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Lipschitz (q, p, E)-summing operators on injective Lipschitz tensor products of spaces

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Abstract. In this paper, we introduce the notion of (q, p)-mixing operators from the injective tensor product space $E\widehat{\otimes}_\epsilon F$ into a Banach space G which we call (q, p, F)-mixing . In particular, we extend the notion of (q, p, E)-summing operators which is a special case of (q, p, F)-mixing operators to Lipschitz case by studying their properties and showing some results for this notion.

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1. Introduction

Farmer and Johnson in [9] have introduced the notion of Lipschitz p-summing operators and justified that it is good generalization of the concept of linear p-summing operators between two Banach spaces (see[9, Theorem 2]). Next, several authors have developed the summability theory for Lipschitz mappings. They studied the class of Lipschitz summing mappings where linear analogue has found its natural place in the linear operator theory (see [2], [8], [7], [18], [1], [3], [13] and the references therein).

By analogy with the notion of mixing operator between Banach spaces was introduced by Pietsch [17], we will define the notion of (q, p)-mixing linear operators from the injective tensor product space $E\widehat{\otimes}_\epsilon F$ into a Banach space G which we call (q, p, F)-mixing operators and give some results concerning this notion. Also, we extend the notion of (q, p, E)-summing (it is

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a special case of (q,p,F)-mixing operators) that introduced by Maurey in [12] and Kislyakov in [11] to Lipschitz case. Furthermore we will show the domination theorem for Lipschitz (p,E)-summing operators. Other results, in the same vein as Blasco and Signes in [4], are also obtained.

The organization of this paper is as follows. After the introductory one, in the second section, we recall some basic definitions and notations concerning the linear and Lipschitz operators, some facts on sequence spaces, tensor product of Banach spaces and Lipschitz tensor product of pointed metric space and Banach space (see [6]). In section 3, the notion of (q,p,F)-mixing operators from the injective tensor product space $E\widehat{\otimes}_{\epsilon}F$ into a Banach space G is introduced, and its fundamental properties are investigated. Finally in the last section we generalize the notion of (q,p,E)-summing, which is a special case of (q,p,F)-mixing operators to the Lipschitz case that we call Lipschitz (q,p,E)-summing operators by giving Pietsch-domination theorem and inclusion type theorems for this class of operators. Some properties related to this class are also presented.

2. Notations and preliminaries

We introduce some concepts and notations that will be used in this paper. The letters E, F and G will denote Banach spaces. The letters X and Y stand for pointed metric spaces with a base point denoted by 0. The closed unit ball of a Banach space E is denoted by B_E. The dual space of E is denoted by E^*. The class of all bounded linear operators between arbitrary Banach spaces E and F will be denoted by $\mathcal{L}(E,F)$. The symbols $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{N}$ stand for the set of all positive real numbers, of all real numbers and of all natural numbers, respectively. For a Banach space E, denote by $\mathrm{Lip}_0(X,E)$ the space of all Lipschitz mappings $T : X \longrightarrow E$ vanishing at 0, that is $T(0)=0$. Equipped with the norm,

$$\mathrm{Lip}\,(T) := \sup_{x\neq y}\frac{\|T(x)-T(y)\|}{d_X(x,y)},$$

$\mathrm{Lip}_0(X,E)$ becomes a Banach space. For $E=\mathbb{K}$, we put $X^\# = \mathrm{Lip}_0(X,\mathbb{K})$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) this Banach space of Lipschitz real functions is called also Lipschitz dual of X (see [19]) and we denote by $B_{X^\#}$ for its closed unit ball which is a compact Hausdorff space for the topology of pointwise convergence on X. The symbols $W(B_{E^*})$ and $W(B_{X^\#})$ stand for the set of all Borel probability measures defined on B_{E^*} and $B_{X^\#}$, respectively. Let $0\leq p\leq\infty$. The Banach space of all absolutely p-summable sequences $\mathbf{x}=(x_j)_{j\in\mathbb{N}}$, where $x_j\in E$, is denoted by $\ell_p(E)$. The norm is defined by

$$\|\mathbf{x}\|_{\ell_p(E)} = \left(\sum_{j=1}^{\infty}\|x_j\|^p\right)^{\frac{1}{p}} < \infty.$$

The Banach space of all weakly absolutely p-summable sequences $\mathbf{x}=(x_j)_{j\in\mathbb{N}}\subset E$, is denoted by $\ell_p^w(E)$. The norm is defined by

$$\|\mathbf{x}\|_{\ell_p^w(E)} = \sup_{a^*\in B_{E^*}}\left(\sum_{j=1}^{\infty}\left|a^*(x_j)\right|^p\right)^{\frac{1}{p}}.$$

If we have $E=\mathbb{K}$, then $\ell_p(\mathbb{K})$ and $\ell_p^{\omega}(\mathbb{K})$ coincide and we denote $\ell_p(\mathbb{K})$ by ℓ_p. If $1<p\leq q\leq\infty$, we determine the real number $q'(p)$ by $\frac{1}{q'(p)}+\frac{1}{q}=\frac{1}{p}$. A family $\mathbf{x}=(x_j)_j$ of elements of E with $j\in\mathbb{N}$, is called mixed (q,p)-summable if it can be written in the form

$$x_j=\tau_j\cdot x_j^0, \tag{2.1}$$

where $\tau=(\tau_j)_j\in\ell_{q'(p)}$ and $x^0=(x_j^0)_j\in\ell_q^{\omega}(E)$. The class of all mixed (q,p)-summable families is denoted by $\ell_{(q,p)}^m(E)$ which is a Banach space with the norm

$$\mathfrak{m}_{(q,p)}(\mathbf{x})=\inf\|\tau\|_{\ell_{q'(p)}}\left\|x^0\right\|_{\ell_q^{\omega}(E)},$$

where the infimum is taken over all possible representations of $\mathbf{x}$ in the form (2.1). The relationships between the various sequence spaces are given by $\ell_p(E)\subset\ell_{(q,p)}^m(E)\subset\ell_p^{\omega}(E)$, with $\|\mathbf{x}\|_{\ell_p^{\omega}(E)}\leq\mathfrak{m}_{(q,p)}(\mathbf{x})\leq\|\mathbf{x}\|_{\ell_p(E)}$, for all $\mathbf{x}\in\ell_p(E)$.

Let $0<p<q<\infty$ where the real number $q'(p)$ satisfying $\frac{1}{p}=\frac{1}{q}+\frac{1}{q'(p)}$. Following [17, Theorem 2.4], a sequence $(x_j)_{j\in\mathbb{N}}$ in E is mixed (q,p)-summable if and only if

$$\left(\sum_{j=1}^{\infty}\left(\int_{B_{E^*}}\left|x^*(x_j)\right|^q d\mu(x^*)\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}<\infty,$$

for every $\mu\in W(B_{E^*})$. In this case

$$\sup_{\mu\in W(B_{E^*})}\left(\sum_{j=1}^{\infty}\left(\int_{B_{E^*}}\left|x^*(x_j)\right|^q d\mu(x^*)\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}=\mathfrak{m}_{(q,p)}((x_j)_{j\in\mathbb{N}}). \tag{2.2}$$

We define $E\widehat{\otimes}_{\epsilon}F$ to be the completion of the algebraic tensor product $E\otimes F$ with the norm

$$\epsilon_{E\otimes F}(u)=\sup_{\substack{x^*\in B_{E^*}\\ y^*\in B_{F^*}}}\left|\sum_{k=1}^{N}x^*(x_k)y^*(y_k)\right|, \tag{2.3}$$

where $u=\sum\limits_{k=1}^{N}x_k\otimes y_k$ and $E\widehat{\otimes}_{\epsilon}F$ is so-called injective tensor product. We denote its weakly p-norm by

$$\left\|(u_j)_{j=1}^m\right\|_{\ell_p^{\omega}(E\widehat{\otimes}_{\epsilon}F)}=\sup_{\substack{x^*\in B_{E^*}\\ y^*\in B_{F^*}}}\left(\sum_{j=1}^{m}\left|\left\langle u_j,x^*\otimes y^*\right\rangle\right|^p\right)^{\frac{1}{p}},$$

where $\left\langle u_j,x^*\otimes y^*\right\rangle=\sum\limits_{k=1}^{N_j}x^*\left(x_{k,j}\right)y^*\left(y_{k,j}\right)$ for any $u_j=\sum\limits_{k=1}^{N_j}x_{k,j}\otimes y_{k,j}$ in $E\otimes F$.

By $u\left[x^*\right] \in F$ is denoted the action of an element u of the tensor product $X \otimes E$ on vectors in X^*. We denote its weakly (p, F)-norm by

$$\left\|\left(u_j\right)_{j=1}^m\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)} = \sup_{x^* \in B_{E^*}} \left(\sum_{j=1}^m \left\|u_j[x^*]\right\|_F^p\right)^{\frac{1}{p}},$$

where $u_j[x^*] = \sum\limits_{k=1}^{N_j} x^*\left(x_{k,j}\right) y_{k,j}$, for $u_j = \sum\limits_{k=1}^{N_j} x_{k,j} \otimes y_{k,j}$, $x_{k,j} \in E$ and $y_{k,j} \in F$. The following relationship proved by Blasco and Signes in [4] as follow

$$\left\|\left(u_j\right)_{j=1}^m\right\|_{\ell_p^{w}(E\widehat{\otimes}_\epsilon F)} \leq \left\|\left(u_j\right)_{j=1}^m\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)}.$$

We will denoted by $X \boxtimes E$ for the Lipschitz tensor product of a pointed metric space X and a Banach space E, it can be constructed as a subspace of the linear functionals on $\mathrm{Lip}_0(X, E^*)$ and their completion denoted by $X\widehat{\boxtimes}_\varepsilon E$ under the injective Lipschitz cross norm ϵ, defined by

$$\epsilon_{X\boxtimes E}(u) = \sup_{\substack{f \in B_{X^\#} \\ y^* \in B_{E^*}}} \left|\sum_{k=1}^N \left(f(x_k) - f(x_k')\right) y^*(y_k)\right|,$$

where $u = \sum\limits_{k=1}^N \delta_{(x_k, x_k')} \boxtimes y_k$ and $X\widehat{\boxtimes}_\epsilon E$ is so-called Lipschitz injective tensor product (see [6]).

The notion of Lipschitz (q, p)-summing operators introduced by Johnson and Schechtman in [10]. For $q, p \geq 1$, a mapping $T \in \mathrm{Lip}_0(X, E)$ is called Lipschitz (q, p)-summing if there exists a constant $C \geq 0$ such that for all $x_1, ..., x_n, x_1', ..., x_n'$ in X and for all $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$ we have

$$\left(\sum_{i=1}^n \lambda_i \left\|T\left(x_i\right) - T\left(x_i'\right)\right\|_E^q\right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n \lambda_i \left|\left(f\left(x_i\right) - f\left(x_i'\right)\right)\right|^p\right)^{\frac{1}{p}}. \tag{2.4}$$

We denoted by $\pi_{q,p}^L(T)$ for the infimum of such constants in the previous inequality and by $\Pi_{q,p}^L(X, E)$ the space of all Lipschitz (q, p)-summing operators in X into E. Note that if we take $q = p$ we find the definition of Lipschitz p-summing operators of Farmer and Johnson (see[9]).

3. (q, p, F)-mixing operators

Recall that, from [17], a normed (Banach) operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components $\mathcal{I}(E, F) := \mathcal{L}(E, F) \cap \mathcal{I}$ satisfy the following conditions:

- (**OI₀**) $\mathcal{I}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$ and the function $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \longrightarrow \mathbb{R}^+$ satisfies that $(\mathcal{I}(E, F), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F.
- (**OI₁**) $\mathcal{I}$ contains the mappings of the form $x^* \otimes y : x \longmapsto \langle x, x^*\rangle y$ where $x \in E$, $x^* \in E^*$ and $y \in F$. In addition, $\|x^* \otimes y\| = \|x^*\| \|y\|$.

($\mathbf{OI_2}$) The ideal property: if $u \in \mathcal{L}(E, K)$, $v \in \mathcal{I}(K, G)$ and $w \in \mathcal{L}(G, F)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(E, F)$ and $\|w \circ v \circ u\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\|$.

The class of $(q, p, F)-$summing operators is studied in [12, 11, 4].

Definition 3.1. *Let E, F and G be Banach spaces and let $1 \leq p \leq q < \infty$. A bounded operator T from the injective tensor product $E\widehat{\otimes}_\epsilon F$ into G is said to be (q, p, F)-summing if there exist a positive constant C such that, for any finite sequence $(u_i)_{1 \leq i \leq n}$ in $E \otimes F$, we have*

$$\left(\sum_{i=1}^{n} \|T(u_i)\|_G^q\right)^{\frac{1}{q}} \leq C \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{n} \|u_i[a^*]\|_F^p\right)^{\frac{1}{p}}, \tag{3.1}$$

where $u_i[a^*] = \sum\limits_{j=1}^{N_i} a^*\left(a_{j,i}\right) b_{j,i}$, for $u_i = \sum\limits_{j=1}^{N_i} a_{j,i} \otimes b_{j,i}, b_{j,i} \in F$ and $a_{j,i} \in E$. We put

$$\pi_{q,p}^F(T) = \{\inf C : \text{ satisfying } (3.1)\},$$

which is called the (q, p, F)-norm of T, and by $\Pi_{q,p}^F\left(E\widehat{\otimes}_\epsilon F, G\right)$ the space of all (q, p, F)-summing operators which is Banach space endowed with such norm. In the case $q = p$ we simply with $\Pi_p^F\left(E\widehat{\otimes}_\epsilon F, G\right)$ and $\pi_p^F(T)$. For $F = \mathbb{K}$ we have $\Pi_{q,p}^{\mathbb{K}}\left(E\widehat{\otimes}_\epsilon \mathbb{K}, G\right) = \Pi_{q,p}(E, G)$ the space of (q, p)-summing operators between E and G.

The next results can be found in [11, 12, 17], and will be used in the sequel.

Theorem 3.1. *Let $1 \leq p < \infty$. An operator $T \in \Pi_p^F\left(E\widehat{\otimes}_\epsilon F, G\right)$ if and only if there are a probability measure μ on (B_{F^*}, ω^*) and a constant $C > 0$ such that for all $u \in E \otimes F$ we have*

$$\|T(u)\|_E^p \leq C \int\limits_{B_{E^*}} \|u[a^*]\|_F^p \, d\mu(a^*). \tag{3.2}$$

Moreover,

$$\pi_p^F(T) = \{\inf C : \textit{ satisfying } (3.2)\}.$$

The definition of (q, p)-mixing operators is due to Pietsch [17].

Definition 3.2. *An operator $T : E \longrightarrow F$ between Banach spaces is called (q, p) -mixing if there is a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and all $x_1, ..., x_n \in E$ we have*

$$\mathfrak{m}_{(q,p)}(\left(Tx_j\right)_j) \leq C \left\|\left(x_j\right)_j\right\|_{\ell_p^\omega(E)}.$$

The class of all (q, p)-mixing operators from E to F is denoted by $\mathbf{M}_{(q,p)}(E, F)$. In this case, the (q, p)-mixing summing norm $\mathbf{m}_{(q,p)}(T)$ of T is the infimum of such constants C.

Now, we introduce the notion of (q, p, F)-mixing operators from the injective tensor product $E\widehat{\otimes}_\epsilon F$ into the Banach space G.

Definition 3.3. *Let $1 < p \leq q \leq \infty$ and $q'(p) > 1$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{q'(p)}$. A bounded operator T from the injective tensor product $E\widehat{\otimes}_\epsilon F$ into G is called (q,p,F)-mixing if there is a constant $C \geq 0$ such that*

$$\mathfrak{m}_{(q,p)}((Tu_j)_{j=1}^m) \leq C \left\| (u_j)_{j=1}^m \right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)},$$

for arbitrary sequence $(u_j)_{j=1}^m$ in $E \otimes F$ and $m \in \mathbb{N}$. Let us denote by $\mathbf{M}_{(q,p)}^F(E\widehat{\otimes}_\epsilon F, G)$ the class of all (q,p,F)-mixing operators from $E\widehat{\otimes}_\epsilon F$ into G with

$$\mathbf{m}_{(q,p)}^F(T) = \inf C.$$

Note that if $F = \mathbb{K}$, then $\mathbf{M}_{(q,p)}^{\mathbb{K}}(E\widehat{\otimes}_\epsilon \mathbb{K}, G) = \mathbf{M}_{(q,p)}(E,G)$. Also, we have the inclusion $\mathbf{M}_{(q,p)}(E\widehat{\otimes}_\epsilon F, G) \subseteq \mathbf{M}_{(q,p)}^F(E\widehat{\otimes}_\epsilon F, G)$, for every $1 < p \leq q \leq \infty$.

The next proposition and its proof are similar to [17, Theorem 20.1.4] and will be omitted.

Proposition 3.1. *The operator $T \in \mathcal{L}(E\widehat{\otimes}_\epsilon F, G)$ is (q,p,F)-mixing if and only if there is a constant $C \geq 0$ such that*

$$\left(\sum_{j=1}^m \left(\sum_{k=1}^n \left| \langle g_k, Tu_j \rangle_{(G^*,G)} \right|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \leq C \left\| (u_j)_{j=1}^m \right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)} \left\| (g_k)_{k=1}^n \right\|_{\ell_q(G^*)},$$

for every sequences $(u_j)_{j=1}^m \subset E \otimes F$, $(g_k)_{k=1}^n \subset G^*$ and $m,n \in \mathbb{N}$. In this case $\mathbf{m}_{(q,p)}^F(T) = \inf C$.

Remark 3.1. *Obviously $\mathbf{M}_{(\infty,p)}^F(E\widehat{\otimes}_\epsilon F, G) = \Pi_p^F(E\widehat{\otimes}_\epsilon F, G)$ and $\mathbf{M}_{(p,p)}^F(E\widehat{\otimes}_\epsilon F, G) = \mathcal{L}(E\widehat{\otimes}_\epsilon F, G)$. In section 4, we generalize this result to the Lipschitz case.*

It is easily seen that

$$\|T\| \leq \mathbf{m}_{(q,p)}^F(T), \tag{3.3}$$

for all $T \in \mathbf{M}_{(q,p)}^F\left(E\widehat{\otimes}_\epsilon F, G\right)$.

Proposition 3.2. *If $1 \leq p \leq q \leq \infty$, then $\left(\mathbf{M}_{(q,p)}^F, \mathbf{m}_{(q,p)}^F\right)$ is a normed operator ideal.*

Proof. It is not difficult to show that the condition $(\mathbf{OI_0})$ is true. To prove the condition $(\mathbf{OI_1})$. Let $x^* \otimes y^* \in (E\widehat{\otimes}_\epsilon F)^*$ and $g \in G$. By using (2.2), for all $(u_j)_{j=1}^m \subset E \otimes F$ we have

$$\left(\sum_{j=1}^m \left(\int_{B_{G^*}} \left| \langle (x^* \otimes y^*) \otimes g(u_j), g^* \rangle \right|^q d\mu(g^*) \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} = \left(\sum_{j=1}^m \left(\int_{B_{G^*}} \left| \langle \langle u_j[x^*], y^* \rangle g, g^* \rangle \right|^q d\mu(g^*) \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}$$

By taking the supremum over all $\mu \in W(B_{G^*})$ we obtain

$$\mathfrak{m}_{(q,p)}\left(\left((x^* \otimes y^*) \otimes g(u_j)\right)_{j=1}^{m}\right) \leq \|g\| \, \|y^*\| \, \|x^*\| \left\|(u_j)_{j=1}^{m}\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)}.$$

Since $(u_j)_{j=1}^{m}$ is arbitrary it follows that

$$\mathbf{m}_{(q,p)}^{F}\left((x^* \otimes y^*) \otimes g\right) \leq \|g\| \, \|y^*\| \, \|x^*\|.$$

Hence $(x^* \otimes y^*) \otimes g \in \mathbf{M}_{(q,p)}^{F}(E\widehat{\otimes}_\epsilon F, G)$. Now, show the reverse inequality, by (3.3) we have

$$\|g\| \, \|y^*\| \, \|x^*\| = \|(x^* \otimes y^*) \otimes g\| \leq \mathbf{m}_{(q,p)}^{F}\left((x^* \otimes y^*) \otimes g\right).$$

To prove the condition $(\mathbf{OI_2})$, let $A \in \mathcal{L}(E_0\widehat{\otimes}_\epsilon F_0, E\widehat{\otimes}_\epsilon F)$, $T \in \mathbf{M}_{(q,p)}^{F}(E\widehat{\otimes}_\epsilon F, G)$, and $B \in \mathcal{L}(G, G_0)$, where $A = R \otimes S$, $R \in \mathcal{L}(E_0, E)$, and $S \in \mathcal{L}(F_0, F)$. Then by applying Proposition 3.1 we have

$$\begin{aligned}
&\left(\sum_{j=1}^{m}\left(\sum_{k=1}^{n}\left|\left\langle g_{0,k}, BTA(u_{0,j})\right\rangle_{(G_0^*,G_0)}\right|^{q}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\
= \; & \left(\sum_{j=1}^{m}\left(\sum_{k=1}^{n}\left|\left\langle g_{0,k} \circ B^*, TA(u_{0,j})\right\rangle_{(G^*,G)}\right|^{q}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \\
\leq \; & \mathbf{m}_{(q,p)}^{F}(T) \left\|(Au_{0,j})_{j=1}^{m}\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)} \left\|(g_{0,k} \circ B^*)_{k=1}^{n}\right\| \ell_q(G^*) \\
\leq \; & \|B\| \, \|R\| \, \|S\| \, \mathbf{m}_{(q,p)}^{F}(T) \left\|(u_{0,j})_{j=1}^{m}\right\|_{\ell_p^{w,F}(E_0\widehat{\otimes}_\epsilon F_0)} \left\|(g_{0,k})_{k=1}^{n}\right\|_{\ell_q(F_0^*)}.
\end{aligned}$$

Hence $BTA \in \mathbf{M}_{(q,p)}^{F}(E_0\widehat{\otimes}_\epsilon F_0, G_0)$ with $\mathbf{m}_{(q,p)}^{F}(BTA) \leq \|B\| \, \|R\| \, \|S\| \, \mathbf{m}_{(q,p)}^{F}(T)$.

We will present in the following proposition a composition formula for (q, p, F)-mixing operators which is a natural generalization of [17, Proposition 20.2.1].

Proposition 3.3. *Let* $1 < p \leq q < \infty$. *Then*

$$\Pi_q(G, H) \circ \mathbf{M}_{(q,p)}^{F}(E\widehat{\otimes}_\epsilon F, G) \subseteq \Pi_p^{F}(E\widehat{\otimes}_\epsilon F, H).$$

Proof. Take $T \in \mathbf{M}_{(q,p)}^{F}(E\widehat{\otimes}_\epsilon F, G)$ and $S \in \Pi_q(G, H)$. For $u_1, ..., u_m \in E \otimes F$ and $\varepsilon > 0$, we have $Tu_j = \tau_j y_j$ where

$$\begin{aligned}
\left\|(\tau_j)_{j=1}^{m}\right\|_{\ell_{q'(p)}} \left\|(y_j)_{j=1}^{m}\right\|_{\ell_q^{w}(G)} &\leq (1+\epsilon)\mathfrak{m}_{(q,p)}((Tu_j)_{j=1}^{m}) \\
&\leq (1+\epsilon)\mathbf{m}_{(q,p)}^{F}(T) \left\|(u_j)_{j=1}^{m}\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)},
\end{aligned}$$

with $\frac{1}{q'(p)} + \frac{1}{q} = \frac{1}{p}$. Using the assumption we have that

$$\begin{aligned}
\left\|(STu_j)_{j=1}^{m}\right\|_{\ell_p(H)} &\leq \left\|(\tau_j)_{j=1}^{m}\right\|_{\ell_{q'(p)}} \left\|(Sy_j)_{j=1}^{m}\right\|_{\ell_q(H)} \\
&\leq \pi_q(S) \left\|(\tau_j)_{j=1}^{m}\right\|_{\ell_{q'(p)}} \left\|(y_j)_{j=1}^{m}\right\|_{\ell_q^{w}(G)}
\end{aligned}$$

$$\leq \quad (1+\epsilon)\pi_q(S)\mathbf{m}^F_{(q,p)}(T)\left\|(u_j)_{j=1}^m\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)}.$$

Which means that $ST \in \Pi_p^F(E\widehat{\otimes}_\epsilon F, H)$ and $\pi_p^F(ST) \leq \pi_q(S)\mathbf{m}^F_{(q,p)}(T)$.

The proof of the following proposition is inspired to [17, Theorem 20.1.7].

Proposition 3.4. *Let $1 \leq p \leq q < \infty$. An operator $T \in \mathcal{L}(E\widehat{\otimes}_\epsilon F, G)$ is (q,p,F)-mixing if and only if there is a constant $C \geq 0$ such that for all $\nu \in W(B_{G^*})$ we can find $\mu \in W(B_{E^*})$ such that*

$$\left(\int\limits_{B_{G^*}} \left|\langle g, Tu\rangle_{(G^*,G)}\right|^q d\nu(g)\right)^{\frac{1}{q}} \leq C\left(\int\limits_{B_{E^*}} \|u[x^*]\|_F^p \, d\mu(x^*)\right)^{\frac{1}{p}} \tag{3.4}$$

For every $u \in E \otimes F$. In this case $\mathbf{m}^F_{(q,p)}(T) = \inf C$.

Proof. If (3.4) is hold we get

$$\begin{aligned}\left(\sum_{j=1}^m \int\limits_{B_{G^*}} \left|\langle g, Tu_j\rangle_{(G^*,G)}\right|^{\frac{p}{q}} d\nu(g)\right)^{\frac{1}{p}} &\leq C\left(\sum_{j=1}^m \int\limits_{B_{E^*}} \|u_j[x^*]\|_F^p \, d\mu(x^*)\right)^{\frac{1}{p}} \\ &\leq C\left\|(u_j)_{j=1}^m\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)}.\end{aligned}$$

This imply $\mathfrak{m}_{(q,p)}\left((Tu_j)_{j=1}^m\right) \leq C\left\|(u_j)_{j=1}^m\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)}$. Hence $T \in \mathbf{M}^F_{(q,p)}(E\widehat{\otimes}_\epsilon F, G)$ and $\mathbf{m}^F_{(q,p)}(T) \leq C$. If $q \geq 1$, for every $\nu \in W(B_{G^*})$ consider the mapping $J_\nu \in \Pi_q\left(G, L_q(B_{G^*}, \nu)\right)$ defined by $J_\nu(x) = f_x$ for all $x \in G$ where $f_x(g) = \langle g, x\rangle$. It follows from Proposition 3.3, $\pi_q(J_\nu) = 1$ and $T \in \mathbf{M}^F_{(q,p)}(E\widehat{\otimes}_\epsilon F, G)$ that $J_\nu T \in \Pi_p^F\left(E\widehat{\otimes}_\epsilon F, L_q(B_{G^*}, \nu)\right)$ and $\pi_p(J_\nu T) \leq \mathbf{m}^F_{(q,p)}(T)$. Then by Theorem 3.1, there exists $\mu \in W(B_{E^*})$ such that

$$\left(\int\limits_{B_{G^*}} \left|\langle g, Tu\rangle_{(G^*,G)}\right|^q d\nu(g)\right)^{\frac{1}{q}} = \|(J_\nu T)u\|_{L_q(B_{G^*},\nu)} \leq \mathbf{m}^F_{(q,p)}(T)\left(\int\limits_{B_{E^*}} \|u[x^*]\|_F^p \, d\mu(x^*)\right)^{\frac{1}{p}},$$

for all $u \in E \otimes F$.

As an immediate consequence of the preceding proposition we arrive an interesting multiplication formula which is a generalization of [17, Proposition 20.1.8].

Proposition 3.5. *If $0 < p \leq q \leq s \leq \infty$, then*

$$\mathbf{M}^F_{(s,q)}(E\widehat{\otimes}_\epsilon F, G) \circ \mathbf{M}^F_{(q,p)}(E\widehat{\otimes}_\epsilon F, G) \subseteq \mathbf{M}^F_{(s,p)}(E\widehat{\otimes}_\epsilon F, G).$$

The following inclusion follows immediately from the Proposition 3.4.

Proposition 3.6. *If $p_1 \leq p_2 \leq q_2 \leq q_1$, then*

$$\mathbf{M}^F_{(q_1,p_1)}(E\widehat{\otimes}_\epsilon F, G) \subseteq \mathbf{M}^F_{(q_2,p_2)}(E\widehat{\otimes}_\epsilon F, G).$$

Proposition 3.7. *If* $\dfrac{1}{q'(p)} + \dfrac{1}{q} = \dfrac{1}{p}$, *then*

$$\mathbf{M}^F_{(q,p)}(E\widehat{\otimes}_\epsilon F, G) \subseteq \Pi^F_{q'(p),p}(E\widehat{\otimes}_\epsilon F, G).$$

Proof. Suppose that $T \in \mathbf{M}^F_{(q,p)}(E\widehat{\otimes}_\epsilon F, G)$. From Proposition [17, Theorem 16.4.5] we obtain

$$\left\| (Tu_j)_{j=1}^m \right\|_{\ell_{q'(p)}(G)} \leq \mathfrak{m}_{(q,p)} \left((Tu_j)_{j=1}^m \right) \leq \mathbf{m}^F_{(q,p)}(T) \left\| (u_j)_{j=1}^m \right\|_{\ell_p^{w,F}(E\widehat{\otimes}_\epsilon F)}.$$

The next proposition and its proof are similar to [17, Proposition 20.1.13]. We write the proof for the aim of completeness.

Proposition 3.8. *We consider* G *is an intermediate space of* $\{G_0, G_1\}$ *admitting* J*-type* θ*. Let* $0 < p \leq q_0, q_1 \leq \infty$ *and* $\dfrac{1}{q} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}$*. If* $T \in \mathcal{L}(E\widehat{\otimes}_\epsilon F, G)$*, then* $T \in \mathbf{M}^F_{(q_0,p)}(E\widehat{\otimes}_\epsilon F, G_0)$ *and* $T \in \mathbf{M}^F_{(q_1,p)}(E\widehat{\otimes}_\epsilon F, G_1)$ *implies that* $T \in \mathbf{M}^F_{(q,p)}(E\widehat{\otimes}_\epsilon F, G)$*. Furthermore*

$$\mathbf{m}^F_{(q,p)} \left(T : E\widehat{\otimes}_\epsilon F \longrightarrow G \right) \leq \mathbf{m}^F_{(q_0,p)} \left(T : E\widehat{\otimes}_\epsilon F \longrightarrow G_0 \right)^{1-\theta} \mathbf{m}^F_{(q_1,p)} \left(T : E\widehat{\otimes}_\epsilon F \longrightarrow G_1 \right)^{\theta}.$$

Proof. Let $r = q'(p)$. We pose $\dfrac{1}{r} = \dfrac{1}{p} - \dfrac{1}{q}$, $\dfrac{1}{r_0} = \dfrac{1}{p} - \dfrac{1}{q_0}$, and $\dfrac{1}{r_1} = \dfrac{1}{p} - \dfrac{1}{q_1}$, then $\dfrac{1}{r} = \dfrac{1-\theta}{r_0} + \dfrac{\theta}{r_1}$. Let $u_1, ..., u_m \in E \otimes F$ where $\left\| (u_j)_{j=1}^m \right\|_{\ell_q^{w,F}(E\widehat{\otimes}_\epsilon F)} \leq 1$. Taken $\varepsilon > 0$, we have $Tu_j = \tau_{0j} y_{0j}$ and $Tu_j = \tau_{1j} y_{1j}$ such that

$$\left\| (\tau_{0j})_{j=1}^m \right\|_{\ell_{r_0}} \leq (1+\epsilon)\mathbf{m}^F_{(q_0,p)}(T : E\widehat{\otimes}_\epsilon F \longrightarrow G_0) \text{ and } \left\| (y_{0j})_{j=1}^m \right\|_{\ell^w_{q_0}(G)} \leq 1,$$
$$\left\| (\tau_{1j})_{j=1}^m \right\|_{\ell_{r_1}} \leq (1+\epsilon)\mathbf{m}^F_{(q_1,p)}(T : E\widehat{\otimes}_\epsilon F \longrightarrow G_1) \text{ and } \left\| (y_{1j})_{j=1}^m \right\|_{\ell^w_{q_1}(G)} \leq 1.$$

Otherwise, we can assume that $\tau_{0j} > 0$, and $\tau_{1j} > 0$. Put $\tau_j = \tau_{0j}^{1-\theta} \tau_{1j}^{\theta}$ and $y_j = \frac{1}{\tau_j} Tu_j$. Hence

$$\left\| (\tau_j)_{j=1}^m \right\|_{\ell_r} \leq \left\| (\tau_{0j})_{j=1}^m \right\|_{\ell_{r_0}}^{1-\theta} \left\| (\tau_{1j})_{j=1}^m \right\|_{\ell_{r_1}}^{\theta}.$$

Conversely, it follows from

$$\left| b(y_j) \right| \leq \left| b(y_{0j}) \right|^{1-\theta} \left| b(y_{1j}) \right|^{\theta} \text{ for all } b \in G^*.$$

That

$$\left\| (y_j)_{j=1}^m \right\|_{\ell^w_q(G)} \leq \left\| (y_{0j})_{j=1}^m \right\|_{\ell^w_{q_0}(G)}^{1-\theta} \left\| (y_{1j})_{j=1}^m \right\|_{\ell^w_{q_1}(G)}^{\theta} \leq 1.$$

Thus

$$\begin{aligned} \mathfrak{m}_{(q,p)}((Tu_j)_{j=1}^m) &\leq \left\| (\tau_j)_{j=1}^m \right\|_{\ell_{q'(p)}} \left\| (y_j)_{j=1^m} \right\|_{\ell^w_q(G)} \\ &\leq (1+\epsilon)\mathbf{m}^F_{(q_0,p)}(T : E\widehat{\otimes}_\epsilon F \longrightarrow G_0)^{1-\theta} \mathbf{m}^F_{(q_1,p)}(T : E\widehat{\otimes}_\epsilon F \longrightarrow G_1)^{\theta}. \end{aligned}$$

As an improvement of Proposition 3.3 and by investing the proof of [17, Theorem 20.3.1] we have the following proposition.

Proposition 3.9. *For* $1 \leq q \leq \infty$. *Then*

$$\mathbf{M}_{(q,p)}^{F}(E\widehat{\otimes}_{\epsilon}F,G) = \Pi_{q}^{-1}(G,\ell_{q}) \circ \Pi_{p}^{F}(E\widehat{\otimes}_{\epsilon}F,G)$$

.

Proof. We consider $T \in \mathcal{L}(E\widehat{\otimes}_{\epsilon}F,G)$ belongs to $\Pi_{q}^{-1}(G,\ell_{q}) \circ \Pi_{p}^{F}(E\widehat{\otimes}_{\epsilon}F,G)$. Let $u_1,...,u_m \in E \otimes F$ and $b_1,...,b_n \in G^*$. Let S be in $\mathcal{L}(G,\ell_q)$ define by

$$Sy = \left(\langle y,b_1\rangle_{(G^*,G)},...,\langle y,b_n\rangle_{(G^*,G)},0,0,0,...\right).$$

Hence $S \in \Pi_q(G,\ell_q)$ where $\pi_q(S) \leq \left\|(b_k)_{k=1}^{m}\right\|_{\ell_q}$ and thus

$$\begin{aligned}\left(\sum_{j=1}^{m}\left(\sum_{k=1}^{n}\left|\left\langle b_k,Tu_j\right\rangle_{(G^*,G)}\right|^{q}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} &= \left(\sum_{j=1}^{m}\left\|STu_j\right\|_{\ell_q}^{p}\right)^{\frac{1}{p}}\\ &\leq \pi_q^F(ST)\left\|(u_j)_{j=1}^{m}\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_{\epsilon}F)}\\ &\leq \pi_q^{-1}\circ\pi_p^F(T)\left\|(u_j)_{j=1}^{m}\right\|_{\ell_p^{w,F}(E\widehat{\otimes}_{\epsilon}F)}\left\|(b_k)_{k=1}^{m}\right\|_{\ell_q}.\end{aligned}$$

Then, relating to Proposition 3.1, we find $T \in \mathbf{M}_{(q,p)}^{F}(E\widehat{\otimes}_{\epsilon}F,G)$ where $\mathbf{m}_{(q,p)}^{F}(T) \leq \pi_q^{-1}\circ\pi_p^F(T)$. This shows that

$$\mathbf{M}_{(q,p)}^{F}(E\widehat{\otimes}_{\epsilon}F,G) \supseteq \Pi_{q}^{-1}(G,\ell_{q}) \circ \Pi_{p}^{F}(E\widehat{\otimes}_{\epsilon}F,G).$$

The reverse inclusion follows from Proposition 3.3.

4. Lipschitz (q,p,E)-summing mappings

In theory of Lipschitz tensor product ([6]) we have that $X \boxtimes E$ is a subspace of $\mathrm{Lip}_0(X,E^*)'$. In [6, Proposition 5.7], it was shown that the mapping $J : X \boxtimes_{\varepsilon} E \longrightarrow \mathcal{F}((X^{\#},\tau_p);E)$, defined by

$$J(u)(f) = \sum_{i=1}^{n}(f(x_i)-f(y_i))e_i,$$

for $u = \sum\limits_{i=1}^{N}\delta_{(x_i,y_i)}\boxtimes e_i \in X\boxtimes E$ and $f \in X^{\#}$, is an isometric isomorphism where $\mathcal{F}\left(\left(X^{\#},\tau_p\right);E\right)$ the space of all finite rank continuous linear operators from $\left(X^{\#},\tau_p\right)$ to E, where τ_p denotes the topology of pointwise convergence of $X^{\#}$ and a consequence $X\widehat{\boxtimes}_{\varepsilon}E$ (the completion of $X\boxtimes_{\varepsilon}E$) is isometrically isomorphic to the closure in the operator norm topology of $\mathcal{F}((X^{\#},\tau_p);E)$. Thus, we can identify u with $J(u)$ and to define Lipschitz (q,p,E)-summing mappings as follows.

Definition 4.1. *Let* X *be a pointed metric space and* E, F *be Banach spaces. Let* $1 \leq p \leq q < \infty$. *A Lipschitz mapping* T *from the injective Lipschitz tensor product* $X\widehat{\boxtimes}_{\varepsilon}E$ *into* F *is said Lipschitz* (q, p, E)*-summing if there exist a positive constant* C *such that, for any finite sequences* $(u_i)_{1\leq i\leq n}, (u_i')_{1\leq i\leq n}$*in* $X \boxtimes E$ *and all positives numbers* $(c_i)_{1\leq i\leq n}$ *we have*

$$\left(\sum_{i=1}^{n} c_i^q \left\|T\left(u_i\right) - T\left(u_i'\right)\right\|_F^q\right)^{\frac{1}{q}} \leq C \sup_{f\in B_{X^\#}} \left(\sum_{i=1}^{n} c_i^p \left\|u_i\left[f\right] - u_i'\left[f\right]\right\|_E^p\right)^{\frac{1}{p}}, \tag{4.1}$$

where

$$u_i\left[f\right] = \sum_{j=1}^{N_i} \left(f\left(x_{i,j}\right) - f\left(y_{i,j}\right)\right) e_{i,j} \text{ and } u_i'\left[f\right] = \sum_{k=1}^{M_i} \left(f\left(x_{i,k}'\right) - f\left(y_{i,k}'\right)\right) e_{i,k}',$$

for $u_i = \sum\limits_{j=1}^{N_i} \delta_{(x_{i,j}, y_{i,j})} \boxtimes e_{i,j},\ u_i' = \sum\limits_{k=1}^{M_i} \delta_{(x_{i,k}', y_{i,k}')} \boxtimes e_{i,k}' \in X \boxtimes E,\ x_{i,j}, x_{i,k}', y_{i,j}, y_{i,k}' \in X$ *and* $e_{i,j}, e_{i,k}' \in E$.

We denote by $\pi_{q,p}^{L,E}(T) = \inf\left\{C : \text{satisfying } (4.1)\right\}$ which is called the Lipschitz (q, p, E)-norm of T, and by $\Pi_{q,p}^{L,E}\left(X\widehat{\boxtimes}_{\varepsilon}E, F\right)$ the space of all Lipschitz (q, p, E)-summing mappings which is a Banach space under the norm $\pi_{q,p}^{L,E}(T)$. In the case $q = p$ we simply write $\Pi_p^{L,E}\left(X\widehat{\boxtimes}_{\varepsilon}E, F\right)$ and $\pi_p^{L,E}(T)$. For $E = \mathbb{K}$ we have $\Pi_{q,p}^{L,\mathbb{K}}\left(X\widehat{\boxtimes}_{\varepsilon}\mathbb{K}, F\right) = \Pi_{q,p}^{L}\left(\left(X^\#\right)^*, F\right)$, since $X\widehat{\boxtimes}_{\varepsilon}\mathbb{K} \equiv \left(X^\#\right)^*$ (see [6, Proposition 5.7]). It is easily seen that $Lip(T) \leq \pi_{q,p}^{L,E}(T)$ for all $T \in \Pi_{q,p}^{L,E}\left(X\widehat{\boxtimes}_{\varepsilon}E, F\right)$.

We present the domination theorem for this class of operators for $p = q$ by using the techniques found in [5, 15, 16]. For the proof of this domination theorem we use the full general Pietsch domination theorem proved in [15], is an improved version of a similar result in [5] (see also [16]).

Let X, Y and V be (arbitrary) non-void sets, $\mathcal{H}(X; Y)$ be a non-void family of mappings from X to Y, G be a Banach space and K be a compact Hausdorff topological space. Let

$$R\colon K \times V \times G \longrightarrow [0, \infty) \text{ and } S\colon \mathcal{H}(X; Y) \times V \times G \longrightarrow [0, \infty)$$

be arbitrary mappings and $1 \leq q < \infty$. According to [5, 15] a mapping $f \in \mathcal{H}(X; Y)$ is RS-abstract q-summing if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{m} S(f, x_j, b_j)^q\right)^{\frac{1}{q}} \leq C \sup_{\varphi\in K} \left(\sum_{j=1}^{m} R\left(\varphi, x_j, b_j\right)^q\right)^{\frac{1}{q}}, \tag{4.2}$$

for all $x_1, \ldots, x_m \in V$, $b_1, \ldots, b_m \in G$ and $m \in \mathbb{N}$.

Suppose that R is such that the mapping

$$R_{x,b}\colon K \longrightarrow [0, \infty) \text{ defined by } R_{x,b}(\varphi) = R(\varphi, x, b) \tag{4.3}$$

is continuous for every $x \in V$ and $b \in G$. The Pietsch Domination Theorem from [15] reads as follows.

Theorem 4.1. *Suppose that S is arbitrary, R satisfies* (**4.3**) *and let $1 \leq p < \infty$. A mapping $f \in \mathcal{H}\,(X;Y)$ is RS-abstract p-summing if and only if there is a constant $C \geq 0$ and a Borel probability measure μ on K such that*

$$S(f,x,b) \leq C\left(\int_K R\,(\varphi,x,b)^q\,d\mu(\varphi)\right)^{1/q}, \tag{4.4}$$

for all $x \in V$ and $b \in G$.

Theorem 4.2. *Let $1 \leq p < \infty$.An mapping T is belongs to $\Pi_p^{L,E}\left(X\widehat{\boxtimes}_\varepsilon E, F\right)$ if and only if there are Borel probability measure μ on $(B_{X^\#}, \omega^*)$ and a constant $C > 0$ such that for all $u, u' \in X \boxtimes E$ we have*

$$\left\|T\left(u\right)-T\left(u'\right)\right\|_F \leq C\left(\int_{B_{X^\#}} \left\|u\left[f\right]-u'\left[f\right]\right\|_E^p d\mu\right)^{\frac{1}{p}}. \tag{4.5}$$

Moreover, $\pi_p^{L,E}\left(T\right)$ is the least of the constants of inequality (**4.5**)**.**

Proof. Choosing the paraments $V = X\widehat{\boxtimes}_\varepsilon E \times X\widehat{\boxtimes}_\varepsilon E$**,** $G = \mathbb{R}$, $K = B_{X^\#}$, which is a compact Hausdorff space in the topology of pointwise convergence on E, $\mathcal{H}$ is the set of all Lipschitz mappings from $X\widehat{\boxtimes}_\varepsilon E$ to F and R, S are defined by

$$R\colon B_{X^\#} \times (X\widehat{\boxtimes}_\varepsilon E \times X\widehat{\boxtimes}_\varepsilon E) \times \mathbb{R} \longrightarrow [0,\infty),\ R(f,(u,u'),c)) = |c|\left\|u\left[f\right]-u'\left[f\right]\right\|_E$$

$$S\colon \mathcal{H} \times (X\widehat{\boxtimes}_\varepsilon E \times X\widehat{\boxtimes}_\varepsilon E) \times \mathbb{R} \longrightarrow [0,\infty),\ S(T,(u,u'),c) = |c|\|T(u)-T(u')\|_F$$

As $R_{(u,u'),c}(\cdot) := R(\cdot,(u,u'),c))$ is continuous for all $((u,u'),c) \in (X\widehat{\boxtimes}_\varepsilon E \times X\widehat{\boxtimes}_\varepsilon E) \times \mathbb{R}$, by Theorem **4.1** we have that $T : X\widehat{\boxtimes}_\varepsilon E \longrightarrow F$ is Lipschitz (p,E)-summing if and only if there is a Borel probability measure μ on $B_{X^\#}$ and a constant $C \geq 0$ such that

$$\left\|T\left(u\right)-T\left(u'\right)\right\|_F \leq \mathbf{C}\left(\int_{B_{X^\#}} \left\|u\left[f\right]-u'\left[f\right]\right\|_E^p d\mu\right)^{\frac{1}{p}},$$

for all $u, u' \in X \boxtimes E$.

Corollary 4.1. *If $1 \leq p \leq q < \infty$, then*

$$\Pi_p^{L,E}\left(X\widehat{\boxtimes}_\varepsilon E, F\right) \subseteq \Pi_q^{L,E}\left(X\widehat{\boxtimes}_\varepsilon E, F\right).$$

In the paper [16] the authors show a version abstract of Inclusion Principle it follows the line of general version of the Pietsch Domination Theorem.

For technical reasons the present abstract setting is slightly different from the one of the noted above. Let X, Y, V, G, W be (arbitrary) non-void sets, Z a vector space and $\mathcal{H}\,(X;Y)$ be a non-void family of mappings from X to Y. Consider the arbitrary mappings

$$R\colon Z \times G \times W \longrightarrow [0,\infty) \text{ and } S\colon \mathcal{H}\,(X;Y) \times Z \times G \times V \longrightarrow [0,\infty).$$

Let $1 \leq p \leq q < \infty$ and $\lambda \in \mathbb{R}$. Suppose that

$$\sup_{w\in W} \sum_{j=1}^{m} R\left(z_j, g_j, w\right)^p < \infty \text{ and } \sup_{v\in V} \sum_{j=1}^{m} S(f, z_j, g_j, v)^q < \infty,$$

for every positive integer m and $z_1, ..., z_m \in Z$ and $g_1, ..., g_m \in G$. A mapping $f \in \mathcal{H}\left(X;Y\right)$ is say (q,p)-abstract (R,S)-summing (notation $f \in RS_{(q,p)}\left(X;Y\right)$) if there is a constant $C > 0$ so that

$$\left(\sup_{v\in V} \sum_{j=1}^{m} S(f, z_j, g_j, v)^q\right)^{\frac{1}{q}} \leq C \left(\sup_{w\in W} \sum_{j=1}^{m} R\left(z_j, g_j, w\right)^p\right)^{1/p},$$

for all $z_1, \ldots, z_m \in Z$, $g_1, ..., g_m \in G$ and $m \in \mathbb{N}$. We will say that S and R are multiplicatives in the variable Z if

$$R\left(\lambda z, g, w\right) = \left|\lambda\right| R\left(z, g, w\right),$$
$$S\left(f, \lambda z, g, v\right) = \left|\lambda\right| S\left(f, z, g, v\right).$$

Theorem 4.3. *If*

$$\begin{array}{l} p_j \leq q_j \text{ for } j = 1,2, \\ 1 \leq p_1 \leq p_2 < \infty, \\ 1 \leq q_1 \leq q_2 < \infty, \\ \frac{1}{p_1} - \frac{1}{q_1} \leq \frac{1}{p_2} - \frac{1}{q_2}, \end{array} \tag{4.6}$$

and S, R are multiplicatives in the variable Z. Then

$$RS_{(q_1,p_1)}\left(X;Y\right) \subset RS_{(q_2,p_2)}\left(X;Y\right).$$

Theorem 4.4. *If p_j and q_j $(j = 1,2)$, be as in (4.6), then*

$$\Pi^{L,E}_{q_1,p_1}\left(X\widehat{\boxtimes}_\varepsilon E, F\right) \subset \Pi^{L,E}_{q_2,p_2}\left(X\widehat{\boxtimes}_\varepsilon E, F\right).$$

Proof. Let X be a pointed metric space, E, F be Banach spaces and $1 \leq p \leq q < \infty$. Then taking

$$Z = \mathbb{R},\ G = X\widehat{\boxtimes}_\varepsilon E \times X\widehat{\boxtimes}_\varepsilon E,\ W = B_{X^\#},\ V = \{0\}$$

and $\mathcal{H}$ is the set of all Lipschitz mappings from $X\widehat{\boxtimes}_\varepsilon E$ to F and R, S are defined by

$$R\colon \mathbb{R} \times (X\widehat{\boxtimes}_\varepsilon E \times X\widehat{\boxtimes}_\varepsilon E) \times B_{X^\#} \longrightarrow [0,\infty),$$
$$R(c,(u,u'),f) = |c| \|u\left[f\right] - u'\left[f\right]\|_E$$

$$S\colon \mathcal{H} \times \mathbb{R} \times (X\widehat{\boxtimes}_\varepsilon E \times X\widehat{\boxtimes}_\varepsilon E) \times \{0\} \longrightarrow [0,\infty),$$
$$S(T,c,(u,u'),0) = |c| \|T(u) - T(u')\|_F.$$

We have that T is Lipschitz (q,p,E)-summing if and only if it is (q,p) -abstract (R,S)-summing. By Theorem 4.3 we obtains inclusion theorem for this class of operators.

Let us give an example of a Lipschitz (q,p,E)-summing mapping.

Example 4.1. *Let X be a pointed metric space, E, F be Banach spaces and $1 \leq p \leq q < \infty$. Let $S \in \mathrm{Lip}_0(E, F)$. Fix $f_0 \in B_{X^\#}$ and consider $T_{f_0,S} : X\widehat{\boxtimes}_\varepsilon E \to F$ given by $T_{f_0,S}(u) = S(u[f_0])$ for all $u \in X \boxtimes E$. Then $T_{f_0,S}$ is Lipschitz (q, p, E)-summing mapping with $\pi^{L,E}_{q,p}(T_{f_0,S}) \leq Lip(S)$. Indeed, we show that $T_{f_0,S}$ is Lipschitz $(1, E)$-summing. First, $T_{f_0,S}$ is Lipschitz mapping because,*

$$\begin{aligned}
& \left\|T_{f_0,S}(u) - T_{f_0,S}(u')\right\|_F \\
= \; & \left\|S(u[f_0]) - S(u'[f_0])\right\|_F \\
\leq \; & \mathrm{Lip}(S) \left\|u[f_0] - u'[f_0]\right\|_E \\
\leq \; & \mathrm{Lip}(S) \sup_{f \in B_{X^\#}} \left\|u[f] - u'[f]\right\|_E \\
\leq \; & \mathrm{Lip}(S) \sup_{f \in B_{X^\#}} \sup_{x^* \in B_{E^*}} \left|\langle x^*, u[f] - u'[f]\rangle\right| \\
\leq \; & \mathrm{Lip}(S)\, \epsilon(u - u').
\end{aligned}$$

Hence $\left\|T_{f_0,S}(u) - T_{f_0,S}(u')\right\|_F \leq \mathrm{Lip}(S)\, \epsilon(u - u')$. Now, we prove that $T_{f_0,S}$ is Lipschitz $(1, E)$-summing mapping,

$$\begin{aligned}
& \sum_{i=1}^{n} c_i \left\|T_{f_0,S}(u_i) - T_{f_0,S}(u_i')\right\|_F \\
= \; & \sum_{i=1}^{n} c_i \left\|S(u_i[f_0]) - S(u_i'[f_0])\right\|_F \\
\leq \; & \mathrm{Lip}(S) \sum_{i=1}^{n} c_i \left\|u_i[f_0] - u_i'[f_0]\right\|_E \\
\leq \; & \mathrm{Lip}(S) \sup_{f \in B_{X^\#}} \sum_{i=1}^{n} c_i \left\|u_i[f] - u_i'[f]\right\|_E .
\end{aligned}$$

So by inclusion theorem $T_{f_0,S}$ is Lipschitz (q, p, E)-summing and $\pi^{L,E}_{q,p}(T_{f_0,S}) \leq Lip(S)$.

We investigate the idea of Montgomery-Smith and Saab in [14], let X be a pointed metric space and E, F be a Banach spaces. Then every Lipschitz mapping $T : X\widehat{\boxtimes}_\varepsilon E \longrightarrow F$ induces a Lipschitz mapping $\Phi(T) = T^\triangle : E \longrightarrow \mathrm{Lip}_0(X, F)$ define by

$$T^\triangle(e)(x) = T(\delta_{(x,0)} \boxtimes e).$$

Since T is Lipschitz map and ε is Lipschitz cross norm (see [6]), This is clearly to see that $T^\triangle$ is Lipschitz mapping. Next we will study the relation between T and $T^\triangle$ such that T is Lipschitz $(q, p, E)-$summing.

Proposition 4.1. *Let X be a pointed metric space, E, F be Banach spaces and $1 \leq p \leq q < \infty$. We have*

(i) $\Phi\left(\Pi^{L,E}_{q,p}\left(X\widehat{\boxtimes}_\varepsilon E, F\right)\right) \subseteq \mathrm{Lip}_0\left(E, \Pi^{L}_{q,p}(X, F)\right)$.

(ii) If $\Phi\left(\Pi^{L,E}_{q,p}\left(X\widehat{\boxtimes}_\varepsilon E, F\right)\right) \subseteq \Pi^{L}_{q,p}(E, \mathrm{Lip}_0(X, F))$. *Then* $\mathrm{Lip}_0(E, F) = \Pi^{L}_{q,p}(E, F)$.

Proof. (i) Let $T : X\widehat{\boxtimes}_\varepsilon E \longrightarrow F$ be a Lipschitz $(q, p, E)-$summing. We must to show that $T^\triangle(e) \in \Pi^{L}_{q,p}(X, F)$ for all $e \in E$. Let $x_1, ..., x_n, x_1', ..., x_n'$ in X and e in E, then

$$\begin{aligned}
& \left(\sum_{i=1}^{n} c_i^q \left\|T^{\triangle}(e)(x_i) - T^{\triangle}(e)\left(x_i'\right)\right\|_F^q\right)^{\frac{1}{q}} \\
= & \left(\sum_{i=1}^{n} c_i^q \left\|T\left(\delta_{(x_i,0)} \boxtimes e\right) - T\left(\delta_{(x_i',0)} \boxtimes e\right)\right\|_F^q\right)^{\frac{1}{q}} \\
\leq & \ \pi_{q,p}^{L,E}(T) \sup_{f \in B_{X^{\#}}} \left(\sum_{i=1}^{n} c_i^p \left\|\left(\delta_{(x_i,0)} \boxtimes e\right)[f] - \left(\delta_{(x_i',0)} \boxtimes e\right)[f]\right\|_E^p\right)^{\frac{1}{p}} \\
= & \ \pi_{q,p}^{L,E}(T) \sup_{f \in B_{X^{\#}}} \left(\sum_{i=1}^{n} c_i^p \left\|\left\langle f(x_i) - f\left(x_i'\right), e\right\rangle\right\|_E^p\right)^{\frac{1}{p}} \\
\leq & \ \pi_{q,p}^{L,E}(T) \left\|e\right\|_E \sup_{f \in B_{X^{\#}}} \left(\sum_{i=1}^{n} c_i^p \left|f(x_i) - f\left(x_i'\right)\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Then $T^{\triangle}(e) \in \Pi_{q,p}^{L}(X,F)$ and $\pi_{q,p}^{L}(T^{\triangle}(e)) \leq \pi_{q,p}^{L,E}(T) \left\|e\right\|_E$.

(ii) Let x_0 in X and $f_0 \in X^{\#}$ such that $f_0(x_0) = 1$. Let $S \in \mathrm{Lip}_0(E,F)$. Fix $f_0 \in X^{\#}$ and we consider $T_{f_0,S} : X\widehat{\boxtimes}_{\varepsilon}E \to F$ given by $T_{f_0,S}(u) = S(u[f_0])$ for all $u \in X \boxtimes E$, which is Lipschitz $(1,E)$-summing. By inclusion theorem, Lipschitz$(1,E)$-summing implies Lipschitz (q,p,E)-summing, hence $T_{f_0,S}^{\triangle} \in \Pi_{q,p}^{L}\left(E, \mathrm{Lip}_0(X,F)\right)$. Therefore $T_{f_0,S}^{\triangle}$ is Lipschitz mapping. Now, we prove that S is Lipschitz (q,p)-summing mapping,

$$\begin{aligned}
& \pi_{q,p}^{L,E}(T^{\triangle}) \sup_{f \in B_{E^{\#}}} \left(\sum_{i=1}^{n} c_i^p \left|\left(f(e_i) - f\left(e_i'\right)\right)\right|^p\right)^{\frac{1}{p}} \\
\geq & \left(\sum_{i=1}^{n} c_i^q \left\|T_{f_0,S}^{\triangle}(e_i) - T_{f_0,S}^{\triangle}\left(e_i'\right)\right\|_{\mathrm{Lip}_0(X,F)}^q\right)^{\frac{1}{q}} \\
\geq & \left(\sum_{i=1}^{n} c_i^q \left\|T_{f_0,S}\left(\delta_{(x_0,0)} \boxtimes e_i\right) - T_{f_0,S}\left(\delta_{(x_0,0)} \boxtimes e_i'\right)\right\|_F^q\right)^{\frac{1}{q}} \\
= & \left(\sum_{i=1}^{n} c_i^q \left\|S\left(\left(\delta_{(x_0,0)} \boxtimes e_i\right)[f_0]\right) - S\left(\left(\delta_{(x_0,0)} \boxtimes e_i'\right)[f_0]\right)\right\|_F^q\right)^{\frac{1}{q}} \\
= & \left(\sum_{i=1}^{n} c_i^q \left\|S\left(e_i f_0(x_0)\right) - S\left(e_i' f_0(x_0)\right)\right\|_F^q\right)^{\frac{1}{q}} \\
= & \left(\sum_{i=1}^{n} c_i^q \left\|S(e_i) - S\left(e_i'\right)\right\|_F^q\right)^{\frac{1}{q}}.
\end{aligned}$$

This yields the result.

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