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Etude des équations intégrales non linéaires de Volterra dans les espaces fonctionnels

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ملخص

في هذه الأطروحة، قمنا بدراسة المعادلات الغيرخطية لفولتيرا ذات الشكل العام

$$\varphi(x) = f(x) + \int_a^x K(x, y, \varphi(y))dy$$

حيث $K(x, y)$ تسمى نواة المعادلة التكاملية. $K(x, y)$, $f(x)$ دوال معطاة

الهدف من هذا البحث هو إثبات وجود ووحدانية حل المعادلة التكاملية الغير خطية لفولتيرا في الفضاءات الدالية، وركزنا بحثنا في فضاءات بناخ وبالضبط في فضاءات الدوال المستمرة وكذا فضاءات الدوال القابلة للقياس لان معظم النتائج في هذه الفضاءات. ثم قدمنا برنامج لحل مثل هذه المعادلات يرتكز على تطوير طريقة نيوتن-كانتوروفيتش لإيجاد الحلول التقريبية لمثل هذه المعادلات.

الكلمات المفتاحية : المعادلات التكاملية, معادلات فولتيرا, نظرية النقطة الصامدة, طريقة نيوتن-كانتوروفيتش المطورة.

Abstract

In this thesis we have studied the nonlinear Volterra integral equations, which have the general form

$$\varphi(x) = f(x) + \int_a^x K(x, y, \varphi(y))dy, \quad x \in [a, b]$$

where K is called the kernel of the integral equation, and both the kernel $K(x, y)$ and the function $f(x)$ in the integral equation are given functions.

The aim of this thesis is to prove the existence as well as the uniqueness of solution of nonlinear Volterra integral equation in function spaces, and we focused our research on Banach spaces such as spaces of continuous functions and spaces of measurable functions because most of the results relate to these spaces. Then provide this study by numerical results using an Adapted Newton-Kantorovich method.

Keywords: Integral equations, Volterra equations, fixed point theory, Nemytskii Operator, Adapted Newton-Kantorovich method.

Resumé

Dans cette thèse, on a étudié les équations intégrales non linéaires de Volterra, qui ont la forme générale

$$\varphi(x) = f(x) + \int_a^x K(x, y, \varphi(y))dy, \quad x \in [a, b]$$

où K est appelée le noyau de l'équation intégrale, et le noyau $K(x, y)$ et la fonction $f(x)$ dans l'équation intégrale sont des fonctions données .

Le but de cette thèse est de prouver l'existence ainsi que l'unicité de la solution de l'équation intégrale non-linéaire de Volterra dans les espaces fonctionnels et on a centré notre recherche sur les espaces de Banach tels que les espaces des fonctions continues et les espaces des fonctions mesurables car la plupart des résultats se rapportent aux ces espaces. Ensuite, fournir cette étude par des résultats numériques en utilisant une méthode de Newton-Kantorovich adaptée.

Mots clés: Equation intégrale, équation de Volterra, la theorie du point fixe, l'opérateur de Nemytskii, la méthode de Newton-Kantorovich Adaptée.

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Introduction

Integral equations are very useful mathematical tools in both pure and applied mathematics, appear in various fields of science and numerous applications such that elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.

An integral equation is defined as an equation in which the unknown function $\varphi(x)$ to be determined appear under the integral sign. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations.

A general form of an integral equation in $\varphi(x)$ is of the form

$$\varphi(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x, y, \varphi(y))dy$$

where $K(x, y)$ is called the kernel of the integral equation , $\alpha(x)$ and $\beta(x)$ are the limits of integration. It is to be noted here that both the kernel $K(x, y)$ and the function $f(x)$ in the integral equation are given functions; and λ is a constant parameter.

If the lower limit of integration is constant and the upper one is variable we are in the case of Volterra integral equations which is our subject. This kind of equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908.

The aim of this thesis is to study the existence as well as the uniqueness of solution of the nonlinear Volterra integral equation in function spaces then provide this study by numerical results i.e prove the existence of the unknown function $\varphi(x)$ that will satisfy the last equation using some fundamental theorems such that fixed point theorems and useful technics like Liapunov functions. it should be noted that most of the results relate to the Banach spaces so we focused our research in these spaces, our work is divided as follows:

In the introductory chapter we recall some notions of compactness in Banach spaces and in particular L^p spaces, and we mention some fundamental theorems such that the Ascoli-Arzela Theorem, Hausdorff's Theorem and The Frechet-Kolmogorov Theorem, in addition to definition of compact operators and its properties in Banach spaces in order to prove the basic Riesz theorems and Fredholm alternative on Banach spaces for compact operators, in this chapter we present a new operator $T = I - \lambda A$ and we discuss the different results when we suppose that A is a compact operator on a Banach space X. Also certain various results and technics will be discuss in this chapter Fréchet derivative in Banach spaces, Nemytskii Operator in L^p spaces and Liapunov functions which well be used in the next chapters.

The second chaptet contained basic theory of fixed point we will recall to some important and different theorems such that Banach fixed point theorem, Brouwer and Schauder fixed point theorems and the Leray-Schauder Principle, these theorems are very useful in the next chapters.

In the third one we present integral equations and we illustrate different criteria of classification of these equations, also we describe the conversion process of initial value problems to Volterra integral equations and finally we mention some theorems to prove existence of solutions of some kind of integral equations.

The last chapter presents the aim of this thesis, which is studying Nonlinear Volterra Integral Equations in Banach spaces, we well use all what we had seen in the previous chapters to prove existence and uniqueness of solution of this kind of equations, then we approximate this solution using Adapted Newton-Kantorovich method, we describe this method then prove convergence, also give some examples solving by this method and compared by others to show the efficiency of this one.

Chapter 1

Some fundamental theorems of functional spaces

In this introductory chapter we recall some notions of compactness in Banach spaces and in particular spaces of measurable functions, and we mention some fundamental theorems such that the Ascoli-Arzela Theorem, Hausdorff's Theorem and The Frechet-Kolmogorov Theorem, in order to study compact operators and its properties in Banach spaces, then prove the basic Riesz theorems and Fredholm alternative on Banach spaces for compact operators, and certain various results and technics will be discuss in this chapter as Fréchet derivative in Banach spaces, Nemytskii Operator in L^p spaces and Liapunov functions which well be used in the next chapters.

1.1 Spaces of continuous functions

The space of continuous functions consists of all continuous maps of the closed interval into $\mathbb{R}^n$ denoted by $C(\Omega, \mathbb{R}^n)$. The norm is the usual supremum norm, given by

$$\|f\| = \sup_{x \in \Omega} |f(x)|$$

If Ω bounded and closed $C(\Omega, \mathbb{R}^n)$ is a Banach space, if Ω is unbounded set, the space $C(\Omega, \mathbb{R}^n)$ is no longer a Banach space, it is a linear metric space, with translation- invariant metric, which is also complete. Such linear spaces are usually called Fréchet spaces.

Definition 1.1.1 *Let* $f : \Omega \to \mathbb{R}^n$ *a continuous function, we say that* f *is bounded if* $f(\Omega)$ *is bounded.*

we denote by $C_b(\Omega, \mathbb{R}^n)$ the subspace of continuous and bounded functions.

Now we recall equicontinuous concept which is a useful notion in the Ascoli-Arzela Theorem.

Definition 1.1.2 *Let* $\mathcal{F} : \Omega \to \mathbb{R}^n$ *a set of continuous functions, we say that* $\mathcal{F}$ *is equicontinuous at a point* $x_0 \in \Omega$ *if and only if for all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\forall\ f \in C(\Omega, \mathbb{R}^n),\ \forall x \in \Omega,\ \|x - x_0\| \leq \delta \textit{ then } \|f(x) - f(x_0)\| \leq \epsilon$$

Proposition 1.1.1 *Let* $\mathcal{F} : \Omega \to \mathbb{R}^n$ *a set of continuous functions,* $\mathcal{F}$ *is equicontinuous at a point* $x_0 \in \Omega$*, then* $\bar{\mathcal{F}}$ *is also equicontinuous at* $x_0 \in \Omega$*..*

Definition 1.1.3 *Let* $\mathcal{F} : \Omega \to \mathbb{R}^n$ *a set of continuous functions,* $\mathcal{F}$ *is equicontinuous if and only if it is equicontinuous at all points* $x \in \Omega$*.*

1.2 Spaces of measurable functions

Let $\Omega \subset \mathbb{R}^N$ be open and $p \in R$, $1 \leq p < \infty$. $f : \Omega \to \mathbb{R}^n$, we say $f \in L^p(\Omega; \mathbb{R}^n)$ if $|f|^p$ is Lebesgue integrable on Ω. So the space $L^p(\Omega; \mathbb{R}^n)$ is endowed with the norm

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}$$

For $p = \infty$ we define $L^\infty(\Omega; \mathbb{R}^n)$ by the space of all measurable functions $f : \Omega \to \mathbb{R}^n$ for which there exists a constant c with $|f(x)| \leq c$ for almost every (a.e. for short) $x \in \Omega$. Where

$$\|f\|_\infty = \sup ess\, |f(x)| = \inf \{c : f(x) < c \text{ for a.e } x \in \Omega\}$$

is a norm on $L^\infty(\Omega; \mathbb{R}^n)$

The space $L^p(\Omega; \mathbb{R}^n)$ proved by the norm $|.|_p$, $L^p(\Omega; \mathbb{R}^n)$ is a Banach space for each $p \in [1, \infty]$.

In the case of functions of two variables $f(x, y)$, the norm is given by

$$\|f\|_p = \begin{cases} \left[\int_\Omega (\int_\Omega |f(x,y)^q|\, dy)^{p/q} dx\right]^{1/p}, & \text{for: } 1 < p < \infty \text{ with } \frac{1}{p} + \frac{1}{q} = 1 \\ \int_\Omega \sup\limits_{y\in\Omega} ess\, |f(x,y)|\, dx & \text{for } p = 1 \\ \sup\limits_{x\in\Omega} ess \int_\Omega |f(x,y)|\, dy & \text{for } p = \infty \end{cases}$$

The space $L^2(\Omega; \mathbb{R}^n)$ with the inner-product

$$\langle f, g\rangle = \int_\Omega (f(x), g(x))\, dx$$

is a Hilbert space.

The conjugate exponent of $p, 1 \leq p \leq \infty$ means as the variable p' which satisfies

$$1/p + 1/p' = 1$$

Now we recall the well known and useful inequality, Holder's inequality.

Theorem 1.2.1 *Holder's inequality if* $f \in L^p(\Omega, \mathbb{R}^n)$,*and* $g \in L^{p'}(\Omega, \mathbb{R}^n)$ *then* $f.g \in L^1(\Omega)$ *and*

$$\int_\Omega |f(x)|\, |g(x)|\, dx \leq \|f\|_p \cdot \|g\|_{p'}$$

This inequality implies that if Ω is bounded then for $1 \leq p < q \leq \infty$, the following inclusions hold

$$C(\bar{\Omega}, \mathbb{R}^n) \subset L^q(\Omega, \mathbb{R}^n) \subset L^p(\Omega, \mathbb{R}^n)$$

A very useful result it well be used in the proof of the next part is: the space $C^o(\Omega, \mathbb{R}^n)$ of all continuous functions from Ω to $\mathbb{R}^n$ with compact support in Ω is dense in $L^p(\Omega; \mathbb{R}^n)$ for every $p \in [1, \infty)$

1.3 Compactness in function spaces

Proposition 1.3.1 *Let* (X, d) *be a metric space. The space* X *is complete if and only if for every sequence of elements of* X *has a convergent subsequence in* X*, furthermore for each* $\epsilon > 0$ *it admits a finite covering by open balls of radius* ϵ*.*

Definition 1.3.1 *A metric space X is said to be compact if X is complete and for each $\epsilon > 0$ it admits a finite covering by open balls of radius ϵ.*

Definition 1.3.2 *(Relatively Compact) Let X be a metric space, $\Omega \subseteq X$ is relatively compact in X, if $\overline{\Omega}$ is compact in X.*

Theorem 1.3.1 *Any set bounded and finite dimension of a normed space is relatively compact.*

Hausdorff's Theorem

Definition 1.3.3 *Let (X, d) be a metric space, Ω a subset of X and $c > 0$, A subset $\Re \subset X$ is said to be an $\epsilon-$net for Ω if for every $x \in \Omega$ there exists $y \in \Re$ such that $d(x, y) < c$.*

Proposition 1.3.2 *A metric space X is compact if and only if it is complete and for every $\epsilon > 0$ it admits a finite $\epsilon-$net.*

Theorem 1.3.2 *(Hausdorff's Theorem)Let (X, d) be a complete metric space and $Y \subset X$ be a subset. The following statements are equivalent:*

(a) Y is relatively compact.

(b) For every $\epsilon > 0$ there exists in X a finite $\epsilon-$net for Y.

(c) For every $\epsilon > 0$ there exists in X a relatively compact $\epsilon-$net for Y.

Definition 1.3.4 *(Precompact) Let X be a metric space; $\Omega \subseteq X$ is precompact (also called totally bounded) if for every $\epsilon > 0$, Ω is covered by finitely-many open balls of radius ϵ.*

In Banach spaces precompactness and relative compactness are equivalent, indeed

Theorem 1.3.3 *Let X be a metric space. If $\Omega \subseteq X$ is relatively compact then it is precompact. Moreover, if X is complete then the converse holds also.*

1.3.1 Compactness in spaces of continuous function

The Ascoli-Arzela Theorem

Let (Ω, d) be a compact metric space and $C(\Omega, \mathbb{R}^n)$ be the Banach space of all continuous functions from Ω to $\mathbb{R}^n$, under the sup-norm $\|.\|_\infty$.

Theorem 1.3.4 *(Ascoli-Arzela) A subset U of $C(\Omega, \mathbb{R}^n)$ is relatively compact if and only if the following conditions are satisfied:*

(i) U is bounded, i. e., there exists a constant $c > 0$ such that

$$|f(x)| \leq c$$

for all $x \in \Omega$ and $f \in U$

(ii) U is equicontinuous, i. e., for every $c > 0$ there exists a constant $\delta > 0$ such that for all $f \in U$

$$|f(x) - f(x')| < c$$

whenever $x, x' \in \Omega$ and $d(x, x') < \delta$

In the next paragraph, we present a very useful corollary of Ascoli-Arzela Theorem in fixed point theory, first we denote by $\|.\|_{k,\infty}$ the norm of $C^k(\bar{\Omega}, \mathbb{R}^n)$, ($k \in \mathbb{N}\backslash\{0\}$, $\Omega \subset \mathbb{R}^n$ bounded open) given by

$$\|.\|_{k,\infty} = Max\left\{\|f\|_\infty, \|f'\|_\infty, ..., \left\|f^{(k)}\right\|_\infty\right\}$$

Corollary 1.3.1 *Let Ω be a bounded open subset of $\mathbb{R}^N$. Every bounded subset of the space $(C^1(\bar{\Omega}, \mathbb{R}^n), \|.\|_{1,\infty})$ is relatively compact in $(C(\bar{\Omega}, \mathbb{R}^n), \|.\|_\infty)$.*

proof. *(see* [36]*)* ■

1.3.2 Compactness in spaces of measurable functions

Definition 1.3.5 *Let Ω is a bounded open set of $\mathbb{R}^N$, $r > 0$, and $f : \mathbb{R}^N \to \mathbb{R}^n$ any locally integrable function satisfy the condition $f(y) = 0$ for $y \notin \Omega$ we define the* ***average function*** *of f with respect to radius r by*

$$m_r(f)(x) = \frac{1}{\omega_r} \int_{B_{r(x)}} f(y)dy \quad (x \in \mathbb{R}^n)$$

Here $\omega_r = \mu(B_r(0))$, the measure of the ball of radius r of $\mathbb{R}^N$.

Lemma 1.3.1 *For every* $f \in L^P(\Omega, \mathbb{R}^n)$ $(1 \leq p < \infty)$ *and every* $r > 0$, *the function* $m_r(f)$ *is continuous on* $\mathbb{R}^N$ *and satisfies*

$$|m_r(f)(x)| \leq \omega_r^{-1/p} \|f\|_{L^p(\Omega,\mathbb{R}^n)}, \quad x \in \mathbb{R}^N$$

and

$$|m_r(f)|_{L^p(\Omega,\mathbb{R}^n)} \leq \|f\|_{L^p(\Omega,\mathbb{R}^n)}$$

The Frechet-Kolmogorov Theorem

Theorem 1.3.5 *(Frechet-Kolmogorov) Let* $\Omega \subset \mathbb{R}^N$ *be bounded open and* $1 \leq p < \infty$. *A subset* U *of* $L^p(\Omega, \mathbb{R}^n)$ *is relatively compact if and only if the following conditions are satisfied:*

(i) U *is bounded, i. e., there exists a constant* $c > 0$ *such that*

$$\|f\|_p \leq c$$

for all $f \in U$;

(ii) $\tau_h(f) = f$ *in* $L^p(\Omega, \mathbb{R}^n)$ *as* $h \to 0$, *uniformly for* $f \in U$, *i.e*

$$\sup_{f \in U} \|\tau_h(f) - f\|_p \to 0 \text{ as } h \to 0$$

proof. *Assume that* U *is relatively compact in* $L^p(\Omega, \mathbb{R}^n)$. *Then clearly (i) holds. Also, for a given* $\epsilon > 0$ *there exists in* $L^p(\Omega, \mathbb{R}^n)$ *a finite* $\epsilon/3-$*net for* U. *Since* $C(\bar{\Omega}, \mathbb{R}^n)$ *is dense in* $L^p(\Omega, \mathbb{R}^n)$ *(recall that* Ω *is bounded), we may assume that the elements of the net belong to* $C(\bar{\Omega}, \mathbb{R}^n)$. *Let these elements be* f_j, $j = 1, 2, ..., m$. *Since* f_j *is uniform continuous on the compact set* $\bar{\Omega}$, *there exists a* $\delta > 0$ *such that*

$$\begin{aligned} |f_j(x) - \tau_h(f_j)(x)| &= |f_j(x) - f_j(x+h)| \\ &< \frac{\epsilon}{(2.3^p \mu(\Omega))^{1/p}} \end{aligned}$$

for all $h \in \mathbb{R}^N$ *with* $|h| < \delta$, $x \in \Omega_h = \Omega \cap (\Omega - h)$ *and* $j = 1, 2, ..., m$. *It follows that*

$$\int_{\Omega_h} |f_j(x) - \tau_h(f_j)(x)|^p \, dx = \frac{\epsilon^p}{2.3^p}$$

In addition, by (i) we may assume that for each j we have consequently

$$|\tau_h(f_j) - f_j|_p < \epsilon/3$$

for $|h| < \delta$ *and* $j = 1, 2, ..., m$*. Now, for a given* $f \in U$ *there is a* $j \in \{1, 2, ..., m\}$ *with* $|f - f_j| < \epsilon/3$*. On the other hand, we have*

$$\tau_h(f) - f = \tau_h(f - f_j) - (f - f_j) + (\tau_h(f_j) - f_j)$$

As a result, if $|h| < \delta$

$$|\tau_h(f) - f|_p = |\tau_h(f - f_j)|_p + |(f - f_j)|_p + |(\tau_h(f_j) - f_j)|_p < \epsilon$$

Thus (ii) holds.

Conversely, assume (i) and (ii) are satisfied. For any $f \in U$ *one has*

$$\begin{aligned} |m_r(f)(x) - m_r(f)(x')| \quad &= \omega_r^{-1} \left| \int_{B_r(x)} f(y)dy - \int_{B_r(x')} f(y)dy \right| \\ &= \omega_r^{-1} \int_{B_r(x)} |f(y) - \tau_{x-x'}(f)(y)|\, dy \\ &\leq \omega_r^{-1/p} \, |u - \tau_{x-x'}(f)|_p \end{aligned}$$

This inequality together with (ii) shows that the set

$$m_r(U) = \{m_r(f) : f \in U\}$$

is equicontinuous. In addition the set $m_r(U)$ *is bounded in* $C(\bar{\Omega}, \mathbb{R}^n)$*. Now the Ascoli-Arzela theorem guarantees that* $m_r(U)$ *is a relatively compact subset of* $C(\bar{\Omega}, \mathbb{R}^n)$*, and so of* $L^p(\Omega, \mathbb{R}^n)$*. On the other hand, from*

$$\begin{aligned} m_r(f)(x) - f(x) \quad &= \omega_p^{-1} \int_{B_r(x)} (f(y) - f(x))dy \\ &\omega_p^{-1} \int_{B_r(0)} (\tau_y(f)(x) - f(x))dy \end{aligned}$$

we deduce

$$|m_r(f) - f|_p \leq \sup_{|y \leq r|} |\tau_y(f) - (f)|_p$$

This shows that U *is the uniform limit (in* $L^p(\Omega, \mathbb{R}^n)$*) of* $m_r(U)$ *as* $r \to 0$*. Then* Y *is relatively compact in* $L^p(\Omega, \mathbb{R}^n)$*.* ■

Now we mention an extension of the Frechet-Kolmogorov theorem

Theorem 1.3.6 *Let* $\Omega \subset \mathbb{R}^N$ *be bounded open and* $1 \leq p \leq \infty$. *Let* $U \subset L^p(\Omega, \mathbb{R}^n)$ *for* $1 \leq p < \infty$ *and* $U \subset C(\bar{\Omega}; \mathbb{R}^n)$ *for* $p = \infty$ *Assume that there exists a function* $g \in L^p(\Omega)$ *such that*

$$|f(x)| \leq g$$

for a.e. $x \in \Omega$ *and all* $f \in U$. *Then* U *is relatively compact in* $L^p(\Omega, \mathbb{R}^n)$ *if and only if*

$$\sup_{f \in U} \|\tau_h(f) - f\|_{L^p(\Omega_h, \mathbb{R}^N)} \to 0 \text{ as } h \to 0$$

Here $\Omega_h = \Omega \cap (\Omega - h)$.

proof. see [36]. ■

1.4 Compact operators on functional spaces

1.4.1 Definitions and properties of compact operators

First recall the basic definitions regarding operators.

Definition 1.4.1 *(Continuous and Bounded Operators). Let* X, Y *be normed linear spaces, and let* $A : X \to Y$ *be a linear operator*

1. *A is continuous at a point* $\varphi \in X$ *if* $\varphi_n \to \varphi$ *in* X *implies* $A\varphi_n \to A\varphi$ *in* Y.
2. *A is continuous if it is continuous at every point, i.e. If* $\varphi_n \to \varphi$ *in* X *implies* $A\varphi_n \to A\varphi$ *in* Y. *for every* $\varphi \in X$.
3. *A is bounded if there exists a finite* $c \geq 0$ *such that*

$$\forall \varphi \in X \ \ \|A\varphi\|_Y \leq c \|\varphi\|_X$$

4. *The operator norm of A is*

$$\|A\| = \sup_{\|\varphi\|=1} \|A\varphi\|$$

5. *we denote by* $B(X, Y)$ *the set of all bounded linear operators mapping* X *into* Y

$$B(X, Y) = \{A : X \to Y : A \text{ is bounded and linear}\}$$

6. If $Y = \mathbb{R}$ (or $\mathbb{C}$) then we say that A is a functional. The set of all bounded linear functional on X is the dual space of X, and is denoted by

$$X' = B(X, \mathbb{R}) = \{A : X \to F : A \textit{ is bounded and linear}\}$$

Definition 1.4.2 *Let X, Y be normed linear spaces, and let $A : X \to Y$ be a linear operator, a subset of X. The operator A is said to be of finite rank if $A(U)$ lies in a finite dimensional subspace of Y.*

(Adjoints) Let X_1, X_2, Y_1, Y_2 be normed linear spaces, $A : X_1 \to X_2$ an operator, then the operator $B : Y_2 \to Y_1$ is said to be the adjoint of A *if*

$$\langle A\varphi, \psi\rangle = \langle \varphi, B\psi\rangle \,, \quad \text{for all } \varphi \in X_1, \psi \in Y_2$$

We usually denote the adjoint of A by A^*, this operator is continuous.

Definition 1.4.3 *(compactness) Let X and Y be two normed linear spaces and $A : X \to Y$ a linear map between X and Y . A is called a compact operator if for all bounded sets $U \subseteq X$, $A(U)$ is relatively compact in Y.*

Definition 1.4.4 *The compact operator A is said to be completely continuous if it is continuous.*

Now we mention other definitions of compactness given in the next theorem.

Theorem 1.4.1 *Let X and Y be two normed linear spaces; suppose $A : X \to Y$, is a linear operator. Then the following are equivalent.*

1. A is compact.

2. The image of the open unit ball under A is relatively compact in Y.

3. For any bounded sequence $\{\varphi_n\}$ in X, there exist a subsequence $\{A\varphi_{n_k}\}$ of $\{A\varphi_n\}$ that converges in Y.

Theorem 1.4.2 *Let A be a bounded operator from X to Y, such that the image $A(X)$ is of a finite dimension. then A is compact.*

proof. (see [15]). ■

Theorem 1.4.3 *Let X, Y, Z be normed spaces.*

1. *Linear combinations of compact operators are compact.*
2. *The linear operator $A : X \to Y$ is of finite-rank then A is compact .*
3. *Let $A_1 \in L(X, Y), A_2 \in L(Y, Z)$. Then $A_1 A_2$ is compact if A_1 or A_2 is compact.*
4. *Let A_1 compact operator and let A_2 continuous operator. Then both right and left compositions, $A_2 \circ A_1$ and $A_1 \circ A_2$ produce compact operators.*
5. *operator-norm limits $A = \lim\limits_{i} A_i$ of compact operators A_i is compact.*

Theorem 1.4.4 *A compact operator is a bounded operator, the converse is false.*

Theorem 1.4.5 *Compact operators invertible only on finite-dimensional space.*

proof. For compact $A : X \to Y$ with continuous inverse A^{-1}, the boundedness of A^{-1} gives a constant c such that $|A^{-1}\psi| \leq c \cdot |\psi|$ for all $\psi \in Y$. Invertibility implies that $AX = Y$, and $|\varphi| \leq c. |A\varphi|$ for all $\varphi \in X$.

Thus, the image by A of the unit ball in X contains an open ball in Y . Compactness implies that Y is finite-dimensional, and invertibility implies that X is finite-dimensional. ■

Theorem 1.4.6 *A Banach-space operator A is compact if and only if A^* is compact.*

proof. (see [15]) ■

1.4.2 Spectral theory of compact operators

Now we present a new operator $T = I - \lambda A$ and we discuss different results when A is a compact operator on a Banach space X.

Theorem 1.4.7 *Let A a compact operator on a Banach space X, for $\lambda \neq 0$, the kernel of $T = (I - \lambda A)X$ given by*

$$\ker T = \{\varphi \in X / \, T\varphi = (I - \lambda A)\varphi = 0\}$$

is closed and finite-dimensional.

proof. It's clear that the kernel $(I - \lambda A)X$ is a vectoriel subspace, Let $\varphi_n \in \ker T$ a convergent subsequence then

$$T\varphi_n = 0 \Rightarrow T\varphi = 0$$

this proves that $\ker T$ is closed.

Let $\varphi \in \ker(T)$ so

$$\begin{aligned} T(\varphi) = 0 \quad &\Rightarrow (I - \lambda A\varphi = 0 \\ &\Rightarrow \lambda A\varphi = \varphi \end{aligned}$$

Then λA coïnsids with the identity in the subspace $\ker(X)$, λA is compact from $\ker(X)$ to $\ker(X)$ as a result the kernel is finite-dimensional ■

Theorem 1.4.8 *Let A a compact operator on a Banach space X, $\lambda \neq 0$ the image $(I - \lambda A)X$ is closed.*

proof. Let $\{\varphi_n\}$ is a bounded sequence. such that $(I - \lambda A)\varphi_n \to f$ .So there exist a convergent subsequence of $\lambda A\varphi_n$, because A compact, and we replace φ_n by this subsequence. Then $-\varphi_n = f - \lambda A\varphi_n$ converges to $f - \lim \ \lambda A\varphi_n$, so φ_n is convergent to $\varphi \in X$, and $\lambda A\varphi = f$.

To reduce the general case to the previous, first reduce to the case that $I - \lambda A$ is injective, as we had seen $\ker(I - \lambda A)$ is finite-dimensional, so we can choose a complementary subspace D to $\ker(I - \lambda A)$. Since $(I - \lambda A)D = (I - \lambda AX$, to prove the image is closed it suffices to consider D, or equivalently, that $I - \lambda A$ is injective on X. Since $I - \lambda A$ is a continuous bijection to its image, by the open mapping theorem it is an isomorphism to its image. Thus, there is $\delta > 0$ such that $|(I - \lambda A)\varphi| < \delta \, |\varphi|$.

Returning to the main argument, suppose that $(I - \lambda A)\varphi_n \to f$. Then $(I - \lambda A)(\varphi_m - \varphi_n)$. We have $\varphi_m - \varphi_n \to 0$ as a result φ_n is bounded, and we are in the previous case. ■

Theorem 1.4.9 *Let A a compact operator on a Banach space X, for $\lambda \neq 0$, injectivity and surjectivity of $I - \lambda A$ are equivalent.*

Theorem 1.4.10 *(Fredholm alternative) Let A a compact operator, either $I - \lambda A$ is bijective, or has non-trivial kernel and non-trivial cokernel, of the same dimension.*

proof. As what we had seen $\ker(I - \lambda A)$ is finite dimensional. Dually, for $\varphi_1, ..., \varphi_n \in X$ linearly independent modulo $(I - \lambda A)X$, by Hahn-Banach there are $g_1, ..., g_n \in X^*$ vanishing on the image $(I - \lambda A)X$ and $g_i(f_j) = h_{ij}$ Such g_i are in the kernel of the adjoint $(I - \lambda A)*$. We know A^* is compact, so $\ker(I - \lambda A)^*$ is finite-dimensional. Now we use the fact that injectivity and surjectivity of $(I - \lambda A)$ are equivalent, and that the kernel and cokernel are finite-dimensional. Let $\varphi_1, ..., \varphi_m$ (for $m \geq 1$) span the kernel, and let the images of $y_1, ..., y_n$ (for $n \geq 1$) span the cokernel, and show that $m = n$.

For $m \leq n$, let X' be a closed complementary subspace to the kernel of $I - \lambda A$. Let F be the finite-rank operator such that F equal to 0 on X' and $\lambda F \varphi_i = f_i$. The adjusted operator $A' = A + F$ is compact. For $(I - \lambda A)x = 0$

$$(I - \lambda A)\varphi = F\varphi \in (I - \lambda A)X \cap span\ f_1, ..., f_n = \{0\}$$

That is, $I - \lambda A'$ is injective, so is surjective, so $m = n$. In the opposite case $m \geq n$, let $F\varphi_i = \lambda f_i$ for $i \leq n$, and $F\varphi_i = \lambda f_n$ for $i \geq n$. With $A' = A + F$ again, in this case $I - \lambda A'$ is surjective, so is injective, and $m = n$. ■

1.5 Integral operators

Definition 1.5.1 *Let $\Omega \subset \mathbb{R}^n$ a compact subset, K a continuous function from $\Omega \times \Omega$ into $\mathbb{R}$ then the linear operator defined from $C(\Omega)$ into itself by*

$$(A\varphi)(x) = \int_\Omega K(x, y)\varphi(y)dy \qquad x \in \Omega$$

is called integral operator, and $K(x, y)$ is the kernel of the integral operator.

A particularly class of integral operators is operators with degenerate kernels. These kernels can be decomposed into a finite sum of separable functions i.e we can write it in the form

$$K(x, y) = \sum_{i=1}^{n} a_i(x)b_i(y)$$

Theorem 1.5.1 *The range of an integral operator A is finite dimensional if it is induced by a degenerate kernel.*

proof. We will show that the image of A is generated by the functions $a_1, a_2, ..., a_n$. Let $\varphi \in L^2(a,b)$, then

$$\begin{aligned} A\varphi(x) &= \int_a^b \sum_{j=1}^{n} a_j(x) b_j(y) \varphi(y) dy \\ &= \sum_{j=1}^{n} (\int_a^b b_j(y)\varphi(y)dy) a_j(x) \end{aligned}$$

which is an element of the vector space generated by $\{a_1, ...a_n\}$. ■

Theorem 1.5.2 *Let A be the integral operator with a kernel $K(x,y)$, then the adjoint A^* is an integral operator with the kernel $K^*(x,y)$ such that*

$$K^*(x,y) = K(y,x)$$

Theorem 1.5.3 *Let A be the integral operator from $C(\Omega)$ into itself with continuous kernel $K(x,y)$, then A is compact.*

Theorem 1.5.4 *Let*

$$(A\varphi)(x) = \int_\Omega K(x,y)\varphi(y)dy \qquad x \in \Omega$$

with kernel $K \in L^2(\Omega \times \Omega)$. Then A is continuous compact operator from $L^2(\Omega)$ into itself.

proof. First we prove that the linear operator A is bounded (continuous), let $\varphi \in L^2(\Omega)$,by the Cauchy-Schwarz inequality we get

$$\begin{aligned} \int_\Omega (A\varphi(x))^2 dx &\leq \int_\Omega (\int_\Omega |K(x,y)|^2 \, dy)(\int_\Omega |\varphi(y)|^2 \, dy) \, dx \\ &\leq M^2 \int_\Omega |\varphi(y)|^2 \, dy < \infty \end{aligned}$$

with $M^2 = \int \int_{]a,b[\times]a,b[} |K(x,y)|^2 \, dydx.$

It remains to prove that A is compact, we recall that the space $L^2(\Omega)$ has a countable orthonormal basis, so we can write the kernel K as a sum of degenerate kernels $(K_n)_{n\in N}$ such that

$$\lim_{n\to\infty} \|K - K_n\| = 0$$

so we get a sequence operators

$$(A_n\varphi)(x) = \int_\Omega K_n(x,y)\varphi(y)dy$$

Obviously, A maps $L^2(\Omega)$ into a finite-dimensional subspace of $L^2(\Omega)$. The range of A_n is finite dimensional and hence A_n is compact. We find

$$\begin{aligned} \|(A_n - A)\varphi\| &= \int_\Omega(\int_\Omega(K_n(x,y) - K(x,y))\varphi(y)dy)^2)dx \\ &\leq (\int\int_{\Omega\times\Omega} |K_n(x,y) - K(x,y)|^2\, dy\,)dx\, \|\varphi\|^2 \\ &= \|K_n - K\|_{L^2(\Omega\times\Omega)} \|\varphi\|^2 \end{aligned}$$

the above expression has to go to zero as $n \to \infty$. We conclude that $A \to A_n$ so A is compact. ■

Theorem 1.5.5 *Let A an integral operator with the kernel $K(x,y)$, assuming that $\|K(x,y)\| < \infty$ for $1 \leq p \leq \infty$,then A maps L^p into itself, furthermore we have*

$$\|A\varphi\|_p \leq \|K\|_p \|\varphi\|_p$$

proof. Let $p \in]1,\infty[$,by applying the Holder's inequality we get

$$\begin{aligned} \int_\Omega \left(\int_\Omega |K(x,y)|\, |\varphi(y)|\, dy\right)^p dx &\leq \int_\Omega \left[\left(\int_\Omega |K(x,y)|^p\, dy\right)^{\frac{p}{q}} \|\varphi\|_p^p\right] dx \\ &= \|K\|_p^p\, \|\varphi\|_p^p \end{aligned}$$

then the operator A exists for almost every, with

$$\|A\varphi\|_p \leq \|K\|_p \|\varphi\|_p$$

Now we discuss the case when $p = 1$, and $p = \infty$ respectively

$$\int_\Omega\int_\Omega |K(x,y)|\, |\varphi(y)|\, dydx \leq \int_\Omega \sup_{y\in\Omega} ess\, |K(x,y)|\, dx \int_\Omega |\varphi(y)|\, dy$$

so

$$\|A\varphi\|_1 \leq \|K\|_1 \|\varphi\|_1$$

from another side

$$\begin{aligned}\sup_{x\in\Omega} ess\, |A\varphi(x)| &= \sup_{x\in\Omega} ess \left| \int_\Omega K(x,y)\varphi(y)dy \right| \\ &\leq \sup_{y\in\Omega} ess\, |\varphi(y)| \sup_{x\in\Omega} ess \int_\Omega |K(x,y)|\, dy\end{aligned}$$

then

$$\|A\varphi\|_\infty \leq \|K\|_\infty \|\varphi\|_\infty$$

■

1.6 The Nemytskii Operator

Nemytskii operators are a class of nonlinear operators on L^p spaces with good continuity and boundedness properties.

Definition 1.6.1 *One says that a function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies Carathéodory conditions if:*

(i) $f(.,y) : \Omega \to \mathbb{R}^n$ is measurable for every $y \in \mathbb{R}^m$;

(ii) $f(x,.) : \mathbb{R}^m \to \mathbb{R}^n$ is continuous for a.e. $x \in \Omega$.

Definition 1.6.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ be a given function. The Nemytskii operator N_f associated to f assigns to each function $u : \Omega \to \mathbb{R}^m$ the function $N_f(u) : \Omega \to \mathbb{R}^n$ defined by*

$$N_f(u)(x) = f(x, u(x))$$

This kind of functions has a lot of properties one of them is that maps $L^p(\Omega, \mathbb{R}^n)$ to $L^q(\Omega, \mathbb{R}^m)$ under some conditions (as we will see in the next paragraph), this property is very useful in our work.

Lemma 1.6.1 *If f satisfies the Carathéodory conditions then N_f maps measurable functions into measurable functions.*

proof. *(see* [36]*)* ■

Definition 1.6.3 *Let* $p \in [1, +\infty]$ *and* $q \in [1, +\infty)$ *A function* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ *is said to be* (p, q)*-Carathéodory if the following condition is satisfied:*

$$\begin{cases} (1) \text{ if } 1 \leq p < \infty \text{ then } |f(x,z)| \leq g(x) + c\,|z|^{p/q} \\ \text{for a.e } x \in \Omega, \text{ all } z \in \mathbb{R}^m \text{ and some } g \in L^q(\Omega; \mathbb{R}^+),\ c \in \mathbb{R}^+ \\ (2) \text{ if } p = \infty \text{ then for every } R > 0 \text{ there is a } g_R \in L^q(\Omega) \text{ with} \\ |f(x,z)| \leq g_R(x) \text{ for a.e } x \in \Omega \text{ and all } z \in R^m \text{ with } |z| \leq R \end{cases}$$

Theorem 1.6.1 *Let* $p \in [1, +\infty]$ *and* $q \in [1, +\infty)$ *Assume that the function* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ *is* $(p, q)-$*Carathéodory. Then the Nemytskii operator* $N_f : L^p(\Omega; \mathbb{R}^m) \to L^q(\Omega; \mathbb{R}^n)$ *associated to* f *is well defined, continuous and satisfies*

$$\begin{cases} (1) \text{ for } 1 \leq p < \infty: \ \|N_f(u)\|_{L^q(\Omega;\mathbb{R}^n)} \leq \|g\|_{L^q(\Omega)} + c\,\|u\|^{p/q}_{L^p(\Omega,\mathbb{R}^m)} \\ \text{for all } u \in L^p(\Omega, \mathbb{R}^m) \\ (2) \text{ for } p = \infty: \ \|N_f(u)\|_{L^q(\Omega;\mathbb{R}^n)} \leq \|g_R\|_{L^q(\Omega)} \\ \text{for all } u \in L^\infty(\Omega, \mathbb{R}^m) \text{ with } \|u\|_\infty \leq R \text{ and every } R > 0 \end{cases}$$

proof. (see [35]) ■

1.7 Fréchet derivative in Banach spaces

The Fréchet derivative is a derivative defined on Banach spaces. Named after Maurice Fréchet, it is commonly used to generalize the derivative of a real-valued function of a single real variable to the case of a vector-valued function of multiple real variables.

Definition 1.7.1 *Let* X, Y *two normed spaces,* U *an open subset of* X*, a mapping* $f : U \to Y$ *is said to be differentiable at* $u \in U$ *if there exists a linear map* $g \in L(X, Y)$ *such that*

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - g_x(h)\|}{\|h\|} = 0$$

If such linear map exists, so it's unique.

We call g_x the Fréchet derivative of f at x,and it will be denoted $Dg(x)$ or $g'(x)$ the element $g'(x)h$ is called the Fréchet differential or differential of f at x in the direction of $h \in X$.

Definition 1.7.2 *The map f is said to be differentiable on U, if it is differentiable at each point of U. In this case the mapping*

$$Df : x \in U \mapsto f'(x) \in L(X, Y)$$

is called the derivative (or differential) of f on U.

Theorem 1.7.1 *Let X, Y be normed spaces and U a non empty open subset of X if $f : U \to Y$ is differentiable at $x \in U$, then there exists $c > 0$ and $\delta > 0$ such that*

$$\|f(x) - f(y)\| \leq C \|x - y\|$$

for $y \in U$, $\|x - y\| \leq \delta$. In particular, it follows that f is continuous at x.

1.8 Liapunov functions

Liapunov functions is tools which describe a nonliear system,they are scalar functions that may be used to prove the stability of an equilibrium of an ODE, and they are suitable for integral equations for more information see [10].

Definition 1.8.1 *Consider the system $\dot{x} = f(x)$ and suppose that $x = 0$ is equilibrium (fixed point) and $\Omega \subset \mathbb{R}^n$ be a domain containing $x = 0$. suppose there exists a continuos differentiable function $V : \Omega \to \mathbb{R}$ such that*

1. *$V(0) = 0$ and $V(x) > 0$ in $\Omega \backslash \{0\}$*
2. *$\dot{V}(x) \leq 0$ in Ω*

Then the equilibrium point $x = 0$ is stable, further more if

$$\dot{V}(x) < 0 \text{ in } \Omega$$

Then the equilibrium point $x = 0$ is asymptotically stable.

Theorem 1.8.1 *(Liapunov's theorem). If there exists a differentiable function $V : \Omega_1 \to \mathbb{R}$, where Ω_1 an open neighborhood of 0, $\Omega_1 \subseteq \Omega$, such that $V(0) = 0$, $V(x) > 0$ for all $x \in \Omega_1 \backslash \{0\}$, and $\langle V'(x), f(x) \rangle < 0$ for all $x \in \Omega_1 \backslash \{0\}$ then asymptotic stability at 0 is guaranteed.*

Chapter 2

Applications theorems of fixed point

This chapter contains basic theory of fixed point we will recall some important and different theorems such that Banach fixed point theorem, Brouwer and Schauder fixed point theorems and The Leray-Schauder Principle, these theorems are very useful in the next chapter.

The theory of fixed point is concerned with the conditions which guarantee that a map $A : X \to X$ of a topological space X into itself admits one or more fixed points, that is , points φ of X for which $\varphi = A(\varphi)$

In this part we will discuss two types of fixed point result. the first type deals with contractions and are referred to Banach's fixed point theorems. the second one deals with compact mappings and is more involved. Names associated with such results are Brouwer and Schauder.

2.1 Banach fixed point theorem

First we look at the problem to find a fixed point for a continuous function in the spirit of Banach's fixed point theorem which is the simplest and the more versatile results in fixed point theory. Being based on an iteration process, it can be implemented on a computer to find a fixed point of a contractive map.

Theorem 2.1.1 *let A be a continuous mapping in Banach space X. Then the following statements hold true*

1- If there exist $\varphi, \psi \in X$ *such that*

$$\lim_{n\to\infty} A^n(\varphi) = \psi$$

then ψ *is a fixed point for* A*, i.e* $A(\psi) = \psi$.

2- If $A(X)$ *is a compact set in* X *and for each* $\epsilon > 0$ *there exists a* $\varphi_\epsilon \in X$ *such that*

$$\|A(\varphi_\epsilon) - \varphi_\epsilon\| < \epsilon$$

then A *has a fixed point.*

proof. *Let* $\psi_n = T^n(\varphi)$*,* $n = 1, 2, ...$ *If* A *is a continuous mapping then*

$$A(\psi) = A(\lim_{n\to\infty} \psi_n) = \lim_{n\to\infty} A(\psi_n) = \lim_{n\to\infty} \psi_{n+1} = \psi \tag{2.1.1}$$

which proves the first statement.

Assume that the assumptions of (2) are satisfied. Then for $n = 1, 2, ...$ *there are* $\varphi_n \in X$ *such that*

$$\|A(\varphi_\epsilon) - \varphi_\epsilon\| < \frac{1}{n}$$

$A(X)$ *is a compact set implies that there exits a convergent subsequence* $(A(\varphi_{n_k}))_{k=1}^{\infty}$ *of* $(A(\varphi_n))_{n=1}^{\infty}$*.Call the limit point* φ*. Then* φ *is a fixed point for* A *since also the sequence* $(\varphi_{n_k})_{k=1}^{\infty}$ *converges to* x *according to (1) and* A *is continuous.* ■

We now formulate one of the main theorems.

Definition 2.1.1 *Let* (X, d) *be a metric space. A mapping* $A : X \to X$ *is a contraction mapping or contraction, if there exists a constant* c *with* $0 < c < 1$*, such that*

$$d(A(\varphi), A(\psi) \leq cd(\varphi, \psi)$$

for all $\varphi, \psi \in X$.

Theorem 2.1.2 *(Banach's fixed point theorem) Let* A *be a contraction on a Banach space* X*. Then* A *has a unique fixed point.*

proof. Fix an arbitrary element $\varphi \in X$ and consider the sequence $(A_n(z))_{n=1}^{\infty}$. Let $\varphi_n = A^n(\varphi)$ for $n = 1, 2, \ldots$ We note that

$$\begin{aligned}
\|\varphi_n - \varphi_m\| &\leq \left\|\varphi_n - \varphi_{n-1}\right\| + \ldots + \left\|\varphi_{m+1} - \varphi_m\right\| \\
&= \left\|A(\varphi_{n-1}) - A(\varphi_{n-2})\right\| + \ldots + \left\|A(\varphi_m) - A(\varphi_{m-1})\right\| \\
&\leq c\left\|\varphi_{n-1} - \varphi_{n-2}\right\| + \ldots + c\left\|\varphi_m - \varphi_{m-1}\right\| \leq \ldots \\
&\leq (c^{n-1} + c^{n-2} + \ldots + c^{m-1})\,\|\varphi_1 - \varphi\| \leq \frac{c^{m-1}}{1-c}\,\|\varphi_1 - \varphi\|
\end{aligned}$$

where we (without loss of generality) have assumed $n > m \geq 1$. This yields $\|\varphi_n - \varphi_m\| \to 0$ as $n, m \to \infty$ and hence $(\varphi_n)_{n=1}^{\infty}$is a Cauchy sequence. Since X is a Banach space the sequence converges, i.e. there is a $\varphi_0 \in X$ such that $\varphi_n \to \varphi_0$ as $n \to \infty$. Here φ_0 is a fixed point for A since

$$\begin{aligned}
\|A(\varphi_0) - \varphi_0\| &\leq \|A(\varphi_0) - A(\varphi_n)\| + \left\|\varphi_{n+1} - \varphi_0\right\| \\
&\leq c\,\|\varphi_0 - \varphi_n\| + \left\|\varphi_{n+1} - \varphi_0\right\|
\end{aligned}$$

where the LHS is independent of and the RHS tends to 0 as $n \to \infty$.

The uniqueness follows from the contraction property for A. If $\varphi_0 \neq \psi_0$ both are fixed points of A then we get

$$\begin{aligned}
\|\varphi_0 - \psi_0\| &= \|A(\varphi_0) - A(\psi_0)\| \\
&\leq c\,\|\varphi_0 - \psi_0\| \\
&< \|\varphi_0 - \psi_0\|
\end{aligned}$$

which yields a contradiction. ■

From the proof it follows that

1- the sequence $(A_n(\varphi))_{n=1}^{\infty}$ converges to the unique fixed point independently of the choice of φ.

2- for an arbitrary element $\varphi \in X$ we have

$$\|\varphi - \varphi_0\| \leq \frac{1}{1-c}\,\|\varphi - A(\varphi)\|$$

where φ_0 denotes the fixed point of A, since

$$\begin{aligned}
\|\varphi - \varphi_0\| &\leq \|\varphi - A(\varphi)\| + \|A(\varphi) - A(\varphi_0)\| \\
&\leq \|\varphi - A(\varphi)\| + c\,\|\varphi - \varphi_0\|
\end{aligned}$$

Banach's fixed point theorem can be generalized in the following way.

Theorem 2.1.3 *Let A be a mapping on a Banach space X such that A^N is a contraction on X for some positive integer N. Then A has a unique fixed point (it is not necessary to assume that A is continuous.).*

proof. Banach's fixed point theorem implies that there exists a unique fixed point for A^N. Call this element φ_0. Now just note that

$$\begin{aligned} \|A(\varphi_0) - \varphi_0\| &= \left\|A^N(A(\varphi_0)) - A^N(\varphi_0)\right\| \\ &\leq c\,\|A(\varphi_0) - \varphi_0\| \end{aligned}$$

implies that $A(\varphi_0) = \varphi_0$ since $0 < c < 1$. The uniqueness is clear since a fixed point for A is also a fixed point for A_N. ■

Corollary 2.1.1 *Let X be a Banach space and $B = B(\varphi_0, r) = \{\varphi / \ \|\varphi - \varphi_0\| < r\}$. Let $A : B \to X$ be contractive with constant $c < 1$. If $\|A(\varphi_0) - \varphi_0\| < (1-c)r$ then A has a fixed point.*

proof. choose $\epsilon < r$ so that $\|A(\varphi_0) - \varphi_0\| \leq (1-c)\epsilon < (1-c)r$. We show that F maps the closed ball $\bar{B}_\epsilon = \{\varphi / \|\varphi - \varphi_0\| \leq \epsilon\}$ into itself. For if $\varphi \in \bar{B}_\epsilon$ then

$$\begin{aligned} \|A(\varphi) - \varphi_0\| &\leq \|A(\varphi) - A(\varphi_0)\| + \|A(\varphi) - \varphi_0\| \\ &\leq \alpha\,\|\varphi - \varphi_0\| + (1-c)\epsilon \leq \epsilon \end{aligned}$$

Since $\bar{B}_\epsilon$ is complete, the conclusion follows from Banach's principle. ■

2.1.1 Extension of the Banach theorem

In this paragraph, we'll study an extension of the Banach theorem in metric spaces.

Theorem 2.1.4 *Let (X, d) be complete metric space and $A : X \to X$ a map not necessarily continuous. Assume*

$$\begin{aligned} \textit{for each } \epsilon &> 0 \textit{there is a } \delta(\epsilon) > 0 \textit{ such that} \\ \textit{if } d(\varphi, A(\varphi)) &< \delta, \textit{ then } A(B(\varphi, \delta)) \subset B(\varphi, \epsilon) \end{aligned}$$

then if $d(A^n(\varphi), A^{n+1}(\varphi)) \to 0$ *for some* $\varphi \in X$ *the sequence* $(A^n(\varphi))$ *converges to a fixed point for* A.

Theorem 2.1.5 *Let* (X, d) *be complete metric space and let*

$$d(A(x), A(y)) \leq \varphi(d(x, y))$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *in any monotone non decreasing (not necessarily continuous) function such that* $\varphi^n(t) \to 0$ *for each fixed point* $t > 0$ *then* A *has a unique fixed point* y *and* $A^n(x) \to y$ *for each* $x \in X$.

Theorem 2.1.6 *let* (X, d) *be complete metric space and let*

$$d(A(\varphi), A(\psi)) \leq \alpha(\varphi, \psi) d(\varphi, \psi)$$

where $\alpha : X \times X \to \mathbb{R}^+$ *has the property: for any closed interval* $[a, b] \subset \mathbb{R}^+ - \{0\}$,

$$\begin{aligned} \sup\{\alpha(\varphi, \psi) \backslash a &\leq d(\varphi, \psi) \leq b\} \\ &= \lambda(a, b) < 1 \end{aligned}$$

Then A *has a fixed point* y *and* $A^n(\varphi) \to y$ *for each* $\varphi \in X$.

Theorem 2.1.7 *Let* (X, d) *be complete and* $f : X \to \mathbb{R}^+$ *an arbitrary (not necessarily continuous) non-negative function. Assume that*

$$\inf\{f(\varphi) + f(\psi) \backslash\ d(\varphi, \psi) > a\} = \lambda(a) > 0 \quad \textit{for all } a > 0 \tag{2.1.2}$$

Then each sequence (φ_n) *in* X *for which* $f(\varphi_n) \to 0$ *converges to one and the same point* $\varphi \in X$

Theorem 2.1.8 *Let* (X, d) *be complete and* $A : X \to X$ *continuous. assume* $f(\varphi)$ *has the property* (2.1.2) *and that* $\inf d(\varphi, A(\varphi)) = 0$. *Then* A *has a unique fixed point*

2.2 Brouwer and Schauder fixed point theorems

We mention first Brouwer fixed point theorem

Theorem 2.2.1 *(Brouwer's fixed point theorem) Assume that* Ω *is a compact convex subset of* $\mathbb{R}^n$ *and that* $A : \Omega \to \Omega$ *is a continuous mapping. Then* A *has a fixed point in* Ω*.*

There are many proofs for Brouwer's fixed point theorem, both analytical and topological. We just sketch one proof (The beautiful one that I found!).

proof. it is enough to prove Brouwer fixed point theorem in the case $\Omega = \overline{B(0,1)}$. Assume that $\Omega = \overline{B(0,1)}$ and that A has no fixed point. Define the mapping $S : \overline{B(0,1)} \to \overline{B(0,1)}$ as follows: For every inner point φ in $\overline{B(0,1)}$ let $\tilde{\varphi}$ denote the point on the boundary $\partial B(0,1)$ that is the intersection of the ray from $A(\varphi)$ through φ and the boundary $\partial B(0,1)$. The ray is always well-defined since A has no fixed point. Now let

$$S(\varphi) = \begin{cases} \tilde{\varphi} \text{ if } \varphi \in B(0,1) \\ \varphi \text{ if } \varphi \in \partial B(0,1) \end{cases}$$

Then A is a continuous mapping from $\overline{B(0,1)}$into $\partial B(0,1)$ such that $S|_{\partial B(0,1)} = I|_{\partial B(0,1)}$. The problem to show that A has no fixed point is now reformulated as to show that there is no continuous mapping $S : B(0,1) \to \partial B(0,1)$ such that $S|_{\partial B(0,1)} = I|_{\partial B(0,1)}$. The statement that there is no such mapping is deep but never the less intuitively obvious.

Consider, for $n = 2$, the case with an elastic membrane fixed on a circular frame. The existence of a mapping S implies that it should be possible to deform the membrane continuously in such a way that it should coincide with the frame without being fractured. For fixed $\varphi \in B(0,1)$ the mapping

$$t \mapsto (1-t)\varphi + tS(\varphi), t \in [0,1]$$

describes how this point on the membrane is moved from φ at $t = 0$ to $A(x) \in \partial B(0,1)$ at $t = 1$, under the deformation. Do not forget that the membrane should be fixed at the frame!!! ∎

Theorem 2.2.2 *(generalization of Brouwer's fixed point theorem) If there exists a homeomorphism, i.e. a continuous bijection with continuous inverse, between a compact convex set*

Ω in R^n and a set $\tilde{\Omega}$, call the homeomorphism ϕ, and $\tilde{A} : \tilde{\Omega} \to \tilde{\Omega}$ is a continuous mapping then $\tilde{A}$ has a fixed point.

The next lemma is Brouwer's theorem for mappings of class C^∞.

Lemma 2.2.1 *Let $A : B \to B$, where $B = \bar{B}_1(0, \mathbb{R}^n)$, be a C^∞ function. Then A has at least one fixed point.*

proof. *(see* [36]*)* ■

In both Brouwer's fixed point theorems we consider a finite-dimentional normed space where we have the equivalence between compactness from side and closedness and boundedness from another side. This is not the case in an infinite-dimentional normed space.

The next example due to Kakutani shows that generalization of Brouwer's fixed point theorem is not available in infinite-dimensional spaces.

Example 2.2.1 *Let B denote the closed unit ball in $l^2(\mathbb{Z})$, where $l^2(\mathbb{Z})$ consists of all elements $x = (..., x_{-1}, x_0, x_1...)$ such that $\|x\| = (\sum\limits_{-\infty}^{+\infty} |x_n|^2)^{1/2} < \infty$. It is clear that B is convex and bounded. Let z be the element in $l^2(Z)$ that satisfies $z_0 = 1$ and $z_n = 0$ for $n \neq 0$ and let S denote the shift operator defined by $(S(x))_n = x_{n-1}$ for $n \in Z$. Now set*

$$A : l_2(Z) \to l_2(Z),$$

where

$$A(x) = S(x) + (1 - \|x\|)z$$

for $x \in B$, i.e. $A(x) \in B$. But A has no fixed point in B since

$$(A(x))_n = x_{n-1}, \ n \neq 0$$

and

$$(A(x))_0 = x_{-1} + (1 - \|x\|),$$

which implies that $x_0 = x_1 = ... = x_n = ...$ and $x_{-1} = x_{-2} = ... = x_{-n} = ...$ This yields a contradiction since $x \in l^2(Z)$.

From this example we see that a generalization of Brouwer's fixed point theorem to infinite-dimensional spaces should have the assumption that $A(\Omega)$ is a compact set. We next formulate versions of Schauder's fixed point theorem. We shall start with Schauder-tychonoff fixed point theorem.

Theorem 2.2.3 *Let X be a locally convex Hausdorff space and assume $A : \Omega \to X$ is a continuous map with $\Omega \subset X$ convex and*

$$A(\Omega) \subset E \subset \Omega$$

where E is compact. Then there exist at least one fixed point for A.

The proof of this theorem is rather lengthy is can be found in a lot of books.

Theorem 2.2.4 *(Schauder's fixed point theorem) Assume that Ω is a convex compact set in a Banach space X and that $A : \Omega \to \Omega$ is a continuous mapping. Then A has a fixed point.*

For applications the following generalization proves to be useful.

Theorem 2.2.5 *(generalization of Schauder's fixed point theorem). Let Ω be a closed convex set in a Banach space X and assume that $A : \Omega \to \Omega$ is a continuous mapping such that $A(\Omega)$ is a relatively compact subset of Ω. Then A has a fixed point.*

To prove Schauder's fixed point theorem we will use Hausdorff's Theorem, in addition to the following proposition.

Definition 2.2.1 *We say that the convex hull of a set F is the set, denoted by coF, that is defined by*

$$\bigcap_{F\subset H \ ,H \ convex} H$$

By a convex combination of the elements $x_1, x_2, \ldots\ , x_n$ we mean a linear combination $\sum_{i=1}^{n} \lambda_i x_i$, where all $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

Proposition 2.2.1 . *The following statements are true:*

1. A set F is relatively compact iff for each $\epsilon > 0$ there exists a finite ϵ–net.

2. A set K is compact iff it is closed and for every $\epsilon > 0$ there exists a finite ϵ–net.

3. The set coF is the same as the set of all convex combination of finitely many elements in F.

4. K compact set implies that coK is compact.

proof. (of the Schauder theorems) The second Schauder theorem is a consequence of the first one. To see this assume that the hypothesis of the second theorem are satisfied. It then follows that the closed hull $\bar{R}$ of $R = A(F)$ is compact and so also $co\ \bar{R}$. Set $K = co\ \bar{R}$. We see that $K \subset F$ since F is closed and convex. Moreover $A : K \to K$ is continuous. Hence the second theorem follows from the first theorem.

It remains to prove the first theorem. This will be done by approximating the compact set K by compact sets $K_n, n = 1, 2, ...$ in finite-dimensional spaces and approximating the mapping A by continuous mappings $A_n : K_n \to K_n$, where the approximation becomes better and better for larger n. Brouwer's fixed point theorem gives a sequence of points (x_n) that are fixed points for the sequence (A_n), from which a converging subsequence of points (x_{n_k}) can be extracted. The limit element of this sequence will be a fixed point for A.

For every positive integer n we define mappings P_n, called Schauder projections, as follows: The compactness of K implies that there are finitely many elements $x_1, ..., x_k \in K$ such that

$$K \subset \bigcup_{i=1}^{k} B(x_i, \frac{1}{n})$$

Set

$$f(x_i) = \max(0, \frac{1}{n} - \|x - x_i\|), \quad i = 1, 2, ..., k$$

For every $x \in K$ there exists an i such that $f_i(x) > 0$. This implies that $\sum\limits_{i=1}^{k} f_i(x) > 0$ for all $x \in K$. Now set $K_n = co\{x_1, ..., x_k\}$ and

$$p_n(x) = \frac{\sum\limits_{i=1}^{k} f_i(x) x_i}{\sum\limits_{i=1}^{k} f_i(x)}, \quad x \in K$$

Finally we define $A_n = P_n T | K_n$. We can now apply Brouwer's theorem to every mapping

$$\|A(x_n) - x_n\| < \frac{1}{n}$$

So Schauder's theorem is proved. ■

Now we mention some additional fixed point theorems. The first one, Schaefer's fixed point theorem, is a version of Schauder's theorem and is an example of the mathematical principle saying "priory estimates implies existence". The second one, Krasnoselskii's fixed point theorem, is a mixture of Banach's and Schauder's fixed point theorems.

Theorem 2.2.6 *(Schaefer's fixed point theorem). Assume that X is a Banach space and that $A : X \to X$ is a continuous compact mapping. Moreover assume that the set*

$$\bigcup_{0 \leq \lambda \leq 1} \{x \in X : x = \lambda A(x)\}$$

is bounded then A has a fixed point.

proof. *Assume that the mapping A satisfies the hypothesis in the theorem. Pick a constant $R > 0$ such that*

$$x = \lambda A(x) \text{ and } 0 \leq \lambda \leq 1$$

implies that

$$\|x\| < R$$

Define the mapping $\tilde{A} : X \to X$ as follows

$$\tilde{A}(x) = \begin{cases} A(x), & \text{if } \|A(x)\| \leq R \\ \dfrac{R}{\|A(x)\|} T(x), & \text{if } \|A(x)\| > R \end{cases}$$

This implies that $\tilde{A} : X \to X$ is a compact operator. To show this take a bounded sequence $(x_n)_{n=1}^{\infty}$ in X. Then there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $\|A(x_{n_k})\| < R$ for all k or $\|A(x_{n_k})\| \geq R$ for all k. In the first case $(\tilde{A}(x_{n_k}))_{k=1}^{\infty}$ has a convergent subsequence since $\tilde{A}(x_{n_k}) = A(x_{n_k})$ and A is a compact mapping.

In the second case we get that $(A(x_{n_k}))_{k=1}^{\infty}$ has a convergent subsequence, denote it by $(A(x_l))_{l=1}^{\infty}$ for convenience. But then it follows that also $(\|A(x_l)\|)_{l=1}^{\infty}$ converges, where also $\|A(x_l)\| \geq R$ for all l. Hence we obtain $\tilde{A}(x_l) = \dfrac{R}{\|A(x)\|} A(x_l)$

Set

$$K = co\overline{\tilde{T}(B(0,R))}$$

Here K is convex (it is the convex hull of a set), compact (the convex hull of a compact set is compact and $\tilde{A}$ is a compact mapping) subset of X such that.

$$\tilde{A} : K \to K$$

Schauder's fixed point theorem implies that $\tilde{A}$ has a fixed point $x_0 \in K$. But x_0 is a fixed point for A if $\|A(x_0)\| \leq R$. Assume that $\|A(x_0)\| > R$. This yields a contradiction since $x_0 = \tilde{A}(x_0) = \lambda(x_0)$, where $\lambda = \dfrac{R}{\|T(x_0)\|} \in (0,1)$, since according to the hypothesis of the theorem it should follow that $\|A(x_0)\| = \|x_0\| < R$. This proves the theorem. ■

In particular, note that to apply Schaefer's theorem we do not need to prove that a certain set is convex or compact. The problem is reformulated as to show certain a priory estimates for the operator A.

Theorem 2.2.7 *(Krasnoselskii's fixed point theorem). Assume that F is a closed bounded convex subset of a Banach space X. Furthermore assume that A_1 and A_2 are mappings from F into X such that*

1. *$A_1(x) + A_2(y) \in F$ for all $x, y \in F$,*
2. *A_1 is a contraction,*
3. *A_2 is continuous and compact.*

Then $A_1 + A_2$ has a fixed point in F

proof. *Assume that the mappings A_1, A_2 satisfies the hypothesis of the theorem. In particular there exists a constant $c \in]0,1[$ such that*

$$\|A_1(x) - A_1(y)\| \leq c\,\|x - y\|, \quad x, y \in F.$$

This yields

$$\begin{aligned} \|(I - A_1)(x) - (I - A_1)(z)\| &\geq \|x - z\| - \|A_1(x) - A_1(z)\| \\ &\geq (1 - c)\,\|x - z\| \end{aligned}$$

and

$$\begin{aligned}\|(I-A_1)(x)-(I-A_1)(z)\| &\leq \|x-z\|+\|A_1(x)-A_1(z)\| \\ &\leq (1+c)\|x-z\|\end{aligned}$$

Consequently $I-A_1 : F \to (I-A_1)(F)$ *is a homeomorphism, and* $(I-A_1)^{-1}$ *exists as a continuous mapping from* $(I-A_1)(F)$*. Furthermore we note that for each* $y \in F$ *the equation*

$$x = A_1(x) + A_2(y)$$

has a unique solution $x \in F$ *according to Banach's fixed point theorem. From this we conclude that* $A_2(y) \in (I-A_1)(F)$ *for every* $y \in F$ *and also that* $(I-A_1)^{-1}A_2 : F \to F$ *is a well-defined continuous mapping. Since* A_2 *is a compact mapping it follows that* $(I-A_1)^{-1}A_2 : F \to F$ *is a compact mapping. Finally the generalization of Schauder's fixed point theorem yields the conclusion of the theorem.* ■

2.3 The Leray-Schauder Principle

In applications one of the drawbacks of Schauder's fixed point theorem is the invariance condition $A(\Omega) \subset \Omega$ which has to be guaranteed for a bounded closed convex subset Ω of a Banach space. The Leray-Schauder principle makes it possible to avoid such a condition and requires instead that a 'boundary condition' is satisfied

Theorem 2.3.1 *(Leray-Schauder) Let* X *be a Banach space,* $\Omega \subset X$ *a closed convex subset,* $U \subset \Omega$ *a bounded set, open in* Ω *and* $\varphi_0 \in U$ *a fixed element. Assume that the operator* $A : \bar{U} \to \Omega$ *is completely continuous and satisfies the boundary condition*

$$\begin{aligned}\varphi &\neq (1-\lambda)\varphi_0 + \lambda T(\varphi), \\ \textit{for all}\ \varphi &\in \delta U, \quad \lambda \in]0,1[\end{aligned} \tag{2.3.1}$$

Then A *has at least one fixed point in* $\bar{U}$*.*

proof. *For the proof we use Granas' fixed point approach Notice the property of* U *of being open as well as the boundary* ∂U *are understood with respect to the topology of* Ω*. We* *may assume that* (2.3.1) *holds on a* ∂U *for all* $\lambda \in [0,1]$. *Indeed, this is obvious for* $\lambda = 0$ *since* $\varphi_0 \in U$, *whilst if* (2.3.1) *does not hold for* $A = 1$, *then the theorem is proved.*

Let

$$S = \{\varphi \in \bar{U} : = (1-\lambda)\varphi_0 + \lambda A(\varphi) \text{ for some } \lambda \in [0,1]\}$$

Obviously S *is non empty (since* $\varphi_0 \in S$*), closed, and* $S \cap \partial U = \emptyset$. *By Urysohn's lemma, there exists a function* $\varphi \in C(\bar{U},[0,1])$ *such that*

$$\phi(\varphi) = \begin{cases} 0 & \text{if } \varphi \in \partial U \\ 1 & \text{if } \varphi \in S \end{cases}$$

We now define the operator $\tilde{A} : \Omega \to \Omega$ *by*

$$\tilde{A}(\varphi) = \begin{cases} (1-\phi(\varphi))\varphi_0 + \phi(\varphi)A(\varphi) & \text{for } \varphi \in U \\ \varphi_0 & \text{for } \varphi \in \Omega \backslash U \end{cases}$$

It is clear that $\tilde{A}$ *is continuous and*

$$A(K) \subset conv\ (\{\varphi_0 \cup A(\bar{U})\})$$

Since A *is completely continuous,* $A(\bar{U})$ *is relatively compact. Hence by ;Mazur's lemma the following subset of* Ω,

$$D = \overline{conv}\ (\{\varphi_0 \cup T(\bar{U})\})$$

is convex and compact. In addition

$$\tilde{A}(D) \in D$$

Hence Schauder's fixed point theorem applies and guarantees the existence of $\varphi \in D$ *with* $\tilde{T}(\varphi) = \varphi$. *By definition of* $\tilde{T}$, φ *must lies to* U. *Then*

$$\varphi = (1-\phi(\varphi))\varphi_0 + \phi(\varphi)T(\varphi)$$

This shows that $\varphi \in S$ *and so* $\phi(\varphi) = 1$. *As a result,* $\varphi = A(\varphi)$. ■

Notice that the essential idea of the Leray-Schauder principle consists in joining the operator A to the constant operator φ_0 by means of the homotopy $H : \bar{U} \times [0,1] \to \Omega$

$$H(\varphi,\lambda) = (1-\lambda)\varphi_0 + \lambda A(\varphi)$$

in a such way that the unique fixed point of $H(.,0)$, namely φ_0, can be 'continued' in a fixed point of $H(.,\lambda)$ for each $\lambda \in [0,1[$ and, in particular, in a fixed point of $H(.,1) = A$. This continuation process is possible if all operators $H(.,\lambda)$ for $\lambda \in [0,1]$ are fixed point free on the boundary of U.

From above we conclude that the difference between applying Schauder's theorem and applying Banach's theorem, namely to apply Banach's theorem we have to show that a mapping is sufficiently small, while to apply Schauder's theorem we have to prove that a mapping is compact. This means that, in $C(\Omega)$ or $L^p(\Omega)$ case, we have to show that the image set for the mapping consists of more "regular" functions.

Chapter 3

Introduction to the theory of integral equations Theory of Integral Equations

In this chapter we present integral equations and we illustrate different criterions of classification of these equations, also we discuss existence of solutions of some kind of integral equations, and finally we describe the conversion process of initial value problems to Volterra integral equations

Integral equation is the equation in which the unknown function $\varphi(x)$ appears inside an integral sign. The most standard type of integral equation in $\varphi(x)$ is of the form

$$\varphi(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x, y, \varphi(y))dy$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x, y)$ is a known function, of two variables x and y, called the kernel or the nucleus of the integral equation. The unknown function $\varphi(x)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $\varphi(x)$ appears inside and outside the integral sign. The functions $f(x)$ and $K(x, y)$ are given in advance. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed. Integral equations appear in many forms.

The classification of integral equations it depends on many characteristics

-The first one in the linearity of the kernel $K(x,y,\varphi(y))$ with respect to the third variable

1.If $K(x,y,\varphi(y))$ is linear with respect to the third variable i.e

$$K(x,y,\varphi(y)) = K(x,y)\varphi(y)$$

the integral equation is called linear equation

2.If $K(x,y,\varphi(y))$ is nonlinear with respect to the third variable i.e if the equation contains nonlinear functions of $\varphi(x)$ the integral equation is called nonlinear equation.

in this case we find two form given by

$$\begin{aligned}\varphi(x) &= f(x) + \lambda \int_\Omega K(x,y,\varphi(y))dy \\ \varphi(x) &= f(x) + \lambda \int_\Omega K(x,y)G(y,\varphi(y))dy\end{aligned}$$

namely Urysohn form and Hammerstein form respectively.

-Two other distinct ways that depend on the limits of integration are used to characterize integral equations, namely:

1. If the limits of integration are fixed, the integral equation is called a Fredholm integral equation given in the form:

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,y)\varphi(y)dy$$

where a and b are constants.

2. If at least one limit is a variable, the equation is called a Volterra integral equation given in the form:

$$\varphi(x) = f(x) + \lambda \int_a^x K(x,y)\varphi(y)dy$$

-Moreover, two other distinct kinds, that depend on the appearance of the unknown function $\varphi(x)$, are defined as follows:

1. If the unknown function $\varphi(x)$ appears only under the integral sign of Fredholm or Volterra equation, the integral equation is called a first kind Fredholm or Volterra integral equation respectively.

2. If the unknown function $\varphi(x)$ appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called a second kind Fredholm or Volterra equation integral equation respectively.

In all Fredholm or Volterra integral equations presented above, if $f(x)$ is identically zero, the resulting equation:

$$\begin{aligned}\varphi(x) &= \int_a^b K(x,y)\varphi(y)dy \\ \varphi(x) &= \int_a^x K(x,y)\varphi(y)dy\end{aligned}$$

is called homogeneous Fredholm or homogeneous Volterra integral equation respectively.

-The last one we should mention is the singularity of the kernel, we will give more details about it in the next paragraph.

3.1 Classification of Integral Equations

Integral equations appear in many types. The types depend mainly on the characteristics which we had been spoke about it. In this text we will classify just the linear equation and it is the same thing for the nonlinear one.

3.1.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $\varphi(x)$ may appear only inside integral equation in the form:

$$f(x) = \int_a^b K(x,y)\varphi(y)dy$$

This is called Fredholm integral equation of the first kind. However, for Fredholm integral equations of the second kind, the unknown function $\varphi(x)$ appears inside and outside the integral sign. The second kind is represented by

the form:

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,y)\varphi(y)dy$$

3.1.2 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the first kind Volterra integral equations, the unknown function $\varphi(x)$ appears only inside integral sign in the form:

$$f(x) = \int_0^x K(x, y)\varphi(y)dy$$

However, Volterra integral equations of the second kind, the unknown function $\varphi(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$\varphi(x) = f(x) + \lambda \int_0^x K(x, y)\varphi(y)dy$$

3.1.3 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modelling of the spatiotemporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$\begin{aligned}\varphi(x) &= f(x) + \lambda_1 \int_a^x K_1(x, y)\varphi(y)dy \\ &\quad +\lambda_2 \int_0^x K_2(x, y)\varphi(y)dy\end{aligned}$$

and

$$u(x, y) = f(x, y) + \lambda \int_0^x \int_\Omega F(x, y, \zeta, \tau, \varphi(\zeta, \tau))d\zeta d\tau, \quad (x, y) \in \Omega \times [0, X]$$

where $f(x, y)$ and $F(x, y, \zeta, \tau, \varphi(\zeta, \tau))$ are analytic functions on $D = \Omega \times [0, X]$, and Ω is a closed subset of $R^n, n = 1, 2, 3$. It is interesting to note that the first equation contains disjoint Volterra and Fredholm integral equations, whereas the second one contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$ and $u(x, t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of the first kind.

3.1.4 Singular Integral Equations

Volterra integral equations of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x, y)\varphi(y)dy$$

or of the second kind

$$\varphi(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,y)\varphi(y)dy$$

are called singular if one of the limits of integration $g(x), h(x)$ or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x,y)$ becomes unbounded at one or more points in the interval of integration, the most famous form is

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}\varphi(y)dy \qquad 0 < \alpha < 1$$

which is of the first kind and it's called generalized Abel's integral equation, the equation of the second kind is given by:

$$\varphi(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha}\varphi(y)dy \qquad 0 < \alpha < 1$$

is called weakly singular integral equations.

In the generalized Abel's integral equation, if $\alpha = \frac{1}{2}$ the equation becomes the Abel's singular integral equation. The singularity of these kind of equations occur from the upper limit when $t = x$ it becomes infinity.

3.2 Existence solutions of integral equations

To prove the existence of solutions of linear integral equations, we will apply Riesz theory and Fredholm alternative, the first two corollaries are direct results for linear equations.

Corollary 3.2.1 *Let A a compact operator of a normed space X into itself, for $\lambda \neq 0$ the nonhomogeneous equation*

$$T\varphi = \varphi - \lambda A\varphi = f$$

has a unique solution $\varphi \in X$, for all $f \in X$, if and only if the homogeneous equation

$$T\varphi = \varphi - \lambda A\varphi = 0$$

has the trivial solution $\varphi = 0$.

proof. Indeed, if the first equation has a solution for all $f \in X$, then T is surjective, as a result T injective what proves that the second equation has the unique solution $\varphi = 0$, and we complete in the same manner the other direction. ■

Furthermore if $\varphi = 0$ is not a solution for the homogeneous equation, then the homogeneous equation has a finite number $m \in N$ of solutions linearly independent, in this case either the nonhomogeneous equation is unsolvable, or has a solution given in the form

$$\varphi = \tilde{\varphi} + \sum_{i=1}^{m} \alpha_i \varphi_i.$$

where $\alpha_1, \alpha_2, ..., \alpha_m$ are arbitrary complex numbers and $\tilde{\varphi}$ a particular solution of the inhomogeneous equation.

Corollary 3.2.2 *Let $\Omega \subset R^n$, and let $K(x, y)$ a continuous function. Then*

either *homogeneous integral equations*

$$\begin{aligned} \varphi(x) - \int_\Omega K(x,y)\varphi(y)dy &= 0, \quad x \in \Omega \\ \psi(x) - \int_\Omega K(y,x)\psi(y)dy &= 0, \quad x \in \Omega \end{aligned}$$

have only the trivial solution $\varphi = 0$ and $\psi = 0$ and in this case the nonhomogeneous equations

$$\begin{aligned} \varphi(x) - \int_\Omega K(x,y)\varphi(y)dy &= f(x), \quad x \in \Omega \\ \psi(x) - \int_\Omega K(y,x)\psi(y)dy &= g(x), \quad x \in \Omega \end{aligned}$$

have a unique solution $\varphi \in C(\Omega)$ and $\psi \in C(\Omega)$ respectively for any $f \in C(\Omega)$ and $g \in C(\Omega)$,

or *the homogeneous integral equations have the same finite number $m \in N$ of solutions linearly independent, and in this case the nonhomogeneous integral equations are solvable if and only if*

$$\int_\Omega f(x)\psi(x)dx = \int_\Omega g(x)\varphi(x)dx = 0$$

for all ψ solution of the adjoint homogeneous equation and for all φ solution of the homogeneous equation.

Now we consider the nonlinear integral equation, in the next theorem we well use the fixed point theory to prove that nonlinear integral equation of the 2nd kind with bounded kernel has a unique solution for sufficiently small $|\lambda|$.

Theorem 3.2.1 *Consider nonlinear integral equation of the second kind*

$$\varphi(x) - \lambda \int_\Omega K(x, y, \varphi(y))dy = f(x), \quad x \in \Omega \tag{3.2.1}$$

such that A is a bounded integral operator and satisfies the Lipschitz condition

$$\|A\varphi_1 - A\varphi_2\| \leq c \|\varphi_1 - \varphi_2\|, \quad c \geq 0$$

with

$$|\lambda| c < 1$$

then (3.2.1) *has a unique solution.*

proof. Rewrite the nonlinear integral equation of the second kind in the form

$$\varphi = T\varphi \tag{3.2.2}$$

with

$$T\varphi = \lambda A\varphi + f$$

so

$$\begin{aligned} \|T\varphi_2 - T\varphi_2\| &= \|\lambda A\varphi_2 + f - (\lambda A\varphi_1 + f)\| \\ &= \|\lambda A\varphi_2 - \lambda A\varphi_1\| \\ &= |\lambda| \|A\varphi_2 - A\varphi_1\| \\ &\leq |\lambda| c \|\varphi_2 - \varphi_1\| \end{aligned}$$

when $|\lambda| c < 1$, the operator T is a contraction and according to Banach fixed point theorem , there exist a unique fixed point of equation (3.2.2). This unique fixed point is also a solution of the nonlinear integral equation. ■

3.3 Some Methods to solve Integral Equations

3.3.1 Method of Successive Approximations for Integral Equations (Neumann series)

Theorem 3.3.1 *Let* A *a linear bounded operator on a Banach space* X *into itsef, such that* $\|\lambda A\| < 1$. *Then* $A_\lambda = I - \lambda A$ *has an inverse bounded operator, given by*

$$(I - \lambda A)^{-1} = \sum_{k=0}^{\infty} \lambda^k A^k$$

which called to be Neumann series, furthermore

$$\left\|(I - \lambda A)^{-1}\right\| \leq \frac{1}{1 - \|\lambda A\|}$$

proof. From the condition$\|\lambda A\| < 1$ we can assume that

$$\sum_{k=0}^{\infty} \left\|\lambda^k A^k\right\| \leq \sum_{k=0}^{\infty} \|\lambda A\|^k = \frac{1}{1 - \|\lambda A\|} < \infty$$

as $L(X)$ is a Banach space then there exists a bounded linear operator S such that

$$S = \sum_{k=0}^{\infty} \lambda^k A^k$$

with

$$\|S\| \leq (1 - \|\lambda A\|)^{-1}$$

this operator is the inverse of $(I - \lambda A)$, indeed

$$\begin{aligned} S(I - \lambda A) &= \lim_{n\to\infty} \sum_{k=0}^{n} \lambda^k A^k (I - \lambda A) \\ &= \lim_{n\to\infty} (I - \lambda^{k+1} A^{k+1}) = I \end{aligned}$$

■

Theorem 3.3.2 *Let* A *a linear bounded operator on a Banach space* X *into itself, such that* $\|A/\lambda\| < 1$, *then for all* $f \in X$ *the successive approximation*

$$\varphi_{n+1} = \lambda A\varphi_n + f \ , \quad n = 0, 1, 2, ...$$

with φ_0 *an arbitrary vector of* X *converge to the unique solution* φ *of the equation*

$$\varphi - \lambda A\varphi = f$$

proof. We use the successive approximation, with the notation $A^0 = I$, we get

$$\begin{aligned}
\varphi_0 &= \lambda^0 A^0 f \\
\varphi_1 &= \lambda A\varphi_0 + f = Af + f \\
\varphi_2 &= \lambda A\varphi_1 + f = \lambda^2 A^2 f + \lambda A f + f \\
. \quad & \quad . \\
. \quad & \quad . \\
. \quad & \quad . \\
\varphi_{n+1} &= \lambda A\varphi_n + f = \lambda A \textstyle\sum_{k=1}^{n} \lambda^k A^k f + f = \sum_{k=0}^{n+1} \lambda^k A^k f
\end{aligned}$$

so

$$\begin{aligned}
\lim_{n\to\infty} \varphi_{n+1} &= \lim_{n\to\infty} (\lambda^{n+1} A^{n+1} f + \sum_{k=0}^{n} \lambda^k A^k f) \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \lambda^k A^k f = \sum_{k=0}^{\infty} \lambda^k A^k f \\
&= Sf = (I - \lambda A)^{-1} f.
\end{aligned}$$

■

Corollary 3.3.1 *Let* A *a linear operator, with continuous kernel such that*

$$\max_{x\in G} \int_G |K(x, y)|\, dy < 1$$

then for any $f \in C(\Omega)$ *the integral equation*

$$\varphi(x) - \int_\Omega K(x, y)\varphi(y)dy = f(x)\,, \quad x \in \Omega$$

has a unique solution $\varphi \in C(\Omega)$, *and furthermore the successive approximation*

$$\varphi_{n+1}(x) = \int_G K(x, y)\varphi_n(y)dy + f(x) \quad n = 0, 1, ...$$

converges uniformly to the solution φ *for any arbitrary vector* $\varphi \in C(\Omega)$.

3.3.2 Method of Successive Substitutions for Integral equations (Resolvent method)

Definition 3.3.1 *Let A an integral operator with continuous kernel K, then the kernel of the repeated operator A^n is called an iterated kernel.*

Function defined by the infinite series

$$R(x,y,\lambda) = \sum_{k=0}^{\infty} \lambda^k K_{n+1}(x,y), \quad K_1(x,y) = K(x,y)$$

is called resolvent.

It's easy to prove that the iterated kernel is given by

$$K_n(x,y) = \int_{\Omega} K(x,z) K_{n-1}(z,y) dz$$

Theorem 3.3.3 *Let K a continuous function on $\Omega \times \Omega$, then*

$$\sum_{k=0}^{\infty} \lambda^k K_{n+1}(x,y)$$

converge uniformly on $\Omega \times \Omega$ if

$$\lambda < (\int_{\Omega}\int_{\Omega} |K(x,y)|^2 \, dxdy)^{-1/2} = \|A\|^{-1}$$

proof. see [29] ■

Corollary 3.3.2 *Let A an integral operator, with continuous kernel then the integral equation*

$$\varphi(x) - \int_{\Omega} K(x,y)\varphi(y)dy = f(x), \quad x \in \Omega$$

has a unique solution given by

$$\varphi(x) = \lambda \int_{\Omega} R(x,y,\lambda) f(y) dy + f(x), \quad x \in \Omega$$

such that

$$\lambda < (\int_{\Omega}\int_{\Omega} |K(x,y)|^2 \, dxdy)^{-1/2} = \|A\|^{-1}$$

if and only if $R(x,y,\lambda)$ is continuous on $\Omega \times \Omega$.

proof. see [29] ■

3.4 Converting IVP to Volterra Integral Equation

In this section, we show how to convert an initial value problem (IVP) to an equivalent Volterra integral equation. For simplicity reasons, we will apply this process to a second order initial value problem given by

$$Z(x) + Z'(x)p(x) + Z''(x)q(x) = g(x) \tag{3.4.1}$$

where z satisfy the initial conditions:

$$Z(0) = \alpha, \quad Z'(0) = \beta$$

where α and β are constants. The functions $p(x)$ and $q(x)$ are analytic functions, and $g(x)$ is continuous through the interval of discussion. We put

$$Z''(x) = \varphi(x) \tag{3.4.2}$$

where $\varphi(x)$ is a continuous function. Integrating both sides of the last equation from 0 to x yields

$$Z'(x) - Z'(0) = \int_0^x \varphi(y)dy$$

or equivalently

$$Z'(x) = \beta + \int_0^x \varphi(y)dy \tag{3.4.3}$$

For the second time, we integrate the last equation from 0 to x we obtain

$$Z(x) - Z(0) = \beta x + \int_0^x \int_0^x \varphi(y)dydy$$

or equivalently

$$y(x) = \alpha + \beta x + \int_0^x (x-t)\varphi(y)dy \tag{3.4.4}$$

obtained upon using the formula that reduce double integral to a single integral [45]. Substituting (2), (3), and (4) into the initial value problem (1) yields the Volterra integral equation:

$$\varphi(x) + p(x)[\beta + \int_0^x \varphi(y)dy] + q(x)[\alpha + \beta x + \int_0^x (x-y)\varphi(y)dy] = g(x)$$

The last equation can be written in the standard Volterra integral equation form:

$$\varphi(x) = f(x) - \int_0^x K(x,y)\varphi(y)dy$$

where

$$K(x,y) = p(x) + q(x)(x-y)$$

and

$$f(x) = g(x) - [\beta p(x) + \alpha q(x) + \beta x q(x)]$$

3.5 Converting Volterra Integral Equation to IVP

A well-known method for solving Volterra integral, converts this equation to equivalent initial value problems. The method is achieved simply by differentiating both sides of Volterra equations with respect to x as many times as we need to get rid of the integral sign and come out with a differential equation. The conversion of Volterra equations requires the use of Leibnitz rule for differentiating the integral at the right hand side. The initial conditions can be obtained by substituting $x = 0$ into $\varphi(x)$ and its derivatives. The resulting initial value problems can be solved easily by using ODEs methods for more details see [45].

Chapter 4

Existence of solutions for Nonlinear Volterra Integral Equations

In this chapter we study Nonlinear Volterra Integral Equations in Banach spaces, we well use all what we had seen in the previous chapters, such that fixed point theorems in order to prove existence and uniqueness of solution of this kind of equations and mention some of its qualitative properties in Banach spaces. Then we present our work which is solving the nonlinear Volterra integral equations using Adapted Newton-Kantorovich method, we describe and prove the convergence of the method. Moreover, we compare the numerical results obtained by this method against ones obtained by another authors. This comparison showed the efficiency of this method.

4.1 Application of Banach principle to Nonlinear Volterra Integral Equations

As we have seen the use of Banach principle requires that a given operator $T : X \to X$ should be contractive relative to some complete norm in X such that the operator T is given by

$$T\varphi = f + A\varphi$$

and A is an integral operator, if the given T is not contractive with respect to one norm it may be possible to find another complete norm with respect to which the given T is contractive as we'll see in proof of the next theorem.

Theorem 4.1.1 *Let $\Omega \subset \mathbb{R}$ bounded and closed subset, $K : \Omega \times \Omega \to \mathbb{R}$ be continuous and satisfy the Lipschitz condition*

$$|K(x, y, z_1) - K(x, y, z_2)| \leq L\,|z_1 - z_2|$$

$$\textit{for all:} \quad (x, y) \in \ \Omega \times \Omega \quad \textit{and} \ z_1, z_2 \in \mathbb{R}$$

Then for any $f \in C(\Omega)$ the equation

$$\varphi(x) = f(x) + \int\limits_0^x K(x, y, \varphi(y))dy \quad (x \in \Omega)$$

has a unique solution $\varphi \in C(\Omega)$. Moreover, defining a sequence of functions(φ_n) inductively by choosing any $\varphi_0 \in C(\Omega)$ and setting

$$\varphi_{n+1}(x) = f(x) + \int\limits_0^x K(x, y, \varphi_n(y))dy \qquad (x \in \Omega)$$

the sequence converges uniformly on $C(\Omega)$ to the unique solution.

proof. *Let X be the Banach space of all continuous real-valued functions on Ω equipped with the norm*

$$|g| = \max_{x \in \Omega} |g(x)| \exp(-Lx)$$

It's easy to prove that this norm is equivalent to the sup norm, since $\exp(-Lx)\,\|g\| \leq |g| \leq \|g\|$, *and moreover, it is complete.*

Define $T : X \to X$ by

$$T(\varphi)(x) = f(x) + \int_0^x K(x, y, \varphi(y))dy$$

to prove that the integral equation has a solution it's enough to show that T has a fixed point. We prove that, in fact , T is contractive: for

$$\begin{aligned}|T(\varphi_1)-T(\varphi_2)| &\le \max_{x\in\Omega}\exp(-Lx)\int_0^x |K(x,y,\varphi_1(y))-K(x,y,\varphi_2(y))|\,dy\\ &\le L\max_{0\le x\le T}\exp(-Lx)\int_0^x |\varphi_1(y)-\varphi_2(y)|\,dy\\ &= L\max_{0\le t\le T}\exp(-Lx)\int_0^x \exp(-Ly)\exp(Ly)\,|\varphi_1(y)-\varphi_2(y)|\,dy\\ &\le L\,|\varphi_1-\varphi_2|\max_{0\le x\le T}\exp(-Lt)\int_0^x \exp(Ly)dy\\ &= L\,|\varphi_1-\varphi_2|\max_{0\le x\le T}\exp(-Lx)\frac{\exp(Lx)-1}{L}\\ &\le (1-\exp(-LT))\,|\varphi_1-\varphi_2|\end{aligned}$$

Because $(1-\exp(-Lx))<1$ for all $x\in\Omega$ the map T is contractive; Banach's principle therefore guarantees first a unique fixed point the sequence (φ_n) determined by the iteration described in the statement of the theorem convergence uniformly in the norm $|\varphi|$ therefore also in the norm $\|\varphi\|$ to that fixed point . ■

In the last theorem we proved the existence and the uniqueness of continuous solution of the nonlinear Volterra integral equation in the Urysohn form using Banach principle, Now discuss the case when the equation is written in the Hammerstein form i.e

$$\varphi(x)=f(x)+\int\limits_0^x k(x,y)g(y,\varphi(y))dy \quad (x\in\Omega)$$

such that the functions f, g, and k are continuous with respect to their arguments.

We well mention four theorems with different conditions which guarantees the existence of continuous solution of the nonlinear Volterra integral equation in the Hammerstein form using the same principle (Banach principle) for more information see [9], [19].

Theorem 4.1.2 *Suppose g satisfies the following properties. $g(x,0)=0$, and satisfies the Lipschitz condition with respect to the second variable i.e*

there is a constant $c>0$ such that for each $(x,y,z)\in R^+\times R\times R$, $|g(x,y)-g(x,z)|\le c\,|y-z|$.

Assume $f(x)$ is bounded and

$$\sup_{x\ge 0} c\int_0^x |k(x,y)|\,dy \le \alpha < 1$$

Then the equation

$$\varphi(x) = f(x) + \int_0^x k(x,y)g(y,\varphi(y))dy \quad (x \in \Omega) \tag{4.1.1}$$

has a unique bounded continuous solution.

proof. *Let* X *be the Banach space of bounded continuous functions on* $[0,+\infty[$ *with the supremum norm,*$\|.\|$, *where* $\|\varphi\| = \sup\limits_{x\geq 0} |\varphi(x)|$. *For each* $\varphi \in X$, *define*

$$(T\varphi)(x) = f(x) + \int_0^x k(x,y)g(y,\varphi(y))dy, \quad x \geq 0$$

We show that $T : X \to X$ *is a contraction map. Therefore a fixed point of* T *is a solution of* (4.1.1). *It follows from the continuity assumptions on* $f, g,$ *and* k *that* $(T\varphi)(x)$ *is continuous in* x.

Now

$$\begin{aligned} |(T\varphi)(x)| &\leq |f(x)| + \int_0^x |k(x,y)|\,|g(y,\varphi(y))|\,dy \\ &\leq |f(x)| + c\alpha\,\|\varphi\| \\ &< \infty \end{aligned}$$

Therefore, $(T\varphi)$ *is bounded and* $T : X \to X$.

For $\varphi, \psi \in X$

$$\begin{aligned} |(T\varphi)(x) - (T\psi)(x)| &\leq \int_0^x |k(x,y)|\,|g(y,\varphi(y)) - g(y,\psi(y))|\,dy \\ &\leq c\int_0^x |k(x,y)|\,dy\,\|\varphi - \psi\| \\ &\leq \alpha\,\|\varphi - \psi\| \end{aligned}$$

Since $\alpha < 1, T$ *is a contraction mapping, which proves* (4.1.1) *has a unique bounded continuous solution.* ■

Now we consider a special case, we assume there exists a function h such that $g(x,y) = y + h(x,y)$ where h satisfies the following properties

$h(x,0) = 0,$ and there is a constant $c > 0$ such that for each $(x,y,z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ we have

$$|h(x,y) - h(x,z)| \leq c\,|y - z| \tag{4.1.2}$$

So (4.1.1) becomes

$$\varphi(x) = f(x) + \int_0^x k(x, y)(\varphi(y) + h(y, \varphi(y)))dy, \quad x \geq 0 \tag{4.1.3}$$

using a technic outlined in the last chapter the resolvent method, suppose $R(x, y)$ satisfies the resolvent equation

$$R(x, y) = k(x, y) + \int_y^x R(x, z)k(z, y)dz \tag{4.1.4}$$

Then the solution $\varphi(x)$ of equation (4.1.3) is given by

$$\varphi(x) = f(x) + \int_0^x R(x, y)(f(y) + h(y, \varphi(y)))dy, \quad x \geq 0$$

Where f, R, and h are all continuous functions.

Theorem 4.1.3 *Suppose h satisfies* (4.1.2). *Assume $f(x)$ is bounded and*

$$\sup_{x\geq 0} c \int_0^x |R(x, y)|\, dy \leq \alpha < 1$$

Then there exists a unique bounded continuous solution of (4.1.1).

proof. Let X be the Banach space of bounded continuous functions on $[0, \infty[$ with the supremum norm. For each $\varphi \in X$, define

$$\varphi(x) = f(x) - \int_0^x R(x, y)(f(y) + h(y, \varphi(y)))dy, \quad x \geq 0$$

It follows from the continuity assumptions on f, h, and R that $(T\varphi)(x)$ is continuous in x. Also, one can easily verify from the given assumptions that $|(T\varphi)(x)| < \infty$, and $|(T\varphi)(t) - (T\psi)(t)| \leq \alpha \|\varphi - \psi\|$ for all $\varphi, \psi \in X$. This shows that T maps X into itself and is a contraction. Therefore (4.1.1) has a unique bounded continuous solution. ■

We remark that the integrability of $R(x, y)$ is itself an important property which is often assumed in the study of qualitative behaviors of integral equations. So we present an example to show it in this case

Example 4.1.1 *Suppose*

$$\sup_{x\geq 0} \int_0^x |k(x, y)|\, dy \leq L < 1$$

Then

$$\sup_{x\geq 0} \int_0^x |R(x, y)|\, dy \leq l < \infty$$

proof. From the resolvent equation (4.1.4), we get

$$\begin{aligned}
\int_0^x |R(x,y)|\,dy &\leq \int_0^x |k(x,y)|\,dy - \int_0^x \int_y^x |R(x,z)|\,|k(z,y)|\,dzdy \\
&= \int_0^x |k(x,y)|\,dy - \int_0^x |R(x,z)| \int_0^z |k(z,y)|\,dydz \\
&\leq L + \int_0^x |R(x,z)|\,Ldu \\
(1-L)\int_0^x |R(x,y)|\,dy &\leq L \\
\sup_{x\geq 0} \int_0^x |R(x,y)|\,dy &\leq \frac{L}{1-L} = l
\end{aligned}$$

■

Now we assume $f'(x)$ and k_x exist and are continuous functions. Differentiating (4.1.3), we get

$$\begin{aligned}
\varphi'(x) &= -k(x,x)\varphi(x) - \int_0^x k_x(x,y)\varphi(y)dy + [f'(x) - k(x,x)h(x,\varphi(x)) \\
&\quad - \int_0^x k_x(x,y)h(y,\varphi(y))dy]
\end{aligned} \tag{4.1.5}$$

So

$$\begin{aligned}
\varphi(x) &= \varphi(0)e^{-\int_0^x k(y,y)dy} + \int_0^x e^{-\int_y^x k(y,y)dy} f'(z)dz \\
&\quad - \int_0^x e^{-\int_y^x k(y,y)dy} \int_0^z k_z(z,y)\varphi(y)dydz \\
&\quad - \int_0^x e^{-\int_y^x k(y,y)dy} k(z,z)h(z,\varphi(z))dz \\
&\quad - \int_0^x e^{-\int_y^x k(y,y)dy} \int_0^z k_z(z,y)h(y,\varphi(y))dydz \\
&= f(0)e^{-\int_0^x k(y,y)dy} + \int_0^x e^{-\int_y^x k(y,y)dy} f'(z)dz \\
&\quad - \int_0^x e^{-\int_y^x k(y,y)dy} \int_0^z k_z(z,y)(\varphi(y) + h(y,\varphi(y)))dydz \\
&\quad - \int_0^x e^{-\int_y^x k(y,y)dy} k(z,z)h(z,\varphi(z))dz
\end{aligned} \tag{4.1.6}$$

where f, k, k_x, and h are continuous functions. In subsequent results, we shall write $f(0) = f_0$.

Theorem 4.1.4 *Suppose h satisfies* (4.1.2). *Assume $f_0(x)$ is bounded and continuous, $\int_0^x k(y,y)dy$ as $x \to \infty$, $\int_0^x e^{-\int_z^x k(y,y)dy}dz$ is bounded, and*

$$\sup_{x\geq 0}(c+1)\int_0^x e^{-\int_z^x k(y,y)dy}\int_0^z |c_z(z,y)|\,dydz + c\int_0^x e^{-\int_z^x k(y,y)dy}\,|c(z,z)|\,dz \leq \alpha < 1$$

Then there exists a unique bounded, continuous solution of (4.1.3).

proof. Let X be the Banach space of bounded continuous functions on $[0,\infty[$.

For each $\varphi \in X$, define

$$\begin{aligned}(T\varphi)(x) = f_0 e^{-\int_0^x k(y,y)dy} + \int_0^x e^{-\int_y^x k(y,y)dy} f'(z)dz \\ - \int_0^x e^{-\int_y^x k(y,y)dy}\int_0^z k_z(z,y)(\varphi(y) + h(y,\varphi(y)))dydz \\ - \int_0^x e^{-\int_y^x k(y,y)dy} k(z,z)h(z,\varphi(z))dz,\ x \leq 0\end{aligned}$$

It follows from the continuity assumptions on f, h, k and k_x that $(T)\varphi(x)$ is continuous in X.

Now

$$|(T\varphi)(x)| \leq |f_0|\,e^{-\int_0^x k(y,y)dy} + \int_0^x e^{-\int_y^x k(y,y)dy}\,|f'(z)|\,dz + \alpha\,\|\varphi\| < \infty$$

So $(T\varphi)$ is bounded and $T : X \to X$.

For $\varphi, \psi \in X$

$$\begin{aligned}|(T\varphi)(x) - (T\psi)(x)| &\leq \left|\int_0^x e^{-\int_z^x k(y,y)dy}\int_0^z k_z(z,y)[\varphi(y) - \psi(y) + h(y,\varphi(y)) - h(y,\psi(y))dy]\right| \\ &\quad + \left|\int_0^x e^{-\int_y^x k(y,y)dy} k(z,z)[h(z,\varphi(z)) - h(z,\varphi(z))]dz\right| \\ &\leq [(c+1)\int_0^x e^{-\int_z^x k(y,y)dy}\int_0^z |k_z(z,y)|\,dydz \\ &\quad + c\int_0^x e^{-\int_y^x k(y,y)dy}\,|k(z,z)|\,dz]\,\|\varphi - \psi\| \\ &\leq \alpha\,\|\varphi - \psi\|\end{aligned}$$

therefore T is a contraction map. So (4.1.6) has a unique bounded solution. Since (4.1.6) is equivalent to (4.1.3), where $\varphi(0) = f(0)$ has a unique bounded continuous solution. ■

One resolvent equation for

$$\varphi'(x) = -k(x,x)\varphi(x) - \int_0^t k_x(x,y)\varphi(y)dy$$

is

$$R_y(x,y) = R(x,y)k(y,y) + \int_y^x R(x,y)k_x(z,y)dz,\ R(x,x) = 1$$

with resolvent $R(x,y)$. Then from (4.1.1), we obtain by the variation of parameters formula

$$\begin{aligned}\varphi(x) &= R(x,0)f_0 + \int_0^x R(x,y)[f'(y) - k(y,y)h(y,\varphi(y)) \\ &\quad - \int_0^y k_y(y,z)h(z,\varphi(z))dz]dy\end{aligned}$$

with resolvent $R(x,y)$. Then from (4.1.1) we obtain by the variation of parameters formula

$$\begin{aligned}\varphi(x) &= R(x,0)f_0 + \int_0^x R(x,y)[f'(y) - k(y,y)h(y,\varphi(y)) \\ &\quad - \int_0^y k_y(y,z)h(z,\varphi(z))dz]dy\end{aligned} \tag{4.1.7}$$

Theorem 4.1.5 *Suppose h satisfies* (4.1.2). *Assume* $f_0(x)$ *is bounded, continuous function,* $R(x,0)$ *is bounded*

$$\sup_{x\geq 0}\int_0^x |R(x,y)|\, dy < \infty$$

and

$$\sup_{x\geq 0} c\int_0^x |R(x,y)|\, [|k(y,y)| + \int_0^y |k_y(y,z)|\, dz]dy \leq \alpha < 1$$

Then there exists a unique bounded, continuous solution of (4.1.3)

proof. Let X be the Banach space of bounded continuous functions on $[0,\infty[$.
For each $\varphi \in X$, define

$$\begin{aligned}(T\varphi)(x) = R(x,0)f_0 + \int_0^x R(x,y)[f'(y) - k(y,y)h(y,\varphi(y)) \\ - \int_0^y k_y(y,z)h(z,\varphi(z))dz]dy,\ x \geq 0\end{aligned}$$

It follows from the continuity assumptions on $f,\ h,\ R,\ k$ and k_x that $(T\varphi)(x)$ is continuous in X.

Now

$$\begin{aligned}
|(T\varphi)(x)| &= \left|R(x,0)f_0 + \int_0^x R(x,y)[f'(y) - k(y,y)h(y,\varphi(y)) - \int_0^y k_y(y,z)h(z,\varphi(z))dz]dy\right| \\
&\leq |R(x,0)|\,|f_0| + \int |R(x,y)|\,|f'(y)|\,dy + c\int_0^x |R(x,y)|\,|k(y,y)|\,dy\,\|\varphi\| \\
&\quad +c\int_0^x |R(x,y)| \int_0^y |k_y(y,z)|\,dzdy\,\|\varphi\| \\
&\leq |R(x,0)|\,|f_0| + \int_0^t |R(x,y)|\,|f'(y)|\,dy\,\|\varphi\| \\
&< \infty
\end{aligned}$$

So $(T\varphi)$ is bounded and $T : X \to X$.

For $\varphi, \psi \in X$

$$\begin{aligned}
|(T\varphi)(x) - (T\psi)(x)| &\leq \int_0^x |R(x,y)|\,|k(y,y)|\,|h(y,\varphi(y)) - h(y,\psi(y))|\,ds \\
&\quad + \int_0^x |R(x,y)| \int_0^y |k_y(y,z)|\,|h(z,\varphi(z)) - h(z,\psi(z))|\,dzdy \\
&\leq c\int_0^x |R(x,y)|\,[|k(y,y)| + \int_0^y |k_y(y,z)|\,dz]dy\,\|\varphi - \psi\| \\
&\leq \alpha\,\|\varphi - \psi\|
\end{aligned}$$

Therefore T is a contraction map, showing (4.1.7) has a unique bounded continuous solution. Since (4.1.7) is equivalent to (4.1.6), which is equivalent to (4.1.5), (4.1.5) has a unique bounded continuous solution. ■

4.2 Continuous Solutions of Nonlinear Volterra Integral Equations via Schauder's Theorem

Theorem 4.2.1 *Let $K : [a,b]^2 \times \mathbb{R}^n \to \mathbb{R}^n$be continuous. Then the Volterra operator associated to K, $T : C([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ given by*

$$T(\varphi(x)) = \int_a^x K(x,y,\varphi(y))dy \quad x \in [a,b]$$

is completely continuous.

proof. We first prove that T is continuous. Let $\varphi_0 \in C([a,b],\mathbb{R}^n)$ and choose any number $R > \|\varphi_0\|_\infty$. Let $\epsilon > 0$. Since K is uniformly continuous on the compact set $[a,b]^2 \times \bar{B}_R(0,\mathbb{R}^n)$, there exists a constant $\delta_\epsilon > 0$ such that for every $\varphi \in C([a,b],\mathbb{R}^n)$ satisfying $\|\varphi - \varphi_0\|_\infty \leq \delta_\epsilon$ one has $\varphi(y) \in \overline{B}_R(0,\mathbb{R}^n)$ and

$$|K(x,y,\varphi(y)) - K(x,y,\varphi_0(y))| \leq \epsilon$$

for all $x, y \in [a,b]$. Then

$$\begin{aligned} |T(\varphi(x)) - T(\varphi_0(x))| &\leq \textstyle\int_a^x |K(x,y,\varphi(y)) - K(x,y,\varphi_0(y))|\, dy \\ &\leq \epsilon\, |b-a| \end{aligned}$$

for every $x \in [a,b]$. Hence

$$\|T(\varphi) - T\,(\varphi_0)\|_\infty \leq \epsilon\, |b-a|$$

Whenever $\|\varphi - \varphi_0\|_\infty \leq \delta_\epsilon$. Therefore T is continuous at φ_0.

Next, given a bounded subset Y of $C([a,b],\mathbb{R}^n)$, we shall prove that $T(Y)$ is relatively compact in $C([a,b],\mathbb{R}^n)$. According to the Ascoli-Arzela theorem, we have to show that $T(Y)$ is bounded and equicontinuous.

Indeed, since Y is bounded there exists a constant $c > 0$ such that

$$|\varphi|_\infty \leq c \text{ for all } \varphi \in Y$$

It follows that for any $\varphi \in Y$ we have

$$|T\,(\varphi)|_\infty \leq M\, |b-a|$$

where

$$M = \underset{[a,b]^2 \times \bar{B}_c(0;R^n)}{Max} |K(x,y,z)|$$

Hence the set $T(Y)$ is bounded in $C([a,b],\mathbb{R}^n)$.

On the other hand, using the uniform continuity of K on the compact $[a,b]^2 \times \bar{B}_c(0,\mathbb{R}^n)$ for each $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$\left|K(x,y,\varphi(y)) - K(x^{'},y,\varphi(y))\right| \leq \epsilon$$

for all $x, x^{'}, y \in [a,b]$ with $\left|x - x^{'}\right| \leq \delta_\epsilon$ and $\varphi \in Y$. This immediately yields

$$\left|T\left(\varphi\left(x\right)\right) - T(\varphi(x^{'}))\right| \leq \epsilon\left|b - a\right|$$

for all $x, x^{'} \in [a,b]$ satisfying $\left|x - x^{'}\right| \leq \delta_\epsilon$ and $\varphi \in Y$. Thus $T(Y)$ is equicontinuous. ■

The next result is a local version of the previous theorem.

Theorem 4.2.2 *Let* $R > 0$ *and* $K : [a,b]^2 \times \bar{B}_R(0, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ *be a continuous mapping. Then the operator* $T : \bar{B}_R(0, C([a,b], \mathbb{R}^n)) \rightarrow C([a,b], \mathbb{R}^n)$ *given in the theorem* (4.2.1) *is completely continuous.*

proof. Essentially the same reasoning as in the proof of the previous Theorem establishes the result. ■

As an application we present an existence theorem for the Volterra integral equation in $\mathbb{R}^n$

$$\varphi(x) = f(x) + \int_a^x K(x, y, \varphi(y))dy, \quad x \in [a,b]$$

Theorem 4.2.3 *Let* $K : [a,b]^2 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ *be continuous and let* $f \in C([a,b], \mathbb{R}^n)$*. Assume that there exists constants* $\alpha, \beta \in \mathbb{R}^+$ *such that*

$$\left|K(x, y, z)\right| \leq \alpha\left|z\right| + \beta$$

for all $x, y \in [a,b],\ z \in \mathbb{R}^n$*. Then the Volterra integral equation has at least one solution* $\varphi \in C([a,b], \mathbb{R}^n)$*.*

proof. Let $T : C([a,b], \mathbb{R}^n) \rightarrow C([a,b], \mathbb{R}^n)$ be given by

$$T(\varphi(x)) = f(x) + \int_a^x K(x, y, \varphi(y))dy$$

According to Theorem (4.2.1), T is completely continuous. We now show that T is a self-mapping of a closed ball of the space $C([a,b], \mathbb{R}^n)$ endowed with a suitable norm, equivalent to the sup-norm $\|.\|_\infty$ Indeed, for any given number $\theta > 0$ we have

$$\begin{aligned}
|T(\varphi(x))| &= \left|\int_a^x K(x,y,\varphi(y))dy + f(x)\right| \\
&\leq \int_a^x |K(x,y,\varphi(y))dy| + \|f(x)\|_\infty \\
&\leq \alpha \int_a^x |\varphi(y)|\, dy + \beta(b-a) + \|f(x)\|_\infty \\
&= \alpha \int_a^x |\varphi(y)|\, e^{-\theta(y-a)} e^{\theta(y-a)} dy + \beta(b-a) + \|f(x)\|_\infty \\
&\leq \alpha \left\|\varphi(.)e^{-\theta(.-a)}\right\|_\infty \int_a^x e^{\theta(y-a)} dy + \beta(b-a) + \|f(x)\|_\infty \\
&= \alpha\theta^{-1} \left\|\varphi(.)e^{-\theta(.-a)}\right\|_\infty (e^{\theta(x-a)} - 1) + \beta(b-a) + \|f(x)\|_\infty \\
&\leq \alpha\theta^{-1} \left\|\varphi(.)e^{-\theta(.-a)}\right\|_\infty (e^{\theta(x-a)}) + \beta(b-a) + \|f(x)\|_\infty
\end{aligned}$$

Since $e^{-\theta(x-a)} \leq 1$ on $[a,b]$, we deduce

$$|T(\varphi(x))|\, e^{-\theta(x-a)} \leq \alpha\theta^{-1} \left\|\varphi(.)e^{-\theta(.-a)}\right\|_\infty + \beta(b-a) + \|f(x)\|_\infty$$

and so

$$\left\|T(\varphi(.))e^{-\theta(.-a)}\right\|_\infty \leq \alpha\theta^{-1} \left\|\varphi(.)e^{-\theta(.-a)}\right\|_\infty + \beta(b-a) + \|f(x)\|_\infty \tag{4.2.1}$$

Now fix any $\theta > a$. Then we can find $R > 0$ such that

$$a\theta^{-1}R + \beta(b-a) + \|f(x)\|_\infty \leq R \tag{4.2.2}$$

Consider a new norm on $C([a,b],\mathbb{R}^n)$, namely

$$\|\varphi\| = \left\|\varphi(.)e^{-\theta(.-a)}\right\|_\infty$$

It is clear that the norm $\|.\|_\infty$ and $\|.\|$ are equivalent (thus T is also completely continuous with respect to $\|.\|_\infty$). On the other hand, (4.2.1) and (4.2.2) show that T maps the closed ball of center 0 and radius R of the space $(C([a,b],\mathbb{R}^n), \|.\|_\infty)$, into itself. Now the conclusion follows from Schauder's Theorem. ■

Now we present another conditions which prove the existence of the Nonlinear Volterra Integral Equation always via Schauder's Theorem.

Theorem 4.2.4 *Consider the Nonlinear Volterra Integral Equation*

$$\varphi(x) = f(x) + \int_a^x K(x, y, \varphi(y))dy, \quad x \in [a, b] \tag{4.2.3}$$

where f is continuous over $[a, b]$. assume that $K(x, y, z)$ satisfies the following conditions

$$|k(x, y, z)| \leq V_1(x)V_2(y)\phi(|z|), \quad \left|\frac{\partial g}{\partial z}(x, y, z)\right| \leq V_1(x)V_2(y)\psi(|z|)$$

where $V_1(.)$ is a positive and continuous function over $[a, b], V_2(.)$ is a positive and integrable function over $[a, b]$,and $\psi(.)$ is a positive and continuous function over $[0, \infty[$. Finally, assume that the function $\phi(.)$ is positive, continuous and satisfies the condition $\lim\limits_{y\to+\infty} \dfrac{\phi(y)}{y} = L < -\infty$ Under these conditions the Volterra Integral Equation has a solution over $[a, b]$.

proof. see [21] ■

Proposition 4.2.1 *consider the nonlinear Volterra integral equation, assume that $K(x, y, z)$ satisfies the conditions of the last theorem with $V_2(.) \in (L^1 \cap L^p)([a, b])$ for some $p > 1$. Then* (4.2.3) *has a unique solution.*

proof. see [21] ■

4.3 Continuous Solutions of Nonlinear Volterra Integral Equations via Leray-Schauder's Theorem

This section presents general existence theorems for the Volterra integral equation via Leray-schauder's theorem. We check for continuous solution $\varphi \in C([a, b], B)$ for the Volterra integral equation in the first kind

$$\varphi(x) = \int_a^x K(x, y, \varphi(y))dy, \quad x \in [a, b] \tag{4.3.1}$$

where $B = \{z \in \mathbb{R}^n : |z| \leq R\}$

The Leray-schauder's theorem yields the following existence principle which can be summarized as follows: 'boundedness yields existence' as we well see in the next theorem.

Theorem 4.3.1 *Let $K : [a,b]^2 \times B \to R^n$ be continuous. Assume that*

$$\|\varphi\| < R \tag{4.3.2}$$

for any solution $\varphi \in C([a,b],B)$ to

$$\varphi(x) = \lambda \int_a^x K(x,y,\varphi(y))dy, \quad x \in [a,b] \tag{4.3.3}$$

for each $\lambda \in]0,1[$. Then (4.3.3) has a solution in $C([a,b],B)$.

proof. Let $K = X = C([a,b],\mathbb{R}^n)$ with norm $\|.\|_\infty$

$$U = \{\varphi \in C([a,b],\mathbb{R}^n) : \|\varphi\|_\infty < R\}$$

φ_0 be the null function and $T : \bar{U} \to C([a,b],\mathbb{R}^n)$ be given by

$$T(\varphi(x)) = \int_a^x K(x,y,\varphi(y))dy, \quad x \in [a,b]$$

The result follows from Leray-Schauder's Theorem. ■

Next we give a sufficient condition for (4.3.2) in the case of the equation in $\mathbb{R}^n$

$$\varphi(x) = \lambda \int_a^x h(x,y)f(y,\varphi(y))dy, \quad x \in [a,b] \tag{4.3.4}$$

Corollary 4.3.1 *Let $K : [a,b]^2 \to \mathbb{R}$ and $f : [a,b] \times B \to \mathbb{R}^n$ be continuous functions. Assume that*

$$|h(x,y)| \leq 1$$

for all $x, y \in [a,b]$, there exists a continuous nondecreasing function $\psi :]0,R] \to (0,\infty)$ and a function $\phi \in C([a,b],\mathbb{R}^+)$ such that

$$|f(y,z)| \leq \phi(y)\psi(|z|)$$

for all $s \in [a,b]$, $z \in B$, and

$$\|\phi\|_{L^1([a,b])} \leq \int_0^R \frac{1}{\psi(\sigma)} d\sigma \tag{4.3.5}$$

Then (4.3.4) has a solution $\varphi \in C([a,b],\mathbb{R}^n)$ with $\|\varphi\|_\infty \leq R$.

proof. Let $\varphi \in C([a,b],B)$ be any solution of (4.3.4) for some $\lambda \in]0,1[$. Here

$$K(x,y,z) = h(x,y)f(y,\varphi(y))$$

Then

$$|\varphi(x)| \leq \lambda \int_a^x |h(x,y)f(y,\varphi(y))|\, dy \leq \lambda \int_a^x \phi(y)\psi(|\varphi(y)|)dy \tag{4.3.6}$$

for all $x \in [a,b]$. Here we have understand that $\psi\ (0) = \lim\limits_{x\to 0}\psi(x)$. Let

$$c(x) = \min\{R,\ \lambda \int_a^x \phi(y)\psi(|\varphi(y)|)dy\}$$

Clearly c is nondecreasing. We claim that $c(b) < R$. Assume the contrary. Then, since $c(a) = 0$, there exists a subinterval $[a',b'] \subset [a,b]$ with

$$c(a') = 0,\ c(b') = R \text{ and } c(x) \in]0,R[\text{ for } x \in]a',b'[$$

since by (4.3.6)

$$|\varphi(x)| \leq c(x) \leq R \ \text{ on } \ [a,b]$$

and ψ is nondecreasing on $[0,R]$, we have

$$c'(y) = \lambda\phi(y)\psi(|\varphi(y)|) \leq \lambda\phi(y)\psi(c(y))$$

for all $y \in [a',b']$. Now integration from a' to b' yields

$$\begin{aligned} \int_{a'}^{b'} \frac{c'(y)}{\psi(c(y))}dy &= \int_0^R \frac{1}{\psi(\sigma)}d\sigma \\ &\leq \lambda \int_{a'}^{b'} \phi(y)dy \\ &\leq \lambda \int_a^b \phi(y)dy \\ &< \int_a^b \phi(y)dy \end{aligned}$$

a contradiction. Notice we may assume $\|\phi\|_{L^1([a,b])} > 0$ since otherwise we have nothing to prove. Hence $c(b) < R$ and so, by (4.3.4), $|\varphi| < R$ for all $x \in [a,b]$. Therefore $\|\varphi\|_\infty < R$ and Theorem (4.3.1) is applied. ∎

Remark 4.3.1 *Notice that this corollary can be directly derived from schauder's fixed point theorem if we observe that* $T(D) \subset D$*, where*

$$\begin{aligned} D &= \{\varphi \in C([a,b],\mathbb{R}^n): \quad |\varphi(x)| \le \gamma(x) \text{ on } [a,b]\} \\ \gamma(x) &= I^{-1}(\int_a^x \phi(y)dy) \\ & I(x) = \int_0^x \frac{1}{\psi(\sigma)} d\sigma \end{aligned}$$

and T *is the Volterra integral operator associated to the right hand side of the equation* (4.3.4)*. We note that* $\gamma(x) \le R$ *for all* $x \in [a,b]$ *because of* (4.3.5)*.*

4.4 Existence theory in spaces of measurable functions

In this section we study the Voterra-Hammerstein integral operator given by

$$T\varphi(x) = f(x) + \int_a^x k(x,y)g(y,\varphi(y))dy \quad (x \in \Omega) \tag{4.4.1}$$

This operator appears as the composition of the linear integral operator A of kernel k with the Nemytskii operator N_g associated to g, as following

$$T = AN_g$$

Theorem 4.4.1 *Let* (a,b) *be a bounded real interval,* $k : (a,b)^2 \to \mathbb{R}$ *and* $g : (a,b) \times \mathbb{R}^n \to \mathbb{R}^n$*. Assume that there exists* $p \in [1,+\infty)$ *and* $q \in [p,+\infty) \cap (1,\infty)$ *such that* $k \in L^p(a,b;L^r(a,b))$*,* $(1/q + 1/r = 1)$ *and* g *is* (p,q)*-Caratheodory. In addition assume that if* $q = p$*, then there exists an* $r' > r$ *such that*

$$\begin{aligned} k(x,.) &\in L^{r'}(a,b) \text{ for a.e } x \in (a,b) \text{ and} \\ \text{the map } x &\mapsto |k(x,.)|_{r'}, \text{ belongs to } L^p(a,b) \end{aligned}$$

then (4.4.1) *has a fixed point* $\varphi \in L^p(a,b,R^n)$*.*

proof. see [36]. ■

It remains to discuss the case when $p = +\infty$.

Theorem 4.4.2 *Let* (a,b) *be a bounded real interval,* $k : (a,b)^2 \to \mathbb{R}$ *and* $g : (a,b) \times \mathbb{R}^n \to \mathbb{R}^n$*. Assume that f satisfies the Caratheodory conditions,there exist* $\alpha, \beta \in (1, +\infty]$, $\gamma \in [1, +\infty)$ *with* $\alpha' < \beta$ $(1/\alpha + 1/\alpha' = 1)$ *and* $1/\alpha + 1/\beta + 1/\gamma = 1, \phi \in L^{\alpha}(a,b)$,*and a continuous nondecreasing function* $\psi : (0, +\infty) \to (0, \infty)$ *such that*

$$k \in L^{\infty}(a,b; L^{\beta}(a,b)),$$

$$|g(y,z)| \leq \phi(y)\psi(|z|^{\gamma})$$

for a.e. $s \in (a,b)$ *and all* $z \in \mathbb{R}^n$, *and*

$$\left[\left| |k(x,.)|_{\beta} \right|_{\infty} |\phi|_{\alpha} \right]^{\gamma} (b-a) \leq \int_0^{\infty} \frac{1}{[\psi(\sigma)]^{\gamma}} d\sigma$$

In addition assume that the linear integral operator A with kernel k is is well defined and completely continuous from $L^{\beta'}(a,b,\mathbb{R}^n)$ *to* $C([a,b],\mathbb{R}^n)$ $(1/\beta + 1/\beta') = 1$*. Then* (4.4.1) *has a fixed point* $\varphi \in C([a,b],\mathbb{R}^n)$*.*

proof. see [35]. ■

In the next paragraph, we find some various L^p properties of solutions which are obtained under appropriate assumptions on f and g, using Liapunov type functions that are suitable for integral equations for more informtions see [9], [19].

Theorem 4.4.3 *Consider the nonlinear Volterra integral equation*

$$T\varphi(x) = f(x) + \int_a^x k(x,y)g(y,\varphi(y))dy \quad (x \in \Omega) \tag{4.4.2}$$

assume that equation (4.4.2) *has a solution* $\varphi(x)$, $x > 0$*. Suppose there exists a constant* $c \geq 0$ *such that*

$$|g(x,y)| \leq c\,|y|$$

and

$$c \int_0^{\infty} |k(u+x,x)|\, du \leq \alpha < 1$$

Then the solution $\varphi \in L^1([0,+\infty[)$

proof. From (4.4.2) it follows that

$$\begin{aligned} |\varphi(x)| \quad &\le |f(x)| + \textstyle\int_0^x |k(x,y)|\,|g(y,\varphi(y))|\,dy \\ &\le |f(x)| + c\textstyle\int_0^x |k(x,y)|\,|\varphi(y)|\,dy \end{aligned}$$

Therefore

$$-c\int_0^x |k(x,y)|\,|\varphi(y)|\,dy \le |f(x)| - |\varphi(x)|$$

Let

$$V(x) = c\int_0^x \int_{x-y}^{\infty} |k(u+y,y)|\,du\,|\varphi(y)|\,dy$$

Then

$$\begin{aligned} V'(x) \quad &= c\textstyle\int_0^\infty |k(u+x,x)|\,du\,|\varphi(x)| - \int_0^x c\,|k(x,y)|\,|\varphi(y)|\,dy \\ &\le \alpha\,|\varphi(x)| - (|\varphi(x)| - |f(x)|) \\ &= (1-\alpha)\,|\varphi(x)| + |f(x)| \end{aligned}$$

Integrating from 0 to t

$$V(x) - V(0) \le (1-\alpha)\int_0^x |\varphi(y)|\,dy + \int_0^x |f(y|\,dy$$

Since $V(t) \ge 0$, $V(0) = 0$ and $(\alpha - 1) < 0$

$$(1-\alpha)\int_0^x |\varphi(y)|\,dy \le \int_0^x |f(y|\,dy$$

This shows that $x \in L^1$ if $f \in L^1$. ■

Theorem 4.4.4 *assume that* (4.1.1) *has a nonnegative solution* $\varphi(x),\ x \ge 0$. *Also, assume there exists a constant* $k > 0$ *such that* $0 \le g(x,y) \le cx$, *for* $y \ge 0,\ x \ge 0$

Let $k(x,y) \ge 0,\ k_{yx}(x,y) \le 0$, *and* $k_x(x,0) \le$. *Then* $\varphi \in L^2[0,+\infty)$ *if* $f \in L^2[0,+\infty)$

proof. For $\varphi(x)$, a nonnegative solution of (4.1.1), let

$$V(x) = \int_0^x k_y(x,y)\Big(\int_y^x g(u,\varphi(u))du\Big)^2 dy + k(x,0)\Big(\int_0^x g(y,\varphi(y))dy\Big)^2$$

Then

$$\begin{aligned} V'(x) \quad = \quad & \int_0^x k_{yx}(x,y)\Big(\int_y^x g(u,\varphi(u))du\Big)^2 dy \\ & + \int_0^x k_y(x,y)2\Big(\int_y^x g(u,\varphi(u))du\Big)g(x,\varphi(x))dy \\ & + k_x(x,0)\Big(\int_0^x g(y,\varphi(y))dy\Big)^2 \\ & + 2k(x,0)\int_0^x g(y,\varphi(y))dy g(x,\varphi(x)) \end{aligned}$$

Integrating the second term of $V'(x)$ by parts, we get

$$\begin{aligned} & 2g(x,\varphi)[k(x,y)\int_y^x g(u,\varphi(u))du|_{y=0}^{y=x} + \int_0^x k(x,y)g(y,\varphi(y))dy] \\ = \ & 2g(x,\varphi)[0 - k(x,0)\int_y^x g(u,\varphi(u))du + \int_0^x k(x,y)g(y,\varphi(y))dy] \end{aligned}$$

Therefore we get

$$\begin{aligned} V'(x) \ = \ & \int_0^x k_{yx}(x,y)(\int_y^x g(u,\varphi(u))du)^2 dy + \\ & 2g(x,\varphi)\int_0^x k(x,y)g(y,\varphi(y))dy + k_x(x,0)(\int_0^x g(y,\varphi(y))dy)^2. \end{aligned}$$

Now from (4.1.1), $f(x) - \varphi(x) = \int_0^x k(x,y)g(y,\varphi(y))dy$. notice that $f(x) - \varphi(x) \geq 0$ by our positivity assumptions on k and g.

So

$$\begin{aligned} V'(x) \ &= \ \int_0^x k_{yx}(x,y)(\int_y^x g(u,\varphi(u))du)^2 dy + k_x(x,0) * \\ & \quad (\int_0^x g(y,\varphi(y))dy)^2 + 2g(x,\varphi)(f(x) - \varphi(x)) \\ &\leq \ 2g(x,\varphi)(f(x) - \varphi(x)) \\ &\leq \ 2c\varphi(t)(f(x) - \varphi(x)) \\ &\leq \ c(f^2(x) + \varphi^2(x) - 2\varphi^2(x)) \\ &\leq \ c(f^2(x) - \varphi^2(x)) \end{aligned}$$

Integrating from 0 to x, we obtain

$$V(x) \leq V(0) + c\int_0^x f^2(y)dy - c\int_0^x \varphi^2(y)dy$$

This implies $\varphi \in L^2([0,+\infty[)$ if $f \in L^2([0,+\infty[)$. ■

Now suppose both k_x and $f\prime(x)$ are continuous. We can then write (4.1.1) as

$$\varphi'(x) = f'(x) - k(x,x)g(x,\varphi) - \int_0^x k_x(x,y)g(y,\varphi(y))dy \tag{4.4.3}$$

Theorem 4.4.5 *assume* (4.4.1) *has a nonnegative solution* $\varphi(x),\ x \geq 0$. *Suppose there exists a constant* $m > 0$ *such that* $g(x,\varphi) \geq m\varphi^p$, *for* $\varphi \geq 0,\ x \geq 0$, *where* p *is a positive integer. Let* $k_x(x,y) < 0$, *and*

$$-k(x,x) + \int_0^\infty |k_1(u+x,x)|\, du \leq -\alpha$$

for some $\alpha > 0$. *Then* $\varphi \in L^p[0, +\infty[$ *if* $f' \in L^1[0, +\infty[$. *Moreover, the solution* $\varphi(x)$ *is bounded.*

proof. see [19] ■

4.5 Approximation solution of nonlinear Volterra integral equations using Adapted Newton-Kantorovich method

4.6 Description of the method

We apply the Newton-Kantorovich method to the nonlinear Volterra integral equation

Let

$$(P\varphi)(x) = \varphi(x) - \int_a^x K(x, y, \varphi(y))dt - f(x).$$

It is known that, the fact where the function $K(x, y, \varphi)$ is continuous and Lipshitzian for the third variable then, $P(\varphi)$ is Fréchet differentiable mapping of a Banach spaces $C(\Omega)$ into itself for all $f(x), \varphi(x) \in C(\Omega)$, say

$$P'(\varphi)h(x) = h(x) - \int_\Omega K_\varphi(x, y, \varphi(y))h(y)dy, \tag{4.6.1}$$

where $K_\varphi(x, y, \varphi(y))$ designates the derivative $\dfrac{\partial K}{\partial \varphi}(x, y, \varphi(y))$.

For the resolution of the functional equation $P(\varphi) = 0$ where P is Fréchet differentiable on a convex set of a Banach space $C(\Omega)$, Kantorovich imitates the Newton method for the equation of the tangent line given by the first two terms of Taylor's formula, written as the method of successive approximation.

$$P(\varphi_{n+1}) = P(\varphi_n) + P'(\varphi_n)(\varphi_{n+1} - \varphi_n) = 0,$$

or equivalently

$$P'(\varphi_n)(\varphi_{n+1} - \varphi_n) = -P(\varphi_n). \tag{4.6.2}$$

The explicit form to the equation (4.6.2) is given as

$$\begin{aligned}\varphi_n(x) - & \int_\Omega K(x,y,\varphi_n(y))dy - f(x) + (\varphi_{n+1}(x) - \varphi_n(x)) \\ - & \int_\Omega K_\varphi(x,y,\varphi_n(y))(\varphi_{n+1}(y) - \varphi_n(y))dy = 0,\end{aligned}$$

or still

$$\varphi_{n+1}(x) = f(x) + \int_\Omega K(x,y,\varphi_n(y))dy + \int_\Omega K_\varphi(x,y,\varphi_n(y))(\varphi_{n+1}(y) - \varphi_n(y))dy \tag{4.6.3}$$

In the Newton-Kantorovich method, we remark that, the kernels $K(x,y,\varphi_n(y))$ and $K_\varphi(x,y,\varphi_n(y))$ of the right-hand side of the equation (4.6.3) are replaced by the ones $K(x,y,\varphi_0(y))$ and $K_\varphi(x,y,\varphi_0(y))$ where φ_0 represents the initial value so that, the equation (4.6.3) becomes a linear integral equation. However, in our work we treat the equation (4.6.3) by adapted a modification, where we replace the expression $(\varphi_{n+1}(x) - \varphi_n(x))$ in the right-hand side by the one $(\varphi_n(x) - \varphi_{n-1}(x))$ so that, the equation (4.6.3) becomes

$$\varphi_{n+1}(x) = f(x) + \int_\Omega K(x,y,\varphi_n(y))dy + \int_\Omega K_\varphi(x,y,\varphi_n(y))(\varphi_n(y) - \varphi_{n-1}(y))dy\ . \tag{4.6.4}$$

The first approximation $\varphi_1(x)$ is obtained by substituting the initial approximation $\varphi_0(x) = f(x)$ into the right hand side of the integral equation, giving

$$\varphi_1(x) = f(x) + \int_\Omega K(x,y,f(y))dy + \int_\Omega K_\varphi(x,y,f(y))(f(x) - 0)dt\ ,$$

and so on, higher iterates may be defined by equation (4.6.3) where we approximate the two integrals presented in equation (4.6.3) by one of the basic numerical integration formulas such as trapezoid method, Simpson methods, or Gauss methods.

4.7 Convergence Theorem

Theorem 4.7.1 *Let P be an operator defined on a Banach space E into a Banach space F and Fréchet differentiable for $\varphi \in \Omega$ an open convex set in E, satisfies the following* *conditions*

$$\begin{aligned}
&(A1) \qquad \|P'(\varphi)-P'(\psi)\| \leq L\,\|\varphi-\psi\|\,, \quad \varphi,\psi\in\Omega\\
&(A2) \qquad \left\|[P'(\varphi_0)]^{-1}\right\| \leq M, \qquad \varphi_0\in\Omega\\
&(A3) \qquad \left\|[P'(\varphi_0)]^{-1}P(\varphi_0)\right\| \leq N, \quad \varphi_0\in\Omega,
\end{aligned}$$

with the constants L, M *and* N *satisfying* $LM<1$, $LMN\leq\dfrac{1}{2}$ *then there exists a domain* $\Omega_1=\left\{\varphi;\ \|\varphi-\varphi_0\|\leq h=\dfrac{(\sqrt{2}-1)}{LM\sqrt{2}}\right\}\subset\Omega$ *such that, the successive approximations*

$$\varphi_{n+1}=\varphi_n-[P'(\varphi_n)]^{-1}P(\varphi_n),$$

are defined for all n, $\varphi_n\in\Omega_1, n=1,2,...$ *and converge to the exact solution* $\varphi\in\Omega_1$ *which satisfies* $P(\varphi)=0$. *Further*

$$\|\varphi_n-\varphi\|\leq\frac{C}{2^n\,(LMN)},\quad C\in\mathbb{R}_+^*,\quad n=1,2,3.....$$

proof. Indeed, it is easy to see that

$$\begin{aligned}
P(\varphi)-P(\psi) &= \int_0^1 P'\left(\varphi+y\left(\varphi-\psi\right)\right)\left(\varphi-\psi\right)dy\\
&= \int_0^1 \left(P'\left(\varphi+y\left(\varphi-\psi\right)\right)-P'(\varphi)\right)\left(\varphi-\psi\right)dy\\
&\quad +\int_0^1 P'(\varphi)\left(\varphi-\psi\right)dy\\
\|P(\varphi)-P(\psi)-P'(\varphi)(\varphi-\psi)\| &= \left\|\int_0^1 \left(P'\left(\varphi+y\left(\varphi-\psi\right)\right)-P'(\varphi)\right)\left(\varphi-\psi\right)dy\right\|\\
\|P(\varphi)-P(\psi)-P'(\varphi)(\varphi-\psi)\| &\leq \int_0^1 \|P'(\varphi+y(\varphi-\psi)-P'(\varphi)\|\,\|(\varphi-\psi)\|\,dy,
\end{aligned}$$

using condition (A1), on Ω we obtain

$$\|P(\varphi)-P(\psi)-P'(\varphi)(\varphi-\psi)\|\leq\frac{L}{2}\,\|(\varphi-\psi)\|^2\,. \tag{4.7.1}$$

Also, for $\varphi\in\Omega_1$,we get

$$\|P'(\varphi)-P'(\varphi_0)\|\leq L\,\|\varphi-\varphi_0\|\leq Lh<\frac{1}{M}, \tag{4.7.2}$$

then, the relation (4.7.2) shows that $P'(\varphi)$ is invertible for all $\varphi\in\Omega_1$ and it comes

$$[P'(\varphi)]^{-1}=\left(I-[P'(\varphi_0)]^{-1}(P'(\varphi)-P'(\varphi_0)\right)^{-1}[P'(\varphi_0)]^{-1},$$

or still

$$\left\|[P'(\varphi)]^{-1}\right\| \leq \frac{M}{(1-LM)\left\|\varphi-\varphi_0\right\|}$$

Given the Newton function as

$$N(\varphi)=\varphi-[P'(\varphi)]^{-1}P(\varphi), \tag{4.7.3}$$

with φ and $N(\varphi)$ in Ω_1 and $\varphi_{n+1}=N(\varphi_n)$, we get

$$\|N(N(\varphi))-N(\varphi)\|=[P'(N(\varphi))]^{-1}P\left(N(\varphi)\right).$$

Hence

$$\|P\left(N(\varphi)\right)\| \leq \frac{L}{2}\left\|N(\varphi)-\varphi\right\|^2. \tag{4.7.4}$$

On the other hand

$$\left\|[P'(N(\varphi))]^{-1}\right\| \leq \frac{M}{(1-LM)\left\|N(\varphi)-\varphi_0\right\|}. \tag{4.7.5}$$

From the relations (4.7.3) and (4.7.4) we obtain

$$\|N(N(\varphi))-N(\varphi)\| \leq \frac{LM\left\|N(\varphi)-\varphi\right\|^2}{2\left(1-LM\right)\left\|N(\varphi)-\varphi_0\right\|},$$

or still

$$\begin{aligned}\left\|\varphi_{n+1}-\varphi_n\right\| &\leq \frac{LM\left\|\varphi_n-\varphi_{n-1}\right\|^2}{2\left(1-LM\right)\left\|\varphi_n-\varphi_0\right\|} \\ &\leq \frac{\left(LM\right)^n\left\|\varphi_1-\varphi_0\right\|^{2^n}}{2^n\left(1-LM\right)^n\left\|\varphi_1-\varphi_0\right\|^n}.\end{aligned} \tag{4.7.6}$$

From the relation $\left\|\varphi_q-\varphi_p\right\| \leq \left\|\varphi_q-\varphi_{q-1}\right\|+\left\|\varphi_{q-1}-\varphi_{q-2}\right\|+....+\left\|\varphi_{p+1}-\varphi_p\right\|$ it comes the sequence φ_n is Cauchy sequence in Banach space. Thus this sequence φ_n represents the Newton iterations are defined and converges to the solution φ in Ω_1[46]. ■

4.8 Illustrating Examples

4.8.1 Example 1.

Consider the nonlinear integral equation of Volterra

$$\varphi(x)-\int_0^x \sin\varphi(y)dy=x+\cos x-1, \qquad 0\leq x,y\leq 1,$$

where the function $f(x_0)$ is chosen so that the exact solution is given by

$$\varphi(x) = x.$$

The approximate solution $\widetilde{\varphi}(x)$ of $\varphi(x)$ is obtained by the adapted Newton-Kantorovich method.

Table 1.1. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for $N = 10$ is compared with the ones treated in [5].

Values of x	Exact solution φ	Approx solution $\widetilde{\varphi}$	Error	Error [5]
0.000000	0.00000e+00	0.00000e+00	0.000e+00	0e+00
0.200000	2.000000e-01	2.000000e-01	7.4733e-10	4e-04
0.400000	4.000000e-01	4.000000e-01	3.1445e-09	6e-04
0.600000	6.000000e-01	6.000000e-01	7.3628e-09	7e-04
0.800000	8.000000e-01	8.000000e-01	1.3385e-08	9e-04
1.000000	1.00000e+00	1.00000e+00	2.0917e-08	1e-03

Table 1. 2 We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for $N = 10$ is compared with the ones treated in [25].

Values of x	Exact solution φ	Approx solution $\widetilde{\varphi}$	Error	Error [25]
0.000000	0.00000e+00	0.00000e+00	0.00e+00	0.00e+00
0.200000	2.000000e-01	2.000000e-01	7.47e-10	4.22e-08
0.400000	4.000000e-01	4.000000e-01	3.14e-09	1.09e-08
0.600000	6.000000e-01	6.000000e-01	7.36e-09	2.35e-08
0.800000	8.000000e-01	8.000000e-01	1.33e-08	1.42e-08
1.000000	1.00000e+00	1.00000e+00	2.09e-08	2.63e-08

4.8.2 Example 2

Consider the nonlinear integral equation of Volterra

$$\varphi(x) - \int_0^x \varphi^2(y)dy = \exp(x) - \frac{1}{2}(\exp(2x) - 1), \qquad 0 \leq x, y \leq 1,$$

where the function $f(x_0)$ is chosen so that the exact solution is given by

$$\varphi(x) = \exp(x).$$

The approximate solution $\widetilde{\varphi}(x)$ of $\varphi(x)$ is obtained by the adapted Newton-Kantorovich method.

Table 2. We present the exact and the approximate solutions of the equation in the example 2 in some arbitrary points, the error for $N = 10$ is compared with the ones treated in [1].

Values of x	Exact solution φ	Approx solution $\widetilde{\varphi}$	Error	Error [1]
0.000000	1.000000e+00	1.000000e+00	0.00e+00	0.00e+00
0.200000	1.221403e+00	1.221919e+00	5.16e-04	9.40e-04
0.400000	1.491825e+00	1.493531e+00	1.70e-03	3.06e-03
0.600000	1.822119e+00	1.826756e+00	4.63e-03	8.16e-03
0.800000	2.225541e+00	2.238233e+00	1.26e-02	2.16e-02
1.000000	2.718282e+00	2.756934e+00	3.86e-02	6.27e-02

4.8.3 Example 3

Consider the nonlinear integral equation of Volterra

$$\varphi(x) - \int_0^x \frac{1}{2}\varphi^2(y)dy = \sin x + \frac{1}{8}\sin 2x - \frac{1}{4}x, \qquad 0 \leq x, y \leq 1,$$

where the function $f(x_0)$ is chosen so that the exact solution is given by

$$\varphi(x) = \sin x.$$

The approximate solution $\widetilde{\varphi}(x)$ of $\varphi(x)$ is obtained by the adapted Newton-Kantorovich method.

Table 3. We present the exact and the approximate solutions of the equation in the example 3 in some arbitrary points, the error for $N = 10$ is compared with the ones treated in [40].

Values of t	Exact solution φ	Approx solution $\widetilde{\varphi}$	Error	Error [40]
0.000000	0.00000e+00	0.00000e+00	0.00e+00	0.0e+00
0.200000	1.986693e-01	1.986672e-01	2.11e-06	8.4e-03
0.400000	3.894183e-01	3.894008e-01	1.75e-05	5.8e-03
0.600000	5.646425e-01	5.645828e-01	5.97e-05	5.0e-03
0.800000	7.173561e-01	7.172144e-01	1.41e-04	7.0e-03
1.000000	8.414710e-01	8.411949e-01	2.76e-04	4.1e-03

Conclusion and prospects

In our work we studied the integral equations in function spaces, we are interesred to Nonlinear Volterra Integral equations which appears in two forms the Uryshon form giving by

$$\varphi(x) = f(x) + \lambda \int_a^x K(x, y, \varphi(y))dy, \quad x \in [a, b]$$

and the Hammerstein form

$$\varphi(x) = f(x) + \lambda \int_a^x K(x, y)g(y, \varphi(y))dy, \quad x \in [a, b]$$

The aim of this study is to prove the existence and the uniquness of solution of the NVIE in Banach spaces in particular spaces of continuous functions and spaces of measurable functions with different conditions on the kernel K, the function f and the operator N_g.

To do this study we should use the compactness concept in Banach spaces, and the fixed point theory which is a very important tool to prove the existence of solutions of nonlinear integral equations, and the The Nemytskii Operator well be one of the useful tools for the Volterra-Hammerstien equations, then we mentioned different cases with different conditions.

A numerical method for solving nonlinear Volterra, based on an adapted Newton-Kontorovich methods is presented. The efficiency of this method is tested by solving some examples for which the exact solution is known. This allows us to estimate the exactness with our numerical results and compare those with another results. Our method is compared with the ones, the Haar wavelets and collocation, the fixed point technique with cubic Bspline scaling function, Adomian decomposition method and block pulse functions by collocation method. treated by (Babolian and Shahsavaran, 2007) Table 1.1, (Maleknejad et al., 2013) Table 1.2, (Awawdeh et al., 2009) Table 2 and (Shahsavaran, 2011) Table 3 respectively.

Consequently, in the direct continuity of our work precisely in the case of our contribution on the methods of numerical resolution of integral equations, we are interested to master the different new techniques of existence approximations in order to get better than what we got.

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