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Mohamed Boudiaf University of Msila
Faculty of Mathematics and Computer Sciences
Department of Mathematics

# Master memory

**Field** : Mathematics and Computer Sciences
**Branch** : Mathematics
**Option**: Partial Differential Equations and applications

# Theme

## *Some Results on the Cauchy Type Problems of Fractional q-Differential Equations*

**Presented by :**
*MESSAD Youcef*

**Before the jury composed of :**

| | | | |
|---|---|---|---|
| *Mr Saadi* Abderachid | M.C.A, | University of M'sila | **President**. |
| *Mr Arioua* Yacine | M.C.A, | University of M'sila | **Supervisor**. |
| *Mr Benmeddour* Mohamed Ourabah | M.A.A, | University of M'sila | **Examiner**. |

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# Contents

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# Symbols

$\mathbb{R}^+ = \{x \in \mathbb{R}; x > 0\}$.

$\mathbb{N} = \{1, 2, 3, \cdots\}$ : The sets of natural numbers.

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

$\mathbb{C}$ : The set of complex numbers.

$[x]$ : The integer part of $x$.

$\|.\|$: Denotes the norm on a Banach space.

$\overline{\Omega}$ : Closure of $\Omega$.

$C[0, 1]$ : Space of continuous functions on $[0, 1]$.

$\Gamma_q$ : The $q$-gamma function.

$B_q$ : The $q$-beta function.

$Re\,(\beta)$ : Real part of the complex number $\beta$.

$P_n(x)$ : The Legendre polynomial.

$I_q^\alpha$ : Fractional $q$-integral of Riemann-Liouville.

$D_q^\alpha$ : Riemann-Liouville $q$-fractional derivative.

${}^cD_q^\alpha$ : Caputo $q$-fractional derivative.

# Introduction

Many branches of mathematics and physics, including number theory, combinatorics, polynomial orthogonal, fundamental hypergeometric functions, quantum theory, and mechanics, all have relevance for the topic of quantum computation. This subject has drawn a lot of interest, thus it is regarded as falling somewhere between physics and mathematics.

AlSalam begin fitting the $q$-fractional calculus notion. He([3],[4]) and Agarwal [1] continued by studying some $q$-fractional integrals and derivatives after that.Recent developments, possibly as a result of the increase in study with the fractional calculus setting,

Particularly, the theory of fractional $q$-difference calculus was developed. The $q$-calculus and the $h$-calculus are the two types of quantum calculus. We are interested in the $q$-calculus in this paper. the $q$-calculus, which is based on the notion of a q-derivative:

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

The $q$-derivative measures the rate of change with respect to a dilatation of the function's argument by a factor $q$, as opposed to the classical derivative, which measures the rate of change of a function of an incremental translation of its argument. It is clear that if $f$ is differentiable at $x = 0$, then it follows that $q$ is a fixed number other than 1, and $x = 0$.

$$\lim_{q\to 1}(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x} = f'(x),$$

This memory is organized as follows:

In the first chapter, we state the basic deffinitions and properties concerning fractional $q$-calculus ($q$-derivative of a function, $q$-Exponential Function, $q$-special Functions, Riemann-Liouville, Caputo, $q$-fractional integral and derivative), which are used throughout the thesis to obtain our results.

In the second chapter, we introduce some fixed point theorems and detail the work submitted by Ferreira [5]), we prove the existence and uniquense of solutions for a $q$-fractional boundary value problem, by tow fixed point theorems, Banach fixed point theorem and Krasnoselskii fixed point theorem

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1 \\ u(0) = 0, u(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2, f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, $D_q^\alpha$ is the Riemann-Liouville fractional $q$-derivatives of order $\alpha$.

Finally, in the last chapter we will study the existence of inverse Probeme solutions for the $q$-Fractional partial differential equation, in the sense of Caputo is written as:

$$\begin{cases} {}^cD_q^\alpha U(t,x) = [(1-x^2)U_x]_x + h(x), & (t,x) \in ]0,T[\times]-1,1[ \\ D_q^k U(t,x)|_{t=0} = v_k(x), & k = 0,1,2,\cdots,m-1. \end{cases}$$

Where $m = [\alpha]+1$ and ${}^cD_q^\alpha U(t,x)$ is the Caputo $q$-fractional derivative of order $\alpha(0 < \alpha < 1)$ of $U(t,x)$ ,$D_q^k$ is the q-fractional derivative of order $k$ and $v_k(x)$ given function.

CHAPTER 1

# Basic definitions of $q$-Fractional Calculus

In this chapter we present some preliminary concepts, definitions, theorems and properties. These concepts will be used in next chapters.

## 1.1 Preliminaries on fractional $q$-calculus

See ([8],[6],[5]), we will give the basic definitions and properties of the quantum calculus. For a real parameter $q \in \mathbb{R}^+ \backslash \{1\}$, we introduce a $q$-real number $[a]_q$ by:

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}. \tag{1.1}$$

For examples:

- If $a = 1$, we have:

$$[1]_q = \frac{1-q}{1-q} = 1.$$

* For $q = 1/2$, we have:

$$[1]_{\frac{1}{2}} = \frac{1-\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

* For $q = 1/3$, we have:

$$[1]_{\frac{1}{3}} = \frac{1-\frac{1}{3}}{1-\frac{1}{3}} = 1.$$

- If $a = 2$, we have:

$$[2]_q = \frac{1-q^2}{1-q} = 1+q.$$

* For $q = 1/2$, we have:

$$[2]_{\frac{1}{2}} = \frac{1-(\frac{1}{2})^2}{1-\frac{1}{2}} = 1+\frac{1}{2} = \frac{3}{2}.$$

* For $q = 1/3$, we have:

$$[2]_{\frac{1}{3}} = \frac{1-(\frac{1}{3})^2}{1-\frac{1}{3}} = 1+\frac{1}{3} = \frac{4}{3}.$$

- If $a = 3$, we have:

$$[3]_q = \frac{1-q^3}{1-q} = 1 + q + q^2.$$

- For $a = n$, we have:

$$\begin{aligned}[n]_q &= \frac{1-q^n}{1-q} \\ &= 1 + q + q^2 + \cdots + q^{n-1}, \qquad n \in \mathbb{R}.\end{aligned}$$

∗ If $q = 1/2$, we have:

$$\begin{aligned}[n]_{\frac{1}{2}} &= \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1} \\ &= \sum_{n=0}^{n-1} \left(\frac{1}{2}\right)^n.\end{aligned}$$

Then:

$$\begin{aligned}[a]_q &= \frac{1-q^a}{1-q} \\ &= 1 + q + q^2 + \cdots + q^{a-1}.\end{aligned}$$

**The $q$-analog factorial**:

For $a \in \mathbb{R}$ and $q \in (0,1)$, we have:

$$\begin{aligned}a &= (1 + 1 + \cdots + 1), \quad a \quad \textit{times} \\ &= \lim_{q\to 1} \left(1 + q + q^2 + \ldots + q^{a-1}\right) \\ &= \lim_{q\to 1} \left(\frac{1-q^a}{1-q}\right) \\ &= \lim_{q\to 1} [a]_q.\end{aligned}$$

So:

The $q$-analog factorial is defined by :

$$\begin{aligned}[a]_q! &= [1]_q \cdot [2]_q \cdots [a-1]_q \cdot [a]_q \\ &= \frac{1-q}{1-q} \cdot \frac{1-q^2}{1-q} \cdots \frac{1-q^{a-1}}{1-q} \cdot \frac{1-q^a}{1-q} \\ &= 1(1+q)\cdots(1+q+q^2+\cdots+q^{a-2})(1+q+q^2+\cdots+q^{a-1}).\end{aligned}$$

**Example 1.1** *Let $a = 3$, Then*

$$3! = 3 \cdot 2 \cdot 1 = 6.$$

*So:*

$$\begin{aligned}[3]_q! &= [3]_q[2]_q[1]_q \\ &= \frac{1-q^3}{1-q}\frac{1-q^2}{1-q}\frac{1-q}{1-q} \\ &= (1+q+q^2)(1+q).\end{aligned}$$

For $k \in \mathbb{N}$, and $a, b \in \mathbb{R}$, The $q$-analog of the power function $(a-b)^k$ is :

$$(a-b)_q^k = (a-b)^{(k)} = \prod_{n=0}^{k-1}(a-bq^n), \quad (a-b)^0 = 1. \tag{1.2}$$

**Example 1.2** *Let $k = 2$, $q = 3$. Then*

$$\begin{aligned}(a-b)_3^2 &= (a-b)^{(2)} \\ &= \prod_{n=0}^{1}(a-3^n b) \\ &= (a-b)(a-3b).\end{aligned}$$

More generally, if $\alpha \in \mathbb{R}$ , then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty}\frac{a-bq^n}{a-bq^{\alpha+n}}, \quad a \neq 0. \tag{1.3}$$

**Remark 1.1** .

1. *If $b = 0$, then $a^{(\alpha)} = a^\alpha$.*

2. $0^{(\alpha)} = 0$ *for $\alpha > 0$.*

**Proposition 1.1** [8] *For $a, b, \alpha \in \mathbb{R}^+$ and $k \in \mathbb{N}$, the following property are valid:*

$$(a-bq^k)^{(\alpha)} = a^\alpha(1-bq^k/a)^{(\alpha)}. \tag{1.4}$$

**Proof.** Using the formula (1.3), we have:

$$\begin{aligned}(a-bq^k)^{(\alpha)} &= a^\alpha \prod_{n=0}^{\infty}\frac{a-(bq^k)q^n}{a-(bq^k)q^{\alpha+n}} \\ &= a^\alpha \prod_{n=0}^{\infty}\frac{1-(bq^kq^n/a)}{1-(bq^kq^{\alpha+n}/a)} \\ &= a^\alpha \prod_{n=0}^{\infty}\frac{1-(bq^k/a)q^n}{1-(bq^k/a)q^{\alpha+n}} \\ &= a^\alpha(1-bq^k/a)^{(\alpha)}.\end{aligned}$$

■

**Proposition 1.2** [8] *For $\alpha \in \mathbb{R}^+$ and $k, n \in \mathbb{N}$, the following property are valid:*

$$(q^n - q^k)^{(\alpha)} = 0, \quad (k \leq n). \tag{1.5}$$

## 1.2 The fractional $q$-derivative

**Definition 1.1** [9] *For a given $q \in \mathbb{R}$, a subset $A \subset \mathbb{R}$ is called q-geometric if $qx \in A$ whenever $x \in A$. That is, $\forall x \in A$, set $A$ includes all geometric sequences $\{xq^n\}_{n=0}^{\infty}$. A typical q-geometric set is $T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $0 < q < 1, \mathbb{Z}$ is the set of integers.*

**Definition 1.2** [9] *Let $f(x)$ be a real-valued function on the set $T_q$ and $0 < q < 1$. Define the q-derivative of $f(x)$ by*

$$(D_q f)(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}, \quad for \quad x \in T_q - \{0\}.$$

$$(D_q f)(0) = \frac{d_q f(x)}{d_q x}|_{x=0} = \lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, \quad x \neq 0.$$

**Remark 1.2** *If $f$ is differentiable at $x \neq 0$, then*

$$\lim_{q\to 1}(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x} = f'(x),$$

**Example 1.3** *The q-derivative of $f(x) = x^2$,*

$$\begin{aligned}(D_q f)(x) &= \frac{x^2 - (qx)^2}{(1-q)x} \\ &= \frac{1-q^2}{1-q}x \\ &= [2]_q\, x.\end{aligned}$$

*If $q = 1/2$, we have:*

$$\begin{aligned}(D_q f)(x) &= \left(\frac{1-(\frac{1}{2})^2}{1-\frac{1}{2}}\right)x \\ &= \left(1+\frac{1}{2}\right)x \\ &= \frac{3}{2}x.\end{aligned}$$

**Proposition 1.3** [6] *The q-derivative of the function product $f(x)$ and the function $g(x)$, East as following:*

$$D_q(f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x). \tag{1.6}$$

If $q \to 1$, we have:

$$\lim_{q\to 1} D_q(f(x)g(x)) = D(f(x)g(x)) = f(x)Dg(x) + g(x)Df(x).$$

**Proof.**

$$\begin{aligned}D_q(f(x)g(x)) &= \frac{f(x)g(x)-f(qx)g(qx)}{(1-q)x}\\ &= \frac{f(x)g(x)+f(x)g(qx)-f(x)g(qx)-f(qx)g(qx)}{(1-q)x}\\ &= f(x)\frac{g(x)-g(qx)}{(1-q)x} - g(x)\frac{f(x)-f(qx)}{(1-q)x},\end{aligned}$$

So:

$$D_q(f(x)g(x)) = f(x)D_qg(x) + g(qx)D_qf(x).$$

■

**Proposition 1.4** [6] *For all integer $n$:*

1. $$D_q(x-a)_q^n = [n]_q\,(x-a)_q^{n-1}. \tag{1.7}$$

2. $$D_q(a-x)_q^n = -\,[n]_q\,(a-qx)_q^{n-1}. \tag{1.8}$$

**Proof.** We reason by induction : The property is true for $n=0$, because $[0]_q=0$, and it is assumed to be true for some rank $k$, i,e :

$$D_q(x-a)_q^k = [k]_q\,(x-a)_q^{k-1}.$$

We check the property for $k+1$, knowing that

$$(x-a)_q^{k+1} = (x-a)_q^k(x-q^ka).$$

Using the derivative of the product (1.6), we get

$$\begin{aligned}D_q(x-a)_q^{k+1} &= D_q(x-a)_q^k(x-q^ka)\\ &= (x-a)_q^k + (qx-q^ka)D_q(x-a)_q^k\\ &= (x-a)_q^k + q(x-q^{k-1}a)\,[k]_q\,(x-a)_q^{k-1}\\ &= (1+q\,[k]_q)(x-a)_q^k\\ &= [k+1]_q\,(x-a)_q^k.\end{aligned}$$

2. According to the formula (1.2), For $n \geq 1$, we have

$$\begin{aligned}(a-x)_q^n &= (a-x)(a-qx)(a-q^2x)\cdots(a-q^{n-1}x)\\ &= (a-x).q(q^{-1}a-x).q^2(q^{-2}a-x)\cdots q^{n-1}(q^{-n+1}a-x)\\ &= (-1)^nq^{n(n-1)/2}(x-q^{-n+1}a)\cdots(x-q^{-2}a)(x-q^{-1}a)(x-a).\end{aligned}$$

From where :

$$(a-x)_q^n=(-1)^n q^{n(n-1)/2}(x-q^{-n+1}a)_q^n. \tag{1.9}$$

Using the formula (1.9) and the proposition (1.7), we get

$$\begin{aligned} D_q(a-x)_q^n &= (-1)^n q^{n(n-1)/2} D_q(x-q^{-n+1}a)_q^n \\ &= (-1)^n q^{n(n-1)/2}\,[n]_q\,(x-q^{-n+1}a)_q^{n-1} \\ &= -\,[n]_q\,q^{n-1}.(-1)^{n-1} q^{(n-1)(n-2)/2}(x-q^{-n+2}(q^{-1}a))_q^{n-1} \\ &= -\,[n]_q\,q^{n-1}.(q^{-1}a-x)_q^{n-1} \\ &= -\,[n]_q\,(a-qx)_q^{n-1}. \end{aligned}$$

■

And the $q$-derivatives of higher order are given by

$$(D_q^0 f)(t)=f(t),$$

and

$$(D_q^k f)(t)=D_q(D_q^{k-1}f)(t), \qquad k\in\mathbb{N}.$$

The q-integral of a function $f$ defined on the interval $[0,b]$ is given by

$$(I_q f)(t)=\int_0^t f(s)d_qs=t(1-q)\sum_{i=0}^{\infty} f(tq^i)q^i, \qquad t\in[0,b]. \tag{1.10}$$

and

$$(I_{q,a}f)(t)=\int_a^t f(s)d_qs=\int_a^t f(s)d_qs-\int_0^a f(s)d_qs. \tag{1.11}$$

If $a=tq^n$ we have:

$$\int_{tq^n}^t f(s)d_qs=t(1-q)\sum_{i=0}^{n-1} f(tq^i)q^i. \tag{1.12}$$

If $a\in[0,b]$ and $f$ is defined in the interval $[0,b]$, then its integral from a to $b$ is defined by

$$\int_a^b f(s)d_qs=\int_0^b f(s)d_qs-\int_0^a f(s)d_qs. \tag{1.13}$$

If $q\to 1$, we have:

$$\lim_{q\to 1}\int_a^b f(s)ds=\int_0^b f(s)ds-\int_0^a f(s)ds.$$

From formula (1.13), it holds that

$$\left|\int_0^b f(s)d_qs\right|\le\int_0^b |f(s)|\,d_qs, \quad b>0. \tag{1.14}$$

Similar to derivatives, an operator $I_q^k$ is given by

$$(I_q^0 f)(t) = f(t),$$

and

$$(I_q^k f)(t) = I_q(I_q^{k-1} f)(t), \qquad k \in \mathbb{N}$$

. The fundamental theorem of calculus applies to these operators $D_q$ and $I_q$, i.e:

$$(D_q I_q f)(t) = f(t),$$

and if $f$ is continuous at $t = 0$, then

$$(I_q D_q f)(t) = f(t) - f(0).$$

**Proposition 1.5** [6] *Are $f$ and $g$ are two $q$-derivable functions on $[a, b]$, $q$-integration by parts is defined by :*

$$\int_a^b f(x) D_q g(x) d_q x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) D_q f(x) d_q x. \tag{1.15}$$

**Proof.** Using the derivative of the product (1.6), we get

$$f(x) D_q g(x) = D_q(f(x)g(x)) - g(qx) D_q f(x),$$

$$\Rightarrow \int_a^b f(x) D_q g(x) d_q x = \int_a^b D_q(f(x)g(x)) - \int_a^b g(qx) D_q f(x),$$

we put $h(x) = f(x)g(x)$, and using the formula (1.10) and definition (1.2) we have:

$$\begin{aligned} \int_a^b D_q h(x) d_q x &= \sum_{i=0}^{\infty} \left(h(bq^i) - h(bq^{i+1})\right) - \sum_{i=0}^{\infty} \left(h(aq^i) - h(aq^{i+1})\right) \\ &= h(b) - h(a) \\ &= f(b)g(b) - f(a)g(a). \end{aligned}$$

Consequently,

$$\int_a^b f(x) D_q g(x) d_q x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) D_q f(x) d_q x.$$

■

## 1.3 The $q$-Exponential Function

**Definition 1.3** [6] *A $q$-analogue of the classical exponential function $e^x$ is*

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}}. \tag{1.16}$$

*Such that,*

$$\frac{1}{(1-x)_q^{\infty}} = \sum_{j=0}^{\infty} \frac{x^j}{(1-q)(1-q^2)\cdots(1-q^j)}.$$

**Definition 1.4** [6] *Another $q$-analogue of the classical exponential function is*

$$E_q^x = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{[j]_q!} = (1+(1-q)x)_q^{\infty}. \tag{1.17}$$

*Such that,*

$$(1+x)_q^{\infty} = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{(1-q)(1-q^2)\cdots(1-q^j)}.$$

And

$$e_q^x E_q^{-x} = 1. \tag{1.18}$$

We note that by (1.17):

$$E_q^0 = 1,$$

and by (1.16),(1.18)

$$E_q^{-\infty} = \lim_{x\to\infty} \frac{1}{e_q^x} = 0.$$

**Proposition 1.6** [6] *We have tow following propriety :*

1.
$$D_q e_q^x = e_q^x. \tag{1.19}$$

2.
$$D_q E_q^x = E_q^{qx}. \tag{1.20}$$

**Proof.**

1. Using definition (1.16) and definition (1.2), we have:

$$
\begin{aligned}
D_q e_q^x &= \sum_{j=0}^{\infty} \frac{D_q x^j}{[j]_q!} \\
&= \sum_{j=1}^{\infty} \frac{[j]_q x^{j-1}}{[j]_q!} \\
&= \sum_{j=1}^{\infty} \frac{x^{j-1}}{[j-1]_q!} \\
&= \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!} \\
&= e_q^x.
\end{aligned}
$$

2. Using definition (1.17) and definition (1.2), we have:

$$
\begin{aligned}
D_q E_q^x &= \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{D_q x^j}{[j]_q!} \\
&= \sum_{j=1}^{\infty} q^{j(j-1)/2} \frac{[j]_q x^{j-1}}{[j]_q!} \\
&= \sum_{j=1}^{\infty} q^{(j-1)(j-2)/2} q^{j-1} \frac{x^{j-1}}{[j-1]_q!} \\
&= \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{q^j x^j}{[j]_q!} \\
&= E_q^{qx}.
\end{aligned}
$$

■

## 1.4 The $q$-special Functions

### 1.4.1 The $q$-Gamma function

**Definition 1.5** [8] *The $q$-gamma function is defined by*

$$\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}, \qquad t \in \mathbb{R}\backslash\{0,-1,-2,\cdots\}. \tag{1.21}$$

*and for any $t > 0$ ,* [6]

$$\Gamma_q(t) = \int_0^{\infty} x^{t-1} E_q^{-qx} d_q x. \tag{1.22}$$

**Example 1.4**

$$\begin{aligned}\Gamma_q(1) &= \int_0^\infty E_q^{-qx} d_q x \\ &= \left[-e_q^{-x}\right]_0^\infty \\ &= \left[-\frac{1}{e_q^x}\right]_0^\infty \\ &= \left[-E_q^{-x}\right]_0^\infty \\ &= E_q^0 - E_q^{-\infty} = 1,\end{aligned}$$

*So:*

$$\Gamma_q(1) = 1.$$

**Proposition 1.7** [6]

1. $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$; $t > 0$.

2. $\forall n \in \mathbb{N}; \Gamma_q(n+1) = [n]_q!$.

**Proof.**

1. For proposition one we have tow method:

   **Method 1**: Using formula (1.22) and $q$-integration by parts (1.15), we have:

$$\begin{aligned}\Gamma_q(t+1) &= \int_0^\infty x^t E_q^{-qx} d_q x \\ &= -\int_0^\infty x^t d_q E_q^{-x} d_q x \\ &= [t]_q \int_0^\infty x^{t-1} E_q^{-qx} d_q x \\ &= [t]_q \Gamma_q(t).\end{aligned}$$

   **Method 2**: The definition is applied (1.21), Then we use the formula (1.2), We obtain:

$$\begin{aligned}\Gamma_q(t+1) &= \frac{(1-q)^{(t)}}{(1-q)^t} \\ &= \frac{(1-q^t)(1-q)^{(t-1)}}{(1-q)(1-q)^{t-1}} \\ &= [t]_q \Gamma_q(t).\end{aligned}$$

2. Using item (1.)of proposition (1.7), We obtain:

$$\begin{aligned}
\Gamma_q(n+1) &= [n]_q \, \Gamma_q(n) \\
&= [n]_q \, [n-1]_q \, \Gamma_q(n-1) \\
&= [n]_q \, [n-1]_q \, [n-2]_q \, \Gamma_q(n-2) \\
&\;\;\vdots \\
&= [n]_q \, [n-1]_q \, [n-2]_q \cdots [1]_q \, \Gamma_q(1) \\
&= [n]_q ! \, .
\end{aligned}$$

■

### 1.4.2 The $q$-Beta function

**Definition 1.6** [6] *For any t,s > 0, The q-Beta function is defined by*

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)^{s-1} d_q x. \tag{1.23}$$

**Proposition 1.8** [6]

1. *if $t > 0$ and $n$ a positive integer, we have*

$$B_q(t,n) = \frac{(1-q)(1-q)_q^{n-1}}{(1-q^t)_q^n}. \tag{1.24}$$

2. *For $t, s > 0$, we have*

$$B_q(t,s) = \frac{(1-q)(1-q)_q^{\infty}(1-q^{t+s})_q^{\infty}}{(1-q^t)_q^{\infty}(1-q^s)_q^{\infty}}. \tag{1.25}$$

3. *For $t, s > 0$, we have*

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}. \tag{1.26}$$

**Proof.**

1. First, using the property (1.8) and $q$-integration by parts,

   we have, for all $t > 1$ , $s > 0$,

$$\begin{aligned}
B_q(t,s) &= -\frac{1}{[s]_q}\int_0^1 x^{t-1} D_q(1-x)_q^s d_q x \\
&= \frac{[t-1]_q}{[s]_q}\int_0^1 x^{t-2}(1-qx)_q^s d_q x,
\end{aligned}$$

Consequently,

$$B_q(t,s) = \frac{[t-1]_q}{[s]_q} B_q(t-1, s+1). \tag{1.27}$$

On the other hand, we have:

$$\begin{aligned} B_q(t,n+1) &= \int_0^1 x^{t-1}(1-qx)_q^{n-1}(1-q^n x)d_q x \\ &= \int_0^1 x^{t-1}(1-qx)_q^{n-1}d_q x - q^n \int_0^1 x^t (1-qx)_q^{n-1} d_q x, \end{aligned}$$

and so:

$$B_q(t,n+1) = B_q(t,n) - q^n B_q(t+1,n). \tag{1.28}$$

Combining (1.27), (1.28) we get

$$\begin{aligned} B_q(t,n+1) &= B_q(t,n) - q^n \frac{[t]_q}{[n]_q} B_q(t,n+1) \\ &= \frac{(1-q^n)}{(1-q^{t+n})} B_q(t,n). \end{aligned}$$

For $t > 0$ and $n$ a positive integer, we have:

$$\begin{aligned} B_q(t,1) &= \int_0^1 x^{t-1} d_q x \\ &= \frac{1}{[t]_q}. \end{aligned}$$

So:

$$\begin{aligned} B_q(t,n) &= \frac{(1-q^{n-1})\cdots(1-q)}{(1-q^{t+n-1})\cdots(1-q^{t+1})[t]_q} \\ &= \frac{(1-q)(1-q)_q^{n-1}}{(1-q^t)_q^n}. \end{aligned}$$

2. Using the following two definitions :

$$(1-q)_q^{n-1} = \frac{(1-q)_q^\infty}{(1-q^n)_q^\infty}. \tag{1.29}$$

$$\frac{1}{(1-q^t)_q^n} = \frac{(1-q^{t+n})_q^\infty}{(1-q^t)_q^\infty}. \tag{1.30}$$

In (1.24), we have:

$$B_q(t,s) = \frac{(1-q)(1-q)_q^\infty(1-q^{t+s})_q^\infty}{(1-q^t)_q^\infty(1-q^s)_q^\infty}.$$

3- Using (1.29) in the definition (1.21), We obtain

$$\Gamma_q(t) = \frac{(1-q)_q^\infty}{(1-q)^{t-1}(1-q^t)_q^\infty}$$

$$\begin{aligned}
\Gamma_q(t)\Gamma_q(s) &= \frac{(1-q)_q^{\infty}}{(1-q)^{t-1}(1-q^t)_q^{\infty}}\frac{(1-q)_q^{\infty}}{(1-q)^{s-1}(1-q^s)_q^{\infty}} \\
&= \frac{(1-q)_q^{\infty}(1-q)_q^{\infty}}{(1-q)^{t+s-2}(1-q^t)_q^{\infty}(1-q^s)_q^{\infty}} \\
&= \frac{(1-q)_q^{\infty}(1-q)_q^{\infty}}{(1-q)^{t+s-2}(1-q^t)_q^{\infty}(1-q^s)_q^{\infty}}\frac{(1-q^{t+s})_q^{\infty}}{(1-q^{t+s})_q^{\infty}} \\
&= \frac{(1-q)(1-q)_q^{\infty}(1-q^{t+s})_q^{\infty}}{(1-q^t)_q^{\infty}(1-q^s)_q^{\infty}}\frac{(1-q)_q^{\infty}}{(1-q)^{t+s-1}(1-q^{t+s})_q^{\infty}} \\
&= B_q(t,s)\Gamma_q(t+s).
\end{aligned}$$

So:

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}.$$

■

### 1.4.3 The $q$-Mittag-Leffler function

**Definition 1.7** *(See[2]) The q-Mittag-Leffler function is defined by*

$$E^q_{\alpha,\beta}(\lambda, z-z_0) = \sum_{k=0}^{\infty}\frac{\lambda^k(z-z_0)_q^{\alpha k}}{\Gamma_q(\alpha k+\beta)}.$$

*Where* $\{\lambda, z, z_0, \alpha, \beta\} \in \mathbb{C}$ *and* $Re\,(\alpha) > 0, Re\,(\beta) \geq 0.$

**Example 1.5** *:*

*1- When* $\beta = 1$ *we simply use :*

$$E^q_{\alpha}(\lambda, z-z_0) = E^q_{\alpha,1}(\lambda, z-z_0) = \sum_{k=0}^{\infty}\frac{\lambda^k(z-z_0)_q^{\alpha k}}{\Gamma_q(\alpha k+1)}.$$

*2-If* $\alpha = 1, \beta = 1, \lambda = 1$ *and* $z_0 = 0, z \in \mathbb{R},$ *we have:*

$$\begin{aligned}
E^q_{1,1}(1,z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma_q(k+1)} \\
&= \sum_{k=0}^{\infty}\frac{z^k}{[k]_q!} \\
&= e_q^z.
\end{aligned}$$

### 1.4.4 The Legendre polynomials

Consider the second-order differential equation with variable coefficient

$$(1-x^2)y''(x) - 2xy'(x) + \lambda y(x) = 0. \tag{1.31}$$

Is called the Legendre equation .

**Lemma 1.1** [7] *Let $-1 \leq x \leq 1$, if $\lambda = n(n+1), (n = 0, 1, 2, \cdots)$, then Legendre's equation admits a solution Under the form*

$$y(x) = p_n(x) = \frac{1}{2^n n!}\frac{d^n (x^2-1)^n}{dx^n}, (n = 0, 1, 2, \cdots). \tag{1.32}$$

*$p_n(x)$ is called the Legendre Polynomial.*

**Example 1.6** *From the relation (1.32) and putting $n = 0, 1, 2, \cdots$. we find*

1. $p_0(x) = 1.$
2. $p_1(x) = x.$
3. $p_2(x) = \frac{1}{2}\left(3x^2 - 1\right).$
4. $p_3(x) = \frac{1}{2}\left(5x^3 - 3x\right).$
5. $p_4(x) = \frac{1}{4}\left(35x^4 - 30x^2 + 3\right).$
6. $p_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right).$

**Proposition 1.9** [7] *Let $P_n(x)$ be the Legendre polynomial, we have:*

1. $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), (n \geq 1),$
2. $\frac{d}{dx}\left[(1-x^2)P'_n(x)\right] + n(n+1)P'_n(x) = 0,$
3. $p_n(1) = 1, p_n(-1) = (-1)^n,$
4. $|p_n(x)| \leq 1, (|x| \leq 1),$
5. $\int_{-1}^{1} P_n(x)P_m(x)dx = 0, (n \neq m),$
6. $\int_{-1}^{1} \left[P_n(x)\right]^2 dx = \frac{2}{2n+1},$

## 1.5 The fractional $q$-integral of Riemann-Liouville

**Definition 1.8** [8] *Let $\alpha \geq 0$ and $f$ be a function defined on $[a, b]$. The fractional $q$-integral of Riemann-Liouville type is given by*

$$(I^{\alpha}_{q,a} f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)^{(\alpha-1)} f(s) d_q s, \qquad \alpha > 0, \quad t \in [a, b]. \tag{1.33}$$

*and*

$$(I^0_{q,a}f)(t) = f(t).$$

**Lemma 1.2** [8] *For $\alpha, \beta \in \mathbb{R}^+$, we have*

$$\int_0^a (t-qs)^{(\beta-1)}(I^\alpha_{q,a}f)(s)d_qs = 0, \quad (0 < a < t < b). \tag{1.34}$$

**Proof.** For $n \in \mathbb{N}$, using definitions (1.12),(1.5) and (1.6), we have

$$\begin{aligned}(I^\alpha_{q,a}f)(aq^n) &= \frac{1}{\Gamma_q(\alpha)}\int_a^{aq^n}(aq^n-qs)^{(\alpha-1)}f(s)d_qs \\ &= \frac{-a}{\Gamma_q(\alpha)}(1-q)\sum_{j=0}^{n-1}(aq^n-aq^{j+1})^{(\alpha-1)}f(aq^j)q^j \\ &= \frac{-a^\alpha}{\Gamma_q(\alpha)}(1-q)\sum_{j=0}^{n-1}(q^n-q^{j+1})^{(\alpha-1)}f(aq^j)q^j \\ &= 0.\end{aligned}$$

Then, according to the definition of the $q$-integral, we have

$$\begin{aligned}\int_0^a (t-qs)^{(\beta-1)}(I^\alpha_{q,a}f)(s)d_qs &= a(1-q)\sum_{n=0}^{\infty}(t-aq^{n+1})^{(\beta-1)}(I^\alpha_{q,a}f)(aq^n)q^n \\ &= 0.\end{aligned}$$

■

**Lemma 1.3** [8] *For $\alpha, \beta, \mu \in \mathbb{R}^+$ we have:*

$$\sum_{n=0}^{\infty}\frac{(1-\mu q^{1-n})^{(\alpha-1)}(1-q^{1+n})^{(\beta-1)}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}}q^{\alpha n} = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}}. \tag{1.35}$$

**Proposition 1.10** [8] *For $\alpha, \beta \in \mathbb{R}^+$, we have*

*The application of the fractional $q$-integral of order $\alpha$ of Riemann-Liouville on the function $f(x) = (x-a)^{(\beta)}$ is given by:*

$$I^\alpha_{q,a}(x-a)^{(\beta)} = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}(x-a)^{(\alpha+\beta)} \; ; \quad x > a, \quad \alpha, \beta > 0.$$

**Proof.** For $\beta \neq 0$, according to definition (1.33) and definition (1.11), we have

$$\begin{aligned}I^\alpha_{q,a}(x-a)^{(\beta)} &= \frac{1}{\Gamma_q(\alpha)}\int_a^x (x-qt)^{(\alpha-1)}(t-a)^{(\beta)}d_qt \\ &= \frac{1}{\Gamma_q(\alpha)}\left[\int_0^x (x-qt)^{(\alpha-1)}(t-a)^{(\beta)}d_qt - \int_0^a (x-qt)^{(\alpha-1)}(t-a)^{(\beta)}d_qt\right].\end{aligned}$$

Also, the following is valid according to the definition (1.10) and the definition (1.5):

$$\begin{aligned}\int_0^a (x-qt)^{(\alpha-1)}(t-a)^{(\beta)}d_qt &= a(1-q)\sum_{k=0}^{\infty}(x-aq^{k+1})^{(\alpha-1)}(aq^k-a)^{(\beta)}q^k \\ &= a^{\beta+1}(1-q)\sum_{k=0}^{\infty}(x-aq^{k+1})^{(\alpha-1)}(q^k-1)^{(\beta)}q^k \\ &= 0.\end{aligned}$$

By the definition (1.10) and the lemma (1.3), we get:

$$\begin{aligned}\int_0^x (x-qt)^{(\alpha-1)}(t-a)^{(\beta)}d_qt &= x(1-q)\sum_{k=0}^{\infty}(x-xq^{k+1})^{(\alpha-1)}(xq^k-a)^{(\beta)}q^k \\ &= x^{\alpha+\beta}(1-q)\sum_{k=0}^{\infty}(1-q^{k+1})^{(\alpha-1)}(q^k-\frac{a}{x})^{(\beta)}q^k \\ &= x^{\alpha+\beta}(1-q)\sum_{k=0}^{\infty}(1-q^{k+1})^{(\alpha-1)}(1-\frac{a}{xq}q^{1-k})^{(\beta)}q^{k(1+\beta)} \\ &= x^{\alpha+\beta}(1-q)\frac{(1-\frac{a}{x})^{(\alpha+\beta)}(1-q)^{(\alpha-1)}(1-q)^{(\beta)}}{(1-q)^{(\alpha+\beta)}} \\ &= (1-q)\frac{(1-q)^{(\alpha-1)}(1-q)^{(\beta)}}{(1-q)^{(\alpha+\beta)}}(x-a)^{(\alpha+\beta)}.\end{aligned}$$

Using the definition (1.21), we get the result.

Particularly, for $\beta = 0$, using a $q$-integration by parts, we have

$$\begin{aligned}I_{q,a}^{\alpha}1 &= \frac{1}{\Gamma_q(\alpha)}\int_a^x (x-qt)^{(\alpha-1)}d_qt \\ &= \frac{-1}{\Gamma_q(\alpha)}\int_a^x \frac{D_q\left((x-t)^{(\alpha)}\right)}{[\alpha]_q}d_qt,\end{aligned}$$

Using Proposition (1.7), we have:

$$\begin{aligned}\frac{-1}{\Gamma_q(\alpha)}\int_a^x \frac{D_q\left((x-t)^{(\alpha)}\right)}{[\alpha]_q}d_qt &= \frac{-1}{\Gamma_q(\alpha+1)}\int_a^x D_q\left((x-t)^{(\alpha)}\right)d_qt \\ &= \frac{-1}{\Gamma_q(\alpha+1)}(x-a)^{(\alpha)}.\end{aligned}$$

■

**Example 1.7** *Let $f(x) = x$, applying a $q$-integration by parts and using the formula:*

$$D_q(x-t)^{(\alpha)} = -[\alpha]_q(x-qt)_q^{(\alpha-1)}.$$

*we get*

$$\begin{aligned}(I_q^\alpha f)(x) &= \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)} t d_q t\\ &= \frac{-1}{\Gamma_q(\alpha+1)}\int_0^x tD_q(x-t)^{(\alpha)} d_q t\\ &= \frac{-1}{\Gamma_q(\alpha+1)}\left([t(x-t)^{(\alpha)}]_0^x - \int_0^x (x-qt)^{(\alpha)} d_q t\right)\\ &= \frac{-1}{\Gamma_q(\alpha+1)}\left(-\int_0^x \frac{-1}{[\alpha+1]_q} D_q(x-t)^{(\alpha+1)} d_q t\right)\\ &= \frac{1}{\Gamma_q(\alpha+2)}\left[-(x-t)^{(\alpha+1)}\right]_0^x\\ &= \frac{x^{(\alpha+1)}}{\Gamma_q(\alpha+2)},\end{aligned}$$

*Then:*

$$(I_q^\alpha f)(x) = \frac{x^{(\alpha+1)}}{\Gamma_q(\alpha+2)}.$$

**Theorem 1.1** [8] *Let $\alpha,\beta \in \mathbb{R}^+$. The q-fractional integration has the following semi- group property*

$$(I_{q,a}^\beta I_{q,a}^\alpha f)(t) = (I_{q,a}^\alpha I_{q,a}^\beta f)(t) = (I_{q,a}^{\alpha+\beta} f)(t)$$

**Proof.** 1.Using the definition (1.33) and the definition (1.11), then we apply the Lemma(1.34)

$$\begin{aligned}(I_{q,a}^\beta I_{q,a}^\alpha f)(t) &= \frac{1}{\Gamma_q(\beta)}\int_a^t (t-qs)^{(\beta-1)} I_{q,a}^\alpha f(s) d_q s\\ &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_a^t (t-qs)^{(\beta-1)}\int_a^t (s-qu)^{(\alpha-1)} f(u) d_q u d_q s\\ &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\left[\int_0^t (t-qs)^{(\beta-1)} - \int_0^a (t-qs)^{(\beta-1)}\right]\\ &\quad\times\left[\int_0^s (s-qu)^{(\alpha-1)} f(u) d_q u - \int_0^a (s-qu)^{(\alpha-1)} f(u) d_q u\right] d_q s\\ &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^t (t-qs)^{(\beta-1)}\int_0^s (s-qu)^{(\alpha-1)} f(u) d_q u d_q s\\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^a (t-qs)^{(\beta-1)}\int_0^s (s-qu)^{(\alpha-1)} f(u) d_q u d_q s\\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^t (t-qs)^{(\beta-1)}\int_0^a (s-qu)^{(\alpha-1)} f(u) d_q u d_q s\\ &\quad + \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^a (t-qs)^{(\beta-1)}\int_0^a (s-qu)^{(\alpha-1)} f(u) d_q u d_q s\\ &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^t (t-qs)^{(\beta-1)}\int_0^s (s-qu)^{(\alpha-1)} f(u) d_q u d_q s\\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^t (t-qs)^{(\beta-1)}\int_0^a (s-qu)^{(\alpha-1)} f(u) d_q u d_q s.\end{aligned}$$

Using the result from [1],

$$(I^{\beta}_{q,0} I^{\alpha}_{q,0} f)(t) = (I^{\alpha+\beta}_{q,0} f)(t)$$

In addition ,

$$I^{\alpha+\beta}_{q,a} f(t) = I^{\alpha+\beta}_{q,0} f(t) - I^{\alpha+\beta}_{q,0} f(a) \tag{1.36}$$

It is concluded that,

$$(I^{\beta}_{q,a} I^{\alpha}_{q,a} f)(t) = I^{\alpha+\beta}_{q,0} f(t) - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^t (t-qs)^{(\beta-1)} \int_0^a (s-qu)^{(\alpha-1)} f(u) d_q u d_q s.$$

After the formula (1.36), we can write

$$\begin{aligned}(I^{\beta}_{q,a} I^{\alpha}_{q,a} f)(t) &= I^{\alpha+\beta}_{q,a} f(t) I^{\alpha+\beta}_{q,0} f(a) \\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^t (t-qs)^{(\beta-1)} \int_0^a (s-qu)^{(\alpha-1)} f(u) d_q u d_q s \\ &= I^{\alpha+\beta}_{q,a} f(t) + \frac{1}{\Gamma_q(\alpha+\beta)} \int_0^a (t-qs)^{(\alpha+\beta-1)} f(s) d_q s \\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^t (t-qs)^{(\beta-1)} \int_0^a (s-qu)^{(\alpha-1)} f(u) d_q u d_q s.\end{aligned}$$

Then,

$$(I^{\beta}_{q,a} I^{\alpha}_{q,a} f)(t) = I^{\alpha+\beta}_{q,a} f(t) + M.$$

$$\begin{aligned}M = &\frac{1}{\Gamma_q(\alpha+\beta)} \int_0^a (t-qs)^{(\alpha+\beta-1)} f(s) d_q s \\ &- \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^t (t-qs)^{(\beta-1)} \int_0^a (s-qu)^{(\alpha-1)} f(u) d_q u d_q s.\end{aligned}$$

Using the definition (1.10), we can write

$$\begin{aligned}M &= \frac{a(1-q)}{\Gamma_q(\alpha+\beta)} \sum_{k=0}^{\infty} (t-aq^{k+1})^{(\alpha+\beta-1)} f(aq^k) q^k \frac{t(1-q)}{\Gamma_q(\alpha)\Gamma_q(\beta)} \sum_{j=0}^{\infty} (t-tq^{j+1})^{(\beta-1)} \\ &\quad \times \sum_{k=0}^{\infty} a(1-q)(tq^j - aq^{k+1})^{(\alpha-1)} f(aq^k) q^k q^j \\ &= a(1-q) \sum_{k=0}^{\infty} \frac{(t-aq^{k+1})^{(\alpha-1)}}{\Gamma_q(\alpha+\beta)} f(aq^k) q^k \\ &\quad - a(1-q) \sum_{k=0}^{\infty} \frac{t(1-q)^{(\alpha-1)}}{\Gamma_q(\alpha)\Gamma_q(\beta)} \sum_{j=0}^{\infty} (t-tq^{j+1})^{(\beta-1)} \sum_{k=0}^{\infty} (tq^j - aq^{k+1})^{(\alpha-1)} f(aq^k) q^k q^j \\ &= a(1-q) \sum_{k=0}^{\infty} \Bigg[ \frac{(t-aq^{k+1})^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} \\ &\qquad - \frac{t(1-q)^{(\alpha-1)}}{\Gamma_q(\alpha)\Gamma_q(\beta)} \sum_{j=0}^{\infty} (t-tq^{j+1})^{(\beta-1)} (tq^j - aq^{k+1})^{(\alpha-1)} q^j \Bigg] f(aq^k) q^k,\end{aligned}$$

Therefore,

$$M = a(1-q)\sum_{k=0}^{\infty} c_k f(aq^k)q^k.$$

$$c_k = \frac{(t-aq^{k+1})^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} - \frac{t(1-q)^{(\alpha-1)}}{\Gamma_q(\alpha)\Gamma_q(\beta)}\sum_{j=0}^{\infty}(t-tq^{j+1})^{(\beta-1)}(tq^j-aq^{k+1})^{(\alpha-1)}q^j.$$

After the definition (1.21) ,we have

$$\Gamma_q(\alpha)\Gamma_q(\beta) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}\frac{(1-q)^{(\beta-1)}}{(1-q)^{\beta-1}} \tag{1.37}$$

$$\Gamma_q(\alpha+\beta) = \frac{(1-q)^{(\alpha+\beta-1)}}{(1-q)^{\alpha+\beta-1}} \tag{1.38}$$

Using the formula (1.37)and the formula (1.38), We therefore obtain

$$\begin{aligned} c_k &= \frac{(1-q)^{(\alpha+\beta-1)}(t-aq^{k+1})^{(\alpha+\beta-1)}}{(1-q)^{\alpha+\beta-1}} \\ &\quad - \frac{t(1-q)(1-q)^{\alpha+\beta-2}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}}\sum_{j=0}^{\infty} x^{\beta-1}(1-q^{j+1})^{(\beta-1)}x^{\alpha-1}(q^j-\frac{a}{x}q^{k+1})^{(\alpha-1)}q^j \\ &= \frac{(1-q)^{\alpha+\beta-1}x^{\alpha+\beta-1}(1-\frac{a}{x}q^{k+1})^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} \\ &\quad - \frac{x^{\alpha+\beta-1}(1-q)^{\alpha+\beta-1}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}}\sum_{j=0}^{\infty}(1-q^{j+1})^{(\beta-1)}(1-\frac{a}{x}q^{k+1-j})^{(\alpha-1)}q^{\alpha j}. \end{aligned}$$

If we take $\mu = \frac{a}{x}q^k$, We find

$$\begin{aligned} c_k &= \frac{(1-q)^{\alpha+\beta-1}x^{\alpha+\beta-1}(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} \\ &\quad - \frac{x^{\alpha+\beta-1}(1-q)^{\alpha+\beta-1}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}}\sum_{j=0}^{\infty}(1-q^{j+1})^{(\beta-1)}(1-\mu q^{1-j})^{(\alpha-1)}q^{\alpha j}. \end{aligned}$$

The Lemma is applied (1.3), we obtain

$$\begin{aligned} c_k &= \frac{(1-q)^{\alpha+\beta-1}x^{\alpha+\beta-1}(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} - \frac{x^{\alpha+\beta-1}(1-q)^{\alpha+\beta-1}(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} \\ &= 0. \end{aligned}$$

■

## 1.6 The fractional $q$-derivative of Riemann-Liouville

**Definition 1.9** ***see***[9]*Suppose that $\alpha \in \mathbb{R}$ and $n = [\alpha], f(t)$ is a real-valued function on $(0,\infty)$. The $\alpha$-order Riemann-Liouville type fractional $q$-derivative of function $f(t)$ is defined* *as follows*

$$(D_q^{\alpha} f)(t) = \begin{cases} (I_q^{-\alpha} f)(t), & \alpha \leq 0, \\ (D_q^n I_q^{n-\alpha} f)(t), & \alpha > 0, \end{cases}$$

*Where* $[\alpha]$ *represents the smallest integer which is greater than or equal to* $\alpha$.

*For* $\alpha > 0$, *we have*

$$D_q^{\alpha} f(t) = \frac{1}{\Gamma_q(n-\alpha)} d_q^n \int_0^t (t-qs)^{n-\alpha-1} f(s) d_q s. \tag{1.39}$$

**Lemma 1.4** [8] *For* $\alpha \in \mathbb{R} \backslash \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, *the following is valid:*

$$(D_q D_{q,a}^{\alpha} f)(t) = (D_{q,a}^{\alpha+1} f)(t), \qquad (0 < a < t < b).$$

**Proof.** We will consider three cases. For $\alpha \leq -1$, according to Theorem (1.1), we have

$$\begin{aligned} (D_q D_{q,a}^{\alpha} f)(t) &= (D_q I_{q,a}^{-\alpha} f)(t) \\ &= (D_q I_{q,a}^{1-\alpha-1} f)(t) \\ &= (D_q I_{q,a} I_{q,a}^{-\alpha-1} f)(t) \\ &= (I_{q,a}^{-(\alpha+1)} f)(t) \\ &= (D_{q,a}^{\alpha+1} f)(t). \end{aligned}$$

In the case $-1 < \alpha < 0$, i.e, $0 < \alpha + 1 < 1$, we obtain

$$\begin{aligned} (D_q D_{q,a}^{\alpha} f)(t) &= (D_q I_{q,a}^{-\alpha} f)(t) \\ &= (D_q I_{q,a}^{1-(\alpha+1)} f)(t) \\ &= (D_{q,a}^{\alpha+1} f)(t). \end{aligned}$$

For $\alpha > 0$, we get

$$\begin{aligned} (D_q D_{q,a}^{\alpha} f)(t) &= (D_q D_q^{[\alpha]} I_{q,a}^{[\alpha]-\alpha} f)(t) \\ &= (D_q^{[\alpha]+1} I_{q,a}^{[\alpha]-\alpha} f)(t) \\ &= (D_{q,a}^{\alpha+1} f)(t). \end{aligned}$$

■

**Lemma 1.5** [8] *Let* $f(t)$ *be a function defined on an interval* $(0, b)$ *and* $\alpha \in \mathbb{R}^+$. *Then the following is valid:*

$$(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f)(t) = f(t), \quad (0 < a < t < b).$$

**Proof.** For $\alpha > 0$, we have:

$$\begin{aligned}(D^{\alpha}_{q,a} I^{\alpha}_{q,a} f)(t) &= (D^{[\alpha]}_{q} I^{[\alpha]-\alpha}_{q,a} I^{\alpha}_{q,a} f)(t) \\ &= (D^{[\alpha]}_{q} I^{[\alpha]-\alpha+\alpha}_{q,a} f)(t) \\ &= (D^{[\alpha]}_{q,a} I^{[\alpha]}_{q,a} f)(t) = f(t).\end{aligned}$$

■

**Lemma 1.6** *(see[9]) Let $f(t) \in C[0,b]$,$0 < q < 1$ and $0 < t \leq b$, then we have:*

1. $D_q I_q f(t) = f(t)$.

2. $I_q D_q f(t) = f(t) - f(0)$.

**Proof.**

1. Using definition (1.2) and the formula (1.10) we have:

$$\begin{aligned}D_q I_q f(t) &= \frac{\int_0^t f(s) d_q s - \int_0^{qt} f(s) d_q s}{(1-q)t} \\ &= \frac{t(1-q)\sum_{i=0}^{\infty} f(tq^i) q^i - t(1-q)\sum_{i=0}^{\infty} f(tq^{i+1}) q^{i+1}}{(1-q)t} \\ &= \sum_{i=0}^{\infty} f(tq^i) q^i - \sum_{i=0}^{\infty} f(tq^{i+1}) q^{i+1} \\ &= f(t).\end{aligned}$$

2. Using the formula (1.10) and definition (1.2) we have:

$$\begin{aligned}I_q D_q f(t) &= \int_0^t \frac{f(s) - f(qs)}{(1-q)s} d_q s \\ &= t(1-q)\sum_{i=0}^{\infty} q^i \frac{f(tq^i) - f(tq^{i+1})}{t(1-q)q^i} \\ &= \sum_{i=0}^{\infty} (f(tq^i) - f(tq^{i+1})) \\ &= f(t) - \lim_{i\to\infty} f(tq^i) \\ &= f(t) - f(0).\end{aligned}$$

■

**Theorem 1.2** [5]*Let $\alpha > 0$ and $n$ be a positive integer. Then, the following equality holds:*

$$(I^{\alpha}_{q} D^{n}_{q} f)(t) = (D^{n}_{q} I^{\alpha}_{q} f)(t) - \sum_{k=0}^{n-1} \frac{t^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (D^{k}_{q} f)(0). \tag{1.40}$$

**Proof.** We prove the theorem by induction. For the case $n = 1$, we have

$$(I_q^\alpha D_q f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} (D_q f)(s) d_q s, \tag{1.41}$$

Using formula (1.8) we get:

$$D_q\left[(t-s)^{(\alpha-1)} f(s)\right] = (t-s)^{(\alpha-1)} D_q f(s) - [\alpha-1]_q (t-qs)^{(\alpha-2)} f(s).$$

So:

$$(t-s)^{(\alpha-1)} D_q f(s) = D_q\left[(t-s)^{(\alpha-1)} f(t)\right] + [\alpha-1]_q (t-qs)^{(\alpha-2)} f(t).$$

Put this on (1.41) we have

$$\begin{aligned}
(I_q^\alpha D_q f)(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} (D_q f)(s) d_q s \\
&= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-2)} f(s) d_q s + \frac{1}{\Gamma_q(\alpha)} \left[(t-s)^{(\alpha-1)} f(s)\right]_{s=0}^{s=t} \\
&= (I_q^{\alpha-1} f)(t) + \left(0 - \frac{t^{(\alpha-1)}}{\Gamma_q(\alpha)} f(0)\right) \\
&= (D_q I_q^\alpha f)(t) - \frac{t^{(\alpha-1)}}{\Gamma_q(\alpha)} f(0).
\end{aligned}$$

Suppose now that (1.40) holds for $n \in \mathbb{N}$, For $n+1$ we have:

$$\begin{aligned}
(I_q^\alpha D_q^{n+1} f)(t) &= (I_q^\alpha D_q^n D_q f)(t) \\
&= (D_q^n I_q^\alpha D_q f)(t) - \sum_{k=0}^{n-1} \frac{t^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (D_q^{k+1} f)(0) \\
&= D_q^n \left[(D_q I_q^\alpha f)(t) - \frac{t^{(\alpha-1)}}{\Gamma_q(\alpha)} f(0)\right] - \sum_{k=0}^{n-1} \frac{t^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (D_q^{k+1} f)(0) \\
&= (D_q^{n+1} I_q^\alpha f)(t) - \frac{t^{(\alpha-1-n)}}{\Gamma_q(\alpha-n)} f(0) - \sum_{k=1}^{n} \frac{t^{\alpha-(n+1)+k}}{\Gamma_q(\alpha+k-(n+1)+1)} (D_q^k f)(0) \\
&= (D_q^{n+1} I_q^\alpha f)(t) - \sum_{k=0}^{n} \frac{t^{\alpha-(n+1)+k}}{\Gamma_q(\alpha+k-(n+1)+1)} (D_q^k f)(0).
\end{aligned}$$

The theorem is proved. ■

## 1.7 The fractional $q$-derivative of caputo type

**Definition 1.10** ***see***[9] *Let $n = [\alpha]$, $f(t)$ is a real-valued function on $(0, \infty)$. The $\alpha$-order caputo type fractional $q$-derivative of function $f(t)$ is defined by*

$${}^c D_q^\alpha f(t) = \begin{cases} (I_q^{-\alpha} f)(t), & \alpha \leq 0, \\ (I_q^{n-\alpha} D_q^n f)(t), & \alpha > 0, \end{cases}$$

*Where $[\alpha]$ represents the smallest integer which is greater than or equal to $\alpha$.*

*For $\alpha > 0$, we have*

$$^{c}D_q^{\alpha} f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_0^t (t-qs)^{n-\alpha-1} d_q^n f(s) d_q s. \tag{1.42}$$

**Lemma 1.7** *(See[9]) Let $0 < \alpha < 1$ and $f(t) \in C[0,b]$ such that $D_q f(t) \in C[0,b]$. Then*

$$^{c}D_q^{\alpha} f(t) = D_q^{\alpha} f(t) - \frac{t^{-\alpha}}{\Gamma_q(1-\alpha)} f(0), \quad t > 0.$$

**Lemma 1.8** *(See[9]) For $0 < \alpha < 1$ and $f(t) \in C[0,b]$, it holds*

$$D_q^{\alpha} I_q^{\alpha} f(t) = f(t), \qquad ^{c}D_q^{\alpha} I_q^{\alpha} f(t) = f(t).$$

**Proof.** By Definition (1.9) and Lemma (1.6), we see that

$$\begin{aligned} D_q^{\alpha} I_q^{\alpha} f(t) &= D_q I_q^{1-\alpha} I_q^{\alpha} f(t) \\ &= D_q I_q f(t) \\ &= f(t). \end{aligned}$$

Then, the first equality holds. From Lemma (1.7) and noting $I_q^{\alpha} f(0) = 0$,

$$\begin{aligned} ^{c}D_q^{\alpha} I_q^{\alpha} f(t) &= D_q^{\alpha} I_q^{\alpha} f(t) \\ &= f(t). \end{aligned}$$

■

**Lemma 1.9** *(**see**[9]) For $0 < \alpha < 1$ and $n-1 < \beta < n$, $n-1 < \alpha+\beta \leq n$ and $n \geq 1$, we have:*

$$^{c}D_q^{\alpha}\ {}^{c}D_q^{\beta} f(t) = {}^{c}D_q^{\alpha+\beta} f(t).$$

**Proof.** Applying Theorem(1.1) and Lemma (1.8), it yields that

$$\begin{aligned} ^{c}D_q^{\alpha}\ {}^{c}D_q^{\beta} f(t) &= {}^{c}D_q^{\alpha} I_q^{n-\beta} D_q^n f(t) \\ &= {}^{c}D_q^{\alpha} I_q^{\alpha} I_q^{n-\alpha-\beta} D_q^n f(t) \\ &= {}^{c}D_q^{\alpha} I_q^{\alpha}\ {}^{c}D_q^{\alpha+\beta} f(t) \\ &= {}^{c}D_q^{\alpha+\beta} f(t). \end{aligned}$$

■

**Lemma 1.10** *(See[9]) The following properties hold*

*(i) If $|f(t)| \leq M$, then $I_q^{\alpha} f(0) = 0, \alpha > 0$,*

*(ii) If* $|D_q f(t)| \leq M$, *then* ${}^cD_q^\alpha f(0) = 0, 0 < \alpha < 1.$

**Proof.** For property $(i)$, using (1.14) we can obtain

$$\begin{aligned}
\left|I_q^\alpha f(t)\right| &\leq \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \left|f(s)\right| d_q s \\
&\leq \frac{M}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} d_q s \\
&= \frac{M}{\Gamma_q(\alpha)} t^\alpha \int_0^1 (t-q\tau)^{(\alpha-1)} d_q \tau \\
&= \frac{M}{\Gamma_q(\alpha)} t^\alpha B_q(\alpha, 1) \\
&= \frac{t^\alpha M}{\Gamma_q(\alpha+1)} \longrightarrow 0, t \longrightarrow 0.
\end{aligned}$$

Next, from $(i)$ and ${}^cD_q^\alpha f(t) = I_q^{1-\alpha} D_q f(t)$, we see that the property $(ii)$ holds. ■

**Theorem 1.3** *(See[2]) Let* $\alpha > 0$ *and* $n = [\alpha]$. *Then, the following equality holds:*

$$(I_q^{\alpha c} D_q^\alpha f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0). \tag{1.43}$$

*and if* $0 < \alpha \leq 1$ *then*

$$(I_q^{\alpha c} D_q^\alpha f)(t) = f(t) - f(0).$$

CHAPTER 2

# Nonlinear $q$-fractional differential equations of Riemann-Liouville type

In this chapter, we will study the existence and the uniqueness of solution of a $q$-fractional boundary value problem (we will detail the work submitted by Ferreira see [5]) by tow fixed point theorems, Banach fixed point theorem and Krasnoselskii fixed point theorem.

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1 \\ \quad u(0) = 0, u(1) = 0, \end{cases} \tag{2.1}$$

Where $1 < \alpha \leq 2, f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, $D_q^\alpha$ is the Riemann-Liouville fractional $q$-derivatives of order $\alpha$.

## 2.1 Fixed point theorems

Fixed point theorems concern maps $f$ of a set $X$ into itself that, under certain conditions, admit a fixed point, that is, a point $x \in X$ such that $f(x) = x$ . The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology.

**Definition 2.1** *Let $(E, d)$ a complete metric space and $F : E \to E$ a continue mapp.*

*i) We say that $u \in E$ is a fixed point of $F$ if $F(u) = u$.*

*ii) We say that $F$ is contractante if $F$ lipschiz raport $0 < L < 1$, i.e, there is $0 < L < 1$, such that*

$$\forall u, v \in E : d(F(u), F(v)) \leq Ld(u, v), 0 < L < 1.$$

**Definition 2.2** *(completely continuous) Let $X$ and $Y$ two Banach space and $F : X \longrightarrow Y$ a mapp define $X$ a value in $F$. We say that $F$ is completely continuous if she is continuous and transform any bounded of $Y$ in a relatively compact set in $Y$. $F$ is said to be compact if $F(X)$ is relatively compact in $Y$.*

**Theorem 2.1** *(Banach ) Let $X$ a Banach space, and a contractant operator $F : X \longrightarrow X$. So $F$ admits a unique fixed point.*

$$i.e.\ \exists!\, u \in X \text{ such that } Fu = u.$$

**Lemma 2.1** *(Krasnoselskii ) Let $E$ be a Banach space, and let $P \subset E$ be a cone. Assume $\Omega_1, \Omega_2$ are open subsets of $E$ with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let $F : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completly continous operator such that*

$$\|Fu\| \geq \|u\|, u \in p \cap \partial\Omega_1, and \|Fu\| \leq \|u\|, u \in p \cap \partial\Omega_2.$$

*Then $F$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.*

## 2.2 Fundamental lemmas

**Lemma 2.2** [5] *Let $1 < \alpha \leq 2$ and $h : [0, 1] \to \mathbb{R}$ is a nonnegative continuous function, then the boundary value problem*

$$\begin{cases} D_q^\alpha u(t) + h(t) = 0, & 0 < t < 1 \\ u(0) = 0, u(1) = 0, \end{cases} \tag{2.2}$$

*has a unique solution*

$$u(t) = \int_0^1 G(t, qs)h(s)d_q s. \tag{2.3}$$

*Where*

$$G(t, s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} [t(1-s)]^{(\alpha-1)} - (t-s)^{(\alpha-1)}, & 0 \leq s \leq t \leq 1, \\ [t(1-s)]^{(\alpha-1)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

**Proof.** Let us put $n = 2$. using Lemma(1.8) and theorem (1.40) we have:

$$D_q^\alpha u(t) = -h(t) \Leftrightarrow (I_q^\alpha D_q^2 I_q^{2-\alpha} u)(t) = -I_q^\alpha h(t),$$

$$\Leftrightarrow u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) d_q s.$$

When certain constants $c_1, c_2 \in \mathbb{R}$ . Using the boundary conditions given in (2.2) we take $c_1 = \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} h(s)d_q s$ and $c_2 = 0$ to get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} t^{\alpha-1} h(s) d_q s - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) d_q s \\ &= \frac{1}{\Gamma_q(\alpha)} \left[ \int_0^t \left( [t(1-qs)]^{(\alpha-1)} - (t-qs)^{(\alpha-1)} \right) h(s) d_q s + \int_t^1 [t(1-qs)]^{(\alpha-1)} h(s) d_q s \right] \\ &= \int_0^1 G(t, qs) h(s) d_q s. \end{aligned}$$

If we define a function $G$ by

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} [t(1-s)]^{(\alpha-1)} - (t-s)^{(\alpha-1)}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{(\alpha-1)}, & 0 \le t \le s \le 1. \end{cases}$$

■

**Lemma 2.3** [5]*Function $G$ defined above satisfies the following conditions:*

*1) $G(t, qs) \ge 0, 0 \le t, s \le 1,$*

*2) $G(t, qs) \le G(qs, qs), 0 \le t, s \le 1.$*

**Proof.** First we define two following functions:

$$g_1(t,s) = (t(1-s))^{(\alpha-1)} - (t-s)^{(\alpha-1)}, 0 \le t \le s \le 1.$$

$$g_2(t,s) = (t(1-s))^{(\alpha-1)}, 0 \le t \le s \le 1.$$

Clear that $g_2(t, qs) \ge 0$, and

$$g_1(t,qs) = (t(1-qs))^{(\alpha-1)} - (t-qs)^{(\alpha-1)}, 0 \le t \le s \le 1.$$

Using the following remark:

**Remark 2.1** [5] *We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$.*

We get:

$$\begin{aligned} g_1(t,qs) &= (t(1-qs))^{(\alpha-1)} - (t-qs)^{(\alpha-1)}, 0 \le t \le s \le 1. \\ &\ge t^{(\alpha-1)}(1-qs)^{(\alpha-1)} - t^{(\alpha-1)}(1-qs)^{(\alpha-1)} \\ &= 0. \end{aligned}$$

Therefore,$G(t, qs) \ge 0$. Moreover, for fixed $s \in [0, 1]$:

$$\begin{aligned} D_q g_1(t,s) &= D_q \left[ (t(1-s))^{(\alpha-1)} - (t-s)^{(\alpha-1)} \right] \\ &= [\alpha-1]_q \, t^{(\alpha-2)}(1-s)^{(\alpha-1)} - [\alpha-1]_q \, (t-s)^{(\alpha-2)} \\ &= [\alpha-1]_q \, t^{(\alpha-2)} \left[ (1-s)^{(\alpha-1)} - (1-s/t)^{(\alpha-2)} \right] \\ &\le [\alpha-1]_q \, t^{(\alpha-2)} \left[ (1-s)^{(\alpha-1)} - (1-s)^{(\alpha-2)} \right] \\ &\le 0. \end{aligned}$$

Which implies that $g_1(t,s)$ is decreasing with respect to $t$ for all $s \in [0,1]$. Therefore,

$$g_1(t,qs) \leq g_1(qs,qs), 0 < t,s \leq 1. \tag{2.4}$$

Now note that $G(0,qs) = 0 \leq G(qs,qs)$ for all $s \in [0,1]$.

Therefore, by (2.4) and the definition of $g_2$ (it is obviously increasing in $t$) we conclude that $G(t,qs) \leq G(qs,qs)$ for all $0 < t,s \leq 1$. ■

Let $X = C[0,1]$ be a Banach space endowed with norm $\|u\|_X = max_{0\leq t\leq 1}|u(t)|$. Define the cone $P \subset \{u \in X : u(t) \geq 0, 0 \leq t \leq 1\}$.

**Remark 2.2** [5] *It follows from the nonnegativeness and continuity of $G$ and $f$ that the operator $F : P \to X$ defined by*

$$Fu = \int_0^1 G(t,qs)f(s,u(s))d_qs,$$

*Satisfies $F(P) \subset P$ and is completely continuous. For our purposes, let us define two constants:*

$$M = \left(\int_0^1 G(qs,qs)d_qs\right)^{-1}$$

$$N = \left(\int_{\tau_1}^{\tau_2} G(qs,qs)d_qs\right)^{-1},$$

*Where $\tau_1 \in \{0, q^m\}$ and $\tau_2 = q^n$ with $m,n \in N_0, m > n$.*

## 2.3 Results of existence and uniqueness

In this section, we will prove the existence and uniqueness of the solution of problem (2.1)in space $C([0,1],\mathbb{R}^+)$, we use contraction principle of Banach.

**Theorem 2.2** *Let $f : C([0,1],\mathbb{R}^+) \longrightarrow \mathbb{R}^+$ a continuous function hold:*
*There exist a constant $K > 0$ such that*

$$|f(s,u(s)) - f(s,v(s))| \leq K|u-v|, \forall s \in [0,1], \forall u,v \in \mathbb{R},$$

*and*

$$\frac{2K}{\Gamma_q(\alpha+1)} < 1,$$

*The problem (2.1)admits unique solution in $[0,1]$.*

**Proof.** We transform problem (2.1) into a fixed point problem Lemma (2.2), by considering the operator :

$$\begin{aligned} F:\ & C([0,1],\mathbb{R}^+) \longrightarrow C([0,1],\mathbb{R}^+) \\ & u \longmapsto Fu(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} t^{\alpha-1} f(s,u(s)) d_q s \\ & \qquad - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s,u(s)) d_q s. \end{aligned}$$

Where $C([0,1],\mathbb{R}^+)$ the Banach space of continuous functions $u$ defined from $[0,1]$ in $\mathbb{R}^+$; equipped with the norm

$$\|u\| = max_{t\in[0,1]} |u(t)|.$$

It is clear that the fixed points of the operator $F$ are the solutions of problem (2.1).
$F$ is well defined, indeed: if $u(t) \in C([0,1],\mathbb{R}^+)$, then $Fu(t) \in C([0,1],\mathbb{R}^+)$.
To show that $F$ admits a fixed point, it suffices to show that $F$ is a contraction, in effect if $u, v \in C[0,1]$, for all $t \in [0,1]$ , using the Lipschitiz condition we get:

$$\begin{aligned} |Fu(t) - Fv(t)| &= \Bigg| \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} t^{\alpha-1} f(s,u(s)) d_q s \\ &\quad - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s,u(s)) d_q s \\ &\quad - \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} t^{\alpha-1} f(s,v(s)) d_q s \\ &\quad + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s,v(s)) d_q s \Bigg| \\ &\le \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} t^{\alpha-1} |f(s,u(s)) - f(s,v(s))| d_q s \\ &\quad + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} |f(s,u(s)) - f(s,v(s))| d_q s \\ &\le k|u-v| \left[ \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} t^{\alpha-1} d_q s + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} d_q s \right] \\ &\le k\|u-v\| \left( \frac{t^{\alpha-1}(1+t)}{\Gamma_q(\alpha+1)} \right) \\ &\le \frac{2K}{\Gamma_q(\alpha+1)} \|u-v\|. \end{aligned}$$

By virtue of $\frac{2K}{\Gamma_q(\alpha+1)} < 1$, we can deduce that $F$ is a contraction, and by the theorem of Banach $F$ admits only one fixed point which is a solution of the problem (2.1). ■

The next existence result is based on the Krasnoselskii fixed point theorem.

**Theorem 2.3** [5] *Let $f(t,u)$ be a nonnegative continuous function on $[0,1]\times[0,\infty)$. If there exists two positive constants $r_2 > r_1 > 0$ such that*

$$f(t,u) \leq Mr_2, for(t,u) \in [0,1]\times[0,r_2], \tag{2.5}$$

$$f(t,u) \geq Nr_1, for(t,u) \in [\tau_1,\tau_2]\times[0,r_1], \tag{2.6}$$

*Then problem (2.1) has a solution $u_0$ satisfying $r_1 \leq \|u_0\| \leq r_2$.*

**Proof.** Since the operator $F : P \longrightarrow P$ is completely continuous we only have to show that the operator equation $u = Fu$ has a solution satisfying $r_1 \leq \|u\| \leq r_2$.
Let $\Omega_1 = \{u \in P : \|u\| < r_1\}$. For $u \in P \cap \partial\Omega_1$ , we have $0 \leq u(t) \leq r_1$ on $[0,1]$. Using Lemma (2.3) and (2.6), and the definitions of $\tau_1$ and $\tau_2$, we obtain
Let

$$Fu = \int_0^1 G(t,qs)f(s,u(s))d_qs,$$

So:

$$\begin{aligned}
\|Fu\| &= max_{0\leq t\leq 1}\int_0^1 G(t,qs)f(s,u(s))d_qs \\
&\geq Nr_1\int_{\tau_1}^{\tau_2} G(qs,qs)d_qs \\
&\geq r_1\left(\int_{\tau_1}^{\tau_2} G(qs,qs)d_qs\right)^{-1}\left(\int_{\tau_1}^{\tau_2} G(qs,qs)d_qs\right) \\
&\geq r_1 \\
&\geq \|u\|.
\end{aligned}$$

Let $\Omega_2 = \{u \in P : \|u\| < r_2\}$. For $u \in P \cap \partial\Omega_2$ , we have $0 \leq u(t) \leq r_2$ on $[0,1]$. Using Lemma (2.3) and (2.5), and the definitions of $\tau_1$ and $\tau_2$, we obtain

$$\begin{aligned}
\|Fu\| &= max_{0\leq t\leq 1}\int_0^1 G(t,qs)f(s,u(s))d_qs \\
&\leq Mr_2\int_0^1 G(qs,qs)d_qs \\
&\leq r_2\left(\int_0^1 G(qs,qs)d_qs\right)^{-1}\left(\int_0^1 G(qs,qs)d_qs\right) \\
&\leq r_2 \\
&\leq \|u\|.
\end{aligned}$$

Now an application of Lemma (2.1) concludes the proof. ■

CHAPTER 3

# Inverse Problem for $q$-Fractional PDEs

In this chapter, we will study the existence of inverse Probeme solutions for the $q$-Fractional partial differential equation, in the sense of Caputo is written as:

$$\begin{cases} {}^cD_q^\alpha U(t,x) = [(1-x^2)U_x]_x + h(x), & (t,x) \in ]0,T[\times]-1,1[ \\ D_q^k U(t,x)|_{t=0} = v_k(x), \quad k = 0,1,2,\cdots,m-1. \end{cases}$$

Where ${}^cD_q^\alpha U(t,x)$ is the Caputo $q$-fractional derivative of order $\alpha (0 < \alpha < 1)$ of $U(t,x)$ ,$D_q^k$ is the $q$-fractional derivative of order $k$ and $v_k(x)$ given function.

## 3.1 Problem with Caputo $q$-derivative

**Lemma 3.1** *Let $\lambda \in \mathbb{R}$, then the cauchy problem:*

$$\begin{cases} {}^cD_q^\alpha u(t) + \lambda u(t) = f(t), \quad 0 < \alpha \leq 1, 0 < t \leq b, \\ D_q^k u(0) = d_k, \quad k = 0,1,2,\cdots,m-1. \end{cases} \tag{3.1}$$

*Where $d_k$ are constants, $m = [\alpha]+1$, admits a solution $u(t)$ in the form:*

$$u(t) = \sum_{k=0}^{m-1} E^q_{\alpha,k+1}(-\lambda,t) d_k t^k + \int_0^t (t-qs)^{(\alpha-1)} E^q_{\alpha,\alpha}(-\lambda, t-q^\alpha s) f(s) d_q s. \tag{3.2}$$

**Proof.** Using Theorem (1.3) we have:

$$(I_q^{\alpha c} D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k u)(0).$$

and

$$\begin{aligned} I_q^\alpha [-\lambda u(t) + f(t)] &= -\lambda I_q^\alpha u(t) + I_q^\alpha f(t) \\ &= \frac{-\lambda}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s) d_q s + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} [-\lambda u(s) + f(s)] d_q s, \end{aligned}$$

So:

$$u(t)-\sum_{k=0}^{m-1}\frac{t^k}{\Gamma_q(k+1)}(D_q^k u)(0)=\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\left[-\lambda u(s)+f(s)\right]d_q s,$$

Then the solution of problem (3.1) satisfies the equivalent integral equation

$$u(t)=\sum_{k=0}^{m-1}d_k\frac{t^k}{\Gamma_q(k+1)}+\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\left[-\lambda u(s)+f(s)\right]d_q s. \tag{3.3}$$

Let $d(t)=\sum_{k=0}^{m-1}d_k\frac{t^k}{\Gamma_q(k+1)}$. Then by a circulative iteration and using Theorem (1.1), we obtain from (3.3) that

$$\begin{aligned}
u(t)&=d(t)-\lambda I_q^\alpha u(t)+I_q^\alpha f(t)\\
&=d(t)-\lambda I_q^\alpha\left[d(t)-\lambda I_q^\alpha u(t)+I_q^\alpha f(t)\right]+I_q^\alpha f(t)\\
&=d(t)-\lambda I_q^\alpha d(t)+\lambda^2 I_q^{2\alpha}u(t)-\lambda I_q^{2\alpha}f(t)+I_q^\alpha f(t)\\
&=d(t)-\lambda I_q^\alpha d(t)+\lambda^2 I_q^{2\alpha}\left[d(t)-\lambda I_q^\alpha u(t)+I_q^\alpha f(t)\right]+I_q^{2\alpha}f(t)+I_q^\alpha f(t)\\
&=d(t)-\lambda I_q^\alpha d(t)+\lambda^2 I_q^{2\alpha}d(t)-\lambda^3 I_q^{3\alpha}u(t)+\lambda^2 I_q^{3\alpha}f(t)-\lambda I_q^{2\alpha}f(t)+I_q^\alpha f(t)\\
&\quad\cdots\cdots\cdots\\
&=\sum_{n=0}^{\infty}(-\lambda)^n I_q^{n\alpha}d(t)+\sum_{n=0}^{\infty}(-\lambda)^n I_q^{(n+1)\alpha}f(t)+\lim_{n\longrightarrow\infty}(-\lambda)^n I_q^{n\alpha}u(t).
\end{aligned} \tag{3.4}$$

Since

$$(-\lambda)^n I_q^{n\alpha}d(t)=(-\lambda)^n I_q^{n\alpha}\left[\sum_{k=0}^{m-1}d_k\frac{t^k}{\Gamma_q(k+1)}\right],$$

Using Proposition (1.10) we have:

$$I_q^{n\alpha}t^k=\frac{\Gamma_q(k+1)}{\Gamma_q(n\alpha+k+1)}t^{n\alpha+k},$$

So:

$$\begin{aligned}
(-\lambda)^n I_q^{n\alpha}\left[\sum_{k=0}^{m-1}d_k\frac{t^k}{\Gamma_q(k+1)}\right]&=\sum_{k=0}^{m-1}d_k\frac{(-\lambda)^n I_q^{n\alpha}t^k}{\Gamma_q(k+1)}\\
&=\sum_{k=0}^{m-1}d_k\frac{(-\lambda)^n t^{n\alpha+k}}{\Gamma_q(n\alpha+k+1)}\\
&=\sum_{k=0}^{m-1}\frac{(-\lambda)^n t^{n\alpha}}{\Gamma_q(n\alpha+k+1)}t^k d_k,
\end{aligned}$$

Then

$$\sum_{n=0}^{\infty}(-\lambda)^n I_q^{n\alpha}d(t)=\sum_{k=0}^{m-1}\left[\sum_{n=0}^{\infty}\frac{(-\lambda)^n t^{n\alpha}}{\Gamma_q(n\alpha+k+1)}\right]t^k d_k=\sum_{k=0}^{m-1}E_{\alpha,k+1}^q(-\lambda,t)d_k t^k. \tag{3.5}$$

Next, by the identity $(t-s)_q^{(\beta+\gamma)} = (t-s)_q^{(\beta)}(t-q^\beta s)_q^{(\gamma)}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}(-\lambda)^n I_q^{(n+1)\alpha}f(t) &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{\Gamma_q(n\alpha+\alpha)}\int_0^t (t-qs)^{(n\alpha+\alpha-1)}f(s)d_qs \\ &= \int_0^t (t-qs)^{(\alpha-1)}\sum_{n=0}^{\infty}\frac{(-\lambda)^n(t-q^\alpha s)^{(n\alpha)}}{\Gamma_q(n\alpha+\alpha)}f(s)d_qs \\ &= \int_0^t (t-qs)^{(\alpha-1)}E^q_{\alpha,\alpha}(-\lambda, t-q^\alpha s)f(s)d_qs.\end{aligned} \tag{3.6}$$

Furthermore, by the argument of Lemma (1.10), and $|u(t)| \leq M, |(-\lambda)b^\alpha| \leq 1$, we have:

$$\left|(-\lambda)^n I_q^{n\alpha}u(t)\right| \leq \frac{((-\lambda)b^\alpha)^{(n)}}{\Gamma_q(n\alpha+1)}M \leq \frac{M}{\Gamma_q(n\alpha+1)} \longrightarrow 0, n \longrightarrow \infty. \tag{3.7}$$

Then, combining (3.4)(3.7), we complete the proof. ■

## 3.2 Inverse Problem for $q$-Fractional PDEs with Caputo $q$-derivative

Let the inverse Problem for $q$-Fractional PDEs with Caputo $q$-derivative given by the following equation:

$$\begin{cases} {}^cD_q^\alpha U(t,x) = [(1-x^2)U_x]_x + h(x), & (t,x) \in ]0,T[\times]-1,1[ \\ D_q^k U(t,x)|_{t=0} = v_k(x), & k = 0,1,2,\cdots,m-1. \end{cases} \tag{3.8}$$

Where ${}^cD_q^\alpha U(t,x)$ is the Caputo $q$-fractional derivative of order $\alpha(0<\alpha<1)$ of $U(t,x)$ and $v_k(x)$ given function.

### 3.2.1 The existence of the Solution

**Theorem 3.1** *The problem (3.8) admits the solution set $U(t,x), h(x)$, given by :*

$$\begin{cases} U(t,x) = \sum_{n=0}^{\infty} U_n(t)P_n(x), \\ h(x) = \sum_{n=0}^{\infty} h_nP_n(x). \end{cases} \tag{3.9}$$

*Where*

$$U_n(t) = \sum_{k=0}^{m-1} E^q_{\alpha,k+1}(-\lambda_n,t)t^k v_{k,n} + \int_0^t (t-qs)^{(\alpha-1)}E^q_{\alpha,\alpha}(-\lambda_n, t-q^\alpha s)h_nd_qs.$$

*Is a function to be determined, $P_n(x)$ the Legendre polynomial and $h_n$ is constant, and*

$$\lambda_n = n(n+1)$$

$$v_{k,n} = \frac{2n+1}{2}\int_{-1}^{1} v_k(x)P_n(x)dx.$$

**Proof.** We seek the solution of Problem (3.8), in the form:

$$\left\{U(t,x) = \sum_{n=0}^{\infty} U_n(t)P_n(x), \quad h(x) = \sum_{n=0}^{\infty} h_n P_n(x)\right\}.$$

then $U(t,x), h(x)$ is to satisfy equation (3.8), C-a-d::

$$\begin{cases} {}^cD_q^\alpha U(t,x) = \sum_{n=0}^{\infty} P_n(x){}^cD_q^\alpha U_n(t) \\ [(1-x^2)U_x]_x = \sum_{n=0}^{\infty}(1-x^2)P_n''(x)U_n(t) - 2x\sum_{n=0}^{\infty} P_n'(x)U_n(t). \end{cases} \tag{3.10}$$

So:

$$\sum_{n=0}^{\infty} P_n(x){}^cD_q^\alpha U_n(t) = \sum_{n=0}^{\infty}(1-x^2)P_n''(x)U_n(t) - 2x\sum_{n=0}^{\infty} P_n'(x)U_n(t) + \sum_{n=0}^{\infty} h_n P_n(x).$$

we obtain

$$P_n(x){}^cD_q^\alpha U_n(t) = (1-x^2)P_n''(x)U_n(t) - 2xP_n'(x)U_n(t) + h_nP_n(x), \quad n = 0,1,2,\cdots.$$

and since $P_n(x)$ is a solution of Legendre equation, we obtain:

$$(1-x^2)P_n''(x)U_n(t) - 2xP_n'(x)U_n(t) = -\lambda P_n(x).$$

So:

$$P_n(x){}^cD_q^\alpha U_n(t) = -\lambda U_n(t)P_n(x) + h_nP_n(x),$$

$${}^cD_q^\alpha U_n(t) = -\lambda U_n(t) + h_n,$$

Then:

$${}^cD_q^\alpha U_n(t) + \lambda U_n(t) = h_n, n = 0,1,2,\cdots. \tag{3.11}$$

and on the other hand:

$$D_q^k U(t,x)_{t=0} = \sum_{n=0}^{\infty} P_n(x)D_q^k U_n(t) = v_k(x) = \sum_{n=0}^{\infty} v_{k,n}P_n(x).$$

So:

$$D_q^k U_n(t) = v_{k,n}.$$

For $n \geq 1$ then:

$$\begin{cases} {}^{c}D_q^{\alpha}U_n(t) + \lambda U_n(t) = h_n, \\ D_q^k U_n(t) = v_{k,n}. \end{cases} \tag{3.12}$$

According to Lemma (3.1) for $0 < \alpha \leq 1$, we have the problem (3.8) admits a solution in the form:

$$U_n(t) = \sum_{k=0}^{m-1} E^q_{\alpha,k+1}(-\lambda_n, t)t^k v_{k,n} + \int_0^t (t-qs)^{(\alpha-1)} E^q_{\alpha,\alpha}(-\lambda_n, t-q^{\alpha}s)h_n d_q s.$$

Then:

$$\begin{aligned} U(t,x) &= \sum_{n=0}^{\infty} U_n(t)P_n(x) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{m-1} E^q_{\alpha,k+1}(-\lambda_n, t)t^k v_{k,n} \right) P_n(x) \\ &\quad + \sum_{n=0}^{\infty} \left( \int_0^t (t-qs)^{(\alpha-1)} E^q_{\alpha,\alpha}(-\lambda_n, t-q^{\alpha}s)h_n d_q s \right) P_n(x). \end{aligned}$$

■

# Conclusion

In this memory we have presented The basic definitions and properties concerning fractional $q$-calculus , and we studied the existence and the uniqueness of solution of a $q$-fractional boundary value problem using tow fixed point theorems, Banach fixed point theorem and Krasnoselskii fixed point theorem, in addition to we studied the existence, of inverse equation problem solution to derivatives partial $q$-fractional in the sense of Caputo.

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# ملخص

الغرض من هذا العمل هو دراسة وجود مشكلة عكسية للمعادلة التفاضلية االجزئية $q-$الكسرية ودراسة وجود وتفرد حل بعض مسائل كوشي المتعلقة بمعادلات $q-$الكسرية المختلفة.
الكلمات المفتاحية: المعادلة التفاضلية $q-$الكسرية،مشكلة كوشي ، الوجود، التفرد، المشكلة العكسية، المعادلة التفاضلية الجزئية $q-$الكسرية.

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# Abstract

The purpose of this work is to study the existence of inverse problem for the partial differential equation q-fractional and study the existence and uniqueness of the solution some Chauchy problems of different q-fractional equations.

**Keywords :** Differential equation q-fractional, problem de Chauchy,Existence, Uniquence, inverse problem,partial differential equation q-fractional.

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# Résumé

Le but de ce travail est d'étudier l'existence d'un problème inverse pour l'équation aux dérivées partielles q-fractionnaire et étudier l'existence et l'unicité de la solution certains problèmes de Chauchy de différentes équations q-fractionnaires.

**Mots clés** : Équation différentielle q-fractionnaire, problèm de Chauchy, 'Existence, Uniquence,problème inverse, équation aux dérivées partielles q-fractionnaire.