PEOPLE's DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf university of Msila
Faculty of Mathematics and computer sciences
Department of Mathematics

Master memory

Field : Mathematics and computer sciences
Branch : Mathematics
Option : Functional analysis

Theme

Interpolative construction for operator ideals

Presented by :
Refice Nawal

In front of the jury composed of :

Abdelaziz Belaada	MCB,	University of Msila	**President**.
Yahi Rachid	MCB,	University of Msila	**Supervisor.**
Belaala Maatougui	MCB,	University of Msila	**Examiner.**

University year 2019/2020

❖شكر وعرفان

قال رسول الله صلى الله عليه وسلم

من اصطنع إليكم معروفا فجازوه، فإن عجزتم عن مجازاته فادعوا له حتى يعلم أنكم شكرتم فإن الله شاكر يحب الشاكرين

الحمد لله على فضله الكبير الذي أجازني بالتوفيق لإتمام هذا العمل ويسعدني أن أتقدم بالشكر الجزيل إلى أولئك الذين ساندوني طيلة فترة دراستي وأقدم لهم هذه الكلمات: تتناثر الكلمات حبرا على صفائح الأوراق لكل من علمني،وأزال غيمة جهل مررت بها برياح العلم الطيبة ولكل من أعاد رسم ملامحي وتصحيح عثراتي، كما أبعث بتحية شكر وتقدير للأستاذ الفاضل المشرف **ياحي رشيد** الذي قدم لي كل الدعم والتوجيه، والذي كان صبره علي وتشجيعه لي أكبر عون وحافز لإتمام هذا العمل خاصة في ظل هذه الظروف الصعبة المصاحبة لجائحة كورونا فله مني جزيل الشكر والتقدير والامتنان. انه لمن دواعي سروري أيضا التقدم بالشكر الجزيل للدكتور **بلعلى المعتوقي** والدكتور **بلاعدة عبد العزيز** لقبولهما مناقشة مذكرة تخرجي. كما أتقدم بالشكر الجزيل لكل من ساهم في إتمام هذا العمل.

❖ إهداء

إلى الينبوع الذي لا يمل العطاء إلى من حاكت سعادتي بخيوط منسوجة من قلبها إلى

والدتي العزيزة.

إلى من سعى وشقي لأنعم بالراحة والهناء، الذي لم يبخل بشيء من أجل

دفعي في طريق النجاح،الذي علمني أن أرتقي سلم الحياة بحكمة وصبر إلى

والدي العزيز.

إلى من حبهم يجري في عروقي ويلهج بذكراهم فؤادي إلى

إخواني وأخواتي.

إلى من سرنا سويا ونحن نشق الطريق معا نحو النجاح

والإبداع إلى من تكاتفنا يدا بيد ونحن نقطف زهرة تعلمنا إلى

صديقاتي وزميلاتي.

إلى من علموني حروفا من ذهب وكلمات من درر وعبارات

من أسمى وأجلى عبارات في العلم، إلى من صاغوا لي من علمهم

حروفا ومن فكرهم منارة تنير لنا مسيرة العلم والنجاح إلى

أساتذتي الكرام.

أهدي هذا العمل المتواضع راجية من المولى عز وجل أن يجد القبول والنجاح.

❄ ارفيس نوال ❄

Abstract

ملخص

في هذه المذكرة قمنا بعرض نظرية الإستقطاب التي تم تناولها من طرف Jarchow and Matter في مقال مشترك لهما ، والتي تستعمل لتوليد فضاءات مثالية جديدة انطلاقا من مثاليان خطيان نذكر منها مثلا فضاء المؤثرات (p,σ) جمعية مطلقا، بالنسبة لنظرية الهيمنة والتفكيك لبيتش الخاصة بهذه الفضاءات قد تم تناولها أيضا في مذكرتنا هذه.

الكلمات المفتاحية: فضاء المؤثرات المثاليات الخطية ، المؤثرات p جمعية. نظرية الهيمنة والتفكيك لبيتش، فضاء المؤثرات (p,σ) جمعية مطلقا.

Abstract

In this memory we study the interpolations method presented by Jarchow and Matter in ordre to get a new scale of operators ideal, as (p,σ)-absolutely continuous operators. Domination and factorisation theorems of Piesch about this class are presented.

Keywords : Linear operator ideals, p-summing operators,Piesch domination and factorization theorems ,(p,σ)-absolutely continuous operators .

Contents

Notaions

$\mathbb{K}$	The field of real or complex numbers.
$\mathbb{R}_+$	The field of non negative real numbers.
p'	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p'} = 1$
E^*	The topological dual of E.
B_E	The closed unit ball of E
$L(E;F)$	The set of all linear operators.
$\mathcal{L}(E;F)$	The sets of all continuous linear operators.
w	The weak topology.
w^*	The weak * topology.
$\mathcal{I}$	The ideal of all linear operator.
$\overline{\mathcal{I}}^{inj}$	The closed injective of the linear operators ideal $\mathcal{I}$.
T^*	The adjoint operator of T.
$\mathcal{K}$	The set of all compact linear operators.
$\mathcal{W}$	The set of all weakly compact linear operators.
$\mathcal{L}_f$	The set of all finite rank linear operators.

Introduction

In this memory we study the interpolative ideal procedure, that was introduced by Matter and Jarchow in 1987 in order to give a new operator ideal from two operators ideals [3]. Let $\mathcal{I}$ and $\mathcal{I}'$ be two linear operators ideals, $0 < \sigma < 1$ and $r = \dfrac{1}{1-\sigma}$. A linear operator $T \in (\mathcal{I}, \mathcal{I}')_\sigma(E, F)$ if there are Banach spaces G, H and two linear operators $S \in \mathcal{I}(E, G)$, and $R \in \mathcal{I}'(E, H)$ such that

$$\|T(x)\| \leq \varepsilon^{-r}.\|S(x)\| + \varepsilon.\|R(x)\|, \ \forall x \in E, \ \forall \varepsilon > 0 \tag{1}$$

It is well known that (1) is equivalent with

$$\|T(x)\| \leq C.\|S(x)\|^{1-\sigma}.\|R(x)\|^{\sigma}, \ \forall x \in E,$$

where C is a constant depending only on σ.

After that, Matter in [9, Definition 3.1] defined the interpolative space $(\mathcal{I})_\sigma$, that is, a linear operator $T \in \mathcal{I}_\sigma(E, F)$ if there exist a Banach space G and an operator $S \in \mathcal{L}(E, F)$ such that

$$\|T(x)\| \leq \|S(x)\|^{1-\sigma}.\|x\|^{\sigma}, \ \forall x \in E.$$

If we take $\mathcal{I} = \Pi_{p,\sigma}$ the Banach space of absolutely p-summing operators, we obtain the injective operator ideal, (p, σ)-absolutely continuous operators $(1 \leq p < \infty\ ,\ 0 \leq \sigma < 1)$, were defined by Matter [9, 8] by applying an interpolative ideal procedure. The interpolated operator ideal $\Pi_{p,\sigma}$ of all (p, σ)-absolutely continuous operators was defined as an intermediate operator ideal between the ideal Π_p of the absolutely p-summing linear operators and the ideal of all continuous operators, and shares similar properties with absolutely p-summing operators.

So our memory is divided into three chapters :

In the first chapter we recall some fundamental notions and definitions concerning the linear operator ideals and Injective Ideal .

In the second chapter , we present the interpolative construction for operator ideals .

The final chapter is devoted to study The ideal of (p, σ)- absolutely continuous linear operators.

Chapter 1

Preliminary

1.1 Basic notions ans terminologies

In this chapter, we present some concepts which will use in the sequel of this Memory.We will write $\mathbb{K}$ for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For $1 \leq p \leq \infty$, by p' we denote the conjugate of p, that is $\frac{1}{p} + \frac{1}{p'} = 1$. Along this memory the letters E, F and G denote Banach spaces. Given a Banach space Z, Z^* denotes its topological dual, and B_Z is its closed unit ball. By $\mathcal{L}(E, F)$ we denote the Banach space of all continuous linear operators between E and F with the norm

$$\|T\| = \sup_{x \in B_E} \|T(x)\| .$$

If $T \in \mathcal{L}(E, F)$, $T^* : F^* \longrightarrow E^*$ is the adjoint operator of T with $\|T\| = \|T^*\|$.

Two norms $\|.\|_1$ and $\|.\|_2$ on a linear space Z are equivalent if there exist positive numbers C, K such that

$$C \|x\|_1 \leq \|x\|_2 \leq K \|x\|_1 , \forall x \in Z.$$

For a linear operator $T \in \mathcal{L}(E, F)$, the linear operator $S : F \longrightarrow E$ that satisfies $T \circ S = id_F$ and $S \circ T = id_E$ is called the inverse of T and is denoted here by T^{-1}.

A linear operator $T : Z_1 \longrightarrow Z_2$ between two normed spaces Z_1 and Z_2 is an *isomorphism* if T is a continuous bijection whose inverse T^{-1} is also continuous. In such case the spaces Z_1 and Z_2 are said to be *isomorphic*. T is an *isometric isomorphism* when $\|T(x)\| = \|x\|$ for all $x \in Z_1$. In particular, any linear operator between normed spaces of the same finite dimension is an isomorphism.

A linear operator T is an *embedding of E into F* if T is an isomorphism onto its image $T(E)$. In this case we say that E *embeds in* F. If $T : E \longrightarrow F$ is an embedding such that $\|T(x)\| = \|x\|$ for all $x \in E$, then T is said to be an *isometric embedding.*

1.2 Linear operator ideals

Recall that a linear operator $T \in \mathcal{L}(E, F)$ is said to have finite rank if $T(E)$ is a finite dimensional subspace of F. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(E, F)$. An operator has rank one if and only it has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle \, y$$

i.e. if $u \in \mathcal{L}_f(E, F)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$ (see [10, Page 25]).

Definition 1.1. *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components*

$$\mathcal{I}(E, F) := \mathcal{L}(E, F) \cap \mathcal{I}$$

satisfy:

i) $\mathcal{I}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$ which contains the finite rank operators.

ii) The ideal property: if $v \in \mathcal{L}(G, E)$, $u \in \mathcal{I}(E, F)$ and $w \in \mathcal{L}(F, H)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(G, H)$. If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies:

i') $(\mathcal{I}(E, F), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F, *ii')* $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

iii') *If* $v \in \mathcal{L}(G, E)$, $u \in \mathcal{I}(E, F)$ *and* $w \in \mathcal{L}(F, H)$, $\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \|v\|_{\mathcal{I}} \|u\|$, *then* $(\mathcal{I}, \|.\|_{\mathcal{I}})$ *is called a normed (Banach) operator ideal.*

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(E, F)$ is a closed subspace of $\mathcal{L}(E, F)$ for the sup norm.

1.2.1 Ideal of p-summing linear operators

The theory of p-summing operators is based on a crucial criterion due to Pietsch [10]. We mention that the linear p-summing operators are the starting point in the study of Lipschitz p-summing mappings.

Let $1 \leq p < \infty$. A linear operator $T : E \longrightarrow F$ is said to be p-summing if there exists a constant $C \geq 0$ such that for all finite sequence$(x_i)_{1 \leq i \leq n}$ in E

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*} \leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p \right)^{\frac{1}{p}}.$$

The infmum of all such constants $C \geq 0$ is denoted by $\pi_p(T)$. The collection of all p-summing operators between E and F is denoted by $\Pi_p(E, F)$.

Theorem 1.1. *If* $1 \leq p \leq q < \infty$, *then* $\Pi_p(E, F) \subset \Pi_q(E, F)$. *Moreover,* $\pi_p(T) \leq \pi_q(T)$ *for every* $u \in \Pi_p(E, F)$.

The following basic result about p-summing operators is due to A. Pietsch, and it characterizes the p-summability by means of a domination theorem.

Theorem 1.2. *(Pietcsh Domination Theorem)*

Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E, F)$. Then T is p-summing if and only if there exist a constant C and a regular Borel probability measure μ on B_{E^*} (with the weak star topology) so that

$$\|T(x)\| \leq C \int\limits_{B_{E^*}} |\langle x, x^* \rangle|^p \, d\mu(x^*), \qquad x \in E. \tag{1.1}$$

In this case, $\pi_p(T)$ is the least of all the constants C such that (1.1) holds.

In order to adapt the previous result into a factorization theorem, we present basic examples of p-summing linear operators.

Example 1.1. *(1) Let K be a compact Hausdorff space, let μ be a positive regular Borel measure on K , and let $1 \leq p < \infty$. The canonical inclusion*

$$J_p : C(K) \longrightarrow L_p(\mu),$$

is p-summing with $\pi_p(J_p) = \|J_p\| = \mu(K)^{\frac{1}{p}}$.

(2) Let (Ω, Σ, μ) be a finite measure space and let $1 \leq p < \infty$. The formal inclusion map

$$I_{\infty,p} : L_\infty(\mu) \longrightarrow L_p(\mu),$$

is p-summing, with $\pi_p(I_{\infty,p}) = \mu(\Omega)^{\frac{1}{p}}$.

We denote by i_E the isometric embedding $E \longrightarrow C(B_{E^*})$ given by $i_E(x) = \langle x, . \rangle$.

Corollary 1.1. *(Pietsch Factorization Theorem)*

Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E, F)$. The following are equivalent

(i) T is p-summing.

(ii) There exist a regular Borel probability measure μ on B_{E^*}(with the weak star topology), a

closed subspace $E_p$of $L_p(\mu)$ and a linear continuous operator $\widetilde{u} : E_p \longrightarrow F$ such that $J_p \circ i_E(E) \subset E_p$ and $\widetilde{u} \circ J_p \circ i_E(x) = T(x)$ for all $x \in E$.

In other words, if $\overline{J_p}$ is the map $i_E(E) \longrightarrow E_p$ induced by J_p, then the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ i_E \downarrow & & \uparrow \widetilde{T} \\ i_E(E) & \xrightarrow{\overline{J_p}} & E_p \\ \cap & & \cap \\ C(B_{E^*}) & \xrightarrow{J_p} & L_p(\mu). \end{array}$$

In addition, we may choose μ and $\widetilde{T}$ so that $\left\|\widetilde{T}\right\| = \pi_p(T)$.

1.3 Injective ideal and the injective of an ideal operator

Definition 1.2. *([2, page 421]) An ideal $\mathcal{I}$ is called injective if , for all Banach spaces E, F, G and every homeomorphic embedding $J \in \mathcal{L}(F, G)$, we have that $T \in \mathcal{L}(E, F)$ belongs to $\mathcal{I}(E, F)$ whenever $J \circ T$ is in $\mathcal{I}(E, G)$.*

Notation 1.1. *Let us introduce the Banach spaces*

$E^\infty := l_\infty(B_{E'})$ *and* $E^1 := l_1(B_E)$.

Recall that there is a canonical linear and isometric embedding $J_E : E \to E^\infty$ and also E^∞ has the extension property that is, if F be a subspace of the normed space $(E, \|.\|)$, then every continuous linear operators $T : F \longrightarrow E^\infty$ extend to a continuous linear operators $\hat{T} : E \longrightarrow E^\infty$ such that $\|\hat{T}\| = \|T\|$

Definition 1.3. *Let $\mathcal{I}$ be an ideal . For arbitrary Banach spaces E and F ,the injective of an* *operator ideal denoted by* $\mathcal{I}^{inj}$ *is defined by*

$$\mathcal{I}^{inj}(E,F) := \{S \in \mathcal{L}(E,F)\, ; J_F \circ S \in \mathcal{I}(E,F^{\infty})\}\, .$$

$\mathcal{I}^{inj}$ *be an ideal and* ,$\mathcal{I}$ *is contained in it .*

Proposition 1.1. $\mathcal{I}^{inj}$ *is the smallest injective ideal .Consequently ,* $\mathcal{I}$ *is injective iff* $\mathcal{I}^{inj} = \mathcal{I}$ *.*

It is for this reason that named $\mathcal{I}^{inj}$ is the injective hull of $\mathcal{I}$.

Proof. Let E, F, G be Banach spaces , and let $J \in (F, G)$ be a homeomorphic embedding .Let $S \in (E, F)$ be such that $J \circ S \in \mathcal{I}^{inj}(E, G)$. This means $J_G \circ J \circ S \in \mathcal{I}(E, G^{\infty})$ .By extension property of F^{∞} , there is a $\widetilde{J}_F \in (G^{\infty}, F^{\infty})$ such that $\widetilde{J}_F \circ J_G \circ J = J_F$. From $J_F \circ S = \widetilde{J}_F \circ J_G \circ J \circ S \in \mathcal{I}(E, F^{\infty})$ we conclude that S is in $\mathcal{I}^{inj}(E, F)$ .Thus $\mathcal{I}^{inj}$ is injective .

If $\mathcal{I}_1$ is injective ideal containing $\mathcal{I}$, then $S \in \mathcal{I}^{inj}(E, F)$ implies $J_F \circ S \in \mathcal{I}(E, F^{\infty}) \subset \mathcal{I}_1(E, F^{\infty})$, hence $S \in \mathcal{I}_1(E, F)$ by injectivity of $\mathcal{I}_1$. This proves $\mathcal{I}^{inj} \subset \mathcal{I}_1$. The remaining assertion is trivial. $\square$

Chapter 2

Interpolative constructions for operator ideals

In this chapter we present the interpolative ideal procedure, that was introduced by Matter and Jarchow in 1987 in order to give a new operator ideal from a linear operators ideal [3]

2.1 Interpolative Constructions for operator ideals

Theorem 2.1. *([2, Theorem 3]) Let $\mathcal{I}$ be an ideal and E and F Banach spaces . $T \in (E,F)$ belongs to $\overline{\mathcal{I}}^{inj}(E,F)$ if for each $\varepsilon > 0$, a Banach space G_ε and an operator $S_\varepsilon \in \mathcal{I}(E,G_\varepsilon)$ exist such that*

$$\|T(x)\| \leqslant \|S_\varepsilon(x)\| + \varepsilon \cdot \|x\| \ , \qquad \forall \varepsilon > 0, \forall x \in E. \tag{2.1}$$

Proof. We take $T \in \overline{\mathcal{I}}^{inj}(E,F) \implies J_F \circ T \in \overline{\mathcal{I}}(E,F^\infty)$, then $\exists (S_n)_n \subset \mathcal{I}(E,F^\infty)$ such that :$S_n \to J_F \circ T$ when $n \to \infty$. that is

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, n \geq N : \|S_n - j_F \circ T\| \leqslant \varepsilon.$$

For all $x \in E$ we have

$$\begin{aligned}
\|T(x)\|_F = \|J_F \circ T(x)\|_{F^\infty} &= \|J_F \circ T(x) - S_n(x) + S_n(x)\|_{F^\infty} \\
&\leqslant \|J_F \circ T(x) - S_n(x)\|_{F^\infty} + \|S_n(x)\|_{F^\infty} \\
&\leqslant \|J_F \circ T - S_n\|_{F^\infty} \cdot \|x\| + \|S_n(x)\|_{F^\infty} \\
&\leqslant \varepsilon \cdot \|x\| + \|S_n(x)\|_{F^\infty} \\
&= \varepsilon \cdot \|x\| + \|S_\varepsilon(x)\|_{F^\infty}.
\end{aligned}$$

$\Longleftarrow$ Let us prove that the validity of (2.1) conversely implies $T \in \overline{\mathcal{I}}^{inj}(E,F)$. Given $\varepsilon > 0$, consider the Banach space $H_\varepsilon := (G_\varepsilon \oplus E)_{\ell_1}$ and define

$$\begin{aligned} R_\varepsilon \quad : E \longrightarrow & \quad H_\varepsilon \\ x \mapsto & \quad (S_\varepsilon(x); \varepsilon.x) \end{aligned}$$

R_ε is linear , continuous, and injective.

By virtue of (2.1), the map

$$\begin{aligned} A_\varepsilon \quad : R(R_\varepsilon) \longrightarrow & \quad F \\ R_\varepsilon(x) \mapsto & \quad T(x) \end{aligned}$$

is a well-defined ; it is again linear and continuous , with $\|A_\varepsilon\| \leq 1$. Since F^∞ has the extension property , $J_F \circ A_\varepsilon$ is the restriction of some $\tilde{A}_\varepsilon \in \mathcal{L}\left(H_\varepsilon, F^\infty\right)$ such that $\left\|\tilde{A}_\varepsilon\right\| = \|J_F \circ A_\varepsilon\|$. Define now $B_\varepsilon \in \mathcal{I}(E, F^\infty)$ by

$$\begin{aligned} B_\varepsilon \quad : E \longrightarrow & \quad F^\infty \\ x \mapsto & \quad \tilde{A}_\varepsilon\left(S_\varepsilon(x), 0\right) \end{aligned}$$

and $C_\varepsilon \in \mathcal{I}\left(E, F^\infty\right)$ by

$$\begin{aligned} C_\varepsilon \quad : E \longrightarrow & \quad F^\infty \\ x \mapsto & \quad \tilde{A}_\varepsilon\left(0, \varepsilon \cdot x\right) \end{aligned}$$

Then $J_F \circ T = B_\varepsilon + C_\varepsilon$ and $\|J_F \circ T - B_\varepsilon\| = \|C_\varepsilon\| \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary , $T \in \overline{\mathcal{I}}^{inj}\left(E, F\right)$ is proved .

$\square$

Remark 2.1. *[2, page 473]*

If $\mathcal{I}$ admits a complete ideal-quasinorm , then we may choose in addition $G_\varepsilon = G$ and

$S_\varepsilon = N(\varepsilon) \cdot S$ *for some* $S \in \mathcal{I}(E,G)$ *and some* $N(\varepsilon) > 0$,*i.e we may replace (2.1) by*

$$\|T(x)\| \leqslant N(\varepsilon) \cdot \|S(x)\| + \varepsilon \cdot \|x\|, \qquad \forall \varepsilon > 0, \forall x \in E. \tag{2.2}$$

Proposition 2.1. *The equality :*

$$\|T(x)\| \leqslant N(\varepsilon) \cdot \|S(x)\| + \varepsilon \cdot \|x\|, \qquad \forall \varepsilon > 0, \forall x \in E.$$

implise :

$$\|T(x)\| \leqslant c \|S(x)\|^{1-\sigma} \cdot \|x\|^{\sigma}, \qquad \forall x \in E.$$

with $0 < \sigma \leq 1$, $N(\varepsilon) = \varepsilon^{-r}$, $r = \frac{\sigma}{1-\sigma}$ *and* $c \in \mathbb{R}_+$.

Proof. Let f the function defined by $f(\varepsilon) = \varepsilon^{-r} \cdot \|S(x)\| + \varepsilon \cdot \|x\|$, the derivative of f is

$$f'(\varepsilon) = -r\varepsilon^{-r-1} \cdot \|S(x)\| + \|x\|.$$

So,

$$\begin{aligned}
f'(\varepsilon) \geq 0 \quad &\Longrightarrow \quad -r\varepsilon^{-r-1} \cdot \|S(x)\| \geq -\|x\| \\
&\Longrightarrow \quad \varepsilon^{-r-1} \leq \frac{\|x\|}{r\|S(x)\|} \\
&\Longrightarrow \quad \ln \varepsilon^{-r-1} \leq \ln\left(\frac{\|x\|}{r\|S(x)\|}\right) \\
&\Longrightarrow \quad -(r+1)\ln\varepsilon \leq \ln\left(\frac{\|x\|}{r\|S(x)\|}\right) \\
&\Longrightarrow \quad \ln\varepsilon \geq \frac{-1}{(r+1)}\ln\left(\frac{\|x\|}{r\|S(x)\|}\right) = \ln\left(\frac{r\|S(x)\|}{\|x\|}\right)^{\frac{1}{r+1}} \\
&\Longrightarrow \quad \varepsilon \geq \left(\frac{r\|S(x)\|}{\|x\|}\right)^{\frac{1}{r+1}} = \varepsilon_1
\end{aligned}$$

Now we construct the sign and variation table .

ϵ	0		ϵ_1		$+\infty$
$f'(\epsilon)$		$-$	0	$+$	
$f(\epsilon)$	$+\infty$	$\searrow$	$f(\epsilon_1)$	$\nearrow$	$+\infty$

We have $\varepsilon_{1=} \left(\frac{r\|S(x)\|}{\|x\|}\right)^{\frac{1}{r+1}}$ then $f\left(\varepsilon_1\right) = \left(\frac{r\|S(x)\|}{\|x\|}\right)^{\frac{-r}{r+1}} \cdot \|S\left(x\right)\| + \left(\frac{r\|S(x)\|}{\|x\|}\right)^{\frac{1}{r+1}} \cdot \|x\|$.

If we put $r = \dfrac{\sigma}{1-\sigma}$, we get :

$$\frac{-r}{r+1} = \frac{\dfrac{-\sigma}{1-\sigma}}{\dfrac{\sigma}{1-\sigma}+1} = \frac{-\sigma}{\sigma+1-\sigma} = -\sigma$$

and

$$\frac{1}{r+1} = \frac{1}{\dfrac{\sigma}{1-\sigma}+1} = \frac{1}{\dfrac{\sigma+1-\sigma}{1-\sigma}} = \frac{1-\sigma}{1} = 1-\sigma$$

So

$$\begin{aligned}
f\left(\varepsilon_1\right) &= r^{-\sigma}\frac{\|S(x)\|^{-\sigma}}{\|x\|^{-\sigma}} \cdot \|S\left(x\right)\| + \frac{r^{1-\sigma}\|S(x)\|^{1-\sigma}}{\|x\|^{1-\sigma}} \cdot \|x\| \\
&= r^{-\sigma}\|x\|^{\sigma} \cdot \|S\left(x\right)\|^{1-\sigma} + r^{1-\sigma}\|S\left(x\right)\|^{1-\sigma} \cdot \|x\|^{\sigma} \\
&= \left(r^{-\sigma} + r^{1-\sigma}\right)\|S\left(x\right)\|^{1-\sigma} \cdot \|x\|^{\sigma} \\
&= \left[\left(\tfrac{\sigma}{1-\sigma}\right)^{-\sigma} + \left(\tfrac{\sigma}{1-\sigma}\right)^{1-\sigma}\right]\|S\left(x\right)\|^{1-\sigma} \cdot \|x\|^{\sigma} \\
&= c\,\|S\left(x\right)\|^{1-\sigma} \cdot \|x\|^{\sigma} .
\end{aligned}$$

where c is a constant depending only on σ

$\left(c = \left(\frac{1-\sigma}{\sigma}\right)^{\sigma} + \left(\frac{\sigma}{1-\sigma}\right)^{1-\sigma}\right)$.

We have $f\left(\varepsilon_1\right) \leqslant f\left(\varepsilon\right)$ and $\|T\left(x\right)\| \leqslant f\left(\varepsilon\right)$ for all $\varepsilon > 0$ and all $x \in E$, then

$$\|T\left(x\right)\| \leqslant f\left(\varepsilon_1\right) = c\,\|S\left(x\right)\|^{1-\sigma} \cdot \|x\|^{\sigma} .$$

□

Definition 2.1. *[9, Definition 1.2] Let $0 \leq \sigma < 1.$ and $(\mathcal{I}, \alpha)$ be a normed operator ideal. An operator $T : E \to F$ belongs to $\mathcal{I}_\sigma (E, F)$ if there exist a Banach space G and an operator $S \in \mathcal{I}(E, G)$ such that*

$$\|T(x)\| \leq \|S(x)\|^{1-\sigma} \cdot \|x\|^{\sigma} \, \forall x \in E. \tag{2.3}$$

For each $T \in \mathcal{I}_\sigma (E, F)$, we set

$$\alpha_\sigma (T) := \inf \alpha (S)^{1-\sigma}$$

where the infimum is taken over all operators S admitted in (2.3). Note that $\|T\| \leq \alpha_\sigma (T)$, by definition .

Theorem 2.2. *[9, Theorem 3.2] $[\mathcal{I}_\sigma, \alpha_\sigma]$ is an injective complete quasinormed operator ideal . If α is even a norm, then the same is true for α_σ.*

Proof. We verify only the triangle inequality .Let $T_1, T_2 \in \mathcal{I}_\sigma (E, F)$ .Given $\varepsilon > 0$,there is a Banach space G_i and an operator $S_i \in \mathcal{I}_\sigma (E, G_i)$ $(i = 1, 2)$ such that

$$\|T_i(x)\| \leqq \left\|\alpha (S_i)^{-\sigma} S_i(x)\right\|^{1-\sigma} \left\|\alpha (S_i)^{1-\sigma} x\right\|^{\sigma} \qquad \forall x \in E$$

and $\alpha (S_i)^{1-\sigma} \leqq (1 + \varepsilon) \alpha_\sigma (T_i)$ $(i = 1, 2)$. Introducing the ℓ_1-sum $G := G_1 \oplus G_2$ and the operator $S := \alpha (S_1)^{-\sigma} J_1 S_1 + \alpha (S_2)^{-\sigma} J_2 S_2 \in \mathcal{I}(E, G)$ $(J_1, J_2$ the canonical injections)and applying Hölder inequality , we get for all $x \in E$

$$\begin{aligned}
\|(T_1+T_2)(x)\| &\leq \|T_1(x)\| + \|T_2(x)\| \\
&\leq \sum_{i=1}^{2} \left\|\alpha(S_i)^{-\sigma} S_i(x)\right\|^{1-\sigma} \left\|\alpha(S_i)^{1-\sigma}(x)\right\|^{\sigma} \\
&\leq \left(\sum_{i=1}^{2} \left\|\alpha(S_i)^{-\sigma} S_i(x)\right\|\right)^{1-\sigma} \left(\sum_{i=1}^{2} \left\|\alpha(S_i)^{1-\sigma}(x)\right\|\right)^{\sigma} \\
&= \left(\alpha(S_1)^{1-\sigma} + \alpha(S_2)^{1-\sigma}\right)^{\sigma} \|S(x)\|^{1-\sigma} \|x\|^{\sigma}.
\end{aligned}$$

Hence $T_1 + T_2 \in \mathcal{I}_\sigma(E,F)$ and furthermore , if β is the quasinorm constant of α,

$$\begin{aligned}
\alpha_\sigma(T_1+T_2) &\leq \left(\alpha(S_1)^{1-\sigma} + \alpha(S_2)^{1-\sigma}\right)^{\sigma} \alpha(S)^{1-\sigma} \\
&\leq \beta^{1-\sigma}\left(\alpha(S_1)^{1-\sigma} + \alpha(S_2)^{1-\sigma}\right) \\
&\leq \beta^{1-\sigma}(1+\varepsilon)\left(\alpha_\sigma(T_1) + \alpha_\sigma(T_2)\right).
\end{aligned}$$

In particular , if α is a norm , then so is α_σ. □

The ideal $\mathcal{I}_0$ is just the injective hull of $\mathcal{I}$ and the quasinorms are equal . Further properties are given in .

Proposition 2.2. *[9, Proposition 3.3] Let $0 \leq \sigma, \sigma_1, \sigma_2 < 1$. Then the following holds :*

(a) $\quad \mathcal{I}_{\sigma_1} \subset \mathcal{I}_{\sigma_2}$ *if* $\sigma_1 \leq \sigma_2$.

(b) $\quad \mathcal{I}^{inj} \subset \mathcal{I}_\sigma \subset \overline{\mathcal{I}}^{inj}$.

(c) $\quad (\mathcal{I}_{\sigma_1})_{\sigma_2} \subset \mathcal{I}_{\sigma_1+\sigma_2-\sigma_1\sigma_2}$.

Here all inclusion maps have norm 1.

Proof. To verify (a), let $T \in \mathcal{I}_\sigma(E,F)$ and $\varepsilon > 0$ .Then

$$\|T(x)\| \leq \|S(x)\|^{1-\sigma_1} \cdot \|x\|^{\sigma_1}, \ \forall x \in E,$$

holds for a suitable Banach space G and an operator $S \in \mathcal{I}(E,G)$ with $\alpha(S)^{1-\sigma_1} \leq (1+\varepsilon)\,\alpha_\sigma(T)$.

Since

$$\|T(x)\| \quad \leq \quad \|S\|^{\sigma_2-\sigma_1} \|S(x)\|^{1-\sigma_2} \cdot \|x\|^{\sigma_2}, \ \ \forall x \in E$$

we get $T \in \mathcal{I}_\sigma(E,F)$ and

$$\alpha_\sigma(T) \quad \leq \quad \|S\|^{\sigma_2-\sigma_1} \alpha(S)^{1-\sigma_2} \quad \leq \quad \alpha(S)^{1-\sigma_1} \quad \leq \quad (1+\varepsilon)\,\alpha_\sigma(T) \ \cdot$$

For $(c)$and (b) see [9]. □

2.2 An generale interpolative ideal procedure

In this section we present An generale interpolative ideal procedure introduced by Matter and Jarchow in [3]. Along of this section we use the reference[7] .

Let us now consider the function $\varphi : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ given by $\varphi(s,t) = s^{1-\sigma}t^\sigma, 0 < \sigma < 1$. The following lemma showing the superadditivity of φ . This property will be frequently used in the sequel.

Lemma 2.1. *For any $a_i, b_i \geq 0$ the following inequality holds*

$$\sum_{i=1}^n \varphi(a_i,b_i) \leq \varphi\left(\sum_{i=1}^n a_i, \sum_{i=1}^n b_i\right).$$

Proof. By the definition of concavity the inequality

$$\sum_{i=1}^n \lambda_i \varphi(a_i,b_i) \leq \varphi\left(\sum_{i=1}^n \lambda_i a_i, \sum_{i=1}^n \lambda_i b_i\right)$$

holds for every $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. The homogeneity of φ gives

$$\sum_{i=1}^n c\lambda_i \varphi(a_i,b_i) \leq \varphi\left(\sum_{i=1}^n c\lambda_i a_i, \sum_{i=1}^n c\lambda_i b_i\right) \quad \text{for every } c > 0.$$

Taking $c\lambda_i = 1$ for every $i = 1, ..., n$ gives the claim . □

The following notion gives a generalization of the procedure introduced by U. Matter in [9].

Definition 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be a two Banach ideals. An operator $T \in \mathcal{L}(E, F)$ belongs to $(\mathcal{A}, \mathcal{B})_\varphi(E, F)$ if there exist Banach spaces Z, W and operators $S \in \mathcal{A}(E, Z)$, $R \in \mathcal{B}(E, W)$ such that*

$$\|T(x)\| \leq \varphi\left(\|S(x)\|, \|R(x)\|\right) \quad \text{for all } x \in E. \tag{2.4}$$

It is easy to see that the above defined class possesses the ideal property.

Proposition 2.3. *Let $V \in \mathcal{L}(X, E), T \in (\mathcal{A}, \mathcal{B})_\varphi(E, F)$ and $U \in \mathcal{L}(F, Y)$, then $V \circ T \circ U \in (\mathcal{A}, \mathcal{B})_\varphi$ and*

$$\|V \circ T \circ U(x)\| \leq \phi\left(\|U\|\|S \circ V(x)\|, \|U\|\|R \circ U(x)\right)\|$$

Proof. We take $U \in \mathcal{L}(X, E), T \in (\mathcal{A}, \mathcal{B})_\varphi(E, F)$ and $V \in \mathcal{L}(F, Y)$, then for all $x \in X$ we get

$$\begin{aligned} \|U \circ T \circ V(x)\| &\leq \|U\|\|T \circ V(x)\| \\ &\leq \|U\|\varphi\left(\|S \circ V(x)\|, \|R \circ V(x)\|\right), S \in \mathcal{A}(E, Z), R \in \mathcal{B}(E, W) \\ &\leq \|U\|\varphi\left(\|S'(x)\|, \|R'(x)\|\right), S' = S \circ V \text{ and } R' = R \circ V \\ &\leq \varphi\left(\|U\|\|S'(x)\|, \|U\|\,\|R'(x)\|\right), \text{ By homogeneity of } \varphi. \end{aligned}$$

So, $U \circ T \circ V \in (\mathcal{A}, \mathcal{B})_\varphi$ and

$$\|U \circ T \circ V(x)\| \leq \phi\left(\|U\|\|S \circ U(x)\|, \|U\|\|R \circ V(x)\right)\|$$

□

In order to see that $(\mathcal{A}, \mathcal{B})_\varphi$ is a linear space, we provide the following alternative characterization of operators for this class.

Proposition 2.4. *An operator T belongs to $(\mathcal{A},\mathcal{B})_\varphi(E,F)$ if and only if there exist some $n \in \mathbb{N}$, Banach spaces Z_i, W_i and operators $S_i \in \mathcal{A}(E,Z_i)$ $R_i \in \mathcal{B}(E,W_i), i = 1,2,...,n,$ such that*

$$\|T(x)\| \leq \sum_{i=1}^{n} \varphi\left(\|S_i(x)\|, \|R_i(x)\|\right) \quad \textit{for all } x \in E. \tag{2.5}$$

Proof. The only if part is obvious. In order to prove the converse implication let

$$Z := \left(\bigoplus_{i=1}^{n} Z_i\right)_{\ell_1} \quad \text{and } W := \left(\bigoplus_{i=1}^{n} W_i\right)_{\ell_1}.$$

We define operators $S : E \longrightarrow Z$ and $R : E \longrightarrow W$ by

$$S := \sum_{i=1}^{n} J_{Z_i}^{Z} S_i \quad \text{and} \quad R := \sum_{i=1}^{n} J_{W_i}^{W} R_i \ ,$$

where $J_{Z_i}^{Z} : Z_i \longrightarrow Z$ denotes the canonical injection. By Lemma (2.1) we obtain

$$\|T(x)\| \leq \sum_{i=1}^{n} \varphi\left(\|S_i(x)\|, \|R_i(x)\|\right) \leq \varphi\left(\sum_{i=1}^{n} \|S_i(x)\|, \sum_{i=1}^{n} \|R_i(x)\|\right) = \varphi\left(\|S(x)\|, \|R(x)\|\right),$$

which finishes our proof. ☐

Remark 2.2. *Now it follows from Proposition (2.4) that with operators $T_1, T_2 \in (\mathcal{A},\mathcal{B})_\varphi$ and numbers λ_1, λ_2 also $\lambda_1 T_1 + \lambda_1 T_1 \in (\mathcal{A},\mathcal{B})_\varphi$. Hence $(\mathcal{A},\mathcal{B})_\varphi$ is an operator ideal.*

From now on, we assume that α, β are quasinorms on $\mathcal{A}, \mathcal{B}$, such that $\mathcal{A}$ and $\mathcal{B}$, respectively, become quasinormed Banach ideals. We now consider the following maps, which are connected to part (2.4) and (2.5), respectively: For each operator $T \in (\mathcal{A},\mathcal{B})_\varphi$ we put

(i)

$$\gamma(T) := \inf \varphi(\alpha(S), \beta(R)),$$

where the infimum ranges over all operators S, R such that inequality (2.4) holds.

(ii)

$$\overline{\gamma}(T) := \inf \sum_{i=1}^{n} \varphi(\alpha(S_i), \beta(R_i)),$$

where the infimum ranges over all $n \in \mathbb{N}$ and all operators S_i, R_i such that inequality (2.5) holds.

We recall that a mapping $\alpha : \mathcal{A} \to R_+$ (in particular we can consider a quasinorm) is said to have the ideal property if for $V \in \mathcal{L}(X_0, X), T \in \mathcal{A}(X, Y)$ and $U \in \mathcal{L}(Y, Y_0)$ the following inequality holds

$$\alpha(UTV) \leq \|U\| \alpha(T) \|V\|.$$

Proposition 2.5. *Both maps γ and $\overline{\gamma}$ possess the ideal property. Moreover, the map $\overline{\gamma}$ is a norm on $(\mathcal{A}, \mathcal{B})_{\varphi}$.*

Proof. We prove the first statement only for γ. The proof for $\overline{\gamma}$ is similar. Let

$V \in \mathcal{L}(X_0, X)$ and $U \in \mathcal{L}(Y, Y_0)$. Let S, R be operators such that

$$\|T(x)\| \leq \varphi(\|S(x)\|, \|R(x)\|)$$

holds. Observe that

$$\begin{aligned} \|(UTV)(x)\| &\leq \|U\| \|T(V(x))\| \leq \|U\| \varphi(\|S(V(x))\|, \|R(V(x))\|) \\ &= \varphi(\|\|U\| SV(x)\|, \|\|U\| RV(x)\|). \end{aligned}$$

So $\widetilde{S} := \|U\| SV$ and $\widetilde{R} := \|U\| RV$ are admissible operators in the definition of $\gamma(UTV)$. We obtain

$$\gamma(UTV) \leq \varphi\left(\alpha\left(\|U\| SV\right), \beta\left(\|U\| RV\right)\right) \leq \|U\| \varphi\left(\alpha\left(S\right), \beta\left(R\right)\right) \|V\|.$$

Taking the infimum over all operators S and R gives the claim.

We only have to show the triangle inequality in the second statement. For that reason, let $T_1, T_2 \in (\mathcal{A}, \mathcal{B})_\varphi$ and assume that $S_1, ..., S_m; R_1, ..., R_m$ are such that

$$\|T_1(x)\| \leq \sum_{i=1}^{n} \varphi\left(\|S_i(x)\|, \|R_i(x)\|\right) \text{ and } \|T_2(x)\| \leq \sum_{i=n+1}^{m} \varphi\left(\|S_i(x)\|, \|R_i(x)\|\right) \text{ for some } n < m.$$

Obviously

$$\|T_1(x) + T_2(x)\| \leq \sum_{i=1}^{m} \varphi\left(\|S_i(x)\|, \|R_i(x)\|\right).$$

We then have

$$\overline{\gamma}\left(T_1 + T_2\right) \leq \sum_{i=1}^{n} \varphi\left(\alpha\left(S_i\right), \beta\left(R_i\right)\right) + \sum_{i=n+1}^{m} \varphi\left(\alpha\left(S_i\right), \beta\left(R_i\right)\right).$$

Turning to the infimum on the right hand side gives the claim. □

Theorem 2.3. *[7, Theorem 6.5] We Assume that α and β are ideal norms. Then*

$$\overline{\gamma}(T) \leq \gamma(T) \leq c\overline{\gamma}(T)$$

In other words, both maps $\overline{\gamma}, \gamma$ are equivalent.

Notation 2.1. *Let us define the class $\boldsymbol{AC}$ consisting of all continuous, positively homogeneous,* *concave functions* $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$, *with* $\varphi(0,0) = 0, \varphi(s,0) = \varphi(0,t) = 0$ *and* $\varphi(1,1) = 1$. *Taking into account the homogeneity of* φ *we put often* $\varphi(s,t) = s\rho(t/s)$ *with* $\rho : [0,\infty) \to \mathbb{R}_+$. *The function* ρ *is also non-decreasing, concave and continuous with* $\rho(0) = 0$ *and* $\rho(1) = 1$.

Theorem 2.4. *Let* $\varphi \in \boldsymbol{AC}$ *and* $\rho : [0,\infty) \longrightarrow \mathbb{R}_+$ *be given by* $\varphi(s,t) = s\rho(t/s)$. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be operators ideals. Then* $(\mathcal{A},\mathcal{B})_\varphi$ *is an operator ideal. If, moreover,* $(\mathcal{A},\alpha)$, $(\mathcal{B},\beta)$ *are quasinormed Banach ideals and* ρ *is submultiplicative, then* $[(\mathcal{A},\mathcal{B})_\varphi, \overline{\gamma}]$ *is a Banach ideal.*

2.2.1 Factoring through interpolation spaces

Let us start this section by recalling some basic notation and results from interpolation theory. *A* pair of Banach spaces (X_0, X_1) is said to be an interpolation couple (or compatible couple) if both spaces are continuously embedded into a certain Hausdorff topological vector space V . For an interpolation couple (X_0, X_1) we

put

$$\triangle(X_0,X_1) = \left\{x \in X_0 \cap X_1 : \|x\|_\triangle = \max\left(\|x_0\|_{X_0}, \|x_1\|_{X1}\right) < \infty\right\},$$

$$\Sigma(X_0,X_1) = \left\{x = x_0 + x_1 \in X_0 + X_1 : \|x\|_\Sigma = \inf_{x=x_0+x_1}\left\{\|x_0\|_{X_0} + \|x_1\|_{X_1}\right\} < \infty\right\}.$$

The linear spaces $\triangle(X_0,X_1)$ and $\Sigma(X_0,X_1)$ equipped with the norms $\|.\|_\triangle$ and $\|.\|_\Sigma$, respectively are Banach spaces. *A* Banach space X is said to be an intermediate space with respect to (X_0,X_1) if $\triangle(X_0,X_1) \hookrightarrow X \hookrightarrow \Sigma(X_0,X_1)$ (continuous inclusion). If additionally for every linear operator $T : X_0 + X_1 \longrightarrow X_0 + X_1$ such that the restrictions $T_0 : X_0 \longrightarrow X_0$ and $T_1 : X_1 \longrightarrow X_1$ are bounded we have that $T : X \longrightarrow X$ is bounded , then we refer X to as interpolation space with respect to (X_0,X_1). Let (X_0,X_1) and (Y_0,Y_1) be interpolation couples. From now on, the notation $T : (X_0,X_1) \longrightarrow (Y_0,Y_1)$ means that $T : \Sigma(X_0,X_1) \longrightarrow \Sigma(Y_0,Y_1)$ is a linear operator such that the restrictions of T given by $T_0 : X_0 \longrightarrow X_1$ and $T_1 : Y_0 \longrightarrow Y_1$ are bounded. A functor $\mathcal{F}$ from the category of all compatible couples into the category of all Banach spaces is called interpolation functor (or interpolation method) if for any couple (X_0, X_1) the Banach space $\mathcal{F}(X_0, X_1)$ is an intermediate space and $T : \mathcal{F}(X_0, X_1) \longrightarrow \mathcal{F}(Y_0, Y_1)$ is bounded for all couples (X_0, X_1), (Y_0, Y_1) and any $T : (X_0, X_1) \longrightarrow (Y_0, Y_1)$. The closed graph theorem implies that for any interpolation functor $\mathcal{F}$ there exists a constant $C > 0$ such that we have

$$\|T : \mathcal{F}(X_0, X_1) \longrightarrow \mathcal{F}(Y_0, Y_1)\| \leq C \max\left(\|T : X_0 \longrightarrow Y_0\|, \|T : X_1 \longrightarrow Y_1\|\right).$$

If C can be chosen equal to one then we say that $\mathcal{F}$ is an exact interpolation functor. Fundamental examples of the exact interpolation methods are the real method of interpolation $(\,\cdot\,,\,\cdot\,)_{\sigma,p}$ with $0 < \sigma < 1$ and $1 \leq p \leq \infty$ which takes its origins from the classical Marcinkiewicz theorem and the complex method of interpolation $[\,\cdot\,,\,\cdot\,]_\sigma$. The idea of this method goes back to the Riesz-Thorin theorem. The most important notion of real interpolation theory is the K-functional. Let (X_0, X_1) be a Banach couple. The K-functional is given by

$$K(x,t) = \inf\left\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1\right\} \text{ for } x \in \Sigma(X_0, X_1) \text{ and } t > 0.$$

Thus $K(x,1)$ is simply the usual norm of the sum space. Moreover the mapping $x \longmapsto K(x,t)$ provides an equivalent norm on $\sum(X_0, X_1)$ for which it becomes an exact interpolation space. In the sequel for $t > 0$ let $t\mathbb{R}$ denote the real line $\mathbb{R}$ equipped with the norm $\|x\|_{t\mathbb{R}} = t\,|x|$ .If $\mathcal{F}$ is an exact interpolation functor, its characteristic function φ is defined by

$$\varphi(s,t)\,\mathbb{R} := \mathcal{F}(s\mathbb{R}, t\mathbb{R}).$$

In particular we may work with a function from the class AC. For a compatible pair (X_0, X_1) of

Banach spaces, we define $(X_0, X_1)_{\varphi,1}$ as the space of all $x \in \sum(X_0, X_1)$ for which there exists a sequence $(x_n) \subset \Delta(X_0, X_1)$ such that $x = \sum_{n=1}^{\infty} x_n$ in $\sum(X_0, X_1)$ and $\sum_{n\geq 1} \varphi\left(\|x_n\|_0, \|x_n\|_1\right) < \infty$. Equipped with the norm

$$\|x\|_{\varphi,1} = \inf\left\{\sum_{n\geq 1} \varphi\left(\|x_n\|_0, \|x_n\|_1\right) : x = \sum_{n=1}^{\infty} x_n\right\}$$

$(X_0, X_1)_{\varphi,1}$ becomes a Banach space. It can be shown, that the space $(X_0, X_1)_{\varphi,1}$ is an interpolation space . Moreover the interpolation functor $(\,\cdot\,,\,\cdot\,)_{\varphi,1}$ turns out to be exact and φ is its characteristic function. In addition, the interpolation functor $(\,\cdot\,,\,\cdot\,)_{\varphi,1}$ possesses a certain minimal property in the following sense. If $\mathcal{F}$ is an arbitrary exact interpolation functor and φ is its characteristic function, then for any Banach couple (X_0, X_1) the following inclusion holds

$$(X_0, X_1)_{\varphi,1} \subseteq \mathcal{F}(X_0, X_1).$$

In addition, the embedding constant is less than one, i.e the above inclusion is a contraction. In the sequel we also consider the interpolation space $(X_0, X_1)_{\varphi,\infty}$ consisting of all $x \in \sum(X_0, X_1)$ such that

$$\|x\|_{\varphi,\infty} = \sup_{t>0} \frac{K\left(t, x\right)}{\varphi^*\left(1, t\right)}$$

is finite. Here $\varphi^* : (0, \infty) \times (0, \infty) \longrightarrow \mathbb{R}_+$ is given by $\varphi^*(s, t) = (\varphi(1/s, 1/t))^{-1}$.Again,

the functor $(\,\cdot\,,\,\cdot\,)_{\varphi,\infty}$ is exact and φ is its characteristic function. Moreover, it possesses the following maximality property. For an arbitrary exact interpolation functor $\mathcal{F}$ with characteristic function φ and for any interpolation couple (X_0, X_1)

$$\mathcal{F}(X_0, X_1) \subseteq (X_0, X_1)_{\varphi,\infty}$$

and the embedding constant is smaller than one.

Chapter 3

The ideal of (p,σ)-absolutely continuous linear operators

3.1 The ideal of (p,σ)-absolutely continuous linear operators

Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. The interpolated operator ideal $\Pi_{p,\sigma}$ of the (p,σ)-absolutely continuous operators was introduced by Matter [9]. Essentially, it is defined to be an intermediate operator ideal between the ideal Π_p of the absolutely p-summing linear operators and the ideal of all continuous operators (see [3]). In the nineties, several papers describing and analysing the properties and applications of this interpolated class appeared. They were mainly devoted to the study of the factorization properties and the trace duality for these operators, finding in particular the class of tensor norms that represent these operator ideals (see [11],[4],[5]). It must be said that, due to the interpolative procedure that was used for defining them, (p,σ)-absolutely continuous operators preserve some of the characteristic properties of the absolutely p-summing linear operators. However, the emergence of new classes whenever $0 < \sigma < 1$, different from absolutely p-summing linear operators, yields that the theory of p-summing linear operators cannot be applied. Therefore, these new classes are a useful tool to deal with summability properties of operators weaker than absolutely p-summability (see for instance [11],[5]).

3.1.1 (p,σ)-absolutely continuous linear operators

Definition 3.1. *[9, Definition 3.1] Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. Recall that a linear operator $T \in \mathcal{L}(E,F)$ between two Banach spaces E and F is called (p,σ)-absolutely continuous if there* *exist a Banach space G and a linear operator $S \in \Pi_p(E,G)$ such that*

$$\|T(x)\| \leq \|x\|^{\sigma} \|S(x)\|^{1-\sigma}, \quad x \in E. \tag{3.1}$$

Let $\pi_{p,\sigma}(T) = \inf \pi_p(S)^{1-\sigma}$, where the infimum is taken over all Banach spaces G and $S \in \Pi_p(E,G)$ such that (3.1) holds. By $\Pi_{p,\sigma}(E,F)$ we denote the Banach space of all (p,σ)-absolutely continuous operators between E and F.

Note that if $T \in \Pi_{p,\sigma}(E,F)$ then $\|T\| \leq \pi_{p,\sigma}(T)$. The class $(\Pi_{p,\sigma}, \pi_{p,\sigma}(.))$ is an injective Banach operator ideal (see [9, Theorem 3.2]). By the inclusion theorem for the class Π_p, (see Theorem 1.1), we have the result

Proposition 3.1. *Let $1 \leq p \leq q < \infty$ and $0 \leq \sigma < 1$. Then the following holds*

$$\Pi_{p,\sigma}(E,F) \subset \Pi_{q,\sigma}(E,F)$$

for all Banach spaces E and F.

Theorem 3.1. *[9, Theorem 4.1]*

For a linear operator $T \in \mathcal{L}(E,F)$ the following statements are equivalent.

(i) $T \in \Pi_{p,\sigma}(E,F)$

(ii) There are a constant $C > 0$ and a regular Borel probability measure μ on B_{E^*}, such that

$$\|T(x)\| \leq C \left(\int_{B_{E^*}} \left(|\langle x, x^* \rangle|^{1-\sigma} \|x\|^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu(x^*) \right)^{\frac{1-\sigma}{p}}, \quad x \in E. \tag{3.2}$$

(iii) There is a constant $C \geq 0$ such that for all $(x_i)_{i=1}^n$ in E we have

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \leq C \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^{n} \left(|\langle x^*, x_i\rangle|^{1-\sigma} \|x_i\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$

In addition, $\pi_{p,\sigma}(T)$ is the smallest number C for which (ii) and (iii) hold.

3.1.2 (p,φ)-absolutely continuous linear operators

By applying the Definition 2.2 to the ideals $[\mathcal{A},\alpha] = [\Pi_p, \pi_p]$ and $[\mathcal{B},\beta] = [\mathcal{L}, \|.\|]$ we may consider the Banach ideal $[\Pi_{p,\varphi}, \pi_{p,\varphi}] := \left[(\Pi_p, \mathcal{L})_{\varphi}, \overline{\gamma}\right]$ of all (p,φ)-absolutely continuous operators. By Theorem 2.3, the ideal norm $\pi_{p,\varphi}$ is equivalent to

$$\widetilde{\pi}_{p,\varphi}(T) = \inf_{\lambda} \left\{\varphi\left(\pi_p(S), \lambda\right) : \|Tx\| \leq \varphi\left(\|Sx\|, \lambda\|x\|\right)\right\}.$$

Definition 3.2. *Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. A linear operator $T \in \mathcal{L}(E,F)$ between two Banach spaces E and F is called (p,φ)-absolutely continuous if there exist a Banach space G and a linear operator $S \in \Pi_p(E,G)$ such that*

$$\|T(x)\| \leq \varphi\left(\|x\| \|S(x)\|\right), \quad x \in E. \tag{3.3}$$

Let

$$\widetilde{\pi}_{p,\varphi}(T) = \inf_{\lambda} \left\{\varphi\left(\pi_p(S), \lambda\right) : \|Tx\| \leq \varphi\left(\|Sx\|, \lambda\|x\|\right)\right\},$$

By $\Pi_{p,\varphi}(E,F)$ we denote the Banach space of all (p,φ)-absolutely continuous operators between E and F.

Remark 3.1. *Using defintions 3.1 and 3.2 we get that $\Pi_{p,\sigma}(E,F) = \Pi_{p,\varphi}(E,F)$*

The follwing theorem characterizes (p,φ)-absolutely continuous operators by a special factorization property through a suitable space given by the interpolation method $(.,.)_{\varphi,1}$.

Theorem 3.2. *[7, Theorem 6.10] For every operator $T \in \mathcal{L}(X,Y)$, the following statements are equivalent*

i) T is (p,φ)-absolutely continuous .

ii) There exist a regular probability measure μ on $B_{X'}$ and a constant $C > 0$ such that

$$\|Tx\| \ \leq \varphi\left(C\|J_\mu x\|, \|x\|\right), \quad \forall x \in X,$$

where $J_\mu : X \longrightarrow L_p(\mu)$ is the restriction of the canonical map $J_p : C(B_{X'}) \longrightarrow L_p(\mu)$ and is given by $x \longrightarrow \langle x, . \rangle$.

iii) There exist a regular probability measure μ on $B_{X'}$ and a constant $C > 0$ and an operator $R : \left(X/\ker(J_\mu), L_p(\mu)\right)_{\varphi,1} \longrightarrow Y$ such that $\|R\| \leq C$ and $T = RJ_{\mu,\varphi}$Q.

In other words , the following diagram commutes :

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[d, "Q"] & & Y \\
X/ker(J_\mu) \arrow[r] & \left(X/\ker(J_\mu), L_p(\mu)\right)_{\varphi,1} \arrow[ru, "R"'] &
\end{tikzcd}$$

Bibliography

[1] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators. Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995.

[2] H. Jarchow, Locally convex spaces, B. G. Teubner, Stuttgart, 1981.

[3] H. Jarchow, U. Matter, Interpolative constructions for operator ideals, NoteMat. **8**, 145-156 (1988).

[4] J.A. López Molina and E.A. Sánchez-Pérez, Ideales de operadores absolutamente continuos, Rev. Real Acad. Ciencias Exactas, F ísicas y Naturales, Madrid. 87 (1993) 349-378.

[5] J.A. López Molina and E.A. Sánchez-Pérez, On operator ideals related to (p, σ)-absolutely continuous operator, Studia Math. 131, 8 (2000) 25-40.

[6] L. Maligranda and L. E. Persson, *Generalized duality of some Banach function spaces*, Indag. Math. **51** (1989), 323-338.

[7] Mariusz Piotrowski, Vector Valued Fourier Transforms and Absolutely Continuous Operators, thesis 2005.

[8] U. Matter, Factoring trough interpolation spaces and super-reflexive Banach spaces, Rev. Roum. Math. Pures Appl. 34 (1989) 147-156.

[9] U. Matter, Absolutely continuous operators and super-reflexivity, Math. Nachr. **130**, 193–216 (1987).

[10] A. Pietsch, Operator ideals. Deutsch. Verlag Wiss., Berlin, 1978; North-Holland, Amsterdam-London-New York-Tokyo, 1980.

[11] E.A. Sánchez-Pérez, On the structure of tensor norms related to (p, σ)-absolutely continuous operators, Collect. Math. 47, 1 (1996) 35-46.