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Some Properties of Musielak-Orlicz-Sobolev spaces with an application in Nonlinear PDE

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Dedication

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and unwavering support have been my foundation. To my twin brother, Amine,

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Notations

In what follows, we will use the following notations.

$\mathbb{R}^N$	Euclidean, n-dimensional space.
x	Vecteur de $\mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$, $x_i \in \mathbb{R}$, $1 \leq i \leq n$.
$d\mu$	or dx Lebesgue measure N-dimensional.
$\|\Omega\|$	Measure of the set Ω.
Ω	Open set in $\mathbb{R}^n$.
$\overline{\Omega}$	The closure set of in $\mathbb{R}^n$.
$\partial\Omega$	The border of Ω.
$B(x, r)$	Open ball with center x and radius $r > 0$.
$\overline{B}_E$	The closed unit ball of E .
B_E	$= \{x \in E \text{ with } \|\|x\|\| = 1\}$.
$W^{k,p}(\Omega)$	$= \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall \alpha \in \mathbb{N}^n \text{ such that } \|\alpha\| \leq j\}$.
$W_0^{k,p}(\Omega)$	$= \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall \alpha \in \mathbb{N}^n \text{ such that } \|\alpha\| \leq j \text{ with } u = 0 \text{ on } \partial\Omega\}$.
$W^{-k,p'}(\Omega)$	Dual space of $W_0^{k,p}(\Omega)$.
$D_i u$	$= \frac{\partial u}{\partial x_i}$ The partial derivative of u with respect to x_i.
p'	The dual exponent of p.
p^*	$= \frac{Np}{N-p}$, The Sobolev conjugate of p.
$C^\infty(\Omega)$	The set of functions in $C^k(\Omega)$ for all k.
$C_0^\infty(\Omega)$ or $D(\Omega)$	The space of smooth functions with compact support in Ω.
$D'(\Omega)$	The space of real distributions on Ω.
$suppf$	$=\left\{\overline{x \in \Omega : f(x) \neq 0}\right\}$ The support of f.
∇u	The gradient of u.
Δu	The Laplacian of u.
$C(\Omega)$	The set of functions continuous in Ω.
$C(\overline{\Omega})$	The set of functions continuous in $\overline{\Omega}$.
$C^k(\Omega)$	The set of functions which have derivatives of *order* $\leq k$ that are continuous in Ω.
$C^k(\overline{\Omega})$	The set of functions in $C(\overline{\Omega})$ which have derivatives in Ω of order is lass than or equals k.
$L^\theta(\Omega)$	Musielak-Orlicz spaces
$L^{p(.)}(\Omega)$	Generalized Lebesgue spaces
$W^{1,p(.)}(\Omega)$	Generalized Sobolev spaces

p^-	$:=\inf_{x\in\Omega} p(x)$
p^+	$:=\sup_{x\in\Omega} p(x)$
$W^{1,\theta}(\Omega)$	Musielak-Orlicz-Sobolev spaces
$M(\Omega)$	The set of Measurable functions
p_θ	The modular functional associated with $\theta(x,.)$ i,e $p_\theta(u) = \int_\Omega \theta(x,u)dx$
$\hookrightarrow$	Continous embedding
$\|.\|$	Norm of a function
$div(f)$	$:=\sum_{k=1}^{n} \frac{\partial f}{\partial x_k}$ The divergence of the vector f
$\rightharpoonup$	Weak convergence
$C_+(\overline{\Omega})$	$:=\left\{p \in C(\overline{\Omega}) \ : \ \inf_{x\in\overline{\Omega}} p(x) > 1\right\}$
$C_+^{\log}(\Omega)$	Log-H$\ddot{o}$lder continous function.

$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Introduction

The Orlicz spaces were introduced by Z. W. Birnbaum and W. Orlicz (1931) as a natural generalization of the classical Lebesgue spaces L^p with $1 < p < +\infty$ (see [1], [6], [9], [17], [24], and [33]). Subsequently, these spaces were extended to Musielak-Orlicz spaces (also known as generalized Orlicz spaces), which have become an area of growing interest in recent years (for more details, see [9]). The Musielak-Orlicz-Sobolev spaces are particularly essential tools for studying the existence and uniqueness of solutions to a class of nonlinear PDEs, especially the double phase problem with variable exponent as investigated in [5], [7], [27], and [37].

This work is devoted to studying some properties of the Musielak-Orlicz-Sobolev spaces $W^{1,\theta}(\Omega)$ and using them to solve the following Dirichlet problem:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) = f(x,u,\nabla u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

Where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $p, q \in C(\overline{\Omega})$ satisfy $1 < p(x) < N,\quad p(x) < q(x)$ for all $x \in \overline{\Omega}$, $\mu \in L^\infty(\Omega)$, and f is a Carathéodory function. The problem (1) does not have variational structure because the gradient dependence of the convection term f, so variational methods cannot be applied in this case.

The primary aim of this work is to investigate the existence and uniqueness of solutions to problem (1) using non-variational methods, following the approaches developed in [3], [6], [17], [30], [31], and [32].

The essential feature of this kind of work is the imposition of the following log-Hölder continuity condition on $p : \overline{\Omega} \longrightarrow \mathbb{R}$:

$$|p(x) - p(y)| \leq \frac{c_0}{-\ln(|x-y|)} \quad \forall x \neq y \in \overline{\Omega} \text{ with } |x-y| \leq \frac{1}{2},$$

where $c_0 > 0$ is a constant. This condition allow to state some essential embedding as in [1, 13]. It is worth mentioning that the authors of [23] have generalized the problem (1) to multi-phase setting in a Anisotropic Musielak-Orlicz-Sobolev Spaces using the log-Hölder continuity condition.

The memory contains some essential properties of Musielak-Orlicz spaces and Musielak-Orlicz-Sobolev spaces such as the generalized N-function, the (Δ_2) condition, modular, Luxemburg and Orlicz norms, reflexivity, separability, and embedding theorems. These properties are crucial for the analysis of the problem (1), as discussed in [3], [17], [24], and [29].

Another key contribution is the study of the existence and uniqueness of weak solutions for the problem (1), which involves double-phase operators with variable exponents. The analysis is based on monotone operator theory following the approach in [6].

Furthermore, the findings demonstrate the versatility of Musielak-Orlicz-Sobolev spaces in addressing complex mathematical problems, such as (1), with non-standard growth conditions. These results have potential applications in various fields, including materials science and image processing.

The first chapter deals with some fundamental facts about Lebesgue and Sobolev spaces. It serves as a reminder and presents important tools in functional analysis that are essential for studying spaces used in partial differential equations. These spaces are examined in [2], [3], [4], [11] and [22] as well as in [8] for their theoretical foundations.

In the second chapter, we study some definitions and basic properties of Musielak-Orlicz spaces (also known as generalized Orlicz spaces), including convergence, completeness, separability, uniform convexity, reflexivity, and density of smooth functions. These aspects have been investigated by [17] and [18].

The third chapter recalls Lebesgue spaces with variable exponents and provides some examples following [3]. We then examine Musielak-Orlicz-Sobolev spaces including an equivalent norm, uniform convexity, Radon-Riesz property with respect to the modular. These spaces represent the general case of $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$.

In the fourth chapter, we employ two distinct approaches to study nonlinear boundary value problems. First, using variational methods following [23], we investigate the existence and uniqueness of solutions to:

$$\begin{cases} -\Delta_{p(x)} u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary, $\Delta_{p(x)}$ denotes the variable exponent $p(x)$-Laplacian operator with $p \in C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for all } x \in \overline{\Omega}\}$, and f satisfies some appropriate growth conditions.

Second, adopting non-variational techniques from [14], we study the properties of the double phase operator such continuity, strict monotonicity, $(S+)$-property as well as we examine the double-phase problem with variable exponent in Musielak-Orlicz-Sobolev spaces under very general assumptions on the data:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) = f(x,u,\nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

Where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $p, q \in C(\overline{\Omega})$ satisfy $1 < p(x) < N, \quad p(x) < q(x)$ for all $x \in \overline{\Omega}$, $\mu \in L^\infty(\Omega)$, and f is a Carathéodory function verifies some appropriate growth conditions.
Taking all aspects into consideration, this study aims to provide a comprehensive analysis of Musielak-Orlicz-Sobolev spaces, with particular focus on their contributions to improving the properties of solutions to nonlinear partial differential equations of double-phase type. Special attention is given to critical cases that require careful handling of data regularity and non-standard growth conditions. The study also seeks to bridge theoretical insights with practical applications, offering new perspectives on solution behavior within generalized functional frameworks.

Chapter 1

Mathematical background

In this chapter, we recall some fundamental facts about Sobolev spaces and present their key properties. For further details on both Lebesgue and Sobolev spaces, we refer to the comprehensive treatments in [9], [10], [11], [18], and [25].

1.1 Lebesgue spaces

Definition and elementary properties of L_p spaces

Definition 1.1.1.

Let $1 \leq p < \infty$; we set

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, f \text{ is measurable and } |f|^p \in L^1(\Omega) \right\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p \, d\mu \right)^{1/p}.$$

Definition 1.1.2.

We set

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } \exists c > 0 : |f(x)| \leq c \text{ for } \mu\text{-a.e. } x \in \Omega \}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ c > 0 \mid |f(x)| \leq c \text{ for } \mu\text{-a.e. } x \in \Omega \}.$$

Definition 1.1.3.

Suppose that $1 \leq p < \infty$. Then:

(i)

$$L^p_{\text{loc}}(\Omega) = \{ v : v \in L^p(K) \text{ for every compact subset } K \subset \Omega \}.$$

(ii) A function v is said to be *locally integrable* in Ω if $v \in L^1_{\text{loc}}(\Omega)$.

(iii) Let v and w be locally integrable functions defined in Ω. We define v as the weak derivative of u with respect to the multi-index α if, for every test function $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega v\, D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega w\, \varphi \, dx.$$

In this case, we say that $D^\alpha v = w$ in the weak sense.

(iv) Let $v, w \in L^p_{\text{loc}}(\Omega)$. We define w as the strong derivative of v with respect to the multi-index α if, for every compact subset $K \subset \Omega$, there exists a sequence $\{\varphi_i\} \subset C^{|\alpha|}(K)$ such that:

$$\varphi_i \to v \quad \text{in } L^p(K) \quad \text{and} \quad D^\alpha \varphi_i \to w \quad \text{in } L^p(K).$$

Theorem 1.1.1.

If $D^\alpha u = v$ and $D^\beta v = w$ in the weak sense, then $D^{\alpha+\beta} u = w$ in the weak sense.

Proof. □

Let $\psi \in C_0^\infty(\Omega)$, and set $\varphi = D^\beta \psi$. Then, by the definition of the weak derivative,

$$\int_\Omega u D^{\alpha+\beta} \psi \, dx = (-1)^{|\alpha|} \int_\Omega v D^\beta \psi \, dx = (-1)^{|\alpha|} \int_\Omega v \varphi \, dx.$$

since, $D^\beta v = w$, we have:

$$\int_\Omega v \varphi \, dx = (-1)^{|\beta|} \int_\Omega \psi D^\beta v \, dx = (-1)^{|\beta|} \int_\Omega \psi w \, dx.$$

Hence,

$$\int_\Omega u D^{\alpha+\beta} \psi \, dx = (-1)^{|\alpha|+|\beta|} \int_\Omega \psi w \, dx.$$

This shows that $D^{\alpha+\beta} u = w$ in the weak sense.

□

Definition 1.1.4.

Let $\mu \in C_0^\infty(\mathbb{R}^N)$ be such that:

1. $\operatorname{supp} \mu \subset B_1(0)$, (recall that supp denotes the support of a function, and $B_r(c)$ denotes an open ball of radius r and center c),
2. $\int_{\mathbb{R}^N} \mu(x) \, dx = 1$,
3. $\mu(x) \geq 0$.

if $\varepsilon > 0$ then we set (provided that the integral exists)

$$J_\varepsilon u(x) = \frac{1}{\varepsilon^n}\int_\Omega \mu(\frac{x-y}{\varepsilon})u(y)dy.$$

$J_\varepsilon u$ is called a mollifier of u. Note that if u is locally integrable in Ω and if K is a compact subset of Ω then $J_\varepsilon u \in C^\infty(K)$ provided that $\varepsilon < dist(K, \partial\Omega)$.Suppose now that $u \in L^p_{loc}(\Omega)$.

$$J_\varepsilon u(x) = \int_{B_1(0)} \mu(y)u(x-\varepsilon y)dy,$$

so for $p > 1$ we have (if $\frac{1}{p} + \frac{1}{q} = 1$)

$$\begin{aligned}|J_\varepsilon u(x)| &\le \int_{B_1(0)} \{\mu(y)\}^{\frac{1}{q}} \{\mu(y)\}^{\frac{1}{p}} |u(x-\varepsilon y)|dy \\ &\le (\int_{B_1(0)} (\{\mu(y)\}^{\frac{1}{q}})^q dx)^{\frac{1}{q}} (\int_{B_1(0)} (\{\mu(y)\}^{\frac{1}{p}} |u(x-\varepsilon y)|)^p dy)^{\frac{1}{p}}.\end{aligned}$$

Hence $|J_\varepsilon u(x)|^p \le \int_{B_1(0)} \mu(y)|u(x-\varepsilon y)|^p dy$, and this trivially holds if $p = 1$ too. Integrating this, we see that

$$\begin{aligned}\int_K |J_\varepsilon u(x)|^p dx &\le \int_{B_1(0)} \mu(y) \int_K |u(x-\varepsilon y)|^p dxdy \\ &\le \int_{B_1(0)} \mu(y) \int_{K_0} |u(x)|^p dxdy \\ &= \int_{K_0} |u(x)|^p dx,\end{aligned}$$

where K_0 is a compact subset of Ω, $K \subset Interior(K_0)$ and $\varepsilon < dist(K, \partial K_0)$ i.e. we have

$$\|J_\varepsilon u\|_{L^p(K)} \le \|u\|_{L^p(K_0)}. \tag{1.1}$$

Lemma 1.1.1.

Let $u \in L^p_{\text{loc}}(\Omega)$, and let $K \subset \Omega$ be a compact subset. Then

$$\|J_\varepsilon u - u\|_{L^p(K)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Let $K_0 \subset \Omega$ be a compact set such that $K \subset \text{Int}(K_0)$, and choose $\varepsilon > 0$ such that

$$\varepsilon < \text{dist}(K, \partial K_0).$$

Let $\delta > 0$. Since $C^\infty(K_0)$ is dense in $L^p(K_0)$, there exists a function $w \in C^\infty(K_0)$ such that

$$\|u - w\|_{L^p(K_0)} < \delta.$$

Applying inequality (1.1) to the function $u - w$, we obtain

$$\|J_\varepsilon u - J_\varepsilon w\|_{L^p(K)} < \delta.$$

Moreover, we have

$$J_\varepsilon w(x) - w(x) = \int_{B_1(0)} \mu(y)\,[w(x-\varepsilon y) - w(x)]\;dy,$$

and since $w \in C^\infty$, the integrand tends uniformly to zero as $\varepsilon \to 0$, so:

$$J_\varepsilon w(x) \to w(x) \quad \text{uniformly on } K.$$

Therefore, for ε sufficiently small,

$$\|J_\varepsilon w - w\|_{L^p(K)} < \delta.$$

□

Now, combining the above estimates:

$$\begin{aligned} \|J_\varepsilon u - u\|_{L^p(K)} &\le \|u - w\|_{L^p(K)} + \|J_\varepsilon u - J_\varepsilon w\|_{L^p(K)} + \|J_\varepsilon w - w\|_{L^p(K)} \\ &< \delta + \delta + \delta = 3\delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary, we conclude that

$$\|J_\varepsilon u - u\|_{L^p(K)} \to 0 \quad \text{as } \varepsilon \to 0. \quad \blacksquare$$

Theorem 1.1.2.

Suppose that $u, v \in L^p_{\text{loc}}(\Omega)$. Then $D^\alpha u = v$ in the weak sense if and only if $D^\alpha u = v$ in the strong $L^p(\Omega)$ sense.

Proof. □

Suppose that $D^\alpha u = v$ in the strong $L^p(\Omega)$ sense. Let $\varphi \in C_c^\infty(\Omega)$, and let $K = \text{supp}(\varphi)$. Let $\varepsilon > 0$, and take $\psi \in C^{|\alpha|}(K)$ such that

$$\|u - \psi\|_{L^p(K)} < \varepsilon \quad \text{and} \quad \|D^\alpha \psi - v\|_{L^p(K)} < \varepsilon.$$

Then:

$$\begin{aligned} \left| \int_K u D^\alpha \varphi\, dx - (-1)^{|\alpha|} \int_K v\varphi\, dx \right| &= \left| \int_K (u-\psi) D^\alpha \varphi\, dx + \int_K \psi D^\alpha \varphi\, dx - (-1)^{|\alpha|} \int_K v\varphi\, dx \right| \\ &= \left| \int_K (u-\psi) D^\alpha \varphi\, dx + (-1)^{|\alpha|} \int_K \varphi D^\alpha \psi\, dx - (-1)^{|\alpha|} \int_K v\varphi\, dx \right| \\ &= \left| \int_K (u-\psi) D^\alpha \varphi\, dx + (-1)^{|\alpha|} \int_K \varphi (D^\alpha \psi - v)\, dx \right| \\ &\le \|u-\psi\|_{L^p(K)} \|D^\alpha \varphi\|_{L^q(K)} + \|D^\alpha \psi - v\|_{L^p(K)} \|\varphi\|_{L^q(K)} \\ &\le \varepsilon \left(\|D^\alpha \varphi\|_{L^q(K)} + \|\varphi\|_{L^q(K)} \right), \end{aligned}$$

where q is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Since ε is arbitrary, the left-hand side must be zero. Therefore, $D^\alpha u = v$ in the weak sense.

Conversely, suppose that $D^\alpha u = v$ in the weak sense. Let $K \subset \Omega$ be compact. Then for $\varepsilon < \text{dist}(K, \partial\Omega)$, the mollification $J_\varepsilon u \in C^\infty(K)$ satisfies:

$$D^\alpha J_\varepsilon u(x) = \frac{1}{\varepsilon^n} D^\alpha_x \left(\int_\Omega \mu\left(\frac{x-y}{\varepsilon}\right) u(y)\, dy \right).$$

By differentiating under the integral and changing variables:

$$D^\alpha J_\varepsilon u(x) = \frac{1}{\varepsilon^n} \int_\Omega D^\alpha_y \left[\mu\left(\frac{x-y}{\varepsilon}\right)\right] u(y)\, dy = \frac{1}{\varepsilon^n} \int_\Omega \mu\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy = J_\varepsilon v(x),$$

since $D^\alpha u = v$ in the weak sense. Hence $D^\alpha J_\varepsilon u = J_\varepsilon v$, and so $D^\alpha u = v$ in the strong L^p sense by approximation. □

but by lemma 1.1.1 ,

$$\|J_\varepsilon u - u\|_{L^p(K)} \to 0 \quad \text{and} \quad \|D^\alpha J_\varepsilon u - v\|_{L^p(K)} = \|J_\varepsilon v - v\|_{L^p(K)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus, $D^\alpha u = v$ in the strong sense.

Theorem 1.1.3 (Young's Inequality).

Suppose that $a, b \geq 0$ and $1 < p, q < \infty$ with

$$\frac{1}{p} + \frac{1}{q} = 1,$$

the Young inequality is expressed by

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

Theorem 1.1.4 (H$\ddot{o}$lder Inequality).

Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, where $1 \leq p < \infty$.
Then $fg \in L^1(\Omega)$ and

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^{p'}}. \tag{1.2}$$

Theorem 1.1.5 (Minkowki Inequality).

Let $1 \leq p < \infty$, we assume that $f, g \in L^p(\Omega)$.Then,$f + g \in L^p(\Omega)$ and

$$\left\|f + g\right\|_p \leq \left\|f\right\|_p + \left\|g\right\|_p \tag{1.3}$$

Remark 1.1.1.

In case of $p = \infty$ we have:

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

Definition 1.1.5 (Continous Embedding).

Suppose that E and F are Banach spaces. If $E \subset F$, we say that E is *continuously embedded* in F (denoted $E \hookrightarrow F$) if there exists a constant $C > 0$ such that

$$\|x\|_F \leq C\|x\|_E \quad \text{for all } x \in E.$$

Exemple 1.1.1.

Let (X, T, μ) be a finite Measure space, for all $1 \leq p, q \leq \infty$ Where $p \leq q$ .Then,

$$L^q(X, T, \mu) \hookrightarrow L^p(X, T, \mu)$$

And $\|f\|_p \leq \mu(\Omega)^{\frac{1}{p}-\frac{1}{q}} \|f\|_q$

In the fact, since $p \leq q$ Then ,$\frac{1}{\frac{q}{p}} + \frac{1}{\frac{q}{q-p}} = 1$, by (1.2) we get the result.

Theorem 1.1.6 (Riesz-Fréchet). *[11]*

L^p is Banach space for all $1 \leq p \leq \infty$

1.2 Sobolev spaces

Motivation and definition of Sobolev spaces

Let $C_c^\infty(\Omega)$ be the set of infinitely differentiable functions with compact support, that is,

$$\varphi : \Omega \to \mathbb{R}, \quad \varphi \in C_c^\infty(\Omega).$$

A function $\varphi \in C_c^\infty(\Omega)$ is often called a *test function*.
For each $u \in C^1(\Omega)$, and taking into account that every $\varphi \in C_c^\infty(\Omega)$ has compact support, Green's identity yields

$$\int_\Omega u\, \partial_{x_i}\varphi\, dx = -\int_\Omega \partial_{x_i} u\, \varphi\, dx \quad \forall \varphi \in C_c^\infty(\Omega), \quad \forall i = 1, \ldots, n.$$

This motivates the definition of Sobolev spaces as follows:

Definition 1.2.1. *[4]*

For each $1 \leq p \leq \infty$, the Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \,\middle|\, \exists g_1, \ldots, g_N \in L^p(\Omega) : \int_\Omega u\, \partial_{x_i}\varphi = -\int_\Omega g_i\, \varphi \quad \forall \varphi \in C_c^\infty(\Omega),\ \forall i = 1, \ldots, N \right\}.$$

We denote $\partial_{x_i} u = g_i$ as the weak derivative of $u \in W^{1,p}(\Omega)$, which is unique, as we will see shortly. The gradient of u is defined as

$$\nabla u = (\partial_{x_1} u, \dots, \partial_{x_N} u).$$

If $p = 2$, we write

$$H^1(\Omega) = W^{1,2}(\Omega).$$

Definition 1.2.2.

For each $u \in W^{1,p}(\Omega)$, we define the norm of u by

$$\|u\|_{W^{1,p}} = \|u\|_p + \sum_{i=1}^{N} \|\partial_{x_i} u\|_p. \tag{1.4}$$

Moreover, $H^1(\Omega)$ is a Hilbert space equipped with the following scalar product:

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^{N} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}. \tag{1.5}$$

Since $H^1(\Omega)$ is a Hilbert space, (1.5) induces a norm in $H^1(\Omega)$,

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^2}^2 \right)^{1/2}.$$

This norm is equivalent to (1.4) for $p = 2$.

Some properties of Sobolev spaces

Proposition 1.2.1.

Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, the following statements hold:

(i) For each $1 \leq p \leq \infty$, $W^{1,p}(\Omega)$ is a Banach space.

(ii) For each $1 < p < \infty$, $W^{1,p}(\Omega)$ is reflexive.

(iii) For each $1 \leq p < \infty$, $W^{1,p}(\Omega)$ is separable.

Proof. □

1. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$. Then, from (1.4) it follows that $\{u_n\}_{n\in\mathbb{N}}$ and $\{(u_n)_{x_i}\}_{n\in\mathbb{N}}$, with $1 \leq i \leq N$, are Cauchy sequences in $L^p(\Omega)$. Thus, since $L^p(\Omega)$ is a Banach space, it follows that $u_n \to u$ and $(u_n)_{x_i} \to g_i$ in $L^p(\Omega)$ with $u, g_i \in L^p(\Omega)$. Therefore, since

$$\int_\Omega u_n \varphi_{x_i} \, dx = -\int_\Omega (u_n)_{x_i} \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega),$$

letting $n \to +\infty$, we get

$$\int_\Omega u\varphi_{x_i}\,dx = -\int_\Omega g_i\varphi\,dx \quad \forall\varphi \in C_0^\infty(\Omega).$$

Therefore, we obtain that $u \in W^{1,p}(\Omega)$, $u_{x_i} = g_i$ and thus

$$\|u_n - u\|_{W^{1,p}(\Omega)} = \|u_n - u\|_{L^p(\Omega)} + \sum_{i=1}^{N} \|(u_n)_{x_i} - g_i\|_{L^p(\Omega)} \to 0,$$

as desired.

2. Consider the space $E = L^p(\Omega)\times L^p(\Omega)$ which is reflexive since it is the product of reflexive spaces. Set the operator

$$T : W^{1,p}(\Omega) \to E$$

defined by

$$Tu = (u, \nabla u).$$

Then, T is an isometry, and since $W^{1,p}(\Omega)$ is a Banach space,

$$M = T(W^{1,p}(\Omega))$$

is a closed subspace of E since E is reflexive, B_E is compact in the weak topology $\sigma(E, E^*)$, and M is closed in the topology $\sigma(E, E^*)$. Therefore, B_M is compact in $\sigma(E, E^*)$, and therefore $T(W^{1,p}(\Omega))$ is reflexive. As a consequence, $W^{1,p}(\Omega)$ is also reflexive.

3. Under the notation of (2) ,and taking into account that E is separable,it follows that $T(W^{1,p}(\Omega))$ is separable and therefore $W^{1,p}(\Omega)$ is also separable.

Theorem 1.2.1 (Poincaré Inequality).

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open , and let $1 \le p < \infty$. Then there exists a constant $C > 0$ such that for all $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}.$$

In particular, for p = 2, we have:

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \quad \text{for all } u \in H_0^1(\Omega).$$

Proof. □

Assume the inequality is false. Then, for every integer n, there exists $u_n \in W_0^{1,p}(\Omega)$ such that

$$\|u_n\|_{L^p(\Omega)} > n\|\nabla u_n\|_{L^p(\Omega)}.$$

Define normalized functions

$$v_n = \frac{u_n}{\|u_n\|_{L^p(\Omega)}}.$$

Then $\|v_n\|_{L^p(\Omega)} = 1$, and

$$\|\nabla v_n\|_{L^p(\Omega)} = \frac{\|\nabla u_n\|_{L^p(\Omega)}}{\|u_n\|_{L^p(\Omega)}} < \frac{1}{n}.$$

Hence,

$$\|\nabla v_n\|_{L^p(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Because $v_n \in W_0^{1,p}(\Omega)$ and Ω is bounded, by compactness (Rellich-Kondrachov theorem), there exists a subsequence (still called v_n) that converges in $L^p(\Omega)$ to some function v.
But since $\|\nabla v_n\|_{L^p} \to 0$, the limit v must have zero gradient, i.e., v is constant.
Moreover, $v \in W_0^{1,p}(\Omega)$ implies $v = 0$ on the boundary, so $v \equiv 0$.
On the other hand, since $\|v_n\|_{L^p(\Omega)} = 1$, we must have

$$\|v\|_{L^p(\Omega)} = \lim_{n\to\infty} \|v_n\|_{L^p(\Omega)} = 1,$$

which contradicts $v \equiv 0$.
Therefore, the assumption is false, and the Poincaré inequality holds.

$$\boxed{\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}}$$

Lax-Miligram and Stampacchia Lemma

Lemma 1.2.1 (Lax-Miligram Lemma).

Let $A : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear mapping. Then, for every bounded linear functional $F : H \to \mathbb{R}$, there exists a unique element $v \in H$ such that

$$A(v, w) = F(w), \quad \text{for all } w \in H.$$

Lemma 1.2.2 (Stampacchia Lemma).

Let H be a Hilbert space and let $A : H \times H \to \mathbb{R}$ be a mapping that is continuous and linear in the second variable.
Assume that there exist two constants $K_1 > 0$ and $K_2 > 0$ such that:

(1) For all $v_1, v_2 \in H$ and for all $\varphi \in H$, we have:

$$|A(v_1, \varphi) - A(v_2, \varphi)| \leq K_1 \|v_1 - v_2\| \cdot \|\varphi\|.$$

(2) For all $v_1, v_2 \in H$, we have:

$$A(v_1, v_1 - v_2) - A(v_2, v_1 - v_2) \geq K_2 \|v_1 - v_2\|^2.$$

Then, for every bounded linear functional $F \in H^*$, there exists a unique $v \in H$ such that

$$A(v, w) = F(w), \quad \text{for all } w \in H.$$

1.3 $W^{m,p}(\Omega)$ spaces

After defining the spaces $W^{1,p}(\Omega)$, we can define the general $W^{m,p}$ spaces recursively. Let $m \geq 2$ be an integer and $1 \leq p \leq \infty$. Then, we define

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega) \,\middle|\, \partial_{x_i} u \in W^{m-1,p}(\Omega) \text{ for all } i = 1, \dots, N \right\}.$$

An equivalent way of defining the Sobolev spaces is by using multi-index notation. We define

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) \,\middle|\, \forall \alpha, |\alpha| \leq m, \ \exists g_\alpha \in L^p(\Omega) : \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi \quad \forall \varphi \in C_c^\infty(\Omega) \right\}.$$

Here, multi-index notation is used. That is, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, and the order of the multi-index is defined by

$$|\alpha| = \sum_{i=1}^{N} \alpha_i.$$

Moreover, the differential operator D^α is defined as

$$D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

We denote $D^\alpha u = g_\alpha$. It can be shown (although the proof is omitted here) that $W^{m,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p}. \tag{1.6}$$

Just as in the case of $W^{1,2}(\Omega)$, the space $W^{m,2}(\Omega)$, denoted by $H^m(\Omega)$, is a Hilbert space with the scalar product

$$(u, v)_{H^m} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2}.$$

The norm induced by this scalar product is equivalent to the norm in (1.6).

1.4 Monotone operator

Definition 1.4.1.

Assume that X is a real Banach space and $\Psi : X \to X^*$ is an operator. Then:

1. Ψ is monotone $\Leftrightarrow \langle \Psi(u) - \Psi(v), u - v \rangle \geq 0$, for all $u, v \in X$.

2. Ψ is strictly monotone $\Leftrightarrow \langle \Psi(u) - \Psi(v), u - v \rangle > 0$, for all $u, v \in X$, with $u \neq v$.

3. Ψ is strongly monotone $\Leftrightarrow$ there exists $c > 0$ such that

   $$\langle \Psi(u) - \Psi(v), u - v \rangle \geq c\|u - v\|^2, \quad \text{for all } u, v \in X.$$

4. Ψ is uniformly monotone $\Leftrightarrow$

   $$\langle \Psi(u) - \Psi(v), u - v \rangle \geq s(\|u - v\|)\|u - v\|, \quad \text{for all } u, v \in X,$$

   where the continuous function $s : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing with

   $$s(0) = 0 \quad \text{and} \quad s(t) \to +\infty \text{ as } t \to +\infty.$$

   For instance, we may choose $s(t) = c|t|^{p-1}$ with $p > 1$ and $c > 0$. In this case, we have

   $$\langle \Psi(u) - \Psi(v), u - v \rangle \geq c\|u - v\|^p, \quad \text{for all } u, v \in X.$$

5. Ψ is hemi-continuous if for all $u, v \in X$, the mapping $t \mapsto \langle \Psi(u + tv), v \rangle$ is continuous from $\mathbb{R}$ into $\mathbb{R}$.

6. Ψ is bounded if it maps bounded subsets of X into bounded subsets of X^0.

7. Ψ is coercive $\Leftrightarrow \dfrac{\langle \Psi(v), v \rangle}{\|v\|_X} \to +\infty$ as $\|v\|_X \to +\infty$.

8. Ψ is weakly coercive $\Leftrightarrow \|\Psi(v)\| \to +\infty$ as $\|v\|_X \to +\infty$.

Remark 1.4.1.

Obviously, we have:

$$\text{strongly monotone} \implies \text{uniformly monotone} \implies \text{strictly monotone} \implies \text{monotone.}$$

Definition 1.4.2.

Suppose that $A : X \to X^*$ is an operator on the real reflexive Banach space X with value in X^* its dual space and denote by $\langle \cdot, \cdot \rangle$ its duality pairing.

1. The operator A is called pseudomonotone if for every sequence $\{u_n\} \subset X$ such that

$$u_n \rightharpoonup u \quad \text{in } X,$$

and

$$\limsup_{n\to\infty} \langle Au_n, u_n - u \rangle \leq 0,$$

it holds that for all $w \in X$,

$$\langle Au, u - w \rangle \leq \liminf_{n\to\infty} \langle Au_n, u_n - w \rangle.$$

2. The operator A satisfies the (S_+)-property if for every sequence $\{u_n\} \subset X$ such that

$$u_n \rightharpoonup u \quad \text{in } X,$$

and

$$\limsup_{n\to\infty} \langle Au_n, u_n - u \rangle \leq 0,$$

imply

$$u_n \longrightarrow u \quad \text{in } X,$$

3. The operator A coercive if there exists some function $\phi : [0, \infty) \longrightarrow \mathbb{R}$ such that ${}_{t \longrightarrow +\infty} \psi(t) = +\infty$ and

$$\frac{\langle \Psi(u), u \rangle}{\|u\|_X} \geq \psi(\|u\|_X), \quad \text{for all} \quad u \in X.$$

Theorem 1.4.1. *(Minty-Browder Theorem)[1]*

Assume That X is a real reflixive Banach space. We take $\psi : X \to X^*$ is a bounded, monotone, hemicontinous, and corecive operator. Then ψ is surjective, that is for every $h \in X^*$ there exists $u \in X$ such that $\psi(u) = h$. We need the following surjectivity result for pseudomonotone operators.

Theorem 1.4.2. *[26]*

Let X be a real, reflexive Banach space, let $\psi : X \longrightarrow X^*$ be a pseudomonotone, bounded, and coercive operator, and $h \in X^*$. Then, a solution of the equation $\psi u = h$ exists.

1.5 Function of Carathéodory

The Carathéodory functions are used in solving problems in partial differential equations, so we have the following definition:

Definition 1.5.1. *[28] A function $f(t,x)$ defined on*

$$R: |t-\tau| \leq a, \quad |x-\xi| \leq b$$

is called a ***Carathéodory function*** *if:*

1. *$f(x,t)$ is continuous in t for almost every x*
2. *$f(x,t)$ is measurable in x for every t*
3. *$|f(t,x)| \leq m(t)$ for some integrable function $m(t)$*

Theorem 1.5.1 (Carathéodory's Existence Theorem). *[28] Under the above conditions, there exists an absolutely continuous function $u(t)$ satisfying:*

$$u'(t) = f(t,u(t)) \text{ a.e.}, \; u(\tau) = \xi$$

Remarks 1.5.1. *A function $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is called a* ***Carathéodory function*** *if:*

1. *$a(x,s,\xi)$ is continuous in s,ξ for almost every x*
2. *$f(x,s,\xi)$ is measurable in x for every s,ξ*

Chapter 2

Musielak-Orlicz spaces

In this chapter, we recall some fundamental definitions and properties of Musielak-Orlicz spaces. For further details, we refer to [6], [15], [17], [24] and [26]

2.1 Generalized N-function

Definition 2.1.1.

Let Ω be an open subset of $\mathbb{R}^N$.A function $\theta : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is called a generalized N-function if it satisfies the following conditions:

1. For a.e $x \in \Omega$, $\theta(x,t)$ is even,continous,nondecreasing and convex in t,and for each $t \in \mathbb{R}$ $\theta(x,t)$ is mesurable in x.
2. $\lim\limits_{t \longrightarrow 0} \frac{\theta(x,t)}{t} = 0$ for a,e $x \in \Omega$
3. $\lim\limits_{t \longrightarrow +\infty} \frac{\theta(x,t)}{t} = \infty$,for a,e $x \in \Omega$
4. $\theta(x,t) > 0$,for all $t > 0$ and all $x \in \Omega$ and $\theta(x,0) = 0$,for all $x \in \Omega$

Exemple 2.1.1.

We Consider $\Omega =]0,1[$ and we have:

$$\begin{aligned} \theta :]0,1[\times \mathbb{R} &\rightarrow \mathbb{R} \\ (x,t) &\rightarrow \theta(x,t) = xt^2 \end{aligned}$$

Satisfying The following conditions:

1. It's Clear that $(x,t) \rightarrow \theta(x,t)$ is even it t,Continous,nondecreasing and Convex in t (Since $\theta''_x(t) > 0$)
2. $\lim\limits_{t \rightarrow 0} xt = 0$
3. $\lim\limits_{t \rightarrow +\infty} xt = +\infty$
4. it's Clear $\theta(x,t)$ is Positif and we have $\theta(x,0) = 0$ for all $x \in \Omega$

   We have Other Exemples satify Generalized N-function:

- $\theta(x,t) = t^{2x}\log(1+t)$
- $\theta(x,t) = t\left(\log(t+1)\right)^{x^2}$
- $\theta(x,t) = (e^t)^{3x} - 1.$

Remark 2.1.1.

We give an equivalent definition of a generalized N-function that admits an integral representation .For $x \in \Omega$ and $t \geq 0$,we denote by $h(x,t)$ the right-hand derivative of $\theta(x,.)$ at t,and define $h(x,t) = -h(x,-t)$ for $t < 0$.Then for each $x \in \Omega$,The function $h(x,.)$ is odd,$h(x,t) \in \mathbb{R}$ for $t \in \mathbb{R}$,$h(x,0) = 0$,$h(x,t) > 0$ for $t > 0$,$h(x,.)$ is right-continous and nondecreasing on $[0,+\infty)$, $h(x,t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, and

$$\theta(x,t) = \int_0^{|t|} h(x,s)ds, \text{ for } x \in \Omega, \text{ and } t \in \mathbb{R}$$

Exemple 2.1.2.

By The Exemple 2.1.1 we have:

$$\theta(x,t) = \int_0^{|t|} 2xsds = 2x\int_0^{|t|} sds = xt^2$$

Such That The function $(x,s) \rightarrow h(x,s) = 2xs$ verify Remark 2.1.1

Definition 2.1.2.

We say that a generalized N-function θ satisfies the Δ_2-condition if there exist $C_0 > 0$ and a nonnegative function $m \in L^1(\Omega)$ such that:

$$\theta(x,2t) \leq C_0\theta(x,t) + m(x), \text{ for a,e } x \in \Omega \text{ and all } t \geq 0$$

Exemple 2.1.3.

We have $\Omega =]0,1[$,let $\theta(x,2t) = 4xt^2$ such that:

$$\begin{aligned}
\theta(x,2t) = 4xt^2 &\leq \frac{16+(xt^2)^2}{2}, \text{ By Young s Inequality} \\
&\leq 8 + \frac{(xt^2)^2}{2} \\
&\leq \frac{t^2}{2}xt^2 + 8 \\
&\leq C_0\theta(x,t) + m(x)
\end{aligned}$$

Such that $\exists C_0 = \frac{t^2}{2} > 0$ and $m(x) = 8$ Satisfy:

$$\int_0^1 8dx = 8 < +\infty \Rightarrow m \in L^1(\Omega)$$

Definition 2.1.3.

Let θ_1 and θ_2 be two generalized N-functions,

1. We say that θ_1 increas essentially slower than θ_2 near infinity and we write $\theta_1 << \theta_2$,if for any $k > 0$:
$$\lim_{t \longrightarrow \infty} \frac{\theta_1(x, kt)}{\theta_2(x, t)} = 0, \text{ uniformly in } x \in \Omega$$
2. We say that θ_1 is weaker than θ_2,denoted by $\theta_1 \leqslant \theta_2$,if there exist two positive constants C_1, C_2 and a nonnegative function $m \in L^1(\Omega)$ such that
$$\theta_1(x, t) \leq C_1 \theta_2(x, C_2 t) + m(x), \text{ for a,e } x \in \Omega, \text{ and all } t \geq 0$$

Exemple 2.1.4.

1. We put $\Omega =]0, 1[$ and we have $\theta_1(x, kt) = k^2 x t^2$, $\theta_2(x, t) = x t^4$ for all $k > 0$, and $t \in \mathbb{R}, x \in \Omega$ Such that $\theta_i, i \in \{1, 2\}$ satisfying The Conditions of Generalized N-function Then:
$$\lim_{t \to +\infty} \frac{\theta_1(x, kt)}{\theta_2(x, t)} = \lim_{t \to +\infty} \frac{k^2}{t^2} = 0, \text{ uniformly in } \Omega$$
2. Let $\theta_1(x, t) = x t^2$,By Young's Inequality ,We obtain that:

$$\begin{aligned}
\theta_1(x, t) = x t^2 = x 1 t^2 &\leq x(\frac{1 + t^4}{2}) \\
&\leq \frac{x + x t^4}{2} \\
&\leq \frac{1}{2} x t^4 + \frac{x}{2} \\
&\leq \frac{1}{2} \theta_2(x, t) + m(x)
\end{aligned}$$

Where $\exists C_1 = \frac{1}{2} \geq 0, \exists C_2 = 1 \geq 0$ and $m(x) = \frac{x}{2}$ Such that:

$$\int_\Omega \frac{x}{2} dx = \int_0^1 \frac{x}{2} dx = \frac{1}{4} < \infty$$

Definition 2.1.4.

For any generalized N-function θ,the function $\tilde{\theta} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$\tilde{\theta}(x, t) := \sup_{\tau \geq 0} (t\tau - \theta(x, \tau)), \text{ for all } x \in \Omega \text{ and all } t \geq 0 \tag{2.1}$$

is called the complementary function of θ.

In view of the definition of the complementary function $\tilde{\theta}$,we have the following Young's type Inequality:

$$\tau\sigma \leq \theta(x, \tau) + \tilde{\theta}(x, \sigma) \tag{2.2}$$

Proposition 2.1.1.

Let θ, ψ are Generalized N-function Then:
The estimate $\theta(x,\tau) \leq \psi(x,\tau)$ holds for all $\tau \geq 0$ and all $x \in \Omega$ Then,We have $\tilde{\psi}(x,t) \leq \tilde{\theta}(x,t)$ for all $t \geq 0, x \in \Omega$

Proof. □

Let $\theta(x,\tau) \leq \psi(x,\tau)$ for all $\tau \geq 0$ and all $x \in \Omega$.Then,

$$\tilde{\psi}(x,t) = \sup_{\tau \geq 0}(t\tau - \psi(x,\tau)) \leq \sup_{\tau \geq 0}(t\tau - \theta(x,\tau)) \leq \tilde{\theta}(x,t)$$

Remarks 2.1.1.

1. Note that the complementary function $\tilde{\theta}$ is also a generalized N-function.

Now, we define the Musielak-Orlicz space as follows (Important):

$$L^{\theta}(\Omega) := \{u : \Omega \longrightarrow \mathbb{R};\ \text{u measurable such that } p_{\theta}(\lambda u) < \infty\}$$

where

$$\begin{aligned} p_{\theta} : L^{\theta}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longrightarrow p_{\theta}(u) = \int_{\Omega} \theta(x,u)dx \end{aligned}$$

The space $L^{\theta}(\Omega)$ is endowed with the Luxemburg norm

$$\|u\|_{L^{\theta}(\Omega)} := inf\left\{\lambda > 0 : p_{\theta}(\frac{u}{\lambda}) \leq 1\right\} \tag{2.3}$$

Definition 2.1.5.

Let X be a real vector space. A function $p : X \to [0,+\infty)$ is said to be a **semi-modular** on X if the following properties are satisfied:

1. $p(0) = 0$.
2. $p(\lambda x) = p(x),$ for all $\lambda > 0$ and all $x \in X$
3. p Satify the condition $p(\alpha x + \beta y) \leq p(x) + p(y), \forall \alpha, \beta > 0$ with $\alpha + \beta = 1$
4. $p(\lambda x) = 0,$ for all $\lambda > 0,$ implies $x = 0$

A semimodular p is called a modular if

- $p(x) = 0,$ implies $x = 0$

On the other hand, if X is a real vector space and the functional $p : X \to [0, \infty)$ is a modular on X, then the space

$$X_p = \left\{ x \in X; \lim_{\lambda \to 0} p(\lambda x) = 0 \right\}$$

is a modular space. The modular space X_p is a subvector space of X.

Exemple 2.1.5.

for $1 \leq s < \infty$,$p_s(f) = \int_\Omega \left| f(x) \right|^s dx$ define a Modular Continous function

Proposition 2.1.2.

The Musielak Orlicz $(L^\theta(\Omega), ||.||_\theta)$ is a norm

Proof. □

1. if $u = 0$, we have $p_\theta(0) = 0 \leq 1$ Then,We obtain That $||u||_\theta = 0$.Conversly

$$\begin{aligned} ||u||_\theta = 0 \implies & \inf \left\{ \lambda > 0; p_\theta(\frac{u}{\lambda}) \leq 1 \right\} = 0 \\ \implies & p_\theta(\frac{u}{\lambda}) \leq 1, \text{ for all } \lambda > 0 \end{aligned}$$

On other hand, we have that

$$p_\theta(u) = p_\theta(\lambda \frac{u}{\lambda}) \leq \lambda p_\theta(\frac{u}{\lambda}) \leq \lambda \to 0, \text{ when } \lambda \to 0$$

And Since $p_\theta(u) \geq 0$,We Conclude That $p_\theta(u) = 0$,Implies that $u = 0$

2. Let $\alpha \in \mathbb{R}, u \in L^\theta(\Omega)$ we have:

$$\begin{aligned} ||\alpha u||_\theta &= \inf \left\{ \lambda > 0; p_\theta(\frac{\alpha u}{\lambda}) \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0; p_\theta(\frac{u}{\frac{\lambda}{|\alpha|}}) \leq 1 \right\} \\ &= \inf \left\{ |\alpha| \lambda' > 0; p_\theta(\frac{u}{\lambda'}) \leq 1 \right\}, \text{ we put } \lambda' = \frac{\lambda}{|\alpha|} \\ &= |\alpha| \inf \left\{ \lambda' > 0; p_\theta(\frac{u}{\lambda'}) \leq 1 \right\} \\ &= |\alpha| \, ||u||_\theta \end{aligned}$$

3. let $f, g \in \mathbb{R}$, and $u, v \in L^\theta(\Omega)$, such that $f > ||u||_\theta$, $g > ||v||_\theta$ and By The Convexity of Modular function we have

$$\begin{aligned} p_\theta(\frac{u+v}{f+g}) = p_\theta(\frac{u}{f+g} + \frac{v}{f+g}) = p_\theta(\frac{f}{f+g}\frac{u}{f} + \frac{g}{f+g}\frac{v}{g}) &\leq \frac{f}{f+g} p_\theta(\frac{u}{f}) + \frac{g}{f+g} p_\theta(\frac{v}{g}) \\ &\leq f + g \end{aligned}$$

Then,$||u+v||_\theta \leq ||u||_\theta + ||v||_\theta$

Definition 2.1.6. *(Luxemburg norm)*

Let $u \in L^{\theta}(\Omega)$ Then:

$$\|u\|_{\theta} = \inf\left\{\lambda > 0; p_{\theta}(\frac{u}{\lambda}) \leq 1\right\}$$

is Called Luxemburg norm

Definition 2.1.7. *(Orlicz norm)*

Let $u \in L^{\theta}(\Omega)$ Then:

$$\|u\|_{\theta} = \sup\left\{\int_{\Omega} |u(x)v(x)|\, dx; v \in L^{\tilde{\theta}}(\Omega)\right\}$$

is called orlicz norm

Lemma 2.1.1. *(Unit ball Property)*

Let θ is Generalized N-function Then

$$\|u\|_{L^{\theta}(\Omega)} < 1 \Rightarrow p_{\theta}(u) \leq 1 \Rightarrow \|u\|_{L^{\theta}(\Omega)} \leq 1$$

If θ is Left Continous,then $p_{\theta}(u) \leq 1 \Longleftrightarrow \|u\|_{L^{\theta}(\Omega)} \leq 1$

Proposition 2.1.3. *Let θ be a generalized N-function satisfy (Δ_2) condition,Then*

$$L^{\theta}(\Omega) := \{u : \Omega \longrightarrow \mathbb{R},\ measurable\ ; p_{\theta}(u) < \infty\}$$

Proposition 2.1.4.

Let θ be a generalized N-function satisfies (1-4),then the following assertions hold:

- $$\min\left\{\lambda^{m}, \lambda^{l}\right\}\theta(x,t) \leq \theta(x,\lambda t) \leq max\left\{\lambda^{m}, \lambda^{l}\right\}\theta(x,t),\ \text{for a,e } x \in \Omega \text{ and all } \lambda, t \geq 0$$

- $$min\left\{\|u\|^{m}_{L^{\theta}(\Omega)}, \|u\|^{l}_{L^{\theta}(\Omega)}\right\} \leq p_{\theta}(u) \leq max\left\{\|u\|^{m}_{L^{\theta}(\Omega)}, \|u\|^{l}_{L^{\theta}(\Omega)}\right\},\ \text{for all} u \in L^{\theta}(\Omega)$$

- $$let(u_n)_{n\in\mathbb{N}} \subseteq L^{\theta}(\Omega) \text{ and } u \in L^{\theta}(\Omega),\ \text{then}$$

  $$\|u_n - u\|_{L^{\theta}(\Omega)} \longrightarrow 0 \Leftrightarrow p_{\theta}(u_n - u) \longrightarrow 0,\ \text{as } n \longrightarrow +\infty$$

Lemma 2.1.2 (Hӧlder Inequality).

Let Ω be an open subset of $\mathbb{R}^N$ and θ be generalized N-function, then:

$$\left|\int_{\Omega} uvdx\right| \leq 2\,\|u\|_{L^{\theta}(\Omega)}\,\|v\|_{L^{\tilde{\theta}}(\Omega)}\,,\ \text{for all } u \in L^{\theta}(\Omega) \text{ and all } v \in L^{\tilde{\theta}}(\Omega) \tag{2.4}$$

Proof. □

Let $u \in L^{\theta}$ and $v \in L^{\tilde{\theta}}$, with $u > f^{\theta}$ and $v > g^{\tilde{\theta}}$. By the unit ball property,

$$p_{\theta}\left(\frac{f}{\|f\|_{\theta}}\right) \leq 1, \text{ and } p_{\theta}\left(\frac{g}{\|g\|_{\tilde{\theta}}}\right) \leq 1$$

Thus, using Young 's inequality (2.2), we obtain

$$\int_{\Omega} |u||v| \frac{1}{\|u\|_{L^{\theta}(\Omega)}\|v\|_{L^{\theta}(\Omega)}}\,d\mu \leq \int_{\Omega}\left[\theta\left(x, \frac{|u|}{\|u\|_{L^{\theta}(\Omega)}}\right) + \tilde{\theta}\left(x, \frac{|v|}{\|v\|_{L^{\theta}(\Omega)}}\right)\right] d\mu \leq p_{\theta}\left(\frac{u}{\|u\|_{L^{\theta}(\Omega)}}\right) + p_{\theta}\left(\frac{v}{\|v\|_{L^{\theta}(\Omega)}}\right) \leq 2.$$

Multiplying both sides by uv, we get The Result

Exemple 2.1.6 (showing the constant 2 cannot be omitted).

Let $\theta(x,t) = \frac{1}{2}t^2$. Then, a short calculation gives

$$\tilde{\theta}(x,t) = \sup_{u\geq 0}(ut - \frac{1}{2}u^2) = \frac{1}{2}t^2.$$

Let $u \equiv v \equiv 1$. Then,

$$\int_0^1 uv\,dy = 1.$$

On the other hand,

$$\inf\left\{\lambda > 0 : \int_0^1 \frac{1}{2}\left(\frac{1}{\lambda}\right)^2 dy \leq 1\right\} = \frac{1}{\sqrt{2}}.$$

Thus,

$$\|u\|_{L^{\theta}(0,1)} = \|v\|_{L^{\tilde{\theta}}(0,1)} = \frac{1}{\sqrt{2}},$$

and

$$\|u\|_{L^{\theta}(0,1)} \cdot \|v\|_{L^{\tilde{\theta}}(0,1)} = \frac{1}{2}.$$

The subsequent proposition deals with some topological properties of the Musielak-Orilicz space

2.2 Uniform convexity and reflexivity

In this section, we prove the reflexivity of L^θ by means of uniform convexity, since it is well known that the latter implies the former.
Note that there is no reason to work with non-convex functions in this context, since mid-point convexity implies convexity for increasing functions.
A uniformly convex function need not be left-continuous: for example, the function $t \mapsto \infty \cdot \chi_{[1,\infty)}(t)$ is uniformly convex.

Definition 2.2.1.

We say that θ is **uniformly convex** if for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that

$$\theta\left(x, \frac{s+t}{2}\right) \leq (1-\delta) \cdot \frac{\theta(x,s) + \theta(x,t)}{2}$$

for μ-almost every $x \in \Omega$, whenever $s, t \geq 0$ and

$$|s-t| \geq \varepsilon \cdot \max\{|s|, |t|\}.$$

Definition 2.2.2.

A vector space X is said to be uniformly convex if it has a norm $\|\cdot\|$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ with

$$\|x-y\| \geq \varepsilon \quad \text{or} \quad \|x+y\| \leq 2(1-\delta)$$

for all $x, y \in X$ with $\|x\| = \|y\| = 1$.
The next technical lemma allows us to have absolute values in the inequality from the definition of uniform convexity.

Lemma 2.2.1.

Let θ is Generalized N-function be uniformly convex. Then for every $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$\theta\left(x, \frac{s+t}{2}\right) \leq (1-\delta_2) \cdot \frac{\theta(x,|s|) + \theta(x,|t|)}{2}$$

for all $s, t \in \mathbb{R}$ with $|s-t| > \varepsilon \cdot \max\{|s|, |t|\}$ and for every $x \in \Omega$.

Proof. □

Fix $\varepsilon \in (0,1)$ and let $\delta > 0$ be as in the Definition 2.2.1 Let $|s-t| > \varepsilon \max\{|s|, |t|\}$. If $|s| - |t| > \varepsilon \max\{|s|, |t|\}$, then the claim follows by the uniform convexity of θ, $|s+t| \leq |s| + |t|$, and the choice $\delta_2 := \delta$. So assume in the following that

$$|s| - |t| \leq \varepsilon \max\{|s|, |t|\}.$$

Since $|s-t| > \varepsilon \max\{|s|,|t|\}$, it follows that s and t have opposite signs, and that

$$\frac{s+t}{2} = \frac{|s|-|t|}{2}.$$

Then, it follows from convexity that

$$\theta\left(x, \frac{s+t}{2}\right) \le \frac{\varepsilon}{2}\max\{|s|,|t|\} \le \frac{\varepsilon}{2}\theta(x, \max\{|s|,|t|\}) \le \varepsilon\theta(x,|s|) + \theta(x,|t|).$$

Therefore, the claim holds with $\delta_2 := \min\{\delta, 1-\varepsilon\}$.

2.3 The weight condition (A_0) and density of smooth functions

In this section we introduce a new assumption on the θ-function and study its implication. The assumption means that we restrict our attention to the essentially unweighted case: if $\theta(x,t) = t^p w(x)$, then (A_0) holds if and only if $w \approx 1$.

Definition 2.3.1.

We say that θ is N-function satisfies (A0) if there exists a constant $\beta \in (0,1]$ such that

$$\beta \le \theta^{-1}(x,1) \le \frac{1}{\beta} \quad \text{for } \mu\text{-almost every } x \in \Omega.$$

Equivalently, this means that there exists $\beta \in (0,1]$ such that

$$\theta(x,\beta) \le 1 \le \theta\left(x, \frac{1}{\beta}\right) \quad \text{for } \mu\text{-almost every } x \in \Omega.$$

Lemma 2.3.1.

Let θ satisfy $(A0)$. Then there exists ψ is Generalized N-function with $\theta \approx \psi$ and

$$\psi(x,1) = \psi^{-1}(x,1) = 1 \quad \text{for } \mu\text{-almost every } x \in \Omega.$$

Proof. □

there exists ψ_1 with $\theta \approx \psi_1$. Since θ satisfies $(A0)$, so does ψ_1. We set

$$\psi_2(x,t) := \psi_1\left(x, \psi_1^{-1}(x,1)t\right).$$

By [34, Lemma 2,5,12], the map $x \mapsto \psi_1^{-1}(x,1)$ is measurable. Since $(x,t) \mapsto \psi_1(x,t)$ is measurable, it follows that $x \mapsto \psi_2(x,t)$ is measurable for fixed t, by the definition of a generalized prefunction. Then ψ_2 satisfies the measurability condition of $s(A,\mu)$ by [34, Theorem 2.5.4]

We show that $\psi_2 \in s(A,\mu)$. The function ψ_2 is increasing since ψ_1 is increasing. By [34, Lemma 2.5.7], ψ_2 is a prefunction. Since $t \mapsto \psi_1(x,t)$ is convex, we obtain

$$\begin{aligned}\psi_2(x, \theta t + (1-\theta)s) &= \psi_1\left(x, \psi_1^{-1}(x,1)\left(\theta t + (1-\theta)s\right)\right)\\ &= \psi_1\left(x, \theta\psi_1^{-1}(x,1)t + (1-\theta)\psi_1^{-1}(x,1)s\right)\\ &\leq \theta\psi_1\left(x, \psi_1^{-1}(x,1)t\right) + (1-\theta)\psi_1\left(x,\psi_1^{-1}(x,1)s\right)\\ &= \theta\psi_2(x,t) + (1-\theta)\psi_2(x,s)\end{aligned}$$

for every $\theta \in [0,1]$ and $s,t \geq 0$.
Since $t \mapsto \psi_1(x,t)$ is continuous into the compactification $[0,\infty]$ for μ-almost every x, and since $\psi_1^{-1}(x,1)$ is independent of t, we obtain that $t \mapsto \psi_2(x,t)$ is continuous for μ-almost every x.
Since ψ_1 satisfies $(A0)$, we have $\psi_1 \approx \psi_2$.

$$\psi_2(x,1) = \psi_1\left(x, \psi_1^{-1}(x,1)\right) = 1 \quad \text{for } \mu\text{-almost every } x \in A.$$

This implies $\psi_2^{-1}(x,1) = 1$ for μ-almost every $x \in A$.

Corollary 2.3.1.

Let θ is Generalized N-function. Then θ satisfies $(A0)$ if and only if there exists $\beta \in (0,1]$ such that

$$\theta(x,\beta) \leq 1 \leq \theta\left(x, \frac{1}{\beta}\right) \quad \text{for } \mu\text{-almost every } x \in A.$$

Proof. □

Assume first that $(A0)$ holds. By Lemma 3.7.3, there exists $\psi \in s(A,\mu)$ with $\psi(x,1) = 1$ and $\psi \approx \theta$. This implies the inequality.
Assume then that the inequality holds. By the definition of θ^{-1}, the inequality $\theta\left(x, \frac{1}{\beta}\right) \geq 1$ yields

$$\theta^{-1}(x,1) \leq \frac{1}{\beta}.$$

By $(\mathrm{aInc})_1$ and $\theta(x,\beta) \leq 1$, we obtain

$$\frac{\theta\left(x, \frac{\beta}{2a}\right)}{\frac{\beta}{2a}} \leq a \cdot \frac{\theta(x,\beta)}{\beta} \leq \frac{a}{\beta},$$

so that

$$\theta\left(x, \frac{\beta}{2a}\right) \leq \frac{1}{2}.$$

This yields

$$\theta^{-1}(x,1) \geq \frac{\beta}{2a}.$$

Corollary 2.3.2.

Let θ is Generalized N-function. If there exists $c > 0$ such that $\theta(x, c) \approx 1$, then θ satisfies $(A0)$.

Proof. □

Let $m \leq \theta(x, c) \leq M$. We may assume that $m \in (0, 1]$ and $M \geq 1$. By (aInc)$_1$ we obtain

$$\theta\left(x, \frac{c}{aM}\right) \leq \frac{c}{aM} \leq a \cdot \frac{\theta(x,c)}{c} \leq aM \quad \text{and} \quad \frac{m}{c} \leq \theta(x,c) \leq a \cdot \frac{\theta\left(x, \frac{ac}{m}\right)}{ac/m}.$$

Thus, $\theta\left(x, \frac{c}{aM}\right) \leq 1$ and $\theta\left(x, \frac{ac}{m}\right) \geq 1$, and the claim follows from corollary 2.3.1

Lemma 2.3.2.

If θ satisfies $(A0)$, then $\tilde{\theta}$ satisfies $(A0)$.

Proof. □

By $(A0)$ of θ, we obtain

$$\frac{1}{(\theta^*)^{-1}(x,1)} \approx \theta^{-1}(x,1) \leq \frac{1}{\beta} \quad \text{and} \quad (\theta^*)^{-1}(x,1) \approx \frac{1}{\theta^{-1}(x,1)} \geq \beta$$

for μ-almost every $x \in \Omega$.

We next characterize the embeddings of the sum and the intersection of generalized Orlicz spaces. Let us introduce the usual notation. Recall that for two normed spaces X and Y (which are both subsets of a vector space Z) we equip the intersection XY and the sum $X + Y := \{g + h : g \in X, h \in Y\}$ with the norms

$$\|f\|_{XY} := \max\{\|f\|_X, \|f\|_Y\} \quad \text{and} \quad \|f\|_{X+Y} := \inf_{f=g+h,\, g\in X,\, h\in Y} \{\|g\|_X + \|h\|_Y\}.$$

In the next lemma, we use the convention that every θ satisfies (aDec)$_\infty$ with constant 1. Hence, we always have $L^1 L^\infty \to L^\theta \to L^1 + L^\infty$.

Notice that the first embedding requires only $\theta(x, \frac{1}{\beta}) \geq 1$, whereas the second requires only $\theta(x, \beta) \leq 1$.

Lemma 2.3.3.

Let θ satisfy $(A0)$, (aInc)$_p$, and (aDec)$_q$, $p \in [1, \infty)$ and $q \in [1, \infty)$. Then

$$L^p(\Omega) L^q(\Omega) \to L^\theta(\Omega) \to L^p(\Omega) + L^q(\Omega).$$

and the embedding constants depend only on $(A0)$.

Proof. □

Let us first study $L^\theta(\Omega) \to L^p(\Omega) + L^q(\Omega)$. Let $f \in L^\theta(\Omega)$ with $f^\theta < 1$ so that $\theta(f) \le 1$ by the unit ball property. We may assume that $f \ge 0$ since otherwise we may study $|f|$. We assume that $p, q \in [1, \infty)$. The case $q = \infty$ follows by simple modifications.
Define $f_1 := f\chi_{\{0 \le f \le \frac{1}{\beta}\}}$ and $f_2 := f\chi_{\{f > \frac{1}{\beta}\}}$. (aInc)$_p$ and (aDec)$_q$, we have

$$\beta^p \le \theta(x, \frac{1}{\beta}) \quad \frac{1}{\beta^p} \le a\theta(x,t) \quad t^p$$

and

$$\beta^q \le \theta(x, \frac{1}{\beta}) \quad \frac{1}{\beta^q} \le a\theta(x,s) \quad s^q$$

for $s \le \frac{1}{\beta} \le t$. Using these, we obtain that

$$\int_A \beta^p a \int_A f_1^p \, dx \le 1 \quad \text{and} \quad \beta^q a \int_\Omega f_2^q \, dx \le 1.$$

Thus, we have $f \in L^p + L^q \le a^{1/p}\beta + a^{1/q}\beta$, and the claim follows by the scaling argument, i.e., by using this result for $f/(f^\theta + \epsilon)$ and then letting $\epsilon \to 0^+$.
Then we consider the embedding $L^p(\Omega)L^q(\Omega) \to L^\theta(\Omega)$ and assume that $f \in L^pL^q \le \frac{1}{a}\min\{\beta^p, \beta^q\}$.
Define $f_1 := f\chi_{\{0 \le f \le \beta\}}$ and $f_2 := f\chi_{\{f > \beta\}}$. (aInc)$_p$ and (aDec)$_q$, we have

$$\theta(x,t) \le a\theta(x,\beta) \quad \beta^p \le a\beta^p$$

and

$$\theta(x,s) \le a\theta(x,\beta) \quad \beta^q \le a\beta^q$$

for $t \le \beta \le s$. Using these, we obtain that

$$\int_A \theta(x, f_1) \, dx \le a\beta^p \int_A f_1^p \, dx \le 1 \quad \text{and} \quad \int_\Omega \theta(x, f_2) \, dx \le a\beta^q \int_A f_2^q \, dx \le 1.$$

Thus, we have $f \in L^\theta \le 1$ and the claim follows by the scaling argument.
Next, we give an example which shows that assumption $(A0)$
Let $\theta(x,t) = t^2|x|^2$. Then $\theta \in s(\mathbb{R})$ satisfies (Inc)$_2$ and (Dec)$_2$ but not $(A0)$.
First, we show that $L^\theta(\mathbb{R}) \to L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ does not hold. For that, let $f(x) := \frac{1}{|x|}\chi_{(-1,1)}$. Then

$$\int_{\mathbb{R}} \theta(x, f) \, dx = \int_{(-1,1)} \frac{1}{|x|}|x|^2 \, dx \approx 1.$$

and thus $f \in L^\theta(\mathbb{R})$. Let $f_1 \in L^2(\mathbb{R})$ and $f_2 \in L^\infty(\mathbb{R})$ be such that $f = f_1 + f_2$. Then we find $r > 0$ such that $f(x) = f_1(x)$ for all $x \in (-r, r)$, and we obtain

$$\int_{(-r,r)} f_1^2 \, dx = \int_{(-r,r)} |x|^2 \, dx$$

and thus such a decomposition does not exist.

Next, we show that $L^1(\mathbb{R})L^2(\mathbb{R}) \to L^\theta(\mathbb{R})$ does not hold. Let $g(x) := \min\{1, |x|^{-5/4}\}$. A short calculation shows that $g \in L^1(\mathbb{R})$. Since $0 < g \leq 1$, this yields that $g \in L^2(\mathbb{R})$. On the other hand, for every $\lambda > 0$, we have

$$\int_{\mathbb{R}\setminus(-1,1)} \theta(x, \lambda g)\, dx \geq \int_{|x|>1} \lambda |x|^{-5/4}|x|^2\, dx \approx \lambda^2 \int_1^\infty |x|^{-1/2}\, dx = \infty,$$

and thus $g/\lambda \notin L^\theta(\mathbb{R})$.

When the set A has finite measure, the previous result simplifies, and we get the following corollaries.

Corollary 2.3.3.

Let A have finite measure and let θ satisfy $(A0)$ and $(aInc)_p$. Then $L^\theta(\Omega) \to L^p(\Omega)$ and there exists β such that

$$\int_A |f|^p\, d\mu \lesssim \int_A \theta(x, |f|)\, d\mu + \mu\{0 < |f| < \frac{1}{\beta}\}.$$

Corollary 2.3.4.

Let A have finite measure and let $\theta \in w(A, \mu)$ satisfy $(A0)$. Then $L^\infty(A, \mu) \to L^\theta(A, \mu)$.

The next example shows that the previous result need not hold if θ does not satisfy $(A0)$.

Exemple 2.3.1.

Let $(0, 1) \subset \mathbb{R}$ and $\theta(x, t) := t|x|$. Then $\theta \in s(0, 1)$ and θ does not satisfy $(A0)$. Let $f \equiv 2 \in L^\infty(A, \mu)$. We obtain

$$\int_0^1 \theta(x, f)\, d\mu < \infty$$

for all $\lambda > 0$ and hence $f \in L^\theta(0, 1)$.

Lemma 2.3.4.

$L_0^\theta(\Omega)$ is dense in $L^\theta(\Omega)$.

Proof. □

Let $f \in L_\theta$ and let $\lambda > 0$ be such that

$$\int \theta(x, \lambda f)\, dx < \infty.$$

Define

$$f_i := f\chi_{B(0,i)}.$$

Then

$$\int_\Omega \theta(x, \lambda|f - f_i|)\, dx = \int_{\Omega\setminus B(0,i)} \theta(x, \lambda|f|)\, dx \to 0.$$

As $i \to \infty$, by the absolute continuity of the integral, we have

$$\int \theta(x, \lambda|f - f_i|)\, dx \to 0.$$

Hence, (f_i) is modular convergent to f.

If we also have (A_0), then a stronger result holds.

2.4 Properties of Musielak-Orlicz spaces

Lemma 2.4.1. *[5],*

Let θ is Generalized N-function.Then,all cauchy sequences $(f_n)_{n\in\mathbb{N}} \subset L^{\theta}(\Omega)$ has a subsequences converge almost everywhere to measurable function f

Theorem 2.4.1.

The Musielak Orlicz $L^{\theta}(\Omega)$ is a Banach space

Proof. □

by Lemma 2.4.1 There exists a subsequenes $(f_{n_k})_{n\in\mathbb{N}}$ and a measurable function $f : \Omega \to \mathbb{R}$ such that $(f_{nk})_{n\in\mathbb{N}}$ converge to f,for all $x \in \Omega$ Then:

$$\lim_{k\to+\infty} \theta(x, \left|f_{n_k}(x) - f(x)\right|) = 0 \text{ a.e}$$

Let $\lambda > 0$,since $(f_n)_{n\in\mathbb{N}}$ is Cauchy sequneces We have that:

$$\forall\epsilon > 0, \exists N_0 > 0; \forall m, n \geq N_0; \left\|\lambda(f_m - m_n)\right\|_{\theta} \leq \epsilon$$

We take that $0 < \epsilon < 1$ by Lemma 2.1.1 we obtain that:

$$p_{\theta}(\lambda(f_m - f_n)) \leq \epsilon$$

And by Fatou's lemma [10],we have:

$$\begin{aligned} p_{\theta}(\lambda(f_m - f_n)) &= \int_{\Omega} \theta(x, \lambda(f_m - f_n))d\mu \\ &= \int_{\Omega} \lim_{k\to+\infty} \theta(x, \lambda(f_m - f_{n_k})d\mu \\ &\leq \liminf_{k\to+\infty} \int_{\Omega} \theta(x, \lambda(f_m - f_{n_k})d\mu \leq \epsilon \end{aligned}$$

Then:

$$\lim_{m\to\infty} p_{\theta}(\lambda(f_m - f)) = 0, \forall\lambda > 0$$

So:

$$\lim_{n\to\infty} \left\|f_n - f\right\| = 0$$

Separability

Theorem 2.4.2.

Let θ be Generalized N-function if,Satifies The (Δ_2) Condition Then, $L^{\theta}(\Omega)$ is a Separable space

Proof. □

To show that $L^{\theta}(\Omega)$ is separable, we need to construct a countable set of functions $\{f_n\} \subset L^{\theta}(\Omega)$ such that for any $f \in L^{\theta}(\Omega)$ and any $\epsilon > 0$, there exists an f_n such that

$$\|f - f_n\|_{\theta} < \epsilon.$$

The strategy involves approximating functions in $L^{\theta}(\Omega)$ by simple functions (functions taking only finitely many values), which are dense in the space, and then showing that we can construct a countable dense subset.

To start with, simple functions are functions of the form

$$\varphi(x) = \sum_{i=1}^{k} a_i \chi_{A_i}(x),$$

where $a_i \in \mathbb{Q}$ (rational numbers), $A_i \subseteq \Omega$ are measurable sets, and χ_{A_i} is the characteristic function of A_i.
The simple functions are dense in $L^{\theta}(\Omega)$ because any measurable function f can be approximated by simple functions. This is a standard result in measure theory: any measurable function can be approximated in the L^p-norm (and hence in the Musielak-Orlicz norm) by simple functions.
Now, we want to construct a countable set of simple functions that are dense in $L^{\theta}(\Omega)$. The idea is to construct simple functions with rational coefficients and rational approximations to the characteristic functions of measurable sets.

a.

Since Ω is a measurable space, we can consider a countable collection of measurable sets $\{A_n\} \subset \Omega$ that form a dense set in the sense that any measurable set in Ω can be approximated (in measure) by some set from this collection.
For example, if $\Omega = [0, 1]$, then the collection of intervals with rational endpoints forms a countable dense set of measurable sets. In general, we can always find a countable collection of measurable sets $\{A_n\}$ that densely approximate any measurable set $A \subseteq \Omega$ in measure.

b.

Next, we approximate the values of the function f by rational numbers. For each function $f \in L^{\theta}(\Omega)$, the values $f(x)$ can be approximated by rational numbers because the rationals $\mathbb{Q}$ are dense in the real numbers $\mathbb{R}$. This means that we can approximate $f(x)$ by a rational-valued function $f_q(x)$, where f_q is a simple function with rational values $\{f_q(x)\} \subset \mathbb{Q}$.
Thus, we approximate f by simple functions f_q that take only rational values.

c.

To approximate the Musielak-Orlicz norm, we use the fact that θ is a continuous, convex function (under mild regularity conditions on θ) that grows as $t \to \infty$. This means that for any function $f \in L^{\theta}(\Omega)$ and for any $\epsilon > 0$, there exists a simple function f_q (with rational coefficients) such that

$$\|f - f_q\|_{\theta} < \epsilon.$$

This approximation works because we can make the simple function f_q arbitrarily close to f both pointwise and in terms of the θ-norm.
In addition, the set of all simple functions of the form

$$\{\varphi_n : \varphi_n(x) = \sum_{i=1}^{k} a_i \chi_{A_i}(x),\ a_i \in \mathbb{Q},\ A_i \in \mathcal{A}\}$$

where $\mathcal{A}$ is the countable collection of measurable sets, is countable. Furthermore, by approximating any function $f \in L^{\theta}(\Omega)$ by these simple functions, we conclude that the Musielak-Orlicz space $L^{\theta}(\Omega)$ is separable.

Remarks 2.4.1.

We see that the separability of the Musielak-Orlicz space follows from the density of simple functions with rational coefficient.

Chapter 3

Musielak-Orlicz-Sobolev spaces

In this chapter, we recall some known results and introduce a new function space needed in our approach, providing some of its essential properties.
For the study of equations with variable exponent double phase phenomena, we need to recall the definition of Lebesgue and Sobolev spaces with variable exponents. Most of these results can be found in the book [15]. See also [13], [14],[16],[17],[18],[21], [34],[35], [36], and [38]. We will present these concepts in a simplified framework that suits our specific needs.

3.1 The variable exponent Lebesgue spaces

Definition 3.1.1.

Let Ω be an open domain in $\mathbb{R}^N$. For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \in M(\Omega), \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\}.$$

The norm on $L^{p(\cdot)}(\Omega)$, called the *Luxemburg norm*, is given by

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \|u\|_{p(\cdot)} = \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

The modular $\sigma_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ of $L^{p(\cdot)}(\Omega)$ is defined by

$$\sigma_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx \quad \text{for all } u \in L^{p(\cdot)}(\Omega).$$

Then, the norm of the space $L^{p(\cdot)}(\Omega)$ can be rewritten using the modular as

$$\|u\|_{p(\cdot)} = \inf\left\{ \lambda > 0 : \sigma_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1 \right\}.$$

Let us recall that the functional $\sigma_{p(\cdot)}$ is a convex modular on the set $L^{p(\cdot)}(\Omega)$, since it satisfies the convex modular conditions. For simplicity, we use the notation

$$\sigma(u) := \sigma_{p(\cdot)}(u).$$

Below is the proposition expressing the relations between the norm and the modular of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$.

Proposition 3.1.1.

Let $u, u_k \in L^{p(\cdot)}(\Omega)$ for $k = 1, 2, \ldots$. Then the following properties hold:

(i)

$$\min\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \leq \sigma(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\},$$

where $p^- := \inf_{x\in\Omega} p(x)$ and $p^+ := \sup_{x\in\Omega} p(x)$.

(ii)

$$\min\left\{\sigma(u)^{1/p^-}, \sigma(u)^{1/p^+}\right\} \leq \|u\|_{p(\cdot)} \leq \max\left\{\sigma(u)^{1/p^-}, \sigma(u)^{1/p^+}\right\}.$$

(iii)

$$\sigma\left(\frac{u}{\|u\|_{p(\cdot)}}\right) = 1 \quad \text{if } \|u\|_{p(\cdot)} \neq 0.$$

(iv)

$$\|u_k\|_{p(\cdot)} \to 0 \quad \Leftrightarrow \quad \sigma(u_k) \to 0.$$

(v)

$$\|u_k\|_{p(\cdot)} \to \infty \quad \Leftrightarrow \quad \sigma(u_k) \to \infty.$$

Here, we give an example of determining whether a function belongs to the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$.

Exemple 3.1.1.

Let $u(x) = 5^{x^2}$, $\Omega = [1, 2]$, and $p(x) = \frac{1}{x}$. Then

$$\begin{aligned}
\sigma_p(\cdot)(u) &= \int_\Omega |u(x)|^{p(x)}\, dx \\
&= \int_1^2 |5^{x^2}|^{\frac{1}{x}}\, dx \\
&= \int_1^2 5^x\, dx \\
&= \left[\frac{5^x}{\log(5)}\right]_1^2 \\
&= \frac{20}{\log(5)}
\end{aligned}$$

Thus, the result is $\frac{20}{\log(5)}$.
Therefore, the function $u(x) = 5^{x^2} \in L^{p(\cdot)}(1, 2)$.
An example related to the Luxembourg norm is given below.

Exemple 3.1.2.

The $\|u\|_{p(\cdot)}$ norm of the function $u(x) = \frac{2}{x}$ for $p(x) = x$, $\Omega = [1, 2]$, is calculated as

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \sigma_{p(x)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \int_1^2 \left(\frac{2}{\lambda x}\right)^x dx \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \int_1^2 (2\lambda^{-1})^x dx \leq 1\right\}$$

$$= \inf\left\{\lambda > 0, \lambda \neq 1 : \int_1^2 2^x \lambda^{-x} dx \leq 1\right\}$$

$$= \inf\left\{\lambda > 0, \lambda \neq 1 : \frac{2^2\lambda^{-2} - 2^1\lambda^{-1}}{\ln(2/\lambda)} \leq 1\right\}$$

(Numerical evaluation or simplification leads to:)

$$= \inf\{\lambda > 0, \lambda \neq 1 : \lambda \geq 1.59708\}$$

$$\approx 1.59708$$

Definition 3.1.2.

Let $\Omega \subset \mathbb{R}^N$ be a domain, and let $p : \Omega \to [1, \infty)$ be a continuous function.

If there exists a constant $C_0 > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_0}{-\log|x - y|}, \quad \text{for all } x, y \in \Omega \text{ with } |x - y| < \tfrac{1}{2},$$

then the function $p(\cdot)$ is said to be **log-H*ö*lder continuous**.

Definition 3.1.3.

A measurable function u in the domain $\Omega \subset \mathbb{R}^N$ is said to be **almost bounded** if there exists a non-negative constant $M \geq 0$ such that

$$|u(x)| \leq M \quad \text{almost everywhere in } \Omega.$$

The space of all almost bounded functions in Ω is denoted by $L^\infty(\Omega)$, and it can be characterized as the limit of Lebesgue spaces $L^p(\Omega)$ as $p \to \infty$, that is,

$$L^\infty(\Omega) = \lim_{p\to\infty} L^p(\Omega).$$

In addition to the previous spaces, we define the space of functions that are elements of the space of almost bounded p-functions in the domain Ω and satisfy the condition

$$\operatorname*{ess\,inf}_{x\in\Omega} p(x) \geq 1.$$

This space is denoted by $L^{\infty}_{+}(\Omega)$, and it is defined as

$$L^{\infty}_{+}(\Omega) = \left\{ u \in L^{\infty}(\Omega) : \operatorname*{ess\,inf}_{x\in\Omega} p(x) \geq 1 \right\}.$$

Proposition 3.1.2.

Let $p \in L^{\infty}_{+}(\Omega)$. The conjugate exponent function $p'(x)$ is defined for almost every $x \in \Omega$ by the relation

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Then, the dual (or conjugate) space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$.

The following inequality holds for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$:

$$\left| \int_{\Omega} u(x)v(x)\,dx \right| \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

which is known as the **Hölder-type inequality** for variable exponent spaces.

(ii) Let $p_1, p_2 \in C_{+}(\overline{\Omega})$ with $p_1(x) \leq p_2(x)$ for every $x \in \overline{\Omega}$. Then, the embedding

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$

is continuous. That is, there exists a constant $C > 0$ such that

$$\|u\|_{p_1(\cdot)} \leq C\|u\|_{p_2(\cdot)} \quad \text{for all } u \in L^{p_2(x)}(\Omega).$$

Proposition 3.1.3.

Assume that $r \in L^{\infty}_{+}(\Omega)$ and $p \in C^{+}(\overline{\Omega})$. If $|u|^{r(\cdot)} \in L^{p(\cdot)}(\Omega)$, then the following inequality holds:

$$\min\left\{ \|u\|^{r^+}_{r(\cdot)p(\cdot)}, \|u\|^{r^-}_{r(\cdot)p(\cdot)} \right\} \leq \left\| |u|^{r(\cdot)} \right\|_{p(\cdot)} \leq \max\left\{ \|u\|^{r^+}_{r(\cdot)p(\cdot)}, \|u\|^{r^-}_{r(\cdot)p(\cdot)} \right\},$$

where

$$r^+ := \operatorname*{ess\,sup}_{x\in\Omega} r(x), \quad r^- := \operatorname*{ess\,inf}_{x\in\Omega} r(x).$$

In particular, if $r(\cdot) \equiv r$ is a constant function, then

$$\||u|^r\|_{p(\cdot)} = \|\,|u|\,\|^r_{rp(\cdot)}.$$

3.2 The variable exponent Sobolev spaces

The corresponding variable exponent Sobolev spaces can be defined in the same way using the variable exponent Lebesgue spaces (for more details see [19]).

Definition 3.2.1.

For $r \in C_+(\overline{\Omega})$ the variable exponent sobolev space $W^{1,r(.)}(\Omega)$ is defined by

$$W^{1,r(.)}(\Omega) = \left\{ u \in L^{r(.)}(\Omega) : |\nabla u| \in L^{r(.)}(\Omega) \right\}$$

endowed with the norm

$$\|u\|_{1,r} = \|u\|_{r(.)} + \|\nabla u\|_{r(.)}$$

Where $\|\nabla u\|_{r(.)} = \||\nabla u|\|_{r(.)}$

Proposition 3.2.1. *[13]*

$$W_0^{1,r(.)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|.\|_{1,r(.)}} \tag{3.1}$$

Proposition 3.2.2.

The spaces $W_0^{1,r(.)}(\Omega)$ and $W^{1,r(.)}(\Omega)$ are both separable and reflixive Banach spaces,in fact uniformly convex Banach spaces.

Proposition 3.2.3. *[14]*

In the space $W_0^{1,r(.)}(\Omega)$,the poincar$\acute{e}$ Inequality holds,namely

$$\|u\|_{r(.)} \leq C_0 \|\nabla u\|_{r(.)} \text{ for all } u \in W_0^{1,r(.)}(\Omega)$$

with some $C_0 > 0$.Therefore,we can consider on $W_0^{1,r(.)}(\Omega)$ the equivalent norm

$$\|u\|_{1,r(.),0} = \|\nabla u\|_{r(.)} \text{ for all } u \in W_0^{1,r(.)}(\Omega)$$

Definition 3.2.2.

We denote by $C^{0,\frac{1}{|\log(t)|}}(\overline{\Omega})$ the set of all functions $h : \bar{\Omega} \longrightarrow \mathbb{R}$ that are log-H$\ddot{o}$lder continous,that is ,there exists $C > 0$ such that

$$\left| h(x) - h(y) \right| \leq \frac{C}{\left| \log \left| x - y \right| \right|} \text{ for all } x, y \in \overline{\Omega} \text{ with } \left| x - y \right| < \frac{1}{2} \tag{3.2}$$

Remark 3.2.1.

We can state the embedding from $W^{1,r(.)}(\Omega)$ into $L^{r^*(.)}(\Omega)$. Let us note that the space $W_0^{1,p(\cdot)}(\Omega)$ is essential when investigating the solutions of Dirichlet problems for elliptic-parabolic type equations involving the $p(\cdot)$-Laplacian.

Definition 3.2.3.

The space $W_0^{1,p(\cdot)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},$$

where ∇u denotes the weak gradient of u, and both u and ∇u belong to $L^{p(\cdot)}(\Omega)$.

Theorem 3.2.1 (Poincaré Inequality).

Let $\Omega \subset \mathbb{R}^N$ be a domain with a smooth boundary and let $p(\cdot) \in L_+^\infty(\Omega)$. Then there exists a positive constant $C > 0$ such that

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

Hence, by the Poincaré inequality, we can define an equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$ by

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} := \|\nabla u\|_{p(\cdot)}.$$

Moreover, the functional

$$\Lambda : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$$

defined by

$$\Lambda(u) = \int_\Omega |\nabla u(x)|^{p(x)}\, dx$$

is the modular corresponding to the norm $\|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)}$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$.

Proposition 3.2.4.

$(L^{p(.)}, \|.\|_{r(.)}),(W^{1,p(.)}, \|.\|_{p(.)})$ are separable, Reflexive, Banach spaces.
The following proposition expresses the relations between the norm and the modular of the variable exponent sobolev space $W^{1,p(\cdot)}(\Omega)$ for $m = 1$

Proposition 3.2.5.

For $u, u_k \in W^{1,p(\cdot)}(\Omega)$ $(k = 1, 2, \ldots)$, we have

(i)

$$\min\left\{\|u\|_{1,p(\cdot)}^{p_-}, \|u\|_{1,p(\cdot)}^{p_+}\right\} \leq \Lambda(u) \leq \max\left\{\|u\|_{1,p(\cdot)}^{p_-}, \|u\|_{1,p(\cdot)}^{p_+}\right\},$$

(ii)

$$\min\left\{\Lambda_1^{p_-}(u), \Lambda_1^{p_+}(u)\right\} \leq \|u\|_{1,p(\cdot)} \leq \max\left\{\Lambda_1^{p_-}(u), \Lambda_1^{p_+}(u)\right\},$$

(iii)

$$\Lambda\left(\frac{u}{\|u\|_{1,p(\cdot)}}\right) = 1 \quad \text{if } \|u\|_{1,p(\cdot)} \neq 0,$$

(iv)

$$\|u_k\|_{1,p(\cdot)} \to 0 \iff \Lambda(u_k) \to 0,$$

(v)

$$\|u_k\|_{1,p(\cdot)} \to \infty \iff \Lambda(u_k) \to \infty.$$

Exemple 3.2.1.

Let

$$u(x) = x^2, \quad \Omega = [1,4], \quad \text{and} \quad p(x) = \begin{cases} 1, & x \in [1,2), \\ 2, & x \in [2,4]. \end{cases}$$

Let us show that $u(x) = x^2 \in W^{1,p(\cdot)}(\Omega)$. Indeed,

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx = \int_1^4 |x^2|^{p(x)}\, dx.$$

Using the definition of $p(x)$, we split the integral as

$$\rho_{p(\cdot)}(u) = \int_1^2 |x^2|^1\, dx + \int_2^4 |x^2|^2\, dx = \int_1^2 x^2\, dx + \int_2^4 x^4\, dx.$$

Calculating the integrals, we have

$$\int_1^2 x^2\, dx = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3},$$

and

$$\int_2^4 x^4\, dx = \frac{x^5}{5}\Big|_2^4 = \frac{4^5}{5} - \frac{2^5}{5} = \frac{1024}{5} - \frac{32}{5} = \frac{992}{5}.$$

Therefore,

$$\rho_{p(\cdot)}(u) = \frac{7}{3} + \frac{992}{5} = \frac{35}{15} + \frac{2976}{15} = \frac{3011}{15} < \infty,$$

which shows that $u \in L^{p(\cdot)}(\Omega)$. Now consider the gradient of $u(x) = x^2$, which is $\nabla u(x) = \frac{du}{dx} = 2x$. We compute the modular:

$$\rho_{p(\cdot)}(\nabla u) = \int_\Omega |\nabla u(x)|^{p(x)}\, dx = \int_1^4 |2x|^{p(x)}\, dx.$$

Using the definition of $p(x)$, we split the integral:

$$\rho_{p(\cdot)}(\nabla u) = \int_1^2 |2x|\, dx + \int_2^4 |2x|^2\, dx = \int_1^2 2x\, dx + \int_2^4 4x^2\, dx.$$

Compute each part:

$$\int_1^2 2x\, dx = x^2\Big|_1^2 = 4 - 1 = 3,$$

$$\int_2^4 4x^2\, dx = 4\int_2^4 x^2\, dx = 4\,\frac{x^3}{3}\Big|_2^4 = 4\left(\frac{64}{3} - \frac{8}{3}\right) = 4 \cdot \frac{56}{3} = \frac{224}{3}.$$

So the total modular becomes:

$$\rho_{p(\cdot)}(\nabla u) = 3 + \frac{224}{3} = \frac{9}{3} + \frac{224}{3} = \frac{233}{3} < \infty.$$

This confirms that $\nabla u \in L^{p(\cdot)}(\Omega)$, and thus

$$u \in W^{1,p(\cdot)}(\Omega).$$

Exemple 3.2.2.

Let us compute the norm $\|u\|_{1,p(\cdot)}$ for $u(x) = x^2$, where

$$p(x) = \begin{cases} 1, & x \in [1,2), \\ 2, & x \in [2,4], \end{cases} \quad \text{on } \Omega = [1,4].$$

We compute:

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

where the modular is

$$\rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) = \int_1^2 \left(\frac{x^2}{\lambda}\right)^1 dx + \int_2^4 \left(\frac{x^2}{\lambda}\right)^2 dx.$$

Compute each part:

$$\int_1^2 \frac{x^2}{\lambda}\, dx = \frac{1}{\lambda}\int_1^2 x^2\, dx = \frac{1}{\lambda}\cdot\left(\frac{8-1}{3}\right) = \frac{7}{3\lambda},$$

$$\int_2^4 \left(\frac{x^2}{\lambda}\right)^2 dx = \frac{1}{\lambda^2}\int_2^4 x^4\, dx = \frac{1}{\lambda^2}\cdot\left(\frac{1024-32}{5}\right) = \frac{992}{5\lambda^2}.$$

Therefore,

$$\rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) = \frac{7}{3\lambda} + \frac{992}{5\lambda^2}.$$

We now solve:

$$\frac{7}{3\lambda} + \frac{992}{5\lambda^2} \leq 1,$$

and take

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \frac{7}{3\lambda} + \frac{992}{5\lambda^2} \leq 1\right\} = \frac{3011}{15}$$

and

$$\|\nabla u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \sigma_{p(x)}\left(\frac{\nabla u}{\lambda}\right) \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \int_1^4 \left(\frac{2x}{\lambda}\right)^{p(x)} dx \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \left(\int_1^2 \left(\frac{2x}{\lambda}\right)^1 dx + \int_2^4 \left(\frac{2x}{\lambda}\right)^2 dx\right) \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \left(\frac{x^2}{\lambda}\Big|_1^2 + \frac{4x^3}{3\lambda^2}\Big|_2^4\right) \leq 1\right\}$$

$$= \inf\left\{\lambda > 0 : \left(\frac{4-1}{\lambda} + \frac{4\cdot(64-8)}{3\lambda^2}\right) \le 1\right\}$$
$$= \inf\left\{\lambda > 0 : \frac{3}{\lambda} + \frac{224}{3\lambda^2} \le 1\right\}$$

Solving this inequality gives:

$$\|\nabla u\|_{p(\cdot)} = \frac{233}{3}$$

Thus, we get:

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} = \frac{3011}{15} + \frac{233}{3} = 1392$$

Proposition 3.2.6.

Let $r \in C^{0,\frac{1}{\log t}}(\overline{\Omega})C_+(\overline{\Omega})$ and let $s \in C(\overline{\Omega})$ be such that

$$1 \le s(x) \le r^*(x) \text{ for all } x \in \overline{\Omega}$$

Then,we have the continous embedding

$$W^{1,r(.)} \hookrightarrow L^{s(.)}(\Omega)$$

if$r \in C_+(\overline{\Omega})$,$s \in C_+(\bar{\Omega})$ and $1 \le s(x) < r^*(x)$ for all $x \in \overline{\Omega}$,Then the embedding above is compact. In the same way we have the embedding into the boundary Lebesgue space,see Fan [17] and Ho-Kim-Winkert-Zhang for the continous and Fan for the compact embedding.

Proposition 3.2.7. *Suppose that $r \in C_+(\bar{\Omega})W^{1,\theta}(\Omega)$ for some $\gamma > N$.Let $s \in C(\bar{\Omega})$ be such that:*

$$1 \le s(x) \le r_*(x) \text{ for all } x \in \overline{\Omega}$$

Then,we have the continous embedding

$$W^{1,r(.)} \hookrightarrow L^{s(.)}(\partial\Omega)$$

if $r \in C_+(\overline{\Omega})$,$s \in C(\overline{\Omega})$ and $1 \le s(x) \le r_*(x)$ for all $x \in \Omega$,then the embedding above is Compact.

Remark 3.2.2.

Note that for a bounded domain $\Omega \subset \mathbb{R}^N$ and $\gamma > N$ we have the following inclusion:

$$C^{0,1}(\overline{\Omega}) \subset W^{1,\gamma}(\bar{\Omega}) \subset C^{0,1-\frac{N}{\gamma}}(\overline{\Omega}) \subset C^{0,\frac{1}{|\log t|}}(\overline{\Omega}).$$

3.3 Musielak-Orlicz-Sobolev spaces

Before addressing this space, it is necessary to first consider the Musielak-Orlicz space, which was studied in Chapter 1

Definition 3.3.1.

Let Ω an bounded open in $\mathbb{R}^N$.The Musielak-orlicz-sobolev spaces defined by:

$$W^{1,\theta}(\Omega) = \left\{ u \in M(\Omega), u, \nabla u \in L^{\theta}(\Omega) \right\},$$

with the norm

$$\|u\|_{W^{1,\theta}(\Omega)} = \|u\|_{1,\theta} = \|u\|_{\theta} + \|\nabla u\|_{\theta},$$

Remark 3.3.1.

$\theta(x,t) = t^{p(x)}$, the space $W^{1,\theta}(\Omega)$ becomes the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$.

Theorem 3.3.1. *Let θ is Generalized N-function be locally integrable such that*

$$\inf_{x \in \Omega} \theta(x,1) > 0 \tag{3.3}$$

Then the spaces $W^{1,\theta}(\Omega)$ and $W_0^{1,\theta}(\Omega)$ are separable Banach spaces which are reflixive if $L^{\varphi}(\Omega)$ is reflixive.

Proposition 3.3.1. *[14]*

Let $p^*(x) := \frac{Np(x)}{N-p(x)}$, $P_*(x) = \frac{(N-1)P(x)}{N-p(x)}$ for all $x \in \Omega$ be the critical exponents to p. Then the following embeddings hold:

(i) $L^{\theta}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, $W^{1,\theta}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega)$, $W_0^{1,\theta}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ for all $r \in C(\Omega)$ with $1 \le r(x) \le p(x)$ for all $x \in \Omega$.

(ii) if $p \in C^+(\Omega)C_{0,1}^{|\log t|}(\Omega)$, then $W^{1,\theta}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ and $W_0^{1,\theta}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ are continuous for $r \in C(\Omega)$ with $1 \le r(x) \le p^*(x)$ for all $x \in \Omega$.

(iii) $W^{1,\theta}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ and $W_0^{1,\theta}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ are compact for $r \in C(\Omega)$ with $1 \le r(x) < p^*(x)$ for all $x \in \Omega$.

(iv) if $p \in C^+(\Omega)W^{1,\gamma}(\Omega)$ for some $\gamma > N$, then $W^{1,\theta}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ and $W_0^{1,H}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ are continuous for $r \in C(\Omega)$ with $1 \le r(x) \le p^*(x)$ for all $x \in \Omega$.

(v) $W^{1,\theta}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ and $W_0^{1,\theta}(\Omega) \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ are compact for $r \in C(\Omega)$ with $1 \le r(x) < p^*(x)$ for all $x \in \Omega$.

Proposition 3.3.2. *[14] We assume that $p(x) \equiv p$ and $q(x) \equiv q$,where p,q are two constants such that $1 < p < q < \infty$.Then the following embedding hold:*

1. *if $p \neq n$,then $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ is continous for all $r \in [1, p^*]$*
2. *if $p = n$, then $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ is continous for all $r \in [1, +\infty]$*

3. *if $p < n$,then $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $r \in [1, p^*)$*

4. *if $p > n$,then $W_0^{1,\theta}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact.*
   Next,we consider the following assumptions:
   $(H_1): p, q \in C(\overline{\Omega}), 1 < p(x) < n, p(x) < q(x) < p^*(x),$ *for all* $x \in \overline{\Omega}$, *and* $0 \leq \mu(.) \in L^\infty(\Omega)$

   $(H_2): p, q \in C^{0,1}(\overline{\Omega}), 1 < p(x) < n, p(x) < q(x),$ *for all* $x \in \overline{\Omega}, \frac{q^+}{p^-} \leq 1 + \frac{1}{d}$, *and* $0 \leq \mu(.) \in C^{0,1}(\overline{\Omega})$

Where $C^{0,1}(\overline{\Omega})$ refers to the space of functions that are Lipschitz continous on $\overline{\Omega}$

Proposition 3.3.3.

[14] Let hypotheses (H_0) be satisfied.Then,the following sobolev embedding hold:

1. $L^\theta(\Omega) \hookrightarrow L^{r(.)}(\Omega), W^{1,\theta}(\Omega) \hookrightarrow, W^{1,r(.)}(\Omega), W_0^{1,\theta}(\Omega) \hookrightarrow, W_0^{1,r(.)}(\Omega)$ are continous for $r(.) \in C(\bar{\Omega})$

2. Let $p \in C^{0,1}(\overline{\Omega})$ and $p^*(x) := \frac{np(x)}{n-p(x)}$, for all $x \in \bar{\Omega}$,then,

   (a) $W^{1,\theta}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ and $W_0^{1,\theta}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ are continous for $r(.) \in C(\bar{\Omega})$ with $1 \leq r(x) \leq p^*(x)$, for all $x \in \overline{\Omega}$

   (b) $W^{1,\theta}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ and $W_0^{1,\theta}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ are compact for $r(.) \in C(\overline{\Omega})$ with $1 \leq r(x) < p^*(x)$, for all $x \in \overline{\Omega}$

3. Let $p \in W^{1,\gamma}(\Omega)$ for same $\gamma > n$ and $p_*(x) := \frac{(n-1)p(x)}{n-p(x)}$,for all $x \in \overline{\Omega}$,then

   (a) $W^{1,\theta}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega)$ and $W_0^{1,\theta}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega)$ are continous for $r(.) \in C(\overline{\Omega})$ with $1 \leq r(x) \leq p_*(x)$, for all $x \in \overline{\Omega}$

   (b) $W^{1,\theta}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega)$ and $W_0^{1,\theta}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega)$ are compact for $r(.) \in C(\overline{\Omega})$ with $1 \leq r(x) \leq p_*(x)$, for all $x \in \overline{\Omega}$

4. if $\mu \in L^\infty(\Omega)$,then $L^{q(.)}(\Omega) \hookrightarrow L^\theta(\Omega)$ is continous.

Chapter 4

Dirichlet problem with variable exponent

The variational method and non-variational method are widely used in the study of Dirichlet problems involving the $p(\cdot)$-Laplacian and partial differential equations (PDEs) with non-standard growth conditions. In this chapter, we investigate the existence and uniqueness of solutions for both the $p(x)$-Laplacian problem using variational method as in [23], and the double-phase (see ,[27, 30, 31, 32, 36, 37]) problem using non-variational method as in [14].

4.1 Variational Approach for the $p(x)$-Laplacian Problem

Problem Setting

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary. We consider the variable exponent Sobolev space:

$$W_0^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega), u|_{\partial\Omega} = 0 \right\}$$

equipped with the norm:

$$\|u\| = \inf \left\{ \lambda > 0 : \int_\Omega \left(\left| \frac{\nabla u}{\lambda} \right|^{p(x)} + \left| \frac{u}{\lambda} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

We study the boundary value problem:

$$\begin{cases} -\Delta_{p(x)} u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ is the variable exponent $p(x)$-Laplacian operator, and $p \in C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for all } x \in \overline{\Omega}\}$.

Assumptions

We make the following assumptions:

1. The exponent $p(\cdot)$ satisfies:

$$p^+ = \sup_{x \in \Omega} p(x) < \infty, \quad p^- = \inf_{x \in \Omega} p(x) > 1$$

2. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, meaning:

- $f(\cdot, s)$ is measurable for all $s \in \mathbb{R}$
- $f(x, \cdot)$ is continuous for a.e. $x \in \Omega$

3. There exists $C > 0$ and $q \in C_+(\overline{\Omega})$ with $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ such that

$$|f(x,s)| \leq C(1 + |s|^{q(x)-1})$$

where $p^*(x)$ is the Sobolev critical exponent:

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N \end{cases}$$

4. The antiderivative $F(x,s) = \int_0^s f(x,t)dt$ satisfies

$$\lim_{|s|\to\infty} \frac{F(x,s)}{|s|^{p^+}} = 0 \quad \text{uniformly in } x \in \Omega$$

Energy Functional

The associated energy functional $\Phi \in C^1(W_0^{1,p(\cdot)}(\Omega), \mathbb{R})$ is given by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \int_\Omega F(x,u)\, dx,$$

for all $u \in W_0^{1,p(\cdot)}(\Omega)$. Its Fréchet derivative at u in direction v is:

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v\, dx - \int_\Omega f(x,u)v\, dx.$$

Definition 4.1.1. *A function $u \in W_0^{1,p(\cdot)}(\Omega)$ is called a weak solution of* (4.1) *if it satisfies*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v\, dx - \int_\Omega f(x,u)v\, dx = 0$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)L^\infty(\Omega)$.

Remarks 4.1.1. *The weak solution of* (4.1) *corresponds to critical points of the energy functional Φ.*

Existence Theorem

Theorem 4.1.1.

[Existence of Weak Solutions] Under assumptions (A1)-(A4), problem (4.1) has at least one weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$.

Proof. We proceed via the direct method of calculus of variations [38].
Coercivity of Φ: From (A4), for any $\epsilon > 0$, there exists $M_\epsilon > 0$ such that

$$|F(x,s)| \leq \epsilon |s|^{p^+} \quad \text{for } |s| \geq M_\epsilon.$$

Combined with (A3), we have for some $C_1 > 0$:

$$|F(x,s)| \leq \epsilon |s|^{p^+} + C_1.$$

Thus,

$$\Phi(u) \geq \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)} dx - \epsilon \int_\Omega |u|^{p^+} dx - C_1 |\Omega|.$$

By the Poincaré inequality and the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^+}(\Omega)$, we obtain for $\|u\|$ large enough:

$$\Phi(u) \geq C_2 \|\nabla u\|_{L^{p(\cdot)}}^{p^-} - C_3.$$

This shows $\Phi(u) \to +\infty$ as $\|u\| \to \infty$.
Weak Lower Semicontinuity: The functional

$$u \mapsto \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx$$

is convex and continuous, hence weakly lower semicontinuous. The term $\int_\Omega F(x,u)dx$ is weakly continuous due to the growth condition (A3) and the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.
Existence of Minimizer: Since Φ is coercive and weakly lower semicontinuous, by the Weierstrass theorem it attains its infimum at some $u \in W_0^{1,p(\cdot)}(\Omega)$.
Critical Point Condition: The minimizer u satisfies $\langle \Phi'(u), v \rangle = 0$ for all $v \in W_0^{1,p(\cdot)}(\Omega) L^\infty(\Omega)$. A density argument extends this to all $v \in W_0^{1,p(\cdot)}(\Omega)$, making u a weak solution. □

The condition $q(x) < p^*(x)$ in (A3) ensures the compact embedding needed for the weak continuity of the nonlinear term. The subcritical growth condition (A4) guarantees the coercivity of Φ.
While the existence theorem guarantees at least one solution, additional conditions are needed to ensure uniqueness. We have the following result:

Theorem 4.1.2.

[Uniqueness of Weak Solutions] Under assumptions (A1)-(A4), if additionally:

1. The nonlinearity $f(x,\cdot)$ is strictly decreasing for almost every $x \in \Omega$, i.e.,

$$(f(x,s) - f(x,t))(s-t) < 0 \quad \text{for all } s \neq t,$$

then the weak solution of (4.1) is unique.

Proof. Suppose $u_1, u_2 \in W_0^{1,p(\cdot)}(\Omega)$ are two weak solutions of (4.1). For each $i = 1, 2$, we have:

$$\int_\Omega |\nabla u_i|^{p(x)-2}\nabla u_i \cdot \nabla v \, dx = \int_\Omega f(x, u_i) v \, dx \quad \text{for all } v \in W_0^{1,p(\cdot)}(\Omega).$$

Take $v = u_1 - u_2$ as a test function for both equations and subtract them to obtain:

$$\int_\Omega \left(|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2\right) \cdot (\nabla u_1 - \nabla u_2) \, dx = \int_\Omega (f(x, u_1) - f(x, u_2))(u_1 - u_2) \, dx.$$

We analyze both sides separately:

Left-hand side: The $p(x)$-Laplacian operator is strictly monotone, meaning there exists $C > 0$ such that for all $\xi, \eta \in \mathbb{R}^N$:

$$(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta) \cdot (\xi - \eta) \geq C|\xi - \eta|^{p(x)} \quad \text{if } p(x) \geq 2,$$

and

$$(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta) \cdot (\xi - \eta) \geq C \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p(x)}} \quad \text{if } 1 < p(x) < 2.$$

In both cases, the left-hand side is non-negative and equals zero if and only if $\nabla u_1 = \nabla u_2$ a.e. in Ω.

Right-hand side: By assumption (A5), we have:

$$(f(x, u_1(x)) - f(x, u_2(x)))(u_1(x) - u_2(x)) \leq 0 \quad \text{a.e. in } \Omega,$$

with equality if and only if $u_1(x) = u_2(x)$.

Combining both sides, we obtain:

$$0 \leq \int_\Omega \left(|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2\right) \cdot (\nabla u_1 - \nabla u_2) \, dx \leq 0.$$

This implies:

1. $\nabla u_1 = \nabla u_2$ almost everywhere in Ω
2. $f(x, u_1(x)) = f(x, u_2(x))$ almost everywhere in Ω

From the first conclusion and the Poincaré inequality (since $u_1 - u_2 \in W_0^{1,p(\cdot)}(\Omega)$), we get:

$$\|u_1 - u_2\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u_1 - \nabla u_2\|_{L^{p(\cdot)}(\Omega)} = 0,$$

which implies $u_1 = u_2$ almost everywhere in Ω.

Alternatively, from the second conclusion and the strict monotonicity (A5), we directly obtain $u_1(x) = u_2(x)$ almost everywhere in Ω. □

4.2 Non-variational Approach for Double phase problem

Assumptions and Problem Setting

Assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with Lipschitz boundary $\partial\Omega$, and we suppose that the exponents p, q satisfy the following hypotheses.

(H_0) : $p, q \in C_+(\bar{\Omega}), 1 < p(x) < N, \quad p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$.

Under the condition (H_0), we consider the following Dirichlet double phase problem:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) = f(x, u, \nabla u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

where f is a Carathéodory function, that is, $x \longmapsto f(x, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $(s, \xi) \longmapsto f(x, s, \xi)$ is continuous for a.a. $x \in \Omega$ and satisfies the following hypothesis:
(H) Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that:

(i) There exists $\alpha \in L^{\frac{q_1}{q_1-1}}(\Omega)$ and constants $a_1, a_2 \geq 0$ such that

$$|f(x, s, \xi)| \leq a_1|\xi|^{p\frac{q_1-1}{q_1}} + a_2|s|^{q_1-1} + \alpha(x) \tag{4.3}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$, where $1 < q_1 < p^*$, and p^* is the critical Sobolev exponent .

(ii) There exists $\omega \in L^1(\Omega)$ and constants $b_1, b_2 \geq 0$ such that

$$f(x, s, \xi)s \leq b_1|\xi|^p + b_2|s|^p + \omega(x) \tag{4.4}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$. Moreover,

$$b_1 + b_2\lambda_{1,p}^{-1} < 1, \tag{4.5}$$

where $\lambda_{1,p}$ is the first eigenvalue of the Dirichlet eigenvalue problem for the p-Laplacian.

Exemple 4.2.1.

We remark that, if

$$f(s, \xi) = -d_1|s|^{q_1-2}s + d_2|\xi|^{p-1}, \quad \text{for all } s \in \mathbb{R} \text{ and all } \xi \in \mathbb{R}^N,$$

with $1 < q_1 < p^*$, $d_1 \geq 0$, and

$$0 \leq d_2 < \frac{p}{p - 1 + \lambda_{1,p}^{-1}}.$$

This function satisfies the hypothesis **(H)**.

Remark 4.2.1.

Note that, due to the gradient dependence of f (often called convection term), problem (4.2) does not have variational structure, so variational methods cannot be applied.

Musielak-Orlicz space $W^{1,\theta}(\Omega)$

To define the Musielak-Orlicz space that related to problem (4.2), we consider the function $\theta : \Omega \times [0, +\infty) \longrightarrow [0, +\infty)$ defined by:

$$\theta(x,t) = t^{p(x)} + \mu(x)t^{q(x)}, \text{ for all } x \in \Omega \text{ and all } t \geq 0$$

It is clear that θ is a generalized N-function which satisfies (3.3), locally integrable, and it fulfills the Δ_2-condition.
First, let us introduce some certain properties in $W^{1,\theta}(\Omega)$.

Proposition 4.2.1. *[14] Let hypothesis (H_0) be satisfied.Then,*

1. *$W^{1,\theta}(\Omega) \hookrightarrow L^{\theta}(\Omega)$ is compact embedding,*
2. *There exists a constant $C > 0$ independent of u such that:*

$$\|u\|_{L^{\theta}(\Omega)} \leq C \|\nabla u\|_{L^{\theta}(\Omega)}, \text{ for all } u \in W_0^{1,\theta}(\Omega). \tag{4.6}$$

Proof. The proof of first assertion follows directly from Proposition 3.3.1.
The proof of second assertion is based on contradiction, we suppose that there exists a sequence

$$\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,H}(\Omega)$$

such that

$$\|u_n\|_{\theta} > n\|\nabla u_n\|_{H}.$$

Let us define

$$y_n := \frac{u_n}{\|u_n\|_{\theta}},$$

then it follows that

$$1 = \|y_n\|_{\theta} > \frac{1}{n}\|\nabla y_n\|_{\theta}, \quad \text{for all } n \in \mathbb{N},$$

i.e., the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\theta}(\Omega)$.
Therefore, there exists a subsequence and a function

$$y \in W_0^{1,\theta}(\Omega) \quad \text{such that} \quad y_n \rightharpoonup y \quad \text{in } W^{1,\theta}(\Omega).$$

Using the weak lower semicontinuity of the mapping $v \mapsto \|\nabla v\|_\theta$ on $W_0^{1,\theta}(\Omega)$ (as it is convex and continuous), we have:

$$\|\nabla y\|_\theta \leq \liminf_{n\to\infty} \|\nabla y_n\|_\theta \leq \lim_{n\to\infty} \frac{1}{n} = 0,$$

thus $y = c \in \mathbb{R}$ is a constant function.
we have that $y \in W_0^{1,p(\cdot)}(\Omega)$, where it is known that the only constant function in $W_0^{1,p(\cdot)}(\Omega)$ is $y = 0$.
However, this leads to a contradiction, since by the compact embedding

$$y_n \to y \quad \text{in } L^\theta(\Omega),$$

hence

$$\|y\|_\theta = \lim_{n\to\infty} \|y_n\|_\theta = 1,$$

so $y \neq 0$, which contradicts the conclusion that $y = 0$. □ □

Proposition 4.2.2. *[14]*

Under the hypothesis (H_0), we have.

(i) The norm $\|\cdot\|_{1,\theta,0}$ on $W_0^{1,\theta}(\Omega)$ is uniformly convex.

(ii) For any sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\theta}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\theta}(\Omega)$ and $\rho_\theta(\nabla u_n) \to \rho_\theta(\nabla u)$, it holds that $u_n \to u$ in $W_0^{1,\theta}(\Omega)$.

Now, according to [38], the compactness-type result reads as:

Lemma 4.2.1. *[14] Let $\{u_n\} \subseteq W_0^{1,\theta}(\Omega)$ and $u \in W_0^{1,\theta}(\Omega)$.Moreover,we assume that:*

$$\begin{gathered} 1 < p^- \leq p_n(x) \leq p^+ < +\infty, \\ 1 < q^- \leq q_n(x) \leq q^+, n \in \mathbb{N} \text{ for a,e } x \in \Omega \end{gathered} \tag{4.7}$$

$$p_n \longrightarrow p, q_n \longrightarrow q \text{ a,e in } \Omega, \text{ as } n \longrightarrow +\infty \tag{4.8}$$

$$\left(|\nabla u_n|^{p_n(x)} + \mu(x)\,|\nabla u_n|^{q_n(x)}\right)_n \text{ is bounded in } L^\theta(\Omega). \tag{4.9}$$

and

$$\nabla u_n \rightharpoonup \nabla u \text{ in } \left(L^1(\Omega)\right), \text{ as } n \longrightarrow +\infty \tag{4.10}$$

Then,$\nabla u \in \left(L^\theta(\Omega)\right)$ *and*

$$\liminf_{n\longrightarrow+\infty} \int_\Omega \left(|\nabla u_n|^{p_n(x)} + \mu(x)\,|\nabla u_n|^{q_n(x)}\right) dx \geq \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)\,|\nabla u|^{q(x)}\right) dx. \tag{4.11}$$

Proof. □

We use some ideas coming from. By Young's inequality one has for $a, b_1, b_2 \in \mathbb{R}^n$ and $1 < r, s < \infty$,

$$\begin{aligned} a.b_1 \leq |a| . |b_1| \leq |a|^r + \frac{|b_1|^{r'}}{r' r^{\frac{r'}{r}}}, \quad \frac{1}{r} + \frac{1}{r'} = 1 \\ a.b_2 \leq |a| . |b_2| \leq |a|^s + \frac{|b_2|^{s'}}{s' s^{\frac{s'}{s}}}, \quad \frac{1}{s} + \frac{1}{s'} = 1 \end{aligned} \tag{4.12}$$

If now b_1 and b_2 are two function in$(L^\infty(\Omega))$ and we take $s = q_n$ and $r = p_n$ in (4.12) and use assumption (4.7) ,one derives

$$\int_\Omega \mu(x) \left(\nabla u_n . b_2 - \frac{|b_2|^{q_n'(x)}}{q_n'(x) \left[q_n(x) \right]^{\frac{q_n'(x)}{q_n(x)}}} \right) dx \leq \int_\Omega \mu(x) \left| \nabla u_n \right|^{q_n(x)} dx. \tag{4.13}$$

and

$$\int_\Omega \left(\nabla u_n . b_1 - \frac{|b_1|^{p_n'(x)}}{p_n'(x) \left[p_n(x) \right]^{\frac{p_n'(x)}{p_n(x)}}} \right) dx \leq \int_\Omega \left| \nabla u_n \right|^{q_n(x)} dx. \tag{4.14}$$

Using assumptions (4.8) and (4.10) and passing to the limit in (4.13) and (4.14) as $n \to \infty$, we deduce that

$$\int_\Omega \mu(x) \left(\nabla u . b_2 - \frac{|b_2|^{q'(x)}}{q_n'(x) \left[q_n(x) \right]^{\frac{q'(x)}{q(x)}}} \right) dx \leq \liminf_{n \to \infty} \int_\Omega \mu(x) \left| \nabla u_n \right|^{q_n(x)} dx := L_1 \tag{4.15}$$

and

$$\int_\Omega \left(\nabla u . b_1 - \frac{|b_1|^{p'(x)}}{p'(x) \left[p(x) \right]^{\frac{p'(x)}{p(x)}}} \right) dx \leq \liminf_{n \to \infty} \int_\Omega \left| \nabla u_n \right|^{p_n(x)} dx := L_2 \tag{4.16}$$

Afterward, we consider the following functions,

$$b_1 = p(x) \left| \nabla u \right|_k^{\frac{1}{p'(x) - 1}} \frac{\nabla u}{|\nabla u|}$$

and

$$b_2 = q(x) \left| \nabla u \right|_k^{\frac{1}{q'(x) - 1}} \frac{\nabla u}{|\nabla u|}$$

with $|\nabla u|_k := min \{ |\nabla u| , k \}$.Inserting b_2 into (4.15),one gets:

$$\int_\Omega \mu(x) \left(|\nabla u|_k \, q(x) \left| \nabla u \right|_k^{\frac{1}{q'(x)-1}} - \left| \nabla u \right|_k^{\frac{q'(x)}{q'(x)-1}} \frac{q(x)}{q'(x)} \right) dx \leq L_1$$

Which Implies:

$$\int_\Omega \mu(x) \left| \nabla u \right|_k^{\frac{q'(x)}{q'(x)-1}} dx \leq L_1.$$

Thus

$$\int_\Omega \mu(x) \left| \nabla u \right|_k^{q(x)} dx \leq L_1. \tag{4.17}$$

Simirlarly,if we insert in b_1 in (4.16),we achieve that:

$$\int_\Omega \left| \nabla u \right|_k^{q(x)} dx \leq L_2. \tag{4.18}$$

Ultimately,the inequality (4.11) follows by letting $k \to \infty$ in (4.17) $-$ (4.18),and $\nabla u \in \left(L^\theta(\Omega) \right)$ due to assumption (4.9).The proof of the lemma is complete.

Properties of the double phase operator

We take the double phase operator, denoted by A, associated to problem (4.2). For this purpose, we define

$$A : W_0^{1,\theta}(\Omega) \to \left(W_0^{1,\theta}(\Omega)\right)^*$$

as follows:

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v + \mu(x)|\nabla u|^{q(x)-2}\nabla u \cdot \nabla v \, dx = \int_\Omega a(x, |\nabla u|)\nabla u \cdot \nabla v \, dx,$$

for all $u, v \in W_0^{1,\theta}(\Omega)$, where $\langle \cdot, \cdot \rangle$ signifies the duality pairing between $W_0^{1,\theta}(\Omega)$ and its dual space $\left(W_0^{1,\theta}(\Omega)\right)^*$.

The energy functional $I : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ related to A is defined by

$$I(u) := \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)} \right) dx, \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Proposition 4.2.3.

Let the hypothesis (H_0) holds true. Then the functional I is well defined and belongs to $C^1\left(W_0^{1,\theta}(\Omega), \mathbb{R}\right)$, with

$$I'(u) = A(u).$$

Proof. First, we have

$$\min\left\{ \|u\|^{p^-}_{W_0^{1,\theta}(\Omega)}, \|u\|^{q^+}_{W_0^{1,\theta}(\Omega)} \right\} \le I(u) \le \max\left\{ \|u\|^{p^-}_{W_0^{1,\theta}(\Omega)}, \|u\|^{q^+}_{W_0^{1,\theta}(\Omega)} \right\}, \quad \forall u \in W_0^{1,\theta}(\Omega).$$

Thus, the functional I is well-defined.

Next, let $u, v \in W_0^{1,\theta}(\Omega)$ and $t \in [-1, 1]$. Note that

$$\frac{I(u + tv) - I(u)}{t} = \int_\Omega \frac{a(x, |\nabla u + t\nabla v|) - a(x, |\nabla u|)}{t} \, dx. \tag{3.23}$$

By the mean value theorem, for some $t \in \mathbb{R}$, there exists $\theta(x, t) \in [0, 1]$ such that

$$a(x, |\nabla u + t\nabla v|) - a(x, |\nabla u|) = a(x, |\nabla u + \theta(x, t)t\nabla v|)(\nabla u + \theta(x, t)t\nabla v) \cdot t\nabla v.$$

Therefore, for almost every $x \in \Omega$, one has

$$\frac{a(x, |\nabla u + t\nabla v|) - a(x, |\nabla u|)}{t} = a(x, |\nabla u + \theta(x, t)t\nabla v|)(\nabla u + \theta(x, t)t\nabla v) \cdot \nabla v \underset{t\to 0}{\longrightarrow} a(x, |\nabla u|)\nabla u \cdot \nabla v.$$

Using the generalized Young inequality, for $0 < |t| \leq t_0$, we obtain

$$\begin{aligned}\frac{a(x,|\nabla u + t\nabla v|) - a(x,|\nabla u|)}{t} &= a(x,|\nabla u + \theta(x,t)t\nabla v|)(\nabla u + \theta(x,t)t\nabla v)\cdot\nabla v\\&\leq a(x,|\nabla u + \theta(x,t)t\nabla v|)|\nabla u + \theta(x,t)t\nabla v||\nabla v|\\&\leq \theta(x,h(x,|\nabla u + \theta(x,t)t\nabla v|)) + \theta(x,|\nabla v|)\\&\leq \theta(x,h(x,|\nabla u| + t_0|\nabla v|)) + \theta(x,|\nabla v|)\\&\leq (q^+ - 1)\theta(x,|\nabla u| + t_0|\nabla v|) + \theta(x,|\nabla v|)\\&\leq \frac{q^+ - 1}{2}(\theta(x,2|\nabla u|) + \theta(x,2t_0|\nabla v|)) + \theta(x,|\nabla v|).\end{aligned}$$

It follows, by (3.23), the pointwise convergence, and the dominated convergence theorem, that

$$\lim_{t\to 0}\frac{I(u+tv) - I(u)}{t} = \int_\Omega a(x,|\nabla u|)\nabla u\cdot\nabla v\,dx,$$

and hence $I'(u) = A(u)$.

For the C^1-property, let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\theta}(\Omega)$ be a sequence such that $u_n \to u$ in $W_0^{1,\theta}(\Omega)$, and let $v \in W_0^{1,\theta}(\Omega)$ with $\|v\|_{W_0^{1,\theta}(\Omega)} = 1$. We have

$$\langle A(u), v\rangle = \int_\Omega a(x,|\nabla u|)\nabla u\cdot\nabla v\,dx.$$

Since

$$|\nabla u_n| \to |\nabla u| \quad \text{in } L^\theta(\Omega),$$

it follows that

$$|\nabla u_n| \to |\nabla u| \quad \text{in measure in } \Omega, a(x,|\nabla u_n|) \to a(x,|\nabla u|) \quad \text{in measure in } \Omega. \tag{4.19}$$

Then, according to the converse of Vitali theorem , the sequence $\{a(x,|\nabla u_n|)\}_{n\in\mathbb{N}}$ is uniformly integrable. On the other hand, by Young inequality, we find that

$$\int_\Omega a(x,|\nabla u_n|)\nabla u_n\cdot\nabla v\,dx \leq \int_\Omega a(x,a(x,|\nabla u_n|)|\nabla u_n|)+a(x,|\nabla v|)\,dx \leq (q^+-1)\int_\Omega a(x,|\nabla u_n|)\,dx + \int_\Omega a(x,|\nabla v|)\,dx. \tag{4.20}$$

It follows, by (4.20), that the sequence of functions

$$\{a(x,|\nabla u_n|)\nabla u_n\cdot\nabla v\}_{n\in\mathbb{N}}$$

is uniformly integrable, and by (4.20), we have convergence in measure:

$$a(x,|\nabla u_n|)\nabla u_n\cdot\nabla v \to a(x,|\nabla u|)\nabla u\cdot\nabla v \quad \text{in measure in } \Omega. \tag{4.21}$$

Therefore, applying Vitali s theorem, in combination with (4.21), we obtain

$$\langle A(u_n), v\rangle = \int_\Omega a(x,|\nabla u_n|)\nabla u_n\cdot\nabla v\,dx \longrightarrow \int_\Omega a(x,|\nabla u|)\nabla u\cdot\nabla v\,dx = A(u), v, \quad \text{as } n\to+\infty.$$

Thus, the proof is completed.

Proposition 4.2.4. *[14] Let hypothesis (H_0) holds true. Then:*

(i) The operator $A : W_0^{1,\theta}(\Omega) \to (W_0^{1,\theta}(\Omega))^*$ is continuous, bounded, and strictly monotone.

(ii) The operator $A : W_0^{1,\theta}(\Omega) \to (W_0^{1,\theta}(\Omega))^*$ satisfies the (S_+)-property, i.e., if

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\theta}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0,$$

then $u_n \to u$ strongly in $W_0^{1,\theta}(\Omega)$.

(iii) The operator $A : W_0^{1,\theta}(\Omega) \to (W_0^{1,\theta}(\Omega))^*$ is a homeomorphism.

(iv) The operator $A : W_0^{1,\theta}(\Omega) \to (W_0^{1,\theta}(\Omega))^*$ is strongly coercive, that is,

$$\lim_{\|u\|_{W_0^{1,\theta}(\Omega)} \to +\infty} \frac{\langle A(u), u\rangle}{\|u\|_{W_0^{1,\theta}(\Omega)}} = +\infty.$$

Proof. **(i)** According to [13, Proposition 3.16], we know that $A = I'$, where I is of class C^1. Hence, the operator A is continuous.
From [13, Lemma 2.5] and [13, Proposition 3.5], we obtain:

$$\begin{aligned}\langle A(u) - A(v), u - v\rangle &= \int_\Omega \left(|\nabla u|^{p(x,|\nabla u|)-2}\nabla u - |\nabla v|^{p(x,|\nabla v|)-2}\nabla v\right)\cdot \nabla(u-v)\,dx \\ &\quad + \int_\Omega \mu(x)\left(|\nabla u|^{q(x,|\nabla u|)-2}\nabla u - |\nabla v|^{q(x,|\nabla v|)-2}\nabla v\right)\cdot \nabla(u-v)\,dx \\ &= \int_\Omega (a(x,|\nabla u|)\nabla u - a(x,|\nabla v|)\nabla v)\cdot \nabla(u-v)\,dx.\end{aligned}$$

Using the growth and monotonicity properties of a, we get:

$$\langle A(u) - A(v), u - v\rangle \ge 4\int_\Omega a\left(x, \frac{|\nabla(u-v)|}{2}\right)dx \ge 4\min\left\{\|u-v\|^{p^-}_{W_0^{1,\theta}(\Omega)}, \|u-v\|^{q^+}_{W_0^{1,\theta}(\Omega)}\right\} > 0,$$

for all $u \not\equiv v$. Thus, the operator A is strictly monotone.
To prove the boundedness of A, let us take $u, v \in W_0^{1,\theta}(\Omega) \setminus \{0\}$. In light of Young inequality. we see that

$$\begin{aligned}\frac{\langle A(u), v\rangle}{\|v\|_{W_0^{1,\theta}(\Omega)}} &= \frac{1}{\|v\|_{W_0^{1,\theta}(\Omega)}}\int_\Omega a(x,|\nabla u|)\nabla u\cdot\nabla v\,dx \\ &\le \frac{1}{\|v\|_{W_0^{1,\theta}(\Omega)}}\int_\Omega a(x, a(x,|\nabla u|)|\nabla u|) + a(x,|\nabla v|)\,dx \\ &\le \frac{1}{\|v\|_{W_0^{1,\theta}(\Omega)}}\int_\Omega (q^+-1)a(x,|\nabla u|) + a(x,|\nabla v|)\,dx \\ &\le (q^+-1)\int_\Omega a(x,|\nabla u|)\,dx + \int_\Omega \frac{a(x,|\nabla v|)}{\|v\|_{W_0^{1,\theta}(\Omega)}}\,dx \\ &\le (q^+-1)I(u) + 1.\end{aligned}$$

Therefore, by [14, Proposition 3.5], it follows that

$$\|A(u)\|_{(W_0^{1,\theta}(\Omega))^*} = \sup_{v\in W_0^{1,\theta}(\Omega)\setminus\{0\}} \frac{\langle A(u), v\rangle}{\|v\|_{W_0^{1,\theta}(\Omega)}} \leq (q^+-1)I(u)+1 \leq (q^+-1)\max\left\{\|u\|^{p^-}_{W_0^{1,\theta}(\Omega)}, \|u\|^{q^+}_{W_0^{1,\theta}(\Omega)}\right\}+1.$$

Hence, A is bounded.

(ii) Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\theta}(\Omega)$ be a sequence such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\theta}(\Omega), \qquad \text{and} \qquad \limsup_{n\to\infty}\langle A(u_n), u_n-u\rangle \leq 0. \tag{4.22}$$

Exploiting the weak convergence $u_n \rightharpoonup u$, we get

$$\lim_{n\to\infty}\langle A(u), u_n-u\rangle = 0.$$

Thus, from (4.22), we obtain

$$\limsup_{n\to\infty}\langle A(u_n)-A(u), u_n-u\rangle \leq 0.$$

Then, by the strict monotonicity of A, we conclude

$$0 \leq \liminf_{n\to\infty}\langle A(u_n)-A(u), u_n-u\rangle \leq \limsup_{n\to\infty}\langle A(u_n)-A(u), u_n-u\rangle \leq 0.$$

Hence,

$$\lim_{n\to\infty}\langle A(u_n)-A(u), u_n-u\rangle = 0 = \lim_{n\to\infty}\langle A(u_n), u_n-u\rangle. \tag{4.23}$$

On the other hand, by (4.23) and [13, Lemma 2.5], we find

$$\lim_{n\to\infty}\int_\Omega \theta\left(x, \frac{|\nabla u_n-\nabla u|}{2}\right)dx = 0. \tag{4.24}$$

Then, it yields

$$|\nabla u_n| \to |\nabla u| \quad \text{in } L^\theta(\Omega). \tag{4.25}$$

Thus, $\{|\nabla u_n|\}_{n\in\mathbb{N}}$ converges in measure to $|\nabla u|$ in Ω. Therefore, there exists a subsequence, still denoted by $\{|\nabla u_n|\}_{n\in\mathbb{N}}$, that converges to $|\nabla u|$ almost everywhere in Ω.
Since $t \mapsto \theta(x,t)$ is convex for all $x\in\Omega$, the functional I is convex. Hence,

$$I(u) \geq I(u_n) + \langle A(u_n), u-u_n\rangle.$$

This is equivalent to

$$\langle A(u_n), u_n-u\rangle \geq I(u_n)-I(u).$$

Combining (4.23), we obtain

$$\lim_{n\to\infty} I(u_n) \leq I(u).$$

On the other hand, by Fatou 's Lemma, we have

$$I(u) \leq \liminf_{n\to\infty} I(u_n).$$

it follows that

$$\lim_{n\to\infty} I(u_n) = I(u) \tag{4.26}$$

This implies, together with the fact that $\{|\nabla u_n|\}_{n\in\mathbb{N}}$ converges in measure to $|\nabla u|$ in Ω, that

$$\{a(x, |\nabla u_n|)\}_{n\in\mathbb{N}} \text{ converges in measure to } a(x, |\nabla u|) \text{ in } \Omega. \tag{4.27}$$

According to the converse of Vitali's theorem [13], we can see that the sequence of functions

$$\{a(x, |\nabla u_n|)\}_{n\in\mathbb{N}} \text{ is uniformly integrable.} \tag{4.28}$$

Using the monotonicity and convexity of the function $t \mapsto a(\cdot, t)$, for all $x \in \Omega$, we get

$$a(x, |\nabla u_n - \nabla u|) \leq a(x, |\nabla u_n| + |\nabla u|) \leq \frac{1}{2}\left(a(x, 2|\nabla u_n|) + a(x, 2|\nabla u|)\right)$$
$$\leq 2q^+ - 1\left(a(x, |\nabla u_n|) + a(x, |\nabla u|)\right).$$

Thus, from (4.28), we have the uniform integrability of the sequence of functions

$$\{a(x, |\nabla u_n - \nabla u|)\}_{n\in\mathbb{N}}.$$

Again, since $\{|\nabla u_n|\}_{n\in\mathbb{N}}$ converges in measure to $|\nabla u|$ in Ω, we see that

$$\{a(x, |\nabla u_n - \nabla u|)\}_{n\in\mathbb{N}} \text{ converges in measure to } 0 \text{ in } \Omega.$$

Applying Vitali's theorem [8, Theorem 4.5.4], it follows that

$$\lim_{n\to\infty} \int_\Omega a(x, |\nabla u_n - \nabla u|)\, dx = 0.$$

Finally, we obtain

$$u_n - u_{W_0^{1,\theta}(\Omega)} \to 0, \quad \text{as} \quad n \to \infty.$$

Namely, $u_n \to u$ in $W_0^{1,\theta}(\Omega)$. Hence, the operator A satisfies the (S+) property. □

Existence and uniqueness results

In this section, we present the main results that can be found in [14].

Definition 4.2.1.

We say that $u \in W_0^{1,\theta}(\Omega)$ is a *weak solution* of problem (4.2) if it satisfies

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \cdot \nabla\varphi\, dx = \int_\Omega f(x, u, \nabla u)\varphi\, dx \tag{4.29}$$

for all test functions $\varphi \in W_0^{1,\theta}(\Omega) \bigcap L^\infty(\Omega)$.

Remark 4.2.2.

Note that, the weak solution in (4.29) is well-defined due to the embedding [3],[12] and the fact that $p < q$, along with conditions.
The main existence result reads as follows.

Theorem 4.2.1.

Under the hypotheses (H_0) and **(H)**, the problem (4.2) admits at least one weak solution

$$u \in W_0^{1,\theta}(\Omega).$$

Proof. □

Let $\widehat{N}_f : W_0^{1,\theta}(\Omega) \subseteq L^{q_1}(\Omega) \to L^{q_1}(\Omega)$ be the Nemytskij operator associated to f, and let

$$i^* : L^{q_1}(\Omega) \to \left(W_0^{1,\theta}(\Omega)\right)^*$$

be the adjoint operator of the embedding $i : W_0^{1,\theta}(\Omega) \hookrightarrow L^{q_1}(\Omega)$. For $u \in W_0^{1,\theta}(\Omega)$, we define

$$N_f := i^* \circ \widehat{N}_f, \quad \text{and set } \mathcal{A}(u) := A(u) - N_f(u).$$

From the growth condition on f, we know that

$$\mathcal{A} : W_0^{1,\theta}(\Omega) \to \left(W_0^{1,\theta}(\Omega)\right)^*$$

maps bounded sets into bounded sets.

Let us now show that $\mathcal{A}$ is pseudomonotone, see [13, Definiton 2,1]. To this end, let $\{u_n\}_{n\geq 1} \subset W_0^{1,\theta}(\Omega)$ be a sequence such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\theta}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle_\theta \leq 0 \tag{4.30}$$

From the compact embedding in [14], we obtain

$$u_n \to u \quad \text{in } L^{q_1}(\Omega), \tag{4.31}$$

since $q_1 < p^*$. Using the strong convergence in $L^{q_1}(\Omega)$, see (4.31), along with Hölder inequality and the growth condition on f, we obtain

$$\lim_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)(u_n - u)\, dx = 0.$$

Therefore, we can pass to the limit in the weak formulation (4.28), replacing u by u_n and θ by $u_n - u$. This gives

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle_\theta = \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle_\theta \leq 0. \tag{4.32}$$

From [13, proposition 2,3] we know that A fulfills the (S_+)-property. Thus, in view of (4.32) and (4.30), we conclude that $u_n \to u$ in $W_0^{1,\theta}(\Omega)$. Hence, because of the continuity of A, we obtain

$$\mathcal{A}(u_n) \to \mathcal{A}(u) \quad \text{in } \left(W_0^{1,\theta}(\Omega)\right)^*,$$

which proves that $\mathcal{A}$ is pseudomonotone.

Next, we show that the operator $\mathcal{A}$ is coercive, that is,

$$\lim_{\|u\|_{1,\theta,0}\to\infty} \frac{\langle \mathcal{A}(u), u\rangle_\theta}{\|u\|_{1,\theta}} = +\infty. \tag{4.33}$$

From the representation of the first eigenvalue of the p-Laplacian, see (4.1), replacing r by p, we have the inequality

$$\|u\|_p^p \leq \lambda_{1,p}^{-1}\|\nabla u\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{4.34}$$

Since $W_0^{1,\theta}(\Omega) \subseteq W_0^{1,p}(\Omega)$, and by applying (4.34), We estimate the operator $\mathcal{A}$ applied to $u \in W_0^{1,\theta}(\Omega)$ as follows:

$$\begin{aligned}
\langle A(u), u\rangle_\theta &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)\cdot \nabla u\, dx - \int_\Omega f(x,u,\nabla u)u\, dx \\
&= \int_\Omega |\nabla u|^p\, dx + \int_\Omega \mu(x)|\nabla u|^q\, dx - \int_\Omega f(x,u,\nabla u)u\, dx \\
&\geq \|\nabla u\|_p^p + \|u\|_{q,\mu}^q - b_1\|\nabla u\|_p^p - b_2\|u\|_p^p - \|\omega\|_1 \\
&\geq \left(1 - b_1 - b_2\lambda_{1,p}^{-1}\right)\|\nabla u\|_p^p + \|u\|_{q,\mu}^q - \|\omega\|_1,
\end{aligned}$$

where we used the inequality (4.34):

$$\|u\|_p^p \leq \lambda_{1,p}^{-1}\|\nabla u\|_p^p,$$

and (4.4). Thus, we obtain:

$$\langle \mathcal{A}(u), u\rangle_\theta \geq \min\left\{1 - b_1 - b_2\lambda_{1,p}^{-1}, 1\right\}\left(\|\nabla u\|_p^p + \|u\|_{q,\mu}^q\right) - \|\omega\|_1.$$

Therefore, since $1 < p < q$ and from assumption (4.5), it follows that

$$\lim_{\|u\|_{1,\theta,0}\to\infty} \frac{\langle \mathcal{A}(u), u\rangle_\theta}{\|u\|_{1,\theta,0}} = +\infty, \tag{3.9}$$

which shows that $\mathcal{A}$ is coercive.

Hence, the operator $\mathcal{A} : W_0^{1,\theta}(\Omega) \to \left(W_0^{1,\theta}(\Omega)\right)^*$ is bounded, pseudomonotone, and coercive. Then, by [13, Theorem 2.2], there exists

$$u \in W_0^{1,\theta}(\Omega) \quad \text{such that} \quad \mathcal{A}(u) = 0.$$

By the definition of $\mathcal{A}$, the function u turns out to be a weak solution of problem (4.2), which completes the proof.

Remark 4.2.3.

The problem (1) has a unique weak solution with the convection term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:

(C1) There exists $c_1 \geq 0$ such that

$$(f(x,s,\xi) - f(x,t,\xi))(s-t) \leq c_1 |s-t|^2,$$

for a.a. $x \in \Omega$, for all $s, t \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$.

(C2) There exists $\rho \in L^r(\Omega)$, with $1 < r < p^*$, and $c_2 \geq 0$ such that $\xi \mapsto f(x,s,\xi) - \rho(x)$ is linear for a.a. $x \in \Omega$, all $s \in \mathbb{R}$, and

$$|f(x,s,\xi) - \rho(x)| \leq c_2 |\xi|,$$

for a.a. $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$. Moreover,

$$c_1 \lambda_{1,2}^{-1} + c_2 \lambda_2^{-1} < 1, \tag{3.11}$$

where $\lambda_{1,2}$ is the first eigenvalue of the Dirichlet eigenvalue problem for the Laplace differential operator.

Theorem 4.2.2.

Under the hypotheses (H_0), **(H)**, $(C1)$, and $(C2)$, the double phase problem (4.2) has a unique solution.

Proof. Let $u, v \in W_0^{1,\theta}(\Omega)$ be two weak solutions of equation (4.35). Testing the corresponding weak formulations with $\varphi = u - v$ and subtracting the resulting equations gives:

$$\int_\Omega |\nabla(u-v)|^2 \, dx + \int_\Omega \mu(x) \left(|\nabla u|^{q(x)-2} \nabla u - |\nabla v|^{q(x)-2} \nabla v \right) \cdot \nabla(u-v) \, dx \tag{4.35}$$
$$= \int_\Omega (f(x,u,\nabla u) - f(x,v,\nabla u))(u-v) \, dx + \int_\Omega (f(x,v,\nabla u) - f(x,v,\nabla v))(u-v) \, dx.$$

The second term on the left-hand side of (4.35) is nonnegative, so we obtain the estimate:

$$\int_\Omega |\nabla(u-v)|^2 \, dx \leq \int_\Omega (f(x,u,\nabla u) - f(x,v,\nabla u))(u-v) \, dx + \int_\Omega (f(x,v,\nabla u) - f(x,v,\nabla v))(u-v) \, dx. \tag{4.36}$$

Using assumptions (H), and Hölder inequality, the right-hand side can be estimated as:

$$\int_\Omega (f(x,u,\nabla u) - f(x,v,\nabla u))(u-v) \, dx \leq c_1 \|u-v\|_{L^2(\Omega)}^2, \tag{4.37}$$

$$\int_\Omega (f(x,v,\nabla u) - f(x,v,\nabla v))(u-v) \, dx \leq c_2 \|u-v\|_{L^2(\Omega)} \|\nabla(u-v)\|_{L^2(\Omega)}.$$

we get:

$$\|\nabla(u-v)\|_{L^2(\Omega)}^2 \leq c_1 \lambda_1^{-1} \|u-v\|_{L^2(\Omega)}^2 + c_2 \lambda_2^{-1} \|u-v\|_{L^2(\Omega)} \|\nabla(u-v)\|_{L^2(\Omega)}. \tag{4.38}$$

Then, by a suitable application of inequality (4.38) it follows that $u = v$. □ □

Conclusion and Further Prospects

This work establishes fundamental properties of Musielak-Orlicz-Sobolev spaces and their application to the $p(x)$-Laplacian problem and double-phase problem with variable exponents, in particular:

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u) = f(x,u,\nabla u), \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases}$$

Where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $p,q \in C(\overline{\Omega})$ satisfy $1 < p(x) < N, \quad p(x) < q(x)$ for all $x \in \overline{\Omega}$, $\mu \in L^\infty(\Omega)$, and f is a Carathéodory function.
Through non-variational analysis, we have studied the existence of weak solutions under log-Hölder continuity of exponents p,q and controlled growth of f. The solutions exhibit enhanced regularity properties, with $\nabla u \in L^{q(\cdot)}(\Omega)$ and partial Hölder continuity depending on the behavior of the weight function $\mu(\cdot)$. The Musielak-Orlicz framework provides a unified functional setting for these non-standard growth problems, extending classical Sobolev space theory. Practical implications include modeling of anisotropic materials with phase transitions and adaptive image processing, where the double-phase structure captures local variations in diffusivity.
Future research, we suggest to study the following double-phase problem:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

Where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $p,q \in C(\overline{\Omega})$ satisfy $1 < p(x) < N, \quad p(x) < q(x)$ for all $x \in \overline{\Omega}$, $\mu \in L^\infty(\Omega)$, and $f \in L^{m(\cdot)}(\Omega)$ with $m(\cdot) > 1$.

Abstract

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This work investigates some properties of Musielak-Orlicz-Sobolev spaces $W^{1,\theta}(\Omega)$ and their application to the nonlinear double-phase problem with variable exponents:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) = f(x, u, \nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.39}$$

Where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $p, q \in C(\overline{\Omega})$ satisfy $1 < p(x) < N, \quad p(x) < q(x)$ for all $x \in \overline{\Omega}$, $\mu \in L^\infty(\Omega)$, and f is a Carathéodory function.
We study the existence and uniqueness results for solutions to problem (4.39) in the Musielak-Orlicz-Sobolev space framework. The variational methods cannot be applied here due to problem (4.39) does not have variational structure. This why our approach employs non-variational method following [20], building on the analytical techniques developed in [12] and [14] for studying partial differential equations with non-standard growth conditions. These methods provide a robust framework for analyzing sequences of approximate solutions and their convergence properties.
The analysis highlights the fundamental role of Musielak-Orlicz-Sobolev spaces in functional analysis, extending the classical variable exponent Lebesgue space theory [3]. We examine several key properties of these spaces that are essential for handling the nonlinearities and variable growth conditions in problem (4.39).

Keywords: Musielak-Orlicz Spaces, Musielak-Orlicz-Sobolev Spaces, Double Phase Operator, Variable Exponent, Nonlinear PDEs, Existence Results, Uniqueness.

في هذا العمل ندرس خصائص فضاءات $Musielak-Orlicz-Sobolev$ و ترمز ب $W^{1,\theta}(\Omega)$ وتطبيقها على المسألة غير الخطية ذات الطور المزدوج بأسّ متغيرة:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) = f(x,u,\nabla u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{4.40}$$

حيث $\Omega \subset \mathbb{R}^N$ (■ \geq ■٢) ميدان محدود و $p,q \in C(\overline{\Omega})$ تحقّان الشرط $1 \leq p(x) \leq q(x)$ لكل $x \in \overline{\Omega}$ ، و $\mu \in L^{\infty}(\Omega)$ ، و f دالة من نوع *carathéordory* .

نثبت نتائج حول وجود وّحدانية الحلول لّمسألة (4.40) في إطار فضاءات $Musielak-Orlicz-Sobolev$، تعتمد دراستنا على طرق غير تغايرية كما استخدمت في [٢٠] ، كما استخدمنا التقنيات التحليلية المطورة التي قدمت في [١٢] - [١٤] لدراسة المعادلات التفاضلية الجزئية ذات شروط النمو غير الاعتيادية. توفر هذه الطرق إطارًا متينًا لتحليل تسلسلات الحلول التقريبية وخصائص تقاربها.

تُظهر هذه الدراسة الدور الأساسي لفضاءات■ پسـ■لكـورلخزـصبلقا■ في التحليل الدالي، حيث توسع النظرية الكلاسيكية لفضاءات لوبيغ ذات الأسّ المتغير [٣] . نستعرض عدة خصائص أساسية لهذه الفضاءات تُعد ضرورية لمعالجة المسائل الغير خطية ذات الاسّ المتغيرة كما في المسألة (4.40) .

\textbfالكلمات المفتاحية : $Musielak-Orlicz$ ،■پسـ■لكـورلخزـصبلقا■ ، مؤثر الطور المزدوج، الأسّ المتغير، معادلات غير الخطية، نتائج وجود، وحدانية..

Résumé

Ce travail étudie les propriétés des espaces de Musielak-Orlicz-Sobolev $W^{1,\theta}(\Omega)$ et leur application au problème non linéaire á double phase avec des exposants variables :

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) = f(x,u,\nabla u), & \text{dans } \Omega, \\ u = 0, & \text{sur } \partial\Omega, \end{cases} \tag{4.41}$$

où $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ est un domaine borné, $p, q \in C(\overline{\Omega})$ satisfont $1 < p(x) < N, \quad p(x) < q(x)$ pour tout $x \in \overline{\Omega}$, $\mu \in L^\infty(\Omega)$, et f est une fonction de Carathéodory.

Nous établissons des résultats d 'existence et d'unicité pour les solutions au problème (4.41) dans le cadre des espaces de Musielak-Orlicz-Sobolev. Notre approche utilise des méthodes non variationnelles suivant [20], s'appuyant sur les techniques analytiques développées dans [12] et [14] pour l 'étude des équations aux dérivées partielles avec des conditions de croissance non standard. Ces méthodes fournissent un cadre robuste pour analyser les séquences de solutions approximatives et leurs propriétés de convergence.

L 'analyse met en évidence le rôle fondamental des espaces de Musielak-Orlicz-Sobolev en analyse fonctionnelle, étendant la théorie classique des espaces de Lebesgue à exposant variable [3]. Nous examinons plusieurs propriétés clés de ces espaces qui sont essentielles pour traiter les non-linéarités et les conditions de croissance variable dans le problème (4.41).

Mots-clés : Espaces de Musielak-Orlicz, Espaces de Musielak-Orlicz-Sobolev, Opérateur à double phase, Exposant variable, Équations PDE non linéaires, Résultats d 'existence, Unicité.

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