x^2 \, dx \int_0^a dy \int_0^a dz = \frac{a^5}{3}.$$

Thus,

$$I_O = 3 \times \frac{a^5}{3} = a^5.$$

**Center of Gravity**

Let $E \subset \mathbb{R}^3$ be a solid region with density function $\rho(x, y, z)$, representing the mass per unit volume at each point $(x, y, z)$.

**Static Moments.**

The static moment of a body with respect to the coordinate axes is defined as follows:

$$M_x = \iiint_E x \, \rho(x, y, z) \, dx \, dy \, dz, \qquad M_y = \iiint_E y \, \rho(x, y, z) \, dx \, dy \, dz, \qquad M_z = \iiint_E z \, \rho(x, y, z) \, dx \, dy \, dz.$$

These quantities represent the "weighted sums" of the coordinates with respect to the density distribution.

**Coordinates of the Center of Gravity.**

The coordinates of the center of gravity (or center of mass) of the solid $E$ are given by

$$x_c = \frac{M_x}{m}, \qquad y_c = \frac{M_y}{m}, \qquad z_c = \frac{M_z}{m},$$

where

$$m = \iiint_E \rho(x, y, z)\, dx\, dy\, dz$$

is the total mass of the body.

**Example**

Find the coordinates of the center of gravity of a homogeneous cube of side $a$, whose edges are parallel to the coordinate axes and occupies the region

$$E = \{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a\},$$

with constant density $\rho(x, y, z) = 1$.

$$m = \iiint_E \rho_0\, dx\, dy\, dz = \rho_0 a^3.$$

$$M_x = \iiint_E x\, dx\, dy\, dz = \int_0^a x\, dx \int_0^a dy \int_0^a dz = \frac{a^2}{2} \cdot a^2 = \frac{a^4}{2}.$$

By symmetry, $M_y = M_z = \dfrac{a^4}{2}$.

Thus,

$$x_c = \frac{M_x}{m} = \frac{a}{2}, \qquad y_c = \frac{M_y}{m} = \frac{a}{2}, \qquad z_c = \frac{M_z}{m} = \frac{a}{2}.$$

$$\boxed{G\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)}$$

Therefore, the center of gravity of the homogeneous cube is located at the center of the cube itself.

# Exercises

## Exercise 1

Calculate the following simple integrals:

$$1 - \int_1^2 \left(x + \frac{2}{3}x^2\right) dx \quad 2 - \int_0^1 \frac{\sin(x)}{\cos^4(x)} dx \quad 3 - \int x\sqrt{x^2+3}dx \quad 4 - \int x \arctan(x) dx$$

## Exercise 2

1. Sketch the domain of integration, then calculate the following double integral: $\iint_D (2x+y)^2 dxdy$, where $D$ is the triangle with vertices $O(0,0)$, $A(1,0)$ and $B(0,2)$.
2. Define and sketch the domain of integration, then calculate the following double integral: $\int_0^1 \int_x^{2-x} x(y+1)dydx$. Reverse the order of integration.

## Exercise 3

Sketch the domain $D$ and calculate the following double integrals:

1. $\iint_D (x+y)dxdy$, $D = \{(x,y) \in \mathbb{R}^2, \quad 0 \leq x \leq 1, x^2 \leq y \leq x\}$
2. $\iint_D y^2 dxdy$, $D = \{(x,y) \in \mathbb{R}^2, \quad 0 \leq x \leq y^2, 0 \leq y \leq 3\}$
3. $\iint_D xydxdy$, $D$: Trapezoid bounded by $y = 0$, $y = 1$, $y = 2 - x$, $y = 1 + \frac{x}{2}$

## Exercise 4

Using polar coordinates, calculate:

1. $\iint_D x^2 dxdy$ with $D = \{(x,y) \in \mathbb{R}^2, x \geq 0, 1 \leq x^2 + y^2 \leq 2\}$
2. $\iint_D \sqrt{x^2+y^2} dxdy$ with $D$: Sector centered at 0 with endpoints $A(\sqrt{3}, 1)$, $B(\sqrt{3}, -1)$

## Exercise 5

1. Calculate the area of the region bounded by the parabola $y^2 = 2x$ and the line $y = x$
2. Calculate the area of the region bounded by the curve $\sqrt{x} + \sqrt{y} = 2$ and the line $y + x = 4$

## Exercise 6

1. Find the coordinates of the center of mass of each figure:
   - Figure (A) $= \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \sin x\}$
   - Figure (B) is a triangular plate with vertices $A(2, 0)$, $B(-2, 0)$, $C(0, 1)$
2. Calculate the moment of inertia of the region $D = \{(x, y) \mid x^2 + y^2 \leq 1, y \geq 0\}$, in the following cases:
   (a) with respect to the $y$-axis
   (b) with respect to the $x$-axis
   (c) with respect to the origin of the coordinates

## Exercise 7

Sketch the region $R$ and Calculate triple integrals over it. .

(a) $\iiint_R (x^2 + y^2 + z^2)dxdydz$, where $R = [0, 1] \times [0, 2] \times [0, 3]$.

(b) $\iiint_R xy \sin z dxdydz$, where $R = \left[\frac{1}{6}, \frac{1}{2}\right] \times [0, 1] \times [0, \pi]$.

(c) $\iiint_R \cos(x + y + z)dxdydz$, where $R = [0, \pi] \times [0, \pi] \times [0, \pi]$.

# Exercise 8

1. Using cylindrical coordinates, calculate: $\iiint_D xyzdxdydz$ with
$$D = \{(x, y, z) \in \mathbb{R}^3, 0 \leq z \leq 1, x^2 + y^2 \leq 1\}$$
2. Using spherical coordinates, calculate: $\iiint_D (x^2 + y^2 + z^2)dxdydz$ where $D$ is the sphere centered at the origin with radius $R$

# Exercise 9

1. Find the area of an ellipse with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
2. Find the volume of a cylinder whose base radius is $R$ and height is $h$

# Additional Exercises

## Exercise 11

Let $D$ be the triangle with vertices $A(1,1)$, $B(2,1)$, and $C(2,3)$.

1. Sketch the domain $D$
2. Using integral calculus, compute the area of domain $D$
3. Compute the following double integral: $I = \iint_D x\cos(y)dxdy$

## Exercise 12

Using cylindrical coordinates, compute: $\iiint_\Delta \frac{dxdydz}{x^2+y^2+z^2}$ where

$$\Delta = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1, 1 \leq z \leq 2\}$$

## Exercise 13

**I.** Let $D$ be the triangle with vertices $A(0,0)$, $B(1,0)$, and $C(1,1)$.

1. Sketch the domain $D$
2. Compute the following double integral: $I = \iint_D \frac{dxdy}{(x^2+1)(y^2+1)}$

**II.** Using cylindrical coordinates, compute the following triple integral:

$$\iiint_\Delta (x^2+y^2+z)dxdydz,$$

where $\Delta$ is the cylinder defined by $\Delta = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 9, -5 \leq z \leq 5\}$

## Exercise 14

Calculate the following integrals over regions bounded by given planes. Sketch each region.

(a) $\iiint_R (x+y)\,dx\,dy\,dz, \quad R : y = x^2, y = 1, z = 0, z = 2$

**Answer:** $\frac{8}{5}$

(b) $\iiint_R (x^2+y^2)\,dx\,dy\,dz, \quad R : x = 0, y = 0, z = 0, x+y+z = 3$

**Answer:** $\frac{81}{10}$

(c) $\iiint_R y\,dx\,dy\,dz, \quad R : x^2+y^2 = 3, z = -1, z = 2$

**Answer:** $6\sqrt{3}$

(d) $\iiint_R (x+y+2z)\,dx\,dy\,dz, \quad R: x^2+z^2=4, y=2, y=3$
**Answer:** $10\pi$

## Exercise 15

Calculate the volume of regions given in Exercise .
**Answers:** (a) $\frac{9}{2}$, (b) $\frac{8}{3}$, (c) $9\pi$, (d) $4\pi$

## Exercise 16

Calculate the mass of solid $V$ bounded by given planes and having density $\rho$. Sketch each solid.

(a) $x=0, x=1, y=0, y=3, z=0, z=2, \quad \rho(x,y,z)=x+y+z$
**Answer:** 18

(b*) $x^2+y^2=4, z=0, z=2\sqrt{x^2+y^2}, \quad \rho(x,y,z)=z$
**Answer:** $16\pi$

(c) $x=0, y=0, z=0, x+y=1, z=x+2y, \quad \rho(x,y,z)=3+2x+2y-2z$
**Answer:** $\frac{1}{6}$

## Exercise 17

Calculate the following integrals using cylindrical coordinates.

(a) $\iiint_R (x^2+y^2)\,dx\,dy\,dz, \quad R: \sqrt{x^2+y^2} \le z \le 1$
**Answer:** $\int_0^{2\pi}\int_0^1\int_r^1 r^3\,dz\,dr\,d\theta = \frac{\pi}{10}$

(b) $\iiint_R r^2\,dx\,dy\,dz, \quad R: 0 \le z \le 9-x^2-y^2$
**Answer:** $\int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^3\cos^2\theta\,dz\,dr\,d\theta = \frac{243\pi}{4}$

## Exercise 18

Calculate the following integrals using spherical coordinates.

(a) $\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz, \quad R: -\sqrt{4-x^2-y^2} \le z \le 0$
**Answer:** $\frac{64\pi}{5}$

(b) $\iiint_R z^2\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz, \quad R: 0 \le z \le \sqrt{4-x^2-y^2}, x \ge 0, y \ge x$
**Answer:** $\frac{8\pi}{9}$

# Chapter 2

# Generalized Integrals

The Riemann integrals studied in the first year concern functions $f$ defined and bounded on a closed and bounded interval $[a, b]$. It is useful to generalize this notion to the case of functions $f$ defined on any interval, possibly undefined at a finite number of points, and not necessarily bounded.

In other words, to define the Riemann integral, we started from a bounded function on a compact interval. How should we deal with the following cases:

1. A function that is unbounded on a bounded interval?
2. A function defined on an unbounded interval?

We shall consider the following situation: $I$ is a non-compact interval and $f : I \to \mathbb{R}$ is an arbitrary function.

## 2.1 Generalities

**Definition 2.1.** We say that $f$ is locally integrable (loc. int.) on $I$ if $f$ is Riemann integrable on every compact subinterval $[a, b]$ contained in $I$.

**Example**

Every continuous or piecewise continuous function is locally Riemann-integrable.

In what follows, all functions considered will be locally integrable. We assume that $I$ is of the form $[a, b[$ (resp. $]a, b]$), where $a, b$ are real numbers with $a < b$; that is, the function is not defined at $b$ (resp. not defined at $a$). This corresponds to the case where the curve $C_f$ of the function $f$ has a vertical asymptote with equation $x = b$ (resp. $x = a$).

Alternatively, $I$ may be of the form $[a, +\infty[$ (resp. $]-\infty, b]$); that is, the domain of integration is unbounded.

**Definition 2.2.** Let $f : [a, b[ \to \mathbb{R}$ be a locally integrable function, where $a < b \leq +\infty$.

1. The integral
$$\int_a^{\to b} f(t)\,dt$$
is called a generalized integral or improper integral of $f$.

2. The integral
$$\int_a^{\to b} f(t)\,dt$$
is said to converge if the limit
$$\lim_{x \to b^-} \int_a^x f(t)\,dt$$
exists and is finite (that is, this limit equals a unique real number). In this case, we write
$$\int_a^{\to b} f(t)\,dt = \lim_{x \to b^-} \int_a^x f(t)\,dt,$$
and we call this number the *value of the generalized integral*. Otherwise, we say that the generalized integral diverges.

Similarly, if $f : ]a, b] \to \mathbb{R}$ is locally integrable, where $-\infty \leq a < b$, then:

1. The integral
$$\int_{\to a}^b f(t)\,dt$$
is called a generalized integral or improper integral of $f$.

2. The integral
$$\int_{\to a}^b f(t)\,dt$$
is said to converge if the limit
$$\lim_{x \to a^+} \int_x^b f(t)\,dt$$
exists and is finite. In this case, we write
$$\int_{\to a}^b f(t)\,dt = \lim_{x \to a^+} \int_x^b f(t)\,dt.$$
Otherwise, we say that the generalized integral diverges.

**Example**

1. $\displaystyle\int_0^{+\infty} \frac{1}{1+t^2}\,dt$. For every $x > 0$,

$$\int_0^x \frac{1}{1+t^2}\,dt = \arctan(x).$$

Hence,

$$\lim_{x\to+\infty} \int_0^x \frac{1}{1+t^2}\,dt = \lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2}.$$

Therefore, the integral $\displaystyle\int_0^{+\infty} \frac{1}{1+t^2}\,dt$ converges.

2. $\displaystyle\int_{\pi/2}^{+\infty} \sin(t)\,dt$. For every $x > \pi/2$,

$$\int_{\pi/2}^x \sin(t)\,dt = -\cos(x).$$

Thus,

$$\lim_{x\to+\infty} \int_{\pi/2}^x \sin(t)\,dt = -\lim_{x\to+\infty} \cos(x),$$

which does not exist since $\cos(x)$ oscillates. Therefore, the integral $\displaystyle\int_{\pi/2}^{+\infty} \sin(t)\,dt$ diverges.

3. $\displaystyle\int_0^5 \frac{1}{t}\,dt$. For every $0 < x < 5$,

$$\int_x^5 \frac{1}{t}\,dt = \ln(5) - \ln(x).$$

Hence,

$$\lim_{x\to0^+} \int_x^5 \frac{1}{t}\,dt = \lim_{x\to0^+} (\ln(5) - \ln(x)) = +\infty.$$

Therefore, the integral $\displaystyle\int_0^5 \frac{1}{t}\,dt$ diverges.

4. $\displaystyle\int_1^{+\infty} \ln t\,dt$

We consider the improper integral $\displaystyle\int_1^{+\infty} \ln t\,dt$. Since the upper limit of integration is infinite, the integral is improper of the *second kind*. We write it as a limit:

$$\int_1^{+\infty} \ln t\,dt = \lim_{b\to+\infty} \int_1^b \ln t\,dt.$$

To compute the integral, we integrate by parts:

$$\begin{cases} u = \ln t & \Rightarrow \quad du = \dfrac{1}{t}\,dt, \\ dv = dt & \Rightarrow \quad v = t. \end{cases}$$

Hence,

$$\int \ln t\,dt = t\ln t - t + C.$$

Therefore,

$$\int_1^b \ln t\,dt = \Big[t\ln t - t\Big]_1^b = (b\ln b - b) - (1\cdot 0 - 1) = b\ln b - b + 1.$$

We now take the limit as $b \to +\infty$:

$$\lim_{b\to+\infty} (b\ln b - b + 1) = +\infty.$$

Then the improper integral $\displaystyle\int_1^{+\infty} \ln t\,dt$ is diverges, since its limit is infinite.

5. $\displaystyle\int_0^1 \ln t\,dt$

We consider the improper integral

$$\int_0^1 \ln t\,dt.$$

Since the lower limit of integration is 0 and the function $\ln t$ is not defined at $t = 0$, the integral is improper of the *first kind*. We write it as a limit:

$$\int_0^1 \ln t\,dt = \lim_{a\to0^+} \int_a^1 \ln t\,dt.$$

To compute the integral, we integrate by parts:

$$\begin{cases} u = \ln t & \Rightarrow \quad du = \dfrac{1}{t}\,dt, \\ dv = dt & \Rightarrow \quad v = t. \end{cases}$$

Hence,

$$\int \ln t\,dt = t\ln t - t + C.$$

Therefore,

$$\int_a^1 \ln t\,dt = \Big[t\ln t - t\Big]_a^1 = (1\cdot 0 - 1) - (a\ln a - a) = -1 - a\ln a + a.$$

We now take the limit as $a \to 0^+$:

$$\lim_{a\to0^+} (-1 - a\ln a + a) = -1,$$

since $a\ln a \to 0$ and $a \to 0$.

Ten the improper integral $\displaystyle\int_0^1 \ln t\,dt$ is converge and its value is $-1$.

## 2.2 Cauchy Criterion and Doubly Improper Integrals

**Theorem 2.1** (Cauchy Criterion for Generalized Integrals)**.**

1. *Let $f$ be a locally integrable function on $[a, +\infty[$. Then the generalized integral*

$$\int_a^{+\infty} f(t)\,dt$$

*converges if and only if*

$$\forall \varepsilon > 0,\ \exists A > 0 : \forall x, x_0 \geq A \implies \left| \int_x^{x_0} f(t)\,dt \right| < \varepsilon.$$

2. *Let $f$ be a locally integrable function on $[a, b[$, where $a, b \in \mathbb{R}$ and $a < b$. Then the generalized integral*

$$\int_a^{\to b} f(t)\,dt$$

*converges if and only if*

$$\forall \varepsilon > 0,\ \exists \delta \in ]0, b-a[ : \forall x, x_0 \in ]b-\delta, b[ \implies \left| \int_x^{x_0} f(t)\,dt \right| < \varepsilon.$$

*For intervals of the type $]a, b]$, we have an analogous criterion.*

**Definition 2.3.** Two generalized integrals are said to be of the same nature if they are either both convergent or both divergent.

**Proposition 2.1.** *Let $f : I \to \mathbb{R}$, where $I = [a, b[$ with $a < b \leq +\infty$, be a locally integrable function on $I$. Then, for every real number $c$ such that $a \leq c < b$, we have*

$$\int_a^{\to b} f(t)\,dt \text{ converges} \iff \int_c^{\to b} f(t)\,dt \text{ converges.}$$

*That is, the two integrals*

$$\int_a^{\to b} f(t)\,dt \quad \text{and} \quad \int_c^{\to b} f(t)\,dt$$

*are of the same nature.*

*Proof.* By the Chasles relation, for any $c$ with $a \leq c < b$, we have

$$\int_a^{\to b} f(t)\,dt = \int_a^c f(t)\,dt + \int_c^{\to b} f(t)\,dt.$$

Since $f$ is integrable on the compact interval $[a, c]$ (because it is locally integrable on $[a, b[$), the integral $\int_a^c f(t)\,dt$ is finite. Thus, the two integrals $\int_a^{\to b} f(t)\,dt$ and $\int_c^{\to b} f(t)\,dt$ are of the same nature. □

### 2.2.1 Doubly Improper Integrals

In this section, we consider intervals of the types $]a, b[$, $]-\infty, b[$, $]a, +\infty[$, and $]-\infty, +\infty[$; that is, the case of doubly improper integrals.

**Proposition 2.2.** *Let $f$ be a locally integrable function on $]a, b[$ and let $c \in ]a, b[$. Then the integral*

$$\int_{\to a}^{\to b} f(t)\,dt$$

*converges if and only if both integrals*

$$\int_{\to a}^{c} f(t)\,dt \quad \text{and} \quad \int_{c}^{\to b} f(t)\,dt$$

*are convergent. In this case, we set*

$$\int_{\to a}^{\to b} f(t)\,dt = \int_{\to a}^{c} f(t)\,dt + \int_{c}^{\to b} f(t)\,dt.$$

*Otherwise, the improper integral $\displaystyle\int_{\to a}^{\to b} f(t)\,dt$ is said to diverge.*

*Remark* 2.1. The previous proposition shows that to study a doubly improper integral, one must split it into two improper integrals.

**Example**

1- We consider the improper integral

$$\int_{-\infty}^{+\infty} \sin(t)\,dt.$$

Since the interval of integration is unbounded on both sides, the integral is *improper at both infinities*. We define it as the sum of two limits:

$$\int_{-\infty}^{+\infty} \sin(t)\,dt = \lim_{a\to-\infty} \int_{a}^{0} \sin(t)\,dt + \lim_{b\to+\infty} \int_{0}^{b} \sin(t)\,dt.$$

We know that

$$\int \sin(t)\,dt = -\cos(t) + C.$$

Hence,

$$\int_{0}^{b} \sin(t)\,dt = [-\cos(t)]_0^b = -\cos(b) + 1 = 1 - \cos(b),$$

and similarly,

$$\int_{a}^{0} \sin(t)\,dt = [-\cos(t)]_a^0 = -\cos(0) + \cos(a) = -1 + \cos(a).$$

Let us take the limits separately.

For $b \to +\infty$:

$$\lim_{b\to+\infty} (1 - \cos(b))$$

does not exist, since $\cos(b)$ oscillates between $-1$ and 1.

For $a \to -\infty$:

$$\lim_{a\to-\infty}(-1+\cos(a))$$

also does not exist, for the same reason.

Neither of the two limits exists, so the improper integral $\int_{-\infty}^{+\infty} \sin(t)\,dt$ is diverges because the integrand oscillates indefinitely and the area under the curve does not approach a finite limit.

2- We consider the improper integral

$$\int_0^{+\infty} \frac{1}{t^2}\,dt.$$

This integral is *improper at both extremities*:

- at $t = 0$, the function $\frac{1}{t^2}$ tends to $+\infty$;
- as $t \to +\infty$, the function $\frac{1}{t^2}$ tends to 0.

Therefore, we write

$$\int_0^{+\infty} \frac{1}{t^2}\,dt = \lim_{a\to0^+}\int_a^1 \frac{1}{t^2}\,dt + \lim_{b\to+\infty}\int_1^b \frac{1}{t^2}\,dt.$$

Let us compute each part separately.

**1. Near $t = 0$**

$$\int_a^1 \frac{1}{t^2}\,dt = \Big[-\frac{1}{t}\Big]_a^1 = (-1) - \Big(-\frac{1}{a}\Big) = \frac{1}{a} - 1.$$

Then,

$$\lim_{a\to0^+}\left(\frac{1}{a} - 1\right) = +\infty.$$

Thus, the integral diverges near $t = 0$.

**2. Near $t = +\infty$**

$$\int_1^b \frac{1}{t^2}\,dt = \Big[-\frac{1}{t}\Big]_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}.$$

Hence,

$$\lim_{b\to+\infty}\left(1 - \frac{1}{b}\right) = 1.$$

This part converges.

Since the integral diverges near $t = 0$, we conclude that $\int_0^{+\infty} \frac{1}{t^2}\,dt$ is diverges.

## 2.3 Falsely Improper Integrals

We consider here the interval $[a, b[$, where $-\infty < a < b < +\infty$.

**Proposition 2.3.** *Let $f$ be a continuous function on $[a, b[$. If $f$ admits a finite limit at $b^-$, then the improper integral*

$$\int_a^{\to b} f(t)\,dt$$

*is convergent. In this case, we say that the integral is* falsely improper *at $b$.*

*Remark* 2.2. There is no falsely improper integral at $\pm\infty$; this notion is reserved for a finite bound.

**Example**

$$\int_0^2 \frac{\sin t}{t}\,dt$$

At first glance, the integral seems improper because the function

$$f(t) = \frac{\sin t}{t}$$

is not defined at $t = 0$.

Let us study the limit of $f(t)$ as $t \to 0$:

$$\lim_{t\to 0} \frac{\sin t}{t} = 1.$$

Therefore, we can define

$$f(0) = 1,$$

so that $f$ becomes continuous on the closed interval $[0, 2]$.

Since the limit exists and is finite, the discontinuity at $t = 0$ is a *removable singularity.* Hence, the integral

$$\int_0^2 \frac{\sin t}{t}\,dt$$

is actually a proper integral.

Although the function $\frac{\sin t}{t}$ is not defined at $t = 0$, the limit exists and is finite. Thus, the integral is falsely improper (or improper in appearance only).

The integral $\displaystyle\int_0^2 \frac{\sin t}{t}\,dt$ converges and is actually proper.

### 2.3.1 Reference Integrals

In this section, we recall some fundamental results concerning improper integrals of special types that can be used in the study of other improper integrals.

**Proposition 2.4.** *1. The improper integral*

$$\int_1^{+\infty} \frac{1}{t^\beta}\,dt$$

*is*

$$\begin{cases} \textit{convergent} & \textit{if } \beta > 1, \\ \textit{divergent} & \textit{if } \beta \leq 1. \end{cases}$$

*2. The improper integral*

$$\int_0^1 \frac{1}{t^\beta}\,dt$$

*is*

$$\begin{cases} \textit{convergent} & \textit{if } \beta < 1, \\ \textit{divergent} & \textit{if } \beta \geq 1. \end{cases}$$

*Proof.* (i) We distinguish two cases: $\beta \neq 1$ and $\beta = 1$.

For $\beta \neq 1$,

$$\int_1^{+\infty} \frac{1}{t^\beta}\,dt = \lim_{x\to+\infty} \int_1^x \frac{1}{t^\beta}\,dt = \lim_{x\to+\infty} \frac{1}{1-\beta}\left(x^{1-\beta} - 1\right).$$

Hence,

$$\int_1^{+\infty} \frac{1}{t^\beta}\,dt = \begin{cases} \dfrac{1}{\beta-1} \in \mathbb{R} & \text{if } \beta > 1, \\ +\infty & \text{if } \beta < 1. \end{cases}$$

For $\beta = 1$,

$$\int_1^{+\infty} \frac{1}{t}\,dt = \lim_{x\to+\infty} \ln x = +\infty.$$

Thus,

$$\int_1^{+\infty} \frac{1}{t^\beta}\,dt \begin{cases} \text{converges if } \beta > 1, \\ \text{diverges if } \beta \leq 1. \end{cases}$$

(ii) Similarly, we obtain

$$\int_0^1 \frac{1}{t^\beta}\,dt \begin{cases} \text{converges if } \beta < 1, \\ \text{diverges if } \beta \geq 1. \end{cases}$$

□

**Example**

The integrals

$$\int_1^{+\infty} \frac{1}{t^2}\,dt \quad \text{and} \quad \int_0^1 \frac{1}{\sqrt{t}}\,dt$$

are convergent, whereas the integrals

$$\int_1^{+\infty} \frac{1}{\sqrt{t}}\,dt \quad \text{and} \quad \int_0^1 \frac{1}{t^2}\,dt$$

are divergent.

**Corollary 2.1.** *If $-\infty < a < b < +\infty$ and $\beta \in \mathbb{R}$, then:*

1. $\displaystyle\int_{\to a}^{b} \frac{1}{(t-a)^{\beta}}\,dt$ *converges if and only if* $\beta < 1$.
2. $\displaystyle\int_{a}^{\to b} \frac{1}{(b-t)^{\beta}}\,dt$ *converges if and only if* $\beta < 1$.

*Proof.* For the first integral, take the change of variable $u = t - a$. For the second, take $u = b - t$. Then apply the previous proposition. □

**Proposition 2.5.** *The improper integral*

$$\int_0^{+\infty} e^{-\beta t}\,dt$$

*is*

$$\begin{cases} \text{convergent} & \text{if } \beta > 0, \\ \text{divergent} & \text{if } \beta \leq 0. \end{cases}$$

*Proof.*

$$\int_0^{+\infty} e^{-\beta t}\,dt = \lim_{x\to+\infty} \int_0^{x} e^{-\beta t}\,dt = \lim_{x\to+\infty} \left[-\frac{1}{\beta} e^{-\beta t}\right]_0^{x} = \lim_{x\to+\infty} \frac{1}{\beta}(1 - e^{-\beta x}).$$

Hence,

$$\int_0^{+\infty} e^{-\beta t}\,dt = \begin{cases} \dfrac{1}{\beta} \in \mathbb{R} & \text{if } \beta > 0, \\ +\infty & \text{if } \beta \leq 0. \end{cases}$$

□

**Example**

The integral

$$\int_0^{+\infty} e^{-t}\,dt$$

is convergent, while

$$\int_0^{+\infty} e^{t}\,dt$$

is divergent.

**Corollary 2.2.** *For every real number $\alpha > 0$ and every integer $n \in \mathbb{N}$,*

$$\int_0^{+\infty} t^n e^{-\alpha t}\, dt = \frac{n!}{\alpha^{n+1}}.$$

*In particular, by setting $\alpha = 1$, we have*

$$\int_0^{+\infty} t^n e^{-t}\, dt = n!.$$

*Proof.* We prove by induction that

$$\int_0^{+\infty} t^n e^{-\alpha t}\, dt = \frac{n!}{\alpha^{n+1}}.$$

**Base case ($n = 0$):**

$$\int_0^{+\infty} e^{-\alpha t}\, dt = \left[-\frac{1}{\alpha} e^{-\alpha t}\right]_0^{+\infty} = \frac{1}{\alpha}.$$

The formula holds since $0! = 1$ and $\frac{0!}{\alpha^1} = \frac{1}{\alpha}$.

**Inductive step:** Assume that for some $n \in \mathbb{N}$,

$$\int_0^{+\infty} t^n e^{-\alpha t}\, dt = \frac{n!}{\alpha^{n+1}}.$$

We prove the result for $n + 1$.

Integrate by parts with

$$\begin{cases} u = t^{n+1}, & du = (n+1)t^n dt, \\ dv = e^{-\alpha t} dt, & v = -\dfrac{1}{\alpha} e^{-\alpha t}. \end{cases}$$

Then,

$$\int_0^{+\infty} t^{n+1} e^{-\alpha t}\, dt = \left[-\frac{t^{n+1}}{\alpha} e^{-\alpha t}\right]_0^{+\infty} + \frac{n+1}{\alpha} \int_0^{+\infty} t^n e^{-\alpha t}\, dt.$$

The boundary term vanishes because $e^{-\alpha t}$ decays faster than $t^{n+1}$, hence

$$\int_0^{+\infty} t^{n+1} e^{-\alpha t}\, dt = \frac{n+1}{\alpha} \cdot \frac{n!}{\alpha^{n+1}} = \frac{(n+1)!}{\alpha^{n+2}}.$$

**Conclusion:** By induction, for every $n \in \mathbb{N}$ and every $\alpha > 0$,

$$\boxed{\int_0^{+\infty} t^n e^{-\alpha t}\, dt = \frac{n!}{\alpha^{n+1}}.}$$

□

# 2.4 Properties of Improper Integrals

## 2.4.1 Linearity

**Theorem 2.2.** *Let $f, g$ be two real locally integrable functions on $[a, b[$ and let $\lambda \in \mathbb{R}$.*

1. *If $\int_a^{\to b} f(t)\,dt$ and $\int_a^{\to b} g(t)\,dt$ both converge, then the integral $\int_a^{\to b} (f+g)(t)\,dt$ also converges, and*

$$\int_a^{\to b} (f+g)(t)\,dt = \int_a^{\to b} f(t)\,dt + \int_a^{\to b} g(t)\,dt.$$

2. *If $\int_a^{\to b} f(t)\,dt$ converges, then $\int_a^{\to b} (\lambda f)(t)\,dt$ also converges and*

$$\int_a^{\to b} (\lambda f)(t)\,dt = \lambda \int_a^{\to b} f(t)\,dt.$$

*Remark* 2.3. If both $\int_a^{\to b} f(t)\,dt$ and $\int_a^{\to b} g(t)\,dt$ diverge, nothing can be concluded about $\int_a^{\to b} (f+g)(t)\,dt$. For example,

$$\int_1^{+\infty} \frac{1}{t^2}\,dt = 1$$

and

$$1 = \int_1^{+\infty} \left(\frac{1+t}{t^2} - \frac{1}{t}\right) dt,$$

but the two improper integrals $\int_1^{+\infty} \frac{1+t}{t^2}\,dt$ and $\int_1^{+\infty} \frac{1}{t}\,dt$ are both divergent, so we have

$$1 = \int_1^{+\infty} \frac{1}{t^2}\,dt \neq \int_1^{+\infty} \frac{1+t}{t^2}\,dt - \int_1^{+\infty} \frac{1}{t}\,dt.$$

## 2.4.2 Integration by Parts

**Theorem 2.3.** *Let $f, g \in C^1([a, b[; \mathbb{R})$. If $fg$ admits a finite limit at $b^-$, then the improper integrals $\int_a^{\to b} (f'g)(t)\,dt$ and $\int_a^{\to b} (fg')(t)\,dt$ are of the same nature. Moreover, if they are both convergent, then*

$$\int_a^{\to b} (f'g)(t)\,dt = [fg]_a^{\to b} - \int_a^{\to b} (fg')(t)\,dt.$$

**Example**

Computation of the Integral by Integration by Parts

$$\int_0^1 \frac{2t \ln t}{(1+t^2)^2}\,dt$$

We set

$$I = \int_0^1 \frac{2t \ln t}{(1+t^2)^2}\, dt.$$

Choose $u$ and $dv$ for Integration by Parts Let

$$\begin{cases} u = \ln t \quad \Rightarrow \quad du = \frac{1}{t} dt, \\ dv = \frac{2t}{(1+t^2)^2} dt \quad \Rightarrow \quad v = \int \frac{2t}{(1+t^2)^2} dt. \end{cases}$$

Use the substitution $s = 1 + t^2 \Rightarrow ds = 2t\, dt$:

$$v = \int s^{-2} ds = -\frac{1}{s} + C = -\frac{1}{1+t^2}.$$

$$\int u\, dv = [uv]_0^1 - \int v\, du.$$

Hence,

$$I = \left[\ln t \cdot \left(-\frac{1}{1+t^2}\right)\right]_0^1 - \int_0^1 \left(-\frac{1}{1+t^2}\right) \cdot \frac{1}{t}\, dt = \left[-\frac{\ln t}{1+t^2}\right]_0^1 + \int_0^1 \frac{1}{t(1+t^2)}\, dt..$$

So

$$\left[-\frac{\ln t}{1+t^2}\right]_0^1 = -\frac{\ln 1}{2} + \lim_{t\to 0^+} \frac{\ln t}{1+t^2}.$$

We know that $\ln 1 = 0$ and $\lim_{t\to 0^+} \frac{\ln t}{1+t^2} = -\infty$. So the integral is improper at $t = 0$. We have

$$\int_0^1 \frac{1}{t(1+t^2)}\, dt = \int_0^1 \left(\frac{1}{t} - \frac{t}{1+t^2}\right) dt,$$

using the decomposition

$$\frac{1}{t(1+t^2)} = \frac{1}{t} - \frac{t}{1+t^2}.$$

Compute each part:

$$\int_0^1 \frac{1}{t}\, dt = \lim_{x\to 0^+} \int_x^1 \frac{1}{t} dt = \lim_{x\to 0^+} [\ln t]_x^1 = -\lim_{x\to 0^+} \ln \epsilon = +\infty,$$

$$\int_0^1 \frac{t}{1+t^2}\, dt = \frac{1}{2} \ln(1+t^2)\Big|_0^1 = \frac{1}{2} \ln 2.$$

Since the term $\int_0^1 \frac{1}{t}\, dt$ diverges, we conclude that the integral $\int_0^1 \frac{2t \ln t}{(1+t^2)^2}\, dt$ is improper and divergent.

### 2.4.3 Change of Variable

**Theorem 2.4.** *Let $a, b \in \mathbb{R}$ with $-\infty \le a < b \le +\infty$, $\varphi : ]a, b[ \to \varphi(]a, b[)$ a $C^1$-bijection, and $f$ a continuous function on $\varphi(]a, b[)$. Then the improper integrals*

$$\int_{\to\varphi(a)}^{\varphi(b)} f(t)\, dt \quad \text{and} \quad \int_a^{\to b} f(\varphi(u))\, \varphi'(u)\, du$$

*are of the same nature, and equal if they converge.*

**Example**

Computation by Substitution $u = \tan(t/2)$ We want to compute

$$I = \int_0^{\pi} \frac{1}{2 + \cos t}\, dt$$

using the substitution

$$u = \tan\frac{t}{2} \quad \Rightarrow \quad dt = \frac{2\,du}{1+u^2}.$$

Recall the Weierstrass substitution formulas: $\cos t = \frac{1-u^2}{1+u^2}$,
Therefore,

$$2 + \cos t = 2 + \frac{1-u^2}{1+u^2} = \frac{2(1+u^2)+1-u^2}{1+u^2} = \frac{u^2+3}{1+u^2}.$$

Hence,

$$\frac{dt}{2+\cos t} = \frac{2\,du}{(1+u^2)} \cdot \frac{1+u^2}{u^2+3} = \frac{2\,du}{u^2+3}.$$

$$t = 0 \Rightarrow u = \tan(0/2) = 0, \qquad t = \pi \Rightarrow u = \tan(\pi/2) = +\infty.$$

Thus, the integral becomes

$$I = \int_0^{+\infty} \frac{2\,du}{u^2+3} = 2\int_0^{+\infty} \frac{du}{u^2 + (\sqrt{3})^2}.$$

We know that

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a} + C.$$

Hence,

$$I = 2 \cdot \frac{1}{\sqrt{3}} \left[\arctan\frac{u}{\sqrt{3}}\right]_0^{+\infty} = \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{\sqrt{3}}.$$

## 2.5 Integrals of Positive Functions

### 2.5.1 Convergence Criteria

**Proposition 2.6.** *Let $f$ be a locally integrable and positive function on $[a, b[$. Then the improper integral $\int_a^{\to b} f(t)\,dt$ converges if and only if there exists $M > 0$ such that*

$$0 \le \int_a^x f(t)\,dt \le M, \quad \forall x \in [a, b[.$$

*Hence, the integral diverges if and only if $\int_a^{\to b} f(t)\,dt = +\infty$.*

### 2.5.2 Comparison Criterion

**Proposition 2.7.** *Let $f, g : [a, b[ \to \mathbb{R}$ be locally integrable positive functions such that $f \leq g$. Then:*

1. *If $\int_a^{\to b} g(t)\,dt$ converges, then $\int_a^{\to b} f(t)\,dt$ also converges.*
2. *If $\int_a^{\to b} f(t)\,dt$ diverges, then $\int_a^{\to b} g(t)\,dt$ also diverges.*

**Proposition 2.8.** *Let $f, g : [a, b[ \to \mathbb{R}$ be locally integrable positive functions such that*

$$\lim_{t \to b^-} \frac{f(t)}{g(t)} = \ell \in \mathbb{R}.$$

1. *If $\ell \neq 0$, then the two integrals are of the same nature.*
2. *If $\ell = 0$ (that is, $f = o(g)$ near $b$), then:*

$$\begin{cases} \int_a^{\to b} g(t)\,dt \text{ convergent } \Rightarrow \int_a^{\to b} f(t)\,dt \text{ convergent,} \\ \int_a^{\to b} f(t)\,dt \text{ divergent } \Rightarrow \int_a^{\to b} g(t)\,dt \text{ divergent.} \end{cases}$$

**Example**

Study of the Improper Integral by Comparison Test

$$\int_0^{\infty} \frac{\sin^2 t}{1+t^2}\,dt$$

We know that for all $t \geq 0$:

$$0 \leq \sin^2 t \leq 1.$$

Therefore,

$$0 \leq \frac{\sin^2 t}{1+t^2} \leq \frac{1}{1+t^2}.$$

Consider the integral

$$\int_0^{\infty} \frac{1}{1+t^2}\,dt.$$

This integral converges because

$$\int_0^{\infty} \frac{1}{1+t^2}\,dt = [\arctan t]_0^{\infty} = \frac{\pi}{2}.$$

Since

$$0 \leq \frac{\sin^2 t}{1+t^2} \leq \frac{1}{1+t^2}$$

for all $t \geq 0$ and $\int_0^{\infty} \frac{1}{1+t^2} dt$ converges, it follows by the Comparison Test that

$$\int_0^{\infty} \frac{\sin^2 t}{1+t^2}\,dt$$

also converges.

**Corollary 2.3.** *1. Let $f$ be a positive locally integrable function on $[a, +\infty[$. If there exists $\beta > 1$ such that $\lim_{t\to+\infty} t^\beta f(t) = 0$, then the improper integral $\int_a^{+\infty} f(t)\,dt$ converges.*

*2. Let $f$ be a positive locally integrable function on $[a, b[$ $(a < b)$. If there exists $\beta < 1$ such that $\lim_{t\to b^-} (b-t)^\beta f(t) = 0$, then the improper integral $\int_a^{\to b} f(t)\,dt$ converges.*

### 2.5.3 Equivalence Criterion

**Theorem 2.5** (Equivalence Criterion)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Let $f, g : [a, b[\to \mathbb{R}$ be two positive and locally integrable functions.*
*Suppose that $f$ and $g$ are equivalent near $b$, which we denote by*

$$f(t) \sim_b g(t), \quad \text{as } t \to b.$$

*Then:*

1. *If the improper integral $\int_a^b f(t)\,dt$ diverges, then $\int_a^b g(t)\,dt$ also diverges.*
2. *If the improper integral $\int_a^b g(t)\,dt$ converges, then $\int_a^b f(t)\,dt$ also converges.*

**Example**

1- Study the convergence of

$$\int_0^1 \sin(\sqrt{t})\,dt.$$

The function $t \mapsto \sin(\sqrt{t})$ is continuous and therefore locally integrable on $]0, 1]$. The only point of concern is $t = 0$, where a potential singularity appears.

Since the function is positive, we study the convergence using the equivalence criterion:

$$\sin(\sqrt{t}) \sim \sqrt{t} \quad \text{as } t \to 0.$$

Now,

$$\int_0^1 \sqrt{t}\,dt$$

converges (its primitive $\frac{2}{3}t^{3/2}$ has a finite limit at 0).

Hence, by the equivalence criterion,

$$\int_0^1 \sin(\sqrt{t})\,dt$$

also converges.

2- Consider the improper integral

$$\int_0^3 \frac{\sin x}{x}\,dx,$$

whose only point of concern is $x = 0$.

We study the behavior of $\frac{\sin x}{x}$ as $x \to 0$. Using the Taylor expansion of $\sin x$ at 0:

$$\sin x = x + o(x), \quad x \to 0,$$

we get

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Hence, $\frac{\sin x}{x}$ is equivalent to 1 near 0. By the equivalence theorem, the integrals

$$\int_0^3 \frac{\sin x}{x}\,dx \quad \text{and} \quad \int_0^3 1\,dx$$

are of the same nature. Since the latter converges, the former converges as well.

3- The equivalence theorem also allows us to conclude that the improper integral

$$\int_1^\infty \frac{x^5 + 3x^2 + 4x + 1}{x^2 + 4} e^{-x}\,dx,$$

whose only point of concern is $+\infty$, converges.

Indeed, as $x \to +\infty$:

$$\frac{x^5 + 3x^2 + 4x + 1}{x^2 + 4} e^{-x} \sim x^3 e^{-x}.$$

Since $\int_1^\infty x^3 e^{-x}\,dx$ converges, our integral converges as well.

4- The function

$$f(x) = \frac{\sqrt{x^5 + 2x}}{x + 1}$$

is equivalent to $x^{3/2}/2$ as $x \to +\infty$.

Since

$$\int_3^\infty x^{3/2}\,dx$$

converges in the sense of Riemann improper integrals, we conclude that

$$\int_3^\infty \frac{\sqrt{x^5 + 2x}}{x + 1}\,dx$$

also converges.

# Bertrand Reference Integrals

**Theorem 2.6** (Bertrand Integrals)**.** *Consider the improper integral*

$$\int_2^{\infty} \frac{1}{t(\ln t)^{\beta}}\, dt,$$

*where $\beta$ is a strictly positive real number ($\beta > 0$).*
*The primitive of this function can be written explicitly as:*

$$\int \frac{dt}{t(\ln t)^{\beta}} = \begin{cases} \frac{(\ln t)^{1-\beta}}{1-\beta} + C, & \text{if } \beta \neq 1, \\ \ln(\ln t) + C, & \text{if } \beta = 1, \end{cases}$$

*where $C$ is the constant of integration.*
*From the explicit primitive, we can deduce the nature of the Bertrand integrals:*

- *If $\beta \leq 1$, then*
$$\int_2^{\infty} \frac{dt}{t(\ln t)^{\beta}} \quad \text{diverges.}$$
- *If $\beta > 1$, then*
$$\int_2^{\infty} \frac{dt}{t(\ln t)^{\beta}} \quad \text{converges.}$$

**Corollary 2.4.** *Let $\alpha$ and $\beta$ be two real numbers. Consider the function*

$$f(t) = t^{-\alpha}(\ln t)^{-\beta}.$$

*Then $f$ is integrable over $[2, +\infty[$ if and only if*

$$\alpha > 1 \quad \text{or} \quad (\alpha = 1 \text{ and } \beta > 1).$$

**Explanation**

- For $\alpha > 1$, the integral
$$\int_2^{\infty} t^{-\alpha}(\ln t)^{-\beta} dt$$
converges regardless of $\beta$, because $t^{-\alpha}$ decays fast enough.
- For $\alpha = 1$, the integral behaves like the Bertrand integral $\int_2^{\infty} \frac{1}{t(\ln t)^{\beta}} dt$, which converges if and only if $\beta > 1$.
- For $\alpha < 1$, the integral diverges because $t^{-\alpha}$ decays too slowly.

## 2.6 Integrals of Functions with Arbitrary Sign

### 2.6.1 Absolute Convergence

**Definition 2.4.** Let $f : [a, b[ \to \mathbb{R}$ be a locally integrable function. The improper integral $\int_a^{\to b} f(t)\,dt$ is said to be *absolutely convergent* if the improper integral $\int_a^{\to b} |f(t)|\,dt$ converges.

**Proposition 2.9.** *If* $\int_a^{\to b} |f(t)|\,dt$ *converges, then* $\int_a^{\to b} f(t)\,dt$ *also converges.*

*Remark* 2.4. To prove absolute convergence of an improper integral, we can use the convergence criteria for positive functions. The interest of the notion of absolute convergence lies in the following theorem.

**Example**

The integral

$$\int_0^{+\infty} e^{-t} \sin\Big(\ln(t^2+1)\Big)\,dt$$

is absolutely convergent, since

$$\Big|e^{-t}\sin(\ln(t^2+1))\Big| \leq e^{-t},$$

and $\int_0^{+\infty} e^{-t}\,dt$ is convergent.

### 2.6.2 Abel's Criterion for Improper Integrals

**Theorem 2.7** (Abel's Theorem)**.** *Let* $f, g : [a, b[ \to \mathbb{R}$ *be two locally integrable functions such that:*

1. *$f$ is monotone and* $\lim_{t \to b^-} f(t) = 0$*;*
2. *there exists* $M > 0$ *such that for every* $x \in [a, b[$*,*

$$\left|\int_a^x g(t)\,dt\right| \leq M.$$

*Then the improper integral*

$$\int_a^{\to b} f(t)g(t)\,dt$$

*converges.*

**Example**

1- Abel's Theorem allows us to easily prove the convergence of the following integrals:

$$\int_1^\infty \frac{\sin t}{t}\,dt \quad \text{and} \quad \int_1^\infty \frac{\cos 2t}{t}\,dt.$$

Consider

$$\int_1^\infty \frac{\sin t}{t}\,dt.$$

Set

$$f(t) = \frac{1}{t}, \quad g(t) = \sin t.$$

For all $t > 1$:

$$\left|\int_1^x \sin t\,dt\right| = |[-\cos t]_1^x| = |\cos 1 - \cos x| \leq 2, \quad M = 2.$$

Hence, by Abel's Theorem, the improper integral

$$\int_1^\infty \frac{\sin t}{t}\,dt$$

converges.

Similarly, consider

$$\int_1^\infty \frac{\cos 2t}{t}\,dt.$$

Take

$$f(t) = \frac{1}{t}, \quad g(t) = \cos(2t),$$

and we can apply Abel's Theorem in the same way to show that the integral converges.

*Remark* 2.5. Let $\int_a^\infty f(t)\,dt$ be a convergent integral. If $f$ has a limit $l$ as $t \to +\infty$, then $l = 0$. However, this is not a sufficient condition for convergence.

2- The integral

$$\int_1^\infty \frac{dt}{\sqrt{t}}$$

diverges even though

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} = 0.$$

*Remark* 2.6. It is possible that $\int_b^\infty f(t)\,dt$ converges even if $f$ does not have a limit as $t \to +\infty$ and is not bounded.

### 2.6.3 Semi-Convergent Integrals

**Definition 2.5.** The improper integral $\int_a^{\to b} f(t)\,dt$ is said to be *semi-convergent* if it converges but is not absolutely convergent.

**Example**

The integral

$$\int_1^{+\infty} \frac{\sin t}{t}\, dt$$

is convergent but not absolutely convergent. Hence, it is semi-convergent.

Indeed, by Abel's theorem, the integral $\int_1^{+\infty} \frac{\sin t}{t}\, dt$ converges.

To study absolute convergence, observe that for all $t \geq 1$,

$$\frac{\sin^2 t}{t} \leq \frac{|\sin t|}{t}, \qquad \frac{\sin^2 t}{t} = \frac{1}{2t} - \frac{\cos(2t)}{2t}.$$

But the integral $\int_1^{+\infty} \frac{1}{t}\, dt$ diverges, and therefore $\int_1^{+\infty} \frac{|\sin t|}{t}\, dt$ also diverges.

# 2.7 Functions defined by an integral

## 2.7.1 Functions defined by an integral

Let

$$f(x,t) : [\alpha,\beta] \times [u,v] \to \mathbb{R}$$

be a function integrable with respect to $t \in [u,v]$, where $u$ and $v$ are functions of $x$ or constants. Let $F : [\alpha,\beta] \to \mathbb{R}$ be defined by

$$F(x) = \int_u^v f(x,t)\, dt.$$

**Property 2.1.** *1- If $f(t,x)$ is continuous on $[\alpha,\beta] \times [u,v]$ and if $u, v$ are continuous on $[\alpha,\beta]$, then the function $F(x)$ is continuous on $[\alpha,\beta]$.*

*2- If $f(t,x)$ and $\frac{\partial f}{\partial x}(x,t)$ are continuous on $[\alpha,\beta] \times [u,v]$ and if $u, v$ are differentiable, then the function $F(x)$ is of class $C^1$ on $[\alpha,\beta]$ and*

$$F'(x) = f(x,v)v'(x) - f(x,u)u'(x) + \int_u^v \frac{\partial f}{\partial x}(x,t)\, dt \quad \textit{(Leibniz).}$$

*3- If $f(t,x)$ is continuous on $[\alpha,\beta] \times [u,v]$, then*

$$\int_\alpha^\beta F(x)\, dx = \int_\alpha^\beta \int_u^v f(x,t)\, dt\, dx = \int_u^v \int_\alpha^\beta f(x,t)\, dx\, dt \quad \textit{(Fubini).}$$

Let

$$\int_a^{+\infty} f(x,t)\, dt$$

be an improper integral depending on a parameter $x \in [\alpha,\beta]$, where $f(x,t)$ is a bounded function on $[\alpha,\beta] \times [a,+\infty]$.

By analogy, this integral converges for $x \in [\alpha,\beta]$ if and only if the limit of

$$\int_a^u f(x,t)\, dt$$

exists when $u \to +\infty$. In other words, for every $\varepsilon > 0$, there exists a number $A(\varepsilon)$ such that for every $u \geq A(\varepsilon)$ and every $x \in [\alpha, \beta]$, one has

$$\left| \int_a^u f(x,t)\,dt - F(x) \right| \leq \varepsilon.$$

($A(\varepsilon, x)$ generally depends on $\varepsilon$ and $x$).

**Definition 2.6.** The integral

$$\int_a^{+\infty} f(x,t)\,dt$$

converges uniformly on $[\alpha, \beta]$ if and only if, for every $\varepsilon > 0$, there exists a number $A(\varepsilon)$ such that for every $u \geq A(\varepsilon)$ and every $x \in [\alpha, \beta]$, one has

$$\left| \int_a^u f(x,t)\,dt - F(x) \right| \leq \varepsilon.$$

($A(\varepsilon)$ does not depend on $x$).
We note that

$$\int_a^u f(x,t)\,dt - F(x) = \int_a^{+\infty} f(x,t)\,dt.$$

**Theorem 2.8.** *(Cauchy criterion). The integral $\int_a^{+\infty} f(x,t)\,dt$ converges uniformly on $[\alpha, \beta]$ if and only if, for every $\varepsilon > 0$, there exists a number $A(\varepsilon)$ such that the inequalities $u > v \geq A(\varepsilon)$ imply*

$$\int_v^u f(x,t)\,dt \leq \varepsilon,$$

*for all $x \in [\alpha, \beta]$.*

### 2.7.2 Continuity

**Theorem 2.9.** *If $f(x,t)$ is continuous on $[\alpha, \beta] \times [a, +\infty[$ and if the integral*

$$\int_a^{+\infty} f(x,t)\,dt$$

*converges uniformly to $F(x)$ on $[\alpha, \beta]$, then $F(x)$ is continuous on $[\alpha, \beta]$.*

## Integration

**Theorem 2.10.** *If $f(x,t)$ is continuous on $[\alpha,\beta]\times[a,+\infty[$ and if the integral*

$$\int_a^{+\infty} f(x,t)\,dt$$

*converges uniformly to $F(x)$ on $[\alpha,\beta]$, then*

$$\int_\alpha^\beta F(x)\,dx = \int_\alpha^\beta \int_a^{+\infty} f(x,t)\,dt\,dx = \int_a^{+\infty}\int_\alpha^\beta f(x,t)\,dx\,dt.$$

## Differentiation

**Theorem 2.11.** *Assume that $f(x,t)$ and $\frac{\partial f}{\partial x}(x,t)$ are continuous on $[\alpha,\beta]\times[a,+\infty[$. If*

$$\int_a^{+\infty} f(x,t)\,dt$$

*converges to $F(x)$ on $[\alpha,\beta]$, and if*

$$\int_a^{+\infty} \frac{\partial f}{\partial x}(x,t)\,dt$$

*converges uniformly on $[\alpha,\beta]$, then:*

1. *$\int_a^{+\infty} f(x,t)\,dt$ converges uniformly on $[\alpha,\beta]$,*
2. *$F$ is of class $C^1$ on $[\alpha,\beta]$,*
3. $$\frac{dF}{dx} = \int_a^{+\infty} \frac{\partial f}{\partial x}(x,t)\,dt.$$

## Weierstrass criterion

**Theorem 2.12.** *If there exists a function $\varphi(t)\geq 0$, integrable on $[a,+\infty[$, such that:*

1. *$|f(x,t)|\leq \varphi(t)$ for all $x\in[\alpha,\beta]$,*
2. *$\int_a^{+\infty}\varphi(t)\,dt$ converges,*

*then the integral*

$$\int_a^{+\infty} f(x,t)\,dt$$

*converges absolutely and uniformly on $[\alpha,\beta]$.*

## Example

The integral

$$\int_1^{+\infty} \frac{\cos(tx)}{t^2}\,dt$$

converges normally since

$$\left|\frac{\cos(tx)}{t^2}\right| \leq \frac{1}{t^2}, \quad \text{and} \quad \int_1^{+\infty} \frac{dt}{t^2} \text{ converges.}$$

### 2.7.3 Abel-Dirichlet criterion

**Theorem 6.6.1.** Let $f, g : [\alpha, \beta] \times [a, +\infty[ \to \mathbb{R}$ be bounded functions, integrable in $t$ on $[a, u]$, $u \geq a$. Suppose that:

1. For every $x \in [\alpha, \beta]$, $f(x, t) \geq 0$ and is decreasing in $t$,
2. $\lim_{t\to+\infty} f(x, t) = 0$ uniformly on $[\alpha, \beta]$,
3. There exists a constant $C$ (independent of $u$ and $x$) such that for all $u \geq a$,

$$\left|\int_a^u g(x, t)\,dt\right| \leq C.$$

Then

$$\int_a^{+\infty} f(x, t)g(x, t)\,dt$$

converges uniformly on $[\alpha, \beta]$.

### 2.7.4 Eulerian Functions

We now study the Gamma function introduced by Euler in 1729. This important function appears in many problems of analysis. We also study another function introduced by Euler in 1730, called the Beta function.

**Problem:**

a) The Euler Gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^{+\infty} e^{-t}t^{x-1}\,dt.$$

Show that this integral converges for $x > 0$. (6.1)

b) Show that the integral (6.1) converges uniformly on $[\alpha, \beta]$ where $0 < \alpha < \beta < +\infty$. Deduce that $\Gamma(x)$ is continuous on $]0, +\infty[$.

c) Show that the function $\Gamma(x)$ is of class $C^\infty$ on $]0, +\infty[$.

d) By integrating $\int_v^u e^{-t}t^{x-1}\,dt$ by parts and passing to the limit as $u \to +\infty$, $v \to 0$, show that

$$\Gamma(x + 1) = x\Gamma(x).$$

This implies, for all $x \in \mathbb{N}^*$, the recurrence relation $\Gamma(x + 1) = x!$ (6.2).

e) Extend the definition of $\Gamma(x)$ to negative values of $x$ using formula (6.2).

f) Define the Euler Beta function by

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx.$$

Show that this integral converges for $p \in ]0,+\infty[$ and $q \in ]0,+\infty[$.

g) Establish the formula

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

**Solution**

a) The integral (6.1) converges if and only if the two integrals

$$\int_0^1 e^{-t}t^{x-1}\,dt \quad \text{and} \quad \int_1^{+\infty} e^{-t}t^{x-1}\,dt$$

converge. The first integral converges for $x > 0$ since, for all $t \geq 0$,

$$e^{-t}t^{x-1} \leq t^{x-1},$$

and by comparison with $\int_0^1 t^{x-1}dt$. The second integral converges for all $x \in \mathbb{R}$ since

$$\lim_{t\to+\infty} \frac{e^{-t}t^{x-1}}{e^{-t^2}} = 0,$$

and $\int_1^{+\infty} e^{-t^2}dt$ converges. Hence (6.1) converges for $x > 0$.

b) For $x \geq \alpha > 0$ and $t \in ]0,1]$, we have

$$e^{-t}t^{x-1} \leq e^{-t}t^{\alpha-1}.$$

Since $\int_0^1 e^{-t}t^{\alpha-1}dt$ converges, the Weierstrass criterion implies uniform convergence on $[\alpha,+\infty[$.

Similarly, for $0 < x \leq \beta$ and $t \in [1,+\infty[$,

$$e^{-t}t^{x-1} \leq e^{-t}t^{\beta-1}.$$

Since $\int_1^{+\infty} e^{-t}t^{\beta-1}dt$ converges, we obtain uniform convergence on $]0,\beta]$. Hence the integral converges uniformly on $[\alpha,\beta]$, and continuity follows from Theorem 6.2.1.

c) Formally,

$$\Gamma'(x) = \int_0^{+\infty} e^{-t}(\ln t)t^{x-1}\,dt.$$

Using estimates on $[0,1]$ and $[1,+\infty[$, one shows uniform convergence on $[\alpha,\beta]$ and obtains

$$\Gamma'(x) = \int_0^{+\infty} e^{-t}(\ln t)t^{x-1}\,dt.$$

Iterating the argument yields

$$\Gamma^{(k)}(x) = \int_0^{+\infty} e^{-t}(\ln t)^k t^{x-1}\,dt,$$

so $\Gamma \in C^\infty(]0,+\infty[)$.

d) By integration by parts,

$$\int_v^u e^{-t}t^{x-1}dt = \frac{e^{-u}u^x}{x} - \frac{e^{-v}v^x}{x} + \frac{1}{x}\int_v^u e^{-t}t^x dt.$$

Passing to the limit gives

$$\Gamma(x) = \frac{1}{x}\Gamma(x+1),$$

hence

$$\Gamma(x+1) = x\Gamma(x), \quad x \in \mathbb{N}^*.$$

Since $\Gamma(1) = 1$, we deduce $\Gamma(n+1) = n!$.

e) For $x \in ]-1, 0[$, the relation

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

extends the definition. Iterating,

$$\Gamma(x) = \frac{\Gamma(x+k+1)}{x(x+1)\cdots(x+k)}, \quad k \in \mathbb{N}.$$

f) The integral is improper at 0 and 1. A comparison argument shows convergence for $p > 0$, $q > 0$.

g) One obtains

$$\Gamma(p)\Gamma(q) = 4\int_0^{+\infty}\int_0^{\pi/2} e^{-r^2} r^{2(p+q)-1}\cos^{2p-1}\theta\sin^{2q-1}\theta\, d\theta\, dr,$$

and using the change of variables $x = \sin^2\theta$, one deduces

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

# Exercises

## Exercise 01

Study the nature of the following integrals:

$$\int_0^{+\infty} \sin t\, dt; \quad \int_0^{+\infty} \frac{2t}{\sqrt{t^5+t+1}}\, dt; \quad \int_0^{+\infty} e^{-t^2}\, dt;$$
$$\int_0^{+\infty} dt; \quad \int_0^{+\infty} \frac{\cos(t^2)}{1+t^2}\, dt; \quad \int_1^{+\infty} \frac{\ln t}{t+2}\, dt;$$
$$\int_1^{+\infty} \tan t\, dt; \quad \int_0^1 \frac{1}{1+t}\, dt; \quad \int_1^{+\infty} \frac{1}{t\sqrt{1+t^2}}\, dt;$$

## Exercise 02

Study the nature of the following integrals using integration by parts:

$$\int_1^{+\infty} \frac{\cos t}{\sqrt{t}}\, dt; \quad \int_0^{+\infty} \frac{1}{t}\cos\left(\frac{1}{t}\right) dt; \quad \int_0^1 \frac{\ln t}{(1+t)^2}\, dt;$$
$$\int_0^{+\infty} \arctan\left(\frac{1}{t}\right) dt; \quad \int_1^{+\infty} \frac{2t\ln t}{(1+t^2)^2}\, dt;$$

## Exercise 03

Compute the following integrals:

$$\int_0^{\pi} \frac{1}{1+\cos t}\, dt; \quad \int_0^{+\infty} \frac{e^{-\sqrt{t}}}{\sqrt{t}}\, dt; \quad \int_{-1}^1 \frac{1}{\sqrt{1-t^2}}\, dt;$$

## Exercise 04

1. Let $\alpha \in \mathbb{R}$, $\beta > 0$. Determine the set of pairs $(\alpha, \beta)$ for which the improper integral

$$\int_0^{+\infty} \frac{t^\alpha \ln t}{1+t^\beta}\, dt$$

converges.

2. Let $I = \int_0^{\frac{\pi}{2}} \ln(\sin t)\, dt$ and $J = \int_0^{\frac{\pi}{2}} \ln(\cos t)\, dt$.

   (a) Show that $I$ and $J$ are convergent and $I = J = -\frac{\pi}{2}\ln 2$.

   (b) Show that $\int_0^{\pi} t\ln(\sin t)\, dt = -\frac{\pi^2}{2}\ln 2$.

## Exercise 05

1. For all $x \in \mathbb{R}$, show that the integral

$$f(x) = \int_{-\infty}^{+\infty} e^{-t^2+itx}\, dt = 2\int_0^{+\infty} e^{-t^2}\cos(tx)\, dt$$

is uniformly convergent, and show that $f$ admits the derivative

$$f'(x) = \int_{-\infty}^{+\infty} it\, e^{-t^2+itx}\, dt.$$

2. Show, by integration by parts, that $f$ satisfies the differential equation

$$2f'(x) + xf(x) = 0,$$

and deduce the value of $f(x)$ knowing that $f(0) = \sqrt{\pi}$.

## Exercise 06

We define, for $x \in \mathbb{R}$,

$$F(x) = \int_0^{+\infty} \frac{\sin(tx)}{t} e^{-t}\, dt.$$

1. Justify that $F$ is well-defined on $\mathbb{R}$.
2. Show that $F$ is of class $C^1$ and give an expression of $F'(x)$ for all $x \in \mathbb{R}$.
3. Compute $F'(x)$.
4. Deduce a simplified expression of $F(x)$.

## Exercise 07

For $x \in \mathbb{R}_+^*$, define

$$f(x) = \int_0^1 \frac{e^t}{t+x}\, dt.$$

1. Show that $f$ is decreasing, continuous, and differentiable.
2. Determine

$$\lim_{x\to+\infty} f(x), \qquad \lim_{x\to 0} f(x),$$

and in both cases give an equivalent. (One will show that $\lim_{x\to 0}\Big[f(x) + \ln x\Big]$ exists.)

# Chapter 3

# Differential equation

## 3.1 Definitions

In what follows, we assume that $I$ is an interval of $\mathbb{R}$.

**Definition 3.1.** A differential equation of order $n$ is any relation $F$ between $x$, $y$, and the derivatives of $y$ given by

$$F(x, y, y', \ldots, y^{(n)}) = 0.$$

Any function $y(x)$ satisfying

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

is called a solution of the differential equation.

**Definition 3.2.**

1. A differential equation of order $n$ is said to be linear if it is of the form:

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^{(n)} = g(x),$$

where $a_i$, for $i = 0, \ldots, n$, and $g$ are real continuous functions on an interval $I \subset \mathbb{R}$.

2. A linear differential equation of order $n$ is said to be homogeneous, or without a second term, if the function $g$ is zero:

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^{(n)} = 0.$$

3. A linear differential equation is said to have constant coefficients if

$$a_0 y + a_1 y' + a_2 y'' + \cdots + a_n y^{(n)} = g(x),$$

where $a_i$, for $i = 0, \ldots, n$, are constants and $g$ is a real continuous function.

**Proposition 3.1.** *If $y_1$ and $y_2$ are solutions of the homogeneous linear differential equation*

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^{(n)} = g(x),$$

*then for all $\lambda, \mu \in \mathbb{R}$, the function $\lambda y_1 + \mu y_2$ is also a solution of this equation. The set of solutions of a homogeneous linear differential equation forms a vector space.*

## 3.2 First-Order Differential Equation

**Definition 3.3.** A first-order differential equation is of the form:

$$y' = f(x, y),$$

where $f$ is a function continuous with respect to $x$ on an interval $I \subset \mathbb{R}$.

### 3.2.1 Differential Equation with Separable Variables

A differential equation with separable variables is an equation of the form:

$$y' f(y) = g(x) \tag{1.1}$$

where $f$ and $g$ are continuous functions on $I$.

#### Method of Solution

Since $y' = \frac{dy}{dx}$, we have

$$\frac{dy}{dx} f(y) = g(x) \quad \Rightarrow \quad f(y)\,dy = g(x)\,dx \quad \Rightarrow \quad \int f(y)\,dy = \int g(x)\,dx.$$

#### Example

1.

$$y' = e^{-y} \quad \Rightarrow \quad e^y dy = dx \quad \Rightarrow \quad e^y = x + c, \quad c \in \mathbb{R} \quad \Rightarrow \quad y = \ln|x + c|.$$

2.

$$(x^2 + 1)\,dy = 2x(y^2 + 1)\,dx \quad \Rightarrow \quad \int \frac{dy}{y^2 + 1} = \int \frac{2x}{x^2 + 1}\,dx$$

$$\Rightarrow \quad \arctan(y) = \ln(x^2 + 1) + c \quad \Rightarrow \quad y = \tan(\ln(x^2 + 1) + c), \quad c \in \mathbb{R}.$$

### 3.2.2 Homogeneous Differential Equation

**1. First case:** A homogeneous differential equation is an equation of the form:

$$y' = F\left(\frac{y}{x}\right).$$

**Method of Solution**

We set

$$t = \frac{y}{x}$$

thus

$$y = tx \quad \Rightarrow \quad y' = t + xt'.$$

Since

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right),$$

we obtain

$$t + xt' = F(t) \quad \Rightarrow \quad F(t) - t = x\frac{dt}{dx} \quad \Rightarrow \quad \frac{dt}{F(t) - t} = \frac{dx}{x}.$$

It remains only to solve the separable differential equation:

$$\frac{dt}{F(t) - t} = \frac{dx}{x}.$$

**Example**

1.

$$y' = \frac{y}{x} + \frac{1}{\cos\left(\frac{y}{x}\right)}.$$

Setting

$$t = \frac{y}{x} \quad \Rightarrow \quad y = tx \quad \Rightarrow \quad y' = t + xt',$$

we get

$$t + xt' = t + \frac{1}{\cos(t)} \quad \Rightarrow \quad x\frac{dt}{dx} = \frac{1}{\cos(t)}.$$

Thus,

$$\cos(t)\,dt = \frac{dx}{x} \quad \Rightarrow \quad \int \cos(t)\,dt = \int \frac{dx}{x}$$

$$\Rightarrow \quad \sin(t) = \ln|x| + c \quad \Rightarrow \quad t = \arcsin(\ln|x| + c), \quad c \in \mathbb{R}.$$

Finally,

$$y(x) = x\,\arcsin(\ln|x| + c).$$

2.

$$xy' = y + x.$$

Setting

$$t = \frac{y}{x} \quad \Rightarrow \quad y = tx \quad \Rightarrow \quad y' = t + xt',$$

we obtain

$$x(t + xt') = tx + x \quad \Rightarrow \quad t'x + t = t + 1 \quad \Rightarrow \quad t'x = 1.$$

Thus,

$$x\frac{dt}{dx} = 1 \quad \Rightarrow \quad dt = \frac{dx}{x}.$$

Hence,

$$t = \ln|x| + c, \quad c \in \mathbb{R},$$

and therefore

$$y(x) = x\ln|x| + cx.$$

**Second case:** A homogeneous differential equation is an equation of the form:

$$(Eh): \quad y' + a(x)y = 0,$$

where $a$ is continuous on $I$.

**Method of Solution**

(Eh) is a separable differential equation. Indeed:

$$y' = -a(x)y \quad \Rightarrow \quad \frac{dy}{y} = -a(x)\,dx \quad \Rightarrow \quad \int\frac{dy}{y} = \int -a(x)\,dx \quad \Rightarrow \quad \ln|y| = \int -a(x)\,dx + c, \quad c \in \mathbb{R}$$

Therefore, the general solution of (Eh) is given by

$$y_h = Ke^{-\int a(x)\,dx}, \quad K = \pm e^c \in \mathbb{R}.$$

**Example**

1.

$$y' - ye^x = 0 \quad \Rightarrow \quad \frac{dy}{y} = e^x dx \quad \Rightarrow \quad \int\frac{dy}{y} = \int e^x dx$$
$$\Rightarrow \quad \ln|y| = e^x + c, \quad c \in \mathbb{R} \quad \Rightarrow \quad y = Ke^{e^x}, \quad K = \pm e^c.$$

2.

$$y' - y\cos(x) = 0 \quad \Rightarrow \quad \frac{dy}{y} = \cos(x)\,dx \quad \Rightarrow \quad \int\frac{dy}{y} = \int\cos(x)\,dx$$
$$\Rightarrow \quad \ln|y| = \sin(x) + c, \quad c \in \mathbb{R} \quad \Rightarrow \quad y = Ke^{\sin(x)}, \quad K = \pm e^c.$$

### 3.2.3 Linear Differential Equation

A linear differential equation is an equation of the form:

$$(E): \quad y' + a(x)y = b(x),$$

where $a$ and $b$ are continuous functions on $I$.

**Method of Solution**

The general solution of (E) is given by

$$y_g = y_h + y_p,$$

where $y_h$ is the solution of the homogeneous equation (Eh), and $y_p$ is a particular solution.

**Method of Variation of the Constant**

Let $y_h = kf(x)$ where $f(x) = e^{-\int a(x)\,dx} = F$. The method consists of letting the constant $k$ vary, that is, replacing the constant $k$ by a function to be determined $k(x)$, with

$$y_p = k(x)F.$$

By substituting into (E), we obtain:

$$y_p' = k'(x)F + F'k(x) \quad \Rightarrow \quad k'(x)F + F'k(x) + a(x)k(x)F = b(x) \quad \Rightarrow \quad k'(x)F = b(x),$$

since

$$F' + a(x)F = 0.$$

Thus we get

$$y_p = \left(\int \frac{b(x)}{F}\,dx\right)F.$$

**Example**

1. Consider the equation

$$y' - y = e^x. \quad (E)$$

We first solve the homogeneous equation

$$(Eh):\ y' - y = 0.$$

We have

$$\frac{dy}{dx} = y \quad \Rightarrow \quad \int \frac{dy}{y} = \int dx \quad \Rightarrow \quad \ln|y| = x + c, \quad c \in \mathbb{R} \quad \Rightarrow \quad y = Ke^x, \quad K = \pm e^c.$$

Next, we look for a particular solution using the variation of constant method (V.C.M.). We set

$$y(x) = k(x)e^x \quad \Rightarrow \quad y' = k'(x)e^x + k(x)e^x.$$

Substituting into (E), we obtain:

$$k'(x)e^x + k(x)e^x - k(x)e^x = e^x \quad \Rightarrow \quad k'(x) = 1.$$

Hence,

$$k(x) = x,$$

and the particular solution is

$$y_p(x) = xe^x.$$

Therefore, the general solution of equation (E) is:

$$y(x) = Ke^x + xe^x.$$

2. Consider the equation
$$y' + 2ty = 2t. \quad (E)$$
We first solve the homogeneous equation
$$(Eh): \ y' + 2ty = 0.$$
We have:
$$\frac{dy}{dt} = -2ty \quad \Rightarrow \quad \int \frac{dy}{y} = -\int 2t\,dt \quad \Rightarrow \quad \ln|y| = -t^2 + c \quad \Rightarrow \quad y = ke^{-t^2}, \quad k = \pm e^c.$$
Next, we look for a particular solution using the variation of constant method. We set
$$y(t) = k(t)e^{-t^2} \quad \Rightarrow \quad y' = k'(t)e^{-t^2} - 2t\,k(t)e^{-t^2}.$$
Substituting into (E), we get:
$$k'(t)e^{-t^2} - 2t\,k(t)e^{-t^2} + 2t\,k(t)e^{-t^2} = 2t \quad \Rightarrow \quad k'(t)e^{-t^2} = 2t.$$
Hence,
$$k(t) = \int 2t\,e^{t^2}\,dt = e^{t^2}.$$
Thus, the particular solution is:
$$y_p = e^{t^2}e^{-t^2} = 1.$$
Finally, the general solution of the equation with second member is:
$$y(t) = ke^{-t^2} + 1.$$
Given the initial condition $y(0) = 0$, we obtain
$$k = -1,$$
and therefore:
$$y(t) = 1 - e^{-t^2}.$$

### 3.2.4 Bernoulli Equation

A Bernoulli equation is an equation of the form:
$$(Eb): \quad y' + a(x)y = b(x)y^n, \qquad n \neq 0,\ n \neq 1.$$

**Method of Solution**

We divide (Eb) by $y^n$, obtaining:
$$y'y^{-n} + a(x)y^{1-n} = b(x).$$
Then we set
$$z = y^{1-n} \quad \Rightarrow \quad z' = (1-n)y'y^{-n}.$$
We obtain a first-order linear differential equation in $z$ of the form:
$$\frac{1}{1-n}z' + a(x)z = b(x).$$

**Example**

$$y' + 2xy = xy^2.$$

This is a Bernoulli differential equation.

We divide (Eb) by $y^2$:

$$y'y^{-2} + 2xy^{-1} = x.$$

Then we set

$$z = y^{-1} \quad \Rightarrow \quad z' = -y'y^{-2}.$$

We obtain a first-order differential equation in $z$:

$$-z' + 2xz = x. \qquad (E)$$

We begin by solving the homogeneous equation:

$$(Eh): \quad -z' + 2xz = 0.$$

We have:

$$\frac{dz}{dx} = 2xz \quad \Rightarrow \quad \int \frac{dz}{z} = \int 2x\,dx \quad \Rightarrow \quad \ln|z| = x^2 + c \quad \Rightarrow \quad z = ke^{x^2}, \quad k = \pm e^c.$$

Next, we search for a particular solution using the method of variation of the constant (V.C.M.). We set:

$$z(x) = k(x)e^{x^2} \quad \Rightarrow \quad z' = k'(x)e^{x^2} + 2xk(x)e^{x^2}.$$

Substituting into (E):

$$-k'(x)e^{x^2} - 2xk(x)e^{x^2} + 2xk(x)e^{x^2} = x \quad \Rightarrow \quad k'(x) = -xe^{-x^2}.$$

Thus,

$$k(x) = \int -xe^{-x^2}\,dx = e^{-2x^2}.$$

Hence, the particular solution is:

$$z_p(x) = e^{-x^2}e^{x^2}/2 = \frac{1}{2}.$$

Therefore, the general solution of the equation with second member is:

$$z(x) = ke^{x^2} + \frac{1}{2}.$$

Finally, since

$$y(x) = \Big(z(x)\Big)^{-1},$$

the solution of (Eb) is:

$$y(x) = \left(ke^{x^2} + \frac{1}{2}\right)^{-1}, \qquad k \in \mathbb{R}.$$

## 3.3 Second-Order Differential Equation

Let the second-order differential equation be given by

$$(E): \quad y'' + a(x)y' + b(x)y = f(x),$$

where $a, b, f$ are continuous functions on an interval $I$.

### Definition

1. $(E)$ is called a *second-order differential equation with constant coefficients* if it is of the form
$$y'' + ay' + by = f(x), \qquad a, b \in \mathbb{R}.$$
2. The *homogeneous equation* associated with $(E)$ is
$$(E_h): \quad y'' + ay' + by = 0.$$
3. The *characteristic equation* associated with $(E_h)$ is
$$(E_r): \quad r^2 + ar + b = 0, \qquad r \in \mathbb{C}.$$

### 3.3.1 Homogeneous Solution $(E_h)$

**Theorem 3.1.** *Let $y_1, y_2$ be two linearly independent solutions of $(E_h)$. Then the general solution of $(E_h)$ is*

$$y = c_1 y_1 + c_2 y_2, \qquad c_1, c_2 \in \mathbb{R}.$$

Let

$$\Delta = a^2 - 4b$$

be the discriminant of the characteristic equation $(E_r)$.

**Theorem 3.2.**
1. *If $\Delta > 0$, the characteristic equation has two distinct real solutions $r_1, r_2$. Then the general solution of $(E_h)$ is*
$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \qquad c_1, c_2 \in \mathbb{R}.$$
2. *If $\Delta = 0$, the characteristic equation has a double root $r_1 = r_2 = r$. The general solution is*
$$y_h = c_1 e^{rx} + c_2 x e^{rx}.$$
3. *If $\Delta < 0$, the characteristic equation has two complex conjugate roots $r_1 = r + i\alpha, r_2 = r - i\alpha$. The general solution is*
$$y_h = c_1 e^{rx}\cos(\alpha x) + c_2 e^{rx}\sin(\alpha x).$$

**Example**

1. Consider
$$y'' + y' - 6y = 0. \quad (E)$$
The characteristic equation associated with $(E_h)$ is
$$(E_r) : r^2 + r - 6 = 0.$$
The solutions of this equation are $r_1 = 2, r_2 = -3$, which are simple roots. Thus, the solution of this second-order equation is:
$$y(x) = c_1 e^{2x} + c_2 e^{-3x}, \qquad c_1, c_2 \in \mathbb{R}.$$

2. Consider
$$y'' + 2y' + y = 0. \quad (E)$$
The characteristic equation is
$$r^2 + 2r + 1 = 0,$$
which admits a double root $r = -1$. Hence, the solution is:
$$y(x) = c_1 e^{-x} + c_2 x e^{-x} = (c_1 + c_2 x)e^{-x}, \qquad c_1, c_2 \in \mathbb{R}.$$

3. Consider
$$y'' + y = 0. \quad (E_h)$$
The characteristic equation is
$$r^2 + 1 = 0,$$
which has simple roots $r_1 = i, r_2 = -i$. Thus, the solution is:
$$y(x) = e^{0\cdot x}[c_1 \cos(x) + c_2 \sin(x)] = c_1 \cos(x) + c_2 \sin(x), \qquad c_1, c_2 \in \mathbb{R}.$$

4. Consider
$$y'' + y' + y = 0. \quad (E_h)$$
The characteristic equation is
$$r^2 + r + 1 = 0,$$
which has simple roots $r_1 = i\sqrt{3}, r_2 = -i\sqrt{3}$. Therefore, the solution is:
$$y(x) = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x), \qquad c_1, c_2 \in \mathbb{R}.$$

### 3.3.2 Solution of a Second-Order Equation with Constant Coefficients

Consider
$$(E) : \quad y'' + ay' + by = f(x).$$

Let
$$y_g = y_h + y_p$$
be the general solution of $(E)$, which is the sum of the homogeneous solution $y_h$ and a particular solution $y_p$.

The method for finding the particular solution is summarized in the following table:

| $f(x)$ | Roots of the characteristic equation $(E_r)$ | Particular solution $y_p$ |
|---|---|---|
| $P_n(x)$ | 0 is not a root of $(E_r)$ | $\tilde{P}_n(x)$ |
| $P_n(x)$ | 0 is a root of multiplicity $\omega$ of $(E_r)$ | $x^\omega \tilde{P}_n(x)$ |
| $P_n(x)e^{\alpha x}$ | $\alpha$ is not a root of $(E_r)$ | $\tilde{P}_n(x)e^{\alpha x}$ |
| $P_n(x)e^{\alpha x}$ | $\alpha$ is a root of multiplicity $\omega$ of $(E_r)$ | $x^\omega \tilde{P}_n(x)e^{\alpha x}$ |
| $P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x)$ | $\pm i\beta$ is not a root of $(E_r)$ | $\tilde{P}_k(x)\cos(\beta x) + \tilde{Q}_k(x)\sin(\beta x)$ |
| $P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x)$ | $\pm i\beta$ is a root of multiplicity $\omega$ of $(E_r)$ | $x^\omega\Big(\tilde{P}_k(x)\cos(\beta x) + \tilde{Q}_k(x)\sin(\beta x)\Big)$ |
| $e^{\alpha x}(P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x))$ | $\alpha \pm i\beta$ is not a root of $(E_r)$ | $e^{\alpha x}\Big(\tilde{P}_k(x)\cos(\beta x) + \tilde{Q}_k(x)\sin(\beta x)\Big)$ |
| $e^{\alpha x}(P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x))$ | $\alpha \pm i\beta$ is a root of multiplicity $\omega$ of $(E_r)$ | $x^\omega e^{\alpha x}\Big(\tilde{P}_k(x)\cos(\beta x) + \tilde{Q}_k(x)\sin(\beta x)\Big),\ k = \max(n, m)$ |

where $P_n(x), \tilde{P}_n(x)$ be polynomials of degree $n$, and $Q_m(x), \tilde{Q}_m(x)$ be polynomials of degree $m$.

*Remark* 3.1. The case where the non-homogeneous term is

$$f(x) = e^{\alpha x}\Big(P_n(x)\cos(\beta x) + Q_m(x)\sin(\beta x)\Big)$$

summarizes all other cases. Indeed:

- For $r = 0 = 0 + 0i$, $\alpha = \beta = 0$, this is the first case where $f(x) = P_n(x)$.
- For $r = \alpha + i \cdot 0$, a real $\beta = 0$, this is the case $f(x) = e^{\alpha x}P_n(x)$.
- For $r = i\beta$, a purely imaginary root, $\alpha = 0$, this is the case $f(x) = P_k(x)\cos(\beta x) + Q_k(x)\sin(\beta x)$.

**Example**

1. Consider
$$y'' + y' - 6y = 6x. \quad (E)$$

The solution of the homogeneous equation

$$(E_h) : y'' + y' - 6y = 0$$

is

$$y_h(x) = c_1 e^{2x} + c_2 e^{-3x}, \quad c_1, c_2 \in \mathbb{R}.$$

The general solution of $(E)$ is $y_g = y_h + y_p$.

Following the previous table, the particular solution depends on $f(x) = 6x$. Here 0 is not a root of $(E_r) : r^2 + r - 6 = 0$, so $y_p$ is a polynomial of the same degree as $f(x)$:

$$y_p = ax + b.$$

Substituting into $(E)$ gives $a = -1$, $b = -1/6$, hence

$$y_p = -x - \frac{1}{6}.$$

Thus, the solution of $(E)$ is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x} - x - \frac{1}{6}.$$

2. Consider

$$y'' + y' - 6y = e^{2x}.$$

The homogeneous solution is the same as before. Here 2 is a root of $(E_r)$, so the particular solution is of the form $x^1 P_0(x) e^{2x}$:

$$y_p = axe^{2x}.$$

Substituting into $(E)$ gives $a = 1$, hence

$$y_g = c_1 e^{2x} + c_2 e^{-3x} + xe^{2x}.$$

3. Consider

$$y'' + y = e^x.$$

The homogeneous solution is

$$y_h = c_1 \cos(x) + c_2 \sin(x), \quad c_1, c_2 \in \mathbb{R}.$$

Following the table, $y_p$ is of the form $ae^x$. Substituting into $(E)$ gives $a = 1/2$, hence

$$y_g = c_1 \cos(x) + c_2 \sin(x) + \frac{1}{2} e^x.$$

4. Consider

$$y'' + y = \sin(x).$$

The homogeneous solution is $y_h = c_1 \cos(x) + c_2 \sin(x)$. Since $i$ is a simple root of $(E_r)$, the particular solution is

$$y_p = x(a \sin(x) + b \cos(x)).$$

Substituting into $(E)$ gives $a = -1/2$, $b = 0$, thus

$$y_g = c_1 \cos(x) + c_2 \sin(x) - \frac{x}{2} \cos(x).$$

5. Find the particular solutions for the following equations:

   (a) $y'' - y' - 2y = e^{2x}$, roots of $(E_r) : r^2 - r - 2 = 0$ are $r_1 = -1, r_2 = 2$, so $y_p = xae^{2x}$.

   (b) $y''-2y'+y = xe^x$, $(E_r) : r^2-2r+1 = 0$, double root $r = 1$, so $y_p = x^2(ax+b)e^x$.

   (c) $y'' + y = x^2e^x \sin(x)$, $(E_r) : r^2 + 1 = 0$, roots $r_1 = i, r_2 = -i$, $1 \pm i$ is not a root, hence

$$y_p = e^x\Big[(ax^2 + bx + c)\sin(x) + (dx^2 + ex + f)\cos(x)\Big].$$

   (d) $y'' + 2y' + y = e^x + \sin(x)$, $(E_r) : (r + 1)^2 = 0$, double root $r = -1$, particular solution is $y_p = y_{p1} + y_{p2}$ by superposition. $y_{p1}$ for $y'' + 2y' + y = e^x$ with 1 not a root: $y_{p1} = ae^x$. $y_{p2}$ for $y'' + 2y' + y = \sin(x)$ with $i$ not a root: $y_{p2} = a\sin(x) + b\cos(x)$.

   (e) $y''+4y = x\sin(2x)+xe^{2x}\cos(x)$, $(E_r) : r^2+4 = 0$, roots $2i, -2i$. $y_p = y_{p1}+y_{p2}$, with $y_{p1}$ for $x\sin(2x)$: $y_{p1} = x\Big((ax + b)\cos(2x) + (cx + d)\sin(2x)\Big)$, $y_{p2}$ for $xe^{2x}\cos(x)$: $y_{p2} = e^{2x}\Big((ax + b)\cos(x) + (cx + d)\sin(x)\Big)$.

# 3.4 Differential Equations

## 3.4.1 Total Differential Equations

**Definition 3.4.** Consider the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \qquad (*)$$

It is called a *total differential equation* if $M(x, y)$ and $N(x, y)$ are continuous and differentiable functions such that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are defined on a certain domain.

**Example**

Solve the differential equation

$$(x^3 + xy^2)\,dx + (x^2y + y^3)\,dy = 0. \qquad (1)$$

**Solution.**
Let

$$M(x, y) = x^3 + xy^2, \qquad N(x, y) = x^2y + y^3.$$

Then

$$\frac{\partial M}{\partial y} = 2xy, \qquad \frac{\partial N}{\partial x} = 2xy,$$

hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus the given equation is a total differential equation.

We seek a function $u(x, y)$ such that

$$M(x, y) = \frac{\partial u}{\partial x}.$$

Integrating with respect to $x$ gives

$$u(x, y) = \int (x^3 + xy^2)\, dx + \varphi(y) = \frac{x^4}{4} + \frac{x^2 y^2}{2} + \varphi(y),$$

where $\varphi(y)$ is an arbitrary function of $y$.

Now differentiate $u$ with respect to $y$:

$$\frac{\partial u}{\partial y} = x^2 y + \varphi'(y).$$

Since $\frac{\partial u}{\partial y} = N(x, y) = x^2 y + y^3$, we obtain

$$x^2 y + y^3 = x^2 y + \varphi'(y) \quad \Longrightarrow \quad \varphi'(y) = y^3.$$

Integrating:

$$\varphi(y) = \frac{y^4}{4} + c_0, \qquad c_0 \in \mathbb{R}.$$

Thus,

$$u(x, y) = \frac{1}{4}(x^4 + y^4) + \frac{1}{2}x^2 y^2 + c_0.$$

Therefore, the general solution of equation (1) is given implicitly by

$$x^4 + y^4 + 2x^2 y^2 = c, \qquad c \in \mathbb{R}.$$

**Example**

Solve the differential equation

$$2tx\, dt + t^2 x^2\, dx = 0. \tag{5}$$

**Solution.**

Let

$$M(t, x) = 2tx, \qquad N(t, x) = t^2 x^2.$$

Then

$$\frac{\partial M}{\partial x} = 2t, \qquad \frac{\partial N}{\partial t} = 2t.$$

Hence

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t},$$

so the given equation is a total differential equation.

We seek a function $F(t,x)$ such that

$$\frac{\partial F}{\partial t} = M(t,x) = 2tx.$$

Integrating with respect to $t$ gives

$$F(t,x) = \int 2tx\,dt + \varphi(x) = t^2x + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function of $x$.

Now differentiate $F$ with respect to $x$:

$$\frac{\partial F}{\partial x} = t^2 + \varphi'(x).$$

Since

$$\frac{\partial F}{\partial x} = N(t,x) = t^2x^2,$$

we obtain

$$t^2 + \varphi'(x) = t^2x^2.$$

Thus,

$$\varphi'(x) = x^2.$$

Integrating:

$$\varphi(x) = \frac{x^3}{3} + C.$$

Therefore,

$$F(t,x) = t^2x + \frac{x^3}{3} + C.$$

The solutions of differential equation (5) are given implicitly by

$$t^2x + \frac{x^3}{3} = C.$$

### 3.4.2 Integrating Factor

**Definition 3.5.** Assume that the expression

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{1}$$

is not a total differential.
If we multiply equation (1) by a function $\varphi(x,y)$ such that

$$\varphi M(x,y)\,dx + \varphi N(x,y)\,dy = 0, \tag{2}$$

becomes a total differential equation, then $\varphi$ is called an *integrating factor*.

*Remark* 3.2. A non-exact equation $M\,dx + N\,dy = 0$ may admit infinitely many integrating factors.

### 3.4.3 Determination of an Integrating Factor

For equation (2) to be a total differential equation, it is necessary and sufficient that

$$\frac{\partial(\varphi M)}{\partial y} = \frac{\partial(\varphi N)}{\partial x}. \tag{3}$$

Condition (3) is equivalent to

$$M\,\frac{\partial \varphi}{\partial y} + \varphi\,\frac{\partial M}{\partial y} = N\,\frac{\partial \varphi}{\partial x} + \varphi\,\frac{\partial N}{\partial x},$$

that is,

$$M\,\frac{\partial \varphi}{\partial y} - N\,\frac{\partial \varphi}{\partial x} = \varphi\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Dividing both sides by $\varphi$, we obtain

$$M\,\frac{\partial(\log \varphi)}{\partial y} - N\,\frac{\partial(\log \varphi)}{\partial x} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{4}$$

Equation (4) is a partial differential equation for the unknown function $\varphi(x, y)$. Its solution is generally difficult, but certain special cases allow us to determine $\varphi$ explicitly.

**1. Case: $\varphi$ depends only on $y$.** If $\varphi = \varphi(y)$, then equation (4) becomes

$$M\,\frac{d(\log \varphi)}{dy} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Thus,

$$\frac{d(\log \varphi)}{dy} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}. \tag{5}$$

Integrating the right-hand side gives $\log \varphi$, and therefore $\varphi$.

*Remark* 3.3. It is clear that this method works only if the expression

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

does not depend on $x$.

**2. Case: $\varphi$ depends only on $x$.** In a manner similar to the previous case, if the expression

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N}$$

does not depend on $y$, then equation (4) becomes

$$N\,\frac{d(\log \varphi)}{dx} = -\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}.$$

Thus,

$$\frac{d(\log \varphi)}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}. \tag{6}$$

From this we can determine $\log \varphi$, and consequently the integrating factor $\varphi$.

**Example**

1- Solve the following differential equation by determining an integrating factor:

$$(x + y^2)\,dx - 2xy\,dy = 0. \tag{5}$$

**Solution.**

Let

$$M(x, y) = x + y^2, \qquad N(x, y) = -2xy.$$

We compute

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = -2y.$$

Since

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

the equation is not exact.

We now seek an integrating factor. Consider

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - (-2y)}{-2xy} = \frac{4y}{-2xy} = -\frac{2}{x}.$$

Thus,

$$\frac{d(\log \varphi)}{dx} = -\frac{2}{x} \quad \Longrightarrow \quad \log \varphi(x) = -2 \ln |x|.$$

Hence,

$$\varphi(x) = x^{-2}.$$

Multiplying equation (5) by the integrating factor gives

$$\frac{x + y^2}{x^2}\,dx - \frac{2xy}{x^2}\,dy = 0,$$

which is now an exact differential equation.

Integrating, we obtain the general solution

$$\frac{y^2}{x} + \ln |x| = c, \qquad c \in \mathbb{R}.$$

2- Solve the differential equation

$$\frac{1}{t^2 x}\,dt + t^2(x - t)\,dx = 0. \tag{14}$$

**Solution.** Let

$$M(t, x) = \frac{1}{t^2 x}, \qquad N(t, x) = t^2(x - t).$$

We compute

$$\frac{\partial M}{\partial x} = \frac{1}{t^2}, \qquad \frac{\partial N}{\partial t} = 2tx - 3t^2.$$

Since

$$\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}, \tag{15}$$

the equation is not exact.

We look for an integrating factor depending only on $t$. Compute

$$\eta(t) = \frac{\dfrac{\partial M}{\partial x} - \dfrac{\partial N}{\partial t}}{N} = \frac{t^2 - (2tx - 3t^2)}{t^2(x-t)} = \frac{2t}{1} = 2t.$$

Thus the integrating factor is

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2} = \frac{1}{t^2}.$$

Multiplying the equation by $\mu(t) = \dfrac{1}{t^2}$ gives the exact equation:

$$\left(\frac{1}{t^4 x}\right) dt + (x - t)\,dx = 0. \tag{16}$$

Now

$$\frac{\partial M}{\partial x} = \frac{1}{t^4}, \qquad \frac{\partial N}{\partial t} = \frac{1}{t^4},$$

so the equation is exact.

Integrating,

$$F(t, x) = \frac{1}{t} - tx + \frac{x^2}{2} = C.$$

We note that multiplying by $\mu(t) = t^{-2}$ removed the solution $t = 0$. Therefore, the general solution of (14) is

$$\frac{1}{t} - tx + \frac{x^2}{2} = C, \qquad t = 0.$$

3- Solve the differential equation

$$(x + tx^2)\,dt - t\,dx = 0. \tag{17}$$

**Solution.** Let

$$M(t, x) = x + tx^2, \qquad N(t, x) = -t.$$

Then

$$\frac{\partial M}{\partial x} = 1 + 2tx, \qquad \frac{\partial N}{\partial t} = -1.$$

Since

$$\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}, \tag{18}$$

the equation is not exact.

We look for an integrating factor $\mu(x)$ depending only on $x$. Compute

$$\eta(x) = \frac{\dfrac{\partial M}{\partial x} - \dfrac{\partial N}{\partial t}}{M} = \frac{1 + 2tx + 1}{x + tx^2} = \frac{2x}{x(1+tx)} = \frac{2}{x}.$$

Thus

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = \frac{1}{x^2}.$$

Multiply equation (17) by $\mu(x) = \dfrac{1}{x^2}$:

$$\left(\frac{1}{x} + t\right) dt - tx^{-2} dx = 0. \tag{19}$$

This equation is exact, since

$$\frac{\partial M}{\partial x} = -\frac{1}{x^2} + 0, \qquad \frac{\partial N}{\partial t} = -\frac{1}{x^2}.$$

Integrating the exact equation gives

$$\frac{t}{x} + \frac{t^2}{2} = C.$$

The integrating factor $\mu(x) = x^{-2}$ removed the solution $x = 0$. Thus the general solution of (17) is

$$\frac{t}{x} + \frac{t^2}{2} = C, \qquad x = 0.$$

4- Solve the differential equation

$$\left(\sqrt{t^2 - x} + 2t\right) dt - dx = 0. \tag{20}$$

**Solution.** Let

$$M(t,x) = \sqrt{t^2 - x} + 2t, \qquad N(t,x) = -1.$$

Compute

$$\frac{\partial M}{\partial x} = -\frac{1}{2\sqrt{t^2 - x}}, \qquad \frac{\partial N}{\partial t} = 0.$$

Thus the equation is not exact.

Let

$$\omega(t,x) = t^2 - x.$$

We seek an integrating factor $\mu = \mu(\omega)$. Compute

$$\eta(\omega) = \frac{\dfrac{\partial M}{\partial x} - \dfrac{\partial N}{\partial t}}{N\frac{\partial \omega}{\partial t} - M\frac{\partial \omega}{\partial x}} = \frac{-\dfrac{1}{2\sqrt{\omega}}}{-2t + \sqrt{\omega} + 2t} = \frac{1}{2\omega}.$$

Thus

$$\mu(\omega) = e^{\int \frac{1}{2\omega}\, d\omega} = e^{\frac{1}{2}\ln|\omega|} = \frac{1}{\sqrt{\omega}} = \frac{1}{\sqrt{t^2 - x}}.$$

Hence the integrating factor is

$$\mu(t, x) = \frac{1}{\sqrt{t^2 - x}}.$$

Multiplying equation (20) by $\mu$ produces an exact differential equation.

# Exercises

## Exercise 01

Differential equation with separable variables

Solve the following differential equations:

1. $y' + x\ln(x) = 0$
2. $xy' = y$
3. $xy' = y\ln(y)$
4. $(x^2+1)y' = y^2 + 1$

## Exercise 02

Homogeneous differential equations

Solve the following differential equations

1. $(x-y)y\,dx - x^2 dy = 0$
2. $xy' = y + x\cos\left(\dfrac{y}{x}\right)$
3. $xy' = y - x$

## Exercise 03

Linear homogeneous differential equations of order 01

Solve the following differential equations

1. $(1-x^2)y' - 2xy = x$
2. $xy' - 2y + x^3e^{-x} = 0$
3. $y' + 2y = x^2$
4. $xy' + y\tan x - \dfrac{1}{\cos x} = 0$
5. $y' + ay = e^{-x}$
6. $y' + y = 2\sin x$
7. $y' - y = (x+1)e^x$
8. $y' + y = 3x - e^x$

## Exercise 04

Equations of Bernoulli

Solve the following differential equations

$$1 - y' + \frac{1}{x}y + y^2 = 0, \quad 2 - xy' + y - xy^3 = 0, \quad 3 - y' - x^2y = x\sqrt{y}.$$

## Exercise 05

Solve the following differential equations

1. $y'' - y' - 2y = 0$
2. $y'' - 4y' + 4y = 0$
3. $y'' - 6y' + 10y = 0$
4. $y'' - 3y' + 2y = 0$
5. $y'' - 5y' + 6y = 0$
6. $y'' - y' = 0$
7. $y'' - 2y' + 5y = 0$
8. $y'' - 6y' + 9y = 0$
9. $y'' + 4y' + 4y = 0$

## Exercise 06

Solve the following differential equations:

1. $y'' - 3y' + 2y = x^2 + 1$
2. $y'' - 3y' = 3x$
3. $y'' - 5y' + 6y = 2xe^x$
4. $y'' - 5y' + 6y = 2xe^{3x}$
5. $y'' - 2y' + y = 2e^x$
6. $y'' - 5y' + 6y = 2xe^{3x} + 2xe^x$
7. $y'' + 4y = 3\sin(2x)$
8. $y'' + 4y = 3\sin(2x) + 5x\cos(2x)$
9. $y'' + 4y = 3\sin(2x) + 5x\cos(3x)$
10. $y'' - 4y' + 5y = \sin(x)e^{2x}$

## Exercise 07

For each of the following differential equations, determine if it is exact. If it is exact, find the implicit solution $F(t, x) = C$. If it is not exact, find an appropriate integrating factor $\mu(t, x)$ and then solve the equation.

1. $4tx\,dt + (2t^2 - 2x)\,dx = 0$
2. $\left(\dfrac{x}{t^2} + xt\right)dt + \left(\dfrac{1}{t} + \ln t\right)dx = 0$
3. $e^x dt - (2x + te^x)dx = 0$
4. $x^2(2t - t^2x)dt + t^2(2x - tx^2)dx = 0$
5. $(x + x^2\cos t)dt + (t + 2x\sin t)dx = 0$
6. $(2 + x^2)dt + tx\,dx = 0$
7. $\left(\dfrac{2x}{t^3}\right)dt + dx = 0$
8. $(x + \ln t)dt - t\,dx = 0$
9. $\left(\dfrac{3}{xt} - 1\right)dt - \dfrac{1}{x^2}dx = 0$
10. $\left(\dfrac{3\sin x}{t} - t\right)dt + \cos x\,dx = 0$
11. $\dfrac{1}{t}dt + \left(4 + \dfrac{\ln t}{x}\right)dx = 0$
12. $(1 + x\cos t)dt + (2tx + 3\sin t)dx = 0$

# Chapter 4

# Numerical series

## 4.1 Numerical Series

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. It is well known that the sum of a finite number of its terms is always finite. However, this is not necessarily the case when we consider infinitely many terms. Therefore, it is natural to give a precise meaning to the expression $\sum_{n=0}^{\infty} u_n$.

To this end, we introduce the sequence of *partial sums*

$$S_n = u_0 + u_1 + u_2 + \cdots + u_n = \sum_{k=0}^{n} u_k,$$

and study the behavior of the sequence $(S_n)_{n\in\mathbb{N}}$.

If the limit $S = \lim_{n\to\infty} S_n$ exists, we write $S = \sum_{n=0}^{\infty} u_n$ and say that the series $\sum_{n=0}^{\infty} u_n$ *converges*, and that $S$ is its sum.

This chapter is devoted to the study of necessary and sufficient conditions for the convergence of the sequence $(S_n)_{n\in\mathbb{N}}$, as well as to the extension of properties known for finite sums to infinite series.

### 4.1.1 Sequence of Partial Sums

**Definition 4.1.** Let $(u_n)_{n\geq 0}$ be a sequence of real or complex numbers. The *partial sums* associated with this sequence are defined by

$$S_n = \sum_{k=0}^{n} u_k = u_0 + u_1 + \cdots + u_n.$$

We are interested in studying the limit of $S_n$ as $n \to \infty$.

**Definition 4.2.** Let $\sum_{n=0}^{\infty} u_n$ be a convergent real or complex numerical series with sum $S$. The *remainder of order $N$*, denoted by $R_N$, is defined by

$$R_N = S - S_N = \sum_{n=0}^{\infty} u_n - \sum_{n=0}^{N} u_n = \sum_{n=N+1}^{\infty} u_n.$$

**Definition 4.3.** Let $\sum_{n=0}^{\infty} u_n$ be a numerical series.

- The series is said to be *convergent* if the limit $S = \lim_{n\to\infty} S_n$ exists. In this case, $S$ is called the sum of the series and we write

$$S = \sum_{n=0}^{\infty} u_n.$$

- The series is said to be *divergent* if the sequence $(S_n)$ does not converge.

**Theorem 4.1.** *If the real or complex numerical series $\sum_{n=0}^{\infty} u_n$ is convergent, then its remainder of order $N$ tends to zero as $N \to \infty$.*

*Proof.* From the definition of the remainder we have

$$R_N = S - S_N = \sum_{n=0}^{\infty} u_n - \sum_{n=0}^{N} u_n = \sum_{n=N+1}^{\infty} u_n.$$

Since the series $\sum_{n=0}^{\infty} u_n$ converges, the sequence of partial sums $(S_n)$ converges to $S$. Hence,

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} (S - S_N) = S - S = 0.$$

□

### 4.1.2 Geometric Series

**Proposition 4.1.** *A* geometric series *is a series whose general term is of the form*

$$u_n = cq^n,$$

*where $c \neq 0$ and $q \in \mathbb{R}$ (or $\mathbb{C}$).*

The partial sum of the geometric series is given by

$$S_n = \begin{cases} \dfrac{c(1-q^{n+1})}{1-q}, & \text{if } q \neq 1, \\ c(n+1), & \text{if } q = 1. \end{cases} \tag{3.1}$$

Consequently,

$$\sum_{n=0}^{\infty} u_n \text{ converges} \iff |q| < 1.$$

In the case where the geometric series converges, its sum is

$$\sum_{n=0}^{\infty} u_n = \frac{c}{1-q}.$$

### 4.1.3 Telescoping Series

**Proposition 4.2.** *A* telescoping series *is a series whose general term can be written in the form*

$$u_n = a_n - a_{n+1}.$$

*The partial sums are given by*

$$S_n = \sum_{k=0}^{n} u_k = a_0 - a_{n+1}.$$

*Consequently, the series* $\sum_{n=0}^{\infty} u_n$ *converges if and only if the limit* $\lim_{n\to\infty} a_n$ *exists.*

**Example**

Determine whether the following series converge or diverge. If they converge, compute their sum.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

   We begin by decomposing the general term into partial fractions:

   $$\frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

   For $n \in \mathbb{N}^*$, the partial sum is

   $$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

   This is a telescoping sum, hence

   $$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

   Letting $n \to \infty$, we obtain

   $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

   Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and its sum is equal to 1.

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

   We factor the denominator:

   $$n^2 - 1 = (n-1)(n+1).$$

Using partial fractions, we obtain

$$\frac{1}{n^2-1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right).$$

The partial sums are therefore

$$S_n = \sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right).$$

After simplification of the telescoping terms, we obtain

$$S_n = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}.$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} S_n = \frac{3}{4}.$$

Hence, the series converges and

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}.$$

**Proposition 4.3** (Necessary Condition for Convergence)**.** *Let $\sum_{n=0}^{\infty} u_n$ be a convergent numerical series. Then $\lim_{n\to\infty} u_n = 0$. The converse statement is false in general.*

*Proof.* If the series $\sum_{n=0}^{\infty} u_n$ converges, then by definition the sequence of partial sums $(S_n)_{n\geq 1}$ converges. Hence, the sequence $(S_n - S_{n-1})_{n\geq 1}$ tends to zero.

Since for every $n \geq 1$ we have

$$S_n - S_{n-1} = u_n,$$

it follows that

$$\lim_{n\to\infty} u_n = 0.$$

□

**Proposition 4.4** (Divergence Test)**.** *Let $(u_n)$ be a sequence of real or complex numbers. If the sequence $(u_n)$ does not converge to 0 as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ diverges. In other words,*

$$\lim_{n\to\infty} u_n \neq 0 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} u_n \text{ diverges.}$$

*Proof.* This result follows from the contrapositive of the previous proposition. □

*Remark* 4.1. The condition $\lim_{n\to\infty} u_n = 0$ is necessary but not sufficient for the convergence of a series. Indeed, there exist divergent series whose general term tends to zero.

**Example**

1. The harmonic series
$$\sum_{n=1}^{\infty} \frac{1}{n}$$
is divergent, although
$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{2n+n^3}{1+4n+n^3}$.
We compute
$$\lim_{n\to\infty} \frac{2n+n^3}{1+4n+n^3} = \lim_{n\to\infty} \frac{n^3\left(\frac{2}{n^2}+1\right)}{n^3\left(\frac{1}{n^3}+\frac{4}{n^2}+1\right)} = 1 \neq 0.$$
Since the limit of the general term is different from zero, the series diverges by the divergence test.

3. $\displaystyle\sum_{n=0}^{\infty} \frac{ne^n}{n^2+4}$.
We compute
$$\lim_{n\to\infty} \frac{ne^n}{n^2+4} = +\infty \neq 0.$$
Hence, by the divergence test, the series diverges.

**Proposition 4.5.** *Let $\sum_{n=0}^{\infty} u_n$ be a numerical series.*

- *If the series $\sum_{n=0}^{\infty} u_n$ converges, then for every $k \in \mathbb{N}$ the series $\sum_{n=k}^{\infty} u_n$ also converges.*
- *Conversely, if there exists $k \in \mathbb{N}$ such that the series $\sum_{n=k}^{\infty} u_n$ converges, then the series $\sum_{n=0}^{\infty} u_n$ also converges.*

**Proposition 4.6** (Convergence of a Complex Series)**.** *A complex series $\sum_{n=0}^{\infty} u_n$ converges if and only if the two real series*
$$\sum_{n=0}^{\infty} \mathrm{Re}(u_n) \quad \textit{and} \quad \sum_{n=0}^{\infty} \mathrm{Im}(u_n)$$
*both converge. In this case,*
$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \mathrm{Re}(u_n) + i \sum_{n=0}^{\infty} \mathrm{Im}(u_n).$$

### 4.1.4 Operations on Series

**Proposition 4.7** (Linearity)**.** *Let $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ be two convergent series. Then for every $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), the series $\sum_{n=0}^{\infty}(\lambda u_n + \mu v_n)$ is convergent and*

$$\sum_{n=0}^{\infty}(\lambda u_n + \mu v_n) = \lambda \sum_{n=0}^{\infty} u_n + \mu \sum_{n=0}^{\infty} v_n.$$

**Corollary 4.1.** *1. If $\sum_{n=0}^{\infty} u_n$ converges but $\sum_{n=0}^{\infty} v_n$ diverges, then the series $\sum_{n=0}^{\infty}(u_n + v_n)$ diverges.*

*2. If both series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ diverge, no general conclusion can be drawn about the convergence of $\sum_{n=0}^{\infty}(u_n + v_n)$.*

**Example**

1. The series $\sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \left(-\frac{1}{n}\right) = 0$ is convergent, although the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
2. Consider the two series $\sum_{n=0}^{\infty} 2$ and $\sum_{n=0}^{\infty}(-2)$. Both series are divergent, but their termwise sum $\sum_{n=0}^{\infty}(2-2)$ is convergent.

**Proposition 4.8.** *Let $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ be two convergent series with sums $S_1$ and $S_2$, respectively. Then $\sum_{n=0}^{\infty}(u_n + v_n)$ is convergent and*

$$\sum_{n=0}^{\infty}(u_n + v_n) = S_1 + S_2.$$

*Moreover, for any scalar $\lambda$, the series $\sum_{n=0}^{\infty} \lambda u_n$ is convergent and*

$$\sum_{n=0}^{\infty} \lambda u_n = \lambda S_1.$$

**Proposition 4.9** (Comparison of Sums)**.** *Let $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ be two convergent real series. If there exists $n_0 \in \mathbb{N}$ such that*

$$u_n \leq v_n \quad \text{for all } n \geq n_0,$$

*then*

$$\sum_{n=0}^{\infty} u_n \leq \sum_{n=0}^{\infty} v_n.$$

## 4.2 Series with Positive Terms

**Definition 4.4.** A series $\sum_{n=0}^{\infty} u_n$ is called a *series with positive terms* if

$$u_n \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

**Definition 4.5.** A series $\sum_{n=0}^{\infty} u_n$ is said to have *positive terms from some index $n_0$ onward* if

$$\exists\, n_0 \in \mathbb{N} \quad \text{such that} \quad u_n > 0 \quad \text{for all } n > n_0.$$

**Proposition 4.10.** *Let $\sum_{n=0}^{\infty} u_n$ be a series with positive terms. Then*

$$\sum_{n=0}^{\infty} u_n \text{ converges} \iff (S_n)_{n\in\mathbb{N}} \text{ is bounded above.}$$

*If the sequence of partial sums is not bounded above, then $S_n \to +\infty$, and we write $\sum_{n=0}^{\infty} u_n = +\infty$.*

*Proof.* The sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ is increasing since

$$S_{n+1} - S_n = u_{n+1} \geq 0.$$

Therefore $(S_n)$ is a monotone increasing sequence. Hence it converges if and only if it is bounded above. □

### 4.2.1 Comparison Tests

**Proposition 4.11** (Comparison Test)**.** *Let $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ be two series such that*

$$0 \leq u_n \leq v_n$$

*for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$.*

- *If the series $\sum_{n=0}^{\infty} v_n$ converges, then the series $\sum_{n=0}^{\infty} u_n$ also converges and*

$$\sum_{n=0}^{\infty} u_n \leq \sum_{n=0}^{\infty} v_n.$$

- *If the series $\sum_{n=0}^{\infty} u_n$ diverges, then the series $\sum_{n=0}^{\infty} v_n$ also diverges.*

**Example**

1. Consider the series $\sum_{n=1}^{\infty} \left|\sin\left(\frac{2n}{3n^2}\right)\right|$. Its general term satisfies

$$u_n = \left|\sin\left(\frac{2n}{3n^2}\right)\right| \leq \frac{1}{n^3}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series (Riemann series with $p = 3 > 1$), it follows by the comparison test that $\sum_{n=1}^{\infty} \left|\sin\left(\frac{2n}{3n^2}\right)\right|$ is convergent.

2. Consider the series $\sum_{n=1}^{\infty} \frac{\ln n}{2^n}$.

   Since $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, the sequence $\left(\frac{\ln n}{n}\right)$ is bounded. Hence there exists a constant $M > 0$ such that
   $$0 \leq \frac{\ln n}{n} \leq M.$$

   Therefore,
   $$0 \leq \frac{\ln n}{2^n} \leq \frac{Mn}{2^n}.$$

   The series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is convergent (it is dominated by a geometric series with ratio $\frac{1}{2}$). Hence, by the comparison test, $\sum_{n=1}^{\infty} \frac{\ln n}{2^n}$ is convergent.

### 4.2.2 Equivalence Test

Suppose that $u_n > 0$ and $v_n > 0$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$. Then we have the following cases:

- If $l = 0$ and the series $\sum_{n=0}^{\infty} v_n$ converges, then the series $\sum_{n=0}^{\infty} u_n$ also converges.
- If $0 < l < +\infty$, then the two series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ have the same nature; that is, they either both converge or both diverge.
- If $l = +\infty$ and the series $\sum_{n=0}^{\infty} v_n$ diverges, then the series $\sum_{n=0}^{\infty} u_n$ also diverges.

**Theorem 4.2.** *Let $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ be two series with positive terms from some index $n_0$ onward. If $u_n \sim v_n$ as $n \to \infty$, then the two series have the same nature (they either both converge or both diverge).*

**Example**

1. Study the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{3+4n}$.

   Since the series has positive terms, we may apply the equivalence test. We observe that
   $$\frac{1}{3+4n} \sim \frac{1}{4n} \quad \text{as } n \to \infty.$$

   The series $\sum_{n=1}^{\infty} \frac{1}{4n}$ is a constant multiple of the harmonic series, which is divergent. Therefore, $\sum_{n=1}^{\infty} \frac{1}{3+4n}$ is also divergent.

2. Show the convergence of the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^4}\right)$.

   First, observe that the terms of the series are positive for sufficiently large $n$. Moreover, using the well-known approximation
   $$\sin x \sim x \quad \text{as } x \to 0,$$

we obtain

$$\sin\left(\frac{1}{n^4}\right) \sim \frac{1}{n^4} \quad \text{as } n \to \infty.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent $p$-series with $p = 4 > 1$, it follows that $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^4}\right)$ also converges.

### 4.2.3 Integral Test

**Theorem 4.3** (Integral Comparison Test)**.** *Let $f : [a, +\infty) \to \mathbb{R}_+$ be a continuous, decreasing, and positive function. Define $u_n = f(n), \quad n \in \mathbb{N}^*$. Then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_a^{+\infty} f(t)\,dt$ have the same nature; that is, they either both converge or both diverge.*

**Example**

1. Show the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(1+n)^3}$.

   Let $f(t) = \frac{1}{(1+t)^3}$ for $t \geq 0$. It is clear that $f$ is positive and decreasing on $[0, +\infty[$. The improper integral

   $$\int_0^{+\infty} \frac{1}{(1+t)^3}\,dt$$

   converges, and by the integral test, it has the same behavior as the series. Hence, the series $\sum_{n=1}^{\infty} \frac{1}{(1+n)^3}$ converges.

2. Consider the function $f : [1, +\infty[ \to \mathbb{R}^+$ defined by $f(x) = \frac{1}{x}$. We have

   $$\int_1^t \frac{dx}{x} = \log t \quad \text{and} \quad \lim_{t \to +\infty} \int_1^t \frac{dx}{x} = +\infty.$$

   Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

3. Let $f : [1, +\infty[ \to \mathbb{R}^+$ be defined by

   $$f(x) = \frac{1}{x(x+1)}.$$

   The function $f$ is continuous, positive, and decreasing. We have

   $$\int_1^t \frac{dx}{x(x+1)} = \int_1^t \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \log 2 - \log\left(1 + \frac{1}{t}\right).$$

   Since $\lim_{t \to +\infty} \int_1^t f(x)dx = \log 2 < +\infty$, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

### 4.2.4 Riemann Series

**Definition 4.6** (Riemann Series)**.** A Riemann series is a numerical series with positive terms whose general term is

$$u_n = \frac{1}{n^\alpha}, \quad \alpha \in \mathbb{R}.$$

**Theorem 4.4** (Convergence of a Riemann Series)**.** *The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.*

**Theorem 4.5.** *Let $\sum_{n\geq 1} u_n$ be a series with positive terms.*

1. *If $\exists \alpha > 1$ such that $\lim_{n\to\infty} n^\alpha u_n = 0$, then the series $\sum_{n\geq 1} u_n$ converges.*
2. *If $\exists \alpha \leq 1$ such that $\lim_{n\to\infty} n^\alpha u_n = +\infty$, then the series $\sum_{n\geq 1} u_n$ diverges.*

### 4.2.5 d'Alembert's Ratio Test

**Theorem 4.6.** *Suppose $u_n > 0$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell.$$

*Then:*

1. *If $\ell < 1$, the series converges.*
2. *If $\ell > 1$, the series diverges.*
3. *If $\ell = 1$, no general conclusion can be drawn.*

**Example**

Determine whether the following series converges or diverges:

$$\sum_{n\in\mathbb{N}} \frac{(n!)^2}{(2n)!}.$$

We use the ratio test:

$$\lim_{n\to\infty} \frac{((n+1)!)^2/(2(n+1))!}{(n!)^2/(2n)!} = \lim_{n\to\infty} \frac{(n+1)^2(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{4} < 1.$$

Hence, the series $\sum_{n\in\mathbb{N}} \frac{(n!)^2}{(2n)!}$ converges.

### 4.2.6 Cauchy Root Test

**Proposition 4.12.** *Suppose $u_n \geq 0$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \ell.$$

*Then:*

1. *If $\ell < 1$, the series $\sum u_n$ converges.*
2. *If $\ell > 1$, the series $\sum u_n$ diverges.*
3. *If $\ell = 1$, no conclusion can be drawn.*

**Example**

1. The series with general term $u_n = \left(\frac{n+1}{n}\right)^{n^2}$ diverges because
$$\lim_{n\to\infty} \sqrt[n]{|u_n|} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n} = e > 1.$$
2. For the series with general term $u_n = \left(\frac{n+1}{n}\right)^{n}$, the Cauchy root test gives
$$\lim_{n\to\infty} \sqrt[n]{|u_n|} = \lim_{n\to\infty} \frac{n+1}{n} = 1,$$
so no conclusion can be drawn.

### 4.2.7 Raabe-Duhamel Test

**Proposition 4.13.** *Let $u_n \geq 0$ for all $n \geq n_0$ and define*

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \ell, \quad \ell \in \mathbb{R}.$$

*Then:*

1. *If $\ell > 1$, the series $\sum u_n$ converges.*
2. *If $\ell < 1$, the series diverges.*
3. *If $\ell = 1$, no conclusion can be drawn.*

**Example**

Consider the series $\sum_{n\geq 1} \frac{1}{n^{12}}$. The ratio test gives $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, which is inconclusive. Applying the Raabe-Duhamel test:

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(n+1)^{12}}{n^{12}} - 1\right) = 12 > 1.$$

Hence, the series converges.

### 4.2.8 Logarithmic Comparison Test

**Theorem 4.7** (Logarithmic Comparison Test)**.** *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two series with strictly positive terms. Suppose that*

$$\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}, \quad \forall n \in \mathbb{N}.$$

*Then:*

1. *If $\sum_{n\in\mathbb{N}} v_n$ converges, then $\sum_{n\in\mathbb{N}} u_n$ converges.*
2. *If $\sum_{n\in\mathbb{N}} u_n$ diverges, then $\sum_{n\in\mathbb{N}} v_n$ diverges.*

*Proof.* 1. For all $n \in \mathbb{N}$,

$$\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n} \quad \Longleftrightarrow \quad \frac{u_{n+1}}{v_{n+1}} \leq \frac{u_n}{v_n} \leq \frac{u_{n-1}}{v_{n-1}} \leq \cdots \leq \frac{u_0}{v_0}.$$

This implies

$$u_n \leq \frac{u_0}{v_0} v_n, \quad \forall n \in \mathbb{N}.$$

Since $\sum_{n\in\mathbb{N}} v_n$ converges, the series $\sum_{n\in\mathbb{N}} \frac{u_0}{v_0} v_n$ also converges. By the standard comparison theorem, $\sum_{n\in\mathbb{N}} u_n$ converges.

2. The second statement follows by contraposition of the first.

□

section Convergence Criteria for Series with Arbitrary Terms

#### Absolute Convergence

**Definition 4.7** (Absolute Convergence)**.** A series $\sum_{n\in\mathbb{N}} u_n$ is said to be *absolutely convergent* if the series of absolute values $\sum_{n\in\mathbb{N}} |u_n|$ converges.

**Definition 4.8** (Conditional (Semi-)Convergence)**.** A real or complex series is called *conditionally convergent* (or semi-convergent) if it converges but does not converge absolutely.

**Theorem 4.8.** *If a numerical series $\sum_{n\in\mathbb{N}} u_n$ is absolutely convergent, then it converges. Moreover,*

$$\left| \sum_{n\in\mathbb{N}} u_n \right| \leq \sum_{n\in\mathbb{N}} |u_n|.$$

**Example**

The series

$$\sum_{n\geq 1} \sin(n(2n^2))$$

converges absolutely since

$$|\sin(n(2n^2))| \leq \frac{1}{2n},$$

and the latter is the general term of a convergent series. Therefore, the series under study is absolutely convergent.

**Definition 4.9** (Alternating Series)**.** A series $\sum_{n\in\mathbb{N}} u_n$ is called *alternating* if it has the form

$$\sum_{n\in\mathbb{N}} (-1)^n a_n \quad \text{with } a_n \geq 0,\ \forall n \in \mathbb{N}.$$

### 4.2.9 Convergence Criterion for Alternating Series

**Theorem 4.9** (Leibniz Criterion)**.** *Let $\sum_{n\in\mathbb{N}} u_n$ be an alternating series such that:*

1. *$(|u_n|)_{n\in\mathbb{N}}$ is a decreasing sequence,*
2. *$\lim_{n\to\infty} u_n = 0$.*

*Then the series $\sum_{n\in\mathbb{N}} u_n$ converges, and its sum has the sign of $u_0$. Moreover, for any $N \in \mathbb{N}$, the remainder $R_N$ satisfies*

$$|R_N| = \left| \sum_{n=N+1}^{\infty} u_n \right| \leq |u_{N+1}|.$$

**Example**

Study the convergence of the series

$$\sum_{n\in\mathbb{N}^*} (-1)^n \frac{1}{\sqrt{3n^3+n+3}}.$$

1. First, check for absolute convergence by examining

$$\sum_{n\in\mathbb{N}^*} \frac{1}{\sqrt{3n^3+n+3}}.$$

Compute

$$\lim_{n\to\infty} \frac{1/n}{1/\sqrt{3n^3+n+3}} = 1.$$

Using the comparison test with the divergent series $\sum_{n\in\mathbb{N}^*} \frac{1}{n}$, we deduce that the series $\sum_{n\in\mathbb{N}^*} 1/\sqrt{3n^3+n+3}$ diverges. Hence, the original series is not absolutely convergent.

2. Since the series is alternating, we apply the Leibniz criterion. Set $a_n = \frac{1}{\sqrt{3n^3+n+3}}$. One can verify that $a_n$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$. Therefore, by Leibniz's test, the series $\sum_{n\in\mathbb{N}^*}(-1)^n a_n$ converges.

### 4.2.10 Rule of Equivalents

**Proposition 4.14** (Equivalence Test)**.** *Let $\sum_{n\in\mathbb{N}} u_n$ and $\sum_{n\in\mathbb{N}} v_n$ be real series such that*

$$u_n \sim v_n \quad \text{as } n \to \infty,$$

*and assume that $(u_n)_{n\in\mathbb{N}}$ has constant sign from some index onward. Then:*

1. *$(v_n)_{n\in\mathbb{N}}$ also has constant sign from some (possibly different) index onward.*
2. *The series $\sum u_n$ converges if and only if $\sum v_n$ converges.*

*More generally, for real or complex series, if $u_n \sim v_n$ or $|u_n| \sim |v_n|$ as $n \to \infty$, then*

$$\sum u_n \text{ converges absolutely} \iff \sum v_n \text{ converges absolutely.}$$

### 4.2.11 D'Alembert's Rule for Series with Arbitrary Terms

**Proposition 4.15** (Generalized Ratio Test)**.** *Let $\sum_{n\geq 0} u_n$ be a series with arbitrary terms, and assume the sequence*

$$\left|\frac{u_{n+1}}{u_n}\right|, \quad n \in \mathbb{N},$$

*is defined for sufficiently large $n$ and has a limit $\ell$ as $n \to \infty$. Then:*

1. *If $\ell < 1$, the series $\sum u_n$ is absolutely convergent.*
2. *If $\ell > 1$, the series $\sum u_n$ diverges.*
3. *If $\ell = 1$, no conclusion can be drawn about the series.*

**Example**

1. Consider the series with general term $u_n = \frac{1}{n^n}$.

   We have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{(n+1)^n}{n^{n+1}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \cdot \frac{1}{n+1} = 0 < 1.$$

   Hence, by the ratio test, the series $\sum_{n\geq 1}\frac{1}{n^n}$ converges.

2. The series with general term

$$u_n = e^n$$

is divergent since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{e^{n+1}}{e^n} = e > 1.$$

3. For the series with general term

$$u_n = \frac{1}{n\log n},$$

the D'Alembert ratio test gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{n\log n}{(n+1)\log(n+1)} = 1,$$

which is inconclusive.

### 4.2.12 Comparison with a Geometric Series

**Proposition 4.16** (Geometric Comparison Test)**.** *Let $\sum_{n>0} u_n$ be a series with arbitrary terms. If there exists a real or complex number $q$ with $0 < |q| < 1$ such that, for sufficiently large $n$,*

$$|u_n| \leq |q|^n,$$

*then the series $\sum_{n>0} u_n$ is absolutely convergent.*

### 4.2.13 Other Convergence Criteria

**Theorem 4.10.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive terms such that $\lim_{n\to\infty} a_n = \ell > 0$. If*

$$u_n = a_n v_n, \quad \forall n \in \mathbb{N},$$

*and $\sum_{n>0} v_n$ is a series with positive terms, then the series $\sum_{n>0} u_n$ and $\sum_{n\geq 0} v_n$ are of the same nature (both converge or both diverge).*

**Example**

The series with general term

$$u_n = \frac{n+4}{n^2+5n}$$

is divergent.

Indeed, decompose $u_n = a_n v_n$ where

$$a_n = \frac{n+4}{n+5} > 0, \quad \forall n \in \mathbb{N},$$

with

$$\lim_{n\to\infty} a_n = 1 > 0,$$

and

$$\sum_{n\geq 1} v_n = \sum_{n\geq 1} \frac{1}{n},$$

which is divergent. Hence, $\sum_{n\geq 1} u_n$ diverges as well.

**Theorem 4.11.** *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two numerical sequences such that*

$$u_n = v_{n+1} - v_n, \quad \forall n \in \mathbb{N}.$$

*Then the series $\sum_{n\geq 0} u_n$ is of the same nature as the sequence $(v_n)_{n\in\mathbb{N}}$.*

## 4.3 Sequences and Series of Functions

### 4.3.1 Sequences of Functions

Let $X \subset \mathbb{R}$ and $K = \mathbb{R}$ or $\mathbb{C}$, and let $\psi(X, K)$ denote the set of functions defined from $X$ into $K$.

**Definition 4.10.** The mapping defined by

$$f : \mathbb{N} \longrightarrow \psi(X, K),$$

$$n \longmapsto f_n(x)$$

is called a sequence of functions on $X$.

### Examples

Let the following two sequences of functions be defined as follows:

$$1 - X = \mathbb{R}, \quad f_n(x) = x^n, \quad 2 - f_n(x) = \frac{nx}{1+nx}, \quad X = \mathbb{R}_+.$$

### Pointwise Convergence

**Definition 4.11.** We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $X$ if for every $x \in X$, the numerical sequence $(f_n(x))_{n\in\mathbb{N}}$ is convergent, that is,

$$\forall x \in X, \quad f(x) = \lim_{n\to+\infty} f_n(x).$$

We say that $f_n$ converges pointwise to $f$ on $X$ and we write

$$f_n \xrightarrow{CS} f.$$

**Remark:**

$$f_n \xrightarrow{CS} f \iff \forall x \in X \ \forall \varepsilon > 0, \ \exists N, \ \forall n \ [n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon].$$

The integer $N$ associated with $\varepsilon$ depends on $\varepsilon$ and in general also on $x$.

## Uniform Convergence

**Definition 4.12.** We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $X$ to $f$ if

$$\lim_{n\to+\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0,$$

and we write

$$f_n \xrightarrow{CU} f \quad \text{on } X.$$

*Remark* 4.2.

$$f_n \xrightarrow{CU} f \iff \forall \varepsilon > 0\ \exists N,\ \forall x\ \forall n\,[n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon]\,.$$

## Example

Let $X = [1, 2]$ and $f_n(x) = 1 + \dfrac{x}{n^2}$.

We want to study the pointwise and uniform convergence of $f_n$ on $X$.

We have

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \left(1 + \frac{x}{n^2}\right) = 1 = f(x).$$

Thus,

$$f_n \xrightarrow{CS} f \quad \text{on } X.$$

Moreover,

$$\sup_{x\in[1,2]} |f_n(x) - f(x)| = \sup_{x\in[1,2]} \left(1 + \frac{x}{n^2} - 1\right) = \sup_{x\in[1,2]} \frac{x}{n^2} = \frac{2}{n^2}.$$

It follows that

$$\lim_{n\to+\infty} \sup_{x\in[1,2]} |f_n(x) - f(x)| = \lim_{n\to+\infty} \frac{2}{n^2} = 0.$$

Hence,

$$f_n \xrightarrow{CU} f \quad \text{on } X.$$

## Example

Let $X = \mathbb{R}_+$ and $f_n(x) = xe^{-nx}$. Study the pointwise and uniform convergence of $f_n$ on $X$.

**Solution:**

We have

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} xe^{-nx} = \lim_{n\to+\infty} \frac{x}{e^{nx}} = 0 = f(x).$$

Thus,

$$f_n \xrightarrow{CS} f \quad \text{on } \mathbb{R}_+.$$

Moreover,

$$\sup_{x\geq 0}|f_n(x)-f(x)| = \sup_{x\geq 0}|xe^{-nx}|.$$

To compute this supremum, we differentiate:

$$f_n'(x) = e^{-nx}(1-nx).$$

Hence,

$$f_n'(x)=0 \iff e^{-nx}(1-nx)=0,$$

so

$$x=\frac{1}{n}.$$

It follows that

$$\sup_{x\geq 0} xe^{-nx} = \frac{1}{ne}.$$

Therefore,

$$\lim_{n\to+\infty}\sup_{x\geq 0} xe^{-nx} = \lim_{n\to+\infty}\frac{1}{ne}=0.$$

Thus,

$$f_n \xrightarrow{CU} f \quad \text{on } \mathbb{R}_+.$$

## Example

Let

$$f_n(x)=\frac{x\sqrt{n}}{1+nx^2}, \quad X=\mathbb{R}_+.$$

Study the pointwise and uniform convergence of $f_n$ on $X$.

**Solution:**

We have

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty}\frac{x\sqrt{n}}{1+nx^2}=0.$$

Thus,

$$f_n \xrightarrow{CS} f=0 \quad \text{on } \mathbb{R}_+.$$

Moreover,

$$\sup_{x\geq 0}|f_n(x)-f(x)| = \sup_{x\geq 0}\frac{x\sqrt{n}}{1+nx^2}.$$

To compute this supremum, we differentiate:

$$f_n'(x)=\frac{\sqrt{n}-2n\sqrt{n}x^2}{(1+nx^2)^2}.$$

Hence,

$$f_n'(x)=0 \quad \Rightarrow \quad x=\frac{1}{\sqrt{n}}.$$

It follows that

$$\sup_{x\geq 0}\frac{x\sqrt{n}}{1+nx^2} = f_n\left(\frac{1}{\sqrt{n}}\right)=\frac{1}{2}.$$

Therefore,

$$\lim_{n\to+\infty} \sup_{x\geq 0} \frac{x\sqrt{n}}{1+nx^2} = \lim_{n\to+\infty} \frac{1}{2} = \frac{1}{2}.$$

Thus,

$$f_n \overset{CU}{\not\longrightarrow} f \quad \text{on } \mathbb{R}_+.$$

### 4.3.2 Graphical Interpretation of Uniform Convergence

The graphs of the functions $f_n$ are entirely contained between the graphs of $f+\varepsilon$ and $f-\varepsilon$.

*Remark* 4.3. To prove that $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly to $f$, it suffices to find a numerical sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $|f_n(\alpha_n) - f(\alpha_n)|$ does not tend to zero. Indeed, since

$$\sup |f_n(\alpha_n) - f(\alpha_n)| \geq |f_n(\alpha_n) - f(\alpha_n)|,$$

then

$$\lim_{n\to+\infty} \sup |f_n(\alpha_n) - f(\alpha_n)| \neq 0.$$

## Example

Consider the sequence of functions

$$f_n(x) = \frac{1}{1+nx^2}, \quad n \in \mathbb{N}.$$

This sequence converges pointwise to the function

$$\lim_{n\to+\infty} f_n(x) = f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

For $x = \alpha_n = \dfrac{1}{\sqrt{n}}$, we have

$$|f_n(\alpha_n) - f(\alpha_n)| = \frac{1}{2}.$$

Thus, $f_n$ does not converge uniformly to $f$.

### 4.3.3 Properties Related to Uniform Convergence

#### Continuity

**Theorem 4.12.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on $X$. If $f_n$ converges uniformly to $f$ on $X$, then $f$ is continuous on $X$.*

*Remark* 4.4. We conclude that

1. $$\lim_{n\to+\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} \lim_{n\to+\infty} f_n(x).$$
2. If $f_n$ is continuous on $X$ and $f$ is not continuous on $X$, then $f_n$ does not converge uniformly to $f$.

## Example

Let $X = [0, 1]$ and

$$f_n(x) = \frac{1}{1 + nx}.$$

We have

$$\lim_{n\to+\infty} f_n(x) = f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in ]0, 1]. \end{cases}$$

Since $f_n$ is continuous on $X$ and $f$ is not continuous on $X$, it follows that $f_n$ does not converge uniformly to $f$ on $X$, that is,

$$f_n \overset{CU}{\not\longrightarrow} f.$$

**Theorem 4.13.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued continuous functions converging to a continuous function $f$ on $[a, b]$. If, in addition, $(f_n)_{n\in\mathbb{N}}$ is increasing, then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f$ on $[a, b]$.*

### Integration

**Theorem 4.14.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on an interval $[a, b] \subset \mathbb{R}$ and integrable on $[a, b]$. If $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f$ on $[a, b]$, then*

1. *The function $f$ is integrable on $[a, b]$.*
2. $$\lim_{n\to+\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to+\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

## Example

Let

$$f_n(x) = n(\cos x)^n \sin x, \quad X = \left[0, \frac{\pi}{2}\right].$$

1. Study the pointwise and uniform convergence of $f_n$ on $X$.
2. Compare
$$\lim_{n\to+\infty} \int_0^{\frac{\pi}{2}} f_n(x)\,dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \lim_{n\to+\infty} f_n(x)\,dx.$$

**Solution:**
1) If $x = 0$ or $x = \dfrac{\pi}{2}$, then

$$\lim_{n\to+\infty} f_n(x) = 0.$$

If $x \in \left]0, \frac{\pi}{2}\right[$, then

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} e^{n\left(\ln n + \ln\cos x + \frac{\ln\sin x}{n}\right)} = 0.$$

Thus,

$$f_n \xrightarrow{CS} f \quad \text{on } X.$$

Moreover,

$$\sup_{x\in\left[0,\frac{\pi}{2}\right]} |f_n(x) - f(x)| = \sup_{x\in\left[0,\frac{\pi}{2}\right]} |f_n(x)| = \sup_{x\in\left[0,\frac{\pi}{2}\right]} f_n(x).$$

We compute

$$f_n'(x) = n(\cos x)^{n-1}\Big(1 - (n+1)\sin^2 x\Big).$$

Hence,

$$f_n'(x) = 0 \Rightarrow x = \frac{\pi}{2} \quad \text{or} \quad x = \arcsin\left(\frac{1}{\sqrt{n+1}}\right) = x_n.$$

Thus,

$$\sup_{x\in\left[0,\frac{\pi}{2}\right]} |f_n(x) - f(x)| = f_n(x_n) = \frac{n}{\sqrt{n+1}}(\cos x_n)^n.$$

Moreover,

$$\cos x_n = \sqrt{1 - \sin^2 x_n} = \sqrt{1 - \frac{1}{n+1}} = \sqrt{\frac{n}{n+1}}.$$

Hence,

$$f_n(x_n) = \frac{n}{\sqrt{n+1}}\left(\frac{n}{n+1}\right)^{\frac{n}{2}} = \frac{n\sqrt{n+1}}{\left(1+\frac{1}{n}\right)^{\frac{n}{2}}}.$$

Therefore,

$$\lim_{n\to+\infty} f_n(x_n) = +\infty,$$

which implies that

$$f_n \xcancel{\xrightarrow{CU}} f \quad \text{on } \left[0, \frac{\pi}{2}\right].$$

2) We have

$$\int_0^{\frac{\pi}{2}} \lim_{n\to+\infty} f_n(x)\,dx = \int_0^{\frac{\pi}{2}} 0\,dx = 0.$$

On the other hand,

$$\lim_{n\to+\infty} \int_0^{\frac{\pi}{2}} f_n(x)\,dx = \lim_{n\to+\infty} \int_0^{\frac{\pi}{2}} n(\cos x)^n \sin x\,dx$$

$$= \lim_{n\to+\infty} \left[-\frac{n(\cos x)^{n+1}}{n+1}\right]_0^{\frac{\pi}{2}} = \lim_{n\to+\infty} \frac{n}{n+1} = 1.$$

Thus,

$$\lim_{n\to+\infty} \int_0^{\frac{\pi}{2}} f_n(x)\,dx \neq \int_0^{\frac{\pi}{2}} \lim_{n\to+\infty} f_n(x)\,dx.$$

**Differentiation**

**Theorem 4.15.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued functions defined on an interval $X$ such that*

1. *For every $n$, $f_n$ is differentiable with continuous derivative on $X$.*
2. *There exists at least one point $x_0 \in X$ such that $(f_n(x_0))$ converges.*
3. *The sequence $(f_n')$ converges uniformly on $X$.*

*Then*

1. *The sequence $(f_n)$ converges uniformly on $X$.*
2. *The limit of $(f_n)$ is differentiable and*

$$\left(\lim_{n\to+\infty} f_n\right)'(x) = \lim_{n\to+\infty} f_n'(x).$$

## 4.4 Series of Functions

Let $f_n(x)$ be a sequence of functions. We set

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

$S_n(x)$ is called the sequence of partial sums.

**Definition 4.13.** We say that the series of functions

$$\sum_{n=1}^{+\infty} f_n(x)$$

converges pointwise (or simply) on $X$ if and only if the sequence of functions $S_n(x)$ converges pointwise on $X$, and we write

$$\lim_{n\to+\infty} S_n(x) = S(x) = \sum_{n=1}^{+\infty} f_n(x).$$

*Remark* 4.5. We can say that the series of functions

$$\sum_{n=1}^{+\infty} f_n(x)$$

converges pointwise on $X$ if and only if for every fixed $x_0 \in X$, the numerical series

$$\sum_{n=1}^{+\infty} f_n(x_0)$$

converges.

**Definition 4.14.** We say that the series of functions

$$\sum_{n=1}^{+\infty} f_n(x)$$

converges uniformly on $X$ if and only if the sequence of functions $S_n(x)$ converges uniformly on $X$, that is,

$$\forall \varepsilon > 0,\ \exists N(\varepsilon),\ \forall n > N(\varepsilon),\ \forall x \in X : |S_n(x) - S(x)| < \varepsilon.$$

**Definition 4.15.** We say that the series of functions

$$\sum_{n=1}^{+\infty} f_n(x)$$

converges absolutely on $X$ if and only if the series of functions

$$\sum_{n=1}^{+\infty} |f_n(x)|$$

converges pointwise on $X$.

### 4.4.1 Uniform Cauchy Criterion for Series of Functions

A necessary and sufficient condition for a series of real or complex functions

$$\sum_{n=1}^{+\infty} f_n(x)$$

to converge uniformly on $X$ is that

$$\forall \varepsilon > 0,\ \exists N(\varepsilon),\ \forall n > N(\varepsilon),\ \forall m > N(\varepsilon),\ \forall x \in X : \left| \sum_{k=n+1}^{m} f_k(x) \right| < \varepsilon.$$

*Remark* 4.6. A necessary condition for the uniform convergence on $X$ of a series of functions

$$\sum_{n=1}^{+\infty} f_n(x)$$

is that the sequence of functions $f_n(x)$ converges uniformly to 0 on $X$.

## Example

Study the pointwise and uniform convergence of the following series of functions

$$\sum_{n=0}^{+\infty} x(1-x)^n, \quad X = [0, 1].$$

**Solution:**

- If $x = 0$ or $x = 1$, then the series is convergent and its sum is equal to 0.
- If $x \in ]0, 1[$, then

$$S_n(x) = \sum_{k=0}^{n} x(1-x)^k = x\frac{1-(1-x)^{n+1}}{1-(1-x)} = 1-(1-x)^{n+1},$$

and therefore

$$\lim_{n\to+\infty} S_n(x) = 1.$$

Hence,

$$\lim_{n\to+\infty} S_n(x) = S(x) = \begin{cases} 1 & \text{if } x \in ]0,1[, \\ 0 & \text{if } x = 0 \text{ or } x = 1. \end{cases}$$

Thus,

$$S_n(x) \xrightarrow{CS} S(x) = 1,$$

consequently, this series converges pointwise.

Since $S_n(x)$ is continuous and $S(x)$ is not continuous, it follows that

$$S_n(x) \not\xrightarrow{CU} S(x) = 1,$$

hence this series does not converge uniformly to $S(x)$ on $X$.

### 4.4.2 Normal Convergence

**Definition 4.16.** Let $\sum_{n=1}^{+\infty} f_n(x)$ be a series of functions.
We say that $\sum_{n=1}^{+\infty} f_n(x)$ is normally convergent on $X$ if and only if $\sum_{n=1}^{+\infty} \sup_{x\in X} |f_n(x)|$ is convergent.

## Example

We want to show that the series of functions

$$\sum_{n=1}^{+\infty} \frac{x}{n(1+nx^2)}$$

converges normally on $\mathbb{R}$.

We look for

$$\sup_{x\in\mathbb{R}} \frac{x}{n(1+nx^2)}.$$

Set

$$f_n(x) = \frac{x}{n(1+nx^2)}.$$

The function $f_n$ is odd, so it suffices to study its variations on $\mathbb{R}_+$.

The derivative is

$$f_n'(x) = \frac{1-nx^2}{n(1+nx^2)^2},$$

which vanishes at

$$x_0 = \frac{1}{\sqrt{n}}.$$

It follows that

$$\sup_{x\in\mathbb{R}} \frac{x}{n(1+nx^2)} = \frac{1}{2n^{3/2}}.$$

Since the numerical series

$$\sum_{n=1}^{+\infty} \frac{1}{2n^{3/2}}$$

converges, it follows that

$$\sum_{n=1}^{+\infty} \frac{x}{n(1+nx^2)}$$

converges normally on $\mathbb{R}$.

**Theorem 4.16.** *If the series of functions*

$$\sum_{n=1}^{+\infty} f_n(x)$$

*is normally convergent on $X$, then it is uniformly convergent on $X$.*

**Proof.** Let $x \in X$. Since

$$\sum_{n=1}^{+\infty} f_n(x)$$

is normally convergent, we have

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} : \forall m > n \geq n_0 : \sum_{p=n+1}^{m} \|f_p\|_\infty < \varepsilon,$$

where

$$\|g\|_\infty = \sup_{x\in X} |g(x)|.$$

On the other hand,

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| \leq \sup_{x\in X} |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)|$$

$$= \sup_{x\in X} |(f_{n+1} + f_{n+2} + \cdots + f_m)(x)| = \|f_{n+1} + f_{n+2} + \cdots + f_m\|_\infty$$

$$\leq \|f_{n+1}\|_\infty + \|f_{n+2}\|_\infty + \cdots + \|f_m\|_\infty < \varepsilon.$$

Therefore, the series

$$\sum_{n=1}^{+\infty} f_n(x)$$

is uniformly convergent.

**Corollary 4.2.** *Let*

$$\sum_{n=1}^{+\infty} f_n(x)$$

*be a series of functions. If there exists a convergent numerical series with positive terms*

$$\sum_{n=1}^{+\infty} U_n$$

*such that*

$$\forall n \in \mathbb{N},\ \forall x \in X :\ |f_n(x)| \leq U_n,$$

*then the series*

$$\sum_{n=1}^{+\infty} f_n(x)$$

*is normally convergent on $X$.*

**Example**

1) We want to show that the two series of functions

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \sin\left(\frac{x}{n}\right) \quad \text{and} \quad \sum_{n=1}^{+\infty} \frac{1}{n^2 + x^2}$$

are uniformly convergent on $\mathbb{R}$.

We have

$$\left|\frac{1}{n^2} \sin\left(\frac{x}{n}\right)\right| \leq \frac{1}{n^2}.$$

Since the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}$$

is convergent (it is a Riemann series with $\alpha = 2 > 1$), it follows that the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \sin\left(\frac{x}{n}\right)$$

is normally convergent. Consequently, it is uniformly convergent on $\mathbb{R}$.

2) The same procedure applies to the second series of functions, observing that

$$\frac{1}{n^2 + x^2} \leq \frac{1}{n^2}.$$

Hence the second series is also normally convergent. Consequently, it is uniformly convergent on $\mathbb{R}$.

### 4.4.3 Properties Related to Uniform Convergence

#### Continuity

**Theorem 4.17.** *Let $\sum_{n=1}^{+\infty} f_n(x)$ be a series of functions continuous on $X$. If this series converges uniformly on $X$, then its sum is continuous on $X$, and*

$$\lim_{x \to x_0} \sum_{n=1}^{+\infty} f_n(x) = \sum_{n=1}^{+\infty} \lim_{x \to x_0} f_n(x).$$

#### Integration

**Theorem 4.18.** *Let $\sum_{n=1}^{+\infty} f_n(x)$ be a series of real-valued functions integrable on $[a, b]$. If this series converges uniformly on $[a, b]$, then*

$$\int_a^b \sum_{n=1}^{+\infty} f_n(x)\, dx = \sum_{n=1}^{+\infty} \int_a^b f_n(x)\, dx.$$

#### Differentiation

**Theorem 4.19.** *Let $\sum_{n=1}^{+\infty} f_n(x)$ be a series of real-valued functions defined on $[a, b]$ such that:*

1. *For every $n \in \mathbb{N}$, $f_n$ is differentiable with continuous derivative on $[a, b]$.*
2. *There exists at least one point $x_0 \in [a, b]$ such that the numerical series $\sum_{n=1}^{+\infty} f_n(x_0)$ converges.*
3. *The series $\sum_{n=1}^{+\infty} f_n'(x)$ converges uniformly on $[a, b]$.*

*Then the series $\sum_{n=1}^{+\infty} f_n(x)$ converges uniformly on $[a, b]$ and we have*

$$\left( \sum_{n=1}^{+\infty} f_n(x) \right)' = \sum_{n=1}^{+\infty} f_n'(x).$$

## Abel's Criterion

**Theorem 4.20.** *Let $(a_n(x))_{n\in\mathbb{N}}$ and $(b_n(x))_{n\in\mathbb{N}}$ be two sequences of functions defined on $X$ such that:*

1. *For every $x \in X$, $a_n(x)$ is monotone.*
2. *The sequence of functions $(a_n(x))_{n\in\mathbb{N}}$ converges uniformly to 0.*
3. *There exists $M > 0$ such that $\forall x \in X,\ \forall n \in \mathbb{N}, \quad |S_n(x)| \leq M$, where*

$$S_n(x) = \sum_{k=0}^{n} b_k(x).$$

Then the series of functions with general term $a_n(x)b_n(x)$ converges uniformly on $X$.

## Example

We want to study the uniform convergence of the series of functions

$$\sum_{n=1}^{+\infty} \frac{(-1)^n x^{2+n}}{n^2} \quad \text{on } [a,b].$$

Set

$$a_n(x) = \frac{x^{2+n}}{n^2}.$$

Then:

1. $a_n(x)$ is decreasing since for every $x \in [a,b]$,

$$\frac{x^{2+n+1}}{(n+1)^2} \leq \frac{x^{2+n}}{n^2}.$$

2.

$$\lim_{n\to+\infty} \sup_{x\in[a,b]} \frac{x^{2+n}}{n^2} = \lim_{n\to+\infty} \frac{M^{2+n}}{n^2} = 0,$$

hence the sequence of functions $\dfrac{x^{2+n}}{n^2}$ converges uniformly to 0.

3.

$$\left|\sum_{n=0}^{+\infty}(-1)^n\right| \leq 2.$$

Therefore,

$$\sum_{n=1}^{+\infty} \frac{(-1)^n x^{2+n}}{n^2}$$

is uniformly convergent on $[a,b]$.

# 4.5 Power Series

## 4.5.1 Generalities

**Definition 4.17.** A power series in the complex variable $z$ (or real variable $x$) is a series of functions whose general term is written as

$$f_n(z) = a_n z^n \quad \text{or} \quad f_n(x) = a_n x^n, \quad n \in \mathbb{N},$$

where $(a_n)$ are called the coefficients of the series.

**Theorem 4.21.** *Let*

$$\sum_{n=0}^{+\infty} a_n z^n$$

*be a power series. There exists a number $R \geq 0$ finite or infinite such that:*

1. *If $|z| < R$, the series is absolutely convergent.*
2. *If $|z| > R$, the series is divergent.*

*$R$ is called the radius of convergence and is defined by*

$$R = \frac{1}{\lim_{n\to+\infty} \sqrt[n]{|a_n|}} \quad \text{or} \quad R = \frac{1}{\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|}.$$

**Proof.** We apply Cauchy's root test to the series

$$\sum_{n=0}^{+\infty} a_n z^n.$$

We have

$$\ell = \lim_{n\to+\infty} \sqrt[n]{|a_n z^n|} = |z| \lim_{n\to+\infty} \sqrt[n]{|a_n|} = |z|\frac{1}{R}.$$

If $|z| < R$, then $\ell < 1$ and the series $\sum_{n=0}^{+\infty} a_n z^n$ is absolutely convergent.
If $|z| > R$, then $\ell > 1$ and the series diverges.

**Lemma 4.1.** *If there exists $z_0$ such that the series $\sum_{n=0}^{+\infty} a_n z_0^n$ is bounded, then the power series $\sum_{n=0}^{+\infty} a_n z^n$ is absolutely convergent for every $z$ such that $|z| < |z_0|$.*

## 4.5.2 Disk of Convergence

Let $R$ be the radius of convergence of a power series. The disk of convergence is defined by:

1. $D_R = \{z \in \mathbb{C} : \ |z| < R\}$ which is the open disk.
2. $D_R = \{z \in \mathbb{C} : \ |z| \leq R\}$ which is the closed disk.

In the real case, we have the interval

$$D_R = ]-R, R[$$

called the open interval of convergence. The interval

$$D_R = [-R, R]$$

is called the closed interval of convergence.

## Example

We want to compute the radii of convergence of the following series:

$$\sum_{n=0}^{+\infty} \frac{z^n}{n!}, \quad \sum_{n=1}^{+\infty} \frac{z^n}{n^2}, \quad \sum_{n=1}^{+\infty} \frac{z^n}{n\,2^n}.$$

1)

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{n!}{(n+1)!} = \lim_{n\to+\infty} \frac{1}{n+1} = 0,$$

thus $R = +\infty$.

2)

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{n^2}{(n+1)^2} = 1,$$

thus $R = 1$.

3)

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{n\,2^n}{(n+1)\,2^{n+1}} = \frac{1}{2},$$

thus $R = 2$.

**Theorem 4.22.** *Let*

$$\sum_{n=0}^{+\infty} a_n z^n$$

*be a power series with radius of convergence $R$. For every $r$ such that $0 \leq r \leq R$, the series is normally convergent in the closed disk $D_r$.*

### 4.5.3 Algebraic Operations

Let

$$\sum_{n=0}^{+\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{+\infty} b_n z^n$$

be two power series with radii of convergence $R$ and $R'$ respectively, and let $f(z)$ and $g(z)$ be their sums.

Consider the series

$$\sum_{n=0}^{+\infty} \alpha a_n z^n + \beta b_n z^n,$$

and denote its radius of convergence by $R''$.

The series

$$\sum_{n=0}^{+\infty} \alpha a_n z^n + \beta b_n z^n$$

converges absolutely for $|z| < \min(R, R')$.

If $R = R'$, then $R'' = \min(R, R')$ and

$$\sum_{n=0}^{+\infty} \alpha a_n z^n + \beta b_n z^n = \alpha f(z) + \beta g(z).$$

### 4.5.4 Analytic Properties

## Continuity

**Theorem 4.23.** *The sum of a power series*

$$\sum_{n=0}^{+\infty} a_n z^n$$

*with radius of convergence $R > 0$ is a continuous function of $z$ inside the disk of convergence (for all $|z| < R$).*

**Theorem 4.24.** *(Abel continuity). Let*

$$\sum_{n=0}^{+\infty} a_n z^n$$

*be a power series with radius of convergence $R$. If the numerical series*

$$\sum_{n=0}^{+\infty} a_n R^n$$

*converges, then*

$$\sum_{n=0}^{+\infty} a_n x^n$$

*converges uniformly on $[0, R]$, its sum is continuous on this interval, and*

$$\lim_{x \to R} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_n R^n.$$

## Derivative

**Theorem 4.25.** *Let*

$$\sum_{n=0}^{+\infty} a_n z^n$$

*be a power series with radius of convergence $R$. Let $f(z)$ be the sum of this series in the open disk of convergence. The function $f$ is differentiable and its derivative is the sum of the power series*

$$\sum_{n=1}^{+\infty} n a_n z^{n-1},$$

*whose radius of convergence is equal to $R$.*

## Primitives

**Theorem 4.26.** *Let*

$$\sum_{n=0}^{+\infty} a_n x^n$$

*be a power series with radius of convergence $R$. Let $f(x)$ be the sum of this series in the open interval of convergence. The power series*

$$\sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{n+1}$$

*obtained by term-by-term integration of the power series*

$$\sum_{n=0}^{+\infty} a_n x^n$$

*has the same interval of convergence $]-R, R[$ as the given series, and the sum function satisfies*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{n+1}.$$

## Example

Consider the series

$$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}, \quad R = 1.$$

Then

$$\sum_{n=1}^{+\infty} n x^{n-1} = \frac{1}{(1-x)^2}, \quad R = 1,$$

and

$$\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x), \quad R = 1.$$

### 4.5.5 Power Series Expansion

**Definition 4.18.** Let $f(z)$ be a function defined in a neighborhood $D$ of the origin, i.e., a subset of $\mathbb{C}$ containing a disk of strictly positive radius centered at the origin. We say that $f$ is expandable into a power series in $D$ if there exists a power series

$$\sum_{n=0}^{+\infty} a_n z^n$$

with nonzero radius of convergence such that its sum equals $f$ in $D$, and we write:

$$\forall z \in \mathbb{C}, \quad \sum_{n=0}^{+\infty} a_n z^n = f(z).$$

**Definition 4.19.** Let $f : ]-\tau, \tau[ \to K$ where $K = \mathbb{R}$ or $\mathbb{C}$. We say that $f$ is expandable into a power series if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that:

$$\forall z \in D_\tau, \quad f(z) = \sum_{n=0}^{+\infty} a_n z^n.$$

**Definition 4.20.** Let $f : ]-\tau, \tau[ \to K$ be such that $f \in C^\infty(]-\tau, \tau[)$, i.e., infinitely differentiable. We call the Taylor series associated with $f$ at 0 the series

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

**Definition 4.21.** Let $f : ]-\tau, \tau[ \to K$. If $f$ is expandable into a power series, i.e.,

$$\forall x \in ]-\tau, \tau[, \quad f(x) = \sum_{n=0}^{+\infty} a_n x^n,$$

then:

1. $f \in C^\infty(]-\tau, \tau[)$,
2. $\forall n \in \mathbb{N}, \quad a_n = \frac{f^{(n)}(0)}{n!}$,

and we have

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

**Theorem 4.27.** *Let $f \in C^\infty(]-\tau,\tau[)$. Suppose that there exists a continuous function $g :]-\tau,\tau[\to K$ such that*

$$\forall n \in \mathbb{N},\ \forall x \in ]-\tau,\tau[, \quad |f^{(n)}(x)| \leq |g(x)|.$$

*Then $f$ is expandable into a power series and*

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

*Remark* 4.7.
1. If $g(x) = M \in \mathbb{R}^+$, then $\forall n, \forall x : |f^{(n)}(x)| \leq M$.
2. The condition is sufficient but not necessary, i.e.,

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n \not\Rightarrow |f^{(n)}(x)| \leq M.$$

**Example**

Let

$$f(x) = \sum_{n=0}^{+\infty} \frac{2^n}{n!} x^n.$$

We have $f_n(0) = 2^n$, therefore $(f_n(0))$ is not bounded.

## 4.6 Fourier Series

### 4.6.1 Trigonometric Series

**Definition 4.22.** We say that a function $f : \mathbb{R} \to \mathbb{C}$ is periodic of period $T > 0$ (or $T$-periodic) if it satisfies:

$$\forall x \in \mathbb{R}, \quad f(x+T) = f(x).$$

**Definition 4.23.** A (real) trigonometric series is a series of functions

$$\sum_{n=0}^{+\infty} a_n \cos(\omega_n x) + b_n \sin(\omega_n x), \quad x \in \mathbb{R},$$

where $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(\omega_n)_{n\in\mathbb{N}}$ are sequences of real numbers.

*Remark* 4.8.

$$|a_n \cos(\omega_n x) + b_n \sin(\omega_n x)| \leq |a_n| + |b_n|.$$

If the series

$$\sum_{n=0}^{+\infty} (|a_n| + |b_n|)$$

converges, then the trigonometric series converges absolutely and uniformly on $\mathbb{R}$.

**Example**

The series

$$\sum_{n=1}^{+\infty} \frac{\cos(nx)}{n^2}$$

converges uniformly on $\mathbb{R}$ since $\sum \frac{1}{n^2}$ converges.

**Lemma 4.2.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a periodic function of period $2\ell$, locally integrable. Then*

$$\forall a \in \mathbb{R}, \quad \int_a^{a+2\ell} g(t)\,dt = \int_0^{2\ell} g(t)\,dt.$$

**Proof.** We have

$$\int_a^{a+2\ell} g(t)\,dt = \int_a^0 g(t)\,dt + \int_0^{2\ell} g(t)\,dt + \int_{2\ell}^{a+2\ell} g(t)\,dt.$$

Setting $t = x + 2\ell$, we get

$$\int_{2\ell}^{a+2\ell} g(t)\,dt = \int_0^a g(x+2\ell)\,dx = \int_0^a g(x)\,dx.$$

Hence

$$\int_a^{a+2\ell} g(t)\,dt = \int_0^{2\ell} g(t)\,dt.$$

**Lemma 4.3.** *Let $g : \mathbb{R} \to \mathbb{R}$ be locally integrable. Then:*

1. *If $g$ is odd, then for all $a > 0$,*

$$\int_{-a}^{a} g(x)\,dx = 0.$$

2. *If $g$ is even, then for all $a > 0$,*

$$\int_{-a}^{a} g(x)\,dx = 2\int_0^a g(x)\,dx.$$

**Definition 4.24.** Let $f : \mathbb{R} \to \mathbb{R}$ be a $2\ell$-periodic locally integrable function. The trigonometric series

$$S_f(x) = \frac{a_0}{2} + \sum_{n=0}^{+\infty} a_n \cos\left(\frac{n\pi}{\ell}x\right) + b_n \sin\left(\frac{n\pi}{\ell}x\right),$$

where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\frac{n\pi}{\ell}t\right)\,dt, \quad n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin\left(\frac{n\pi}{\ell}t\right)\,dt, \quad n = 1, 2, \ldots$$

is called the Fourier series of $f$. The numbers $a_n$ and $b_n$ are called Fourier coefficients of $f$.

**Corollary 4.3.** *If $f$ is $2\pi$-periodic, then its Fourier series is*

$$S_f(x) = \frac{a_0}{2} + \sum_{n=0}^{+\infty} a_n \cos(nx) + b_n \sin(nx),$$

*where*

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt, \quad n = 0, 1, 2, \ldots$$

*and*

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt, \quad n = 1, 2, \ldots$$

*Remark* 4.9.

1. If $f$ is even, then $b_n = 0$ and
$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos(nt)\, dt.$$
2. If $f$ is odd, then $a_n = 0$ and
$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin(nt)\, dt.$$

**Theorem 4.28.** *If $f$ is the sum of a trigonometric series uniformly convergent on $\mathbb{R}$, then it coincides with its Fourier series, i.e.*

$$S_f(x) = f(x).$$

## Convergence

**Definition 4.25.** Let $f$ be a function defined on $[a, b] \subset \mathbb{R}$. We say that $f$ is piecewise continuous (resp. piecewise monotone) if:

1. It is defined on $[a, b]$ except possibly at a finite number of points $(a_k)_{0 \le k \le n}$ with $a_0 = a$, $a_n = b$, forming a subdivision.
2. On each interval $]a_k, a_{k+1}[$, the restriction coincides with a continuous (resp. monotone) function on $[a_k, a_{k+1}]$.

*Remark* 4.10. The function $f$ admits a right and left limit at every point $y \in [a, b]$:

$$\lim_{x \to y^+} f(x) = f(y + 0), \quad \lim_{x \to y^-} f(x) = f(y - 0).$$

**Definition 4.26.** The right and left derivatives at $x_0$, if they exist, are:

$$f'(x_0{+}0) = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0 + 0)}{h}, \quad f'(x_0{-}0) = \lim_{h \to 0^-} \frac{f(x_0 - h) - f(x_0 - 0)}{h}.$$

**Theorem 4.29.** *Let $f$ be a $2\pi$-periodic integrable function on $\mathbb{R}$. If it satisfies one of the following: $f$ is piecewise monotone, and $f$ is piecewise continuous and admits left and right derivatives at every point,*
*then:*

1. *The Fourier series converges for every $x$ and*
$$S_f(x) = \frac{f(x+0) + f(x-0)}{2}.$$
2. *The convergence is uniform on any interval $[a, b]$ where $f$ is continuous or has bounded derivative.*

**Corollary 4.4.** *If $f$ is $2\pi$-periodic, continuous, and piecewise continuously differentiable, then its Fourier series converges normally on $\mathbb{R}$ and*
$$S_f(x) = f(x).$$

### 4.6.2 Parseval–Plancherel Formula

**Theorem 4.30.** *Let $f$ be a $2\pi$-periodic integrable function. Its Fourier coefficients satisfy:*
$$\frac{|a_0|^2}{2} + \sum_{n=1}^{+\infty}(|a_n|^2 + |b_n|^2) = \frac{1}{\pi}\int_0^{2\pi} |f(x)|^2\, dx.$$

**Example**

1. Develop the $2\pi$-periodic Fourier series of
$$f(x) = \begin{cases} 0 & x \in ]-\pi, 0] \\ 1 & x \in ]0, \pi] \end{cases}$$
2. Study the convergence and deduce the values of the following series:
$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1}, \quad \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^2}, \quad \sum_{k=1}^{+\infty} \frac{1}{k^2}, \quad \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k^2}.$$

**Solution:** We first compute the Fourier coefficients.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx = \frac{1}{\pi}\int_0^{\pi} \cos(nx)\, dx = \frac{\sin(nx)}{n\pi}\Big|_0^{\pi} = 0, \quad n \geq 1.$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx = 1.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx = \frac{1-(-1)^n}{n\pi}.$$

Hence the Fourier series:

$$S_f(x) = \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{1-(-1)^n}{n\pi}\sin(nx).$$

We obtain:

$$S_f(x) = \frac{1}{2} + \sum_{k=0}^{+\infty} \frac{2}{(2k+1)\pi}\sin((2k+1)x).$$

At $x = \frac{\pi}{2}$:

$$1 = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1},$$

hence

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Using Parseval:

$$1 = \frac{1}{2} + \sum_{k=0}^{+\infty} \frac{4}{(2k+1)^2\pi^2}.$$

Thus:

$$\sum_{k=0}^{+\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}, \quad \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

### 4.6.3 Complex Fourier Series

Let $f$ be a $2\pi$-periodic integrable function on $\mathbb{R}$. We have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n\cos(nx) + b_n\sin(nx)\right).$$

It follows that

$$a_n\cos(nx) + b_n\sin(nx) = a_n\frac{e^{inx}+e^{-inx}}{2} + b_n\frac{e^{inx}-e^{-inx}}{2i}$$

$$= \frac{a_n - ib_n}{2}e^{inx} + \frac{a_n+ib_n}{2}e^{-inx}.$$

If we set

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n+ib_n}{2},$$

then the complex Fourier series of $f$ is

$$f(x) = c_0 + \sum_{n=1}^{+\infty} c_n e^{inx} + c_{-n}e^{-inx}$$

or equivalently

$$S_f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx},$$

where

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx, \quad n \in \mathbb{Z}.$$

The numbers $C_n$ are called the complex Fourier coefficients of $f$.

*Remark* 4.11. The Fourier coefficients satisfy the relation:

$$\sum_{n=-\infty}^{+\infty} |C_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx,$$

or equivalently

$$|C_0|^2 + \sum_{n=1}^{+\infty}\left(|C_n|^2 + |C_{-n}|^2\right) = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx.$$

**Example**

1. Develop the Fourier series of the $2\pi$-periodic function defined by
$$f(x) = e^{i\alpha x}, \quad |x| < \pi, \quad \alpha \in \mathbb{C}/\mathbb{Z}.$$
2. Study the convergence of the Fourier series to the function, and if $\alpha$ is real, show that
$$\frac{\pi^2}{\sin^2(\pi\alpha)} = \sum_{n=-\infty}^{+\infty} \frac{1}{(\alpha-n)^2}.$$

**Solution:**

1) The function $f$ is complex-valued, $2\pi$-periodic and piecewise continuously differentiable. We compute its complex Fourier coefficients.

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\alpha x}e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\alpha-n)x}\,dx.$$

Thus

$$C_n = \frac{e^{i(\alpha-n)x}}{2\pi i(\alpha-n)}\Big|_{-\pi}^{\pi} = \frac{e^{i\pi(\alpha-n)} - e^{-i\pi(\alpha-n)}}{2\pi i(\alpha-n)}.$$

Hence

$$C_n = \frac{\sin(\pi(\alpha-n))}{\pi(\alpha-n)}.$$

Since

$$\sin(\pi(\alpha-n)) = (-1)^n\sin(\pi\alpha),$$

we obtain

$$C_n = \frac{(-1)^n\sin(\pi\alpha)}{\pi(\alpha-n)}.$$

2) The Fourier series converges uniformly on any interval where $f$ is continuous; by Theorem 4.2.1 it converges to $f$ whenever $x \neq k\pi$.

We have

$$S_f(x) = e^{i\alpha x} = \frac{\sin(\pi\alpha)}{\pi}\sum_{n=-\infty}^{+\infty}\frac{e^{inx}}{\alpha-n}.$$

If $\alpha \in \mathbb{R}$, Parseval's identity gives

$$\frac{\pi^2}{\sin^2(\pi\alpha)} = \sum_{n=-\infty}^{+\infty} \frac{1}{(\alpha-n)^2}.$$

# Exercises

## Numerical Series

**Exercise 01.** Determine whether the following series converge or diverge. If a series converges, compute its sum.

$$\sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)}, \quad \sum_{n=1}^{\infty} \frac{3^{-n+2}}{2^{n+1}}, \quad \sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}.$$

**Exercise 02.** Test the convergence of the following series using the *ratio test (D'Alembert test)*.

$$\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\cdots, \quad 1+\frac{1\cdot 2}{3}+\frac{1\cdot 2\cdot 3}{3\cdot 5}+\cdots, \quad \frac{1}{2}+\frac{2}{3!\,4}+\frac{2\cdot 5}{4!\,6}+\cdots.$$

**Exercise 03.** Test the convergence of the following series using the *integral test*.

$$\frac{2}{1^3}+\frac{3}{2^3}+\frac{4}{3^3}+\cdots, \quad \frac{1+1}{1^2}+\frac{1+2}{2^2}+\frac{1+3}{3^2}+\cdots, \quad \frac{1}{2(\ln 2)^2}+\frac{1}{3(\ln 3)^2}+\frac{1}{4(\ln 4)^2}+\cdots.$$

**Exercise 04.** By comparison with a harmonic series, study the convergence of the following series.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n}, \quad \sum_{n=2}^{\infty} \frac{1}{\ln n}, \quad \sum_{n=1}^{\infty} \frac{n}{5^{n-1}}.$$

**Exercise 05.** Study the convergence of the following series.

$$\sum_{n\geq 0} \frac{n^n}{(n+3)^{4n}}, \quad \sum_{n=0}^{\infty} \frac{5n^3+1}{n^2}, \quad \sum_{n=1}^{\infty} \cos\!\left(\frac{1}{n^2+3}\right), \quad \sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{5n+4}}, \quad \sum_{n=1}^{\infty} \frac{(n!)^3}{4^n}, \quad \sum_{n=1}^{\infty} \left(\frac{4n^2+1}{n}\right)^n.$$

**Exercise 06.** Show that the series with general terms

$$u_n = \frac{(-1)^n}{\sqrt{2n+1}}, \qquad v_n = \frac{(-1)^n}{\sqrt{2n+1}} + \frac{1}{n}$$

are not of the same nature, although $u_n \sim v_n$ as $n \to \infty$.

## Sequences and Series of Functions

**Exercise 01.** Let

$$f_n(x) = x^n \ln x, \quad n \in \mathbb{N}^*,\ x \in ]0,1], \qquad \text{and } f_n(0) = 0.$$

Study the pointwise and uniform convergence of $(f_n)$ on $[0,1]$.

**Exercise 02.** Study the pointwise and uniform convergence of the sequence of functions

$$f_n(x) = \frac{n^2 x}{1+n^2x^2}$$

on $[1, +\infty[$.

**Exercise 03.** Let the sequence of functions

$$f_n(x) = \frac{2^n x}{1 + n\,2^n x^2}.$$

Study its pointwise and uniform convergence.

**Exercise 04.** Let the sequence of functions

$$f_n(x) = e^{-nx}\cos(nx).$$

1) Study the pointwise and uniform convergence of $(f_n)$ on $\mathbb{R}_+$.

2) Study the pointwise and uniform convergence on $]0, +\infty[$.

3) Study the pointwise and uniform convergence on $[1, +\infty[$.

4) More generally, study the pointwise and uniform convergence on $[a, +\infty[$, with $a > 0$.

**Exercise 05.** Study the pointwise and uniform convergence of the series of functions

$$\sum_{n=0}^{+\infty} \frac{xe^{-nx}}{\ln n}$$

on $X = [a, b]$, where $a > b > 0$.

**Exercise 06.**

1) Study the pointwise convergence on $\mathbb{R}$ of the series

$$\sum_{n=0}^{+\infty} e^{-x\sqrt{n}}.$$

2) Study the pointwise, uniform, and normal convergence on $\mathbb{R}_+$ of the series

$$\sum_{n=1}^{+\infty} \frac{1 - x^n}{1 + x^n}\,\frac{1}{n^2}.$$

**Exercise 07.**

1) Study the pointwise, uniform, and normal convergence on $\mathbb{R}_+$ of the series

$$\sum_{n=1}^{+\infty} \frac{nx^2}{n^3 + x^2}.$$

2) Study the pointwise, uniform, and normal convergence on $[0, a]$, with $a > 0$.

**Exercise 08.**

1) Show that the series with general term $ne^{-nx}$ converges uniformly on $[1, +\infty[$.

2) Let $S(x)$ be the sum of this series. Compute

$$\int_a^b S(x)\,dx \quad \text{with } 1 < a < b.$$

## Power Series

**Exercise 01.** Compute the radius of convergence.

$$f_n(z) = \frac{1+2ni}{n+2i}\, z^n, \quad f_n(z) = (1+in)z^n.$$

**Exercise 02.**

1) Determine the radius of convergence of the following power series:

$$\sum_{n=1}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n-1)(2n+1)}.$$

2) If $f(x)$ denotes the sum of this series inside the disk of convergence, determine the domain of continuity of the function $f$.

3) Compute $f(x)$ and deduce the value of the sum

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{4n^2-1}.$$

**Exercise 03.**

1) Determine the radius of convergence of the following power series and compute their sums:

$$\sum_{n=0}^{+\infty} n^2 x^n, \quad \sum_{n=0}^{+\infty} \frac{n^2+1}{n!} x^n.$$

**Exercise 04.**

1) Show that the series $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{n3^n}$ is convergent.

2) Compute the sum of this series.

**Exercise 05.** Develop into power series the following functions:

$$f(x) = \ln(2+x), \quad f(x) = e^{2-x^2}, \quad f(x) = \frac{1}{x^2-3x+2}.$$

**Exercise 06.**

1) Determine the power series of the form $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ whose sum is a solution of the differential equation

$$xy'' + y' + xy = 0.$$

2) Give the radius of convergence of the obtained series.

3) Let $S_0$ be the series obtained for $a_0 = 1$. Show that

$$S_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta)\, d\theta.$$

## Fourier Series

**Exercise 01.**

1) Determine the Fourier series of the $2\pi$-periodic odd function equal to $x$ on the interval $]-\pi,\pi]$. Study the convergence.

2) Find the value of the sum $\displaystyle\sum_{k=0}^{+\infty}\frac{(-1)^k}{2k+1}$.

**Exercise 02.**

1) Determine the Fourier series of the $2\pi$-periodic function equal to $|x|$ on the interval $]-\pi,\pi]$. Study the convergence.

**Exercise 03.** Let $f$ be the $2\pi$-periodic function equal to $x^2$ on the interval $[-\pi,\pi[$.

1) Determine its Fourier series and give the sum of the series for all real $x$.

2) Deduce the relation $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

3) Deduce the following relations:

$$\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8},\quad \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12},\quad \sum_{n=1}^{+\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

**Exercise 04.** Let $f$ be the $2\pi$-periodic function equal to $x^2$ on $[0,2\pi[$.

1) Determine its Fourier series and compute its sum at $x=0$.

2) Discuss the convergence of $S_f$.

**Exercise 05.** Let $f$ and $g$ be two $2\pi$-periodic functions defined by

$$f(x)=x,\quad g(x)=\frac{\sin x}{2}$$

on $[0,2\pi]$.

1) Determine the complex Fourier series of $f$ and give the domains of uniform convergence to $f$.

2) Determine the complex Fourier series of $g$, study its convergence, and deduce the value of the sum

$$\sum_{n=1}^{+\infty}\frac{1}{4n^2-1}.$$

**Exercise 06.** Let $f$ be a $2\pi$-periodic odd function defined on $[0,\pi[$ by

$$f(x)=x(\pi-x).$$

1) Determine the Fourier series of $f$, study its convergence, and deduce the value of the sum

$$\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}.$$

# Chapter 5

# Transformation de Fourier

he Fourier transform decomposes a function into its frequency components. It is essential in signal processing, physics, and differential equations. This chapter introduces the necessary function spaces $L^1$, $L^2$, $\mathcal{S}$, and $\mathcal{D}$. We then define the Fourier transform and establish its main properties: linearity, scaling, modulation, convolution, and Parseval's theorem. Finally, we apply these results to solve linear differential equations with constant coefficients. The Fourier transform converts differential equations into algebraic equations, making explicit solutions accessible.

**Definition 5.1.** $L^1$ is the set of summable functions.

**Definition 5.2.** $L^2$ is the set of square-summable functions.

**Definition 5.3.** $L^1$ is the set of functions defined on $\mathbb{R}$, except possibly at a finite number of points, continuous outside these points, and absolutely integrable on $\mathbb{R}$.

**Definition 5.4.** Let $f : \mathbb{R} \to \mathbb{C}$. $f$ is said to be rapidly decreasing, and we write $f \in \mathcal{S}$, if $f \in C^\infty$ and

$$\forall k, n \in \mathbb{N},\ \exists C_{k,n} : \left| x^k f^{(n)}(x) \right| \leq C_{k,n}, \quad \forall x \in \mathbb{R}. \tag{1}$$

**Definition 5.5.** Let $f : \mathbb{R} \to \mathbb{C}$. $f$ is a test function if it is infinitely differentiable with compact support, and we write $f \in \mathcal{D}$. In other words,

$$f \in C^\infty \quad \text{and} \quad \exists a, b \in \mathbb{R} : a \leq b,\ \forall x \in \mathbb{R} \setminus [a, b],\ f(x) = 0.$$

## 5.1 The Fourier Transform Formula.

The Fourier transform is a powerful mathematical tool that allows us to decompose a periodic or rapidly decreasing function into an infinite sum of sines and cosines. This decomposition is very useful in many fields such as signal processing, electrical circuit analysis, optics, etc.
The Fourier transform formula is as follows:

Let $f(t)$ be a complex-valued function defined on $\mathbb{R}$ and satisfying certain regularity conditions (for example, $f$ must be integrable on every bounded interval). Then its Fourier transform, denoted $\mathcal{F}[f](\omega)$ or $\hat{f}(\omega)$, is defined by

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt$$

where $\omega$ is the frequency variable.

**Differentiation in the Time Domain**

$$\mathcal{F}\left[\frac{d^n f}{dt^n}\right](\omega) = (i\omega)^n \hat{f}(\omega)$$

**Differentiation in the Frequency Domain**

$$\frac{d^n \hat{f}}{d\omega^n}(\omega) = \mathcal{F}\Big((it)^n f(t)\Big)(\omega)$$

**Translation in Time**

$$\mathcal{F}\Big(f(t-t_0)\Big)(\omega) = e^{-i\omega t_0}\hat{f}(\omega)$$

*Remark:* A term corresponding to a linear phase shift as a function of frequency ($e^{-i\omega t_0}$) is translated into a time shift. The modulus of the Fourier transform is unchanged by this operation.

## 5.2 Properties of the Fourier Transform

**Linearity of $\mathcal{F}^{-1}$**

Suppose $\lambda$ and $\mu$ are any two complex numbers, $\hat{f}(\omega)$ and $\hat{g}(\omega)$ are two locally integrable and absolutely integrable functions on $\mathbb{C}$. Then

$$\mathcal{F}^{-1}\Big(\lambda\hat{f} + \mu\hat{g}\Big)(\omega) = \lambda f(t) + \mu g(t)$$

**Scaling (Dilation in Time)**

$$\mathcal{F}\Big(f(at)\Big)(\omega) = \frac{1}{|a|}\,\hat{f}\Big(\frac{\omega}{a}\Big)$$

### Modulation

$$\mathcal{F}\Big(e^{i\omega_0 t} f(t)\Big)(\omega) = \hat{f}(\omega - \omega_0)$$

*Remark* 5.1. In the time domain, multiplication by a sinusoid results in a translation in the frequency domain.

### Conjugation

$$\mathcal{F}\Big(\overline{f(t)}\Big)(\omega) = \overline{\hat{f}(\omega)}$$

### Convolution

**Definition 5.6.** The function $f*g$ is the convolution product of the real or complex functions $f$ and $g$, defined as

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt$$

The product of the transforms is the Fourier transform of the convolution product:

$$\mathcal{F}(f * g)(\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega)$$

### Continuity

$$f \in L^1 \;\Rightarrow\; \mathcal{F}(f) \in C(\mathbb{R})$$

### Behavior at Infinity

$$\lim_{|\omega| \to +\infty} \hat{f}(\omega) = 0$$

*Remark* 5.2.

- If $f(t)$ is real and even, then $\mathcal{F}(f)$ is real and even.
- If $f(t)$ is real and odd, then $\mathcal{F}(f)$ is purely imaginary and odd.
- If $f(t)$ is imaginary and even, then $\mathcal{F}(f)$ is imaginary and even.
- If $f(t)$ is imaginary and odd, then $\mathcal{F}(f)$ is real and odd.

**Theorem 5.1.** *Let $f, g \in \mathcal{S}$. Then*

$$\int_{-\infty}^{+\infty} f(t)\, g(t)\, dt = \int_{-\infty}^{+\infty} \hat{f}(\omega)\, \overline{\hat{g}(\omega)}\, d\omega$$

**Theorem 5.2.** *(Parseval's Theorem). Let $f(t)$ be a finite-energy function ($f \in L^2$) and $\hat{f}(\omega)$ its Fourier transform. Then*

$$\int_{-\infty}^{+\infty} |f(t)|^2 \, dt = \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 \, d\omega$$

## 5.3 Application to the Resolution of Differential Equations

Thanks to the Fourier transform, it is possible to explicitly solve a linear differential equation by transforming it into an algebraic equation.

**Proposition 5.1.** *Let $f$ be an integrable function on $\mathbb{R}$ with values in $\mathbb{C}$, and denote its Fourier transform by $\mathcal{F}(f) = F$. Then we have*

$$\mathcal{F}\left(\frac{df}{dt}(t)\right)(\omega) = (2i\pi\omega)F(\omega),$$

$$\mathcal{F}\left(\frac{d^2f}{dt^2}(t)\right)(\omega) = (2i\pi\omega)^2F(\omega).$$

*More generally, for every integer $n \geq 1$,*

$$\mathcal{F}\left(\frac{d^nf}{dt^n}(t)\right)(\omega) = (2i\pi\omega)^nF(\omega).$$

Consider now the following linear differential equation with constant coefficients:

$$\sum_{j=0}^{n} a_{n-j} \, y^{(n-j)}(t) = g(t),$$

where $y^{(0)}(t) = y(t)$.

Applying the Fourier transform to both sides of the equation and using its linearity, we obtain

$$\sum_{j=0}^{n} a_{n-j} \, \mathcal{F}\Big(y^{(n-j)}(t)\Big)(\omega) = \mathcal{F}(g(t))(\omega).$$

Using Proposition 5.1, this becomes

$$\left(\sum_{j=0}^{n} a_{n-j}(2i\pi\omega)^{n-j}\right) F(\omega) = \mathcal{F}(g)(\omega),$$

where $\mathcal{F}(y) = F$.

Hence,

$$F(\omega) = \left(\sum_{j=0}^{n} a_{n-j}(2i\pi\omega)^{n-j}\right)^{-1} \mathcal{F}(g)(\omega).$$

If we define $h$ by

$$\mathcal{F}(h)(\omega) = \left( \sum_{j=0}^{n} a_{n-j} (2i\pi\omega)^{n-j} \right)^{-1},$$

then

$$F(\omega) = \mathcal{F}(h)(\omega)\, \mathcal{F}(g)(\omega) = \mathcal{F}(h * g)(\omega).$$

Therefore,

$$y(t) = (h * g)(t).$$

This shows that solving a linear differential equation with constant coefficients reduces to computing a convolution after applying the Fourier transform.

# Exarcices

## Exercise 1

Compute the Fourier transform of the function $f(x) = e^{-ax^2}$ where $a > 0$.

## Exercise 2

Find the Fourier transform of the gate function $g(x) = \text{rect}(x)$, defined by

$$\text{rect}(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

## Exercise 3

Give the expression of the Fourier transform of the function $h(x) = \sin(x)$.

## Exercise 4

Compute the Fourier transform of the function $k(x) = \cos(ax)$, where $a$ is a real constant.

## Exercise 5

Find the Fourier transform of the function $m(x) = e^{-|x|}$.

# Chapter 6

# Transformation de Laplace

## 6.1 Direct Laplace Transform

The direct Laplace transform is a method used to convert a function from the time domain to the complex Laplace domain. The Laplace transform of a function $f(t)$ is defined as follows:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

where $F(s)$ is the Laplace transform of $f(t)$, $s$ is a complex variable, and $t$ is time.

**Example**

Let us find the Laplace transform of the unit step function $u(t)$. Using the definition, we have:

$$\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} u(t)\, dt = \int_0^\infty e^{-st}\, dt = \frac{1}{s}.$$

**Existence of the Laplace Transform**

The Laplace transform exists for any function $f(t)$ that is locally integrable on the interval $[0, \infty)$. This means that the function must be integrable on every bounded interval of the form $[0, T]$, where $T$ is a positive real number.

**Example**

The function $f(t) = e^{-at}$ is locally integrable for any $a > 0$, so its Laplace transform exists.

**Uniqueness of the Laplace Transform**

If two functions $f(t)$ and $g(t)$ have the same Laplace transform $F(s)$, then $f(t) = g(t)$ for almost every $t \geq 0$. In other words, the Laplace transform is injective.

**Example**

Let $f(t) = t$ and $g(t) = 2t$. Their Laplace transforms are the same, $\mathcal{L}\{t\} = \mathcal{L}\{2t\} = \frac{1}{s^2}$, but $f(t)$ is not equal to $g(t)$ for all $t$.

## 6.2 Properties of the Laplace Transform

The Laplace transform possesses several useful properties, such as linearity, time shift, differentiation, and convolution.

**Linearity**

The Laplace transform is linear, which means that for constants $a$ and $b$ and functions $f(t)$ and $g(t)$, we have:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

**Example**

Let

$$f(t) = e^{-at} \quad \text{and} \quad g(t) = \sin(\omega t).$$

The Laplace transform of $f(t) + g(t)$ is:

$$\mathcal{L}\{f(t) + g(t)\} = \frac{1}{s+a} + \frac{\omega}{s^2+\omega^2}.$$

**Derivative**

The derivative of the Laplace transform of $f(t)$ with respect to $s$ is equal to the Laplace transform of the derivative of $f(t)$ with respect to $t$. This can be expressed as:

$$\frac{d}{ds}\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{d}{dt}f(t)\right\}$$

**Example**

Let us find the Laplace transform of the first derivative of $f(t) = e^{-at}$. We have:

$$\frac{d}{dt}e^{-at} = -ae^{-at}.$$

The Laplace transform of $-ae^{-at}$ is $-aF(s)$, therefore the Laplace transform of $f'(t)$ is $-aF(s)$ (or more precisely, $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$).

## 6.3 Application to Solving Differential Equations

The Laplace Transform is a very powerful tool for solving linear differential equations with constant coefficients. It transforms a differential equation into an algebraic equation, which is much easier to solve.

Consider a second-order linear differential equation of the form:

$$a_2y''(t) + a_1y'(t) + a_0y(t) = f(t), \qquad y(0) = y_0,\ y'(0) = y_0'$$

where $a_2, a_1, a_0$ are real constants and $f(t)$ is a piecewise continuous function.

To solve this equation using the Laplace Transform, we follow these steps:

**Step 1: Apply the Laplace Transform**

Applying the Laplace Transform to the differential equation, we obtain:

$$a_2\mathcal{L}\{y''(t)\} + a_1\mathcal{L}\{y'(t)\} + a_0\mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}$$

Using the properties of the Laplace Transform, we have:

$$a_2\Big(s^2Y(s) - sy_0 - y_0'\Big) + a_1\Big(sY(s) - y_0\Big) + a_0Y(s) = \mathcal{L}\{f(t)\}$$

where $Y(s) = \mathcal{L}\{y(t)\}$.

Collecting terms and rewriting this equation in algebraic form:

$$a_2s^2Y(s) + (a_1s + a_0)Y(s) = a_2sy_0 + a_2y_0' + a_1y_0 + \mathcal{L}\{f(t)\}$$

**Step 2: Solve the resulting algebraic equation**

The obtained algebraic equation is a second-degree equation in $Y(s)$:

$$a_2s^2Y(s) + (a_1s + a_0)Y(s) = a_2sy_0 + a_2y_0' + a_1y_0 + \mathcal{L}\{f(t)\}$$

We can solve this equation to obtain the expression for $Y(s)$:

$$Y(s) = \frac{a_2sy_0 + a_2y_0' + a_1y_0 + \mathcal{L}\{f(t)\}}{a_2s^2 + a_1s + a_0}$$

**Step 3: Determine the solution $y(t)$ by inverse Laplace Transform**

To find the solution $y(t)$, we need to compute the inverse Laplace transform of $Y(s)$. This can be done by decomposing $Y(s)$ into partial fractions, and then applying known inverse Laplace transform formulas.

Let $Y(s) = \dfrac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$ are polynomials in $s$. We decompose $Y(s)$ into partial fractions:

$$Y(s) = \sum_i \frac{A_i}{s - s_i}$$

where $A_i$ are constants and $s_i$ are the roots of the denominator $D(s)$.

Then, we apply the inverse Laplace transform to each term of the partial fraction decomposition to obtain the solution $y(t)$ in the time domain.

In summary, applying the Laplace transform to solve a linear differential equation with constant coefficients involves three main steps:

1. Apply the Laplace transform to the differential equation to obtain an algebraic equation in $Y(s)$.
2. Solve the resulting algebraic equation to find the expression for $Y(s)$.
3. Determine the solution $y(t)$ in the time domain by computing the inverse Laplace transform of $Y(s)$.

This method efficiently solves linear differential equations with constant coefficients by transforming the problem into the Laplace domain, where it is easier to handle.

**Example**

(Homogeneous second-order differential equation) Consider the following differential equation:

$$y''(t) + 4y'(t) + 3y(t) = 0, \qquad y(0) = 1,\ y'(0) = -1$$

Let us apply the three-step method:

**Step 1 - Apply the Laplace transform:**

$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 3\mathcal{L}\{y(t)\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 4\Big(sY(s) - y(0)\Big) + 3Y(s) = 0$$

$$(s^2 + 4s + 3)Y(s) = s + 1$$

**Step 2 - Solve the algebraic equation:**

$$Y(s) = \frac{s+1}{s^2 + 4s + 3}$$

**Step 3 - Find the solution $y(t)$ by inverse Laplace transform:**

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2 + 4s + 3}\right\} = e^{-t}\Big(\cos(2t) + \tfrac{1}{2}\sin(2t)\Big)$$

**Example**

(Non-homogeneous second-order differential equation) Consider the following differential equation:

$$y''(t) + 4y'(t) + 3y(t) = e^{-t}, \qquad y(0) = 0,\ y'(0) = 0$$

Let us apply the three-step method:

**Step 1 - Apply the Laplace transform:**

$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 3\mathcal{L}\{y(t)\} = \mathcal{L}\{e^{-t}\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4\Big(sY(s) - y(0)\Big) + 3Y(s) = \frac{1}{s+1}$$

$$(s^2 + 4s + 3)Y(s) = \frac{1}{s+1}$$

**Step 2 - Solve the algebraic equation:**

$$Y(s) = \frac{1}{(s+1)(s^2 + 4s + 3)}$$

**Step 3 - Find the solution $y(t)$ by inverse Laplace transform:**

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2 + 4s + 3)}\right\} = \frac{1}{2}e^{-t} - \frac{1}{4}e^{-2t}\sin(2t)$$

# Exercises

**Exercise 1**

Compute the Laplace transform of the following functions:

$$f(t) = t^n e^{-at} \quad \text{for } a > 0 \text{ and } n \in \mathbb{N}$$

$$f(t) = \cos(\omega_0 t)$$

**Exercise 2**

Let $f(t) = u(t) - u(t-a)$ where $u(t)$ is the unit step function.

1. Compute the Laplace transform of $f(t)$.
2. Interpret the result obtained.

**Exercise 3**

Let $f(t) = te^{-at}$ with $a > 0$.

1. Compute the Laplace transform of $f(t)$.
2. Deduce the Laplace transform of $\dfrac{df}{dt}(t)$.

**Exercise 4**

Let $f(t) = \text{rect}(t/T)$ with $\text{rect}(x) = 1$ if $|x| \leq 1/2$ and 0 otherwise.

1. Compute the Laplace transform of $f(t)$.
2. Deduce the Laplace transform of $\dfrac{df}{dt}(t)$.

**Exercise 5**

Let $f(t) = \text{tri}(t/T)$ with $\text{tri}(x) = 1 - |x|$ if $|x| \leq 1$ and 0 otherwise.

1. Compute the Laplace transform of $f(t)$.
2. Deduce the Laplace transform of $\dfrac{df}{dt}(t)$.

**Exercise 6**

Let $f(t) = \text{sgn}(t)$, where $\text{sgn}(x) = 1$ if $x > 0$, 0 if $x = 0$, and $-1$ if $x < 0$.

1. Compute the Laplace transform of $f(t)$.
2. Deduce the Laplace transform of $\dfrac{df}{dt}(t)$.

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