PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESERACH

Mohamed Boudiaf University of M'sila
Faculty of Mathematics and Informatics
Departement of Mathematics

Master of Mathematics

Domain: Mathematics and camputer science
Specialty: Mathematics
Option : EDPs

Titled

Approch solution for the free surface flow over polygonal obstacle with a large froude number

---

Presented by :
HADJ HAFSI SOMIA

Publicly presented on : 05/06/2024.

Before Jury :

Chair	N.Benhamidouche	Pr.	University of M'sila .
Supervisor	N.BOUNAB	M.C.B	University of M'sila.
Examiner	W.Delloum	M.A.A,	University of M'sila .

Academic Year: 2023/2024

Thank

All praise and glory to Almighty **Allah** (Subhanahu Wa Taalaa) who gave me courage and patience to carry out this work .Peace and blessing of Allah be upon last Prophet Muhammad (Peace Be Upon Him)

I would like to express my sincere thanks and appreciation to my **parents** and my `family` for all the moral and material support they have given me.

I warmly thank Professeur **Bounab Noura** for accepting the supervision of this dissertation .I am very grateful for the help her given me through the many discussions . In addition ,her availability , her precious advice , and guidance for my supervision , as well as her relevant comments and encouragement.

I would also like to thank Prof **Benhamidouche Nourdine** for the honor of accepting to chair the jury.And ,I would like to express our sincere thanks to professeur **Dellom Wahiba** for the honor of accepting she has as a member of the jury and examining this work.

I would like to give thank to all the professors of the department of Mathematics , more precisely the specialty of Partial Differential Equation and its Application.

Finally, my thank all those who have contributed directly or indirectly to the development and devlopment of this work.

Thank you all

Dedication

I see my college journey has already ended today after a long time tiring .*A*nd here I am today concluding my graduation thesis with all my determination and activity,and within me all appreciation and gratitude to every person who has been credited with my study and provided me with assistance,even if it was few.

I dedicate this humble work to my father *Abdellah*,who has watched throughout my life to encourage me,helps and protects me .*T*o the one who gave me life,the symbol of love who sacrificed herself for my life to encourage me,gives me help and protects meter.

To my brothers:Mohamed,Ibrahim,Aboubaker-el-sedik and Ismail.

*T*o my sisters: Mariem and Amriya.

*T*o my familly,*T*o my friends(Amal,Fatima,Iman,Kenza).

*T*o my colleagues.

*T*o all those who are dear to me

*T*o everyone *I* love.

I dedicate this work.

Somia

Contents

List of Figures

Notation

ρ	volumic mass
$\vec{u}$	the speed vector
m	the mass
$\vec{a}$	acceleration vector
P	pressure
S	the surface
V	volume
n	the normal
f	complex function
g	gravity acceleration
F	Froude number
z	the complex variable
(u,v)	components of the velocity vector
ψ	Stream function
ϕ	potentiel function
$M_0(x_0, y_0, z_0)$	initial position
Γ	diffusion coefficient
S_ϕ	source term
f'	The dimensionless complex velocity

INTRODUCTION

The study of fluid mechanics dates back at least to the times of ancient Greece wit the famous *Archimedes* soap,known for its principle which was at the origin of the statics of fluids.Today, fluid dynamics is an active area of research with many unresolved or partially resolved issues.
In fluid machanics ,the problems of free surface flows of a perfect fluid are studied thanks to their importance of application in several fields.The first ones travers in this sector are characterized by the use of the hodograph method and the Schwartz-Christoffel transformation ,which can treat flows that have a polygonal geometry.
In fluid mechanics the problems of free surface flows fluids are studied thanks to their importance of application in serval fields , analytical and numerical methods using various technique such as method based on the conformal mapping transformation like Schwartz-Cristoffel transformation , the series truncation technique and the boundary integral method which consist to determine the form of free surface for many problems of potential flow over different giving shapes of obstacle .It were used by sereval authors ,such as Elcrat.A.R and L.N.T[12],Boutros[8] ,N.bounab[4],A.Gasmi[5] and Bounif.M,Abd-el-Malek.M.B and S.Z.Masoud[1] studies Linearized solution of a flow over a ramp,and others.....
In this work, we studied numericaly a problem of two-dimensional potential flow of an ideal non-viscous,incompressible ,irrotational fluid over an infinite open channel with a symmetrical triangular obstacle .
This work organized of three chapters,supplemented by an appendix.
The first chapter,we presented some fondamental notions and equations of fluid mechanics.
The second chapter,consistes to the position of our problem ,which is a two dimensional flow of a potential and incompressible fluid over a symetric triangular obstacle in the bottom forms an angle $\alpha = \frac{\pi}{4}$ with the axe $(x'ox)$.Where we have taken in account the force of gravity **The third chapter** ,contains the formulation of our problem presented in the chapter 2 and its resolution using the Hilbert transformation with the discussion and results based on the large Froude number
Finally,we have terminited this work by general conclusion and appendix.

1

Preliminary Concepts and Definitions

Certain assymptions about the behavior and physical properties of the fluid are made to simplify the equations of motions which are complex,analytical solutions do not can only be found in certain simple situations. The simplest flow equations are obtained by considering a perfect fluid . In this chapter we present some definitions and properties of fluids:*kinetics* and *dynamic* of a perfect incompressible fluid.

This chapter contains :

- Fluids.
- Description of fluid motion
- Use of Complex Variable Theory
- Fluid Flows
- Some Equations of Fluid Mechanics
- Energy Conservation Equation
- Stream Ligne Theory
- Froude Number

1.1 Fluids

A fluid (liquid or gas),according to physicists,is a simple body composed of identical atoms or molecules .From a mechanical point of view ,the definition of a material is linked to its deformation as a function of stresses :"a fluid is something that flows" under the action of a given stress and even if the deformation is large ,this does not cause loss cohesion between its molecules. The two states of matter :
A liquid has "a proper volume ,but no proper shape",while a gas has no "volume clean but tends to occupy all the space available to it .Under normal conditions pressure and temperature,the distinction between liquid and gas is obvious but because of the continuity of the fluid state the transition from the gas phase to the liquid phase can be done simple modification (raising the temperature for example).

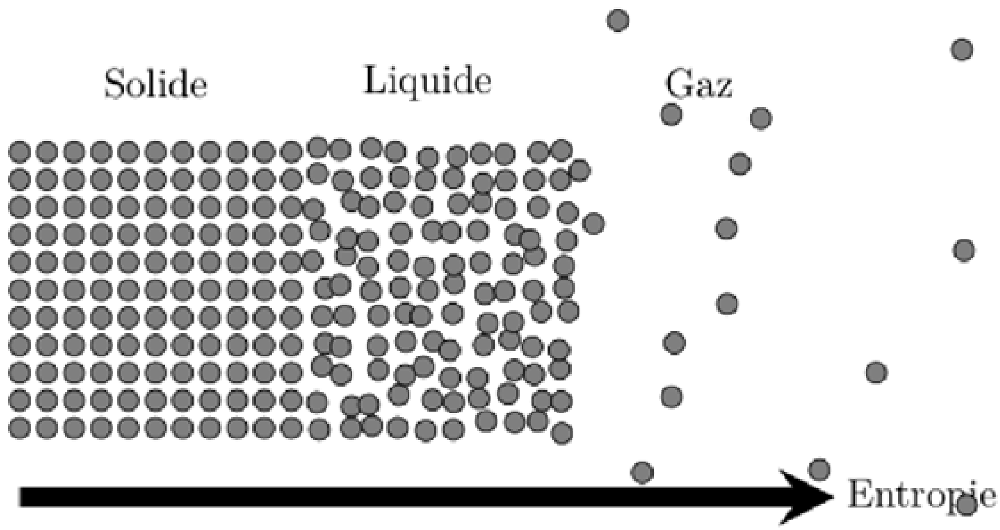

Figure 1.1: Microscopic interpretation of fluid types

- **Perfect fluid**: A perfect fluid,whose flow occurs "without internal resistante ",is a fluid considered to be non-viscous(is the characteristicof resistance to sliding or deformation of a fluid .

- **Incompressible fluid**: A fluid is said to be incompressible when the volume occupied

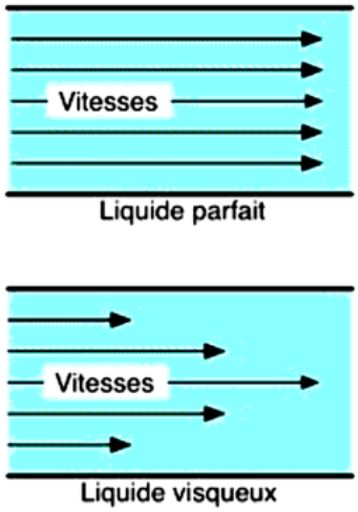

Figure 1.2: Representation of velocity in perfect flow and viscous flow

by a given mass does not vary as a function external pressure .Liquids can be considered as incompressible fluids (water,oil,etc.).

☞ **Compressible fluid**: A fluid is said to be compressible when the volume occupied by a given mass varies in function of external pressure.Gases are compressible fluids.For example,air,hydrogen and methane in the gaseous state are considered compressible fluids.

1.2 Description of fluid motion

1.2.1 Lagrangian Description

This method consists of studying the different quantities $(P, \rho, temperature...etc)$ of each particle individually during its movement . In the lagrangian description , we describe the movement by the trajectories of particles of determined identities . The identity of a particle is given by its initial position $M_0(x_0, y_0, z_0)$.
The description of movement is therefore to determine the position vector $\vec{r}(M_0, t)$ at any time t for all the particles of the fluid .

$$\vec{r} = \vec{r}(M_0, t) ou \vec{r} = \vec{r}(x_0, y_0, z_0, t)$$

. That is to say:

$$x_i = x_i(x_0, y_0, z_0, t).$$

And

$$\vec{u} = \vec{u}(M_0, t) = \frac{\partial \vec{r}}{\partial t}(M_0, t), \vec{a} = \vec{a}(M_0, t) = \frac{\partial \vec{u}}{\partial t}$$

1.2.2 Eulerian Description

Euler's method consists of describing the flow by giving the components of the velocity vector and other physical quantities at each point in space .That is to say ,we fix a point in space and we notice the variations in the quantities linked to the particles of the fluid passing through this point .

At time t_1,we determine in **M** a particle P_1 with speed $\vec{u}$ and other characteristics physical **K**. And at the instant $t_2 = t_1 + \Delta t$ we find at the same point **M** in space, another particle P_2 of different speed and physical characteristics.

So,we have at **M** and at time t_1:

$$\vec{u} = \vec{u}(P_1, t_1) = \vec{u}(x, y, z, t_1).$$

And at time t_2,we have at the same point **M**:

$$\vec{u} = \vec{u}(P_2, t_2) = \vec{u}(x, y, z, t_2).$$

☞ **Trajectory:**

We define the trajctory as the path followed by a moving fluid particele .

$$\frac{dx}{u} = \frac{dy}{v}$$

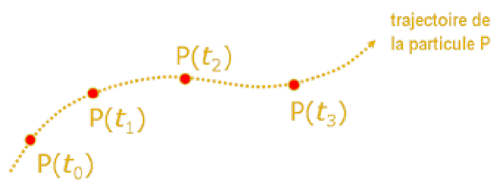

Figure 1.3: Trajectory of the P particle.

1.3 Use of Complex Variable Theory

Let φ and ψ be the potential function and the stream function respectively of a two-dimensional potential flow. We relate the flow plan to the complex plan by writing $z = x + iy$,then we define the complex function $f(z)$ by:

$$f(z) = \phi(x, y) + i\psi(x, y) \tag{1.1}$$

$f(z)$ is called the potential function .Since the real part and the imaginary part of $f(z)$ satisfy the Laplace's equation,as following

we have:

$$\begin{cases} u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \\ v = -\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{cases}$$

Then the Cauchy-Riemann relations:

$$\begin{cases} \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{cases} \tag{1.2}$$

The theory of complex variables offers a very powerful method for obtaining solutions some flow.If the plane (x, y) is considered as the plane of $z = x + iy$ the function $f(z)$ will be analytical in the flow domain.Additionally,the complex speed is defined by:

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial y} \\ &= u - iv \end{aligned} \tag{1.3}$$

The flow plan will also be analytical.This very important property will allow us to use the theory of complex analytical functions to solve a problem considered.

1.4 Fluid Flows

1.4.1 Steady Flows

Stationary flows(also called permanent)are flows whose speed components are independent of the time variable .In this type of flow, we have:

$$\partial_t u = \partial_t \rho = \partial_t T = \partial_t p = 0$$

Such a flow is possible when the domain, the applied mass forces,the sources heat and boundary conditions are also time independent .For example the stationary Navier-stokes equations are written:

$$\begin{cases} -v\Delta u + \rho(u.\nabla)u + \nabla p = f \\ divu = 0 \end{cases} \tag{1.4}$$

1.4.2 Incompressible Flow

A flow is said to be incompressible when the volume occupied by a given mass does not vary depending on the external pressure its density is constant.

$$\rho = cte.$$

1.4.3 Potential Flow

We say that the flow is potential if its the derivative vector, written in the form

$$\vec{u} = \nabla\phi$$

$$u = \frac{d\phi}{dx}, v = \frac{d\phi}{dy}$$

The function $\phi(x, y)$ is the potential of the speeds.

1.4.4 Irrotational Flow

A flow is called irrotational flow if:

$$rot\vec{u} = 0 \tag{1.5}$$

Naturally , a flow that is not irrotational is said to be rotational .A flow potential is an irrotational flow.In fact,we have:

$$u = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$rotu = \left(\frac{\partial\phi}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial\phi}{\partial z}\left(\frac{\partial\phi}{\partial y}\right), \frac{\partial\phi}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial\phi}{\partial x}\left(\frac{\partial\phi}{\partial z}\right), \frac{\partial\phi}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial\phi}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)\right)$$

1.4.5 Uniform flow

A flow is said to be uniform if the velocities of all the particuls are the time at every point of the fluid.

1.5 Some Equations of Fluid Mechanics

1.5.1 Continuity Equation

Let a part of a fluid of density ρ be delimited by a closed surface S(of volume V).
Let dS be an elementary vector of this surface ,oriented outwards to the closed surface .the fluid part has a mass

$$m = \int\int_V\int \rho dV$$

The mass flow leaving the surface S is equal to

$$\int\int_S \rho u dS$$

.

The conservation of mass is written as:

$$\frac{dm_s}{dS} - \int\int_S \rho\vec{u}dS = \int\int_V\int \frac{\partial\rho}{\partial t}dV$$

Where $\frac{dm_s}{dS}$ represents the mass flow of fluid internal to the volume considered,counted positively if it is a source and negatively if it is a well .Considering the theoremof Ostrogradsky to transform the surface integral into a volume integral,

$$\int \int_S \vec{u} dS = \int \int_V \int div(\rho\vec{u}) dV.$$

The conservation of mass equation written:

$$\frac{dm_s}{dS} = \int \int_V \int \left\{ div(\rho\vec{u}) + \frac{\partial \rho}{\partial t} \right\} dV$$

The equality written above is valid whatever the volume V considered and the integral is zero,which leads to the local expression of the conservation of mass:

$$div(\rho\vec{u} + \frac{\partial \rho}{\partial t}) = 0 \tag{1.6}$$

Two particular cases must then be considered:

1. if **the fluid is incompressible**,the density does not change over time and the mass conservation equation reduces to :

   $$div\vec{u} = 0 \tag{1.7}$$

   For stationary or non-stationary flow.This flow is said *isovolume*

   Equation (1.7)expresses the conservation of the volume of a fluid element during of its deformation by the flow .

2. The case of a **stationary flow** $\frac{\partial \rho}{\partial t} = 0$ then:

   $$div(\rho u) = 0 = \rho divu + (u.\nabla)\rho$$

   A part from case 1,there is the possibility of isovolume flows such that $(u.\nabla)\rho = 0$,that is to say the variations in density are orthogonal ,at all points ,to the velocity vector .

1.5.2 Conservation of Fluid Energy

We will evaluate the temporal evolution of the kinetic energy of a fluid element of unit volume and mass,limiting ourselves to the flows of incompressible fluids:

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right) = \rho u_i \frac{\partial u_i}{\partial t} \tag{1.8}$$

Using the equation of motion to express the Eulerian derivative of the velocity(1.8) becomes:

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right) = \rho u_i u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial \sigma_{i,j}}{\partial x_j} + u_i f_i$$

Or,by decomposing the stress tensor as previously into an isotropic part $-p\delta_{i,j}$,and in a deviator $d_{i,j}$

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right) = u_j \frac{\partial}{\partial x_j}\left(\frac{\rho u^2}{2} - p\right) + \frac{\partial u_i d_{i,j}}{\partial x_j} - d_{i,j}\frac{\partial u_i}{\partial x_j} + u_i f_i \tag{1.9}$$

Or,in vector notation:

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right) = u.\nabla\left(\frac{\rho u^2}{2} - p\right) + \nabla.\left(u.d\right) - d.\nabla u + u.f \tag{1.10}$$

Finally,taking into account the incompressibility condition ($\nabla.u = 0$),we can put the first term on the right hand side of (1.10)in the form of a divergence ,i.e:

$$\frac{\partial e_c}{\partial t} = \nabla.\left[u\nabla\left(\frac{\rho u^2}{2} - p\right) + u.d\right] - d.\nabla u + u.f \tag{1.11}$$

Let us rewrite this equation for the evolution of kinetic energy in integral form,integrating each of the terms on a fixed volume V and using the divergence theorem:

$$\frac{\partial}{\partial t}\left(\int_V e_c dV\right) = \int_S \frac{\rho u^2}{2} u.ndS + \int_S \left(\sigma.u\right).ndS + \int_V u.fdV - \int_V \sigma.\nabla udV \tag{1.12}$$

What is the physical meaning of the different terms in (1.12)

1. The first term of the right hand side is the flow of kinetic energy"convected by the flow through the surface S.

2. The second term is the work ,per unit of time ,of the stresses exerted on the surface S.

3. The third term is the work,per unit of time ,of the forces in volume .

4. Finally , the fourth term is associated with the deformation of the volume V.It represents the energy dissipated by viscosity during this deformation.

1.5.3 Ligne and Stream Function

stream ligne

We call the streamline the curve which at each of its points is tangent to the speed vector.Its differential equation is written:

$$\frac{dx}{u(x.y.z)} = \frac{dy}{v(x.y.z)}$$

Stream Function

If we consider the flow is incompressible $(i.e.\frac{D\rho}{Dt} = 0)$then the continuity equation will be given:

$$div\vec{u} = 0$$

Or

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial t} \tag{1.13}$$

We present a new function ψ of x and y which we call current function ,checking:

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \tag{1.14}$$

The surfaces defined by $(\psi = cte)$ are current lines , in fact the exact differential of given:

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy \tag{1.15}$$

Since $\psi = cte$,then $\partial\psi = 0$ we find the equation of the stremline according to (1.14). Let ,C be a fine curve which goes from one streamline to another characterized by $\psi = \psi_1$ and $\psi = \psi_2$ respectively. Let $\vec{n}$ a unit vector normal to C and oriented in the direction of the flow , the flow at through C given by

$$Q = \int_C \vec{u}.\vec{n} = \int_C \left(-u\frac{\partial y}{\partial t} + v\frac{\partial x}{\partial t}\right) = \int_C (vdx - udy)\, dt$$

From where

$$Q = \int_C \left(\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy\right) = \int_C d\psi$$

Therefore

$$Q = \psi_1 + \psi_2 \tag{1.16}$$

1.5.4 Equation of Motion of Fluids

By the fundamental relation of dynamics,the temporal variation of the quantity of motion of an element of volume V is equal to the sum of the forces exerted on this volume element,i.e.:

$$\frac{d}{dt}\left(\int_V \rho u dx\right) = \int_V \rho \frac{du}{dt} dx$$

The integral of the surface forces cen be wrritten using Ostrogradsky's theorem under the form $\int_V div\sigma dx$.By making the volume V tend towards zero the equation of motion becomes:

$$\rho\left(\partial_t u + (u.\nabla)\, u\right) = f + div\sigma \tag{1.17}$$

1.5.5 Differential Equations of The Function ϕ and ψ

Consider a two-dimensional,irrotational and stationary flow of an incompressible fluid non-viscous .since:

$$\vec{u} = grad\vec{\phi}$$

And

$$div\vec{u} = 0$$

It turns out that:

$$div\left(grad\vec{\phi} = 0\right)$$

From where

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

That's to say:

$$\Delta\phi = 0 \tag{1.18}$$

Likewise,according to :

$$\vec{u} = (u, v) = \left(-\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right)$$

And

$$rot\vec{u} = 0$$

We find:

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

From where

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi^2}{\partial y^2} = 0$$

That's to say

$$\Delta \psi = 0$$

Hence, the potential function ϕ and the streamline function $\psi$and verify the equation of the place .

A two-dimensional ,irrotational and stationary flow of an incompressible fluid,not viscous is potential flow.

1.5.6 Stokes Equations

The navier-stokes equation:

$$\begin{cases} -v\nabla u + \rho(u.\nabla)u + \nabla p = f \\ divu = 0 \end{cases} \tag{1.19}$$

By neglecting in the stationary incompressible Navier-Stokes equation the terms proportional to the density of the fluid $(u.\nabla)u$, We obtain the Stokes aquation

$$\begin{cases} -v\nabla u + \nabla p = f \\ \ divu = 0 \end{cases} \tag{1.20}$$

The smaller the flow velocity in relation to the dimensions of Ω and the viscosity value ,the more the stokes model is a valid approximation of the equation of Navier-Stokes.The fundamental difference between the two equations is that the non linear term in velocity has gone,the Stokes equation is a linear partial differential equation .

1.5.7 Bernoulli's Theorem

Bernoulli's theorem is an application of the conservation of energy to the case of fluids in motion .

Bernoulli's first theorem

In a stationary flow , along a trajectory we have conservation of charge

$$H = \frac{\hat{p}}{pg} + \frac{u^2}{2g} = z + \frac{p}{pg} + \frac{u^2}{2g} = Cont.$$

$\hat{p} = p + pgz$

Bernoulli's second theorem

In a potential flow the Euler equation is written:

$$p\left[\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\frac{u^2}{2}\right] = -\nabla\hat{p}$$

$$\Longleftrightarrow \frac{\partial\phi}{\partial t} + \frac{u^2}{2} + \frac{\hat{p}}{\rho} = \frac{\partial\phi}{\partial t} + \frac{u^2}{2} + \frac{p}{\rho} + gz = \frac{\partial\phi}{\partial t} + gH$$

1.6 Energy Conservation Equation

The energy conservation equation is obtianed from the first law of thermodynamics.This principle connects the different forms of energy,either:

$$\rho C_p\frac{dT}{dt} = div\left[\lambda.gr\vec{a}dT\right] + T.\beta\frac{dP}{dt} + \varphi + P_s \tag{1.21}$$

1.7 Stream Ligne Theory

The theory of dtream lignes consists of studying the problems of potential flow,limited by rectilinear rigid walls and free current lines of unknow shapes,on which the pressure is assumed to be constant .

If free stream lignes are not present and gravity effects are neglected ,the flow region in the physical plane is a polygon.

Also the free stream lines present and the effects of gravity as well as the effects of surface tension are neglected,the flow region can be transformed by a conformal transformation to a polygonal region.

This region is a perfect of the defined hodograph plan

$$\Omega = \log\left(1/\frac{df}{dz}\right)$$

In the case where the flow is partially delimited by free surfaces we give the resolution methode introduced by kirchhoff (1869).

The idea is to introduce the complex function defined by :

$$\Omega = \log\left(u/\frac{df}{dz}\right) = \log\left(\frac{u}{u-iv}\right) = \log\left(\frac{u}{q}\right) + i\theta \tag{1.22}$$

Or $f = \phi + i\psi$,$\frac{df}{dz} = u - iv$,$q = \sqrt{u^2+v^2}$,(u,v) are the components of the following velocity vector of the x axis and the y axis respectively,θ is the angle that the speed vector makes with the horizontal and U the reference speed .

- Real part of Ω is constant on the free current line ,i.e.$\log\left(\frac{u}{q}\right) = cte$.
- The imaginary part of Ω is constant on each rectilinear wall, i.e.$\theta = cte$.

Therefore , the flow is represented by a plan figure with straight sides (polygon) denoted Ω.

Using the Schwarz-Christoffel transformation , the polygonal domain Ω is transformed into a upper half plane of the auxiliary variable λ the flow is uniform represented by the potential function $F(\lambda) = c\lambda$.To illustrate the above, we give some properties of the Schwarz-Christoffel conformal transformation.

1.7.1 Schwartz-Christoffel Transformation

We consider a polygon [Figure(1.4)]in the plan Ω,having vertices $A_1, A_2, ..., A_n$ and for interior angles respectively $\alpha_1, \alpha_2, ...\alpha_n$.

Let $A_1, A_2, ..., A_n$,An the points corresponding respectively to $\alpha_1, \alpha_2, ...\alpha_n$ of the real axis of the plan of λ[Figure(1.5)]. Schwarz-Christoffel transformation ,transforms the interior of a polygon into the upper (or lower)half-plan of another plane.The transformation is given by :

$$\frac{d\Omega}{d\lambda} = \alpha(\lambda - \lambda_1)^{\frac{\alpha_1}{\pi}-1}(\lambda - \lambda_2)^{\frac{\alpha_2}{\pi}-1}....(\lambda - \lambda_n)^{\frac{\alpha_n}{\pi}-1} \tag{1.23}$$

Or

$$\Omega = \alpha \int (\lambda - \lambda_1)^{\frac{\alpha_1}{\pi}-1}(\lambda - \lambda_2)^{\frac{\alpha_2}{\pi}-1}....(\lambda - \lambda_n)^{\frac{\alpha_n}{\pi}-1} + \beta \tag{1.24}$$

Or α and β are complex constants.It will be noted that:

- ✈ Among the points $\lambda_1, \lambda_2, ...\lambda_n$ we can choose three arbitrarily.
- ✈ The constants αandβ determine the orientation and position of the polygon .
- ✈ It is convenient to choose a point,for example λ_n,at infinity,case in which factor of (2.10) does not exist.
- ✈ Infinite non-closed polygons can be considered as limited cases of polygons.

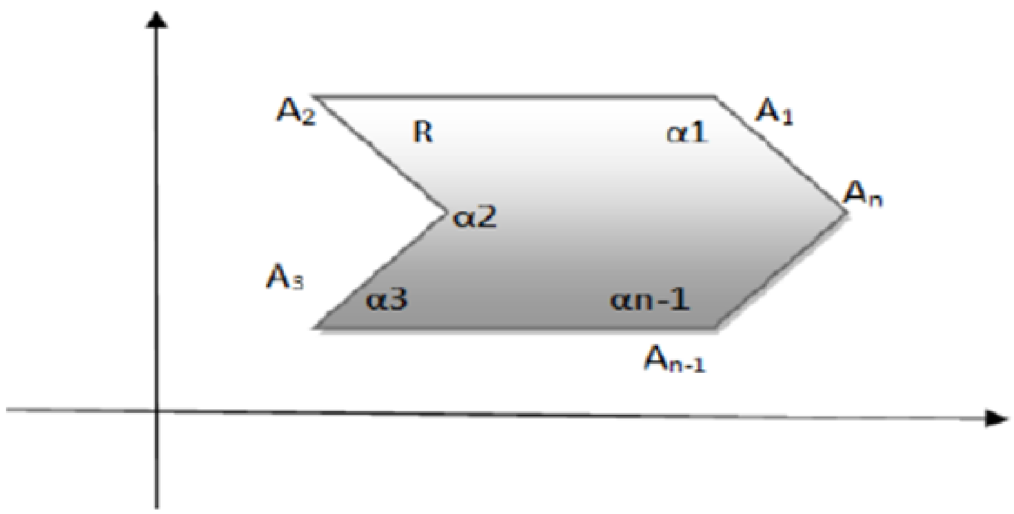

Figure 1.4: Plan of Ω

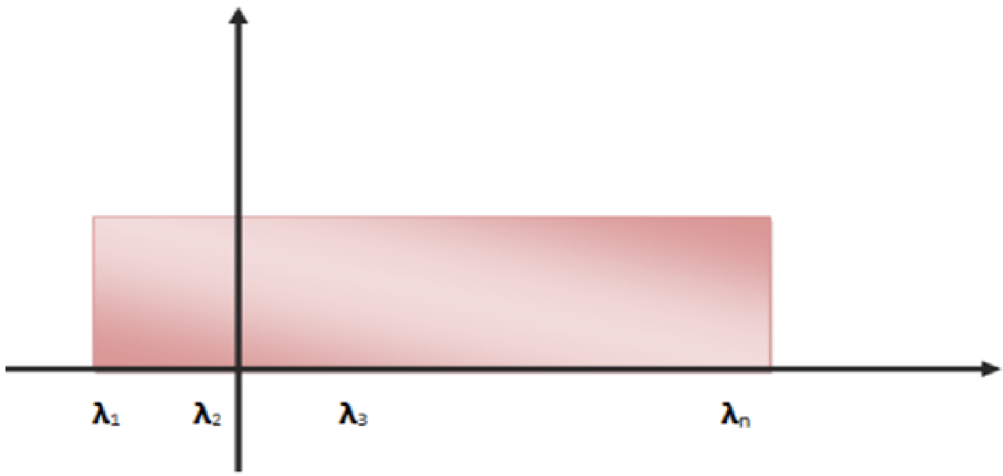

Figure 1.5: Plane of variable λ

1.8 Froude Number

The froud number ,being the ratio of kinetic energy $(mU^2/2)$ to gravitational potential energy (mgr),is defined as follows:

$$F = \frac{U}{\sqrt{gr}}$$

2

Position of The Problem

In this chapter ,we presented and treated numerically nonlinear two-dimensional flow of an ideal fluid over a polygonal obstacle in bottom of the channel where the force of gravity is taken in account ,using the schwartz-chrisstoffel transformation and hilbert's method of mixed boundary valie problem in apper half-plan. This chapter contains :

- Position of problem
- Formulation of problem

2.1 Position of Problem

We have considered a two dimentional potential flow over a symetric triangular obstacle in the bottom wich forms an angle $\alpha = \frac{\pi}{4}$ with the axis (x'ox)

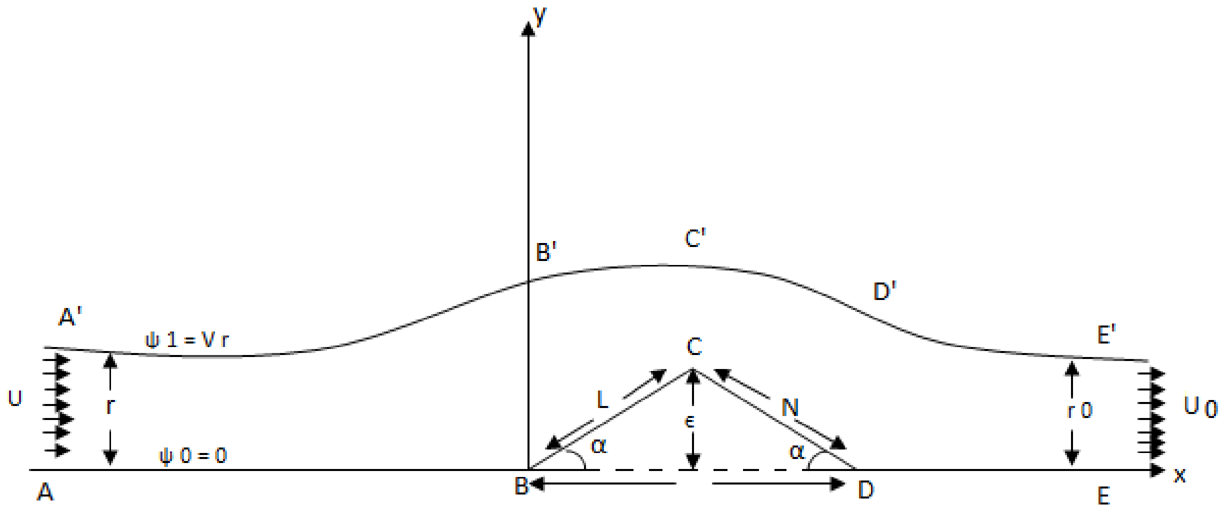

Figure 2.1: The sketch of the flow plan(z-plan).

2.2 Formulation of the Problem

The two-dimensional motion of an inviscid ,irrotational , and incompressible fluid is governed by the place's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{2.1}$$

Where $\phi(x, y)$ is the velocity potential .

The complex velocity potential

$$f(z) = \phi(x, y) + i\psi(x, y)$$

is an analytic function of $z = x + iy$,where ψ is the stream function .The complex conjugate velocity is given by

$$\frac{df(z)}{dz} = u(x, y) - iv(x, y) = qe^{-i\theta} \tag{2.2}$$

Where q is the speed of the flow and θ its direction. If the variables are dimensionless

$$z' = \frac{z}{r}, q' = \frac{q}{U}, f' = \frac{f}{Vh} \tag{2.3}$$

In the free surface , Bernoulli's equation is defined in terms of the Froude number, which we denote F

$$q'^2 + \frac{2}{F^2}(y' - 1) = 1 \tag{2.4}$$

And

$$F = \frac{U}{\sqrt{gr}} \tag{2.5}$$

In Figure 2.1 we have the height of the isosceles triangle ϵ

$$\epsilon = \frac{Lsin^2\alpha}{sin(2\alpha)} \tag{2.6}$$

And the complex dimensionless conjugate velocity has the form

$$\xi = \frac{df'}{dz'} = q'e^{-i\theta} \tag{2.7}$$

Respectively.with the logarithmic hodograph variable w is defined by

$$w = ln\xi = ln(q'e^{-i\theta}) = lnq' + lne^{-i\theta} = lnq' + i(-\theta) \tag{2.8}$$

From equations 2.7 and 2.8we arrive at the following statement

$$z' = \int e^{-w} df' \tag{2.9}$$

In f'-plan the fluid region is an infinite strip of unit width with a free surface **A'E'** located at $\Psi' = 1$, and at infinity it can be considered a two-vertexed polygon.If we can express *f'* and w in statement (2.9) as functions of the same single variable t,Then the integral(2.9) can be evaluated .For the first part of the problem .To express *f'* as function of t ,We make use of the *Schwarz-Christoffel* transform,Which ,through explicit mappings on the upper half of the t-plan,gives *f'* in parametric form

$$f' = f'(t)$$

The general form of the **Schwarz-Christoffel transformation** is:

$$\frac{df'}{dt} = A \prod_i (t - t_i)^{\frac{\alpha_i}{\pi} - 1} \tag{2.10}$$

where t_i is the t-plane coordinate related to a vertex of the polygon ,α_i is the corresponging internal angle in the f'-plane ,and A is constant .The conformal mapping (2.10)should map the fluid region in the f'-plane (figure2.2)into the upper half-plane ,the t-plane ,and the boundary of the fluid region onto the real axis ,the boundary of the t-plane(see figure 2.2).

To ensure the uniquenessof the mapping ,we choose three corresponding points:

$$\begin{cases} B : f' = 0 & t = 0 \\ E, E_\infty : f' \longrightarrow +\infty & t = 1 \\ A, A : f' \longrightarrow -\infty & \quad t \longrightarrow +\infty \end{cases} \tag{2.11}$$

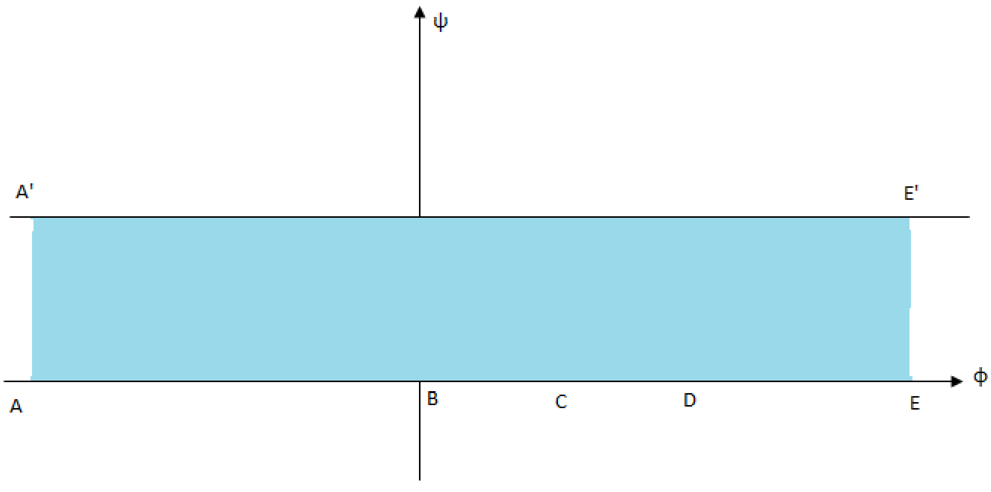

Figure 2.2: f-plan for a flow over symetric triangular obstacle

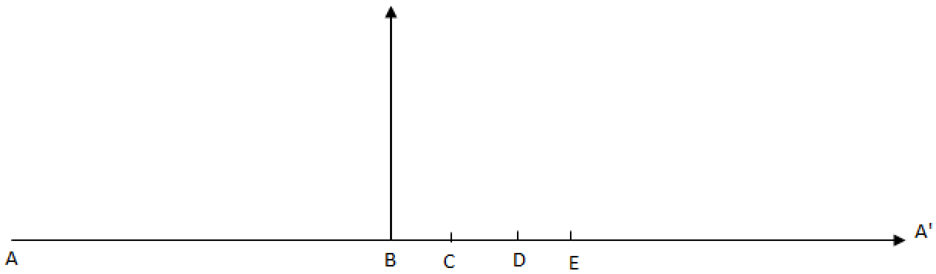

Figure 2.3: t-plane

The mapping is

$$f' = -\frac{1}{\pi}\ln(t-1) + i \qquad 0 \leq arg(t-1) \leq \pi \tag{2.12}$$

The hilbert metod for the mixed boundary value problem is introduced in the upper half to express was a function of a single variable t.The general solution to the hilbert problem in the upper half is well known .

$Q(t)$ is an analytic function defined in the upper half of t-plane,and we assume that $Im[Q(t)]$ satisfies Holder's condition on the boundary,the real axis of t-plane.

If the imaginary part of $Q(t)$ is known on the boundary,then $Q(t)$ is given by the formule integral poisson :

$$Q(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{Im[Q(v)]}{v-t}dv + \sum_{j=0}^{+\infty} A_j t^j \tag{2.13}$$

Where A_j are real constants

We introduce an auxiliary function $H(t)$ to obtaina relationship between $Q(t)$ and $w(t)$,which makes $Im[Q(t)] = Im[\frac{w(t)}{H(t)}]$ known.On the t-plane boundary of statement (2.8)the real and imaginary parts of $w(t)$ on the boundary are

$$\begin{cases} Im[w(t)] = -\theta(t) \quad t < 1 \\ Re[w(t)] = \frac{1}{2} ln\left[1 - \frac{2}{\eta'(t)}\right] \quad t > 1 \end{cases} \tag{2.14}$$

where

$$\eta'(t) = y'(t) - 1 \tag{2.15}$$

And

$$1 - \frac{2}{F^2}\eta'(t) = q'^2$$

So

$$Re[w(t)] = \frac{1}{2} ln[1 - \frac{2}{F^2}\eta'(t)] = \frac{ln(q'^2)}{2} = \frac{2lnq'}{2} = lnq'$$

The general form of the auxiliary function H(t) is defined as follows

$$H(t) = a \prod_j (t - b_j)^{\pm \frac{1}{2}} \tag{2.16}$$

Where b_j are constants ,and $a = \pm \sqrt{\pm 1}$

So the value of $H(t)$ is

$$H(t) = \begin{cases} \sqrt{1-t} : \ t < 1 \\ -i\sqrt{t-1} : \ t > 1 \end{cases} \tag{2.17}$$

The final solution depends on an appropriate choice of H(t)

$$Im[Q(t)] = \begin{cases} 0, \quad t < 0 \\ -\frac{\pi}{4\sqrt{1-t}}, \quad 0 < t < t_c \\ -\frac{\pi}{4\sqrt{1-t}}, \quad t_c < t < t_d \\ 0, \quad t_d < t < 1 \\ \frac{ln[1 - \frac{2}{F^2}\eta']}{2\sqrt{t-1}}, \quad t > 1. \end{cases} \tag{2.18}$$

The upstream condition implies that

$$A_j = 0, j = 0, 1, 2, ... \tag{2.19}$$

Then from 2.13

$$Q(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{Im[Q(v)]}{v - t} dv \tag{2.20}$$

With

$$Im[Q(t)] = Im[\frac{w(t)}{H(t)}]$$

We also have from 2.8 and 2.18

$$\begin{aligned} Q(t) &= \tfrac{1}{\pi}\int_{-\infty}^{+\infty} Im[\tfrac{w(t)}{H(t)}].\tfrac{1}{v-t}dv \\ &= \tfrac{1}{\pi}\left[\int_{-\infty}^{1} Im[\tfrac{lnq'-i\theta}{\sqrt{1-t}}].\tfrac{1}{v-t}dv + \int_{1}^{+\infty} Im[\tfrac{lnq'-i\theta}{-i\sqrt{t-1}}].\tfrac{1}{v-t}dv\right] \\ &= \tfrac{1}{\pi}\left[\int_{-\infty}^{1} \tfrac{i\theta}{\sqrt{t-1}(v-t)}dv + \int_{1}^{+\infty} \tfrac{lnq'}{\sqrt{t-1}(v-t)}dv\right] \\ &= i\left[\tfrac{1}{\pi}\int_{-\infty}^{+\infty} \tfrac{R(v)}{v-t}dv\right] + \tfrac{1}{\pi}\int_{-\infty}^{+\infty} \tfrac{S(v)}{v-t}dv \end{aligned}$$

$$Q(t) = K(t) + iM(t) \tag{2.21}$$

With

$$K(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{S(v)}{v-t}dv \tag{22a}$$

And

$$M(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{R(v)}{v-t}dv \tag{22b}$$

By using (22a),(2.8)and (2.17),and after $Q(t) = \frac{w(t)}{H(t)} \Longrightarrow w(t) = Q(t).H(t)$.

On the other hand $w(t) = lnq' + i(-\theta)$

$$\begin{aligned} w(t) &= Q(t).H(t) \\ lnq' + i(-\theta) &= \left[\tfrac{1}{\pi}\int_{-\infty}^{+\infty} \tfrac{Im[Q(v)]}{v-t}dv\right].H(t) \\ &= -\tfrac{\sqrt{1-t}}{\pi}\left[\int_{0}^{t_d} \tfrac{\pi}{2\sqrt{1-v}(v-t)}dv - \int_{1}^{+\infty} \tfrac{lnq'}{\sqrt{v-1}(v-t)}dv\right] + i.\tfrac{\sqrt{t-1}}{\pi}\left[\int_{0}^{t_d} \tfrac{\pi}{2\sqrt{1-v}(v-t)}dv - \int_{1}^{+\infty} \tfrac{lnq'}{\sqrt{v-1}(v-t)}dv\right] \end{aligned}$$

Compare (2.8) with the last one

$$\theta(t) = -\frac{\sqrt{t-1}}{\pi}\left[\int_{0}^{t_d} \frac{\pi}{2\sqrt{1-v}(v-t)}dv - \int_{1}^{+\infty} \frac{lnq'}{\sqrt{v-1}(v-t)}dv\right] \tag{2.22}$$

$$lnq'_j(t) = -\frac{\sqrt{1-t}}{\pi}\left[\int_{0}^{t_d} \frac{\pi}{2\sqrt{1-v}(v-t)}dv - \int_{1}^{+\infty} \frac{lnq'}{\sqrt{v-1}(v-t)}dv\right], j = 1,2,3,4 \tag{2.23}$$

Where $q'_1(t)$ is defined in $t < 0$,$q'_2(t)$ in $0 < t < t_c$,$q'_3(t)$ in $t_c < t < t_d$,and $q'_4(t)$ in $t_d < t1$

The coordinates (x',y')of a point on the free surface can be obtained from (2.9) and (2.12):

$$Z'(t) = (x'_0 + i) - \frac{1}{\pi}\int_{t}^{\infty} \frac{e^{i\theta(v)}}{(1-v)q'(v)}dv, t > 1 \tag{2.24}$$

Separating the real and imaginary parts implies ,and from $Z'(t) = x'(t) + iy'(t)$

$$x'(t) = x'_0 - \frac{1}{\pi}\int_{t}^{\infty} \frac{cos\theta(v)}{(1-v)q'(v)}dv, t > 1 \tag{2.25}$$

$$y'(t) = 1 - \frac{1}{\pi}\int_t^{\infty} \frac{sin\theta(v)}{(1-v)q'(v)}dv, t > 1 \tag{2.26}$$

$$\epsilon' = \frac{1}{\pi}\int_0^{t_c} \frac{dv}{(1-v)q_b'(v)} \tag{2.27}$$

Now the problem is completely described by (2.4),(2.22),(2.23),(2.25),(2.26) and (2.27),which will be solved in the third chapter.

3

Discussion of the Results

In this chapter, we have the approximate solution using successive approximations and presented the numerical results with some discussion .

This chapter contains :

- The Approximate Equations
- Numerical Results and Discussion

3.1 The Approximat Equations

Most physics problems confront engineers,applied mathematicians,and physicists .Nonlinear equations(as the system of equation described in (2.4)(2.22),and (2.25)-(28) shows).Hence,to obtain information about the solutions of the equations,we are forced to resort to approximations.

In the introduction to approximation methods .We assumed that the difference in $\theta(t)$ is small,due to the small slope of the interior angles of the triangle.So $sin(t)$ can be approximated by $\theta(t)$,and $cos(t)$by unity.According to the perturbation technique ,the solution is represented by the first few terms of the asymptotic expansion in terms of small parameters α of ordre 10^{-1}.This expansion is called parameter disturbance.Since these small parameters tend to zero ,it must be assumed that the flow approaches the limit ,which can be called the fundamental solution .In parameter perturbations ,one usually views the fundamental solution as the zero approximation and calls it the principal perturbation.In fact ,the number of terms in the asymptotic expansion does not exceed two.

We consider the case of a large froude number ,and since $\eta'(t)$ is of ordre 10^{-2},$\frac{\eta'(t)}{F^2}$ will be small enough so that at

$$ln(1-\frac{1}{F^2}\eta'(t)) \approx -\frac{\eta'(t)}{F^2} \tag{X}$$

Having done this ,we begin to drop the prime numbers ,and the approximate equations take the form

$$q(t) \approx 1-\frac{1}{F^2}\eta(t),\ t>1 \tag{3.1}$$

By (2.2)and by (2.26)

$$\begin{aligned}\eta(t) = y'(t)-1 \\ &= 1-\frac{1}{\pi}\int_t^{+\infty}\frac{sin\theta(v)}{(1-v)q'(v)}-1 \\ &\approx -\frac{1}{\pi}\int_t^{+\infty}\frac{\theta(v)}{1-v}.\frac{1}{\left[1+\frac{\eta(v)}{F^2}\right]}dv,\ t>1\end{aligned} \tag{3.2}$$

From (2.22), (3.1) and (X) we obtain

$$\begin{aligned}\theta(t) &= -\frac{\sqrt{t-1}}{\pi}\left[\int_0^{td}\frac{\pi}{2\sqrt{1-v}(v-t)}-\int_1^{+\infty}\frac{lnq'(v)}{(v-t)\sqrt{v-1}}dv\right] \\ &= -\frac{\sqrt{t-1}}{\pi}\left[\int_0^{td}\frac{\pi}{2\sqrt{1-v}(v-t)}-\int_1^{+\infty}\frac{ln(1-\frac{\eta(v)}{F^2})}{(v-t)\sqrt{v-1}}dv\right] \\ &\approx -\frac{\sqrt{t-1}}{\pi F^2}\left[tan^{-1}\frac{\sqrt{1-v}}{t-1}\right]_0^{t_d}+\frac{\sqrt{t-1}}{\pi F^2}\int_1^{+\infty}\frac{\eta(v)}{(v-t)\sqrt{v-1}}dv,\ t>1\end{aligned}$$

Using integration by part,we get:

$$\theta(t) \approx -\sqrt{t-1}tan^{-1}\frac{\gamma}{t-1} + \frac{\sqrt{t-1}}{\pi F^2}\int_t^{+\infty}\frac{\eta(v)}{(v-t)\sqrt{v-1}}dv,\ t>1 \tag{3.3}$$

Where

$$\gamma = 1-\sqrt{1-t_d}$$

From the previous words,we know that $cos\theta(t) \approx 1$,and $sin\theta(t) \approx \theta(t)$

$$\begin{array}{c} x'(t) = x'_0 - \frac{1}{\pi}\int_t^{+\infty}\frac{cos\theta(v)}{(1-v)q(v)}dv \\ x(t) \approx x_\infty - \frac{1}{\pi}\int_t^{+\infty}\frac{dv}{(1-v)\left[1-\frac{\eta(v)}{F^2}\right]},\ t>1 \end{array} \tag{3.4}$$

We use same steps as (3.3) on $lnq_b(t)$,we get

$$lnq_b(t) \approx -\sqrt{t-1}tan^{-1}\frac{\gamma}{t-1} + \frac{\sqrt{1-t}}{\pi F^2}\int_t^{+\infty}\frac{\eta(v)}{(v-t)\sqrt{v-1}}dv,\ 0<t<t_c \tag{3.5}$$

$$L = \frac{1}{\pi}\int_0^{t_c}\frac{dv}{(1-v)q_b(v)}dv \tag{3.6}$$

To be able to solve equations (3.1)-(3.4),we use the successive approximation method ,so α and β are equal and we assume that they are small,we can be written q(t),$\eta(t)$,$\theta(t)$ and x(t) as:

$$\begin{array}{c} q(t) = q_0(t) + \alpha q_{1,\alpha}(t) + \beta q_{1,\beta}(t) + ... = q_0(t) + 2\alpha q_{1,\alpha}(t) \\ \eta(t) = \eta_0(t) + \alpha\eta_{1,\alpha}(t) + \beta\eta_{1,\beta}(t) + ... = \eta_0(t) + 2\alpha\eta_{1,\alpha}(t) \\ \theta(t) = \theta_0(t) + \alpha\theta_{1,\alpha}(t) + \beta\theta_{1,\beta}(t) + ... = \theta_0(t) + 2\alpha\theta_{1,\alpha}(t) \\ x(t) = x_0(t) + \alpha x_{1,\alpha}(t) + \beta x_{1,\beta}(t) + ... = x_0(t) + 2\alpha x_{1,\alpha}(t) \end{array} \tag{3.7}$$

If we substitute the expressions (3.7) in equation (3.1)-(3.4) and equate terms of similar powers of α and β,the zero-and first-order approximation take the following form:

1. **Zero-order approximation:**

$$q_0(t) = 1 - \frac{\eta_0(v)}{F^2},\ t > 1 \tag{3.8}$$

$$\eta_0(t) = -\frac{1}{\pi}\int_t^{+\infty} \frac{\theta_0(v)}{(1-v)[1-\frac{\eta_0(v)}{F^2}]},\ t > 1 \tag{3.9}$$

$$\theta_0(t) = \frac{\sqrt{t-1}}{\pi F^2}\int_t^{+\infty} \frac{\eta_0(v)}{(v-t)\sqrt{v-1}}dv,\ t > 1 \tag{3.10}$$

$$x(t) = x_\infty - \frac{1}{\pi}\int_t^{+\infty} \frac{dv}{(1-v)\left[1-\frac{\eta_0(v)}{F^2}\right]},\ t > 1 \tag{3.11}$$

2. **First-order approximation:**

$$\begin{aligned} q_{1,\alpha} &= \tfrac{1}{2\alpha}\left[q(t) - q_0(t)\right] \\ &= \tfrac{1}{2\alpha}\left[1 - \tfrac{\eta(t)}{F^2} - 1 + \tfrac{\eta_0(t)}{F^2}\right] \\ &= \tfrac{2}{\pi}\left[\tfrac{-\eta(t)+\eta_0(t)}{F^2}\right] = \tfrac{-\eta_{1,\alpha}(t)}{F^2},\ t > 1 \end{aligned} \tag{3.12}$$

$$\begin{aligned} \eta_{1,\alpha} &= \tfrac{1}{2\alpha}\left[\eta(t) - \eta_0(t)\right] \\ &= \tfrac{2}{\pi}\left[\tfrac{1}{\pi}\int_t^{+\infty} \tfrac{-\theta(v)}{(1-v)\left[1-\frac{\eta(v)}{F^2}\right]} + \tfrac{\theta_0(v)}{(1-v)\left[1-\frac{\eta 0(v)}{F^2}\right]}\right] \\ &= \tfrac{2}{\pi}.\tfrac{1}{\pi}\int_t^{+\infty} \tfrac{-\theta(v)+\theta_0(v)}{(1-v)}dv \\ &= -\tfrac{1}{\pi}\int_t^{+\infty} \tfrac{\theta_{1,\alpha}(v)}{(1-v)}dv,\ t > 1 \end{aligned} \tag{3.13}$$

$$\begin{aligned} \theta_{1,\alpha} &= \tfrac{1}{2\alpha}\left[\theta(t) - \theta_0(t)\right] \\ &= \tfrac{2}{\pi}\tfrac{\sqrt{t-1}}{\pi F^2}\int_1^{+\infty} \tfrac{\eta(v)-\eta_0(v)}{(v-t)\sqrt{v-1}}dv \\ &= \tfrac{\sqrt{t-1}}{\pi F^2}\int_1^{+\infty} \tfrac{\eta_{1,\alpha}(v)}{(v-t)\sqrt{v-1}}dv,\ t > 1 \end{aligned} \tag{3.14}$$

$$x_{1,\alpha} = -\tfrac{1}{\pi F^2}\int_t^{+\infty} \tfrac{\eta_{1,\alpha}(v)}{1-v}dv \tag{3.15}$$

The solution of equations (3.8)-(3.11) is given by

$$q_0(t) = 1,\ t > 1 \tag{3.16}$$

$$\eta_0(t) = 0,\ t > 1 \tag{3.17}$$

$$\theta_0(t) = 0,\ t > 1 \tag{3.18}$$

$$x_0(t) = x_\infty - \frac{1}{\pi}\int_t^{+\infty} \frac{dv}{1-v},\ t > 1 \tag{3.19}$$

Which corresponds to the solution of a flow over a flat plate.We write

$$x_\infty = \frac{1}{\pi}\int_0^{\infty} \frac{dv}{1-v}$$

Hence

$$x_0 = -\frac{1}{\pi}ln(t-1),\ t>1 \tag{3.20}$$

For large froude number F the first-ordre approximation can be obtained by neglecting the $\frac{1}{F^2}$ term.

Therefore equations (3.12)-(3.15) can be approximated by

$$\theta_{1,\alpha} \approx \frac{2}{\pi}tan^{-1}\frac{\gamma\sqrt{t-1}}{t-\gamma},\ t>1 \tag{3.21}$$

When these equations are substituted,equations (3.13) can be written as

$$\eta_{1,\alpha}(t) \approx \frac{2}{\pi^2}\int_t^{+\infty}\left[tan^{-1}\frac{\gamma\sqrt{t-1}}{t-\gamma}\right]\frac{dv}{v-1},\ t>1 \tag{3.22}$$

Further simplifications can be obtained by assuming that $\gamma << 1$.In this case ,the expressions in (3.22),after the integration is carried out,take the form

$$\eta_{1,\alpha}(t) \approx \frac{4}{\pi^2}\frac{\gamma}{\sqrt{1-\gamma}}tan^{-1}\sqrt{\frac{1-\gamma}{t-1}},\ t>1 \tag{3.23}$$

By using (**??**) in (3.12)and (3.14) the following equations are obtained:

$$q_{1,\alpha}(t) \approx -\tfrac{4}{\pi^2F^2}\tfrac{\gamma}{\sqrt{1-\gamma}}tan^{-1}\sqrt{\tfrac{1-\gamma}{t-1}},\ t>1 \tag{3.24}$$

$$x_{1,\alpha}(t) \approx \tfrac{8}{\pi^3F^2}\tfrac{\gamma}{\sqrt{t-1}},\ t>1 \tag{3.25}$$

For large values of F^2 and after some manipulations in $lnq_b(t)$ an approximation of L can be written as

$$L \approx -\frac{1}{\pi}ln(1-t_c) \tag{3.26}$$

Hence,up to first-order approximation in α and β,the functions $q(t)$,$y(t)$,$\theta(t)$,and $x(t)$ for $t>1$ are then given by

$$q(t) \approx 1+\frac{4}{\pi^2F^2}\left[-\frac{2\alpha\gamma}{\sqrt{1-\gamma}}tan^{-1}\sqrt{\frac{1-\gamma}{t-1}}\right] \tag{3.27}$$

$$y(t) \approx 1+\frac{4}{\pi^2}\left[\frac{2\alpha\gamma}{\sqrt{1-\gamma}}tan^{-1}\sqrt{\frac{1-\gamma}{t-1}}\right] \tag{3.28}$$

$$\theta(t) \approx \frac{2}{\pi}\left[2\alpha tan^{-1}\frac{\gamma\sqrt{t-1}}{t-\gamma}\right] \tag{3.29}$$

$$x(t) \approx -\frac{1}{\pi}ln(t-1)+\frac{8}{\pi^3F^2}\left[\frac{2\alpha}{\sqrt{t-1}}\right] \tag{3.30}$$

3.2 Numerical Results and Discussion

Conformal mapping and the Hilbert solution to a well-posed boundary-value problem are combined to find the approximate profile of the two-dimensional flow of an ideal in open channel with symmetrical triangle obstacle in the bottom for large Froude number,presented above used to solve our problem described in the chapter 2.We found the shape of the free surface for $F = \sqrt{10}$ such us in the Figure(3.1)

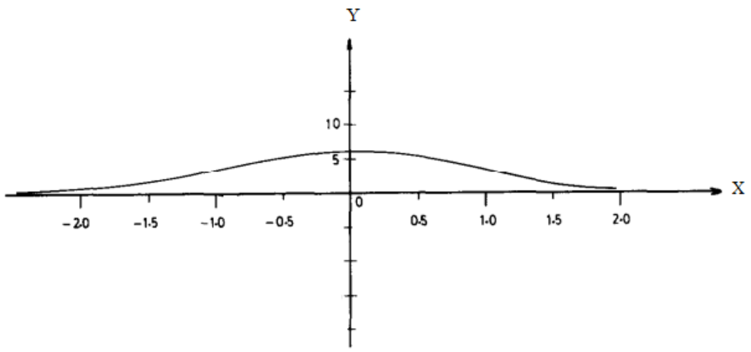

Figure 3.1: The forme of the free surface for $F = \sqrt{10}$.

Figure(3.2),shows that the depth of the free surface varied when the value of Froude number varied between $F = \sqrt{2}$;$F = \sqrt{5}$;$F = \sqrt{10}$.It is clear that the minimum of the depth of the free surface profile,corresponding to $F = \sqrt{10}$,it is about 1.2 and the maximum depth is about 4.95 corresponding to $F = \sqrt{2}$. Then ,we can say that the depthof the shape of the free surface

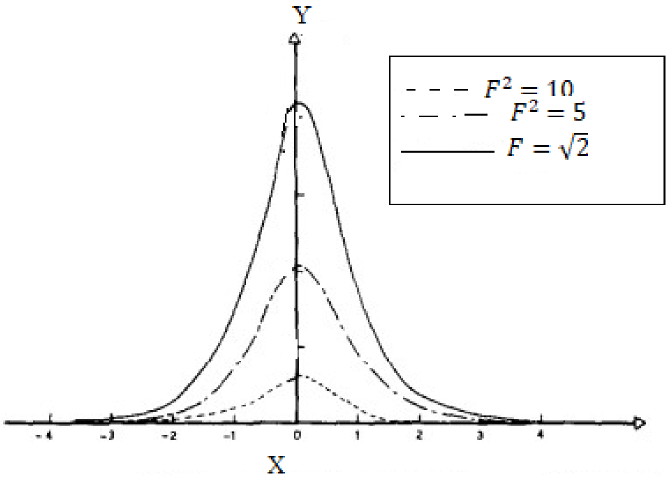

Figure 3.2: The forme of the free surface for differenets valie of Froude number.

increases when the Froude number decreases,also we can notice that the profile of the free surface is symmetrical with respect to point C

4

GENERAL CONCLUSION

The two-dimensional potential flows problems appear in many fileds like:industry ,urbain planning for example:jet pump,dams,sources or wells .The numerical solution of these problems is becoming very important and difficult,esspitially when the force of gravity and the surface tension are considered ,due to the apperance of the nonlinear term in the Bernouli equation.

In our work , we have studied numerically a problem of a two dimensional potential flow of an ideal fluid above a symmetric triangular obstacle in the bottom of an open channel.The obstacle forms an angle $\alpha = \frac{\pi}{4}$ with the horizontal axis $(x'ox)$ .The effect of the force of gravity is taken in account but the superficial tension is neglected.

We have used a conformal mapping especially Schwartz-Christoffel transformation and the Hilbert method of mixed boundary problem in the upper-half plan to find the shape of free surface,the calculus had found for a large Froude number.A numerical solution is succefully found depended to the Froude number,where we have obtained it for each number of Froude vafied between:$F = \sqrt{2}$,$F = \sqrt{5}$,$F = \sqrt{10}$.

In a forthcoming work ,we will presente and study other problems in the domain of potential flows over different geometry of the obstacle,with a new boundary conditions by adopting other numerical methods like finit difference and finits volumes to find the approch solution and the shape of the free surface.

5
Appendix

☞ HILBERT TRANSFORMATION

1. *The hilbert transform on $L^2(R)$:*

   Let $f \in \mathbf{S(R)}$. Its hilbert transform $Hf = \frac{1}{\pi}vp(\frac{1}{x}) * f$ is a function of class $\mathbb{C}^\infty$ and for all $x \in \mathbb{R}$,we have

   $$Hf(x) = \frac{1}{\pi} \lim_{0<\epsilon \longrightarrow 0} \int_{|y|>\epsilon} \frac{f(x-y)}{y} dy = \frac{1}{\pi} \lim_{0<\epsilon \longrightarrow 0} \int_{|y-x|>\epsilon} \frac{f(y)}{x-y} dy$$

   Moreover, $\mathscr{F}(hf) = \mathscr{F}(vp(\frac{1}{x}))\mathscr{F}(f)$; we will determine $\mathscr{F}(vp(\frac{1}{x}))$.

   Lemma:

   $\mathscr{F}(vp(\frac{1}{x}))$ is the distribution given by the bounded function $x \longrightarrow -i\pi sign(x)$

2. *Hilbert transformation and holomorphic functions:*

   Let $f \in S(\mathbb{R})$ be real-valued.A function U is a solution to the problem of *Dirichlet* in the upper half-plane $P = R \times 0, \infty[$ if U is $\mathbb{C}^2$ on P and if

   $\Delta u(x, y) = 0$ on P and $\lim_{y \longrightarrow 0} u(x, y) = f(x)$ for any $x \in \mathbb{R}$.

   This problem admits as a solution $u(x, y) = (P_y * f)(x)$,,where P_y is the kernel of poisson defined by $P_y = \frac{1}{\pi} \frac{y}{x^2+y^2}$ for $x \in \mathbb{R}$ and $y > 0$.

   The Cauchy integral of f is defined on P by $F_f(z) = \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{z-t} dt$. There function F_f is holomorphic in P,its real part is

   $$u_f(x, y) = ReF_f(x + iy) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{f(x-t)}{t^2 + y^2} dt = (P_y * f)(x)$$

its imaginary part is

$$v_f(x,y) = ImF_f(x+iy) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(x-t)t}{t^2+y^2}dt = (Q_y * f)(x)$$

With $Q_y(x) = \frac{1}{\pi}\frac{x}{x^2+y^2}$.Since $F_f = u_f + iv_f$,the functions U and V are functions conjugated harmonics.As mentioned above ,we have $\lim_{y\longrightarrow 0} u_f(x,y) = f(x)$ for any $x \in \mathbb{R}$.Concerning v_f,we have the following result:

Proposition: We have $\lim_{y\longrightarrow 0} v_f(x,y) = Hf(x)$ for all $x \in \mathbb{R}$

☞ METHOD OF SUCCESSIVE APPROXIMATION

1. **THE METHOD OF SUCCESSIVE APPROXIMATION:**

   Now consider the general problem of finding solutions of equation,noted `IVP` :
   $IVPy' = f(x,y)$
   $y(x_0) = y_0$
   $f(x,y)$ is any continuons reel-valued function defined on some rectangle:

   $$R : |x - x_0| \leq a, |y - y_0| \leq b$$

   (a,b>0),In the real (x,y)-plane.

   What is meant by solution ? There exist a real-valued differentiable function ϕ satisfysing :

   - $\phi(x_0) = y_0$
   - $(x, \phi(x_0) \in \mathbb{R}$,for$x \in I$
   - $\phi'(x) = f(x, \phi(x)), \forall x \in I$

   Such a function $\phi(x)$ is called the solution of the IVP(**??**).

2. **EQUIVALENCE OF IVP INTEGRAL EQUATION :**

   ϕ is a solution of IVP $\Longleftrightarrow$ ϕ is solution of the integral equation :$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$

   Now instead of solving IVP we solue the integral equation by the method of successive approximate:

   As a first approximation to a solution we consider the function ϕ_0 defined by:

   $$\phi_0 = y_0$$

   This function satisfies the initial condition but not the integraleen.

   However ,if we compute a second approximation:
   $\phi_1 = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))dt$
   $= y_0 + \int_{x_0}^{x} f(t, y_0)dt$
   Continue the process and define successively $\phi_0 = y_0$ For $(k = 0, 1, 2, ...)$

   $$\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))dt$$

We expect on taking the limits as $k \longrightarrow \infty$,that we would obtain

$$\phi_k(x) \longrightarrow \phi(x)$$

Where ϕ would satisfy:

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt$$

Thus ϕ would be our desired solution .

We call the function $\phi_0, \phi_1, ...$ defined in (2)successive approximations to a solution of the integral solution .Or the IVP.

Theorem:

The successive approximation ϕ_k,defined by $\phi_{k+1(x)} = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt$ exist as continuous functions on

$$I : |x - x_0| \leq \alpha = Min(a, \frac{b}{M})$$

And $(x, \phi_k(x))$ is in $\mathbf{R}$ for $x \in I$.

In indect,the ϕ_k is satisfy

$$|\phi_k(x) - y_0| \leq M\,|x - x_0|\,, \forall x \in I$$

Remark:

Since for $x \in I$,$|x - x_0| \leq \frac{b}{M}$,

$$|\phi_k(x) - y_0| \leq M\,|x - x_0| \leq b, for x \in I \tag{II}$$

Which shows that the points $(x, \phi_k(x)) \in \mathbf{R}$,for $x \in I$.

The precise geometric interpretation of the inquality (II) in the above is that the graph of each ϕ_k lies in the region T in $\mathbf{R}$ bounded by the two lines :

$$y - y_0 = M(x - x_0), y - y_0 = -M(x - x_0)$$

And the lines

$$x - x_0 = \alpha, x - x_0 = -\alpha$$

Bibliography

[1] Abd-el-Malek.B.M,and Masoud.S.Z,Linearized solution of a flow over a ramp,Appl.math.Modelling,1988,Vol.12,406-410.

[2] Abd-el-Malek.B.M,Flow in a waterfall with large Froude number,Mathematics Unit,Science Department ,The American University in Cairo,P.O.Box 2511,Cairo,Egypt,2june 1992,25 October 1992

[3] Abd-el-malek.B.M,S.N.Hanna, and M.T.Kamel,Approximate solution of gravity flow from a uniform channel over triangular bottom for large Froude number ,Appl.Math.Modelling,1991,Vol.15,January.

[4] Bounab.N,Simulation Numerique de Quelques ecoulements potentiels en 2D"Curved Nozzle"These de doctorat,Universit Mohamed Bodiaf M'sila,(2015).

[5] Bounif.M,A.GASMI,First order perturbation approach for the free surface flow over a step with large Weber number,INCAS BULLETIN,Vol.13,Issue2/2021,pp.11-19.

[6] Bounif.M,A.GASMI,The application of the hodograph method to free surface flow problem,J,Sofia,Vol.52(2022)pp.118-128.

[7] Bounif.M,Abd-el-malek,S.N.Hanna.Approximate solution of a flow over a ramp for large Froude number.J.Comput.Appl.Math.1989,28,105-117.

[8] Boutros.Y.Z,Abd-el-Malek,M.B.and Masoud,S.Z.Linearized solution of a flow over a non-uniform bottom.J.Comput.Appl.Math.1986,16,105-116.

[9] Churchill,R.V.Complex Variables and Applications,2nd ed .McGraw-Hill.New York,1960 .

[10] DABBENE.F et H.PAILLER,Initiation a la simulation numerique en mecanique des fluides :Element d'analyse numerique,Cours ensta MF307,6juin 2003.

[11] Damou.M.Mecanique des fluides,(O.P.U),(1993).

[12] Elcrat.A.R and L.N.Trefthen,Classical free-streamline flows over a polygonal obstacle,J.Comp and Applied Math.14(1986),256-265.

[13] Emmanuel.P Mines Nancy,Mecanique des fluides,Version du 21 octobre 2016.

[14] Forbes,L.K.and Schwartz,L.M.Free-surface flow over a semicircular obstruction.J.Fluid Mech.1982,114,299-314.

[15] Sekhri.H,Ecoulement a surface libre sur une depression de forme triangulaire.Mmoire de Magistere,Universit de Batna.

[16] SMITH.A.C and T.H.LIM,Symmetric supercritical free surface flow over a polygonal obstacle,Int J Engng Sct Vol.23,No 3,pp289-306,1985.

ملخص

في هذا العمل قمنا بدراسة تدفق كموني ثنائي الأبعاد في قناة ذات عائق ذو شكل مثلثي موضوع في الأسفل مع إهمال تأثير السطح والأخذ بعين الإعتبار الجاذبية .

ولإيجاد الحل التقريبي العددي إستعملنا طرق التحويلات المطابقة و طريقة هيلبرت بالإستعانة بالتقريبات المتتالية .

الكلمات المفتاحية

التحويلات المطابقة , رقم فرود , طريقة هيلبرت , التدفق الكموني

ABSTRACT

In this work ,we studied a two-dimensional potential flow in a open channel with symmetrical triangular obstacle placed at the bottom ,neglecting the surface effect and taking the forces of gravity into account .

To find the approximate numerical solution ,we used coformal transformation methods ,the Hilbert method and successive approximations.

K EYWORDS

Conformal mapping ,Froude Number ,Hilbert method ,Potential Flow.

Résumé

Dans ce travail , nous avons étudié un écoulement potential bidimensionnel dans un canal avec un obstacle de forme triangulaire placé au fond , en négligeant l’effet de surface et en tenant compte de force de la gravité.

Pour trouver la solution numérique approchée , nous avons utilisé des méthodes de transformation conforme et la méthode de Hilbert et les approximations successives.

L es mots clés

transformation conforme , Nombre de Froude , Méthode de Hilbert , Ecoulement potentiel.