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Theme

Bessel Functions in differential equations

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بسم الله الرحمن الرحيم

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Dedication

I dedicate this thesis to:

My father, who guided me and advised me on the path of knowledge and urged me to search for glory and higher degrees,Thank you for your support and encouragement.

My mother, for her support, her sacrifice, her precious advice, her assistance and her presence in my life.

A big thank you to my parents for the values instilled and their permanent support.

My sisters, Imane, Hanine and Rafif .

My whole big family.

All my friends.

To all departement teachers of mathematical and numerical analysis, to all Master batch students of 2020.

Contents

Chapter 1

INTRODUCTION

Bessel functions are named for Friedrich Wilhelm Bessel (1784 - 1846), however, Daniel Bernoulli is generally credited with being the first to introduce the concept of Bessels functions in 1732.

We used to use diferential equations in mathematical modelling when we were pursuing Engineering. Infact mathematical modelling is based on diferential equation. You want to convert any physical system into a mathematical model you must have its diferential equation with you.

Bessel functions are found to appear in practical problems of real situationand are extensively investigated by many scholars in many diverse applications

to a real life situation, where they surface more frequently.

Bessel Equation is something which often occurs in Engineering and Physics which deals with the Cylindrical Coordinates such as Circular Plates, Circular Membranes etc. Bessel Equation actually tried to deal with the singularities happening most of the times. Neuman problems, Vibration of Circular Mem- branes, Heating Equation in plates has a good deal of use of Bessel Equations in them. We do not usually solve Bessel, but the properties of Bessel are the ones which are most useful for us.

In this thesis we have:

In chapter 1 we overview of functional analysis and then in chapter 2 we defined Basic Concepts,and in chapter 3 we shall present the notion of Bessel's equation as a special kind of differential equation and also present their special solution as Bessel functions of different types and their properties. In chapter 4, we used Bessel function to applications in Some Differential Equations Reducible to Bessel's Equation and the Polynomial Approximation of

Bessel Functions and Orthogonality, Finally In chapter 5 Conclusion.

1.1 Objectives

After studying this Unit you should be able to:

- identify Bessel's differential equation
- Knowledge of Bessel functions
- The characteristics of each of the functions of Bessel
- use Bessel functions to solve some equations

1.2 Functional spaces:

in this chapters we need to:

1.2.1 Inner product:

Definition 1 ***(Inner product)****:Let H be a complex vector space. A complex bilinear function/mapping $\langle .,. \rangle : H \times H \to \mathbb{C}$ is called an inner product if for any $f, f_1, f_2, g, g_1, g_2 \in H$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ the following conditions are satisfied:*

$$\langle f, g \rangle = \overline{\langle g, f \rangle}$$

$$\langle \alpha_1 f_1 + \alpha_2 f_2, g \rangle = \alpha_1 \langle f_1, g \rangle + \alpha_2 \langle f_2, g \rangle$$

$$\langle f, f \rangle \geqslant 0 \text{ and } \langle f, f \rangle = 0 \Longleftrightarrow f = 0$$

where $\overline{\langle . \rangle}$ denotes complex conjugate

Definition 2 ***(Induced norm):****Given an inner product $\langle .,. \rangle$ in a vector space H, the norm can be defined by*

$$\|f\| = \langle f, f\rangle^{1/2}$$

or more succinctly

$$\|.\| = \langle ., .\rangle^{1/2}$$

1.2.2 Vector space:

A vector x of $\mathbb{R}^n$ is an ordered collection $x = (x_1, x_2, , ..., x_n)\ '$ of n reals x_j, $j = 1, 2, ..., n$ called components of x. The number n is called the dimension of the n vector.The $\mathbb{R}^n$ space is the set of all collections of this type. It has two operations basic linear:

- Addition: The sum of the two vectors $x, y \in \mathbb{R}^n$ is a vector of $\mathbb{R}^n$, defined by:

$$x + y = (x_1 + y_1; x_2 + y_2; .....; x_n + y_n)' \tag{1.1}$$

- Multiplication by reals: Let $x = (x_1, x_2, , ..., x_n)\ '$ be a vector of $\mathbb{R}^n$. The multiplication of the vector x by the real λ is also a vector of $\mathbb{R}^n$, defined as follows:

$$\lambda x = (\lambda x_1, \lambda x_2, ..., \lambda x_n)' \tag{1.2}$$

The structure we get (the set of all n-dimensional vectors with the two operations that we have just defined) is called the real vector space $\mathbb{R}^n$ $n-$dimensional.

1.2.3 normed vector spaces:

It is an important class of metric spaces, of which Euclidean spaces are the model basic. In general, a vector normed space is a vector space in which there is a metric compatible with the vector space structure.

Definition 3 *(normed vector spaces)*

A space $(E, \|.\|)$ is said to be vector space normed Body $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$ if it has application $\|.\| : E \to \mathbb{R}$ who checks:

1. $\forall x \in E; \|x\| \geqslant 0$ and $\|x\| = 0 \Leftrightarrow x = 0$

2. $\forall \lambda \in \mathbb{k}, x \in E, \|\lambda x\| = |\lambda| \|x\|$ or $|\lambda|$ respectively denotes the absolute value if: $\mathbb{k} = \mathbb{R}$ or the module $\mathbb{k} = \mathbb{C}$

3. $\forall x, y \in E, \|x + y\| \leq \|x\| + \|y\|$ (triangular inequality).

Example 4 *if $(E, \|.\|)$ is a normed vector space, we define the distance associated with a norm by :* $d_{\|.\|}(x, y) = \|x - y\|$

1. on $\mathbb{R}^n$ we can define several normed

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$
$$\|x\|_\infty = \max_{1 \leq i \leq n} \{|x_i|\}$$
$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

2. Vector space $C([0, 1], \mathbb{R})$ can be provided with standards:

$$\|f\|_\infty = \max_{t \in [0,1]} |f(t)|$$
$$\|f\|_1 = \int_0^1 |f(t)| \, dt$$
$$\|f\|_2 = \sqrt{\int_0^1 (f(t))^2 dt}$$

3. On the space of bounded numerical sequences (with value in $\mathbb{R}$ or $\mathbb{C}$), we can define norm

$$\|u\| = \sup_{n \geqslant 0} |u_n|$$

4. Product Norm: if $(E, \|.\|_E)$ and $(F, \|.\|_F)$ are two normed spaces, we can define a norm on the vector space $E \times F$ by

$$\forall (x, y) \in E \times F, \|(x, y)\| = \max \{\|x\|_E, \|y\|_F\}$$

Definition 5 *Let E be a normalized vector space. Two norms $\|.\|_1$ and $\|.\|_2$ of E are said to be equivalent if there is $c_1, c_2 > 0$ such that, for all $x \in E$:*

$$c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1$$

Remark 6 *:*

a. *on* $\mathbb{R}^n$*, norms* $\|.\|_1, \|.\|_2$ *and* $\|.\|_3$ *are equivalent.*

b. on $C([0,1],\mathbb{R})$, norms $\|.\|_1, \|.\|_2$ and $\|.\|_3$ are not equivalent.

c. Two equivalent norms induce two equivalent distances.

1.2.4 The space $L^2(\Omega)$:

Consider the set of square integrable functions defined on Ω. Unless otherwise stated,Ω refers to the closed interval on the real interval $[\alpha, \beta]$. The limits α and β can be finite, but $\alpha = -\infty$ or $\beta = +\infty$ or both are also allowed, in which case the appropriate open interval is meant. A functions $f(x) \in L^2(\Omega)$ satisfies

$$\|f\|_2 = (\int_\Omega |f(x)|^2 \, dx)^{\frac{1}{2}} < \infty$$

1.2.5 Inner product and orthogonality with weighting in $L^2(\Omega)$:

Sometimes it is useful to work with a slightly different inner product in $L^2(\Omega)$, than that given by incorporating a weight function into the definition of the inner product. The definition below is a generalization of the inner product in $L^2(\Omega)$

$$\langle f, g \rangle_w = \int_\Omega w(x) f(x) \overline{g(x)} dx \tag{1}$$

where $w(x) \geq 0$ is a real, non-negative weighting function and reduces to the normal (unweighted) inner product when $w(x) = 1$.In addition, $w(x)$ should be such that the inner product implicit in (1) satisfies all the conditions of an inner product. Two functions φ_m and φ_n in $L^2(\Omega)$ are said to be orthonormal with weight $w(x)$ if

$$\langle\varphi_m,\varphi_n\rangle_w=\int\limits_{\Omega} w(x)\varphi_m(x)\varphi_n(x)dx=\delta_{n,m};\forall m,n\in\mathbb{N}$$

1.3 Hilbert Space

the Hilbert Space H : is a vectorial normed and complet $\Longrightarrow$ inner product integral.

$$\begin{aligned} L^2(\Omega,\|.\|_2) &= \{f:\Omega\rightarrow\mathbb{R}/\int_\Omega |f(x)dx|^2<\infty\} \\ \|f\|_2 &= (\int_\Omega |f(x)|^2\,dx)^{\frac{1}{2}} \end{aligned}$$

1.4 Banach space

Definition 7 *:A complete normed space is said to be a Banach space, B.*

Example 8 *The space $C[a,b]$ of continuous functions defined in a closed*

interval $a\leq x\leq b$ with the (uniform) norm

$$\|f\|_c=\max_{a\leq x\leq b}|f(x)|$$

and the metric

$$\rho(\phi_1,\phi_2)=\|\phi_1-\phi_2\|=\max_{a\leq x\leq b}|\phi_1(x)-\phi_2(x)|$$

is a normed and complete space. Indeed, every fundamental (convergent) functional sequence $\{\phi_n(x)\}$ of continuous functions converges to a continuous function in the $C[a,b]$-metric.

Chapter 2

Basic Concepts and Definitions

2.1 Cauchy's problem

We can limit our selve to the first order equations, because an equation of order $p > 1$ can always lead to a system of p equations of order1. An ordinary differential equation generally admits an unevenness of solutions. select one, or impose an additional level corresponding to the value taken by the solution at one point of the integration interval. $y(t_0) = y_0$, onselection'suniqueolution corresponding to the value of the condition.

Definition 9 *We will therefore consider problems, called Cauchy, of the following form:*

find $y : I \subset \mathbb{R}$ such:

$$\begin{cases} y^{'}(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases} \forall t \in I \tag{PVI}$$

Where $f : I \times \mathbb{R} \to \mathbb{R}$ a given function and$y^{'}$is the derivative of y compared to t finally, t_0 is a point of I and y_0 is a value called initial data. Recalled in the proposal following a conventional analysis result.

Proposition 10 *We assume that the function $f(t, y)$ is:*

* Continue with its two variables.

* Lipchitzienne compared to its second variable, that is to say that there is a constant

positive L (called Lipchitz constant) such as:

$$|f(t,y_1) - f(t,y_2)| < |y_1 - y_2|, \forall t \in I, \forall y_1, y_2 \in \mathbb{R}$$

So the solution $y = y(t)$ of the Cauchy problem exists, is unique and belongs to $C^1(I)$.

Definition 11 *Function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^m$ is lipchitzian in y and uniformen t if there exists a constant $L > 0$ such that:*

$$\forall t \in [a,b], \forall y_1, y_2 \in \mathbb{R}^m \;\; ||f(t,y_1) - f(t,y_2)|| \leq L\,||y_1 - y_2||$$

L is called the lipschitz constant f.

Theorem 12 *(Cauchy-Lipschitz)*

We assume that the function f is continuous on $\mathbb{R} \times \mathbb{R}^+$ and that there exists a real L such that for all $x, y \in \mathbb{R}$ and for all $t \in \mathbb{R}$

we are

$$|f(x,t) - f(y,t)| \leq L(x - y)$$

Then Problem **(PVI)** admits a unique global solution.

2.2 Boundary problem

Boundary problems are differential problems posed over an interval $]a,b[$ over an opening with several dimensions $\Omega \subset \mathbb{R}^d (d = 2,3)$, for which values of the unknown (or its derivatives) are fixed to the ends a and b or on the $\partial\Omega$ in the multidimensional case. Here are some examples of boundary problems Is Ω a domain of $\mathbb{R}$

1. The problem of Dirichlet or First problem of limits:

$$\begin{cases} lu = f & x \in [0,1] \\ u = g & x \in \{0,1\} \end{cases}$$

2. The Neumann problem or second boundary problem:

$$\begin{cases} lu = f & x \in [0,1] \\ D_n u = g & x \in \{0.1\} \end{cases}$$
$$D_n u = \partial u / \partial n = \nabla u.n$$

3. The Dirichlet-Neumann Problem Or Third Boundary Problem:

$$\begin{cases} lu = f & x \in [0,1] \\ D_n u + \alpha u = g & x \in \{0,1\} \end{cases}$$

Such as α a function of x

2.3 Ordinary differential equations

General definitions: differential equation is called an equation establishing a relationship between the independent variable x and the unknown function $y = \varphi(x)$ and its derived $y, y^{'}, ...y^{(n)}$ Symbolically

Definition 13 *the differential order equation n is represented as follows*

$$F(x, y, y^{'}, ..y^{(n)}) = 0 \tag{EDO}$$

where F is a function of $(n+2)$ variables.We only consider the case where x and y have values in $\mathbb{R}$. A solution for such a differential equation over the interval $I \subset \mathbb{R}$ is a function $y \in C^n(I, \mathbb{R})$ (a function $y : I \to \mathbb{R}$ which is n times continuously differentiable) as for everything $x \in I$ we are $F(x, y(x), y^{'\prime}(x), y^{''}(x), ...y^{(n)}(x)) = 0$

Definition 14 *We call the order of a differential equation the most order high of the derivative in this equation*

So,

$$y^{'} - 2xy^2 + 5 = 0$$

is a first order equation. and

$$y^{''} - ky^{'} - by - \sin x = 0$$

is a second order equation, etc.

2.3.1 Linear Differential Equations

A differential equation of order n is linear if and only if it is of the form

$$l(y) = f(x) \tag{1.2}$$

With:

$$l(y) = a_0(x)y + a_1(x)y^{'} + a_2(x)y^{''} + ...a_n(x)y^{(n)}$$

Proposition 15 *The application* $l : C^n \rightarrow C^0$, $l(y)$ *is a linear application. The differential equation*

$$l(y) = 0 \tag{1.3}$$

Is called homogeneous equation

Proposition 16 *The set* S_0 *of the solutions to homogeneous equation is the core of the linear application* l*. it is therefore a vector subspace of* $C^n(\mathbb{R})$*. The set* S *of the solutions to* ***(1.2)*** *is given by:*

$$\begin{aligned} S &= y_p + S_0 \\ &= \{y_p + y_h; y_h \in S_0\} \end{aligned}$$

with

$$l(y_p) = f(x)$$

that is, the solutions are of the form$y = y_p + y_h$, or y_p is a particular solution of (**1.2**), and y_h is solution of the homogene equation (**1.3**)

Theorem 17 *(existence of Peano's)*

let $R(a, b)$ is a rectangle in rectangle in xy plane and the point (x_0, y_0) is inside such that

$$R(a, b) = \{(x, y); |x - x_0| < a. |y - y_0| < b\}$$

if $F(x, y)$ is continous and $|F(x, y)| < M$ at all point $(x, y) \in \mathbb{R}$ then the problem with initial valeus has $y(x)$ a solution that is dèfined for all x over the interval $|x - x_0| < c$ where $c = \min\{a, b/M\}$

Theorem 18 *(uniqueness of Picard and Lindelöf)*

Under the hypotheses of theorem Peano's and if $\frac{\partial f}{\partial y}(x,y)$ is continuous and bounded for all points (x,y) in R, then the problem with initial values has a unique solution $y(x)$

which is defined for all x over an interval $|x-x_0|<c$

2.3.2 Linear Differential Equations of the First Order

A linear differential equation (LDE) of the 1^{st}order is a differential equation which ;can write in the form

$$a(x)y^{'}+b(x)y=c(x)$$

where a,b,c continuous functions to $I\subset\mathbb{R}$ and $\forall x\in I:a(x)\neq 0$;To this differential equation, we can associate the same equation with $c=0$:

$$a(x)y^{'}+b(x)y=0 \tag{1.4}$$

This is the homogeneous equation associated with linear (EDO), or equations without a second member.

2.3.3 Linear Differential Equations of the Second Order

We are now interested in 2nd order differential equations

Definition 19 *Linear differential equations of the 2^{nd}order with coefficients*

constants is a differential equation of the form

$$ay^{''}+by^{'}+cy=f(x)$$

where $a,b,c\in\mathbb{R}$ $(a\neq 0)$ and $f\in C^0(I)$

Definition 20 *The general linear homogeneous second order ODE is given by*

$$y'' + p(x)y' + q(x)y = 0 \tag{1.1.1}$$

has coefficients$p$ and q that are not both constants. However, sometimes we can write a solution $y(x)$ as a power series:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where we use ODE (1.1.1) to where we determine the coefficie a_n .with $p(x) = \frac{1}{x}$ and $q(x) = 1 - \frac{n^2}{x^2}$ We get a Bessel equations

Theorem 21 *.Ordinary Points Theorem*

If x_0 is an ordinary point of ODE (**1.1.1**) that is, if $p(x)$ and $q(x)$ are both analytic at x_0, then the general solution of ODE (**1.1.1**) is given by the series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x) \qquad ((1.1.2))$$

Wherea_0 and a_1 are arbitrary and for each $n \geq 2$, an can be written in terms ofa_0 and a_1.When this is done, we get the right- hand term in formula (1.1.2), where $y_1(x)$ and $y_2(x)$ are

linearly independent solutions of ODE (1.1.1) that are analytic at x_0.

Theorem 22 *Frobenius' Theorem*

If x_0 is a regular singular point of ODE (1.1.1), then there is at least one series solution at x_0 of the form

$$y_1(x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{n + r_1}$$

where r_1 is the larger of the two roots r_1 and r_2 of the indicial equation.
Proof. Here are a few things to keep in mind when finding a Frobenius series.

*The roots of the indicial equation may not be integers, in which case the series representation of the solution would not be a power series, but is still a valid series.

$$y(x) = \sum a_n x^{n+r}$$

*If $r_1 = r_2$ in not an integer, then the smaller root r_2 of the indicial equation generates a second solution of the form

$$y_2(x) = (x - x_0)^{r_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

which is linearly independent of the first solution y_1.

*When $r_1 = r_2$is an integer, a second solution of the form

$$y_2(x) = cy_1(x) \ln(x - x_0) \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$$

exists, where the values of the coefficients bn are determined by finding a recurrence formula, and C is a constant. The solution $y_2(x)$ is linearly independent of $y_1(x)$. ■

Notation 23 *We use Frobenius series to solve Bessel equation We get solutions Bessel functions*

Chapter 3

Bessel Equation and Bessel Functions

we shall present the notion of Bessel's equation as a special kind of differential equation and also present their special solution as Bessel functions of different kinds and their properties

3.1 Bessel Equations

Bessel's equation and Bessel's function occurs in relation with many problems of engineering and physics also there is an extensive literature that deals with the theory and application of this equation and its solution.

Definition 24 *Bessel Functions are solutions of the differential equation*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \qquad (1)$$

Equation (1) is Called the Bessel Differential equation of order n
where n is arbitraryis
the general solution $y(x)$ depends on the number n Further we consider separately two cases:

1. The order$n$ is non-integer

2. The order n is an integer

case1: The order n is non-integer

Assuming that the number n is non-integer and positive, the general solution of the Bessel equation can be written as

$$y(x) = C_1 J_n(x) + C_2 J_{-n}(x)$$

where C_1, C_2 are arbitrary constants and $J_n(x), J_{-n}(x)$ are Bessel functions of the first kind.

The Bessel function can be represented by:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\Gamma(n+k+1)}$$

The Bessel functions of the negative order $(-n)$ are written in similar way:

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{-n+2k}}{k!\Gamma(-n+k+1)}$$

case2: The order n is an integer

If the order n of the Bessel differential equation is an integer, the Bessel functions $J_n(x)$ and $J_{-n}(x)$ can become dependent from each other. In this case the general solution is described by another formula:

$$y(x) = C_1 J_n(x) + C_2 Y_n(x)$$

where $Y_n(x)$ is the Bessel function of the second kind. Sometimes this family of functions is also called Neumann functions

The Bessel function of the second kind $Y_n(x)$ can be expressed in terms of the Bessel functions of the first kind $J_n(x)$ and $J_{-n}(x)$

$$Y_n(x) = \frac{(\cos n\pi) J_n(x) - J_n(x)}{\sin n\pi}$$

Example 25 *the Bessel equation of order* $\frac{1}{2}$:

$$x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0$$

the general solutions is

$$y(x) = C_1 J_{\frac{1}{2}} + C_2 J_{-\frac{1}{2}}$$

Example 26 *the Bessel equation of order* 3

$$x^2y^{''} + xy^{'} + (x^2 - 9)y = 0$$

the general solutions is

$$y(x) = C_1 J_3(x) + C_2 Y_3(x)$$

we need the Gamma Functions we shall defined below;

3.2 Gamma Functions

Definition 27 *The gamma function is defined for $x > 0$ in integral form by the improper integral known as Euler's integral of the second kind.*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

Proposition 28 ***of the gamma function***

For any positive real number α

1. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$

2. $\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$ for $\lambda > 0$

3. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$

4. $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3.......$

5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

3.3 Bessel Functions

In this chapter, we are going to introduce the Bessel functions of the first,second and third kind In order to define these functions,we will solve a differential equation known as Bessel's equation

a) First Kind : $J_n(x)$ in the solution to Bessel's equation is referred to as a Bessel function of the first kind.

b) Second Kind: $Y_n(x)$ in the solution to Bessel's equation is referred to as a Bessel function of the second kind or sometimes the Weber function or the Neumann function.

c) Third Kind: The Hankel function or Bessel function of the third kind

3.3.1 Bessel functions of the first kind

Definition 29 *$J_n(x)$ in the solution to Bessel's equation is referred to as a Bessel function of the first kind of order n and argument x.*

consider the function $J_n(x)$ defined by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\Gamma(n+k+1)}$$

$\forall n \in \mathbb{R}$, $\Gamma$is gamma finction

Some Special Values

In this section we will examine what the Bessel functions look like when some particular values are chosen.

the most use of Bessel functions are

1. $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

2. $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

For proof, see [Book Bessel and Related Functions] p.8

Proposition 30 *:*

$$J_{-n}(x) = (-1)^n J_n(x)$$

Proof.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\Gamma(n+k+1)}$$

for $n = -n$, we get

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k (x/2)^{-n+2k}}{k!\Gamma(-n+k+1)}$$

let $k = n + s$, we get

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} (x/2)^{n+2s}}{\Gamma(n+s+1)\Gamma(s+1)} = (-1)^n J_n(x)$$

■

Proposition 31 *The relationship between the Bessel functions of the first kind of orders zero and one*

$$J_0'(x) = -J_1(x)$$

Proof. We compute the derivative of $J_0(x)$ directly

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\Gamma(n+k+1)}$$

for $n = 0$, we get

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(n!)^2 2^{2n}}$$

$$\Rightarrow J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(n!)^2 2^{2n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(n-1)!n!2^{2n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(n-1)!n!2^{2n-1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2(n+1)-1}}{(n+1-1)!(n+1)!2^{2(n+1)-1}}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

$$= -J_1(x)$$

■

Recurrence Relations for Bessel Functions

Bessel function of the first kind $J_n(x)$ satisfies the following recurrence relations :

(P1) $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$

(P2) $\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$

(P3) $J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$

(P4) $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

(P5) $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

(P6) $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

Proof. For proof (p1)
from the expression of Bessel function

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\Gamma(n+k+1)}$$

we have

$$\frac{d}{dx}\{x^n J_n(x)\} = \frac{d}{dx}\{x^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\Gamma(n+k+1)}\}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{n+2k} k! \Gamma(n+k+1)} \frac{d}{dx} x^{2n+2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k 2(n+k) x^{2n+2k-1}}{2^{n+2k} k! \Gamma - n + k + 1)}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k-1}}{k! \Gamma(n+k)} = x^n J_{n-1}(x)$$

For proof(P2) from expression of Bessel function

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k! \Gamma(n+k+1)}$$

we have

$$\frac{d}{dx}\{x^{-n} J_n(x)\} = \frac{d}{dx}\{x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k-1}}{k! \Gamma(n+k+1)}\}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{n+2k} k! \Gamma(n+k+1)} \frac{d}{dx} x^{2k} = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{2^{n+2k-1} (k-1)! \Gamma(n+k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{n+2k+1} k! \Gamma(n+k+2)} = -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k+1}}{k! \Gamma(n+k+2)}$$

$$= -x^{-n} J_{n+1}(x)$$

For proof (P3) carrying out the differentiation in (P1), we obtain

$$n x^{n-1} J_n(x) + x^n J_n'(x) = x^n J_{n-1}(x)$$

Dividing by x^n throughout and rearranging , we get

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

For proof(P4) carrying out the differentiation in (P2) ,we obtain

$$-n x^{-n-1} J_n(x) + x^{-n} J_n'(x) = -x^{-n} J_{n+1}(x)$$

Mutiplying by x^n and rearranging , we get

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

For proof(P5) Adding (P3) and (P4)the result follows immediately. For proof (P6) subtracting (P4) from (P3), the result follows immediately ■

Integrals Involving Bessel Functions

some integrals involving $J_n(x)$ are obtained

$$\int x^n J_{n-1}(x)dx = x^n J_n(x) + c \qquad ....(1)$$

$$\int x^{-n} J_{n+1}(x)dx = -x^{-n} J_n(x) + c \qquad ....(2)$$

where C is arbitrary constants.

Generating Function For Bessel Functions

Bessel functions of the first kind $J_n(x)$ satisfy the generating expression

$$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Proof. to prove this,we proceed as followes.provided that t is not zero,the exponential function $e^{xt/2}$ and $e^{-xt/2}$ can be expanded in powers of t using Maclaurin's expansion to give

$$e^{\frac{x}{2}(t-1/t)} = e^{xt\ /\ 2}e^{-x\ /\ 2t} = \sum_{k=0}^{\infty} \frac{(xt/2)^k}{k!} \sum_{m=0}^{\infty} \frac{(-x/2t)^m}{m!}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m(1/2)^{k+m}\frac{x^{k+m}t^{k-m}}{k!m!}$$

It will be sufficient to show that the coefficient of t^n in this double summation is $J_n(x)$. First if n is a positive integer or zero ,the coefficient of t^n can be foundby taking $k = n+m$ and letting m vary form 0 to ∞. then the coefficient of $t^n$is

$$\sum_{m=0}^{\infty} \frac{(-1)^m(x/2)^{n+2m}}{(n+m)!m!} = J_n(x)$$

the coefficient of t^{-n} is foud by taking $k = -n + m$ and letting m vary from n to ∞, then the coefficient of t^{-n} is

$$\sum_{m=0}^{\infty} \frac{(-1)^m(x/2)^{-n+2m}}{(-n+m)!m!} = -J_{-n}(x) = (-1)^n J_n(x)$$

then, the total coefficient of t^n for n varying from $-\infty$ to ∞ is $J_n(x)$ and

$$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

■

Proposition 32 *if n is an integer ,*

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y)$$

Proof. From the generating function, we have

$$e^{\frac{(x+y)}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x+y)$$

then $J_n(x+y)$ is the coefficient of t^n in the expression

$$e^{\frac{(x+y)}{2}(t-1/t)}$$

Now,

$$e^{\frac{(x+y)}{2}(t-1/t)} = e^{\frac{x}{2}(t-1/t)} e^{\frac{y}{2}(t-1/t)}$$

$$= \sum_{k=-\infty}^{\infty} t^k J_k(x) \sum_{m=-\infty}^{\infty} t^m J_m(y)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_k(x) J_m(y) t^{k+m}$$

for a particular value of k, the coefficient of t^n is foud by taking $m = n - k$,then the total coefficient of $t^n$is obtained by letting k vary from $-\infty$ to ∞ to get: coefficient of

$$t^n = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y)$$

then

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y)$$

■

Proposition 33 *:*

$$\sum_{n\in\mathbb{Z}} J_n(x) = 1$$

Proof. let $t = 1$ in the generating function then we have

$$e^{\frac{x}{2}(1-1/1)} = 1 = \sum_{n=-\infty}^{+\infty} J_n(x)(1)^n$$

$$= \sum_{n\in\mathbb{Z}} J_n(x)$$

■

Integral Form of Bessel Functions

Using some of the properties seen previously, we can deduce several identities of Bessel functions, known as Bessel's integral formulas

Theorem 34 *(Bessel's Integral Formulas). Let z be a complex number and n*

an integer, Then

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta - n\theta)d\theta$$

moreover
if n is even

$$J_n(z) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(z\sin\theta)\cos n\theta d\theta$$

if n is odd

$$J_n(z) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin(z\sin\theta)\sin n\theta d\theta$$

Proof. For proof we use the generating function ■

3.3.2 Bessel functions of the second kind

Definition 35 *$Y_n(x)$ in the solution to Bessel's equation is referred to as a Bessel function of the second kind of order n and argument x .*

consider the function $Y_n(x)$ defined by
$\forall n \in \mathbb{R}$

$$Y_n(x) = \frac{(\cos n\pi)J_n(x) - J_n(x)}{\sin n\pi} \qquad ((\text{i}))$$

whenever n is not an integer or zero

when n is an integer or zero, the expression on the right of (i) takes the indetermined form(0/0), but

$$\lim_{k\to n} \frac{(\cos k\pi)J_k(x) - J_{-k}(x)}{\sin k\pi}$$

exists

Proposition 36 *We have a*

1. $Y_{n+1/2}(x) = (-1)^{n+1} J_{-n-1/2}(x)$
2. $Y_{-n-1/2}(x) = (-1)^{n+1} J_{n+1/2}(x)$

Proof. we have

$$Y_n(x) = \frac{(\cos n\pi)J_n(x) - J_{-n}(x)}{\sin n\pi}$$

1.Replace n by $(n + 1/2)$, we get

$$Y_{n+1/2}(x) = \frac{(\cos(n\pi + \pi/2)J_{n+1/2}(x) - J_{-n-1/2}(x)}{\sin(n\pi + \pi/2)}$$

$$= (-1)^{n+1} J_{-n-1/2}(x)$$

2.Replace n by $-(n + 1/2)$, we get

$$Y_{-n-1/2}(x) = \frac{\cos(n\pi + \pi/2)J_{-n-1/2}(x) - J_{n+1/2}(x)}{-\sin(n\pi + \pi/2)}$$

$$= (-1)^{n+1} J_{n+1/2}(x)$$

■

Proposition 37 *:*

$$Y_{\frac{1}{2}}(x) = J_{\frac{-1}{2}}(x)$$

Proof. we have

$$Y_n(x) = \frac{(\cos n\pi)J_n(x) - J_{-n}(x)}{\sin n\pi}$$

Replace n by $(-1/2)$, we get

$$Y_{\frac{1}{2}} = \frac{\cos(\frac{\pi}{2})J_{\frac{1}{2}}(x) - J_{\frac{-1}{2}}(x)}{\sin(\frac{\pi}{2})}$$

$$= (0)J_{\frac{1}{2}}(x) - J_{\frac{-1}{2}}(x)$$

$$= J_{\frac{-1}{2}}(x)$$

■

Recurrence Relations for Bessel Functions

the Bessel function of the second kind $Y_n(x)$ satisfies the same recurrence relations as that of the first kind, namely

(P1) $\frac{d}{dx}\{x^n Y_n(x)\} = x^n Y_{n-1}(x)$

(P2) $\frac{d}{dx}\{x^{-n} Y_n(x)\} = -x^n Y_{n+1}(x)$

(P3) $Y_n'(x) = Y_{n-1}(x) - \frac{n}{x} Y_n(x)$

(P4) $Y_n'(x) = \frac{1}{2}[Y_{n-1}(x) - Y_{n+1}(x)]$

(P5) $Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$

(P6) $Y_n'(x) = \frac{n}{x} Y_n(x) - Y_{n+1}(x)$

3.3.3 Bessel functions of the third kind

Definition 38 ***Hankel Functions***

Hankel Functions, often called Bessel functions of the third kind of order n, are defined as

$\forall n \in \mathbb{R}$

$$H_n^{(1)}(x) = J_n(x) + iY_n(x)$$

$$H_n^{(2)}(x) = J_n(x) - iY_n(x)$$

Recurrence Relations for Bessel Functions

Both Hankel functions are infinite at $x = 0$; and their are especially useful fortheir behavior for large values of x. they satisfy recurrence relations identical to $J_n(x)$, i.e.

(P1) $\frac{d}{dx}\{x^n H_n^{(1)}(x)\} = x^n H_{n-1}^{(1)}(x)$

$\frac{d}{dx}\{x^n H_n^{(2)}(x)\} = x^n H_{n-1}^{(2)}(x)$

(P2) $\frac{d}{dx}\{x^{-n} H_n^{(1)}(x)\} = -x^n H_{n+1}^{(1)}(x)$

$\frac{d}{dx}\{x^{-n} H_n^{(2)}(x)\} = -x^n H_{n+1}^{(2)}(x)$

(P3) $\frac{d}{dx} H_n^{(1)}(x) = H_{n-1}^{(1)}(x) - \frac{n}{x} H_n^{(1)}(x)$

$\frac{d}{dx} H_n^{(2)}(x) = H_{n-1}^{(2)}(x) - \frac{n}{x} H_n^{(2)}(x)$

(P4) $\frac{d}{dx} H_n^{(1)}(x) = \frac{1}{2}[H_{n-1}^{(1)}(x) - H_{n+1}^{(1)}(x)]$

$\frac{d}{dx} H_n^{(2)}(x) = \frac{1}{2}[H_{n-1}^{(2)}(x) - H_{n+1}^{(2)}(x)]$

(P5) $H_{n-1}^{(1)}(x) + H_{n+1}^{(1)}(x) = \frac{2n}{x} H_n^{(1)}(x)$

$H_{n-1}^{(2)}(x) + H_{n+1}^{(2)}(x) = \frac{2n}{x} H_n^{(2)}(x)$

(P6) $\frac{d}{dx} H_n^{(1)}(x) = \frac{n}{x} H_n^{(1)}(x) - H_{n+1}^{(1)}(x)$

$\frac{d}{dx} H_n^{(2)}(x) = \frac{n}{x} H_n^{(2)}(x) - H_{n+1}^{(2)}(x)$

Hankel transformation:

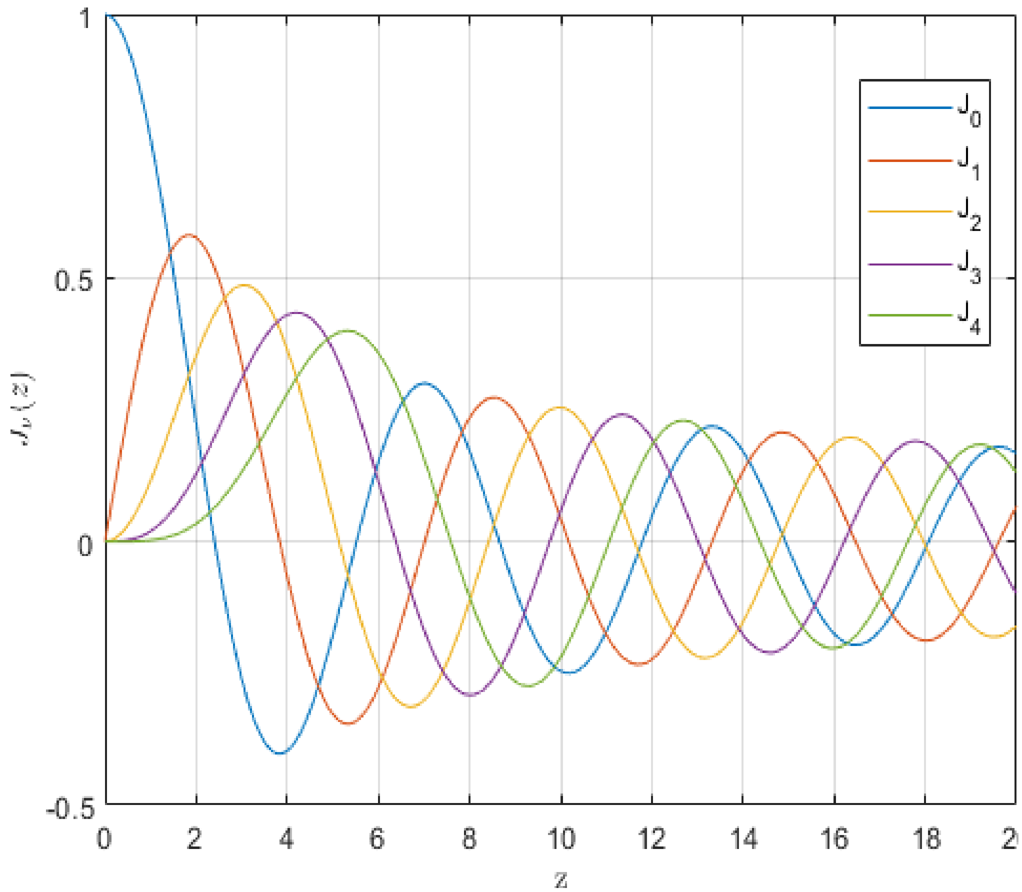
In mathematics, the Hankel transformation, or Fourier-Bessel transformation,

express a given function $f(r)$ as the weighted integral of Bessel functions of

first type $J_v(kr)$. such that $F_v(k) = \int_0^R f(r) J_v(kr) dr$ is the Hankel transform of $f(r)$.

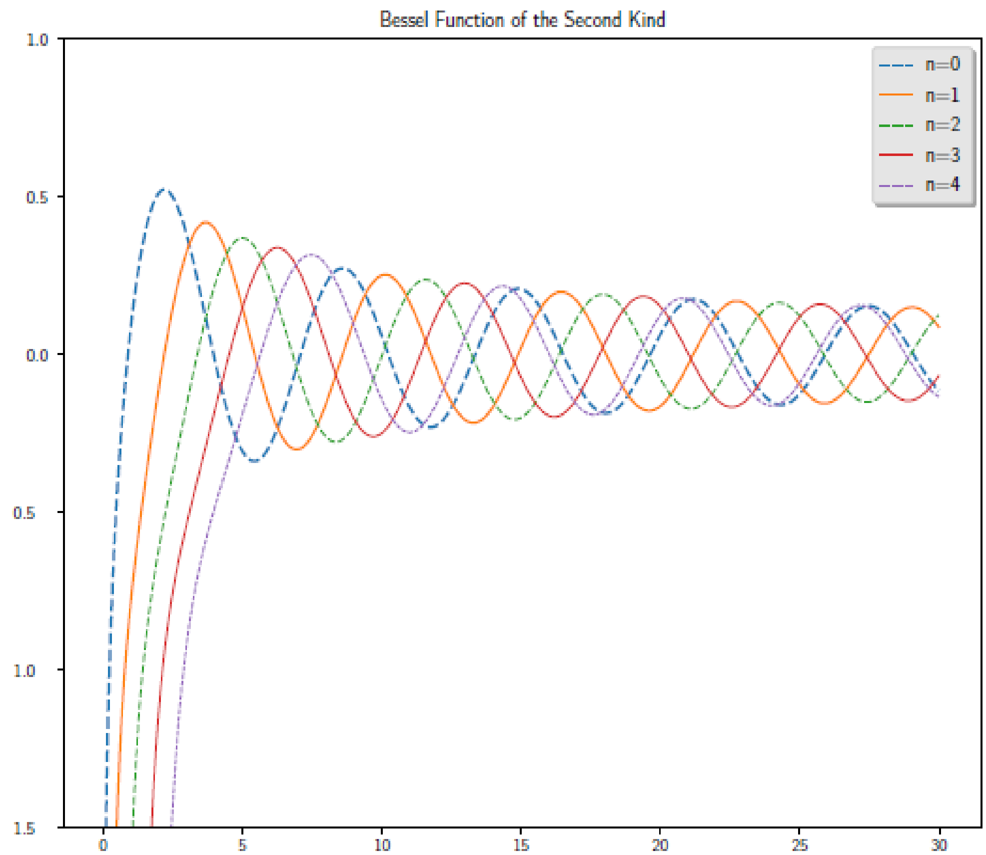
3.4 Graphs of Bessel Function

The following code was implemented in Python to create graphs of the Bessel functions of orders $n = 0; 1; 2; 3; 4$



Graph of First Bessel Function

The following code was implemented in Python to create graphs of the Bessel functions of orders $n = 0; 1; 2; 3; 4$



Graph of Second Bessel Function

Chapter 4

Application

in this chapter . We address Some Differential Equations Reducible to Bessel's Equation and Additionally,we discuss how to use Polynomial Approximation of Bessel Functions and Orthogonality.

4.1 Some Differential Equations Reducible to Bessel's Equation

1. The differential equation of type

$$x^2y^{''} + xy^{'} + (\alpha^2x^2 - n^2)y = 0$$

differs from the Bessel equation only by a factor α^2before x^2 and has the general solution in the form:

$$y(x) = C_1J_n(\alpha x) + C_2Y_n(\alpha x)$$

2. The Airy differential equation known in astronomy and physics has the form:

$$y^{''} - xy = 0$$

It can also be reduced to the Bessel equation. Its solution is given by the Bessel functions of the fractional order $\pm\frac{1}{3}$:

$$y(x) = C_1\sqrt{x}J_{\frac{1}{3}}(\frac{2}{3}ix^{\frac{3}{2}}) + C_2\sqrt{x}J_{\frac{-1}{3}}(\frac{2}{3}ix^{\frac{3}{2}})$$

3. The similar differential equation

$$x^2y^{''} + \alpha xy^{'} + (x^2 - v^2)y = 0$$

is reduced to the Bessel equation

$$x^2z^{''} + xz^{'} + (x^2 - n^2)z = 0$$

by using the substitution

$$y(x) = x^{\frac{1-\alpha}{2}}z(x)$$

Here the parameter n^2 denotes:

$$n^2 = v^2 + \frac{1}{4}(\alpha - 1)^2$$

As a result, the general solution of the differential equation is given by:

$$y(x) = x^{\frac{1-\alpha}{2}}[C_1J_n(x) + C_2Y_n(x)]$$

The special Bessel functions are widely used in solving problems of theoretical physics, for example in investigating

* wave propagation;

* heat conduction;

* vibrations of membranes

in the systems with cylindrical or spherical symmetry.

EXP 1: Solve the differential equation

$$x^2y^{''}+xy^{'}+(3x^2-2)y=0$$

SOL 1: This equation has order $\sqrt{2}$ and differs from the standard Bessel equation only by factor 3

before x^2 Therefore, the general solution of the equation is expressed by the formula

$$y(x)=C_1J_{\sqrt{2}}(\sqrt{3}x)+C_2Y_{\sqrt{2}}(\sqrt{3}x)$$

where C_1, C_2 are constants, $J_{\sqrt{2}}(\sqrt{3}x)$,$Y_{\sqrt{2}}(\sqrt{3}x)$are Bessel functions of the $1st$ and $2nd$ kind, respectively.

EXP 2: the differential equation

$$x^2y^{''}+2xy^{'}+(x^2-1)y=0$$

SOL 2: We make the substitution:

$$\begin{aligned}
y &= x^{\frac{1-2}{2}}z=x^{\frac{-1}{2}}z\Rightarrow y^{'}=-\frac{1}{2}x^{\frac{-3}{2}}z+x^{\frac{-1}{2}}z^{'}\\
\Rightarrow y^{''} &= \frac{3}{4}x^{\frac{-5}{2}}z-\frac{1}{2}x^{\frac{-3}{2}}z^{'}-\frac{1}{2}x^{\frac{-3}{2}}z^{'}+x^{\frac{-1}{2}}z^{''}\\
&= \frac{3}{4}x^{\frac{-5}{2}}z-x^{\frac{-3}{2}}z^{'}+x^{\frac{-1}{2}}z^{''}
\end{aligned}$$

Put these expressions back into the equation:

$$x^2y^{''}+2xy^{'}+(x^2-1)y=0$$

$$\Rightarrow x^2(\frac{3}{4}x^{\frac{-5}{2}}z-x^{\frac{-3}{2}}z^{'}+x^{\frac{-1}{2}}z^{''})+2x(\frac{-1}{2}x^{\frac{-3}{2}}z+x^{\frac{-1}{2}}z^{'})+(x^2-1)x^{\frac{-1}{2}}z=0$$

$$\Rightarrow \frac{3}{4}x^{\frac{-1}{2}}z-x^{\frac{-1}{2}}z^{'}+x^{\frac{3}{2}}z^{''}-x^{\frac{-1}{2}}z+2x^{\frac{1}{2}}z^{'}+x^{\frac{3}{2}}z-x^{\frac{-1}{2}}z=0$$

$$\Rightarrow x^{\frac{3}{2}}z^{''}+x^{\frac{-1}{2}}z^{'}+(-\frac{5}{4}x^{\frac{-1}{2}}+x^{\frac{3}{2}})z=0$$

$$\Rightarrow x^2z^{''}+xz^{'}+(x^2-\frac{5}{4})z=0$$

Indeed, we see that

$$n^2 = v^2 + \frac{1}{4}(\alpha - 1)^2 = 1 + \frac{1}{4}(2-1)^2 = 1 + \frac{1}{4} = \frac{5}{4}$$

Thus, the general solution for the function $z(x)$ can be written in the form

$$z(x) = C_1 J_{\frac{\sqrt{5}}{2}}(x) + C_2 Y_{\frac{\sqrt{5}}{2}}(x)$$

Then the solution for the original function $y(x)$ is given by

$$y(x) = x^{\frac{-1}{2}} z(x) = \frac{1}{\sqrt{x}}[C_1 J_{\frac{\sqrt{5}}{2}}(x) + C_2 Y_{\frac{\sqrt{5}}{2}}(x)]$$

where$C_1$and C_2 are arbitrary constants.

4.2 Polynomial Approximation of Bessel Functions

In this section we will use polynoms for aproximat Bessel function

Method: We imitate Bessel function to polynoms

For $x \in [1, +\infty[$, and $\forall n \in \mathbb{R}$ we have

$$J_n(x) \sim p(x)$$

and

$$Y_n(x) \sim q(x)$$

such that

$$p(x) = \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4} - \frac{n\pi}{2})$$

and

$$q(x) = \sqrt{\frac{2}{\pi x}} \sin(x - \frac{\pi}{4} - \frac{n\pi}{2})$$

Proof. to proof used integrals in both of the integral represenations ■

4.3 Orthogonality

We now explore the orthogonality property of $J_n(\lambda x)$and $J_n(\rho x)$, the Bessel functions of the first kind of order n where n and λ are distinct positive roots of $J_n(x) = 0$. We do this by proving

$$\int_0^1 x J_n(\lambda x) J_n(\rho x) dx = 0$$

A few remarks are in order. Recall the dot product or inner product defined for$\mathbb{R}_n$. Two vectors $X, Y \in \mathbb{R}^n$ are said to be orthogonal if

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i = 0$$

As a generalization we de ne the inner product for the space of Rieman n integrable functions $R[a, b]$ on a closed and bounded interval by

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx$$

We say f, g are orthogonal if $\langle f, g \rangle = 0$. Sometimes, as in the case of Bessel functions, inner products are de ned with a weight function $w(x)$. In this case, we say $f, g \in R[a, b]$ are orthogonal if

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x) dx = 0$$

In the case of Bessel functions, we have $w(x) = x$. We are now ready to prove the following theorem.

Theorem 39 *If* λ*and* ρ *are positive roots of* $J_n(x)$*, then*

$$\int_0^1 x J_n(\lambda x) J_n(\rho x) dx = \begin{cases} 0 & \text{if } \lambda \neq \rho \\ \frac{1}{2} J_{n+1}^2(\lambda) & \text{if } \lambda = \rho \end{cases}$$

Proof. The proof of the Theorem justfies division by

$$\int_0^1 rJ_0(j_n r)J_0(j_n r)dr$$

in the computation of the Fourier Coefficient

$$A_n = \frac{\int_0^1 rJ_0(j_n r)f(r)dr}{\int_0^1 rJ_0(j_n r)J_0(j_n r)dr}$$

for the wave equation in polar coordinates. ■

Chapter 5

CONCLUSION

The Bessel functions appear in many diverse scenarios, particularly situations involving cylindrical symmetry. The most difficult aspect of working with the Bessel function is first determining that they can be applied through reduction of the system equation to Bessel's differential.The constants in each of the solution are to be determined via application of boundary conditions. This shows that the Bessel function appeared in many diverse scenarios, more especially in a situation involving cylindrical symmetry .the general formula for the Bessel function of order n of the second kind $(Y_n(x))$ is extremely complicated. This topic can be greatly expanded upon, and the reader is highly encouraged to review the applications and developme applications of bessel function in the third kind Also.

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Abstract:

In this thesis we studied Bessel functions and series of solutions to a second order differential equation (bessel equation) And some characteristics of applications of Bessel functions in mathematical equations as well as in the physical fields whereas that arise in many diverse situations, Develops the different kinds of Bessel functions equation as applied to that the solution in each case which is present in the solution of Bessel differential equation are the same.

Keywords: Bessel differential equation, Bessel functions, Hankel functions, Neumann functions, Bessel-type functions, linear ordinary differential equations, Hilbert space, gamma function.

Résumé:

Dans cette thèse, nous avons étudié les fonctions de Bessel et les séries de solutions à une équation différentielle du second ordre (équation de bessel) Et certaines caractéristiques des applications des fonctions de Bessel dans les équations mathématiques ainsi que dans les domaines physiques alors que celles qui se posent dans de nombreuses situations diverses, Développe les différents les types d'équation de fonctions de Bessel appliqués à ce que la solution dans chaque cas qui est présente dans la solution de l'équation différentielle de Bessel sont les mêmes.

Mots-clés: équation différentielle de Bessel, fonctions de Bessel, fonctions de Hankel, fonctions de Neumann, fonctions de type Bessel, équations différentielles ordinaires linéaires, espace de Hilbert, fonction gamma.

الملخص:

في هذه الاطروحة قمنا بدراسة دوال بيسل و سلسلة من الحلول لمعادلة تفاضلية من الدرجة الثانية (معادلات بيسل) و بعض خصائص و تطبيقات دوال بيسل في المعادلات الرياضية التي تستعمل في المجال الفيزيائي و التي تنشا في العديد من الوظائف و المجالات المختلفة التي تحدث . في معادلات بيسل الحل المطبق دائما هو دوال بيسل بحيث دائما يكون الحل موجود

الكلمات المفتاحية:

دالة جاما, فضاء هيلبرت, معادلة تفاضلية الخطية , دوال من نوع بيسل , دوال نيومان,.دوال هانكل,.دوال بيسل,معادلات بيسل.