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Radial basis function method for Solving Integral equations

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بسم الله الرحمن الرحيم

Dedication

Firstly, we thank God for His great guidance and immense favor upon us. And the best thing to say is: 'O Allah, we praise You until You are pleased, and we praise You when You are pleased, and we praise You after You are pleased.

I dedicate this work specifically to

My beloved **parents** my **mother** and my **father**, who supported me every step of my academic journey, You are my role models, thank you for everything.

I want to say thank you and send my love to all my **sisters**, mentioning each of them by name, and also to my **brothers Ibrahime** and **Haroune**.

I thank my **grandmother** and **grandfather** for all that they have given us, and pray that God preserves them.

I also dedicate this work to all my **friends** who shared our academic journey, each by their place and name, and I can't forget my loyal friend and roommate **Chaima Messelmi**.

Finally I dedicate this work to everyone who knows me, whether near or far, and to everyone who supported me and advised me throughout this academic journey.**Thank you all**.

Acknowledgement

Thank God always and ever.

I start by thanking and appreciating all the honorable teachers:

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▶Thanks to Mr. Mostefa.Nadir,to have accepted the presidency of the jury of this work.

▶I would like to thank Mrs. Amina.Khirani, for agree to review this work.

At the end, I would like to express my sincere thanks and respect to all the teachers who taught me throughout my academic journey, especially the math teachers, each one by name.

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Introduction

Integral equations are a type of equations that contain an unknown function under the integral sign, these equations are used to model many mathematical and physical problems, and they are an essential part of applied mathematics, given in their general form

$$\mathbf{\Phi(x)u(x)} - \lambda \int_{g(x)}^{h(x)} \mathbf{k(x,t)u(t)dt} = \mathbf{f(x)}. \tag{1}$$

We have also many types of these equations, the most famous of which are the **Volterra** and **Fredholm** equations of the **first** and **second** kind, based on that, we want to solve this type of equations.

There are numerous techniques available for approximating the solution of integral equations, with one of the most renowned being **the radial basis function (RBF)** method, which was first used in **2006** and has proven its accuracy in approximating the solution.

In this work, the RBF technique was applied to the Fredholm equation of the second kind, our work included three chapters.

In **first chapter** we defined the integral equations and their types with discussing their solvability, **second chapter** we introduced the concept of radial basis function and its types, we also described principle of RBF's interpolation, **in the third chapter** we provided theoretical and practical explanations of Multiquadric (MQ)-RBF technique in approximating the solution to Fredholm's second kind integral equation.

Lastly, we'll evaluate the accuracy of this method by comparing it to the exact solution.

Chapter 1

Preliminaries and Integral equations

1.1 Preliminaries

1.1.1 Normed space

Definition 1.1.1 *Let E be a vector space on K, a norm on E is an application*

$$N : E \longrightarrow \mathbb{R}_+$$

$$x \longrightarrow N(x),$$

satisfies the following properties:

▶ $N(x) = 0 \iff x = 0$ for all $x \in E$, (Positively).

▶ $N(\lambda x) =\mid \lambda \mid N(x)$ for all $x \in E$, $\lambda \in \mathbb{R}$, (Homogeneity).

▶ $N(x + y) \leq N(x) + N(y)$ for all $x, y \in E$, (Triangle inequality).

The norm N will be designated by the symbol $\| \cdot \|$ and $(E, \| \cdot \|)$ is called normed vector space.

1.1.2 Metric space

Definition 1.1.2 *Let* $(E, \| \, . \, \|)$ *a norm space, the following application*

$$d : E \times E \longrightarrow \mathbb{R}_{+}.$$

$$(x, y) \longrightarrow d(x, y) = \| \, x - y \, \| \, ,$$

is called the distance associated with the norm, it checks the following properties:

▶ $\forall (x, y) \in E^2 : d(x, y) = 0 \Leftrightarrow x = y,$ (Positivity).

▶ $\forall (x, y) \in E^2 : d(x, y) = d(y, x),$ (Symmetry).

▶ $\forall (x, y, z) \in E^2 : d(x, z) \leq d(x, y) + d(y, z),$ (Triangle inequality).

We then call (E, d) Metric space.

1.1.3 Bounded and Compact Operator

Definition 1.1.3 *Let* A *be operator from* X *a linear space into a linear space*Y *is called linear if*

$$A(\alpha\varphi + \beta\psi) = \alpha A\varphi + \beta A\psi, \quad \varphi, \psi \in X, \;\; \alpha, \beta \in \mathbb{R}.$$

Theorem 1.1.1 *A linear operator such that* $A : X \to Y$ *we say* A *bounded operator if there exists a positive number* C *such that*

$$\|A\varphi\| \leq C\|\varphi\|.$$

In other words, the aid of the linearity of$A$,we say A is bounded if and only if

$$\|A\| := \sup \|A\varphi\| < \infty.$$

Theorem 1.1.2 *[4] (finite dimensional domain)*

Let A be a bounded operator defined from E into F, with the domain E has a finite dimension, $dimE < \infty$ then the operator$A$ is compact.

Proof. ■

The space E has a finite dimension, $dimE < \infty$ implies the finite dimensional range $A(E)$, say

$$dimA(E) \leq dimE \ ,$$

it follows that, A is a compact operator.

Theorem 1.1.3 *[4](Continuous Kernel)*

The integral operator A defined from $C(\Omega)$ into $C(\Omega)$

$$A\varphi(x) = \int_{\Omega} K(x,t)\varphi(t)dt, \quad t, x \in \Omega,$$

with continuous kernel $k(x;t)$ is a compact operator.

1.1.4 Numerical Integral

There are many ways that calculate the approximate values of their integration

Trapeze Method

Let f a continuous function over an interval $[a, b]$, the approximate value of the integral of $\int_a^b f(t)dt$ such as $x_0 = 0$, $h = (b-a)/N$, $x_{i+1} = x_0 + ih$, $i = 1, ...N$.

By Trapeze method is then given by

$$I_N = \int_a^b f(t)dt = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{N-1} f(t_i)\right).$$

Trapeze Method error

If $f \in C^2$, error is increased

$$\mid I_N - \int_a^b f(t)dt \mid \leq (b-a) * h^2 \frac{M_2}{12}, \quad M_2 = \max \mid f(t) \mid .$$

Quadratic Gauss Method

The quadratic Gauss formula that is given in the form

$$\int_{-1}^{1} g(\varepsilon)d\varepsilon \approx \sum_{q=1}^{Q} W_q g(\varepsilon_q),$$

the integral range can be transformed with formula $t = \frac{(a+b)}{2} + \frac{(b-a)}{2}\varepsilon = h(\varepsilon)$.

1.1.5 Fourier and Laplace Transform

Fourier transform

Definition 1.1.4 *Let $f \in L^1(\mathbb{R})$, we call the Fourier transform of f the complex function of the real variable ξ such that*

$$\mathcal{F}(f(x))_{(\xi)} = \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)dx, \quad \xi \in \mathbb{R},$$

with $f \in L^1(\mathbb{R}) \Longleftrightarrow \int_{\mathbb{R}} \mid f(x) \mid dx < \infty$.

Laplace transform

Laplace transform of a function is defined by

$$\mathcal{L}(f(x))_{(s)} = F(s) = \int_0^{+\infty} e^{-sx} f(x)dx,$$

such that s is a complex variable, and $F(s)$ is a complex function.

1.1.6 Conditioning and stability of matrix

Conditioning

Definition 1.1.5 *Let $\mathbb{R}^n$ be attached with a norm $\| \cdot \|$ and $M_n(\mathbb{R})$ with the induced norm, let $A \in M_n(\mathbb{R})$ be an invertible matrix, the conditioning of A with respect to the norm $\| \cdot \|$ is the positive real number $Cond(A)$ defined as:*

$$Cond(A) = \| A \| \| A^{-1} \| .$$

Stability

when using radial basis functions on dense data sets, with small fill distance, the condition of the matrix used will get extremely large if the separation distance of points $X = \{x_1, ..., x_n\}$ gets small:

$$S(X) := \frac{1}{2} \min_{1 \leq i \leq j \leq n} \|x_i - x_j\|_2.$$

1.2 Integral equations

1.2.1 Definition

An integral equation is characterized by having the unknown function $u(x)$ that needs to be solved for appearing within the integral sign, and the most standard form of integral equation is given as follows:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt. \tag{1.1}$$

In this context, $k(x,t)$ represents a familiar function with two variables x and t termed as the kernel of the integral equation, the limits of integration $g(x)$ and $h(x)$ can be constants, or a combination of constants and variable, λ is a real and a constant value, $f(x)$ is a known function.

The unknown function $u(x)$ that will be determined appears inside and outside the integral sign.

1.2.2 Classifications of integral equations

Integral equations can be classified into the following classes:

►Volterra integral equation (VIE).

►Fredholm integral equation (FIE).

►Volterra-Fredholm integral equations.

Let is outline these equations using fundamental definition and properties of each type.

Volterra integral equation (VIE)

We say about each equation written from the shape

$$\Phi(x)u(x) = f(x) + \lambda \int_a^{h(x)} k(x,t)u(t)dt, \;\; x \in [a,b]$$

It's Volterra equation, so that the last limit of integration are function of x, and there are two cases:

1-First kind(VIE)

- $f(x) = \lambda \int_a^{h(x)} k(x,t)u(t)dt, \;$ If $\Phi(x) = 0$.

2-Second kind(VIE)

- $u(x) = f(x) + \lambda \int_a^{h(x)} k(x,t)u(t)dt, \;$ If $\Phi(x) = 1$.

Fredholm integral equation (FIE)

When the integration limits are fixed, the integral equation is called a Fredholm integral equation given in the form

$$\Phi(x)u(x) = f(x) + \lambda \int_a^{b} k(x,t)u(t)dt, \qquad x \in [a,b] \,.$$

There are two cases

1-First kind(FIE)

- $f(x) = \lambda \int_a^{b} k(x,t)u(t)dt$, when $\Phi(x) = 0$.

2-Second kind(FIE)

- $u(x) = f(x) + \lambda \int_a^{b} k(x,t)u(t)dt$, when $\Phi(x) = 1$.

Volterra-Fredholm integral equations

An integral equation of Volterra-Fredholm comprises separate Volterra and Fredholm integrals, which appear together within the context of an integral equations defined as

$$u(x) = f(x) + \lambda_1 \int_a^{h(x)} k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt, \quad x \in [a,b]\,.$$

1.2.3 Linearity and Homogeneity

Linearity

Definition 1.2.1 *[4] Let the integral equation be defined as given in the following equation*

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)F(u(t))dt. \tag{1.2}$$

Equation 1.2 is classified as **a linear** integral equation when the function $F(u(t))$ is linear, which means it takes the form of either $F(u(t)) = a$ (where 'a' is a constant) or $F(u(t)) = u(t)$.

Equation 1.2 becomes **nonlinear** when the function F is nonlinear, such as in the case where $F(u(t)) = u^n(t)$ for $n \geq 2$.

Homogeneity

We say about Volterra and Fredholm integral equations of the second kind are

- **homogeneous:** if $f(x) = 0$, like $u(x) = \int_0^{b|x} (1 - x - t)u^2(t)dt$.
- **inhomogeneous:** If $f(x) \neq 0$, like $u(x) = \sin x + \int_0^{b|x} xtu(t)dt, \;\; f(x) = \sin x$.

1.2.4 Existence and Uniqueness of solution of integral equations

For operator integral equations of second kind

$$\varphi - A\varphi = f.$$

The Neumann series can confirm both the existence and uniqueness of a solution when provided that A is a contraction $\| A \| < 1$.

Theorem 1.2.1 *[7](Neumann series)*

In a Banach space X, suppose $A : X \to X$ is a bounded linear operator, with $\| A \| < 1$, and $I : X \to X$ represent the identity operator.
Then $(I - A)$ has a bounded inverse on X that is given by the Neumann series

$$(I - A)^{-1} = \sum_{K=0}^{\infty} A^k$$

and satisfies

$$\| (I - A)^{-1} \| \leq \frac{1}{1 - \| A \|}.$$

Given the assumptions of the theorem above, for any $f \in X$ the consecutive approximations

$$\varphi_{n+1} := A\varphi_n + f$$

Converge to the unique solution φ of $\varphi - A\varphi = f$.

Corollary 1.2.1 *Let k be a continuous kernel satisfying*

$$\max_{x \in G} \int_G | k(x,t) | \, dt < 1.$$

The integral equation of the second kind

$$\varphi(x) - \int_{\Omega} k(x,t)\varphi(t)dt = f(x) \text{ , for each } f \in C(G), \text{ and } x \in G \text{ ,}$$

has a unique solution $\varphi \in C(G)$.

Example

Consider Fredholm's integral equation of the second kind.

$$\lambda\varphi(x) - \int_{G} k(x,t)\varphi(t)dt = f(x). \tag{1.3}$$

The function $K : \Omega \times \Omega \to \mathbb{R}$ is continuous we rewrite this equation as follows

$$(\lambda I - k)\varphi = f$$

and from it

$$(I - A)\varphi = \frac{1}{\lambda}f, \quad \text{such that } A = \frac{1}{\lambda}k,$$

and based on **Naiman's theory**

$$\| A \| = \frac{1}{| \lambda |} \| k \| < 1$$
$$\| k \| = \max_{x \in \Omega} \int_{\Omega} | k(x,t) | \, dt < | \lambda |,$$

in this case the operator $(\lambda I - k)^{-1}$exist, and we have

$$\| (\lambda I - k)^{-1} \| \leq \frac{1}{| \lambda | - \| k \|},$$

we have for all $f \in C(\Omega)$ Fredholm's integral equation of the second kind (1.3) admits a unique solution $\varphi \in C(\Omega)$.

Chapter 2

Radial basis function

Radial basis function are areal valued function that depends on calculating the distance between inputs and some fixed points called centers, it serves as a computational method to approximate complex and multi-variable functions into simpler and more understandable functions.

2.1 Radial function

Definition 2.1.1 *[2] A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be radial if there exist a function $\varphi : [0, \infty) \to \mathbb{R}$ such that $\Phi(x) = \varphi(\|x\|_2)$, for all $x \in \mathbb{R}^d$ where $\| \cdot \|$ is the Euclidean norm.*

2.2 Radial basis function

Definition 2.2.1 *[3]*

A radial basis functions on $\mathbb{R}^d$ is a function of the form $\Phi(\| x - x_i \|)$ where x, $x_i \in \mathbb{R}^d$ and $r = \| x - x_i \|$ notes the Euclidean distance between x and x_i, if one chooses N

points$\{x_i\}_{i=1}^N$ in $\mathbb{R}^d$ then by custom

$$S(x) = \sum_{i=1}^{N} C_i \Phi(\| x - x_i \|), \quad C_i \in \mathbb{R},$$

is called a radial basis functions as well.

2.3 Types of RBF and their classifications

Name of RBF	$\Phi(r)$, $r \geqslant 0$
Gaussian(GA)	$e^{-(\alpha r)^2}$
Multiquadric(MQ)	$\sqrt{\alpha^2 + r^2}$
Inverse Multiquadric(IMQ)	$\frac{1}{\sqrt{\alpha^2 + r^2}}$
Linear(LI)	r
Cubic(CU)	r^3
Thin plate spline(TPS)	$r^2 \log(r)$

Table 2.1: some types of RBFs

There are many functions that generate RBF, some of them are explained in the table above, and we categorize them based on the parameter "α" into two classes:

2.3.1 First Class:Infinitely smooth RBFs

In this class the basis function $\Phi(r)$ heavily depends on the free shape parameter "α", and that is what we notice in these functions like:Gaussian (GA), and Multiquadric (MQ), Inverse Multiquadric (IMQ),...etc.

2.3.2 Second Class:Piecwise smooth RBFs

The key advantage is that the basis function of this category are shape parameter free, and that is what we notice in these functions like: Linear (LI), Thin plate spline

(TPS),...etc.

2.4 The scattered data interpolation problem

Scattered data problem can be described as follows:

Let $X \in \mathbb{R}^d$, consider m distinct points as $X = \{x_1, x_2, ...x_m\}$ in this space at which the function to be approximated is known and real scalers $(f(x_1), f(x_2), ...f(x_m))$, we desire to construct a continuous function S:

$$S : \mathbb{R}^N \to \mathbb{R}.$$

$$S(x_i) = f_i, \quad i = \overrightarrow{1 : m}.$$

And that's what we notice in these curves

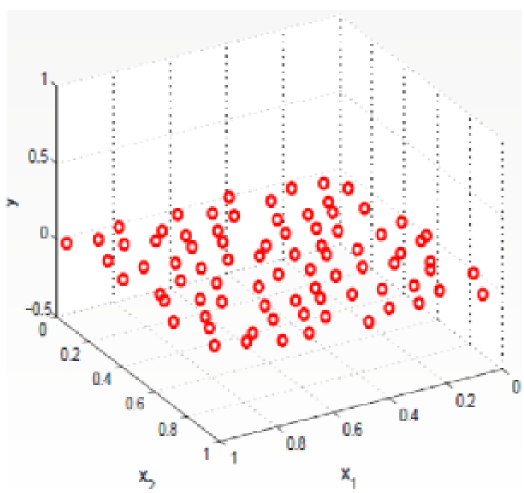


Figure 2.1: Data points

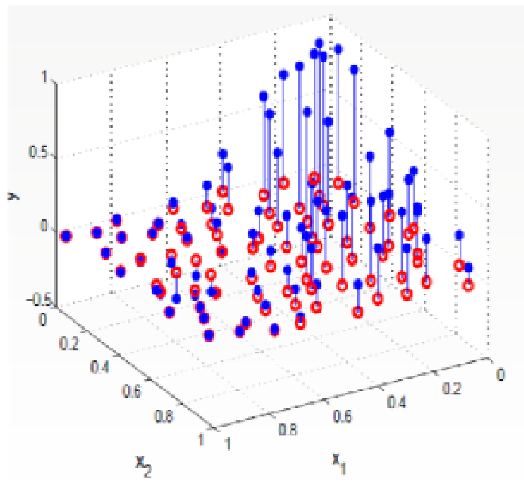


Figure 2.2: Data values

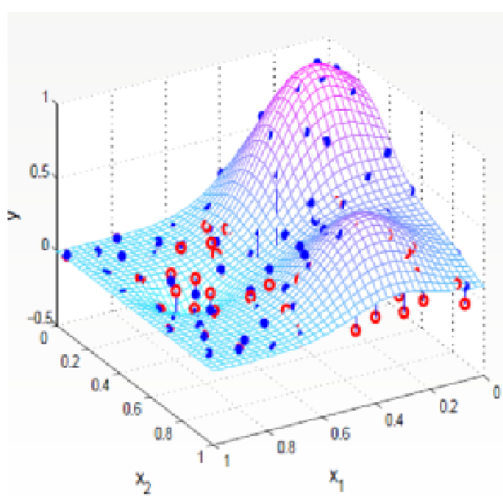


Figure 2.3: Interpolation function

2.4.1 Radial basis function interpolation

Using a Radial basis function (RBF) expansion for scattered data interpolation in $\mathbb{R}^d$ can offer improved results compared to simple distance matrices.

The RBF approach involves expressing the interpolation function as a sum of radial basis functions centered at scattered data points, then S can be as the following form

$$S(x) = \sum_{i=1}^{N} C_i \Phi(\|x - x_i\|_2), \qquad x \in \mathbb{R}^d.$$

Where C_i are parameters which should be chosen so that sapproximates f in point x_i based on the interpolation, we determine the following linear system $AC = F$ such that

$$\begin{bmatrix} \Phi(\|x_1 - x_1\|_2) & \Phi(\|x_1 - x_2\|_2)........\Phi(\|x_1 - x_N\|_2) \\ \Phi(\|x_2 - x_1\|_2) & \Phi(\|x_2 - x_2\|_2)........\Phi(\|x_2 - x_N\|_2) \\ . & . \quad . \quad . \\ . & . \quad . \quad . \\ . & . \quad . \quad . \\ \Phi(\|x_N - x_1\|_2) & \Phi(\|x_N - x_2\|_2)........\Phi(\|x_N - x_N\|_2) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ . \\ . \\ . \\ C_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ . \\ . \\ . \\ f(x_N) \end{bmatrix}$$

$A_{ij} = \Phi(\|x - x_i\|_2)$.

$C = [C_1, C_2, ..., C_N]^T$.

$F = [f(x_1), f(x_2), ...f(x_N)]^T$.

The solution to the problem will exist and be unique, if and only if The matrix A is non singular, this is related to the positively of the matrix A.

2.4.2 The Haar Maurhuber -Curtis Theorem

Definition 2.4.1 *(**Haar space**) Suppose that $\Omega \subseteq \mathbb{R}^d$ contains at least N points, let $V \subseteq C(\Omega)$ be an N dimensional linear space, then V is called a **Haar space** of* *dimension N on Ω if for arbitrary distinct points* $\{x_1, x_2, ..., x_N\} \in \Omega$ *and arbitrary* $\{f_1, f_2, ..., f_N\} \in \mathbb{R}$, *there exists exactly one function* $S \in V$ *with*

$$S(x_i) = f_i, \quad 1 \leq i \leq N.$$

Theorem 2.4.1 *[1]* ***(Haar-Mairhuber-Curtis)***

If $\Omega \subset \mathbb{R}^d$,$d \geqslant 2$ contains an interior point, then there exist no Haar spaces of continuous functions except for the 1-dimensional case.

Remark 2.4.1 *So as a result of this theorem, if we pick basis functions without considering the data, we can not ensure a well-defined problem.*
The Haar-Mairhuber-Curtis theorem states that for a successful multivariate scattered data interpolation, we can not pre-determine the basis functions, they must vary based on the data's location.

2.4.3 Positive definite matrices and functions

Positive definite matrix

Definition 2.4.2 *[1]*

A real symmetric matrix A is called positive semi-definite, if its associated quadratic form $C^T AC \geq 0$ for all $C \in \mathbb{R}^N$,
that is

$$\sum_{i=1}^{N} \sum_{j=1}^{N} C_i C_j A_{ij} \geqslant 0 \ , \tag{2.1}$$

If the quadratic form (2.1) is zero only for $C \equiv 0$, then A is called positive definite.

Positive definite functions

Definition 2.4.3 *[1]*

A continuous function $\Phi : \mathbb{R}^d \rightarrow \mathbb{C}$ is called positive semi- definite if for all $N \in \mathbb{N}$, all sets of pairwise distinct centers $X = \{x_1, x_2, ..., x_N\} \subseteq \mathbb{R}^d$ and all $C \in \mathbb{C}^N$, the quadratic form

$$\sum_{i=1}^{N}\sum_{j=1}^{N} C_i\overline{C_j}\Phi(x_i - x_j) \geqslant 0, \tag{2.2}$$

is non negative.

The function Φ is called positive definite if the quadratic form (2.2) is positive for all $C \in \mathbb{C}^N \setminus \{0\}$.

2.5 Example

We take two simple examples in 1Dimension to explain the phenomenon of interpolation using the RBF function.

Consider three distinct points as $X = \{x_1, x_2, x_3\} = \{1, 3, 3.5\}$ in this space at which the function to be approximated is known and real scalers $(f(x_1), f(x_2), f(x_3)) = (1, 0.2, 0.1)$ respectively.

►We choose Multiquadric (MQ) function $\Phi(r) = \sqrt{\alpha^2 + r^2}$, $\alpha = 1$.

Firstly to calculate our interpolant $S(x)$, we need to calculate C for the matrix as

$$S(x) = \sum_{i=1}^{N} C_i\Phi(\|x - x_i\|_2).$$

We have

$$\begin{bmatrix} 1 & \sqrt{5} & \sqrt{7.25} \\ \sqrt{5} & 1 & \sqrt{1.25} \\ \sqrt{7.25} & \sqrt{1.25} & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2 \\ 0.1 \end{bmatrix}.$$

After calculating C values of this system, we find that

$$\begin{cases} C_1 = -0.118798 \\ C_2 = 0.015169 \\ C_3 = 0.402913 \end{cases}$$

This means that our interpolant S is

$$S(x) = -0.118798\Phi(\mid x-1\mid) + 0.015169\Phi(\mid x-3\mid) + 0.402913\Phi(\mid x-3\mid).$$

►We choose Gaussian (GA) function $\Phi(r) = e^{-(\alpha r)^2}$, $\alpha = 1$ with the same previous data,

we have

$$S(x) = \sum_{i=1}^{N} \lambda_i \Phi(\|x - x_i\|_2).$$

we need to calculate λ for the matrix as

$$\begin{bmatrix} \Phi(0) & \Phi(2) & \Phi(2.5) \\ \Phi(2) & \Phi(0) & \Phi(0.5) \\ \Phi(2.5) & \Phi(0.5) & \Phi(0) \end{bmatrix} \lambda = \begin{bmatrix} 1 & 0.02 & 0.002 \\ 0.02 & 1 & 0.78 \\ 0.002 & 0.78 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.2 \\ 0.1 \end{bmatrix}.$$

Solving this equation directly we find that $\begin{cases} \lambda_1 = 0.995 \\ \lambda_2 = 0.268 \\ \lambda_3 = -0.111 \end{cases}$

This means that our interpolant is

$$S(x) = 0.995\Phi(|x-1|) + 0.268\Phi(|x-3|) - 0.111\Phi(|x-3.5|)$$

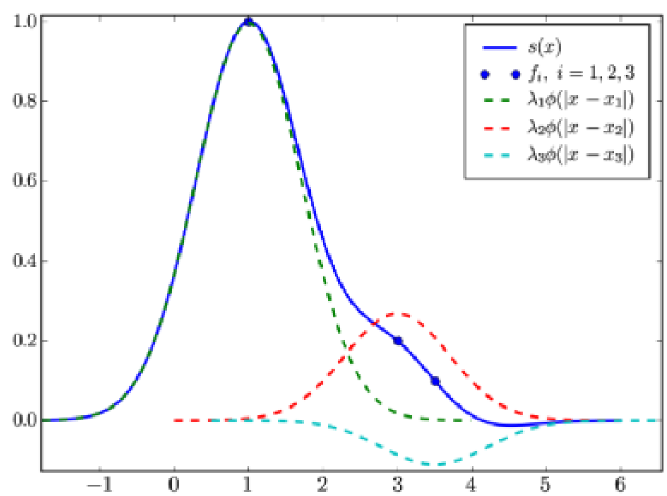


Figure 2.4: Gaussian(GA) Interpolation

Chapter 3

Application of RBF to integral equations

In this chapter, we show how to solve Fredholm's integral equations using Radial basis function method, we also demonstrate the method with an example.

3.1 Radial basis function for Fredholm integral equation

3.1.1 Theoretical model

Let's examine this integral equation of Fredholm type

$$u(x) - \lambda \int_a^b k(x,t)u(t)dt = f(x), \quad x \in [a,b]\,. \tag{3.1}$$

We're trying to approximate the function $u(x)$ using the RBF approximation, which can be represented as a linear combination like this.

$$u(x) \approx \sum_{j=1}^{N} C_j \Phi_j(x). \tag{3.2}$$

We compensate 3.2 in 3.1 we find

$$\sum_{j=1}^{N} C_j \Phi_j(x) - \lambda \int_a^b k(x,t) \sum_{j=1}^{N} C_j \Phi_j(t) dt = f(x),$$

and for any $x_i \in [a,b]$ we choose M collocation points i.e:$i = \overrightarrow{1:M}$, moreover by using the linear property of integral, equation becomes

$$\sum_{j=1}^{N} C_j \Phi_j(x_i) - \lambda \int_a^b k(x_i,t) \sum_{j=1}^{N} C_j \Phi_j(t) dt = f(x_i)$$

$$\sum_{j=1}^{N} C_j \left[\Phi_j(x_i) - \lambda \int_a^b k(x_i,t) \Phi_j(t) dt \right] = f(x_i).$$

So we translate the above equation into a linear system of shape

$$[\Phi - \lambda K][C] = [F],$$

such that $C = [C_1, C_2, ...C_N]^T$, $F = [f(x_1), f(x_2), ...f(x_N)]^T$.

we also know the matrix as follows $[\Phi - \lambda K]$ such as:

$$\Phi_{ij} = \Phi_j(x_i)$$

$$K_{ij} = \int_a^b k(x_i,t) \Phi_j(t) dt.$$

And to simplify the term K we put

$$K_{ij} = \int_a^b k(x_i, t)\Phi_j(t)dt = \int_a^b g(t)dt$$

So that, the integration of the g function can be calculated using **Trapeze's Method** that is given in the form

$$I_N = \int_a^b g(t)dt = \frac{h}{2}\left(g(a) + g(b) + 2\sum_{q=1}^{N-1} g(t_q)\right)$$

we find

$$K_{ij} = \frac{h}{2}\left(k(x_i, a)\Phi_j(a) + k(x_i, b)\Phi_j(b) + 2\sum_{q=1}^{N-1} k(x_i, t_q)\Phi_j(t_q)\right).$$

3.1.2 Numerical model

After we applied the radial basis function to second kind Fredholm's integral equation, we're going to do a numerical experiment using two types of RBF on this equation.

Example

Consider the following linear Fredholm integral equation of the second kind

$$u(x) + A(x)u(h(x)) + \lambda\int_a^b k(x,t)u(t)dt = f(x), \quad a = 0, b = 1, \text{ and } \lambda = 1.$$

Where $A(x) = x$, $k(x,t) = x - t$, $h(x) = x$, and $f(x) = x^3 + x^2 + \frac{1}{3}x - \frac{1}{4}$.

The exact solution is $u(x) = x^2$, the collocation points are the same as the center points $N = 10$.

Thus, the Fredholm integral equation is given the following form

$$u(x) + xu(x) + \int_0^1 (x-t)u(t)dt = x^3 + x^2 + \frac{1}{3}x - \frac{1}{4}$$

First, we take Multiquadratic-RBF's $\Phi(x) = \sqrt{\alpha^2 + \mid x - x_j \mid^2}$**,** $\alpha = 0.9$.

$$u(x) \approx \sum_{j=1}^{N} C_j \sqrt{\alpha^2 + \mid x - x_j \mid^2}$$

We find

$$\sum_{j=1}^{N} C_j \sqrt{\alpha^2 + \mid x - x_j \mid^2} + x \sum_{j=1}^{N} C_j \sqrt{\alpha^2 + \mid x - x_j \mid^2} + \int_0^1 (x-t) \sum_{j=1}^{N} C_j \sqrt{\alpha^2 + \mid t - x_j \mid^2} dt$$
$$= x^3 + x^2 + \frac{1}{3}x - \frac{1}{4}$$

and for any $x_i \in [0,1]$

$$\sum_{j=1}^{N} C_j \left[\sqrt{(0.9)^2 + \mid x_i - x_j \mid^2} + x_i \sqrt{(0.9)^2 + \mid x_i - x_j \mid^2} + \int_0^1 (x_i - t) \sqrt{(0.9)^2 + \mid t - x_j \mid^2} dt \right]$$
$$= x_i^3 + x_i^2 + \frac{1}{3}x_i - \frac{1}{4}.$$

we get a system

$$AC = F.$$

Such that $A = \Phi_{ij} + x\Phi_{ij} + K_{ij}$.

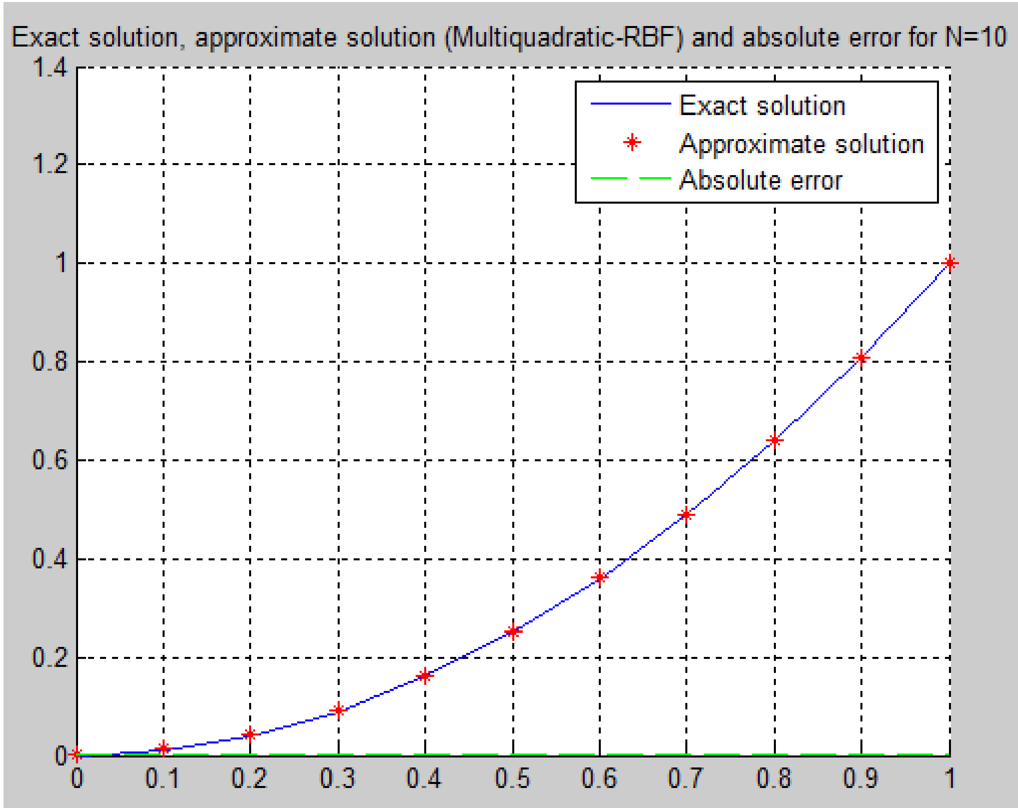
We can approximate the integral in the above equation by using **Trapeze Method**, so we get

$$K_{ij} = \int_0^1 (x_i - t)\sqrt{(0.9)^2 + \mid t - x_j \mid^2} dt$$
$$= \frac{1}{20}\left(x_i\sqrt{\alpha^2 + \mid -x_j \mid^2} + (x_i - 1)\sqrt{\alpha^2 + \mid 1 - x_j \mid^2} + 2\sum_{q=1}^{9}(x_i - t_q)\sqrt{\alpha^2 + \mid t_q - x_j \mid^2} \right)$$

After solving the system we get the numerical results in the table and figure following

$\mathbf{x}_i$	$\mathbf{u}_{exact}$	$\mathbf{u}_{app}$	**error**
0	0	$2.8276e-03$	$2.8273e-03$
0.1	$1.0000e-02$	$1.2318e-02$	$2.3178e-03$
0.2	$4.0000e-02$	$4.1893e-02$	$1.8931e-03$
0.3	$9.0000e-02$	$9.1534e-02$	$1.5338e-03$
0.4	$1.6000e-01$	$1.6123e-01$	$1.2258e-03$
0.5	$2.5000e-01$	$2.5096e-01$	$9.5891e-04$
0.6	$3.6000e-01$	$3.6073e-01$	$7.2535e-04$
0.7	$4.9000e-01$	$4.9052e-01$	$5.1927e-04$
0.8	$6.4000e-01$	$6.4034e-01$	$3.3608e-04$
0.9	$8.1000e-01$	$8.1017e-01$	$1.7234e-04$
1	$1.0000e+00$	$1.0000e+00$	$2.4643e-05$

Table 3.1: Exact solution,approximaet solution and error

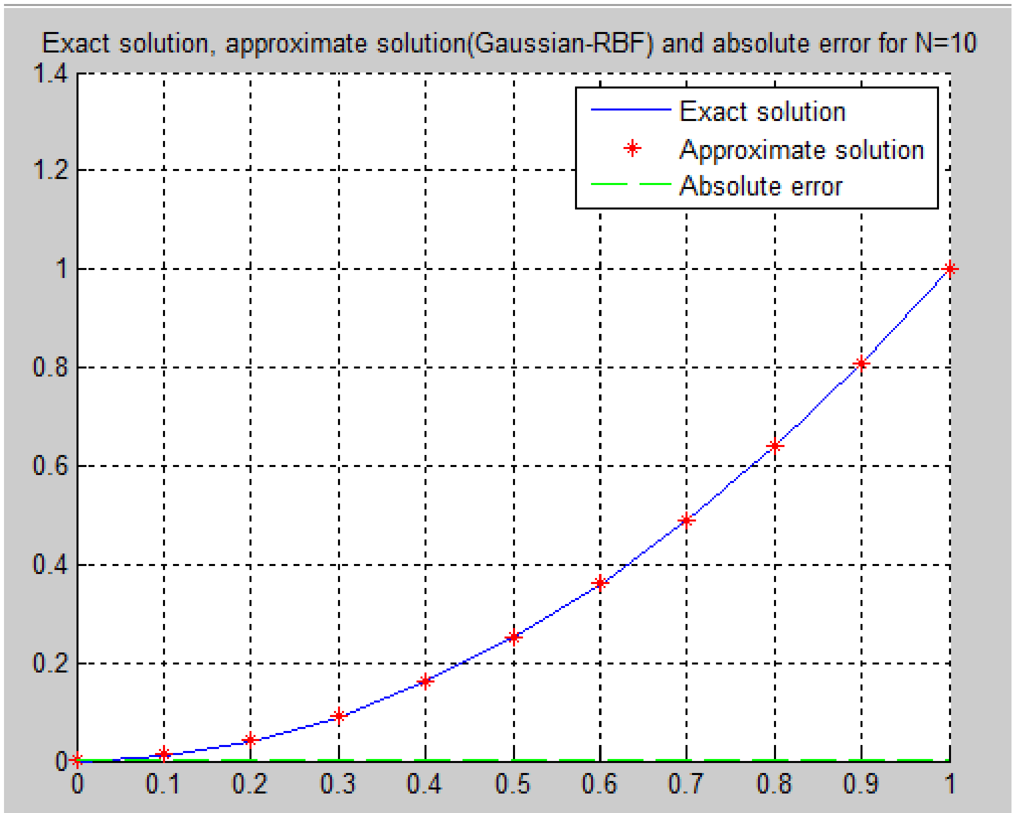


Second, we take Gaussian-RBF's $\Phi(x) = e^{-\alpha^2 . |x-x_j|^2}$, $\alpha = 3$.

we get the numerical results in the table and figure following

$\mathbf{x}_i$	$\mathbf{u}_{exact}$	$\mathbf{u}_{app}$	**error**
0	0	$2.8277e-03$	$2.8277e-03$
$1.0000e-01$	$1.0000e-02$	$1.2318e-02$	$2.3181e-03$
$2.0000e-01$	$4.0000e-02$	$4.1893e-02$	$1.8934e-03$
$3.0000e-01$	$9.0000e-02$	$9.1534e-02$	$1.5341e-03$
$4.0000e-01$	$1.6000e-01$	$1.6123e-01$	$1.2261e-03$
$5.0000e-01$	$2.5000e-01$	$2.5096e-01$	$9.5918e-04$
$6.0000e-01$	$3.6000e-01$	$3.6073e-01$	$7.2562e-04$
$7.0000e-01$	$4.9000e-01$	$4.9052e-01$	$5.1954e-04$
$8.0000e-01$	$6.4000e-01$	$6.4034e-01$	$3.3635e-04$
$9.0000e-01$	$8.1000e-01$	$8.1017e-01$	$1.7245e-04$
$1.0000e+00$	$1.0000e+00$	$1.0000e+00$	$2.4938e-05$

Table 3.2: Exact solution,approximaet solution and error


Exact solution, approximate solution(Gaussian-RBF) and absolute error for N=10
Exact solution
Approximate solution
Absolute error
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Conclusion

In this study, we relied on radial basis function(RBF) interpolation, using two types of radial basis functions to solve the second kind Fredholm integral equation.

According to the numerical results we obtained, we conclude that the radial basis function collocation method is an effective tool for approximating the solution to this type of equation.

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Abstract:

In this work, an approximate solution was found for a type of integral equation, specifically a second kind Fredholm equation.

We used the radial basis function interpolation method to approximate the solution, the results demonstrated the accuracy and effectiveness of this method.

Résumé:

Dans ce travail, une solution approximative d'un certain type **d'équations intégrales a été trouvée, à savoir une équation de Fredholm du deuxième type.**

Nous avons utilisé la méthode d'interpolation par fonction de base radiale pour approcher la solution, et les résultats ont démontré de manière claire la précision et l'efficacité de cette méthode.

ملخص:

في هذا العمل، تم إيجاد الحل التقريبي لنوع من المعادلات التكاملية، وهي معادلة فريدهولم من النوع الثاني.

استخدمنا طريقة استقطاب دالة الأساس الشعاعي لتقريب الحل، وقد أثبتت النتائج دقة وفعالية هذه الطريقة.