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كلية التكنولوجيا
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في يوم الثلاثاء السابع والعشرين من شهر ماي عام ألفين وخمسة وعشرين، وعلى الساعة الواحدة ونصف (13:30) بعد الزوال، إنعقد إجتماع للمجلس العلمي لكليّة التّكنولوجيا، في جلسة عادية بقاعة المناقشات بالكلية، ومن بين النقاط المطروحة للنقاش في جدول الأعمال:

المصادقة على تشكيل لجنة الخبراء للموافقة على المطبوعة البيداغوجية الجامعية

- بعد الإطلاع على طلب الذي تقدمت به الأستاذة : **حرايز توفيق– أستاذ محاضر قسم ب** بالقاعدة المشتركة، بكلية التكنولوجيا بجامعة محمد بوضياف بالمسيلة للموافقة على المطبوعة البيداغوجية ، واستنادا للفصل الثاني للقانون الداخلي في مادتة الرابعة في فقرته الخاصة بسير المطبوعة والدروس على الخط، تمّ تقرير التّالي:

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Mathematics 1
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وتمت الموافقة بالاجماع على هذه المطبوعة.

رئيس المجلس العلمي للكلية

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Lecture Notes on Mathematics 1

Domain: Exact Sciences and Engineering

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Lecture Notes on Mathematics 1 for First-Year STEM Students

Technology, Computer Science, and Natural Sciences

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Abstract

These lecture notes are specifically designed for first-year undergraduate students pursuing degrees in Technology, Computer Science, or Natural Sciences. The material covers foundational concepts essential for STEM disciplines, with a focus on building both theoretical understanding and practical problem-solving skills.

These notes are primarily intended for:

- Freshmen in Computer Science and Information Technology
- First-year Engineering and Technology students
- Natural Sciences undergraduates

Contents

Introduction

The journey through higher education in technology, computer science, and natural sciences demands a robust foundation in mathematical thinking. These lecture notes, Mathematics I, are meticulously crafted to equip first-year students with the essential tools for rigorous problem-solving, logical analysis, and abstract reasoning. Over six carefully structured chapters, we bridge the gap between secondary education and the advanced mathematics required in STEM disciplines. From mastering formal proof techniques in Reasoning Methods and Logic to exploring the algebraic frameworks that underpin modern computing in Algebraic Structures, this course serves as both a toolkit and a language for scientific inquiry.

The progression of topics reflects the interconnected nature of mathematics and its applications. Set Theory and Binary Relations will introduce you to the universal language of modern mathematics, critical for database design (in computer science) and modeling physical systems (in engineering). The study of Real Functions provides the analytical backbone for understanding phenomena ranging from electrical signal processing to biological growth rates. Finite Expansion techniques, often overlooked in early education, will empower you to approximate complex systems—a skill vital for algorithm optimization and experimental physics.

As future innovators in technology and science, you will discover that these concepts are not abstract exercises but the very scaffolding of your disciplines. The Algebraic Structures chapter, for instance, directly informs cryptography (group theory) and machine learning (vector spaces). Approach each chapter as both a theoretical challenge and a practical workshop—every proof technique mastered and every function analyzed strengthens your capacity to engineer solutions, model natural processes, or develop computational systems. Let these notes be your compass in navigating the mathematical landscape that shapes our technological world.

Chapter 1

Mathematical Reasoning Methods

Logic, the bedrock of structured thought, governs how we derive truths, validate arguments, and navigate complexity across disciplines. This chapter delves into the methods of reasoning deductive, inductive, and abductive that underpin mathematical proofs, algorithmic design, and philosophical inquiry. By exploring formal systems (propositional and predicate logic), truth tables, and logical fallacies, we illuminate the mechanisms that distinguish valid inferences from flawed assertions. Through case studies in mathematics, computer science, and everyday decision-making, we unravel how logical frameworks empower clarity, resolve paradoxes, and bridge abstract principles to real-world problem-solving. Prepare to sharpen analytical rigor and master the universal language of reason.

1.0.1 Definitions and Properties

Definition 1.1. *(**Assertion (Proposition)**) An assertion (proposition) is a statement that can be either "true" or "false", which are the truth values, sometimes denoted as "T", "F" or "1", "0" respectively.*

Logical Connectives

1 Negation: Let P be an assertion. The negation of P, denoted as $\neg P$ or $\overline{P}$, is true if P is false, and false if P is true.

p	1	0
$\overline{p}$	0	1

Truth table

2 Conjunction ("and"): Let p and q be two assertions. The conjunction "$p \wedge q$" is true only when both p and q are simultaneously true.

p	1	0	0	1
q	0	1	0	1
$p \wedge q$	0	0	0	1

Truth table

3 Disjunction ("or"): Let p and q be two assertions. The disjunction "$p \vee q$" is true when at least one of the assertions is true.

p	1	0	0	1
q	0	1	0	1
$p \vee q$	1	1	0	1

Truth table

Properties

. De Morgan's Laws:

- $\overline{p \wedge q} = \overline{p} \vee \overline{q}$
- $\overline{p \vee q} = \overline{p} \wedge \overline{q}$

4 **Implication:** For assertions p and q, "$p \implies q$" means "if p then q", equivalent to $\overline{p} \vee q$.

Example: For $x \in \mathbb{R}$, $x > 2 \implies x^2 > 4$.

Exercise 1.2. *Construct the truth table for* $p \implies q$.

a Negation of implication: $\overline{(p \implies q)} \equiv p \wedge \overline{q}$

b The converse $q \implies p$ is not equivalent to the original implication

Example 1.3. *For real numbers:* $x = 1 \implies x^2 = 1$ *(true), but its converse* $x^2 = 1 \implies x = 1$ *(false).*

5 **Logical Equivalence:** $p \iff q$ means both $p \implies q$ and $q \implies p$ hold ("if and only if").

1.1 Quantifiers

Let E be a non-empty set.

1. **Universal Quantifier ($\forall$):** "$\forall x \in E, p(x)$" means $p(x)$ holds for all x in E.

   Example 1.4. $\forall x \in \mathbb{R},\ x^2 \geq 0$ *(true)*

   Example 1.5. $\forall x \in \mathbb{R},\ x^2 \geq x$ *(false; counterexample:* $x = 0.5$*)*

2 **Existential Quantifier ($\exists$):** "$\exists x \in E, p(x)$" means there exists at least one x in E satisfying $p(x)$.

   Example 1.6. $\exists x \in \mathbb{R},\ x(x-1) < 0$ *(true, e.g.,* $x = 0.8$*)*

   Remark 1.7. *Unique existence:* $\exists! x \in \mathbb{R},\ f(x) = 0$ *means exactly one solution exists.*

Quantifier Negation Rules

- $\neg(\forall x \in E, p(x)) \equiv \exists x \in E, \neg p(x)$

- $\neg(\exists x \in E, p(x)) \equiv \forall x \in E, \neg p(x)$

Exercise 1.8. *Formalize these statements using quantifiers:*

1. *Every positive integer is greater than any negative integer*
2. *Commutativity of real number addition*

Solutions:

1. $\forall n \in \mathbb{N},\ \forall m \in \mathbb{Z}^-,\ n > m$
2. $\forall x, y \in \mathbb{R},\ x + y = y + x$

1.2 Mathematical Reasoning Methods

1.2.1 Direct Reasoning

To show that the implication "$p \implies q$" is true, we assume p is true and demonstrate that q must then be true. This is the most familiar method.

Example 1.9. *Show that if $a, b \in \mathbb{Q}$, then $a + b \in \mathbb{Q}$.*

Proof. Let $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$. Then $a = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$. Similarly, $b = \frac{p'}{q'}$ for some $p' \in \mathbb{Z}$ and $q' \in \mathbb{N}^*$. Now:

$$a + b = \frac{p}{q} + \frac{p'}{q'} = \frac{pq' + p'q}{qq'}$$

The numerator $pq' + p'q \in \mathbb{Z}$, and the denominator $qq' \in \mathbb{N}^*$. Thus, $a + b$ can be expressed as $\frac{p''}{q''}$ with $p'' \in \mathbb{Z}$ and $q'' \in \mathbb{N}^*$. Therefore, $a + b \in \mathbb{Q}$. ☐

1.2.2 Contrapositive Reasoning

Contrapositive reasoning relies on the equivalence:

$$\text{The implication "}p \implies q\text{" is equivalent to "}\overline{q} \implies \overline{p}\text{"}$$

Thus, to prove "$p \implies q$", we instead show that if $\overline{q}$ holds, then $\overline{p}$ must hold.

Example 1.10. *Let $n \in \mathbb{N}$. Show that if n^2 is even, then n is even.*

Proof. Assume n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{N}$. Squaring both sides:

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2l + 1 \quad (l \in \mathbb{N})$$

Hence n^2 is odd. By contraposition, if n^2 is even, then n must be even. ☐

1.2.3 Proof by Contradiction

To prove "$p \implies q$" by contradiction, assume both p is true and q is false, then derive a logical contradiction.

Example 1.11. *Let $a, b \geq 0$. Show that if $\dfrac{a}{1+b} = \dfrac{b}{1+a}$, then $a = b$.*

Proof. Assume $a \neq b$. Cross-multiplying:

$$\begin{gathered} a(1+a) = b(1+b) \\ a^2 + a = b^2 + b \\ (a-b)(a+b) = -(a-b) \end{gathered}$$

Since $a \neq b$, we divide by $(a-b)$ to get $a + b = -1$. But $a, b \geq 0$ implies $a + b \geq 0$—a contradiction. Thus, $a = b$. ☐

1.2.4 Counterexample Method

To disprove a universal statement "$\forall x \in E, p(x)$", it suffices to find one $x \in E$ where $p(x)$ fails.

Example 1.12. *Is the statement "$\forall x \in \mathbb{R}, x^2 \geq x$" true or false? Justify.*

Proof. The statement is false. Take $x = 0.5$: then $x^2 = 0.25$, and $0.5 > 0.25$. ☐

1.2.5 Mathematical Induction

To prove an assertion $P(n)$ holds for all $n \in \mathbb{N}$:

- **Base case:** Verify $P(0)$.
- **Inductive step:** Assume $P(n)$ holds for some $n \geq 0$; prove $P(n+1)$.
- **Conclusion:** By induction, $P(n)$ holds $\forall n \in \mathbb{N}$.

Example 1.13. *Prove that $\forall n \in \mathbb{N},\ 2^n > n$.*

Proof. **Base case:** For $n = 0$, $2^0 = 1 > 0$.
Inductive step: Assume $2^n > n$. Then:

$$2^{n+1} = 2 \cdot 2^n = 2^n + 2^n > n + 2^n \geq n + 1 \quad (\text{since } 2^n \geq 1)$$

Thus, $2^{n+1} > n + 1$. By induction, the inequality holds for all $n \geq 0$. ☐

Remark 1.14. *If proving a property for $n \geq n_0$, initialize at n_0.*

1.3 Exercises

Exercise 01

Fill in the blanks with the appropriate logical connective: $\Leftrightarrow$, $\Leftarrow$, $\Rightarrow$

1. $x \in \mathbb{R}$ $\quad x^2 = 4$ ______ $x = 2$
2. $z \in \mathbb{C}$ $\quad z = \overline{z}$ ________ $z \in \mathbb{R}$
3. $x \in \mathbb{R}$ $\quad x = \pi$ ________ $e^{2ix} = 1$

Exercise 02

Consider the four statements:

$$(a)\ \exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ x + y > 0; \qquad (b)\ \forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x + y > 0;$$
$$(c)\ \forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ x + y > 0; \qquad (d)\ \exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ y^2 > x.$$

1. Determine the truth value of each statement.
2. Write their negations.

Exercise 03

Let $a, b \in \mathbb{R}_+$. Show that if $a \leq b$, then $a \leq \dfrac{a+b}{2} \leq b$ and $a \leq \sqrt{ab} \leq b$.

Exercise 04

Prove that for all $n \in \mathbb{N}$, $n(n+1)$ is divisible by 2.
Hint: Consider even and odd n.

Exercise 05

Let $k, k' \in \mathbb{N}^*$. Show that $kk' = 1 \implies k = k' = 1$.

Exercise 06

Using proof by contradiction, show that $\sqrt{2}$ is irrational.

Exercise 07

1. Is the implication $x < 2 \implies x^2 < 4$ true for all $x \in \mathbb{R}$?
2. Is the statement "$\forall x \in \mathbb{R}, x^2 \geq x$" true or false? Justify.

Exercise 08

1. Prove the following formula:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{N}$$

2. Let $x \geq 0$. Show that for all integers $n \geq 1$, $(1+x)^n \geq 1+nx$.

Chapter 2

Sets, Binary Relations, and Applications

Set theory, the universal language of mathematical abstraction, provides the framework for organizing elements into collections and defining relationships between them. At its core lie binary relations, which formalize connections between pairs of elements from one or two sets, enabling the study of order, equivalence, and functional mappings. This chapter explores the foundational concepts of sets, Cartesian products, and key relation types (such as equivalence relations, partial orders, and applications), while illustrating their applications in computer science, database design, and social network analysis. By bridging abstract structures like the power set or graph representations to real-world phenomena such as optimization and classification, we uncover how relations between sets serve as versatile tools for modeling interdependence, solving combinatorial problems, and encoding the logic of systems.

2.1 Set Theory

Definitions and Properties

Definition 2.1. *A set E is by definition a collection of objects called elements of E.*

Examples: $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are sets (both can have their elements enumerated). The real numbers also form a set (denoted $\mathbb{R}$) but cannot be enumerated.

- A special set is the empty set, denoted $\varnothing$, containing no elements.

We write $(x \in E)$ if x is an element of E, and $x \notin E$ otherwise.

- Alternative set definition: A collection of elements satisfying a property.

Example 2.2. $A = \{x \in \mathbb{R} \mid |x-2| < 1\}$, $B = \{z \in \mathbb{C} \mid z^5 = 1\}$.

Subset ($\subset$): For sets E and F, if every element of E belongs to F (i.e., $\forall x \in E,\ x \in F$), we say E is a subset of F, written $E \subset F$.

Equality: $E = F$ if and only if $E \subset F$ and $F \subset E$.

Power Set: Denoted $\mathcal{P}(E)$, it contains all subsets of E. For example, if $E = \{1, 2, *\}$, then:

$$\mathcal{P}(E) = \{\{1\}, \{2\}, \{*\}, \{1, 2\}, \{1, *\}, \{2, *\}, \{1, 2, *\}\}$$

Complement: For $A \subset E$, the complement of A in E is:

$$A_E^c = \{x \in E \mid x \notin A\}$$

Also written $E \setminus A$ or simply C_A (sometimes A^c or $\overline{A}$).

Example 2.3. *If* $E = \{0, 1, 2, 3, 4, 5, 6\}$ *and* $A = \{2, 3, 5\}$*, then:*

$$A^c = \{0, 1, 4, 6\}$$

Clearly $A \cap A^c = \varnothing$ and $A \cup A_E^c = E$.

Union, Intersection, Symmetric Difference

For subsets A, B of E:

- Union: $A \cup B = \{x \in E \mid x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x \in E \mid x \in A \text{ and } x \in B\}$
- Symmetric Difference: $A \triangle B = (A \setminus B) \cup (B \setminus A) = A \cup B \setminus A \cap B$

Properties: For subsets A, B, C of E:

1. Commutativity: $A \cap B = B \cap A$
2. Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$
3. Identity: $A \cap \varnothing = \varnothing$, $A \cap A = A$
4. Subset Relation: $A \subset B \iff A \cap B = A$
5. Union Identity: $A \cup \varnothing = A$, $A \cup A = A$
6. Subset Union: $A \subset B \iff A \cup B = B$
7. Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$
8. Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
10. Involution: $(A^c)^c = A$; thus $A \subset B \iff B^c \subset A^c$
11. De Morgan: $(A \cap B)^c = A^c \cup B^c$
12. De Morgan: $(A \cup B)^c = A^c \cap B^c$

Exercise 2.4. *Prove properties (11) and (12).*

Cartesian Product: For sets E and F, the Cartesian product $E \times F$ is:

$$E \times F = \{(x, y) \mid x \in E \text{ and } y \in F\}$$

2.2 Binary Relations

2.2.1 Order Relations

Definition 2.5. *A binary relation* $\mathcal{R}$ *on a non-empty set* E *is an order relation if:*

i) **Reflexive:** $\forall x \in E,\ x\mathcal{R}x$

ii) **Antisymmetric:** $\forall x, y \in E,\ x\mathcal{R}y$ and $y\mathcal{R}x \implies x = y$

iii) **Transitive:** $\forall x, y, z \in E,\ x\mathcal{R}y$ and $y\mathcal{R}z \implies x\mathcal{R}z$

Example 2.6. $\leq$, $\geq$, *and* $=$ *are order relations on* $\mathbb{R}$ *(and* $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$*). The inclusion relation* $\subset$ *is an order on* $\mathcal{P}(\Omega)$.

Total Order vs. Partial Order: $\mathcal{R}$ is a total order if any two elements are comparable: $\forall x, y \in E,\ (x\mathcal{R}y$ or $y\mathcal{R}x)$. Otherwise, it's a partial order.

Example 2.7. $\leq$ *is a total order on* $\mathbb{R}$. $\subset$ *is a partial order on* $\mathcal{P}(\Omega)$.

Vocabulary for Ordered Sets:
Let $\leq$ denote an order relation on E.

1. **Maximum/Minimum:** An element $a \in A$ is the maximum (resp. minimum) of $A \subset E$ if $\forall x \in A,\ x \leq a$ (resp. $a \leq x$). Denoted $\max A$ and $\min A$.

   Example 2.8. *For* $\leq$ *on* $\mathbb{R}$, *if* $A = \{2, -4, 0, 2, 5, 1\}$, *then* $\max A = 5$, $\min A = -4$.

2 **Upper/Lower Bounds:** An element $z \in E$ is an upper bound (resp. lower bound) of $A \subset E$ if $\forall x \in A,\ x \leq z$ (resp. $z \leq x$).

   Example 2.9. *For* $\leq$ *on* $\mathbb{Z}$, *if* $A = \{2, -4, 0, 2, 5, 1\}$, *upper bounds are* $\{5, 6, 7, \ldots\}$, *lower bounds are* $\{-4, -5, -6, \ldots\}$.

3 **Supremum/Infimum:** If A is bounded above and the set of upper bounds has a minimum, this is the supremum ($\sup A$). Analogously for infimum ($\inf A$).

   Remark 2.10. *If* A *has a maximum, then* $\sup A = \max A$.

   Proof. If $a = \max A$, then a is the least upper bound. For any upper bound b, $a \leq b$, making $a = \sup A$. □

Exercise 2.11. *For the ordered set* $(E, \leq)$, *determine (if they exist)* $\max A$, $\min A$, $\sup A$, $\inf A$ *for:*

1. $E = \mathbb{R},\ A = \{0, 1, -5, 3, 5, -2\}$
2. $E = \mathbb{R},\ A = [-4, 2[$
3. $E = \mathbb{N},\ A = \{0, 1, 5, 3, 6\}$
4. $E = \mathbb{R},\ A =]-1, 1[$
5. $E = [-1, 1],\ A = \{\cos \frac{7n\pi}{2} \mid n \in \mathbb{Z}\}$
6. $E = \mathbb{R},\ A = \{x^2 - 1 \mid x \in \mathbb{R}\}$

2.2.2 Equivalence Relations

Definition 2.12. *Let E be a set and $\mathcal{R}$ a binary relation on E. $\mathcal{R}$ is an equivalence relation if:*

- $\forall x \in E$, $x\mathcal{R}x$ (reflexivity)
- $\forall x, y \in E$, $x\mathcal{R}y \implies y\mathcal{R}x$ (symmetry)
- $\forall x, y, z \in E$, $x\mathcal{R}y$ and $y\mathcal{R}z \implies x\mathcal{R}z$ (transitivity)

Example 2.13. *The "parallel to" relation is an equivalence relation on the set E of affine lines in a plane.*

The "same age" relation on a set of people is an equivalence relation.
Equivalence Class

Definition 2.14. *Let E be a set equipped with an equivalence relation $\mathcal{R}$. The equivalence class of an element $x \in E$ is the subset of E defined by:*

$$\mathrm{cl}(x) = \{y \in E \mid y\mathcal{R}x\}$$

Also denoted $\overset{o}{x}$ or $\overline{x}$. If $y \in \mathrm{cl}(x)$, y is called a representative of $\mathrm{cl}(x)$.

Definition 2.15. *A **partition** of a set E is a collection $\{E_i\}$ of subsets of E such that $E = \bigcup_i E_i$ and $E_i \cap E_j = \varnothing$ for $i \neq j$.*

Proposition 2.16. *Let E be a set with an equivalence relation $\mathcal{R}$. We have:*

1. $\mathrm{cl}(x) = \mathrm{cl}(y) \iff x\mathcal{R}y$
2. For all $x, y \in E$, either $\mathrm{cl}(x) = \mathrm{cl}(y)$ or $\mathrm{cl}(x) \cap \mathrm{cl}(y) = \varnothing$
3. Let D be a set of representatives for all classes. Then $\{\mathrm{cl}(x) \mid x \in D\}$ forms a partition of E.

The Set $\mathbb{Z}/n\mathbb{Z}$
Let $n \geq 2$ be an integer. Define the relation on $\mathbb{Z}$:

$$a \equiv b \pmod{n} \iff a - b \text{ is a multiple of } n$$

Examples for $n = 6$: $26 \equiv 2 \pmod 6$, $34 \equiv 4 \pmod 6$, $-3 \equiv 21 \pmod 6$.
This is an equivalence relation because:

- Reflexive: $a - a = 0 = 0 \cdot n$
- Symmetric: $a \equiv b \pmod n \implies b \equiv a \pmod n$
- Transitive: $a \equiv b \pmod n$ and $b \equiv c \pmod n \implies a \equiv c \pmod n$

Equivalence Classes

The equivalence class of $a \in \mathbb{Z}$ is:

$$\begin{aligned}\overset{o}{a} &= \{b \in \mathbb{Z} \mid b \equiv a \pmod{n}\} \\ &= \{b \in \mathbb{Z} \mid b = a + nk,\ k \in \mathbb{Z}\}\end{aligned}$$

Noting that:

$$\overset{o}{n} = \overset{o}{0},\ \overset{o}{n+1} = \overset{o}{1},\ \overset{o}{n+2} = \overset{o}{2},\ \ldots$$

The set of equivalence classes is:

$$\mathbb{Z}/n\mathbb{Z} = \{\overset{o}{0}, \overset{o}{1}, \overset{o}{2}, \ldots, \overset{o}{n-1}\}$$

Example for $n = 5$:

. $\overset{o}{0} = 5\mathbb{Z} = \{\ldots, -15, -10, -5, 0, 5, 10, \ldots\}$

. $\overset{o}{1} = \{\ldots, -14, -9, -4, 1, 6, 11, \ldots\}$

. $\overset{o}{2} = \{\ldots, -13, -8, -3, 2, 7, 12, \ldots\}$

. $\overset{o}{3} = \{\ldots, -12, -7, -2, 3, 8, 13, \ldots\}$

. $\overset{o}{4} = \{\ldots, -11, -6, -1, 4, 9, 14, \ldots\}$

This partitions $\mathbb{Z}$ into 5 classes.

Exercise 2.17. *Define addition in $\mathbb{Z}/7\mathbb{Z}$ by $\overset{o}{a} + \overset{o}{b} = \overset{o}{a+b}$. Construct the addition table for $\mathbb{Z}/7\mathbb{Z}$. Repeat for multiplication $\overset{o}{a} \times \overset{o}{b} = \overset{o}{a \times b}$.*

2.3 Applications

2.3.1 General Concepts

Definition 2.18. *An application f from set E to set F assigns to each $x \in E$ a unique element $f(x) \in F$ (written $f : E \to F$).*

E is the domain, F is the codomain.

Example 2.19. *The identity application $Id_E : E \to E$ is defined by $Id_E(x) = x$ for all $x \in E$.*

Equality: Two applications $f, g : E \to F$ are equal if $f(x) = g(x)$ for all $x \in E$, denoted $f = g$.

Graph: The graph of $f : E \to F$ is the subset of $E \times F$:

$$\Gamma_f = \{(x, f(x)) \in E \times F \mid x \in E\}$$

Composition: For $f : E \to F$ and $g : F \to G$, the composition $g \circ f : E \to G$ is defined by:

$$g \circ f(x) = g(f(x))$$

$$E \xrightarrow{f} F \xrightarrow{g} G$$

Example 2.20. *Consider the applications:*

$$f :]-\infty, 0[\to]0, \infty[, \quad x \mapsto -\frac{1}{x}$$

$$g :]0, \infty[\to \mathbb{R}, \quad x \mapsto \sqrt{x}$$

Then $g \circ f :]-\infty, 0[\to \mathbb{R}$ *is:*

$$g \circ f(x) = \sqrt{-\frac{1}{x}}$$

2.3.2 Direct Image, Inverse Image

Let E and F be two non-empty sets, and let U be an application from E to F ($U : E \longrightarrow F$).

Definition 2.21. *Let A be a subset of E. The* ***direct image*** *of A under U is the subset of F denoted by $U(A)$, defined as:*

$$U(A) = \{U(x) \in F \mid x \in A\}$$

Definition 2.22. *Let B be a subset of F. The* ***inverse image*** *of B under U is the subset of E denoted by $U^{-1}(B)$, defined as:*

$$U^{-1}(B) = \{x \in E \mid U(x) \in B\}$$

Let $E = [0, 1]$ and $F = [-1, 0]$ be intervals of $\mathbb{R}$. Consider the application $U : E \to F$ defined by $U(x) = x^2 - 1$.

1. Determine $U\left(\left]0, \frac{1}{2}\right[\right)$ and $U^{-1}\left(\left]-\frac{1}{2}, 0\right[\right)$.

   Solution:
   For $U\left(\left]0, \frac{1}{2}\right[\right)$:

$$\begin{aligned} x \in \left]0, \tfrac{1}{2}\right[&\implies 0 < x < \tfrac{1}{2} &&\implies 0 < x^2 < \tfrac{1}{4} \\ &\implies -1 < x^2 - 1 < -\tfrac{3}{4} \end{aligned}$$

Thus, $U\left(\left]0, \frac{1}{2}\right[\right) = \left]-1, -\frac{3}{4}\right[$.

For $U^{-1}\left(\left]-\frac{1}{2}, 0\right[\right)$:

$$\begin{aligned} U(x) \in \left]-\tfrac{1}{2}, 0\right[&\implies -\tfrac{1}{2} < x^2 - 1 < 0 &&\implies \tfrac{1}{2} < x^2 < 1 \\ &\implies \tfrac{1}{\sqrt{2}} < |x| < 1 &&\implies \begin{cases} \frac{1}{\sqrt{2}} < x < 1 \\ -1 < x < -\frac{1}{\sqrt{2}} \end{cases} \end{aligned}$$

Since $x \in E = [0, 1]$, we get $U^{-1}\left(\left]-\frac{1}{2}, 0\right[\right) = \left]\frac{1}{\sqrt{2}}, 1\right[$.

2.3.3 Injective, Surjective, and Bijective applications

Let E and F be two non-empty sets, and let U be an application from E to F ($U : E \longrightarrow F$).

Definition 2.23. *U is* ***injective*** *if for all $x, x' \in E$, $U(x) = U(x') \implies x = x'$. Equivalently:*

$$\forall x, x' \in E,\ U(x) = U(x') \implies x = x'$$

Definition 2.24. *U is* ***surjective*** *if for every $y \in F$, there exists $x \in E$ such that $y = U(x)$. Equivalently:*

$$\forall y \in F,\ \exists x \in E \text{ such that } y = U(x)$$

1. U is injective if and only if every $y \in F$ has at most one preimage.
2. U is surjective if and only if every $y \in F$ has at least one preimage.

Example 2.25. *1. The application*

$$U : \mathbb{N} \longrightarrow \mathbb{Q}, \quad x \longmapsto U(x) = \frac{1}{2+x}$$

is injective. If $U(x) = U(x')$, then:

$$\frac{1}{2+x} = \frac{1}{2+x'} \implies x = x'.$$

Example 2.26. *2. The application*

$$f : \mathbb{Z} \longrightarrow \mathbb{Q}, \quad x \longmapsto f(x) = \frac{1}{2+x^2}$$

is not injective. For example, $f(-2) = f(2)$, but $-2 \neq 2$.

Example 2.27. *3. The function*

$$g : \mathbb{R} \longrightarrow \mathbb{R}_+, \quad x \longmapsto g(x) = x^2$$

is surjective. For every $y \in \mathbb{R}_+$, $x = \sqrt{y}$ satisfies $g(x) = y$.

Example 2.28. *4. The application*

$$h : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto h(x) = x^2$$

is not surjective because negative real numbers have no preimage.

Bijection:

Definition 2.29. *U is* ***bijective*** *if it is both injective and surjective. Equivalently:*

$$\forall y \in F,\ \exists!\, x \in E \text{ such that } y = U(x).$$

Proposition 2.30. *Let $U : E \longrightarrow F$ be an application.*

1. *U is bijective if and only if there exists an application $g : F \longrightarrow E$ such that:*

$$U \circ g = Id_F \quad \text{and} \quad g \circ U = Id_E.$$

2. *If U is bijective, the inverse $g = U^{-1}$ is unique and bijective.*

Application Example

Let $U :]0, +\infty[\longrightarrow]0, 1[$ be defined by $U(x) = \frac{1}{\sqrt{x+1}}$.

1. Determine $U^{-1}\left(\left]\frac{1}{2}, \frac{\sqrt{3}}{2}\right]\right)$ and $U\left(]2, 4]\right)$.
2. Prove that U is bijective and find U^{-1}.

Solution:
1. For $U^{-1}\left(\left]\frac{1}{2}, \frac{\sqrt{3}}{2}\right]\right)$:

$$U^{-1}(B) = \left\{x \in]0, +\infty[\mid \frac{1}{2} < \frac{1}{\sqrt{x+1}} \leq \frac{\sqrt{3}}{2}\right\} = \left[\frac{1}{3}, 3\right[.$$

For $U\left(]2, 4]\right)$:

$$2 < x \leq 4 \implies \frac{1}{\sqrt{5}} \leq \frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{3}} \implies U\left(]2, 4]\right) = \left[\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{3}}\right[.$$

2. **Bijectivity:** - *Injectivity:* If $U(x) = U(x')$, then $\frac{1}{\sqrt{x+1}} = \frac{1}{\sqrt{x'+1}} \implies x = x'$. - *Surjectivity:* For $y \in]0, 1[$, solve $y = \frac{1}{\sqrt{x+1}} \implies x = \frac{1}{y^2} - 1$.
The inverse application is:

$$U^{-1} :]0, 1[\longrightarrow]0, +\infty[, \quad U^{-1}(y) = \frac{1}{y^2} - 1.$$

Chapter 3

Real Functions

Real-valued functions form the cornerstone of mathematical analysis, bridging abstract theory with practical applications across science, engineering, and economics. Defined as mappings from a subset of the real numbers to another, these functions assign a unique output $f(x)$ to each input (x) in their domain. From simple linear relationships like $f(x) = 2x+1$ to complex transcendental expressions, they encapsulate phenomena such as growth, motion, and optimization. This chapter explores their fundamental properties (domain, range, continuity, and differentiability), while examining graphical interpretations and algebraic manipulations. By analyzing these functions, we unlock tools to model real-world systems, solve equations, and understand the behavior of dynamic processes through the lens of calculus and beyond.

3.1 Definitions and properties

Definition 3.1. *A* ***real-valued function*** *defined on a domain X is any mapping f that assigns to each point x in X a unique element y in $\mathbb{R}$. We write:*

$$f : X \to \mathbb{R}$$

$$x \mapsto y = f(x)$$

Here, X is the domain of definition of f, and the set of values of f (or the image of f) is:

$$f(X) = Im\ f = \{y \in \mathbb{R} \mid \exists x \in X; y = f(x)\}$$

Definition 3.2. *Graph of a Function The* ***graph*** *of a function f is the set of points $M(x, y)$ where $x \in X$ and $y = f(x)$. We write:*

$$G_f = \{M(x, y) \mid x \in X \text{ and } y = f(x)\}$$

Operations on Real-Valued Functions

Let $f, g : X \to \mathbb{R}$.

Equality and Inequality

1. f is equal to g, written $f = g$, if:
$$f(x) = g(x) \quad \forall x \in X$$
2. f is less than or equal to g, written $f \leq g$, if:
$$f(x) \leq g(x) \quad \forall x \in X$$
3. f is greater than or equal to g, written $f \geq g$, if:
$$f(x) \geq g(x) \quad \forall x \in X$$

Arithmetic Operations

- Sum: $(f + g)(x) = f(x) + g(x)$, $\forall x \in X$
- Difference: $(f - g)(x) = f(x) - g(x)$, $\forall x \in X$
- Product: $(f \cdot g)(x) = f(x)g(x)$, $\forall x \in X$
- Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $\forall x \in X$, $g(x) \neq 0$

Composition of Functions

Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ such that $f(X) \subset Y$. The **composition** of f and g, denoted $g \circ f$, is defined on X by:

$$(g \circ f)(x) = g(f(x)); \quad \forall x \in X$$

Example 3.3. *Let $f(x) = \cos x$ and $g(x) = x^2$, where $x \in \mathbb{R}$. Then:*

$$(f \circ g)(x) = \cos(g(x)) = \cos(x^2)$$
$$(g \circ f)(x) = g(f(x)) = (f(x))^2 = \cos^2 x$$

Clearly, $g \circ f \neq f \circ g$.

General Properties of Functions

Even and Odd Functions

A set $X \subset \mathbb{R}$ is called **symmetric with respect to the origin** if:

$$\forall x \in X \Rightarrow -x \in X$$

Definition 3.4. *A function f defined on a symmetric set X is:*

1. ***Even** if:*
$$\forall x \in X, f(-x) = f(x)$$
2. ***Odd** if:*
$$\forall x \in X, f(-x) = -f(x)$$

Periodicity

Definition 3.5. *Let $f : X \to \mathbb{R}$. The function f is called* ***periodic*** *if there exists $\alpha \in \mathbb{R}_+$ such that:*

1. $x + \alpha \in X$
2. $f(x + \alpha) = f(x), \forall x \in X$

It is evident that:

$$f(x + k\alpha) = f(x)$$

Definition 3.6. *The* ***period*** *of f is the smallest positive number T such that:*

$$f(x + T) = f(x)$$

Monotonicity

Definition 3.7. *Let $f : X \to \mathbb{R}$. The function f is called:*

1. ***Increasing*** *if:*
$$\forall x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$
2. ***Strictly increasing*** *if:*
$$\forall x_1, x_2 \in X, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$
3. ***Decreasing*** *if:*
$$\forall x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$
4. ***Strictly decreasing*** *if:*
$$\forall x_1, x_2 \in X, x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Bounded Functions

Definition 3.8. *A function f is called:*

1. ***Bounded above*** *on X if there exists $M \in \mathbb{R}$ such that:*
$$f(x) \leq M, \forall x \in X$$
2. ***Bounded below*** *on X if there exists $m \in \mathbb{R}$ such that:*
$$f(x) \geq m, \forall x \in X$$
3. ***Bounded*** *on X if it is both bounded above and below, i.e., there exist $m, M \in \mathbb{R}$ such that:*
$$m \leq f(x) \leq M, \forall x \in X$$
or equivalently, there exists $c \in \mathbb{R}_+$ such that:
$$|f(x)| \leq c, \forall x \in X$$
4. ***Unbounded*** *if for every $c \in \mathbb{R}_+$, there exists $x' \in X$ such that:*
$$|f(x')| > c$$

Supremum and Infimum of Functions

Definition 3.9. *The* ***supremum*** *(resp.* ***infimum****) of f on X is the smallest upper bound (resp. largest lower bound) of f, denoted by:*

$$\sup f \quad (\textit{resp. } \inf f)$$

For a function $f : X \to \mathbb{R}$:

$$M = \sup_{x\in X} f(x) \iff \begin{cases} 1/\forall x \in X, f(x) \leq M \\ 2/\forall \varepsilon > 0, \exists x_0 \in X, f(x) > M - \varepsilon \end{cases}$$

$$m = \inf_{x\in X} f(x) \iff \begin{cases} 1/\forall x \in X, f(x) \geq m \\ 2/\forall \varepsilon > 0, \exists x_1 \in X, f(x) < m + \varepsilon \end{cases}$$

Theorem 3.10. *Every function that is bounded above (resp. below) admits a supremum (resp. infimum).*

Maximum and Minimum of a Function

Let $f : X \to \mathbb{R}$.

Definition 3.11. *The function f is said to have a* ***maximum*** *(resp.* ***minimum****) at a point $x_0 \in X$ if:*

$$\forall x \in X, f(x) \leq f(x_0) \quad (\textit{resp. } f(x) \geq f(x_0))$$

Inverse Functions

Definition 3.12. *Let $f : X \to \mathbb{R}$. The function f is called:*

1. ***Injective*** *if:*

$$\forall x_1, x_2 \in X, (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)) \quad \textit{or} \quad (x_1 = x_2 \Rightarrow f(x_1) = f(x_2))$$

2. ***Surjective*** *if:*

$$\forall y \in \mathbb{R}, \exists x \in X, y = f(x)$$

3. ***Bijective*** *if it is both injective and surjective.*

Definition 3.13. *A function $f : X \to Y$ is called* ***invertible*** *if there exists a function $g : Y \to X$ such that:*

$$(g \circ f)(x) = x \quad \textit{and} \quad (f \circ g)(y) = y$$

The function g is called the ***inverse*** *of f and is denoted by $g = f^{-1}$. We have:*

$$y = f(x) \iff x = f^{-1}(y)$$

Properties 3.14. *Let $f : X \to Y$ be invertible (bijective). Then:*

1. *The inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.*

2. *If f is odd (resp. even), then f^{-1} is also odd (resp. even).*

3. *If f is strictly monotone, then f^{-1} is also strictly monotone.*

Graph of an Inverse Function

Proposition 3.15. *Let G_f be the graph of an invertible function f, and let $G_{f^{-1}} = \{(y, f^{-1}(y)) \mid y \in Y\}$ be the graph of f^{-1}. In the Cartesian coordinate system Oxy, we have:*

$$(y,x) \in G_{f^{-1}} \iff x = f^{-1}(y), y \in Y \iff y = f(x), x \in X \iff (x,y) \in G_f$$

Thus, $G_{f^{-1}}$ is symmetric to G_f with respect to the first bisector $y = x$.

Elementary Functions

The following functions are called **elementary functions**:

1. **Constant functions**: $f(x) = c, \forall x \in X$

Power Functions

The **power function** is defined as:

$$f(x) = x^\alpha, \quad \alpha \in \mathbb{R}$$

The domain D_f of the power function depends on the value of α:

$$\begin{cases} \alpha = n : & D_f = \mathbb{R} \\ \alpha = -n : & D_f = \mathbb{R}^* \\ \alpha = \frac{p}{q} : & f(x) = x^{\frac{p}{q}} = \sqrt[q]{x^p}; \\ D_f = \begin{cases} \mathbb{R}, & \text{if } q \text{ is odd} \\ \mathbb{R}_+, & \text{if } q \text{ is even} \end{cases} \end{cases}$$

Exponential Functions

The **exponential function** with base a ($a > 0$, $a \neq 1$) is defined as:

$$f(x) = a^x, \quad D_f = \mathbb{R}, \quad \text{Im } f = \mathbb{R}_+$$

The function f is:

- **Increasing** if $a > 1$
- **Decreasing** if $0 < a < 1$

Additionally, $f(0) = 1$.

Logarithmic Functions

The **logarithmic function** with base a ($a > 0$, $a \neq 1$) is defined as:

$$y = \log_a x \iff x = a^y, \quad D_f = \mathbb{R}^*, \quad \operatorname{Im} f = \mathbb{R}$$

The logarithmic function satisfies:

$$x = 1 \Rightarrow y = 0$$

The function $y = \log_a x$ is:

- **Increasing** if $a > 1$
- **Decreasing** if $0 < a < 1$

The graph of $y = \log_a x$ is symmetric to the graph of $y = a^x$ with respect to the first bisector $y = x$.

Trigonometric Functions

The **trigonometric functions** include:

- Sine (sin)
- Cosine (cos)
- Tangent (tan)
- Cotangent (cot)

Inverse Trigonometric Functions

The **inverse trigonometric functions** include:

- Arcsine (arcsin)
- Arccosine (arccos)
- Arctangent (arctan)

Trigonometric Functions

Sine Function

The **sine function** is defined as:

$$y = \sin x$$

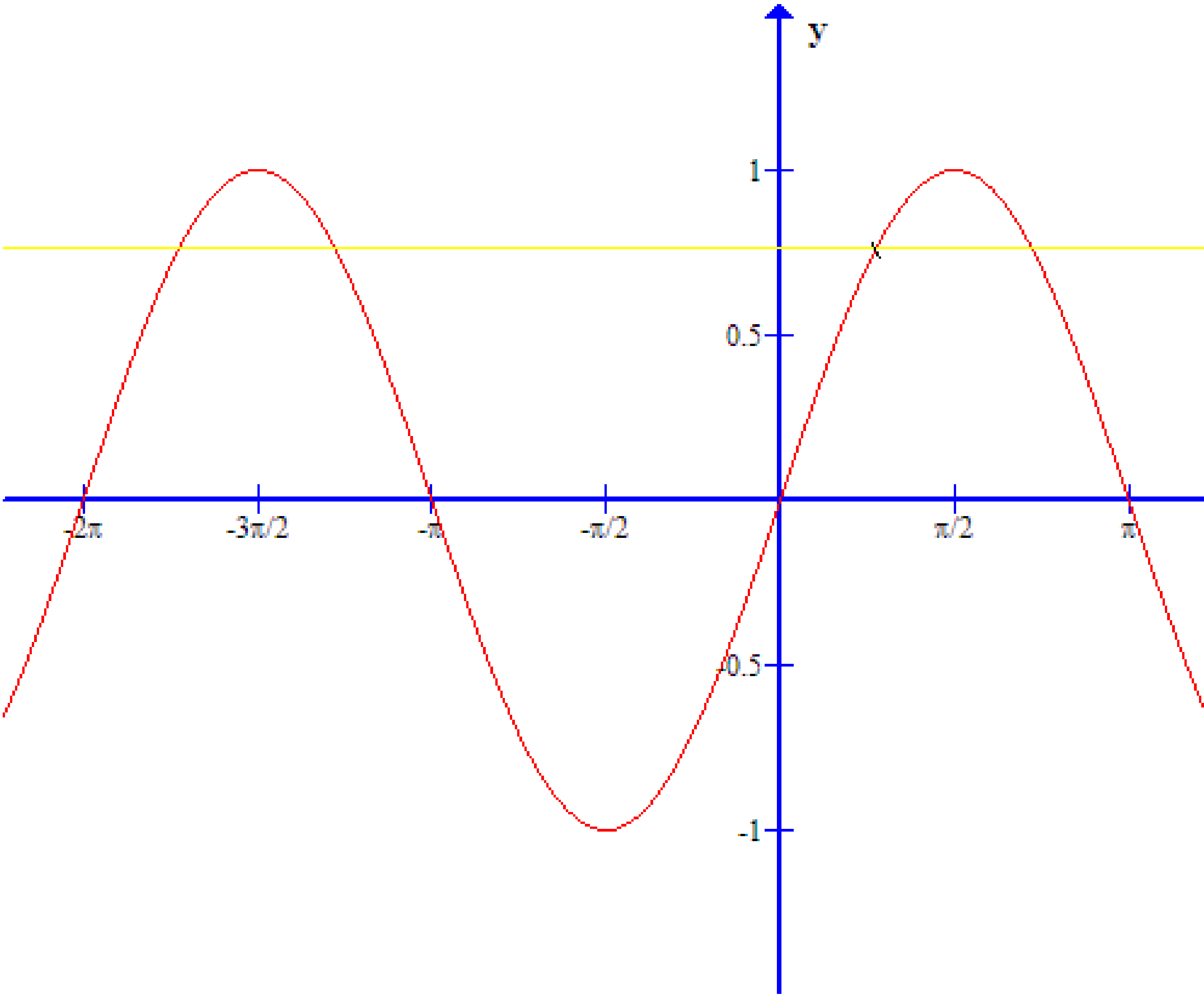
Properties 3.16. *1. Domain:* $D_f = \mathbb{R}$

2. The sine function is ***odd*** *and* ***periodic*** *with period* 2π*.*

3. $|\sin x| \leq 1$

4. $\sin x = 0 \iff x = k\pi$, *where* $k \in \mathbb{Z}$

5. *In the interval* $[0, 2\pi]$, *the sine function is:*
   - ***Increasing*** *on* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
   - ***Decreasing*** *on* $\left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$



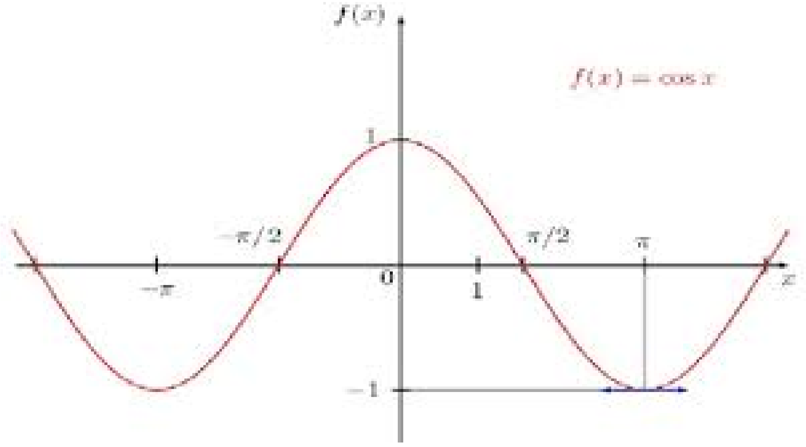
Cosine Function

The **cosine function** is defined as:

$$y = \cos x$$

Properties 3.17. *1. Domain:* $D_f = \mathbb{R}$

2. *The cosine function is* ***even*** *and* ***periodic*** *with period* 2π.
3. $|\cos x| \leq 1$
4. $\cos x = 0 \iff x = \frac{\pi}{2} + k\pi$, *where* $k \in \mathbb{Z}$
5. *The cosine function is:*
   - ***Increasing*** *on* $\left[-\frac{\pi}{2}, 0\right] \cup \left[\pi, \frac{3\pi}{2}\right]$
   - ***Decreasing*** *on* $[0, \pi]$



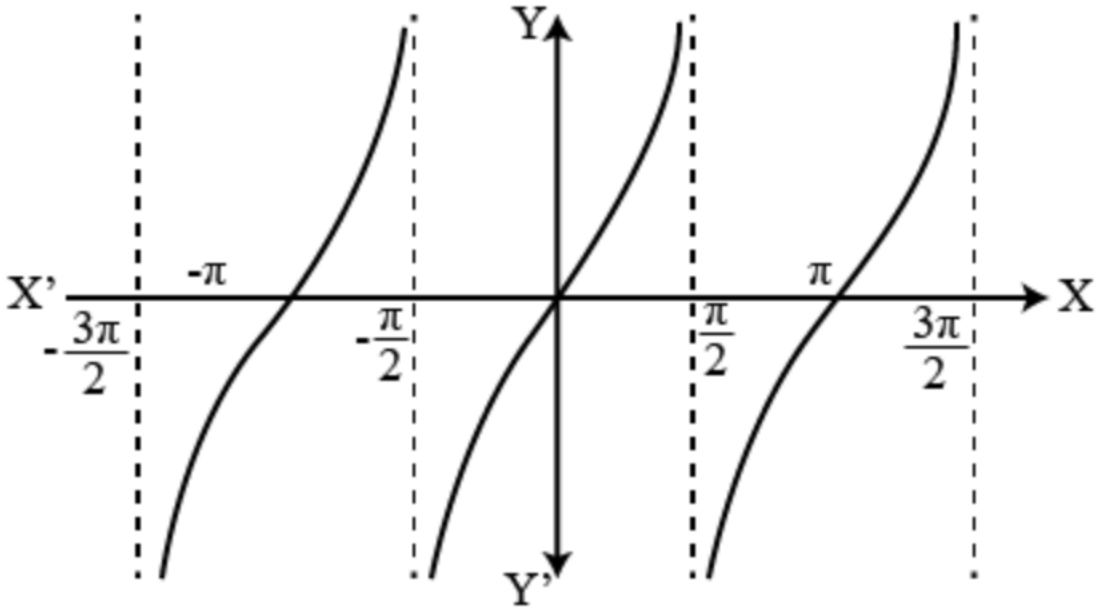
Tangent Function

The **tangent function** is defined as:

$$y = \tan x = \frac{\sin x}{\cos x}$$

Properties 3.18. 1. *Domain:* $D_f = \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$

2. $\tan(-x) = -\tan x$, *so the tangent function is* ***odd***.
3. *The tangent function is* ***periodic*** *with period* $T = \pi$.
4. *The tangent function is* ***increasing*** *on its domain.*
5. $\tan x = 0 \iff x = k\pi$, *where* $k \in \mathbb{Z}$



Cotangent Function

The **cotangent function** is defined as:

$$y = \cot x = \frac{\cos x}{\sin x}$$

Properties 3.19. 1. *Domain:* $D_f = \mathbb{R} - \{k\pi, k \in \mathbb{Z}\}$

2. $\cot x = 0 \iff x = \frac{\pi}{2} + k\pi$, *where* $k \in \mathbb{Z}$
3. *The cotangent function is* ***odd*** *and* ***decreasing*** *on its domain.*
4. *The cotangent function is* ***periodic*** *with period* $T = \pi$.

Inverse Trigonometric Functions

Arcsine Function

The **arcsine function** is the inverse of the sine function restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The sine function:

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$$

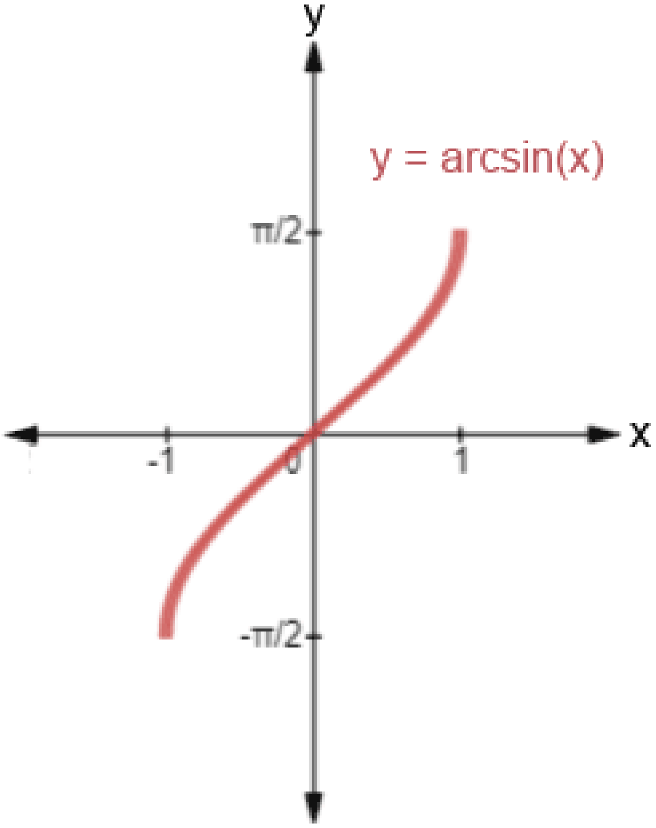
is continuous and monotonic, hence bijective and invertible. Its inverse is the arcsine function:

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$y = \sin x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \iff x = \arcsin y, \quad y \in [-1, 1]$$

Properties 3.20. *Let* $y = \arcsin x$*, where* $x \in [-1, 1]$*. Then:*

1. $\arcsin x = 0 \iff x = 0$
2. *The arcsine function is* ***odd****:*
$$\arcsin(-x) = -\arcsin x, \quad \forall x \in [-1, 1]$$
3. *The arcsine function is* ***increasing*** *for all* $x \in [-1, 1]$*.*
4. $\sin(\arcsin x) = x, \quad \forall x \in [-1, 1]$
5. $\arcsin(\sin x) = x, \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
6. $\cos(\arcsin x) = \sqrt{1 - x^2}, \quad \forall x \in [-1, 1]$



Example 3.21. *Solve the equation:*

$$2\sin x = 1 \iff \sin x = \frac{1}{2} \Rightarrow x = (-1)^k \frac{\pi}{6} + k\pi, \quad k \in \mathbb{Z}$$

Arccosine Function

The **arccosine function** is the inverse of the cosine function restricted to the interval $[0, \pi]$. The cosine function:

$$\cos : [0, \pi] \to [-1, 1]$$

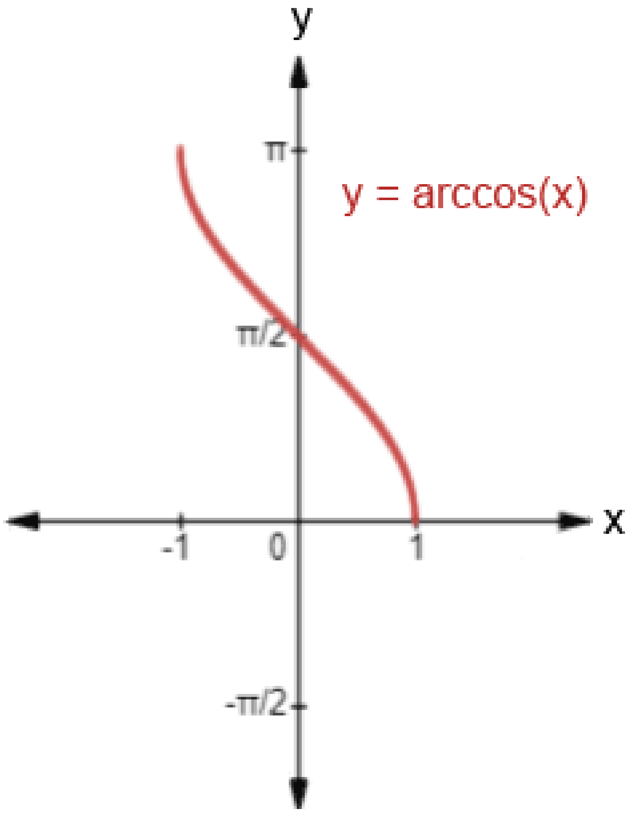
is continuous and decreasing, hence bijective and invertible. Its inverse is the arccosine function:

$$\arccos : [-1, 1] \to [0, \pi]$$

$$y = \cos x, \quad x \in [0, \pi] \iff x = \arccos y, \quad y \in [-1, 1]$$

Properties 3.22. *Let* $y = \arccos x$*, where* $x \in [-1, 1]$*. Then:*

1. $\arccos x = 0 \iff x = 1$
2. $y = \arccos x$ *is decreasing for all* $x \in [-1, 1]$
3. $\cos(\arccos x) = x$*, for all* $x \in [-1, 1]$
4. $\arccos(\cos x) = x$*, for all* $x \in [0, \pi]$
5. $\sin(\arccos x) = \sqrt{1 - x^2}$
6. $\arcsin x + \arccos x = \frac{\pi}{2}$*, for all* $x \in [-1, 1]$
7. $y = \cos x,\ x \in \mathbb{R} \iff x = \pm \arccos y + 2k\pi,\ k \in \mathbb{Z}$



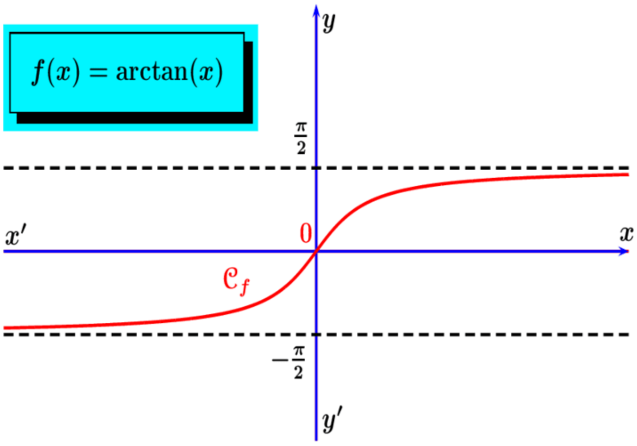
arctangente function

The function $\tan\colon \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\longrightarrow \mathbb{R}$ is one to one then it's invertible, its inverse is the function arctan*gente*, noted arctan, and defined by

$$\begin{array}{l} \arctan\colon \mathbb{R} \longrightarrow \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\\ y=\tan x,\ x\in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\iff x=\arctan y,\ y\in\mathbb{R} \end{array}$$

Proposition 3.23. *if* $y=\arctan x$, $x\in\mathbb{R}$ *then*

1. $\arctan x = 0 \iff x=0$
2. $\arctan(-x)=\arctan(x)$, *for all* $x\in\mathbb{R}$
3. arctan *is increasing function for all* $x\in\mathbb{R}$
4. $\tan(\arctan x)=x$, *for all for all* $x\in\mathbb{R}$, *and* $\arctan(\tan x)=x$, *for all* $x\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$
5. $\sin(\arctan x)=\dfrac{x}{\sqrt{1+x^2}}$, $x\in\mathbb{R}$ *and* $\tan(\arcsin x)=\dfrac{x}{\sqrt{1-x^2}}$, $x\in\left]-1,1\right[$
6. $y=\tan x$, *for* $x\in\mathbb{R}\backslash\left\{\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\right\} \iff x=\arctan y+k\pi,\ k\in\mathbb{Z}$



arccotangente function

The function $\cot\colon \left]0,\pi\right[\longrightarrow \mathbb{R}$ is one to one (bijection) then it's invertible, it's inverse is arccotangente function, noted by arccot, then we have

$$\begin{array}{l} \text{arccot}\colon \mathbb{R} \longrightarrow \left]0,\pi\right[\\ y=\cot x,\ x\in\left]0,\pi\right[\iff x=\text{arccot}y,\ y\in\mathbb{R} \end{array}$$

Proposition 3.24. *Let* $y=arccotx$, $x\in\mathbb{R}$, *then*

1. *The arccotangent function is* ***decreasing*** *for all* $x\in\mathbb{R}$.

2. *arccot*$(-x) = \pi -$ *arccot* x*, for all* $x \in \mathbb{R}$.
3. $\cot($*arccot* $x) = x$*, for all* $x \in \mathbb{R}$*, and* *arccot*$(\cot x) = x$*, for all* $x \in [0, \pi]$.
4. $\arctan x +$ *arccot* $x = \frac{\pi}{2}$*, for all* $x \in \mathbb{R}$.

Elementary Functions

An **elementary function** is any real-valued function obtained from basic functions using a finite number of arithmetic operations and function compositions.

Example 3.25.

$$y = \frac{x^2 + \sin 3x + \arcsin x}{e^x + \log x}$$

Rational Functions

A **rational function** is the ratio of two polynomials:

$$y = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}{b_0 + b_1x + b_2x^2 + \ldots + b_nx^n}$$

where the domain $D_f = \{x \mid Q(x) \neq 0\}$.

Irrational Functions

An **irrational function** is any function $y = f(x)$ where $f(x)$ is composed of arithmetic operations on x and non-integer powers.

Example 3.26.

$$y = f(x) = \sqrt[3]{1 - x^2}$$

Hyperbolic Functions

The **hyperbolic functions** are defined as follows:

1. **Hyperbolic sine**:
$$y = \sinh x = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbb{R}$$
2. **Hyperbolic cosine**:
$$y = \cosh x = \frac{e^x + e^{-x}}{2}, \quad x \in \mathbb{R}$$
3. **Hyperbolic tangent**:
$$y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{2e^x - 1}{2e^x + 1}, \quad x \in \mathbb{R}$$
4. **Hyperbolic cotangent**:
$$y = \coth x = \frac{\cosh x}{\sinh x} = \frac{2e^x + 1}{2e^x - 1}, \quad x \in \mathbb{R}$$

Hyperbolic Functions

Hyperbolic Sine (sinh)

The **hyperbolic sine** function $\sinh x$ has the following properties:

1. $\sinh x = 0 \iff x = 0$
2. $\sinh x$ is **odd**.
3. $\sinh x$ is **strictly increasing**.
4. $\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \sinh x_2 \cosh x_1$

Hyperbolic Cosine (cosh)

The **hyperbolic cosine** function $\cosh x$ has the following properties:

1. $\cosh x \neq 0$ for all $x \in \mathbb{R}$.
2. $\cosh x$ is **even**.
3. $\cosh x$ is **increasing** on $\mathbb{R}_+$ and **decreasing** on $\mathbb{R}_-$.
4. $\cosh^2 x - \sinh^2 x = 1$
5. $\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2$

Hyperbolic Tangent (tanh)

The **hyperbolic tangent** function $\tanh x$ has the following properties:

1. $\tanh x = 0 \iff x = 0$
2. $\tanh x$ is **odd**.
3. $\tanh x$ is **strictly increasing**.

Hyperbolic Cotangent (coth)

The **hyperbolic cotangent** function $\coth x$ has the following properties:

1. $\coth x \neq 0$ for all $x \in \mathbb{R}^*$.
2. $\coth x$ is **odd**.
3. $\coth x$ is **strictly decreasing**.

Inverse Hyperbolic Functions

Inverse Hyperbolic Sine (argsinh)

The **inverse hyperbolic sine** function, denoted argsinh, is the inverse of the hyperbolic sine function sinh. The hyperbolic sine function:

$$\sinh : \mathbb{R} \to \mathbb{R}$$

is bijective, hence invertible. Its inverse is:

$$\operatorname{argsinh} : \mathbb{R} \to \mathbb{R}$$

$$y = \operatorname{argsinh} x, \quad x \in \mathbb{R} \iff x = \sinh y, \quad y \in \mathbb{R}$$

Properties 3.27. *:*

1. $y = argsinh\ x$ *is* ***odd***.
2. $y = argsinh\ x$ *is* ***strictly increasing***.
3. $argsinh\ x = 0 \iff x = 0$

Inverse Hyperbolic Cosine (argcosh)

The **inverse hyperbolic cosine** function, denoted argcosh, is the inverse of the hyperbolic cosine function cosh. The hyperbolic cosine function:

$$\cosh : [0, +\infty) \to [1, +\infty)$$

is bijective, hence invertible. Its inverse is:

$$\operatorname{argcosh} : [1, +\infty) \to [0, +\infty)$$

$$y = \cosh x, \quad x \in [0, +\infty) \iff x = \operatorname{argcosh} y, \quad y \in [1, +\infty)$$

Properties 3.28. *:*

1. $argcosh\ x = 0 \iff x = 1$
2. $y = argcosh\ x$ *is* ***strictly increasing*** *on its domain.*

:

Inverse Hyperbolic Tangent (argtanh)

The **inverse hyperbolic tangent** function, denoted argtanh, is the inverse of the hyperbolic tangent function tanh. The hyperbolic tangent function:

$$\tanh : \mathbb{R} \to [-1, 1]$$

is bijective, hence invertible. Its inverse is:

$$\text{argtanh} : [-1, 1] \to \mathbb{R}$$

$$y = \tanh x, \quad x \in \mathbb{R} \iff x = \text{argtanh } y, \quad y \in [-1, 1]$$

Properties 3.29. *:*

1. *argtanh* $x = 0 \iff x = 0$
2. $y =$ *argtanh* x *is* ***odd*** *and* ***strictly increasing*** *on its domain.*

:

Inverse Hyperbolic Cotangent

The **inverse hyperbolic cotangent** function, denoted argcoth, is the inverse of the hyperbolic cotangent function coth. The hyperbolic cotangent function:

$$\coth : \mathbb{R}^* \to (-\infty, -1) \cup (1, +\infty)$$

is bijective, hence invertible. Its inverse is:

$$\text{argcoth} : (-\infty, -1) \cup (1, +\infty) \to \mathbb{R}^*$$

$$y = \coth x, \quad x \in \mathbb{R}^* \iff x = \text{argcoth } y, \quad |y| > 1$$

Properties 3.30. *:*

1. $y =$ *argcoth* x *is* ***odd****.*
2. $y =$ *argcoth* x *is* ***strictly decreasing*** *on its domain.*

:

Theorem 3.31. *The inverse hyperbolic functions can be expressed as logarithms:*

1. *argsinh* $x = \log(x + \sqrt{x^2 + 1})$*, for all* $x \in \mathbb{R}$
2. *argcosh* $x = \log(x + \sqrt{x^2 - 1})$*, for all* $x \geq 1$
3. *argtanh* $x = \frac{1}{2} \log \frac{1+x}{1-x}$*, for all* $x \in (-1, 1)$

4. argcoth $x = \frac{1}{2}\log\frac{1+x}{1-x}$*, for all* $|x| > 1$

Proof. For argsinh x:

$$y = \operatorname{argsinh} x \iff x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{e^{2y}-1}{e^y} \iff e^{2y} - 2xe^y - 1 = 0$$

$$\iff e^y = x \pm \sqrt{x^2+1} \Rightarrow e^y = x + \sqrt{x^2+1} \iff y = \log(x + \sqrt{x^2+1})$$

Similarly, the expressions for argcosh x, argtanh x, and argcoth x can be derived. ☐

Limits of Functions

Definition 3.32. *Definition of Limit Let* f *be defined in a neighborhood of* x_0*, except possibly at* x_0*. The number* l *is called the limit of* f *as* x *approaches* x_0*, written:*

$$l = \lim_{x\to x_0} f(x)$$

if:

$$\forall\varepsilon > 0, \exists\delta = \delta(\varepsilon) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon)$$

Example 3.33. *Prove that:*

$$\lim_{x\to 0} \sin x = 0$$

Indeed:

$$|\sin x - 0| = |\sin x| \le |x|$$

Let $\delta(\varepsilon) = \varepsilon$*. Then:*

$$\forall\varepsilon > 0, \exists\delta(\varepsilon) > 0, \text{ such that } |x| < \delta(\varepsilon) \Rightarrow |\sin x - 0| = |x| < \varepsilon$$

Definition 3.34. *Alternative Definition The number* l *is called the limit of* f *as* x *approaches* x_0 *if for every sequence* (x_n) *in* $V^0(x_0)$ *(a punctured neighborhood of* x_0*) converging to* x_0*, the sequence* $y_n = f(x_n)$ *converges to* l*. We write:*

$$\forall x_n \in V^0(x_0)$$

$$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = l$$

Remark 3.35. *According to definition 2, if there exist two sequences* (u_n)*,* (v_n) *converging to* x_0 *such that*

$$\lim_{n\to\infty} f(u_n) \ne \lim_{n\to\infty} f(v_n)$$

then the limit of f *does not exist at* x_0*.*

Example 3.36. *Study the limit of* $y = \sin\frac{\pi}{x}$ *as* $x \to 0$*. Let* $u_n = \frac{1}{n} \implies 0$ *and* $v_n = \frac{1}{\frac{1}{2}+2n} \implies 0$*. Then we have:*

$$f(u_n) = \sin n\pi = 0$$

and

$$f(v_n) = \sin\left(\frac{\pi}{2} + 2n\right) = 1$$

Hence, we conclude that the limit of f *does not exist.*

Extension of the Limit:

1. One-Sided Limits at a Point:

Let $V \subset \mathbb{R}$ be a set containing the interval $]a, x_0[$ (or $]x_0, b[$).

Definition 3.37. *The number $l \in \mathbb{R}$ is said to be the right-hand limit (resp. left-hand limit) of f at x_0 if:*

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) \in \mathbb{R}_+^*, \quad \forall x \in V, \quad (0 < x_0 - x < \delta) \Rightarrow |f(x) - l| < \varepsilon$$

Respectively:

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) \in \mathbb{R}_+^*, \quad \forall x \in V, \quad (0 < x - x_0 < \delta) \Rightarrow |f(x) - l| < \varepsilon$$

We denote: Right-hand limit:

$$l = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0 + 0)$$

Left-hand limit:

$$l = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} f(x) = f(x_0 - 0)$$

Theorem 3.38.

$$\lim_{n \to x_0} f(x) = l \Leftrightarrow f(x + 0) = f(x - 0)$$

2. Limit at Infinity:

$$\lim_{x \to +\infty} f(x) = l \Leftrightarrow (\forall \varepsilon > 0, \exists A > 0, \forall x \in V(+\infty), (x > A \Rightarrow |f(x) - l| < \varepsilon))$$

And

$$\lim_{x \to -\infty} f(x) = l \Leftrightarrow (\forall \varepsilon > 0, \exists A > 0, \forall x \in V(-\infty), (x < -A \Rightarrow |f(x) - l| < \varepsilon))$$

3. Infinite Limit:

Definition 3.39.

$$\lim_{x \to x_0} f(x) = +\infty \Leftrightarrow (\forall A > 0, \exists \delta = \delta(A) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow f(x) > A))$$

$\lim_{x \to x_0} f(x) = -\infty$ *if and only if:*

$$\forall A > 0, \exists \delta = \delta(A) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow f(x) < -A)$$

2. $\lim_{x \to +\infty} f(x) = +\infty$ *if and only if:*

$$\forall A > 0, \exists B > 0, \forall x \in V(+\infty), (x > B \Rightarrow f(x) > A)$$

Theorem 3.40. *Uniqueness of Limits If a function f has a limit at x_0, then the limit is unique.*

Proof. Assume f has two limits l_1 and l_2 at x_0. Then:

$$\lim_{x\to x_0} f(x) = l_1 \iff \forall\epsilon > 0, \exists\delta_1 = \delta_1(\epsilon) > 0, \forall x \in V(x_0), (0 < |x-x_0| < \delta_1 \Rightarrow |f(x)-l_1| < \frac{\epsilon}{2})$$

$$\lim_{x\to x_0} f(x) = l_2 \iff \forall\epsilon > 0, \exists\delta_2 = \delta_2(\epsilon) > 0, \forall x \in V(x_0), (0 < |x-x_0| < \delta_2 \Rightarrow |f(x)-l_2| < \frac{\epsilon}{2})$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for $0 < |x - x_0| < \delta$, both conditions are satisfied, and:

$$|l_1 - l_2| = |(l_1 - f(x)) + (f(x) - l_2)| \le |f(x) - l_1| + |f(x) - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, $l_1 = l_2$. □

Local Properties

Theorem 3.41. *If $\lim_{x\to x_0} f(x) = l$, then there exists a neighborhood of x_0 in which f is bounded. That is:*

$$\exists V(x_0) \textit{ such that } \forall x \in V(x_0), |f(x)| \le M$$

Passage to the Limit in Inequalities

Theorem 3.42. *Let f and g be two functions defined in a neighborhood of x_0 such that:*

$$\lim_{x\to x_0} f(x) = l_1, \quad \lim_{x\to x_0} g(x) = l_2, \quad \textit{and} \quad l_1 < l_2$$

Then there exists a punctured neighborhood of x_0 in which $f(x) \le g(x)$.

Proof.

$$\lim_{x\to x_0} f(x) = l_1 \iff \forall\epsilon > 0, \exists\delta_1 = \delta_1(\epsilon) > 0, \forall x \in V^0(x_0), (0 < |x-x_0| < \delta_1 \Rightarrow |f(x)-l_1| < \epsilon)$$

$$\lim_{x\to x_0} g(x) = l_2 \iff \forall\epsilon > 0, \exists\delta_2 = \delta_2(\epsilon) > 0, \forall x \in V(x_0), (0 < |x-x_0| < \delta_2 \Rightarrow |f(x)-l_2| < \epsilon)$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for $\forall x \in V(x_0), 0 < |x - x_0| < \delta$, both conditions are satisfied. Moreover:

$$|f(x) - l_1| < \epsilon \iff l_1 - \epsilon < f(x) < l_1 + \epsilon$$

Let $l_1 < l < l_2$. Given $\varepsilon = l - l_1 > 0 \Rightarrow l_1 - (l - l_1) < f(x) < l_1 + (l - l_1) \Rightarrow \forall x \in V(x_0), \quad 0 < |x - x_0| < \delta, \quad f(x) < l \ldots (3)$

$$|g(x) - l_2| < \varepsilon \Leftrightarrow l_2 - \varepsilon < g(x) < l_2 + \varepsilon$$

Given $\varepsilon = l_2 - l > 0 \Rightarrow l_2 - (l_2 - l) < g(x) < l_2 + (l_2 - l) \Rightarrow \forall x \in V(x_0), \quad 0 < |x - x_0| < \delta, \quad g(x) > l \ldots (4)$

From (3) and (4), we deduce $\forall x \in V(x_0), \quad 0 < |x - x_0| < \delta, \quad f(x) < g(x)$ □

Corollary 3.43. *Let f be a function defined in a neighborhood of x_0 such that $\lim_{x\to x_0} f(x) = l$ and $f(x) > a$*
then $l \geq a$

Theorem 3.44. ***Intermediate Function Criterion***
Let f, g, h be three functions defined in a punctured neighborhood of x_0 such that

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in V^0(x_0)$$

If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l$ then $\lim_{x\to x_0} g(x) = l$

Proof.

$$\lim_{x\to x_0} f(x) = l \Leftrightarrow \text{for every sequence } x_n \text{ converging to } x_0, \quad \lim_{n\to\infty} f(x_n) = l$$

$$\lim_{x\to x_0} h(x) = l \Leftrightarrow \text{for every sequence } x_n \text{ converging to } x_0, \quad \lim_{n\to\infty} h(x_n) = l$$

Hence $f(x_n) \leq g(x_n) \leq h(x_n), \quad \forall (x_n) \in V^0(x_0)$ From the theorem of three sequences, we deduce that $\lim_{x_n\to\infty} g(x_n) = l$ Which is equivalent to saying $\lim_{x\to x_0} g(x) = l$ □

Example 3.45. *Study the limit of $f(x) = x\sin\frac{1}{x}; \quad x_0 = 0$*

We have $\forall x \in \mathbb{R} - \{0\}, \quad -1 \leq \sin\frac{1}{x} \leq +1 \Leftrightarrow \begin{cases} -x \leq x\sin\frac{1}{x} \leq x & \textit{if } x > 0 \\ -x \geq x\sin\frac{1}{x} \geq x & \textit{if } x < 0 \end{cases}$

Taking the limit, we obtain

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

Operations on Limits

Proposition 3.46. *Let f, g be two functions defined in a neighborhood of x_0 such that $\lim_{x\to x_0} f(x) = l_1$,*

$$\lim_{x\to x_0} g(x) = l_2 \textit{ and then we have:}$$

1/ $\lim_{x\to x_0}(f(x) \pm g(x)) = l_1 \pm l_2$
2/ $\lim_{x\to x_0}(f(x)g(x)) = l_1 l_2$
3/ $\lim_{x\to x_0}(Af(x)) = \lambda l_1$
4/

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}$$

if

$$\lim_{x\to x_0} g(x) \neq 0$$

5/

$$\lim_{x\to x_0} |f(x)| = |l_1|$$

Proof. Let's prove (4) for example:

$$\lim_{x\to x_0} f(x) = l_1 \iff \text{for every sequence } x_n \text{ converging to } x_0, \quad \lim_{n\to+\infty} f(x_n) = l_1$$

$$\lim_{x\to x_0} g(x) = l_2 \iff \text{for every sequence } x_n \text{ converging to } x_0, \quad \lim_{n\to+\infty} g(x_n) = l_2$$

By the theorem of the limit of the ratio of two sequences, we find

$$\lim_{n\to+\infty} \frac{f(x_n)}{g(x_n)} = \frac{\lim_{n\to+\infty} f(x_n)}{\lim_{n\to+\infty} g(x_n)} = \frac{l_1}{l_2}$$

□

Example 3.47. *Calculate*

$$\lim_{n\to+\infty}(\sqrt{x^2+3x-4}-x) = \lim_{n\to+\infty}\frac{3x-4}{\sqrt{x^2+3x-4}+x} = \lim_{n\to+\infty}\frac{x(3-\frac{4}{x})}{\sqrt{1+\frac{3}{x}-\frac{4}{x^2}}+1} = \frac{3}{2}$$

Infinitely Large and Infinitely Small Functions

Definition 3.48. *Let f be a function defined in a punctured neighborhood of x_0, then:*

If

$$\lim_{x\to x_0} f(x) = 0$$

, then f is said to be infinitely small.

If

$$\lim_{x\to x_0} f(x) = +\infty$$

, then f is said to be infinitely large.

Remark 3.49. *If f is an infinitely small function at x_0 and if*

$$\lim_{x\to x_0} f(x) \neq 0$$

in

$$V^0(x_0)$$

, then the function

$$g(x) = \frac{1}{f(x)}$$

is infinitely large.

Remarkable Limits:

1. Let's prove that

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

The function

$$f(x)=\frac{\sin x}{x}$$

is an even function on

$$\mathbb{R}^*$$

, so it suffices to calculate the limit at

$$0^+$$

. (To be seen!)

2.

$$\lim_{x\to 0}\cos x=1$$

indeed:

$$0<|1-\cos x|=\left|2\sin^2\frac{x}{2}\right|=2\sin^2\frac{x}{2}\Rightarrow\lim_{x\to 0}\cos x=1$$

3. a)

$$\lim_{x\to 0}\frac{\tan x}{x}=1;\quad b)\lim_{x\to 0}\frac{\arcsin x}{x}=1;\quad c)\lim_{x\to 0}\frac{\arctan x}{x}=1$$

For (b), let

$$\arcsin x=y\iff\sin y=x;\quad x\to 0\implies y\to 0$$

, so we have:

$$\lim_{x\to 0}\frac{\arcsin x}{x}=\lim_{y\to 0}\frac{\sin y}{y}=1$$

4.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)=\frac{1}{2}$$

5.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$

Let $x\to+\infty$, let $E(x)=n$, then:

$$n\le x<n+1\Rightarrow\frac{1}{n+1}<\frac{1}{x}\le\frac{1}{n}\Rightarrow\frac{1}{n+1}+1<1+\frac{1}{x}\le 1+\frac{1}{n}\Rightarrow\left(1+\frac{1}{n+1}\right)^{n+1}<\left(1+\frac{1}{x}\right)^x\le\left(1+\right.$$

$$\Rightarrow\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e$$

This formula is also true when $x\to-\infty$

Generalization:

Let's compute

$$\lim_{x\to x_0}(f(x))^{g(x)}$$

, $f(x) > 0$, then we have the following cases:
i. If

$$\lim_{x\to x_0} f(x) = l_1$$

and

$$\lim_{x\to x_0} g(x) = l_2$$

, then

$$\lim_{x\to x_0} (f(x))^{g(x)} = (l_1)^l$$

ii. If

$$\lim_{x\to x_0} f(x) = 1$$

and

$$\lim_{x\to x_0} g(x) = \infty$$

, then

$$\lim_{x\to x_0} (f(x))^{g(x)} = e^{\lim_{x\to x_0}(f(x)-1)g(x)}$$

Example 3.50. *1.*

$$\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} = e^{\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}} = e^{\lim_{x\to 0}\frac{\sin x}{x}} = e$$

2.

$$\lim_{x\to +\infty}\left(\frac{2x+5}{x+3}\right)^{x+3} = 2^1 = 2$$

because

$$\lim_{x\to +\infty}\frac{2x+5}{x+3} = 2$$

and

$$\lim_{x\to +\infty}\frac{x+3}{x-5} = 1$$

3.

$$\lim_{x\to +\infty}\frac{a^0 + a^1x + a^2x^2 + \cdots + a^nx^n}{b^0 + b^1x + b^2x^2 + \cdots + b^mx^m} = \lim_{x\to +\infty}\frac{a^nx^n}{b^mx^m} = \begin{cases} \frac{a}{b} & \textit{if } n = m \\ \infty & \textit{if } n > m \\ 0 & \textit{if } n < m \end{cases}$$

Change of Variables in Limit Calculation:

Proposition 3.51. *Let $y = f(x)$ be defined in a neighborhood $V^0(x_0)$, $z = g(y)$ be defined in $W^0(y_0)$ satisfying the relations:*

$$f(x) \neq y_0, \quad \forall x \in V^0(x_0), \quad \lim_{x\to x_0} f(x) = y_0 \quad \& \quad \lim_{y\to y_0} g(y) = l$$

Then

$$\lim_{x\to x_0} g(f(x)) = l$$

Proof.

$$\left(\lim_{y\to y_0} g(x) = l\right) \Leftrightarrow (\forall \epsilon > 0, \exists \delta_1 > 0, \forall y \in V(y_0), \quad (0 < |y - y_0| < \delta_1 \Rightarrow |g(x) - l| < \epsilon)$$

$$\left(\lim_{x\to x_0} f(x) = y_0\right) \Leftrightarrow (\text{for } \delta_1 > 0, \exists \delta_2 > 0, \forall x \in V(x_0), \quad (0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - y_0| < \delta_1)$$

Then we obtain:

$$\forall x \in V(x_0), \quad (0 < |x - x_0| < \delta_2 \Rightarrow |y - y_0| < \delta_1 \Rightarrow |g(x) - l| < \epsilon)$$

That is:

$$\forall \epsilon > 0, \exists \delta_2 > 0, \forall x \in V(x_0), \quad (0 < |x - x_0| < \delta_2 \Rightarrow |g(f(x)) - l| < \epsilon) \Rightarrow \lim_{x\to x_0} g(f(x)) = l$$

□

Example 3.52. *Calculate*

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{1}{\cos x} - \tan x\right)$$

Let

$$x - \frac{\pi}{2} = y \Leftrightarrow x = y + \frac{\pi}{2}, \quad x \to \frac{\pi}{2} \Leftrightarrow y \to 0$$

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{1}{\cos x} - \tan x\right) = \lim_{y\to 0} \left(\frac{1}{\cos\left(\frac{\pi}{2} + y\right)} - \tan\left(\frac{\pi}{2} + y\right)\right) = \lim_{y\to 0} \left(\frac{1}{-\sin y} + \cot y\right)$$

$$= \lim_{y\to 0} \left(\frac{\cos y}{\sin y} - \frac{1}{\sin y}\right) = \lim_{y\to 0} \left(\frac{\cos y - 1}{\sin y}\right) = \lim_{y\to 0} \left(\frac{2\sin^2\frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}\right) = \lim_{y\to 0} \tan\frac{y}{2} = 0$$

Limit of a Monotone Function:

Theorem 3.53. *Let f be an increasing (respectively decreasing) function on the interval $I = (a, b)$.*

If f is bounded above (resp. bounded below) on the interval I, then the limit of f exists, and moreover:

$$\lim_{x\to b^-} f(x) = \sup_{(a,b)} f(x) \quad \textit{if} \quad f \quad \textit{is bounded above.}$$

$$\lim_{x\to a^+} f(x) = \inf_{(a,b)} f(x) \quad \textit{if} \quad f \quad \textit{is bounded below.}$$

Proof. f is bounded above on $I = (a, b) \Rightarrow$

$$\exists M_0 = \sup_{(a,b)} f(x) \Leftrightarrow \begin{cases} 1. \quad \forall x \in (a, b),\ f(x) \le M_0 \\ 2. \quad \forall \epsilon > 0,\ \exists x' \in (a, b),\ f(x) > M_0 - \epsilon \end{cases}$$

f is increasing on $I = (a, b) \Rightarrow$

$$(x' < x \Rightarrow f(x') < f(x))$$

Let

$$b - x' = \delta \quad \text{then} \quad x' = b - \delta \quad \text{and :}$$

$$x' < x < b \Rightarrow (b - \delta < x < b) \Rightarrow M_0 - \epsilon < f(x') \leq f(x) < M_0 < M_0 + \epsilon$$

In this way:

$$b-\delta < x < b \Rightarrow M_0-\epsilon < f(x) < M_0+\epsilon \Leftrightarrow |f(x)-M_0| < \epsilon \Leftrightarrow \lim_{x \to b^-} f(x) = M_0 = \sup_{(a,b)} f(x)$$

□

Cauchy Criterion:

Let f be a function defined in a punctured neighborhood of x_0.

Definition 3.54. *We say that f satisfies the Cauchy criterion at the point x_0 if:*

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \forall x', x'' \in V^0(x_0)$$

$$(0 < |x' - x_0| < \delta \quad \& \quad 0 < |x'' - x_0| < \delta \Rightarrow |f(x') - f(x'')| < \epsilon)$$

Theorem 3.55. *For the limit of a function f to exist at x_0, it is necessary and sufficient that f satisfies the Cauchy criterion at x_0.*

Comparison of Functions – Landau Notation:

Let f, g be two functions defined in a neighborhood of x_0, except possibly at x (x_0 can be equal to $+\infty$).

Definition 3.56. *Negligible Functions:*

We say that f is negligible compared to g in the neighborhood of x_0 if there exists a function $h(x)$ that is infinitely small at x_0 such that:

$$f(x) = h(x)g(x), \quad \forall x \in V^0(x_0)$$

We write:

$$f = o(g)(x \to x_0) \quad (\textit{Landau notation})$$

Read as: f equals little-o of g in the neighborhood of x_0.

Properties 3.57. *1/ If $g(x) \neq 0$, $\forall x \in V^0(x_0)$, then:*

$$f = o(g)(x \to x_0) \iff \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

2/ $f = l + o(l)(x \to x_0) \iff \lim_{x \to x_0} f(x) = l$. In particular, $f = o(l)(x \to x_0) \iff \lim_{x \to x_0} f(x) = 0$.

Example 3.58.

$$\sin^2 x = o(x), \quad x \to 0$$

Indeed:

$$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \sin x \right) = 1 \cdot 0 = 0$$

Definition 3.59. *Dominant Functions: We say that the function f is dominated by the function g at x_0 if there exists $k > 0$ such that:*

$$|f(x)| \leq k|g(x)|, \quad \forall x \in V^0(x_0)$$

And we write: $f = O(g)(x \to x_0)$ (Landau notation).
Read as: f equals big-O of g in the neighborhood of x_0.

Properties 3.60. *1/ If $g(x) \neq 0$, $\forall x \in V^0(x_0)$, then:*

$$f = O(g)(x \to x_0) \iff \exists k > 0, \quad \left| \frac{f(x)}{g(x)} \right| \leq k$$

2/ $f = O(l)(x \to x_0) \iff \exists k > 0, \quad |f(x)| \leq k$.

Example 3.61.

$$2x \sin x = O(x^2), \quad x \to 0$$

Definition 3.62. *Equivalent Functions: We say that f is equivalent to g in the neighborhood of x_0 if there exists a function h such that:*

$$\lim_{x\to x_0} h(x) = 1 \quad \text{and} \quad f(x) = h(x)g(x), \quad \forall x \in V^0(x_0)$$

We write $f \sim g$ (as $x \to x_0$).
Read as: f is equivalent to g in the neighborhood of x_0.

Properties 3.63. *1/ If $g(x) \neq 0$, $\forall x \in V^0(x_0)$, then:*

$$f \sim g \iff \lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$$

2/ If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = l \neq 0$, then $f \sim g$.
3/ If $\lim_{x\to x_0} f(x) = l$ and $f \sim g$, then $\lim_{x\to x_0} g(x) = l$.
4/ $f \sim g \iff f(x) = g(x)(1 + \alpha(x))$, where $\lim_{x\to x_0} \alpha(x) = 0$.

Theorem 3.64.

$$f \sim g \iff f = g + o(g), \quad x \to x_0$$

Theorem 3.65. *Let $f \sim f_1$ and $g \sim g_1$ as $x \to x_0$, then:*

$$f \cdot g \sim f_1 \cdot g_1 \quad \text{and} \quad \frac{f}{g} \sim \frac{f_1}{g_1} \quad \text{with} \quad (g \neq 0; \quad g_1 \neq 0)$$

Proof.

$$f \sim f_1 \iff f(x) = f_1(x)h_1(x) \quad \text{and} \quad \lim_{x\to x_0} h_1(x) = 1$$

$$g \sim g_1 \iff g(x) = g_1(x)h_2(x) \quad \text{and} \quad \lim_{x\to x_0} h_2(x) = 1$$

$$f(x)g(x) = h_1(x)h_2(x)f_1(x)g_1(x) = h(x)f_1(x)g_1(x) \quad \text{and} \quad \lim_{x\to x_0} h(x) = 1$$

This shows that $f \cdot g \sim f_1 \cdot g_1$. □

Remark 3.66.

$$f \sim f_1 \quad \textit{and} \quad g \sim g_1 \quad \textit{does not imply} \quad (f \pm g) \sim f_1 \pm g_1$$

Example 3.67.

$$\cos x \sim 1 + x^3 \quad \textit{and} \quad -1 \sim -1 \quad \textit{But} \quad \cos x - 1 \sim x^3 \quad \textit{because}$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x^3} = \lim_{x\to 0} \frac{-2\sin^2 \frac{x}{2}}{x^3} = -\frac{2}{4}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 \frac{1}{x} = \infty$$

Proposition 3.68. *Composition of Equivalences: Let $t = \varphi(x)$ be defined in $V^0(x_0)$.*
Let $f(t), g(t)$ be defined in $V^0(a)$.
If $f(t) \sim g(t)$ and $\varphi(x) \to a$ with $\varphi(x) \neq 0$, then $f(\varphi(x)) \sim g(\varphi(x))$.

Proof.

$$f \sim g \iff f(t) = g(t)h(t) \quad \text{and} \quad \lim_{t\to a} h(t) = 1$$

$$\Rightarrow f(\varphi(x)) = h(\varphi(x))g(\varphi(x)) \quad \text{and} \quad \lim_{x\to x_0} h(\varphi(x)) = \lim_{t\to a} h(t) = 1$$

Let $h(\varphi(x)) = h_1(x)$, then $\lim_{x\to x_0} h_1(x) = 1$ and we obtain:

$$f(\varphi(x)) = h_1(x)g(\varphi(x)) \iff f(\varphi(x)) \sim g(\varphi(x))$$

□

Remark 3.69. *In general, $f \sim g$ does not imply $\varphi(f) \sim \varphi(g)$.*

Example 3.70.

$$f(x) = e^{x^2+x}, \quad g(x) = e^{x^2}; \quad \varphi(x) = \ln(x), \quad x_0 = 0$$

We have $e^{x^2+x} \sim e^{x^2}$.

$$\varphi(f) = \ln(e^{x^2+x}) = x^2 + x; \quad \varphi(g) = \ln(e^{x^2}) = x^2.$$

$$\lim_{x\to 0} \frac{\varphi(f)}{\varphi(g)} = \lim_{x\to 0} \frac{x^2 + x}{x^2} = \infty$$

Thus,

$$\varphi(f) \nsim \varphi(g)$$

Table of Equivalences for Elementary Functions: The notion of equivalence is very useful in calculating limits. For this purpose, we provide a table of equivalences for some commonly used elementary functions in the neighborhood of $x_0 = 0$: 1. $\ln(1+x) \sim x \iff \ln(1+x) = x + o(x)$ 2. $e^x - 1 \sim x \iff e^x - 1 = x + o(x)$ 3. $\sin x \sim x \iff \sin x = x + o(x)$ 4. $\cos x - 1 \sim -\frac{x^2}{2} \iff \cos x - 1 = -\frac{x^2}{2} + o(x^2)$ 5. $\tan x \sim x \iff \tan x = x + o(x)$ 6. $(1+x)^r - 1 \sim rx \iff (1+x)^r - 1 = rx + o(x), \forall r \in \mathbb{Q}^+$ 7. $\arcsin x \sim x \iff \arcsin x = x + o(x)$ 8. $\arctan x \sim x \iff \arctan x = x + o(x)$ 9. $\sinh x \sim x \iff \sinh x = x + o(x)$ 10. $\cot x \sim \frac{1}{x} \iff \cot x = \frac{1}{x} + o\left(\frac{1}{x}\right)$

Scales of Comparison for Infinitesimal Functions – Principal Parts:

Let g be an infinitesimal function in the neighborhood of x_0 with $g(x) \neq 0$.

Definition 3.71. *1. f is said to be an infinitesimal of higher order with respect to g at x_0 if g is negligible compared to f, i.e.,*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \iff f = o(g)(x \to x_0)$$

2. f is said to be an infinitesimal of lower order with respect to g at x_0 if f is negligible compared to g, i.e.,

$$\lim_{x \to x_0} \frac{g(x)}{f(x)} = 0 \iff g = o(f)(x \to x_0)$$

3. f is said to be an infinitesimal of order k with respect to g at x_0 if

$$\lim_{x \to x_0} \frac{f(x)}{g^k(x)} = l \neq 0$$

i.e.,

$$\lim_{x \to x_0} \frac{f(x)}{l \cdot g^k(x)} = 1 \iff f \sim l \cdot g^k \iff f(x) = l \cdot g^k(x) + o(g^k(x))$$

In this case, the function $l \cdot g^k(x)$ is called the principal part of f at x_0, and we write:

$$p.p.f(x) = l \cdot g^k(x)$$

Scales of Comparison for Infinitely Large Functions:

Let g be an infinitely large function at x_0.

Definition 3.72. *1. f is said to be infinitely large of higher order with respect to g at x_0 if g is negligible compared to f, i.e.,*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \infty \iff g = o(f)(x \to x_0)$$

2. f is said to be infinitely large of lower order with respect to g at x_0 if f is negligible compared to g, i.e.,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \iff f = o(g)(x \to x_0)$$

3. f is said to be infinitely large of order k with respect to g at x_0 if

$$\lim_{x\to x_0}\frac{f(x)}{g^k(x)}=l\neq 0$$

i.e.,

$$\lim_{x\to x_0}\frac{f(x)}{l\cdot g^k(x)}=1\iff f\sim l\cdot g^k\iff f(x)=l\cdot g^k(x)+o(g^k(x))$$

Example 3.73. *Determine the principal part of the function $f(x)=1-\cos x$ in the neighborhood of $x_0=0$ of the form $\alpha\cdot x^\beta$. Indeed:*

$$\lim_{x\to 0}\frac{1-\cos x}{\alpha\cdot x^\beta}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\alpha\cdot x^\beta}=\lim_{x\to 0}\frac{2\cdot\frac{x^2}{4}}{\alpha\cdot x^\beta}=\frac{1}{2\alpha}\lim_{x\to 0}x^{2-\beta}=1\quad \textit{iff}\quad \begin{cases}\beta=2\\ 2\alpha=1\end{cases}\Rightarrow\begin{cases}\beta=2\\ \alpha=\frac{1}{2}\end{cases}$$

Thus,

$$p.p.(1-\cos x)=\frac{1}{2}x^2$$

Examples of Limit Calculations Using Equivalences and Little "o":

1. Calculate

$$\lim_{x\to 2}\frac{\sin(x^2-4)}{x^2-5x+6}$$

We have

$$\sin x\sim x\Rightarrow\sin u(x)\sim u(x)\quad\text{when}\quad\lim_{x\to x_0}u(x)=0\quad\text{and thus:}$$

$$\lim_{x\to 2}\frac{\sin(x^2-4)}{x^2-5x+6}=\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x-3)}=-4$$

2. Calculate

$$\lim_{x\to 0}\frac{e^x-\sqrt[3]{1+x}}{2\arctan x-\arcsin x}$$

$$\sqrt[3]{1+x}=1+\frac{1}{3}x+o(x)\quad\text{and}\quad o(x)\pm o(x)=o(x)\quad\text{and}\quad e^x-1=x+o(x)$$

$$\arctan x=x+o(x)\quad\text{and}\quad\arcsin x=x+o(x)$$

Thus:

$$\lim_{x\to 0}\frac{e^x-\sqrt[3]{1+x}}{2\arctan x-\arcsin x}=\lim_{x\to 0}\frac{1+(e^x-1)-\sqrt[3]{1+x}}{2\arctan x-\arcsin x}=\lim_{x\to 0}\frac{1+(x+o(x))-(1+\frac{1}{3}x+o(x))}{2(x+o(x))-(x+o(x))}$$

$$=\lim_{x\to 0}\frac{\frac{2}{3}x+o(x)}{x+o(x)}=\lim_{x\to 0}\frac{x\left(\frac{2}{3}+o(1)\right)}{x(1+o(1))}=\frac{\frac{2}{3}+\lim_{x\to 0}o(1)}{1+\lim_{x\to 0}o(1)}=\frac{2}{3}$$

Upper and Lower Limits:

Definition 3.74. *Let f be a function defined in $V^0(x_0)$. The following values* $\lim_{\delta\to 0}\sup f(x)$ *and* $\liminf_{\delta\to 0} f(x)$*, for* $0<|x-x_0|<\delta$*, are called the upper limit and lower limit of f at x_0, respectively. They are denoted by:*

$$\limsup_{x\to x_0} f(x) \quad \textit{and} \quad \liminf_{x\to x_0} f(x)$$

Theorem 3.75.

$$\lim_{x\to x_0} f(x)=l \iff \limsup_{x\to x_0} f(x)=\liminf_{x\to x_0} f(x)=l$$

Definition 3.76. *The number $\lambda\in\mathbb{R}$ is called a partial limit of f at x_0 (or a cluster value) if there exists a sequence (x_n) different from x_0 such that* $\lim_{n\to\infty} x_n = x_0$ *and* $\lim_{n\to\infty} f(x_n)=\lambda$.

We denote by $\Lambda(x_0)$ the set of cluster values of f at x_0.

Then:

$$\limsup_{x\to x_0} f(x)=\sup\Lambda(x_0) \quad \textit{and} \quad \liminf_{x\to x_0} f(x)=\inf\Lambda(x_0)$$

Example 3.77. *Determine* $\limsup_{x\to 0} f(x)$ *and* $\liminf_{x\to 0} f(x)$ *where* $f(x)=\cos\frac{1}{x}$ *and* $x_0=0$.

For $x_n=\frac{1}{2\pi n}\to 0$*:*

$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty}\cos(2\pi n)=1 \quad \textit{since} \quad |\cos x|\le 1$$

For $x'_n=\frac{1}{\pi+2\pi n}\to 0$*:*

$$\lim_{n\to\infty} f(x'_n)=\lim_{n\to\infty}\cos(\pi+2\pi n)=-1$$

Thus:

$$\Lambda(0)=\{-1,+1\} \implies \limsup_{x\to 0} f(x)=1 \quad \textit{and} \quad \liminf_{x\to 0} f(x)=-1$$

Continuous Functions

Definitions and Properties

Definition 3.78. *We say that the function f is continuous at $x_0\in\mathbb{R}$ if: 1. f is defined in a neighborhood of x_0. 2.*

$$\lim_{x\to x_0} f(x)=f(x_0)$$

In other words:

$(f$ *continuous at* $x_0) \Leftrightarrow (\forall\epsilon>0,\exists\delta=\delta(\epsilon)>0,\forall x\in V^0(x_0),\ (|x-x_0|<\delta\Rightarrow|f(x)-f(x_0)|<\epsilon))$

Definition 3.79. *Increment of the Function: Let f be a function defined in a neighborhood of x_0. The value $x - x_0 = \Delta x$ is called the increment of the argument x at the point x_0. The value $f(x) - f(x_0) = \Delta f(x)$ is called the increment of the function $f(x)$ at the point x_0.*

Definition 3.80. *Using Sequences: Let f be a function defined in a neighborhood V of x_0. We say that f is continuous at x_0 if:*

$$\forall (u_n) \subset V;\ \lim_{n\to\infty} u_n = x_0 \Rightarrow \lim_{n\to\infty} f(u_n) = f(x_0)$$

Continuity from the Right – Continuity from the Left:

Definition 3.81. *i. f is said to be continuous from the right at $x_0 \in \mathbb{R}$ if: 1. f is defined on $[x_0, b]$. 2.*

$$\lim_{x\to x_0^+} f(x) = f(x_0 + 0) = f(x_0)$$

ii. f is said to be continuous from the left at $x_0 \in \mathbb{R}$ if: 1. f is defined on $[a, x_0]$. 2.

$$\lim_{x\to x_0^-} f(x) = f(x_0 - 0) = f(x_0)$$

Theorem 3.82. *For f to be continuous at x_0, it is necessary and sufficient that:*

$$f(x_0 + 0) = f(x_0 - 0) = f(x_0)$$

Theorem 3.83. *Continuity on a Set: f is continuous on an interval I if it is continuous at every point of this interval.*

Discontinuity – Classification of Points of Discontinuity:

Definition 3.84. *The function f is said to be discontinuous at a point x_0 if it is not continuous at x_0, i.e.: 1. f is not defined at x_0. 2. The limit of f at x_0 exists but is different from $f(x_0)$. 3. The limit of f at x_0 does not exist.*

For the first case: If f is not defined at x_0 but

$$\exists \lim_{x\to x_0} f(x) = l$$

, then by setting $f(x_0) = l$, we make f continuous at x_0 (i.e., we have extended f by continuity at x_0). The point x_0 is called a removable discontinuity in this case.

Example 3.85. *The function*

$$f(x) = \frac{\sin x}{x}$$

is not defined at $x_0 = 0$, but

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

. Thus, by setting $f(0) = 1$, we obtain a function continuous at $x_0 = 0$:

$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

For the second case:

$$\exists f(x_0+0), \quad \exists f(x_0-0), \quad \text{but} \quad f(x_0+0) \neq f(x_0-0)$$

The point x_0 is called a discontinuity of the first kind.

Example 3.86.

$$f(x) = E(x), \quad x_0 = m; \quad m \in \mathbb{Z}$$

$$\lim_{x \to m^+} f(x) = \lim_{x \to m^+} E(x) = m \quad \textit{and} \quad \lim_{x \to m^-} f(x) = \lim_{x \to m^-} E(x) = m-1$$

Thus:

$$f(m+0) \neq f(m-0)$$

For the third case: If one of the limits $f(x_0+0)$ or $f(x_0-0)$ does not exist or is infinite, the point x_0 is called a discontinuity of the second kind.

Example 3.87.

$$f(x) = \frac{1}{x-2}; \quad x_0 = 2, \quad \lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{1}{x-2} = \pm\infty$$

Discontinuity Points of Monotone Functions

Theorem 3.88. *Every monotone function on an interval (a,b) can only have discontinuities of the first kind.*

Proof. Let f be an increasing function on (a,b) and let $x_0 \in (a,b)$. For every point $x \leq x_0$, we have $f(x) \leq f(x_0)$. By the theorem on the limit of a monotone function, it follows that:

$$f(x_0-0) = \lim_{x \to x_0^-} f(x) \leq f(x_0)$$

Thus: - If $f(x_0-0) = f(x_0)$, then f is continuous from the left at x_0. - If $f(x_0-0) < f(x_0)$, then x_0 is a discontinuity of the first kind. □

3.1.1 Operations on Continuous Functions

Proposition 3.89. *Let f, g be two functions continuous at a point x_0. Then:*

$$f \pm g; \quad fg; \quad Af(\lambda \in \mathbb{R}); \quad \frac{f}{g}(g(x_0) \neq 0) \quad \textit{and} \quad |f|$$

are functions continuous at x_0.

Proof. f is continuous at $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$. g is continuous at $x_0 \iff \lim_{x \to x_0} g(x) = g(x_0)$. Let $\varphi(x) = f(x) + g(x)$, then:

$$\lim_{x \to x_0} \varphi(x) = \lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = f(x_0) + g(x_0) = \varphi(x_0)$$

□

Continuity of Composite Functions:

Proposition 3.90. *Let $y = f(x)$ be continuous at x_0, and $z = g(y)$ be continuous at $y_0 = f(x_0)$. Then the composite function $(g \circ f)(x) = g(f(x))$ is continuous at x_0.*

Proof. Let (x_n) be an arbitrary sequence in the domain of $y = f(x)$ such that:

$$\lim_{n\to\infty} x_n = x_0$$

Since f is continuous at x_0 and g is continuous at $y_0 = f(x_0)$, we have:

$$\lim_{n\to\infty} (g \circ f)(x_n) = \lim_{n\to\infty} g(f(x_n)) = g\left(\lim_{n\to\infty} f(x_n)\right) = g(f(x_0)) = (g \circ f)(x_0)$$

This shows that $(g \circ f)$ is continuous at x_0. □

Proposition 3.91. *Continuity of Monotone Functions: If f is strictly monotone on an interval $I = (a, b)$ and the image $f(I) = J$ is an interval, then f is continuous on I.*

**

Proof. We prove by contradiction. Assume that f has a discontinuity at x_0 in $I = (a, b)$. x_0 can only be a discontinuity of the first kind (since f is monotone). For every point $x < x_0$, we have $f(x) < f(x_0 - 0) < f(x_0)$ (assuming f is increasing).

$$\begin{array}{c} f(x_0) \\ f(x_0 - 0) \\ \hline \end{array}$$

Consider the interval $(f(x_0 - 0), f(x_0))$. For every $y \in (f(x_0 - 0), f(x_0))$, there is no point $x \in (a, b)$ such that $y = f(x)$. This contradicts the hypothesis of the theorem that $f(I) = J$ is an interval. . □

Continuity of Elementary Functions:

Theorem 3.92. *All elementary functions are continuous in their respective domains..*

Example 3.93. *1. $y = \sin x$, $x_0 \in \mathbb{R}$, arbitrary.*

$$|\sin x - \sin x_0| = \left|2 \sin \frac{x - x_0}{2} \cos \frac{x - x_0}{2}\right| \leq 2 \left|\frac{x - x_0}{2}\right| = |x - x_0|$$

since $|\sin \alpha| \leq |\alpha|$ and $\cos \alpha \leq 1$.

Choose $\delta(\epsilon) = \epsilon$. Then $|x - x_0| < \delta = \epsilon$ implies $|\sin x - \sin x_0| < \epsilon$. Thus, $y = \sin x$ is continuous at x_0.

2. $y = \cos x = \sin\left(\frac{\pi}{2} - x\right)$, $y = \tan x = \frac{\sin x}{\cos x}$, and $y = \cot x = \frac{\cos x}{\sin x}$ are continuous because they are composite functions.

3. $y = a^x$ $(a > 0,\ a \neq 1)$ is continuous because it is monotone, etc.

Properties of Continuous Functions

Theorem 3.94. *Bolzano-Cauchy: Let f be a function defined and continuous on a closed interval $[a, b]$ such that $f(a)f(b) < 0$ (i.e., $f(a)$ and $f(b)$ have opposite signs). Then there exists at least one point $c \in [a, b]$ such that $f(c) = 0$.*

Theorem 3.95. *Bolzano-Cauchy Intermediate Value Theorem: Let f be a function defined and continuous on an interval I of $\mathbb{R}$, and let $a, b \in I$ $(a < b)$. Then for every point α between $f(a)$ and $f(b)$, there exists a number c in $[a, b]$ such that $f(c) = \alpha$.*

Proof. Let $g(x) = f(x) - \alpha$. Then:

$$g(a) = f(a) - \alpha < 0$$

$$g(b) = f(b) - \alpha > 0$$

$$\Rightarrow g(a)g(b) < 0$$

By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $g(c) = 0$, which implies $f(c) = \alpha$. □

Corollary 3.96. *The image of an interval under a continuous function is an interval.*

Theorem 3.97. *First Weierstrass Theorem: Every function defined and continuous on a closed interval $[a, b]$ is bounded.*

Proof. By contradiction, assume that f is not bounded on $[a, b]$. Then there exists a sequence (x_n) in $[a, b]$ such that $f(x_n) > n$. Since (x_n) is a bounded sequence, we can extract a convergent subsequence $x_{n_k} \to x_0$. Then:

$$f(x_{n_k}) \to f(x_0)$$

This implies that $f(x_{n_k})$ is bounded, which contradicts the assumption $f(x_{n_k}) > n_k$. □

Theorem 3.98. *Second Weierstrass Theorem: Let f be a function defined and continuous on a closed interval $[a, b]$. Then f attains its supremum and infimum.*

Proof. Let $M = \sup_{[a,b]} f(x)$. We need to show that there exists $x_i \in [a, b]$ such that $f(x_i) = M$. Assume, for contradiction, that $f(x) < M$ for all $x \in [a, b]$. Consider the function:

$$\varphi(x) = \frac{1}{M - f(x)}$$

Since $\varphi(x)$ is continuous on $[a, b]$, it is bounded. Thus, there exists $c > 0$ such that:

$$\varphi(x) < c \Rightarrow \frac{1}{M - f(x)} < c \Rightarrow M - f(x) > \frac{1}{c} \Rightarrow f(x) < M - \frac{1}{c}$$

This contradicts the fact that M is the least upper bound of f. □

Theorem 3.99. *Inverse Function Theorem: Let f be a function defined and continuous on $I = [a, b]$, strictly increasing (resp. decreasing). Then f has an inverse function defined on the interval $J = f(I)$, which is strictly increasing (resp. decreasing). Moreover:*

$$f([a, b]) = [f(a), f(b)] \quad \text{if } f \text{ is increasing, and } f([a, b]) = [f(b), f(a)] \quad \text{if } f \text{ is decreasing.}$$

3.1.2 Uniform Continuity:

Definition 3.100. *A function f defined on an interval I is said to be uniformly continuous on I if:*

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) > 0; \quad \forall x, x' \in I; \quad (|x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon)$$

Remark 3.101. *It is clear that uniform continuity on an interval I implies simple continuity at every point x_0 of I, by taking x' instead of x_0 in the definition.*

The converse is generally not true, but the following theorem holds:

Theorem 3.102. *(Cantor):** Every function defined and continuous on a closed interval $[a, b]$ is uniformly continuous on this interval.*

Proof. We prove by contradiction. Assume that f is not uniformly continuous on $[a, b]$, i.e.:

$$\exists \varepsilon > 0, \quad \forall \delta = \delta(\varepsilon) > 0; \quad \exists x, x' \in I; \quad (|x - x'| < \delta \wedge |f(x) - f(x')| \geq \varepsilon)$$

Consider a sequence (δ_n) converging to 0. Then:

$$\exists \varepsilon > 0, \quad \forall \delta_n > 0; \quad \exists x_n, x'_n \in I; \quad (|x_n - x'_n| < \delta_n \wedge |f(x_n) - f(x'_n)| \geq \varepsilon)$$

The sequences x_n, x'_n are bounded, and from any bounded sequence, we can extract a convergent subsequence (Bolzano-Weierstrass theorem). Thus, there exist $(x_{n_k}) \subset (x_n)$ and $(x'_{n_k}) \subset (x'_n)$ such that:

$$|x_{n_k} - x'_{n_k}| < \delta_{n_k} \implies \lim_{k \to \infty} |x_{n_k} - x'_{n_k}| = 0$$

Therefore:

$$\lim_{k \to \infty} x_{n_k} = \lim_{k \to \infty} x'_{n_k} = x_0$$

By the continuity of f on $[a, b]$, we obtain:

$$\lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(x'_{n_k}) = f(x_0) \implies \lim_{k \to \infty} (f(x_{n_k}) - f(x'_{n_k})) = 0$$

This contradicts the fact that $|f(x_n) - f(x'_{n_k})| \geq \varepsilon$ for all $n = 1, 2, \ldots$. ☐

Example 3.103. *The function $f(x) = \frac{3}{x^2 - 25}$ is uniformly continuous on* $[1, 3]$.

3.2 Differentiable Functions

3.2.1 Derivatives at a Point:

Let f be a function defined in a neighborhood $V(x_0)$ of the point x_0 in $\mathbb{R}$.

Definition 3.104. *We say that f is differentiable at x_0 if the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. This limit is called the derivative of f at x_0 and is denoted by $f'(x_0)$.

Alternative Notations: - Let $x - x_0 = \Delta x = h$. Then:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Example 3.105. *Find the derivative of $y = x^3$ at a point x_0 in $\mathbb{R}$:*

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^3 - x_0^3}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x^2 + xx_0 + x_0^2)}{x - x_0} = x_0^2 + x_0^2 + x_0^2 = 3x_0^2$$

3.2.2 Differentiability at a Point:

Definition 3.106. *We say that a function f is differentiable at x_0 if: 1. f is defined in a neighborhood of x_0. 2. There exists a number A in $\mathbb{R}$ and an infinitesimal function $\alpha(x)$ such that the increment Δf of f corresponding to the increment Δx of x can be written as:*

$$\Delta f(x) = f(x + \Delta x) - f(x) = A\Delta x + \alpha(\Delta x)\Delta x; \quad \lim_{\Delta x \to 0} \alpha(\Delta x) = 0 \quad (*)$$

The following theorem holds:

Theorem 3.107. *For a function to be differentiable at a point x_0, it is necessary and sufficient that it be differentiable at that point.*

Proof. **Necessary Condition:** If f is differentiable at x in $\mathbb{R}$, then (*) holds, and thus:

$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} (A + \alpha(\Delta x)) = A \Rightarrow f'(x) = A$$

Sufficient Condition: If f is differentiable at x in $\mathbb{R}$, then f is differentiable at that point, which follows from the definition. □

Theorem 3.108. *Every function differentiable at a point x_0 is continuous at that point.*

Proof.

$$f \text{ differentiable at } x_0 \Rightarrow \exists \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$\Leftrightarrow \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \alpha(x - x_0) \quad \text{where} \quad \lim_{x \to x_0} \alpha(x - x_0) = 0$$

$$\Rightarrow f(x) - f(x_0) = f'(x_0)(x - x_0) + \alpha(x - x_0)(x - x_0)$$

$$\Rightarrow f(x) = f(x_0) + f'(x_0)(x - x_0) + \alpha(x - x_0)(x - x_0)$$

$$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

□

Remark 3.109. *The converse is not true.*

Example 3.110. *The function $y = |x|$, $x_0 = 0$ is continuous at $x_0 = 0$, but:*

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Thus, it is not differentiable at x_0.

3.2.3 Left and Right Derivatives:

Definition 3.111. *We say that f is differentiable at x_0 from the right (resp. from the left) if: 1. f is defined on $[x_0, b]$ (resp. $[a, x_0]$).*

2. $\exists \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = f'_+(x_0)$ $\left(\text{resp.} \quad \exists \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = f'_-(x_0)\right)$

Theorem 3.112. *For a function f to be differentiable at a point x_0, it is necessary and sufficient that it be differentiable at x_0 from the left and from the right and that $f'_+(x_0) = f'_-(x_0)$.*

3.2.4 Differential:

Let f be a function differentiable at x_0. Then:

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \alpha(x - x_0)(x - x_0)$$

Let $x - x_0 = \Delta x \Rightarrow x = x_0 + \Delta x$:

$$f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0) = \alpha(\Delta x)\Delta x \Rightarrow \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

Thus:

$$\Delta f(x_0) = f'(x_0)\Delta x + \alpha(\Delta x)\Delta x$$

Definition 3.113. *The expression $f'(x)\Delta x$ is called the differential of f at x and is denoted by $df(x) = f'(x)\Delta x$.*

In particular, for $f(x) = x$, we have $f'(x) = 1$, so $dx = 1 \cdot \Delta x = \Delta x$. Therefore:

$$df(x) = f'(x)\Delta x \Rightarrow f'(x) = \frac{df(x)}{\Delta x}$$

3.2.5 Application of the Differential to Approximate Calculations:

Find an approximate value of $\sqrt{1.0005}$. If f is differentiable at x_0 in D_f, then:

$$\Delta f(x) = df(x) + \alpha(\Delta x)\Delta x \Rightarrow \Delta f(x) \approx df(x) \Rightarrow f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x$$

$$x - x_0 = \Delta x \Leftrightarrow x = x_0 + \Delta x$$

If $x_0 = 0$, then $f(x) = f(0) + f'(0)x$. In our example: $f(x) = (1 + x)^{\alpha} \Longrightarrow f'(x) = \alpha\,(1 + x)^{\alpha - 1}$

$f(x) = \sqrt{1 + x} = (1 + x)^{\frac{1}{2}} \Longrightarrow f'(x) = \frac{1}{2}\,(1 + x)^{-\frac{1}{2}}$

$f(0) = 1$, $f'(0) = \frac{1}{2} \Longrightarrow f(x) = 1 + \frac{1}{2}x$

Then $\sqrt{1.0005} = \sqrt{1 + 0.0005} = 1 + \frac{1}{2}\,(0.0005) = 1.00025$

Geometric interpretation of the derivative

let f a derivable function at x_0 and $f'(x_0) \neq 0$ and lat C the graph of the function f, $M(x_0, y_0)$ a point of C that is $f(x_0) = y_0$ and $M_0\left(x_0 + \triangle x,\ y_0 + \triangle y\right)$ a variable point on C that is, $f\left(x_0 + \triangle x\right) = y_0 + \triangle y$. i.e MM_0 is a secant of the curve C. When $\triangle x \longrightarrow 0$, $M \longrightarrow M_0$ and the line (MM_0) take the position of the tangent; that is, the derivative of y with respect to x is the slope of the line

$$\begin{aligned}
\lim_{x \longrightarrow x_0} \frac{\triangle y}{\triangle x} &= \lim_{x \longrightarrow x_0} \tan\beta \Longrightarrow \lim_{x \longrightarrow x_0} \frac{f\left(x_0 + \triangle x\right) - f(x_0)}{x - x_0} \\
&= \lim_{\beta \longrightarrow \alpha} \tan\beta = \tan\alpha \Longrightarrow f'(x_0) = \tan\alpha.
\end{aligned}$$

That is, the tangent line to the graph of $y = f(x)$ at $x = a$ is the line with equation $y = f\prime(a)(x - a) + f(a)$

Hence a tangent line to the graph of a function f is a line through a point on the graph of f whose slope is equal to the slope of the graph at that point.

Arithmetic operation on the derivatives

Let f and g be two derivable functions at x_0 then

$f \pm g$, $f.g$, $\lambda f\ (\lambda \in \mathbb{R})$, $\dfrac{f}{g}$ (si $g(x) \neq 0$ on $V(x_0)$ the neighberhood of x_0) are derivable at x_0 and we have

1. $(f \pm g)' = f' \pm \pm g'$
2. $(f.g)' = f'.g + f.g'$
3. $(\lambda f)' = \lambda f'$
4. $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

Let's show the first statement

$$\begin{aligned}
(f + g)'(x_0) &= \lim_{x \longrightarrow x_0} \frac{(f + g)(x) - (f + g)(x_0)}{x - x_0} \\
&= \lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0) + g(x) - g(x_0)}{x - x_0} = f'(x_0) + g'(x_0)
\end{aligned}$$

Derivative of a Composite Function:

Proposition 3.114. *If $y = f(x)$ is differentiable at x_0 and $z = g(y)$ is differentiable at $y_0 = f(x_0)$, then the composite function $g \circ f(x) = g(f(x))$ is differentiable at x_0, and we have:*

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0)$$

Proof.

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{y\to y_0} \frac{g(y)-g(y_0)}{y-y_0} \cdot \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = g'(y_0)f'(x_0)$$

This holds because $y=f(x)$ is differentiable at x_0 and thus continuous at that point. □

3.2.6 Derivative of an Inverse Function:

Theorem 3.115. *If f is bijective in a neighborhood of x_0 and $f'(x_0)$ exists and is nonzero, then f^{-1} is differentiable at $y_0=f(x_0)$, and moreover:*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

Proof.

$$\left(f^{-1}\right)'(y_0) = \lim_{y\to y_0} \frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \lim_{y\to y_0} \frac{1}{\frac{y-y_0}{f^{-1}(y)-f^{-1}(y_0)}} = \lim_{y\to y_0} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)}$$

□

Theorem 3.116. *The following elementary functions are differentiable in the indicated domains, and we have: 1. $y=constant$, $y'=0$, $\forall x\in\mathbb{R}$ 2. $y=x^a$, $y'=ax^{a-1}$, $x>0$ 3. $y=a^x$, $y'=a^x\ln a$, $\forall x\in\mathbb{R}$, $(a>0$ and $a\neq 1)$ 4. $y=\log_a x$, $y'=\frac{1}{x}\log_a e$, $x>0$, $(a>0$ and $a\neq 1)$ 5. $y=\sin x$, $y'=\cos x$, $\forall x\in\mathbb{R}$ 6. $y=\cos x$, $y'=-\sin x$, $\forall x\in\mathbb{R}$ 7. $y=\tan x$, $y'=\frac{1}{\cos^2 x}$, $x\neq\frac{\pi}{2}+k\pi$ 8. $y=\cot x$, $y'=-\frac{1}{\sin^2 x}$, $x\neq k\pi$ 9. $y=\arcsin x$, $y'=\frac{1}{\sqrt{1-x^2}}$, $x\in[-1,+1]$ 10. $y=\arccos x$, $y'=-\frac{1}{\sqrt{1-x^2}}$, $x\in[-1,+1]$ 11. $y=\arctan x$, $y'=\frac{1}{1+x^2}$, $x\in\mathbb{R}$ 12. $y=arccot\ x$, $y'=\frac{-1}{1+x^2}$, $x\in\mathbb{R}$*

13. $y=\sinh x$, $y'=\cosh x$, $x\in\mathbb{R}$
14. $y=\cosh x$, $y'=\sinh x$, $x\in\mathbb{R}$
15. $y=\tanh x$, $y'=\frac{1}{\cosh^2 x}$, $x\in\mathbb{R}$
16. $y=\coth x$, $y'=\frac{-1}{\sinh^2 x}$, $x\neq 0$
17. $y=argsinh\ x$, $y'=\frac{1}{\sqrt{1+x^2}}$, $x\in\mathbb{R}$
18. $y=argcosh\ x$, $y'=\frac{1}{\sqrt{x^2-1}}$, $x>1$
19. $y=argtanh\ x$, $y'=\frac{1}{1-x^2}$, $x\in[-1,+1]$
20. $y=argcoth\ x$, $y'=\frac{-1}{1-x^2}$, $|x|>1$

Proof. 3. $(a^x)'=\lim_{h\to0}\frac{a^{x+h}-a^x}{h}=a^x\lim_{h\to0}\frac{a^h-1}{h}=a^x\ln a$

4. $(\log_a x)'=\frac{1}{(a^y)'}=\frac{1}{a^y\ln a}=\frac{1}{x\ln a}$; $(y=\log_a x\Leftrightarrow x=a^y)$

5. $(\cos x)'=\lim_{h\to0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}=-\lim_{h\to0}\sin\left(x+\frac{h}{2}\right)\lim_{h\to0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}=-\sin x$

9. $(\arcsin x)'=\frac{1}{(\sin y)'}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$; $(y=\arcsin x\Leftrightarrow x=\sin y)$ □

Example 3.117. *Calculate the derivative of the composite function $y=\ln^4\left(\arctan\frac{x}{x^2+1}\right)$.*

$$y'=4\left(\arctan\frac{x}{x^4+1}\right)\ln^3\left(\arctan\frac{x}{x^4+1}\right)=4\left(\frac{x}{x^4+1}\right)\ln^3\left(\arctan\frac{x}{x^4+1}\right)$$

Remark 3.118. *If f is differentiable at x_0 and $f(x_0) \neq 0$, then the function $y = \log|f(x)|$ is differentiable at x_0, and we have:*

$$(\log|f(x)|)' = \frac{f'(x)}{f(x)}$$

3.2.7 Higher-Order Derivatives:

Let f be a differentiable function on an interval I. Then f' is called the first-order derivative of f.

Definition 3.119. *If the derivative $y' = f'(x)$ is differentiable on I, then its derivative is called the second-order derivative of f, denoted by f'', and we have:*

$$f''(x) = (f'(x))'$$

Similarly, we define the n-th order derivative of f, denoted by $f^{(n)}(x)$, as:

$$f^{(n)}(x) = (f^{(n-1)}(x))'$$

Definition 3.120. *The function f is said to be of class $C^{(n)}(I)$ if the n-th derivative of f exists and is continuous on I.*

Example 3.121.

$$\begin{gathered} y = \sin x,\ x \in \mathbb{R},\ y^{(n)} = ? \\ y' = \cos x = \sin\left(x + \frac{\pi}{2}\right) \\ y'' = -\sin x = \sin(x+\pi) = \sin\left(x + 2\cdot\frac{\pi}{2}\right) \\ \vdots \\ y^{(n)} = \sin\left(x + n\cdot\frac{\pi}{2}\right) \quad \textit{(proof by induction)} \end{gathered}$$

3.2.8 Leibniz Formula:

Proposition 3.122. *Let f and g be two functions n times differentiable at x. Then:*

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)$$

Proof. by induction For $n = 1$:

$$(f\cdot g)'(x) = \sum_{k=0}^{1} \binom{1}{k} f^{(1-k)}(x) g^{(k)}(x) = \binom{1}{0} f'(x)g(x) + \binom{1}{1} f(x)g'(x) = f'(x)g(x) + f(x)g'(x)$$

Thus, $P(1)$ is true. Assume that $P(n)$ is true and prove $P(n+1)$:

$$(f\cdot g)^{(n+1)} = \left[\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)\right]' = \sum_{k=0}^{n} \binom{n}{k} \left[f^{(n-k+1)}(x) g^{(k)}(x) + f^{(n-k)}(x) g^{(k+1)}(x)\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k+1)}(x) g^{(k)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k+1)}(x)$$

$$= \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)}(x) g^{(k)}(x) + \binom{n}{0} f^{(n+1)}(x) g(x) + \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)}(x) g^{(k)}(x) + \binom{n}{n} f(x) g^{(n+1)}(x)$$

$$= \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) f^{(n-k+1)}(x) g^{(k)}(x) + f^{(n+1)}(x) g(x) + f(x) g^{(n+1)}(x)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)}(x) g^{(k)}(x)$$

Hence, the proposition $P(n+1)$ holds. □

Example 3.123. *Calculate* $(x^3 \cos 4x)^{(4)}$.

$$(x^3 \cos 4x)^{(4)} = \sum_{k=0}^{4} \binom{4}{k} (\cos 4x)^{(4-k)} (x^3)^{(k)}$$

$$= \binom{4}{0} (\cos 4x)^{(4)} (x^3) + \binom{4}{1} (\cos 4x)^{(3)} (x^3)' + \binom{4}{2} (\cos 4x)^{(2)} (x^3)^{(2)} + \binom{4}{3} (\cos 4x)' (x^3)^{(3)} + \binom{4}{4} (\cos 4x)(x^3$$

$$= 4^4 x^3 \cos 4x - \binom{4}{1} 4^3 \cdot 3x^2 \sin 4x + \binom{4}{2} 4^2 \cdot 6x \cos 4x - \binom{4}{3} \cdot 24 \cdot \sin 4x + \binom{4}{4} \cos 4x \cdot 0$$

3.2.9 Higher-Order Differentials:

Let f be a differentiable function on a set X. Then it is differentiable on X, and we have:

$$df(x) = f'(x) dx$$

This is the first-order differential of f.

If f' is differentiable on X, then $df(x)$ is differentiable on X, and we have:

$$d^2 f(x) = d(df(x)) = d(f'(x) dx) = (f''(x) dx) dx = f''(x) dx^2$$

This is the second-order differential of f.

Similarly, if f is n times differentiable on X, then it is n times differentiable on X, and we have:

$$d^n f(x) = f^{(n)}(x) dx^n$$

3.2.10 Fundamental Theorems on Differentiable Functions:

Rolle's Theorem:

Theorem 3.124. *Rolle's Theorem: Let f be a function satisfying:*

1. *Defined and continuous on the closed interval $[a, b]$*
2. *Differentiable on the open interval (a, b)*
3. *$f(a) = f(b)$ Then there exists a point c in (a, b) such that $f'(c) = 0$.*

**

Proof. Since f attains its maximum and minimum on $[a, b]$ (at least one of these extrema is in $[a, b]$), let:

$$\sup_{[a,b]} f(x) = f(c), \quad c \in [a, b]$$

This gives:

$$\forall \Delta x > 0 \quad (\text{or } \Delta x < 0); \quad f(c + \Delta x) \leq f(c)$$

$$\Rightarrow \begin{cases} \frac{f(c+\Delta x)-f(c)}{\Delta x} \geq 0 & \text{if } \Delta x < 0 \\ \frac{f(c+\Delta x)-f(c)}{\Delta x} \leq 0 & \text{if } \Delta x > 0 \end{cases}$$

Taking the limit as $\Delta x \to 0$:

$$\begin{cases} \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} \geq 0 \implies f'(c) \geq 0 \\ \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} \leq 0 \implies f'(c) \leq 0 \\ \Rightarrow f'(c) = 0 \end{cases}$$

□

Geometric Interpretation:

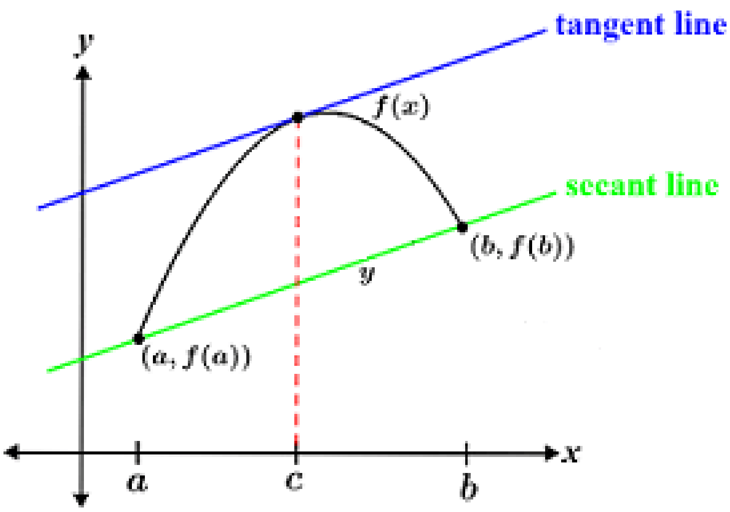
Remark 3.125. *All the conditions of Rolle's Theorem are necessary.*

Lagrange's Theorem (Mean Value Theorem):

Theorem 3.126. *Let f be a function satisfying: 1. Defined and continuous on $[a, b]$ 2. Differentiable on (a, b)*

Then there exists a point c in (a, b) such that:

$$f(b) - f(a) = (b - a)f'(c)$$



Proof. Consider the function:

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

The function $g(x)$ is: 1. Continuous on $[a,b]$ because it is composed of continuous functions. 2. Differentiable on (a,b). 3. $g(a)=0$ and $g(b)=0$. By Rolle's Theorem, there exists a point c in (a,b) such that $g'(c)=0$. Now:

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

Thus:

$$g'(c)=0 \Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \Rightarrow f(b)-f(a) = (b-a)f'(c)$$

□

Cauchy's Theorem:

Theorem 3.127. *Let f and g be two functions satisfying: 1. Defined and continuous on $[a,b]$ 2. Differentiable on (a,b) 3. $g'(x) \neq 0$ for all $x \in (a,b)$ Then there exists c in (a,b) such that:*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof. Consider the function:

$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$$

It is clear that the function F satisfies the conditions of Rolle's Theorem, and thus there exists a point c in (a,b) such that $F'(c)=0$. Now:

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(x)$$

$$F'(c) = 0 \implies f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0 \implies \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

□

3.2.11 Derivative of an Implicit Function:

Let $y = f(x)$ be an implicit function given by the equation $F(x, y) = 0$. That is, $F(x, f(x)) = 0$ for all $x \in (a, b)$.

Derivative of a Function Given in Parametric Form:

Let $y = f(x)$ such that

$$\begin{cases} x = \varphi(t) \\ y = \phi(t) \end{cases} \qquad t_0 \le t \le T.$$

Assume that φ and ϕ are differentiable on (t_0, T) and that $t = \mathcal{G}(x)$ (where $\mathcal{G}$ is the inverse of φ). Then $y = \phi(t) = \phi(\mathcal{G}(x))$. Thus, by the chain rule, we obtain:

$$y'_x = y'_t \cdot t'_x = \frac{y'_t}{x'_t} \implies y'_x = \frac{y'_t}{x'_t}$$

Example 3.128. *Let*

$$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases} \qquad 0 \le t \le \frac{\pi}{2} \quad \textit{then:}$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{a\cos t}{-a\sin t} = -\cot t; \quad t \neq k\pi$$

Equation of the Tangent and Normal to a Curve:

Let $y = f(x)$, $x \in I$, with the curve C and let $M_0(x_0, y_0) \in C$. We want to write the equations of the tangent and normal to the curve C at the point $M_0(x_0, y_0)$. The equation of the tangent passing through the point $M_0(x_0, y_0)$ is given by:

$$y - y_0 = k(x - x_0), \quad y_0 = f(x_0) \quad \text{and} \quad k = \tan\alpha = f'(x_0)$$

Thus:

$$y_T = f'(x_0)(x - x_0) + f(x_0)$$

Equation of Normal

It is of the form $y_N = k_1(x - x_0) + y_0$.

$$MT \perp NA \Rightarrow k_1 = -\frac{1}{k}$$

Thus:

$$y_N = -\frac{1}{k}(x - x_0) + f(x_0)$$

3.2.12 Limit of the Ratio of Two Infinitesimals

First L'Hôpital's Rule:

Theorem 3.129. *Let f and g be two functions defined and differentiable in a punctured neighborhood $V^0(x_0)$ satisfying the following conditions: 1.*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

2.

$$g'(x) \neq 0 \quad on \quad V^0(x_0)$$

If there exists:

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = I \quad then \quad \exists \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} \quad and\ we\ have: \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = I$$

Proof. Condition 1 implies that f and g can be extended to x_0 and that $f(x_0) = g(x_0) = 0$.

Let x be an arbitrary point in $V^0(x_0)$. Then f and g satisfy the conditions of Cauchy's Theorem on the interval $[x_0, x]$, i.e.:

$$\exists c \in [x_0, x];$$

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(c)}{g'(c)} = I$$

□

Remark 3.130. *L'Hôpital's Rule is valid if:*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad exists,\ because\ sometimes \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} \quad exists$$

while:

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad does\ not\ exist.$$

Example 3.131.

$$f(x) = x^2 \cos \frac{1}{x}, \quad g(x) = x; \quad x \to 0$$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \cos \frac{1}{x}}{x} = \lim_{x \to 0} x \cos \frac{1}{x} = 0$$

However:

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{2x \cos \frac{1}{x} + \sin \frac{1}{x}}{1} \quad does\ not\ exist!$$

Remark 3.132. *L'Hôpital's Rule remains valid when $x \to \infty$ because:*

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{t \to 0} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t \to 0} \frac{f'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}{g'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)} = \lim_{t \to 0} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

Remark 3.133. *If $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \frac{0}{0}$ and the functions $f'(x)$ and $g'(x)$ satisfy the conditions of L'Hôpital's Rule, then we can apply L'Hôpital's Rule again.*

Limit of the Ratio of Two Infinitely Large Functions: Second L'Hôpital's Rule:

Theorem 3.134. *Let* f *and* g *be two functions defined and differentiable in a punctured neighborhood* $V^0(x_0)$ *satisfying the following conditions: 1.* $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \infty$ *2.* $g'(x) \neq 0$ *on* $V^0(x_0)$ *If there exists* $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l$*, then* $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ *exists and we have:*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = l$$

Remark 3.135. *Remarks 1-3 remain valid for Theorem 2.*

Example 3.136. *Calculate* $\lim_{x\to\infty} \frac{x^n}{e^x}$.

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0$$

Other Indeterminate Forms:

$\alpha 0, \omega, \gamma, \omega + \infty - \infty, \omega, \gamma, \omega^0, \gamma, \omega^0$. 1. If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = \infty$, then:

$$\lim_{x\to x_0} [f(x)g(x)] = \lim_{x\to x_0} \frac{f(x)}{\frac{1}{g(x)}} = \left(\frac{0}{0}\right)$$

2. If $\lim_{x\to x_0} f(x) = \infty$ and $\lim_{x\to x_0} g(x) = \infty$, then:

$$\lim_{x\to x_0} [f(x) - g(x)] = \lim_{x\to x_0} \frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{g(x)}} = \left(\frac{0}{0}\right)$$

3. If $\lim_{x\to x_0} f(x) = 1$ and $\lim_{x\to x_0} g(x) = \infty$, then by setting $y = (f(x))^{g(x)}$, we obtain:

$$\ln y(x) = g(x) \ln f(x)$$

$$\lim_{x\to x_0} (\ln y(x)) = \lim_{x\to x_0} g(x) \ln f(x) = l \quad \text{(of the form } \infty \cdot 0)$$

$$\ln\left(\lim_{x\to x_0} y(x)\right) = l \Rightarrow \lim_{x\to x_0} y(x) = e^l$$

4. If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = \infty$, then by setting $y = (f(x))^{g(x)}$, we obtain a limit of the form $(\infty \cdot 0)$, as mentioned in the first case. 5. If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$, then by setting $y = (f(x))^{g(x)}$, we obtain a limit of the form 1.

3.2.13 Criterion for Monotonicity of a Real Function:

Theorem 3.137. *Let* f *be a function from* $I \to \mathbb{R}$ *differentiable on* I*. Then the following equivalences hold: 1.* $f' \geq 0$ *on* $I \iff f$ *is increasing on* I*.*

2. $f' \leq 0$ *on* $I \iff f$ *is decreasing on* I*.*

Proof. $(\Rightarrow)$ Suppose f is increasing on I and prove that $f' \geq 0$. Indeed, let x_0 be an arbitrary point in I. Then for all $h \in \mathbb{R}^*$, we have:

$$\frac{f(x_0+h) - f(x_0)}{h} \geq 0 \Rightarrow \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \geq 0 \Rightarrow f'(x_0) \geq 0$$

($\Leftarrow$) Suppose $f' \geq 0$ for all $x \in I$ and prove that f is increasing.
Let x_1, x_2 be two arbitrary points in I such that $x_1 < x_2$.
By the Mean Value Theorem on $[x_1, x_2]$, there exists a point $c \in [x_1, x_2]$ such that:

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$$

$f'(c) \geq 0 \quad \text{and} \quad (x_2 - x_1) \geq 0 \Rightarrow f'(c)(x_2 - x_1) \geq 0 \Rightarrow f(x_2) - f(x_1) \geq 0 \Rightarrow f(x_2) \geq f(x_1)$

Thus, f is increasing on I. □

Example 3.138. *Show that* $\tan x > x + \frac{x^3}{3}$ *for all* $x \in \left[0, \frac{\pi}{2}\right]$.
Let $y = \tan x - \left(x + \frac{x^3}{3}\right)$.

$$y(0) = 0$$

$y'(x) = \dfrac{1}{\cos^2 x} - (1 + x^2) = 1 + \tan^2 x - 1 - x^2 = \tan^2 x - x^2 > 0 \quad \forall x \in \left[0, \dfrac{\pi}{2}\right] \Rightarrow y(x)$ *is increasing.*

Thus:

$$\tan x - \left(x + \frac{x^3}{3}\right) > 0 \iff \tan x > x + \frac{x^3}{3}$$

3.2.14 Extrema:

Let f be a function defined on I, and $x_0 \in I$.

Definition 3.139. *We say that f has a local maximum (resp. local minimum) at the point x_0 if there exists a neighborhood V of x_0 such that: For all $x \in V$, $f(x) \leq f(x_0)$ [resp. $f(x) \geq f(x_0)$]. "max f" and "min f" are called the extrema of the function f.*

Maximum local **Minimum local**

x_0 x_1

Definition 3.140. *We say that f has an absolute maximum (resp. absolute minimum) at the point x_0 of I if*

$$\forall x \in I, f(x) \leq f(x_0) \quad \textit{[resp. } f(x) \geq f(x_0)\textit{]}$$

Necessary Condition for an Extremum:

Theorem 3.141. *If f has an extremum at x_0 and is differentiable at this point, then $f'(x_0) = 0$.*

Proof. Analogous to the proof of Rolle's Theorem. □

Remark 3.142. *This theorem is not sufficient for the existence of an extremum.*

Example 3.143. *The function $y = x^3$ does not have an extremum at $x_0 = 0$ even though $y'(0) = 0$.*

Remark 3.144. *A function f may have an extremum at a point x_0 without being differentiable at that point.*

Example 3.145. *The function* $y = |x|$ *is not differentiable at* $x_0 = 0$*, but it has a minimum at this point. The points where* f' *is zero or does not exist are called critical points of* f*.*

First Sufficient Condition:

Proposition 3.146. *Let* f *be defined and continuous in a neighborhood* V *of the point* x_0*, and differentiable in this neighborhood except possibly at* x_0*. If the derivative* f' *changes sign from left to right of the point* x_0*, then* f *has an extremum at this point. Moreover:*

$$\begin{cases} f' > 0 \quad \text{if } x < x_0 \quad \text{and } f' < 0 \quad \text{if } x > x_0 \Rightarrow x_0 \quad \text{is a maximum point of } f \\ f' < 0 \quad \text{if } x < x_0 \quad \text{and } f' > 0 \quad \text{if } x > x_0 \Rightarrow x_0 \quad \text{is a minimum point of } f \end{cases}$$

Proof. Assume that $f' > 0$ if $x < x_0$ and $f' < 0$ if $x > x_0$, and show that x_0 is a maximum of f. By the Mean Value Theorem, there exists a point c such that:

$$f(x) - f(x_0) = (x - x_0)f'(c)$$

If $x < x_0$, $f'(c) > 0$ and $(x - x_0) < 0$, then $f'(c)(x - x_0) < 0 \Rightarrow f(x) < f(x_0)$.
If $x > x_0$, $f'(c) < 0$ and $(x - x_0) > 0$, then $f'(c)(x - x_0) < 0 \Rightarrow f(x) < f(x_0)$.
Thus, x_0 is a maximum point of f. □

Example 3.147. *Determine the extrema of* $y = 3x^2 - 10x + 5$*. We have:*

$$y'(x) = 0 \Leftrightarrow 6x - 10 = 0 \Leftrightarrow x = \frac{5}{3}$$

And:

$$y' : \quad \underset{\frac{5}{3}}{-} \quad \underset{\frac{5}{3}}{|} \quad +$$

Second Sufficient Condition:

Proposition 3.148. *Let* f *be a function twice differentiable at the point* x_0 *with* $f'(x_0) = 0$*. Then:*

- If $f''(x_0) < 0$*, then* x_0 *is a maximum of* f*. - If* $f''(x_0) > 0$*, then* x_0 *is a minimum of* f*.*

Proof. Assume that $f'(x_0) = 0$ and $f''(x_0) < 0$, and prove that x_0 is a maximum point. Indeed:

$$f''(x_0) < 0 \Rightarrow (f')'(x_0) < 0 \Rightarrow f \text{ is decreasing in the neighborhood of } x_0.$$

Moreover, $f'(x_0) = 0$, so:

$$f' > 0 \quad \text{if} \quad x < x_0 \quad \Rightarrow \quad x_0 \text{ is a maximum point of } f.$$

$$f' < 0 \quad \text{if} \quad x > x_0.$$

□

Remark 3.149. *If $f'(x_0) = 0$ and $f''(x_0) = 0$, we cannot conclude anything.*

Third Sufficient Condition:

Proposition 3.150. *Let f be a function n times differentiable at $x = x_0$. If $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$, then:*

1. When n is an even number, the function f has an extremum at x_0, and moreover: - a) $f^{(n)}(x_0) < 0 \Rightarrow x_0$ is a maximum. - b) $f^{(n)}(x_0) > 0 \Rightarrow x_0$ is a minimum.

2. When n is an odd number, the function f does not have an extremum.

Convexity – Concavity

Definition 3.151. *The function $f : I \to \mathbb{R}$ with the representative curve is said to be convex (resp. concave) on the interval I if any chord passing through arbitrary points of C lies above the arc AB (resp. below AB) of this curve.*

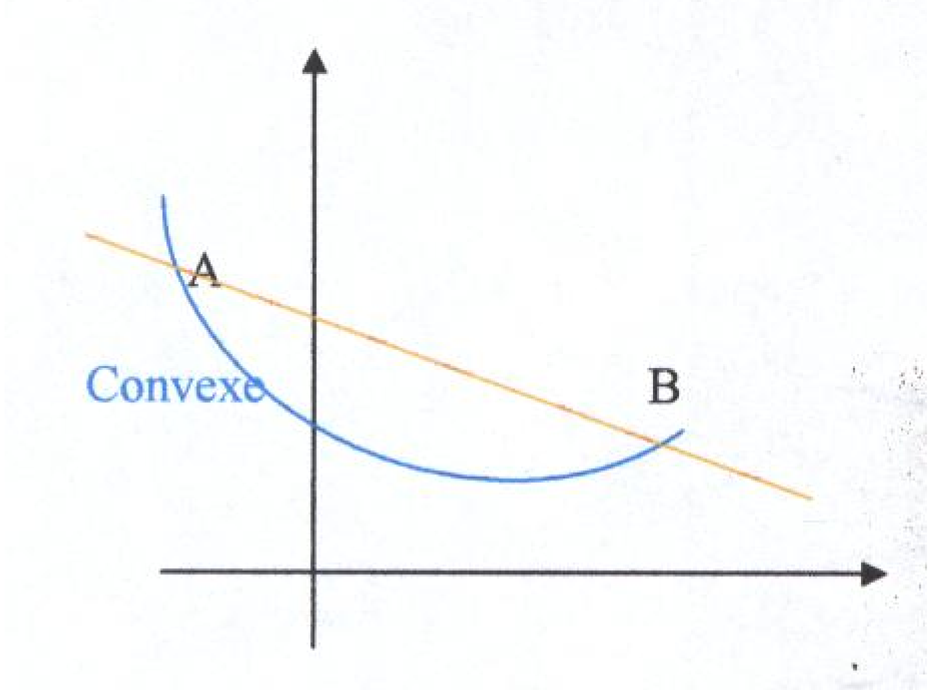
Definition 3.152. *The function $f : I \to \mathbb{R}$ with the representative curve is said to be convex (resp. concave) on the interval I if:*

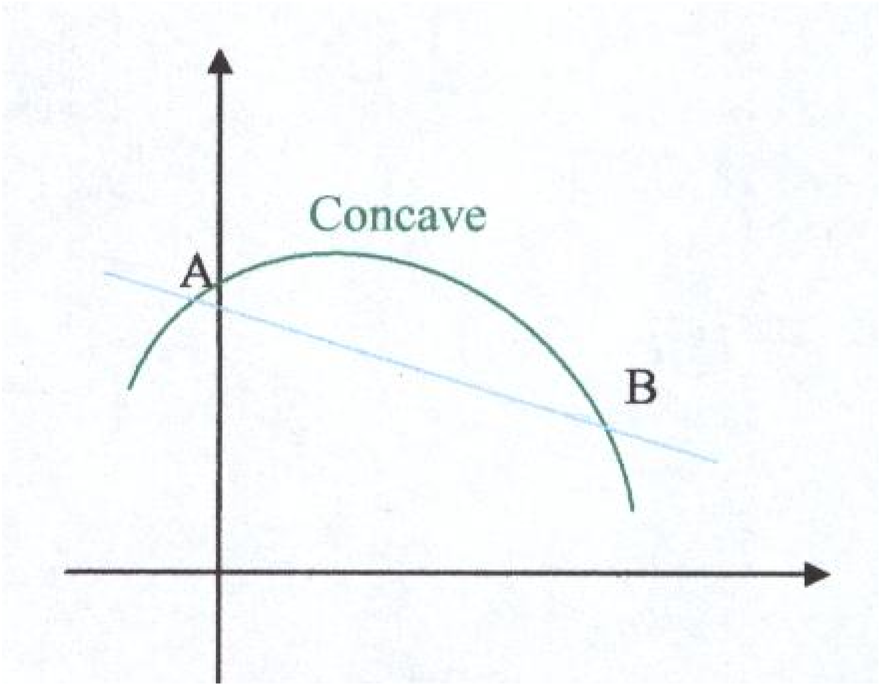
$$\forall x_1, x_2 \in I, \quad \forall q_1 > 0, \quad q_2 > 0 \quad \text{with} \quad q_1 + q_2 = 1 \quad \text{we have}$$
$$f(q_1x_1 + q_2x_2) \leq q_1 f(x_1) + q_2 f(x_2) \, [resp. \cdot f(q_1x_1 + q_2x_2) \geq q_1 f(x_1) + q_2 f(x_2)]$$

Definition 3.153. *The function $f : I \to \mathbb{R}$ with the representative curve is said to be convex (resp. concave) on the interval I if any tangent to the curve C at the point with abscissa x_0 of I lies above the curve (resp. below the curve).*

Theorem 3.154. *If $f : I \to \mathbb{R}$ is twice differentiable on I, then the following equivalences hold:*

1. *f is convex on $I \Leftrightarrow f'' \geq 0$ on I*
2. *f is concave on $I \Leftrightarrow f'' \leq 0$ on I*





Proof. Let $f : I \to \mathbb{R}$ be twice differentiable on I and x_0 an arbitrary point of I. The equation of the tangent to the curve at the point x_0 is given by $y_T = f'(x_0)(x-x_0)+f(x_0)$. The equation of the curve is $y = f(x)$. We have:

$y-y_T = f(x)-f(x_0)-f'(x_0)(x-x_0)$ By Lagrange's Theorem: $f'(c)(x-x_0)-f'(x_0)(x-x_0), \quad (x < c <$

$= [f'(c)-f'(x_0)](x-x_0)$ By Lagrange's Theorem: $f''(c)(c-x_0)(x-x_0), \quad (c < c < x_0)$

The sign of $y - y_T$ is the same as that of $f''(c)$ because $(c - x_0)(x - x_0) > 0$. ☐

3.2.15 Points of Inflection

Definition 3.155. *Let $f : I \to \mathbb{R}$ with the curve C.*

The point $M_0(x_0, y_0)$ is called a point of inflection of the curve C at x_0 if the curve C is convex on one side of x_0 and concave on the other side.

Necessary Condition: If f is twice differentiable at x_0 and f' has a point of inflection at x_0, then $f''(x_0) = 0$.

Sufficient Condition: Let f be twice differentiable in a neighborhood of the point x_0, except possibly at x_0. If f'' changes sign from left to right of the point x_0, then f has a point of inflection at x_0.

3.2.16 Asymptotes:

Let $f : I \to \mathbb{R}$ with the curve C.

Definition 3.156. *The line $x = x_0$ is called a vertical asymptote of the curve C if:*

$$\lim_{x \to x_0} f(x) = \infty$$

Example 3.157. *The line $x = 0$ is an asymptote of the curve $y = \log x$.*

Definition 3.158. *The line $y = y_0$ is called a horizontal asymptote of the curve C if:*

$$\lim_{x \to \infty} f(x) = y_0$$

Example 3.159. *The line $y = \frac{\pi}{2}$ is an asymptote of the curve $y = \arctan x$.*

Definition 3.160. *The line $y = kx + b$ is called an oblique asymptote of the curve C if there exists a function $h = h(x)$ such that $f(x) = kx + b + h(x)$ and $\lim_{x \to \infty} h(x) = 0$. We have:*

$$f(x) = kx + b + h(x) \Rightarrow k = \frac{f(x)}{x} - \frac{b}{x} - \frac{h(x)}{x}$$

Taking the limit as x tends to infinity, we obtain:

$$k = \lim_{x \to \infty} \frac{f(x)}{x} \quad \textit{and} \quad b = \lim_{x \to \infty} [f(x) - kx]$$

General Scheme for Plotting a Curve:

The study of functions generally involves determining:

1. The domain of definition of the function. 2. The points of discontinuity of the function. 3. The intervals of increase and decrease of the function. 4. The points of extrema as well as the maximum and minimum values of the function. 5. The intervals of convexity and concavity as well as the points of inflection. 6. The graphical asymptotes of the function.

Chapter 4

Taylor Formula and Finite Expansion

In the case of a polynomial P of degree n, knowing the value of P and its derivatives up to order n at a point a allows us to determine P at any point $x = a + h$ in $\mathbb{R}$. However, this property no longer holds for arbitrary functions. For a general function f, knowing its value and derivatives up to order n at a point a provides only an n^{th} order approximation of $f(a + h)$ near $a$achieved by equating it to the value of a polynomial P_n, called the "regular part" of its Taylor expansion. In analysis, the Taylor-Lagrange formula, named after mathematician Brook Taylor (1685–1731), who established it in 1712, approximates a sufficiently differentiable function near a point using a polynomial whose coefficients depend solely on the function's derivatives at that point. This chapter begins with the theory of function comparison near a point and Landau notation, followed by a detailed presentation of Taylor's formula and its core concepts, including finite expansions and the Taylor-Young theorem. We will explore Finite expansions of common elementary functions and conclude with practical applications of these expansions, demonstrating their utility in simplifying limits, asymptotic analysis, and error estimation.

4.1 Taylor's Formula for a Polynomial:

Let $P(X)$ be a polynomial of degree n expressed in terms of increasing powers of X:

$$P(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$$

We want to express this polynomial in terms of powers of $(X - x_0)$, where $x_0 \in \mathbb{R}$. For example, $X^2 + 2X + 5 = 4 + (X + 1)^2$. To do this, we set:

$$P(X) = b_0 + b_1(X - x_0) + b_2(X - x_0)^2 + \cdots + b_n(X - x_0)^n \quad (1)$$

where the coefficients $b_0, b_1, \dots, b_n$ are unknowns. We have:

$$X = x_0 \Rightarrow P(x_0) = b_0$$

$$P'(X) = b_1 + 2b_2(X - x_0) + 3b_3(X - x_0)^2 + \cdots + nb_n(X - x_0)^{n-1} \Rightarrow P'(x_0) = b_1$$

$$P''(X) = 2b_2 + 6b_3(X - x_0) + \cdots + n(n - 1)(X - x_0)^{n-2} \Rightarrow P''(x_0) = 2b_2 \Rightarrow b_2 = \frac{P''(x_0)}{2!}$$

$$\vdots$$

Similarly, we obtain:

$$b_n = \frac{P^{(n)}(x_0)}{n!}$$

Substituting into equation (1), we get: Taylor's formula for a polynomial is given by:

$$P(X) = P(x_0) + \frac{P'(x_0)}{1!}(X - x_0) + \frac{P''(x_0)}{2!}(X - x_0)^2 + \cdots + \frac{P^{(n)}(x_0)}{n!}(X - x_0)^n \quad \ldots \ (2)$$

4.2 Taylor's Formula for a Function

Let f be a function n times differentiable at the point x_0. We aim to construct a polynomial of degree n in increasing powers of x_0 satisfying the following conditions:

$$P(x_0) = f(x_0)$$

$$P'(x_0) = f'(x_0)$$

$$P''(x_0) = f''(x_0)$$

$$P^{(n)}(x_0) = f^{(n)}(x_0)$$

From the previous formula, it follows that:

$$P(X) = f(x_0) + \frac{f'(x_0)}{1!}(X - x_0) + \frac{f''(x_0)}{2!}(X - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(X - x_0)^n \quad \ldots \ (3)$$

Let:

$$f(x) - P(x) = R_n(x, x_0)$$

Then we have:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x, x_0) \quad \ldots \ (4)$$

Expression (4) is called Taylor's formula for the function f in the neighborhood of the point x_0 with the remainder $R_n(x, x_0)$.

4.3 Estimation of the Remainder

Theorem 4.1. *If the function $r = r(x)$ is differentiable up to order n at the point x_0 and moreover*

$$r(x_0) = r'(x_0) = r''(x_0) = \cdots = r^{(n)}(x_0) = 0$$

Then $r(x)$ is a function negligible compared to $(x - x_0)^n$ in the neighborhood of x_0, $c - a - d$:

$$r(x) = o((x - x_0)^n) \quad x \to x_0$$

Proof. We prove that

$$\lim_{x\to x_0}\frac{r(x)}{(x-x_0)^n}=0$$

By applying L'Hôpital's rule n times, we obtain:

$$\lim_{x\to x_0}\frac{r(x)}{(x-x_0)^n}=\lim_{x\to x_0}\frac{r'(x)}{n(x-x_0)^{n-1}}=\lim_{x\to x_0}\frac{r''(x)}{n(n-1)(x-x_0)^{n-2}}=\cdots=\lim_{x\to x_0}\frac{r^{(n)}(x)}{n!(x-x_0)^0}=0$$

☐

4.3.1 Taylor's Formula with Peano's Remainder

Let f be a function differentiable up to order n at the point x_0. Then there exists a neighborhood of the point x_0 in which the following formula holds:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n)$$

Proof. We have

$$f(x)-P_n(x)=R_n(x,x_0)$$

where

$$P_n(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Hence, it follows that $R_n(x,x_0)$ satisfies the conditions of the previous theorem and

$$R_n(x,x_0)=o((x-x_0)^n)\quad x\to x_0$$

☐

4.3.2 Taylor's Formula with General Remainder

Theorem 4.2. *Let f be a function differentiable up to order $n+1$ in the neighborhood of the point x_0. Then, for all $\delta>0$ and for all $x\in V(x_0)$ with $[x_0,x]\subseteq V(x_0)$ (or $[x,x_0]\subseteq V(x_0)$), there exists $c\in[x_0,x]$ such that:*

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{n!}\left(\frac{x-x_0}{x-c}\right)^{\delta}\frac{(x-c)^{n+1}}{n!\delta}$$

4.3.3 Taylor's Formula with Lagrange Remainder

Remark 4.3. *In the general formula, if we take $\delta=n+1$, we find:*

$$R_n(x,x_0)=f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{(x-c)^{n+1}}\frac{(x-c)^{n+1}}{n!(n+1)}=f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{n!(n+1)}=f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{(n+1)}$$

Hence, Taylor's formula with Lagrange remainder is given by:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

4.3.4 Determining an Approximate Value of c

Let:

$$\frac{c - x_0}{x - x_0} = \theta, \quad 0 < \theta < 1$$

$$c - x_0 = \theta(x - x_0) \Rightarrow c = \theta(x - x_0) + x_0$$

Thus, Taylor's formula with Lagrange remainder can be written as:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1}$$

4.4 Taylor's Formula with Cauchy Remainder

In the general formula, if we take $\delta = 1$, we find:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1}(1 - \theta)^n$$

Because we have:

$$c - x_0 = \theta(x - x_0) \Leftrightarrow x - c + x_0 = x - \theta(x - x_0) \Leftrightarrow x - c = (x - x_0) - \theta(x - x_0)$$

$$\Leftrightarrow (x - c)^n = (x - x_0)^n (1 - \theta)^n$$

$$\Rightarrow (x - x_0)(x - c)^n = (x - x_0)^{n+1}(1 - \theta)^n$$

4.5 Maclaurin Formulas

Maclaurin formulas are obtained from Taylor's formulas for the special case $x_0 = 0$. thus, by sunstitution, we can get the following formulas

1. **Maclaurin with Peano Remainder**

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n)$$

2. **Maclaurin with Lagrange Remainder**

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

3. **Maclaurin with Cauchy Remainder**

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\alpha x)}{n!}x^{n+1}(1 - \theta)^n$$

4.6 Application of Maclaurin's Formula for Common Functions

4.6.1 Maclaurin Series for e^x

For $f(x) = e^x$, we have $f^{(n)}(x) = e^x$ and thus $f^{(n)}(0) = 1$. Also, $f^{(n+1)}(\theta x) = e^{\theta x}$.

Therefore, the Maclaurin series for e^x is:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!}x^{n+1}$$

4.6.2 Maclaurin Series for $\sin x$

For $f(x) = \sin x$, we have:

$$f^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right) \Rightarrow f^{(n)}(0) = \sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2p \\ (-1)^p & \text{if } n = 2p+1 \end{cases}$$

Also:

$$f^{(2p+2)}(\theta x) = \sin\left(\theta x + (2p+2)\frac{\pi}{2}\right) = \sin(\theta x + (p+1)\pi) = (-1)^{p+1}\sin(\theta x)$$

Therefore, the Maclaurin series for $\sin x$ is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^p x^{2p+1}}{(2p+1)!} + \frac{(-1)^{p+1}\sin(\theta x)}{(2p+2)!}x^{2p+2}$$

4.6.3 Maclaurin Series for $\cos x$

For $f(x) = \cos x$, we have:

$$f^{(n)}(x) = \cos\left(x + n\frac{\pi}{2}\right) \Rightarrow f^{(n)}(0) = \cos\left(n\frac{\pi}{2}\right)$$

Similarly, we obtain the Maclaurin series for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^p x^{2p}}{(2p)!} + \frac{(-1)^{p+1}\sin(\theta x)}{(2p+1)!}x^{2p+1}$$

4.6.4 Maclaurin Series for $\log(1+x)$

For $f(x) = \log(1+x)$ with $x > -1$, we have:

$$f'(x) = \frac{1}{1+x}; \quad f''(x) = \frac{-1}{(1+x)^2}; \quad f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

Thus:

$$f^{(n)}(0) = (-1)^{n+1}(n-1)!$$

Therefore, the Maclaurin series for $\log(1+x)$ is:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \frac{(-1)^n x^{n+1}}{(n+1)(1+\theta x)^{n+1}}$$

Chapter 5

Algebraic Structures

Algebraic structures, built upon internal composition laws, provide the framework for unifying symmetry, operations, and mathematical abstraction. At their foundation lie groups sets equipped with a single associative operation, an identity element, and inverses that formalize symmetry in mathematics and science. When this operation is commutative, we encounter abelian (commutative) groups, essential in number theory and linear algebra. Expanding to structures with two compatible operations, rings generalize arithmetic with addition and multiplication, while fields (such as $\mathbb{R}, \mathbb{Q}$, or $\mathbb{C}$) enrich this duality by ensuring every non-zero element has a multiplicative inverse. This chapter explores the axiomatic definitions and interconnections of these structures, from the simplicity of group axioms to the layered constraints of rings and fields. Through examples like modular arithmetic, polynomial rings, and finite fields. By mastering these concepts, we unlock the language of modern algebra, bridging discrete operations to continuous systems and empowering applications across mathematics and its sciences.

5.1 Internal Composition Laws

Definition 5.1. *Let E be a non-empty set. An internal composition law on E is any mapping from $E \times E \to E$. If E is equipped with an internal composition law $*$, then $(E, *)$ is called a magma.*

Example 5.2.
- $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \to x + y$
- $\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \to x \times y$

Proposition 5.3. *Let E be a set and $(**)$, $(*)$ be two internal composition laws on E, then:*

1. *$(**)$ is commutative $\Leftrightarrow \forall x, y \in E, x * y = y * x$*
2. *$(**)$ is associative $\Leftrightarrow \forall x, y, z \in E, x * (y * z) = (x * y) * z$*
3. *$(**)$ has an identity element $\Leftrightarrow \exists e \in E, \forall x \in E, x * e = e * x = x$*
4. *If $(**)$ has an identity element e and $x, x' \in E$, x' is the inverse of x for $* \Leftrightarrow x*x' = x' * x = e$.*

5. $(**)$ *is distributive over* $(**) \Leftrightarrow \forall x, y, z \in E$,

$$\begin{cases} (x * y) \cdot z = (x * z) * (y * z) \\ \text{and } x \cdot (y * z) = (x * y) * (x * z) \end{cases}$$

6. *An element* a *is said to be regular or cancellable if:* $\forall (x, y) \in E^2$

$$a * x = a * y \Rightarrow x = y \quad \& \quad x * a = y * a \Rightarrow x = y$$

Remark 5.4.

1. *If* $(**)$ *is a commutative law, then to prove that it has an identity element (resp. inverse element), it suffices to prove the existence of the latter on the left or on the right.*
2. *When the law is denoted* $(**)$ *or* $(**)$*, the inverse of an element* x *is denoted* x^{-1}*. When it is denoted* $(**)$*, the inverse of* x *is denoted* $(-x)$*.*

5.1.1 Groups

Definition 5.5. *Let* G *be a set equipped with a law* $(**)$*. We say that* $(G, *)$ *is a group if and only if:*

1. $(**)$ *is internal in* G*.*
2. $(**)$ *is associative.*
3. $(**)$ *has an identity element in* G*.*
4. *Every element of* G *has an inverse for the law* $(**)$*.*

If, in addition, $(**)$ is commutative, we say that G is an abelian (or commutative) group. If the number of elements in G is card $G = n$, we say that G is a finite group of order n.

The most commonly used notations are as follows

Law	Composition of two elements	Identity	Inverse of an element x	Composition of x and th
$*$	$x * y$	e	x^{-1}	$x * y^{-1}$
$\times$	$x \times y$	1	x^{-1} or $\frac{1}{x}$	$x \times y^{-1}$
$\circ$	$f \circ g$	Id	f^{-1}	$f \circ g^{-1}$
T	xTy	e	x^{-1}	xTy^{-1}
$+$	$x + y$	0	$-x$	$x - y$

Example 5.6. $(\mathbb{R}, +)$, $(\mathbb{R}^*, \times)$, $(\mathbb{L}, +)$ *are abelian groups.*

Remark 5.7. $(\mathbb{N}, +)$ *is not a groups, because the nonexistence of inverse.*

Exercise 5.8. *Consider the set* $E = \left\{ \frac{1}{1+ab} \mid a, b \in E \right\}$ *with the operation defined by:*

$$a * b = \frac{a + b}{1 + ab}.$$

Show that E *is an abelian group.*

Solution:

1. **Closure:** We need to show that $\forall a, b \in E, a * b \in E$, i.e., $-1 < a * b < 1$. This is equivalent to:

$$\left|\frac{a+b}{1+ab}\right| < 1 \iff |a+b| < |1+ab| \iff |a+b| < 1+ab \iff |a+b| - 1 - ab < 0.$$

Since $a, b \in [-1, 1]$, we have $|ab| < 1$ and $1 + ab > 0$.

- **Case 1:** If $a + b \leq 0$, then:

$$|a+b| - 1 - ab = -a - b - 1 - ab = -(1+a)(1+b) < 0.$$

- **Case 2:** If $a + b > 0$, then:

$$|a+b| - 1 - ab = a + b - 1 - ab = -(1-a)(1-b) < 0.$$

From both cases, we conclude that the operation is closed.

2. **Commutativity:** The operation $*$ is commutative because addition and multiplication in $\mathbb{R}$ are commutative:

$$a * b = \frac{a+b}{1+ab} = \frac{b+a}{1+ba} = b * a.$$

3. **Associativity:** The operation $*$ is associative (left as an exercise for the student).

4. **Identity Element:** The element 0 is the identity element for $*$:

$$a * 0 = \frac{a+0}{1+a \cdot 0} = a = 0 * a.$$

5. **Inverse Element:** For each $a \in [-1, 1]$, the inverse a' satisfies $a * a' = 0$:

$$a * a' = 0 \iff a + a' = 0 \iff a' = -a.$$

Since $-a \in [-1, 1]$, every element in E has an inverse in E.

Therefore, $(E, *)$ is an abelian group.

5.1.2 Subgroups

Definition 5.9. *Let $(G, *)$ be a group and $\emptyset \neq H \subseteq G$. We say that H is a subgroup of G if and only if:*

$$\forall x, y \in H, \quad x * y^{-1} \in H,$$

where y^{-1} is the inverse of y with respect to $*$.

Example 5.10. *1. If G is a group with identity element e, then $\{e\}$ and G are subgroups of G, with $\{e\}$ being the smallest and G being the largest.*

2. For any $n \in \mathbb{N}$, the set $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{Z}, +)$.

Proposition 5.11. *Let $(G, *)$ be a group with identity element e, and let $\emptyset \neq H \subseteq G$. Then:*

$$\begin{cases} e \in H, \\ \forall x, y \in H, \quad x * y \in H, \\ \forall x \in H, \quad x^{-1} \in H. \end{cases}$$

Proof. "$\Rightarrow$" 1. Since H is a subgroup of G, $e \in H$ because H is non-empty.
2. Let $x \in H$. Then x^{-1} exists in G. We need to show that $x^{-1} \in H$. In fact
Let H be a subgroup of a group G.
3. If $e \in H$ and $x \in H$, then $e * x^{-1} \in H$ (since H is a subgroup) $\Rightarrow x^{-1} \in H$.
4. Let $x, y \in H$. Then $y^{-1} \in H$, and since H is a subgroup of G:

$$x * (y^{-1})^{-1} \in H \Rightarrow x * y \in H$$

The converse is obvious. □

Proposition 5.12. *Let G be a group, and $(H_i)_{i\in I}$ a family of subgroups of G. Then:*

$$\bigcap_{i\in I} H_i$$

is a subgroup of G.

Proof. Let $H = \bigcap_{i\in I} H_i$.
1. For all $i \in I$, $e \in H_i \Rightarrow e \in \bigcap_{i\in I} H_i = H$.
2. Let $x, y \in H$. Then for all $i \in I$, $x \in H_i$ and $y \in H_i \Rightarrow$ for all $i \in I$, $x * y \in H_i$ $\Rightarrow x * y \in \bigcap_{i\in I} H_i$.
3. Let $x \in H$. Then for all $i \in I$, $x \in H_i \Rightarrow$ for all $i \in I$, $x^{-1} \in H_i \Rightarrow x^{-1} \in \bigcap_{i\in I} H_i$ $\Rightarrow x^{-1} \in H$.
Therefore, $H = \bigcap_{i\in I} H_i$ is a subgroup of G. □

Remark 5.13. *In general, the union of subgroups is not a subgroup.*

Example 5.14. *For $G = \mathbb{Z}$, $H_1 = 2\mathbb{Z}$, and $H_2 = 3\mathbb{Z}$, we have $H_1 \cup H_2 = 2\mathbb{Z} \cup 3\mathbb{Z}$.*
$2 \in 2\mathbb{Z}$ *and* $3 \in 3\mathbb{Z}$, *but* $2 + 3 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$.

Definition 5.15. *Let $(G, *)$ be a group and $A \subset G$. The intersection of all subgroups of G containing A is a subgroup of G called the subgroup generated by A and denoted $\langle A\rangle$.*

Proposition 5.16. *Let $(G, *)$ be a group and $A \subset G$, and $\langle A\rangle$ the subgroup generated by A. Then $\langle A\rangle$ is the smallest subgroup of G containing A (with respect to inclusion).*

Group Homomorphisms

Definition 5.17. *Let $(G, *)$ and $(G', \perp)$ be two groups. A group homomorphism (or simply homomorphism) is a function $f : (G, *) \to (G', \perp)$ such that:*

$$\forall x, y \in G, f(x * y) = f(x) \perp f(y).$$

Terminology
1. An endomorphism of groups is a homomorphism $f : G \to G$.
2. An isomorphism of groups is a bijective homomorphism.
3. An automorphism of groups is a bijective endomorphism.

Proposition 5.18. *Let $(G, *)$ be a group with identity element e, and $(G', \perp)$ a group with identity element e'. Let $f : (G, *) \to (G', \perp)$ be a group homomorphism. Then:*
1. $f(e) = e'$.
2. $\forall x \in G, f(x^{-1}) = (f(x))^{-1}$.

Proof. 1. Let z be the inverse of $f(e)$ in G'. Since $f(e) = f(e * e) = f(e) \perp f(e)$, we have:

$$e' = z \perp f(e) = z \perp (f(e) \perp f(e)) = (z \perp f(e)) \perp f(e) = e' \perp f(e) = f(e).$$

2. Since $e' = f(e) = f(x * x^{-1}) = f(x) \perp f(x^{-1})$ for all $x \in G$, it follows that:

$$f(x) \perp f(x^{-1}) = e' \iff f(x^{-1}) = (f(x))^{-1}.$$

□

Proposition 5.19. *Let $(G, *)$, $(G', \perp)$, and (G'', ∇) be three groups. Let $f : (G, *) \to (G', \perp)$ and $g : (G', \perp) \to (G'', \nabla)$ be two group homomorphisms. Then the composition $g \circ f : (G, *) \to (G'', \nabla)$ is also a group homomorphism.*

Proof. Let x, x' be two elements of G. Then:

$$(g \circ f)(x * x') = g(f(x * x')) = g(f(x) \perp f(x')) = g(y \perp y') = g(y) \nabla g(y') = g(f(x)) \nabla g(f(x')) = (g \circ f)(x) \nabla$$

This shows that $g \circ f$ is a group homomorphism from G to G''. □

Proposition 5.20. *Let $(G, *)$ and $(G', \perp)$ be two groups, and let $f : (G, *) \to (G', \perp)$ be a group homomorphism. If f is bijective, then its inverse f^{-1} exists and is also a group homomorphism.*
In particular, the identity map $Id_G : (G, *) \to (G, *)$ is a group automorphism.

Example 5.21. *The maps:*

$$f : (\mathbb{R}, +) \to (\mathbb{R}, +), \quad x \mapsto 2x$$

and

$$g : (\mathbb{R}, +) \to (\mathbb{R}^*, \times), \quad x \mapsto e^x$$

are group homomorphisms.

5.1.3 Kernel of a Homomorphism

Definition 5.22. *Let $f : (G, *) \to (G', \perp)$ be a group homomorphism, and let e' be the identity element of G'. The inverse image of $\{e'\}$ under f is called the kernel of f and is denoted by* $\ker f$. *That is:*

$$\ker f = f^{-1}(\{e'\}) = \{x \in G \mid f(x) = e'\}.$$

Proposition 5.23. *Let $f : (G, *) \to (G', \perp)$ be a group homomorphism. Then the kernel of f is a subgroup of G.*

Proof. Let $f : (G, *) \to (G', \perp)$ be a group homomorphism, and let $\ker f$ be its kernel.

1. $\ker f \neq \emptyset$ because $e \in \ker f$ (since $f(e) = e'$).
2. Let $x, y \in \ker f$. Then $f(x) = e'$ and $f(y) = e'$. Therefore:

$$f(x * y) = f(x) \perp f(y) = e' \perp e' = e' \Rightarrow x * y \in \ker f.$$

3. Let $x \in \ker f$. We show that $x^{-1} \in \ker f$:

$$f(x^{-1}) = (f(x))^{-1} = (e')^{-1} = e' \Rightarrow x^{-1} \in \ker f.$$

Thus, $\ker f$ is a subgroup of G. □

Example 5.24. *1. Consider the map $f : \mathbb{R}^2 \to (\mathbb{R}, +)$ defined by $f(x, y) = x$. The kernel of f is:*

$$\ker f = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\} = \{(x, y) \in \mathbb{R}^2 \mid x = 0\} = \{0\} \times \mathbb{R}.$$

2. Consider the map $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = x - y$. The kernel of g is:

$$\ker g = \{(x, y) \in \mathbb{R}^2 \mid g(x, y) = 0\} = \{(x, y) \in \mathbb{R}^2 \mid x - y = 0\} = \{(x, x) \mid x \in \mathbb{R}\}.$$

Image of a Group Homomorphism

Definition 5.25. *Let $f : (G, *) \to (G', \perp)$ be a group homomorphism. The set $f(G) = \{f(x) \mid x \in G\}$ is called the image of f and is denoted by Im f.*

Proposition 5.26. *Let $f : (G, *) \to (G', \perp)$ be a group homomorphism. Then Im f is a subgroup of G'.*

Proof. 1. $\mathrm{Im}\, f \neq \emptyset$ because $e' = f(e) \in \mathrm{Im}\, f$. 2. Let $y_1, y_2 \in \mathrm{Im}\, f$. Then there exist $x_1, x_2 \in G$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. We have:

$$y_1 \perp y_2 = f(x_1) \perp f(x_2) = f(x_1 * x_2) = f(x) \in \mathrm{Im}\, f.$$

3. Let $y \in \mathrm{Im}\, f$. We show that $y^{-1} \in \mathrm{Im}\, f$:

$$y \in \mathrm{Im}\, f \Rightarrow \exists x \in G, y = f(x) \Rightarrow y^{-1} = (f(x))^{-1} = f(x^{-1}) \in \mathrm{Im}\, f.$$

Thus, Im f is a subgroup of G'. □

Example 5.27. *1. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x$. The image of f is:*

$$Im\ f = \{f(x, y) \mid (x, y) \in \mathbb{R}^2\} = \{x \mid (x, y) \in \mathbb{R}^2\} = \mathbb{R}.$$

2. Consider the map $g : \mathbb{I} \to \mathbb{I}$ defined by $g(x) = 3x$. The image of g is:

$$Im\ g = \{g(x) \mid x \in \mathbb{I}\} = \{3x \mid x \in \mathbb{I}\} = 3\mathbb{I}.$$

3. Consider the map $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x, y) = (y, 0)$. The image of h is:

$$Im\ h = \{h(x, y) \mid (x, y) \in \mathbb{R}^2\} = \{(y, 0) \mid (x, y) \in \mathbb{R}^2\} = \mathbb{R} \times \{0\}.$$

5.1.4 Rings

Definition 5.28. *Let A be a set equipped with two binary operations $*$ and $\bullet$. We say that $(A, *, \bullet)$ is a ring if and only if:*
1. $(A, *)$ is an abelian group. 2. $\bullet$ is associative. 3. $\bullet$ is distributive over $*$.

If $\bullet$ is commutative, the ring is called commutative. If $\bullet$ has an identity element, the ring is called unital.

Example 5.29. *The rings $(\mathbb{Z}, +, \times)$, $(\mathbb{R}, +, \times)$, $(\mathbb{Q}, +, \times)$, and $(\mathbb{C}, +, \times)$ are commutative and unital.*

5.1.5 Subrings

Definition 5.30. *Let $(A, *, \bullet)$ be a ring, and let $A' \subset A$. We say that A' is a subring of A if and only if:*

1. $(A', *)$ is a subgroup of A. 2. A' is closed under the operation $\bullet$, i.e., $\forall x, y \in A', x \bullet y \in A'$.

Example 5.31. *The set $(\mathbb{Z}, +, \times)$ is a subring of $(\mathbb{R}, +, \times)$.*

Proposition 5.32. *Let $(A, *, \bullet)$ be a ring and A' a subset of A. Then A' is a subring of A if and only if:*

$$\forall x, y \in A', \quad \begin{cases} x * y \in A', \\ x \bullet y \in A'. \end{cases}$$

5.1.6 Integral Domains

Definition 5.33. *Let $(A, *, \bullet)$ be a ring, and let $a \in A$. Denote by 0_A the identity element of A for the operation $*$. We say that a is a zero divisor if and only if:*

1. $a \neq 0_A$.

2. There exist $x, y \in A$ such that $x \bullet a = a \bullet y = 0_A$.

The ring A is called an integral domain if:

$$\forall x, y \in A, \quad x \bullet y = 0_A \Rightarrow x = 0_A \text{ or } y = 0_A.$$

In other words, A has no zero divisors.

The ring A is not an integral domain if:

$$\exists x, y \in A, \quad x \neq 0_A \text{ and } y \neq 0_A \text{ and } x \bullet y = 0_A.$$

Example 5.34. *Consider the ring $\mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, 4, 5\}$. For $x = 2$ and $y = 3$, we have:*

$$x \times y = 2 \times 3 = 6 \equiv 0 \pmod 6.$$

Thus, the ring $\mathbb{Z}/6\mathbb{Z}$ is not an integral domain.

5.1.7 Ring Homomorphisms

Definition 5.35. *Let $(A, *, \bullet)$ and $(A', +, \times)$ be two rings, and let $f : A \to A'$ be a function. We say that f is a ring homomorphism if and only if:*

$$\forall x, y \in A, \quad \begin{cases} f(x * y) = f(x) + f(y), \\ f(x \bullet y) = f(x) \times f(y). \end{cases}$$

Example 5.36. *1. Let $n \in \mathbb{N}$, and consider $\mathbb{Z}/n\mathbb{Z} = \{1, 2, 3, \ldots, n\}$, the set of integers modulo n. The function:*

$$f : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, \quad x \mapsto x \pmod n$$

is a ring homomorphism from $(\mathbb{Z}, +, \times)$ to $(\mathbb{Z}/n\mathbb{Z}, +, \times)$.

2. The function:

$$\Psi : C^1_{\mathbb{R}} \to C^0_{\mathbb{R}}, \quad f \mapsto f'$$

is not a ring homomorphism because $(f \times g)' \neq f' \times g'$.

Proposition 5.37. *1. If $f : A \to A'$ and $g : A' \to A''$ are ring homomorphisms, then $g \circ f : A \to A''$ is also a ring homomorphism.*

2. If $f : A \to A'$ is a ring isomorphism, then $f^{-1} : A' \to A$ is also a ring isomorphism.

3. The identity map $Id_A : A \to A$ is a ring automorphism.

5.1.8 Ring and Field

Let $f : (A, *, \bullet) \to (A', +, \times)$ be a ring homomorphism. Denote by $0_{A'}$ the identity element of A' for the operation $+$. We define the kernel of f as:

Definition 5.38. *Let $f : (A, *, \bullet) \to (A', +, \times)$ be a ring homomorphism. The kernel of f is defined as:*

$$\ker f = f^{-1}(\{0_A\}) = \{x \in A \mid f(x) = 0_A\}.$$

The image of f is defined as:

$$Im\ f = \{f(x) \mid x \in A\}.$$

Proposition 5.39. *1. In general, $\ker f$ is not a subring of A. 2. $Im\ f$ is a subring of A'.*

5.1.9 Fields

Definition 5.40. *A set K equipped with two operations $*$ and $\bullet$ is called a field if: 1. $(K, *, \bullet)$ is a unital ring. 2. Every element of K except 0_K has an inverse with respect to $\bullet$.*

If $\bullet$ is commutative, the field is called a commutative field.

Example 5.41. *1. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are commutative fields with the usual addition and multiplication.*

2. $(\mathbb{Z}, +, \times)$ is not a field because not every element of $\mathbb{Z}$ has a multiplicative inverse.

Subfields

Definition 5.42. *Let $(K, +, \times)$ be a field, and let $K' \subset K$. We say that K' is a subfield of K if:*

1. $\forall x, y \in K', \quad x - y \in K'$.

2. $\forall x \in K', \quad z \in K' - \{0\}, \quad x \times z^{-1} \in K'$.

Example 5.43. *$\mathbb{Q}$ is a subfield of $\mathbb{R}$, and $\mathbb{R}$ is a subfield of $\mathbb{C}$ with the usual operations.*

Remark 5.44. *The definitions of ring homomorphisms, kernel, and image remain valid for fields, along with their properties.*

Lemma 5.45. *Let $f : K \to K'$ be a homomorphism from $(K, *, \bullet)$ to $(K', +, \times)$. Then f is injective if and only if:*

$$\forall x \in K, \quad f(x) = 0_K \Rightarrow x = 0_K.$$

Proof. Assume the hypothesis $\forall x \in K, f(x) = 0_K \Rightarrow x = 0_K$ holds. We show that f is injective, i.e., $f(x) = f(y) \Rightarrow x = y$. Let $x, y \in K$ such that $f(x) = f(y)$. Then:

$$f(x) = f(y) \Rightarrow f(x)+(-f(y)) = 0_K \Rightarrow f(x)+f(-y) = 0_K \quad \text{(since } f \text{ is a homomorphism).}$$

$$\Rightarrow f(x * (-y)) = 0_K \Rightarrow x * (-y) = 0_K \quad \text{(by hypothesis).}$$

$$\Rightarrow x = y.$$

This shows that f is injective. The converse is immediate. □

Theorem 5.46. *Let $f : K \to K'$ be a homomorphism from $(K, *, \bullet)$ to $(K', +, \times)$. Then either f is the zero map or f is injective.*

Proof. Assume f is not injective. Then, by the previous lemma, there exists $x \in K$ such that $x \neq 0_K$ and $f(x) = 0_K$. We have $\forall y \in K - \{0\}$:

$$f(y) = f(x \bullet x^{-1} \bullet y) = f(x) \times f(x^{-1}) \times f(y)$$

$$= 0_K \times f(x^{-1}) \times f(y) = 0_K$$

Then f is nul. □

Exercise 5.47. Part I

Define a binary operation $\otimes$ on $\mathbb{N}^*$ by:

$$\forall n, m \in \mathbb{N}^*, \quad n \otimes m = m^n.$$

Is this operation commutative? Is it associative?

Part II

Let $A = \mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$. For $z = a + b\sqrt{3} \in A$, define:

$$\overline{z} = a - b\sqrt{3} \quad \textit{and} \quad q(z) = z\overline{z} = a^2 - 3b^2.$$

1. *Show that $(A, +, \times)$ is a subring of the ring $(\mathbb{R}, +, \times)$.*
2. *Assuming that $\sqrt{3} \notin \mathbb{Q}$, show that:*

$$a + b\sqrt{3} = 0 \iff a = b = 0.$$

3. *Show that:*

$$q(z) = 0 \iff z = 0,$$

and that:

$$q(z \cdot z') = q(z) \cdot q(z') \quad \forall z, z' \in A^2.$$

Exercise 5.48. *Let H and K be two subgroups of a group $(G, \cdot)$. Define a relation $\mathcal{H}$ on G by:*

$$\forall x, y \in G, \quad x\mathcal{H}y \iff \exists h \in H, \exists k \in K, \quad y = h \cdot k.$$

1. *Show that $\mathcal{H}$ is an equivalence relation.*
2. *For $x \in G$, define:*

$$G_x = \{(h, k) \in H \times K \mid h \cdot k^{-1} = x\}.$$

Show that G_x is a subgroup of $H \times K$.

3. If H and K are finite, show that each equivalence class is finite and its cardinality divides card$(H) \times$ card(K).

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