General Decay for a Coupled System of Viscoelastic Wave Equation of Infinite Memory with Acoustic Boundary Conditions

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***Abstract*—A coupled system of viscoelastic wave equation of infinite memory is considered. Our system is coupled with the acoustic boundary conditions. Under a very general assumption on the relaxation function, we establish a uniform decay rate. This work substantially improves the earlier results in cases of acoustic boundary conditions.**

***Index Terms*—viscoelastic damping, convex functions, general decay**

I. INTRODUCTION

In this paper, we consider the following viscoelastic coupled system with mixed boundary conditions

$$\begin{cases} u_{tt}-\ell\operatorname{div}(\mathrm{A}\nabla u) & \\ \quad -\int_0^{+\infty} g(s)\operatorname{div}(\mathrm{A}\nabla\eta(s))\mathrm{d}s=0 & \text{in} \quad \Omega\times\mathbb{R}_+ \\ \eta_t+\eta_s-u_t=0 & \text{in} \quad \Omega\times\mathbb{R}_+\times\mathbb{R}_+ \\ hz_{tt}+fz_t+mz+u_t=0 & \text{on} \quad \Gamma_1\times\mathbb{R}_+ \\ u=0 & \text{on} \quad \Gamma_0\times\mathbb{R}_+ \\ \ell\frac{\partial u}{\partial\nu_{\mathrm{A}}}+\int_0^{+\infty} g(s)\frac{\partial\eta}{\partial\nu_{\mathrm{A}}}(s)\mathrm{d}s=z_t & \text{on} \quad \Gamma_1\times\mathbb{R}_+ \\ u(x,-t)=u_0(x,t) & \text{for} \quad (x,t)\in\Omega\times\mathbb{R}_+ \\ u_t(x,0)=u_1(x) & \text{for} \quad x\in\Omega \\ \eta(x,s,0)=\eta_0(x,s) & \text{for} \quad (x,s)\in\Omega\times\mathbb{R}_+ \\ z(x,0)=z_0(x),\ z_t(x,0)=z_1(x) & \text{for} \quad x\in\Gamma_1 \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^n$; $n\geq 1$, with a smooth boundary $\Gamma=\Gamma_0\cup\Gamma_1$, such that Γ_0 and Γ_1 are closed and disjoint and ν represents the unit outward normal to Γ. Note that $\eta(x,s,t)=u(x,t)-u(x,t-s)$ for each $x\in\Omega$ and $t,s\in\mathbb{R}_+$. The integral term $\int_0^{+\infty} g(s)\operatorname{div}(\mathrm{A}\nabla\eta(s))\mathrm{d}s$ is the infinite memory (past history) responsible for the viscoelastic damping. The function g is called the relaxation function and ℓ is a positive constant. The functions $h,f,m:\Gamma_1\to\mathbb{R}^+$ are bounded. The initial conditions $u_0,u_1:\Omega\to\mathbb{R}$, $z_0,z_1:\Gamma_1\to\mathbb{R}$ are given functions. The operator $\mathrm{A}=(a_{ij}(x))_{i,j};\ i,j=1,\ldots,n$; and $\frac{\partial u}{\partial\nu_{\mathrm{A}}}=\sum\limits_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i$.

The above model would be to describe the motion of fluid particles from rest in the domain Ω into part of the surface at a given point $x\in\Gamma_1$, which can be expressed by the pressure at that point. The relationship between the velocity potential $u_t=u_t(x,t)$ at a point on the surface and the displacement $z=z(x,t)$ is proportional to the pressure. The acoustic impedance may be complex in the case of the velocity potential was not in phase with the pressure. The coupling of our model (1) is via the acoustic boundary condition $(1)_3$ and the impenetrability boundary condition $(1)_5$.

The partial differential equation (PDE) system of a viscoelastic equation with acoustic boundary conditions was first considered by Morse and Ingard [14] and developed by Beale [5]. In [5], the semigroup theory was used to solve an initial value problem in a Hilbert space. The loss of decay has obtained where the term z_{tt} was included. Boukhatem and Benabderrahmane [6] eliminated the second derivative z_{tt} and studied existence of solution and exponential decay of system (1). The absence of the term z_{tt} brings some difficulties in the study because of the abnormality of the system. It can not apply directly the semigroups theory or Faedo-Galerkin's procedure. The term εz_{tt} when $\varepsilon\to 0$ has been added in the arguments to overcome the difficulty. See also [8], [10] and [13].

The primary discussion touched upon by several authors is to use the integral term of relaxation function g instead of the frictional damping term u_t. The question that has been focused their attention as an important work is when the viscoelastic damping of memory effect should be strong enough to procreate the decay of the system.

For a very wider class of relaxation functions

$$g'(t)\leq-\xi(t)\mathcal{H}(g(t)), \tag{2}$$

for all $t\geq 0$, where $\mathcal{H}$ is an increasing and a strictly convex $\mathcal{C}^2$ function and ξ is a positive nonincreasing differentiable function, many papers were interested by a viscoelastic wave equation in the case of finite memory. We refer to previous studies established a stability result [7], [13] and [15]. Recently, problems related to viscoelasticity with infinite memory have attracted many researchers for accuracy the general decay rates, see for example [9], [12]. Guesmia [11] showed the stability results of the following two viscoelastic wave equations

$$u_{tt} - \Delta u + \int_0^{\infty} g(s)\Delta u(t-s)\mathrm{d}s = 0, \quad \text{in } \Omega \times \mathbb{R}_+,$$

and

$$u_{tt} - \Delta u - \int_0^{\infty} g(s) u(t-s)\mathrm{d}s = 0, \quad \text{in } \Omega \times \mathbb{R}_+.$$

Al-Mahdi [1] obtained the same result of the following plate problem

$$u_{tt} - \sigma \Delta u_{tt} + \Delta^2 u - \int_0^{\infty} g(s)\Delta^2 u(t-s)\mathrm{d}s = 0, \quad \text{in } \Omega \times \mathbb{R}_+,$$

where $\sigma \geq 0$. For a coupled system of viscoelastic wave equations of viscoelastic wave equations acting two kernals, Al-Mahdi et al. [2] treated the following

$$\begin{cases} u_{tt} - \Delta u - \int_0^{+\infty} g_1(s)\Delta u(t-s)\mathrm{d}s + f_1(u,v) = 0 \\ v_{tt} - \Delta v - \int_0^{+\infty} g_2(s)\Delta v(t-s)\mathrm{d}s + f_2(u,v) = 0 \end{cases}$$

in $\Omega \times \mathbb{R}_+$, where f_1 and f_2 are nonlinear functions describing the interacation between the two waves. They established the decay rate of solutions by a relation to the growth of $g_i; (i = 1,2)$ at infinity satisfying (2) with specific properties of ξ and $\mathcal{H}_i$. In one dimensional Timoshenko system, Al-Mahdi et al. [3] considered

$$\begin{cases} \rho_1 \varphi_{tt} + K(\varphi_x + \psi)_x = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \int_0^{+\infty} g(s)\psi_{xx}(t-s)\mathrm{d}s = 0 \end{cases}$$

where $(x,t) \in (0,L) \times \mathbb{R}_+$, L, b, K, ρ_1 and ρ_2 are positive physical constants. Under suitable conditions on ξ and $\mathcal{H}$ satisfying (2), they established some new decay results in the equal speeds of propagation case, as well as the nonequal-speed case. Their results generalize and improve many earlier results in the literature such as [15] and [11].

This work is organized as follows, in Sect. II, we give some notations and we state some assumptions needed for our work. The decay rate, in Sect. III, is improved explicitly by some properties on the convex function $\mathcal{H}$ and on the relaxation function. The proof is based on the construction of a suitable Lyapunov functional.

II. PRELIMINARY

Let $\mathrm{H}(\mathrm{div}, \Omega) = \{u \in \mathrm{H}^1(\Omega); \mathrm{div}(\mathrm{A}\nabla u) \in \mathrm{L}^2(\Omega)\}$ be the Hilbert space endowed with the norm

$$\|u\|_{\mathrm{H}(\mathrm{div},\Omega)} = \left(\|u\|^2_{\mathrm{H}^1(\Omega)} + \|\mathrm{div}(\mathrm{A}\nabla u)\|_2^2 \right)^{1/2},$$

where $\mathrm{H}^1(\Omega)$ is the Sobolev space of first order, $\|.\|_2$ is a L^2-norm and $\|.\|_{2,\Gamma_1}$ is a L^2-norm on Γ_1, and $(.,.)$, $\langle .,. \rangle_{\Gamma_1}$ are the scalar product in $\mathrm{L}^2(\Omega)$, $\mathrm{L}^2(\Gamma_1)$, respectively.

Denoting $\gamma_0 : \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Gamma)$ and $\gamma_1 : \mathrm{H}(\mathrm{div},\Omega) \to \mathrm{L}^2(\Gamma)$ defined by $\gamma_0(u) = u_{|\Gamma}$ and $\gamma_1(u) = \left(\frac{\partial u}{\partial \nu_{\mathrm{A}}}\right)_{\Gamma}$ for all u in $\mathrm{H}(\mathrm{div}, \Omega)$. Some times to simplify the notations we write u and $\frac{\partial u}{\partial \nu_{\mathrm{A}}}$ instead $\gamma_0(u)$ and $\gamma_1(u)$, respectively.

We denote by

$$\mathrm{H}^1_{\Gamma_0}(\Omega) = \{u \in \mathrm{H}^1(\Omega) \mid \ \gamma_0(u) = 0 \text{ on } \Gamma_0\}$$

the closure subspace of $\mathrm{H}^1(\Omega)$ equipped with the norm equivalent to the usual norm in $\mathrm{H}^1_0(\Omega)$. The Poincaré inequality holds in $\mathrm{H}^1_{\Gamma_0}(\Omega)$; then, there exists a positive constant C_* depending on Ω only such that

$$\|u(t)\|_2 \leq C_* \|\nabla u(t)\|_2, \qquad \forall u \in \mathrm{H}^1_{\Gamma_0}(\Omega), \tag{3}$$

and a positive constant $\overline{C}_*$ satisfying

$$\|u(t)\|_{2,\Gamma_1} \leq \overline{C}_* \|\nabla u(t)\|_2, \qquad \forall u \in \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{4}$$

To present our result, we need some assumptions

(A.1) The coefficients a_{ij} in $\mathcal{C}^1(\overline{\Omega})$ are symmetric and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \geq a_0|\zeta|^2, \qquad \forall x \in \overline{\Omega}, \ \ \forall \zeta \in \mathbb{R}^n. \tag{5}$$

(A.2) The relaxation function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded $\mathcal{C}^1$ function such that, for some $\beta_0 > 0$

$$g'(t) \geq -\beta_0 g(t), \ \ g(0) > 0, \ \ 1 - \int_0^{\infty} g(s)\mathrm{d}s = \ell > 0, \tag{6}$$

which exists a strictly increasing and strictly convex $\mathcal{C}^2$ function $\mathcal{H} : \mathbb{R}_+ \to \mathbb{R}_+$ on $(0,r]$, for a positive constant $r \leq g(0)$, with $\mathcal{H}(0) = \mathcal{H}'(0) = 0$ such that

$$g'(t) \leq -\xi(t)\mathcal{H}(g(t)), \quad \forall t \geq 0, \tag{7}$$

where $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing $\mathcal{C}^1$ function.

(A.3) There exist three positive constants f_0, m_0 and h_0 such that

$$f(x) \geq f_0, \ \ m(x) \geq m_0, \ \text{ and } h(x) \geq h_0,$$

for almost everywhere x in Γ_1.

Furthermore, we define

$$a(u(t), v(t)) = (\mathrm{A}\nabla u(t), \nabla v(t)) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \frac{\partial u(t)}{\partial x_i} \frac{\partial v(t)}{\partial x_i} \mathrm{d}x.$$

Using the hypothesis **(A.1)**, we can verify that the bilinear form $a(.,.) : \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathrm{H}^1_{\Gamma_0}(\Omega) \to \mathbb{R}$ is continuous, and from (5), we deduce that a is coercive.

Let $\mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))$ be the Hilbert space of $\mathrm{H}^1_{\Gamma_0}(\Omega)-$valued functions on $\mathbb{R}_+$, equipped with the inner product

$$\langle \eta(t), \tilde{\eta}(t) \rangle_{\mathrm{L}^2_g(\mathbb{R}_+; \mathrm{H}^1_{\Gamma_0}(\Omega))} = \int_0^{+\infty} g(s) a(\eta(s,t), \tilde{\eta}(s,t))\mathrm{d}s. \tag{8}$$

We mention some additional remarks that will be used in arguments and proofs

Remark 1: Let $\mathcal{H}$ be a strictly convex on $(0,r]$ with $\mathcal{H}(0) = 0$, then

$$\mathcal{H}(\gamma x) \leq \gamma \mathcal{H}(x), \quad \forall \gamma \in [0,1], \quad \forall x \in (0,r]. \tag{9}$$

Remark 2 ([15]): Assume ϕ is a strictly positive measurable function satisfying $\int_\Omega \phi(x)\mathrm{d}x = 1$. For any f a real-valued measurable function and Ψ a convex over the range of f, then Jensen's inequality states that

$$\Psi\left(\int_\Omega f(x)\phi(x)\mathrm{d}x\right) \leq \int_\Omega \Psi\left(f(x)\right)\phi(x)\mathrm{d}x. \tag{10}$$

Remark 3 ([15]): If (**A.2**) holds. Then $\overline{\mathcal{H}}$ is an extension of $\mathcal{H}$, which is a strictly increasing convex $\mathcal{C}^2$ function on $\mathbb{R}_+$. For example, setting $\mathcal{H}(r) = \alpha$, $\mathcal{H}'(r) = \beta$ and $\mathcal{H}''(r) = \gamma$, thus $\overline{\mathcal{H}}$ defined by, for all $t > r$,

$$\overline{\mathcal{H}}(t) = \frac{\gamma}{2}t^2 + (\beta - \gamma r)t + \left(\alpha + \frac{\gamma}{2}r^2 - \beta r\right). \tag{11}$$

We state, in the following proposition without proof, the existence of solution for the system (1) which can be established by means of the semigroup approach and combining the results from [12] with the ones from [16].

Proposition 2.1: Let $u_0(x,.) \in \mathrm{H}^1_{\Gamma_0}(\Omega)$,$u_1 \in \mathrm{L}^2(\Omega)$ and $z_0, z_1 \in \mathrm{L}^2(\Gamma_1)$ be given. Suppose that (**A.1**)-(**A.3**) hold. Then, the system (1) has a unique regular global (weak) solution

$$u \in \mathcal{C}(\mathbb{R}_+;\mathrm{H}^1_{\Gamma_0}(\Omega)) \cap \mathcal{C}^1(\mathbb{R}_+;\mathrm{L}^2(\Omega)); \quad u \in \mathrm{H}(\mathrm{div},\Omega);$$
$$z \in \mathcal{C}^1(\mathbb{R}_+;\mathrm{L}^2(\Gamma_1)).$$

III. MAIN RESULT

A. Modified Lyapunov functional

We are going to construct the modified Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) = NE(t) + N_1\varphi(t) + N_2\psi(t), \tag{12}$$

where N and $N_i,\ i = 1, 2$; are positive constants to be fixed later, and

$$(g \diamond u)(t) = \int_0^{+\infty} g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s,$$

$$\begin{aligned} E(t) &= \frac{1}{2}\Big(\|u_t(t)\|_2^2 + \ell a(u(t),u(t)) + (g \diamond u)(t) \\ &\qquad + \|m^{1/2}z(t)\|^2_{2,\Gamma_1} + \|h^{1/2}z_t(t)\|^2_{2,\Gamma_1}\Big), \end{aligned} \tag{13}$$

$$\begin{aligned} \varphi(t) &= \int_\Omega u_t(t)u(t)\mathrm{d}x + \int_{\Gamma_1} h(x)z_t(t)z(t)\mathrm{d}\Gamma \\ &\qquad + \int_{\Gamma_1} z(t)u(t)\mathrm{d}\Gamma + \frac{1}{2}\|f^{1/2}z(t)\|^2_{2,\Gamma_1}, \end{aligned} \tag{14}$$

$$\psi(t) = -\int_\Omega u_t(t)\int_0^{+\infty} g(s)(\eta(t,s))\mathrm{d}s\mathrm{d}x. \tag{15}$$

Lemma 3.1: For N large enough, the Lyapunov functional (12) is equivalent to the energy functional (13), and the inequality

$$\kappa_1 E(t) \leq \mathcal{L}(t) \leq \kappa_2 E(t), \tag{16}$$

holds for two positive constants κ_1 and κ_2.

Proof: We can estimate each terms of (12). Indeed, we have

$$|\mathcal{L}(t) - NE(t)| \leq N_1|\varphi(t)| + N_2|\psi(t)|. \tag{17}$$

Using the inequalities of Cauchy-Schwarz and Young, using (3) and (4), we obtain

$$\int_{\Gamma_1} z(t)u(t)\mathrm{d}\Gamma \leq \frac{\|m\|_\infty}{2m_0^2}\|m^{1/2}z(t)\|^2_{2,\Gamma_1} + \frac{\overline{C}_*^2}{2a_0}a(u(t),u(t)),$$

$$\begin{aligned} &\int_{\Gamma_1} h(x)z_t(t)z(t)\mathrm{d}\Gamma \\ &\leq \frac{\|h\|_\infty}{2}\|h^{1/2}z_t(t)\|^2_{2,\Gamma_1} + \frac{\|m\|_\infty}{2m_0^2}\|m^{1/2}z(t)\|^2_{2,\Gamma_1}, \end{aligned}$$

and

$$\frac{1}{2}\|f^{1/2}z(t)\|^2_{2,\Gamma_1} \leq \frac{\|f\|_\infty}{2m_0}\|m^{1/2}z(t)\|^2_{2,\Gamma_1}.$$

Then,

$$\begin{aligned} |\varphi(t)| &\leq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{C_*^2 + \overline{C}_*^2}{2a_0}a(u(t),u(t)) \\ &\quad + \frac{1}{2m_0}\left(\frac{2\|m\|_\infty}{m_0} + \|f\|_\infty\right)\|m^{1/2}z(t)\|^2_{2,\Gamma_1} \\ &\quad + \frac{\|h\|_\infty}{2}\|h^{1/2}z_t(t)\|^2_{2,\Gamma_1}. \end{aligned}$$

Also,

$$|\psi(t)| \leq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{C_*^2(1-\ell)}{2a_0}(g \diamond u)(t).$$

From (17), we have

$$|\mathcal{L}(t) - NE(t)| \leq CE(t),$$

where $C \geq 0$. By choosing N large enough, we conclude (16). ■

Lemma 3.2: The energy (13) is a nonincreasing function satisfying, for all $t \geq 0$

$$E'(t) = \frac{1}{2}(g' \diamond u)(t) - \|f^{1/2}z_t(t)\|^2_{2,\Gamma_1}. \tag{18}$$

Proof: Taking the scalar product of $(1)_1$ with u_t and $(1)_3$ with z_t in $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^2(\Gamma_1)$, respectively, then adding it to the inner product (8) of $(1)_2$ with η. Using Green's formula, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u_t(t)\|_2^2 + \ell a(u(t),u(t)) + (g \diamond u)(t) \\ &\qquad + \|m^{1/2}z(t)\|^2_{2,\Gamma_1} + \|h^{1/2}z_t(t)\|^2_{2,\Gamma_1}\Big) \\ &= -\int_0^{+\infty} g(s)a(\eta_s(t,s),\eta(t,s)\mathrm{d}s - \|f^{1/2}z_t(t)\|^2_{2,\Gamma_1}. \end{aligned} \tag{19}$$

Using (6) and the properties of η, we have

$$\int_0^{+\infty} g(s)a(\eta(s),\eta_s(s))\mathrm{d}s = -\frac{1}{2}\int_0^{+\infty} g'(s)a(\eta(s),\eta(s))\mathrm{d}s. \tag{20}$$

Combining (20) with (19), we get (18). ■

Lemma 3.3: The functional φ defined by (14) satisfies, for any $0 < \alpha < 1$, for all $t \geq 0$, the estimate

$$\begin{aligned} \varphi'(t) &\leq \|u_t(t)\|_2^2 - \ell a(u(t),u(t)) + C_1C_\alpha(k \diamond u)(t) \\ &\quad - \|m^{1/2}z(t)\|^2_{2,\Gamma_1} + C_2\|h^{1/2}z_t(t)\|^2_{2,\Gamma_1}, \end{aligned} \tag{21}$$

where

$$C_\alpha = \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)} \mathrm{d}s, \quad k(t) = \alpha g(t) - g'(t),$$

$$C_1 = \frac{n a_1}{4 a_0 C_*^2} \quad \text{and} \quad C_2 = 1 + \frac{\|h\|_\infty}{h_0^2}.$$

Proof: We differentiate (14) with respect to t, we arrive by using Green's formula and the boundary conditions $(1)_{4,5}$ at

$$\begin{aligned}\varphi'(t) = -a(u(t),u(t)) + \int_0^{+\infty} g(s) a(u(t-s),u(t)) \mathrm{d}s \\ + \|u_t(t)\|_2^2 + 2\int_{\Gamma_1} z_t(t) u(t) \mathrm{d}\Gamma - \|m^{1/2} z(t)\|_{2,\Gamma_1}^2 \\ + \|h^{1/2} z_t(t)\|_{2,\Gamma_1}^2. \end{aligned} \tag{22}$$

Now, thanks to Cauchy-Schwarz's inequality

$$\begin{aligned} &\int_\Omega \left(\int_0^{+\infty} g(s) \nabla \eta(t,s) \mathrm{d}s \right)^2 \mathrm{d}x \\ &= \int_\Omega \left(\int_0^{+\infty} \frac{g(s)}{\sqrt{k(s)}} \sqrt{k(s)} \nabla \eta(t,s) \mathrm{d}s \right)^2 \mathrm{d}x \\ &\le \frac{C_\alpha}{a_0} (k \diamond u)(t). \end{aligned}$$

Using Young's inequality, (5) and (6), we obtain, for some $\mu > 0$

$$\begin{aligned} &\int_0^{+\infty} g(s) a(u(t-s),u(t)) \mathrm{d}s \\ &\le \left((1-\ell) + \mu \frac{a_1}{a_0} \right) a(u(t),u(t)) + \frac{n C_\alpha}{4 a_0 \mu} (k \diamond u)(t), \end{aligned} \tag{23}$$

and,

$$2\int_{\Gamma_1} z_t(t) u(t) \mathrm{d}\Gamma \le \frac{\|h\|_\infty}{h_0^2} \|h^{1/2} z_t(t)\|_{2,\Gamma_1}^2 + \frac{\overline{C}_*^2}{a_0} a(u(t),u(t)). \tag{24}$$

Substituting (23)-(24) into (22). By choosing $\mu = \dfrac{C_*^2}{a_1}$, we find (21). ■

Lemma 3.4: The functional ψ defined by (15) satisfies, for $\mu > 0$, for all $t \ge 0$, the estimate

$$\begin{aligned} \psi'(t) \le &-(1-\ell-\mu) \|u_t(t)\|_2^2 + \mu \frac{a_1}{a_0} a(u(t),u(t)) \\ &+ C_3 \|h^{1/2} z_t(t)\|_{2,\Gamma_1}^2 + C_4 C_\alpha (k \diamond u)(t), \end{aligned} \tag{25}$$

where $C_3 = \dfrac{\overline{C}_*^2 \|h\|_\infty}{h_0^2}$, $k_1 = \int_0^{+\infty} k(s) \mathrm{d}s$ and

$$C_4 = \frac{1}{4a_0} \max\left\{ \frac{2\overline{C}_*^2 h_1}{\mu}, 1 + 4\sqrt{a_1 n} + \frac{n + 2\alpha^2 C_*^2}{\mu} \right\}.$$

Proof: By exploiting $(1)_1$, we have

$$\begin{aligned} &\psi'(t) \\ &= -(1-\ell) \|u_t(t)\|_2^2 - \int_{\Gamma_1} z_t(t) \int_0^{+\infty} g(s) \eta(t,s) \mathrm{d}s \mathrm{d}\Gamma \\ &+ \ell \int_\Omega \mathrm{A} \nabla u(t) \int_0^{+\infty} g(s) \nabla \eta(t,s) \mathrm{d}s \mathrm{d}x \\ &+ \int_\Omega \left(\int_0^{+\infty} g(s) \mathrm{A} \nabla \eta(s) \mathrm{d}s \right) \left(\int_0^{+\infty} g(s) \nabla \eta(s) \mathrm{d}s \right) \mathrm{d}x \\ &- \int_\Omega u_t(t) \int_0^{+\infty} g'(s) \eta(t,s) \mathrm{d}s \mathrm{d}x. \end{aligned} \tag{26}$$

Similar calculations in (23) and (24), we obtain, for $\mu > 0$

$$\begin{aligned} &\int_\Omega \mathrm{A} \nabla u(t) \int_0^{+\infty} g(s) \nabla \eta(t,s) \mathrm{d}s \mathrm{d}x \\ &\le \mu \frac{a_1}{a_0} a(u(t),u(t)) + \frac{n C_\alpha}{4 a_0 \mu} (k \diamond u)(t), \end{aligned} \tag{27}$$

$$\begin{aligned} &- \int_{\Gamma_1} z_t(t) \int_0^{+\infty} g(s) \eta(t,s) \mathrm{d}s \mathrm{d}\Gamma \\ &\le \frac{C_\alpha}{4a_0} (k \diamond u)(t) + \frac{\overline{C}_*^2 \|h\|_\infty}{h_0^2} \|h^{1/2} z_t(t)\|_{2,\Gamma_1}^2, \end{aligned} \tag{28}$$

and, for $\mu_1 > 0$

$$\begin{aligned} &- \int_\Omega u_t(t) \int_0^{+\infty} g'(s) \eta(t,s) \mathrm{d}s \mathrm{d}x \\ &\le \mu_1 \|u_t(t)\|_2^2 + \frac{(\alpha^2 C_\alpha + k_1) C_*^2}{2 \mu_1 a_0} (k \diamond u)(t). \end{aligned} \tag{29}$$

We arrive to estimate

$$\begin{aligned} &\int\limits_\Omega \left(\int\limits_0^{+\infty} g(s) \mathrm{A} \nabla \eta(s) \mathrm{d}s \right) \left(\int\limits_0^{+\infty} g(s) \nabla \eta(s) \mathrm{d}s \right) \mathrm{d}x \\ &\le \frac{\sqrt{a_1 n} C_\alpha}{a_0} (k \diamond u)(t). \end{aligned} \tag{30}$$

Inserting (27)-(30) into (26), we obtain (25). ■

Lemma 3.5: For each $i = 1, 2;$ N_i large enough while N so large that the functional $\mathcal{L}$ defined by (12) satisfies, for all $t \ge 0$

$$\begin{aligned} &\mathcal{L}'(t) \\ &\le -\|u_t(t)\|_2^2 - (\rho+1)(1-\ell) a(u(t),u(t)) \\ &\quad + \frac{1}{4} (g \diamond u)(t) - \|m^{1/2} z(t)\|_{2,\Gamma_1}^2 - \|h^{1/2} z_t(t)\|_{2,\Gamma_1}^2, \end{aligned} \tag{31}$$

where $\rho = \dfrac{2 a_1 n}{a_0^2} + 1$ is a positive constant.

Proof: By Combining (12), (18), (21) and (25), taking $\mu = \ell a_0/(a_1 N_2)$, we obtain

$$
\begin{aligned}
&\mathcal{L}'(t) \\
&\leq -\left((1-\ell)N_2 - N_1 - \frac{\ell a_0}{a_1}\right)\|u_t(t)\|_2^2 - \|m^{1/2}z(t)\|_{2,\Gamma_1}^2 \\
&\quad - (\ell N_1 - \ell)\, a(u(t),u(t)) + \frac{\alpha N}{2}(g \diamond u)(t) \\
&\quad - \left(\frac{N}{2} - C_\alpha\left(C_1 N_1 + C_4 N_2\right)\right)(k \diamond u)(t) \\
&\quad - \left(\frac{\|f\|_\infty}{h_0^2}N - C_2 N_1 - C_3 N_2\right)\|h^{1/2}z_t(t)\|_{2,\Gamma_1}^2,
\end{aligned}
$$

We choose N_1 large enough, so

$$\ell N_1 - \ell > (\rho+1)(1-\ell),$$

and therefore N_2 large enough such that

$$(1-\ell)N_2 - N_1 - \frac{\ell a_0}{a_1} > 1.$$

Let us choose N so large, that's to say

$$\frac{\|f\|_\infty}{h_0^2}N - C_2 N_1 - C_3 N_2 > 1.$$

As $\dfrac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$, using the Lebesgue dominated convergence theorem to show

$$\alpha C_\alpha = \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)}\mathrm{d}s \to 0 \quad \text{as} \quad \alpha \to 0.$$

Then, there is $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$, hence

$$\alpha C_\alpha < \frac{1}{C_1 N_1 + C_4 N_2} \quad \text{and} \quad \alpha = \frac{1}{2N} < \alpha_0,$$

which means

$$\frac{N}{2} - C_\alpha(C_1 N_1 + C_4 N_2) > 0.$$

So, we conclude (31). ■

Lemma 3.6: There exists a positive constant M_0 such that

$$\int_t^{+\infty} g(s) a(\eta(t,s), \eta(t,s))\mathrm{d}s \leq M_0 h_0(t), \tag{32}$$

where $h_0(t) = \int_t^{+\infty} g(t+s)\left(1 + ||\nabla u_0(s)||_2^2\right)\mathrm{d}s$.

Proof: As in [1] and [11]. Using (5), (13) and the definition of η, we obtain, for all $t \geq 0$

$$
\begin{aligned}
&\int_t^{+\infty} g(s) a(\eta(t,s), \eta(t,s))\mathrm{d}s \\
&\leq 2C||\nabla u(t)||_2^2 \int_t^{+\infty} g(s)\mathrm{d}s + 2C\int_t^{+\infty} g(s)||\nabla u(t-s)||_2^2 \mathrm{d}s \\
&\leq \frac{4C}{a_0\ell}E(0)\int_0^{+\infty} g(t+s)\mathrm{d}s + 2C\int_0^{+\infty} g(t+s)||\nabla u_0(s)||_2^2\mathrm{d}s,
\end{aligned}
$$

which gives (32) where $M_0 = C\max\left\{\frac{4E(0)}{a_0\ell}, 2\right\}$. ■

Remark 4 ([11]): The function h_0 is of class $\mathcal{C}^1$ on $\mathbb{R}_+$. Indeed, since $\eta_0 \in \mathrm{L}_g^2(\mathbb{R}_+; \mathrm{H}_{\Gamma_0}^1(\Omega))$, then, for all $t \geq 0$

$$
\begin{aligned}
h_0(t) &\leq (1+\ell) + \int_0^{+\infty} g(t+s)\|\nabla u_0(0) - \nabla \eta_0(s)\|_2^2\mathrm{d}s \\
&\leq (1-\ell)\left(1 + 2\|\nabla u_0(0)\|_2^2\right) \\
&\quad + \frac{2}{a_0^2}\int_0^{+\infty} g(s) a(\eta_0(s), \eta_0(s))\mathrm{d}s = M_1,
\end{aligned}
$$

and, using the left assumption in (6)

$$
\begin{aligned}
|h_0'(t)| \leq \beta_0 \int_0^{+\infty} g(t+s)\left(1 + \|\nabla u_0(s)\|_2^2\right)\mathrm{d}s &= \beta_0 h_0(t) \\
&\leq \beta_0 M_1.
\end{aligned}
$$

Next, we use the functional

$$\Phi(t) = \int_0^t p(t-s) a(u(s), u(s))\mathrm{d}s,$$

where $p(t) = \int_t^{+\infty} g(s)\mathrm{d}s$.

Lemma 3.7: The functional Φ satisfies the estimate

$$\Phi'(t) \leq -\frac{1}{2}(g \diamond u)(t) + \rho(1-\ell)a(u(t), u(t)) + \frac{M_0}{2}h_0(t). \tag{33}$$

Proof: The proof of (33) is based on similar proceedings used in the one of Lemma 3.4 in [3] with suitable changes. In fact, we have

$$p'(t) = -g(t),$$

and

$$
\begin{aligned}
\int_0^t g(t-s)\mathrm{d}s &= \int_0^t g(s)\mathrm{d}s \\
&= \int_0^{+\infty} g(s)\mathrm{d}s - \int_t^{+\infty} g(s)\mathrm{d}s = p(0) - p(t).
\end{aligned}
$$

Now, differentiating Φ, we get

$$
\begin{aligned}
\Phi'(t) &= p(0)a(u(t), u(t)) - \int_0^t g(t-s) a(u(s), u(s))\mathrm{d}s \\
&= -\int_0^t g(s) a(\eta(t,s) - u(t), \eta(t,s) - u(t))\mathrm{d}s \\
&\quad + p(0) a(u(t), u(t)) \\
&= -\int_0^t g(s) a(\eta(t,s), \eta(t,s))\mathrm{d}s + 2\int_0^t g(s) a(\eta(t,s), u(t))\mathrm{d}s \\
&\quad + \left(p(0) - \int_0^t g(s)\mathrm{d}s\right) a(u(t), u(t)) \\
&= p(t) a(u(t), u(t)) + \int_t^{+\infty} g(s) a(\eta(t,s), \eta(t,s))\mathrm{d}s \\
&\quad - (g \diamond u)(t) + 2\int_0^t g(s) a(\eta(t,s), u(t))\mathrm{d}s.
\end{aligned}
$$

Using Young's inequality for some $\delta > 0$, the last term can be estimated as follows

$$\begin{aligned}
&\int_0^t g(t-s)a(\eta(t,s),u(t))\mathrm{d}s\\
&= \int_0^t g(s)\int_\Omega \mathrm{A}\left(\nabla u(t) - \nabla u(t-s)\right).\nabla u(t)\mathrm{d}x\mathrm{d}s\\
&\le n\delta \int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2 + \frac{a_1}{4\delta}\int_0^t g(s)\|\nabla\eta(t,s)\|_2^2\mathrm{d}s\\
&\le \delta\frac{n\int_0^t g(s)\mathrm{d}s}{a_0^2}a(u(t),u(t)) + \frac{a_1}{4a_0^2\delta}(g\diamond u)(t)\\
&\quad -\frac{a_1}{4a_0^2\delta}\int_t^{+\infty} g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s.
\end{aligned}$$

By choosing $\delta = \dfrac{a_1(1-\ell)}{a_0^2\int_0^t g(s)\mathrm{d}s}$. Then, as $p(t)\le p(0) = (1-\ell)$ and $\int_0^t g(s)\mathrm{d}s \le (1-\ell)$, from (32), we conclude (33). ■

B. Stability estimate

At this Position, we introduce the class D of functions $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying, for some constants $\alpha_1, \alpha_2 > 0$

$$\vartheta \in \mathcal{C}^1(\mathbb{R}_+), \quad \vartheta \le 1, \quad \vartheta' \le 0, \tag{34}$$

and

$$\alpha_2\mathcal{H}\left(M_0 h_1(t)h_0(t)\right) \le \alpha_1\left(\mathcal{H}_2\left(\frac{\mathcal{H}_3(t)}{\vartheta(t)}\right) - \frac{\mathcal{H}_2\left(\mathcal{H}_3(t)\right)}{\vartheta(t)}\right), \tag{35}$$

where

$$\mathcal{H}_1(t) = \int_t^r \frac{\mathrm{d}s}{\mathcal{H}_2(s)}, \quad \mathcal{H}_2(t) = t\mathcal{H}'(\varepsilon_0 t),$$

$$\mathcal{H}_3(t) = \mathcal{H}^{-1}\left(\alpha_1\int_0^t \xi(s)\mathrm{d}s\right),$$

and ε_0, α_1 and α_2 are positive constants to be chosen properly later, and the functions h_0 and h_1 are defined in (32) and (38)(below), respectively.

Remark 5: Accordingly, $\mathcal{H}_1$ is strictly decreasing and convex on $(0,r]$ with $\lim\limits_{t\to 0}\mathcal{H}_1(t) = +\infty$, $\mathcal{H}_2$ is convex increasing on $\mathbb{R}_+$ and $\mathcal{H}_3$ is of class $\mathcal{C}_1$ on $\mathbb{R}_+$.

The following theorem gives us the stability result.

Theorem 3.8: Suppose that (**A.1**)-(**A.3**) hold. Then, there exists a constant $C > 0$ such that, for all $t \ge 0$

$$E(t) \le C\frac{\mathcal{H}_3(t)}{\vartheta(t)h_1(t)}. \tag{36}$$

Proof: We start by using (31) and (33) to obtain

$$\mathcal{L}_1'(t) \le -\beta_1 E(t) + \frac{M_0}{2}h_0(t),$$

where β_1 is positive constant and $\mathcal{L}_1(t) = \mathcal{L}(t) + \Phi(t)$. Therefore, for all $t \ge 0$

$$\beta_1\int_0^t E(s)\mathrm{d}s \le \mathcal{L}_1(0) + \frac{M_0}{2}\int_0^t h_0(s)\mathrm{d}s,$$

which means

$$\int_0^t E(s)\mathrm{d}s \le \beta_2\left(1 + \int_0^t h_0(s)\mathrm{d}s\right), \tag{37}$$

where $\beta_2 = \max\left\{\frac{\mathcal{L}_1(0)}{\beta_1}, \frac{M_0}{2\beta_1}\right\}$. From (37), we have, for a constant $C > 0$

$$\begin{aligned}
&\int_0^t a(\eta(t,s),\eta(t,s))\mathrm{d}s\\
&\le 2C\int_0^t \left(||\nabla u(t)||_2^2 + ||\nabla u(t-s)||_2^2\right)\mathrm{d}s\\
&\le \frac{8C}{a_0\ell}\int_0^t E(t)\mathrm{d}s \le \frac{8\beta_2 C}{a_0\ell}\left(1 + \int_0^t h_0(s)\mathrm{d}s\right).
\end{aligned}$$

Then, for $0 < c_2 < \min\left\{1, \frac{a_0\ell}{8\beta_2 C}, \frac{r}{M_1}\right\}$, we obtain, for all $t \ge 0$

$$\begin{cases} h_1(t) = \dfrac{c_2}{1+\int_0^t h_0(s)\mathrm{d}s} < 1,\\[2ex] h_1(t)\displaystyle\int_0^t a(\eta(t,s),\eta(t,s))\mathrm{d}s < 1. \end{cases} \tag{38}$$

Now, we define the function, for a constant $\beta_3 > 0$

$$\lambda(t) = -\int_0^t g'(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s \le -\beta_3 E'(t). \tag{39}$$

From the hypothesis (**A.2**), the using of (9), (38) and Jensen's inequality (10) leads to

$$\begin{aligned}
\lambda(t) &= \frac{1}{h_1(t)}\int_0^t h_1(t)(-g'(s))a(\eta(t,s),\eta(t,s))\mathrm{d}s\\
&\ge \frac{1}{h_1(t)}\int_0^t h_1(t)\xi(s)\mathcal{H}(g(s))a(\eta(t,s),\eta(t,s))\mathrm{d}s\\
&\ge \frac{\xi(t)}{h_1(t)}\mathcal{H}\left(\int_0^t h_1(t)g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s\right)\\
&\ge \frac{\xi(t)}{h_1(t)}\overline{\mathcal{H}}\left(h_1(t)\int_0^t g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s\right).
\end{aligned} \tag{40}$$

This implies that

$$\int_0^t g(s)a(\eta(t,s),\eta(t,s))\mathrm{d}s \le \frac{1}{h_1(t)}\overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right),$$

and (31) gives

$$\mathcal{L}'(t) \le -\beta_1 E(t) + \frac{\beta_4}{h_1(t)}\overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right) + \beta_4 h_0(t), \tag{41}$$

where $\beta_4 = \max\{\frac{1}{4}, M_0\}$.

Now, for $\varepsilon_0 > 0$, we define the functional

$$\mathcal{F}(t) = \overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right)\mathcal{L}(t).$$

From (41) and the fact that $E' < 0$, $\overline{\mathcal{H}}' > 0$, $\overline{\mathcal{H}}'' > 0$, and $h_1' < 0$, we conclude that $\mathcal{F} \sim E$ and

$$\begin{aligned}
\mathcal{F}'(t) \le& -\beta_1 E(t)\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right)\\
&+\frac{\beta_4}{h_1(t)}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right)\overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right)\\
&+\beta_4 h_0(t)\overline{\mathcal{H}}'\left(\varepsilon_0\frac{h_1(t)E(t)}{E(0)}\right).
\end{aligned} \tag{42}$$

On the other hand, dew to the argument given in [4, page 61-64], we have

$$\overline{\mathcal{H}}^*(s) = s(\overline{\mathcal{H}}')^{-1}(s) - \overline{\mathcal{H}}[(\overline{\mathcal{H}}')^{-1}(s)], \quad \forall s > 0, \tag{43}$$

where $\overline{\mathcal{H}}^*$ is the convex conjugate of $\mathcal{H}$ such that

$$\overline{\mathcal{H}}^*(s) = \sup_{t \in \mathbb{R}_+} \{st - \overline{\mathcal{H}}(t)\},$$

and $\overline{\mathcal{H}}^*$ satisfies the following generalized Young inequality

$$AB \le \overline{\mathcal{H}}^*(A) + \overline{\mathcal{H}}(B). \tag{44}$$

In view of (43) and (44) with $A = \overline{\mathcal{H}}'\left(\varepsilon_0 \frac{h_1(t)E(t)}{E(0)}\right)$ and $B = \overline{\mathcal{H}}^{-1}\left(\frac{h_1(t)\lambda(t)}{\xi(t)}\right)$, (42) gives

$$\begin{aligned} \mathcal{F}'(t) \le{}& -(\beta_1 E(0) - \beta_4 \varepsilon_0)\frac{E(t)}{E(0)}\overline{\mathcal{H}}'\left(\varepsilon_0 \frac{h_1(t)E(t)}{E(0)}\right) \\ &+\beta_4 \frac{\lambda(t)}{\xi(t)} + \beta_4 h_0(t)\overline{\mathcal{H}}'\left(\varepsilon_0 \frac{h_1(t)E(t)}{E(0)}\right). \end{aligned} \tag{45}$$

So, multiplying (45) by $\xi(t)$. From (39) and the fact that

$$\varepsilon_0 \frac{h_1(t)E(t)}{E(0)} < r,\ \overline{\mathcal{H}}'\left(\varepsilon_0 \frac{h_1(t)E(t)}{E(0)}\right) = \mathcal{H}'\left(\varepsilon_0 \frac{h_1(t)E(t)}{E(0)}\right),$$

we find

$$\begin{aligned} \xi(t)\mathcal{F}'(t) \le{}& -(\beta_1 E(0) - \beta_4 \varepsilon_0)\xi(t)\frac{E(t)}{E(0)}\mathcal{H}'\left(\varepsilon_0 \frac{h_1(t)E(t)}{E(0)}\right) \\ &-\beta_4 c_3 E'(t) + \beta_4 \frac{\xi(t)}{h_1(t)} h_1(t)h_0(t)\overline{\mathcal{H}}'\left(\varepsilon_0 \frac{h_1(t)E(t)}{E(0)}\right). \end{aligned}$$

As above, with $A = \overline{\mathcal{H}}'\left(\varepsilon_0 \frac{h_1(t)E(t)}{E(0)}\right)$ and $B = h_1(t)h_0(t)$ in the fact that $B < r$ and $\overline{\mathcal{H}}(B) = \mathcal{H}(B)$, we obtain

$$\begin{aligned} \xi(t)\mathcal{F}'(t) \le{}& -\left(\beta_1 E(0) - 2\beta_4 \varepsilon_0\right)\xi(t)\frac{E(t)}{E(0)}\mathcal{H}'\left(\varepsilon_0 \frac{h_1(t)E(t)}{E(0)}\right) \\ &-\beta_4 \beta_3 E'(t) + \beta_4 \frac{\xi(t)}{h_1(t)}\mathcal{H}\left(h_1(t)h_0(t)\right). \end{aligned}$$

Let us choose ε_0 small enough such that $\beta_5 := \beta_1 E(0) - 2\beta_4\varepsilon_0 > 0$. Then, we obtain, for all $t \ge 0$

$$\mathcal{F}_1'(t) \le -\beta_5 \frac{\xi(t)}{h_1(t)}\mathcal{H}_2\left(\frac{h_1(t)E(t)}{E(0)}\right) + \beta_4 \frac{\xi(t)}{h_1(t)}\mathcal{H}\left(h_1(t)h_0(t)\right), \tag{46}$$

where $\mathcal{F}_1 = \xi\mathcal{F} + \beta_4\beta_3 E \sim E$ and satisfies, for some constants $\kappa_3, \kappa_4 > 0$

$$\kappa_3 \mathcal{F}_1(t) \le E(t) \le \kappa_4 \mathcal{F}_1(t). \tag{47}$$

Since $\mathcal{H}_2'(t) = \mathcal{H}'(\varepsilon_0 t) + \varepsilon_0 t \mathcal{H}''(\varepsilon_0 t)$. Using the strict convexity of $\mathcal{H}$ on $(0, r]$, we find that $\mathcal{H}_2, \mathcal{H}_2' > 0$ on $[0, 1)$. Hence, using (47) with $c_1 = \dfrac{\kappa_3}{E(0)} > 0$, we get

$$\mathcal{H}_2\left(\frac{h_1(t)E(t)}{E(0)}\right) \ge \mathcal{H}_2(c_1\mathcal{F}_1(t)h_1(t)). \tag{48}$$

Let $\mathcal{F}_2(t) = c_1\mathcal{F}_1(t)h_1(t)$. As $c_1h_1(t)$ is nonincreasing and from (48) for some constants $\alpha_1 = c_1\beta_5 > 0$ and $\alpha_2 = c_1 > 0$, we arrive at

$$\mathcal{F}_2'(t) \le -\alpha_1\xi(t)\mathcal{H}_2\left(\mathcal{F}_2(t)\right) + \alpha_2\xi(t)\mathcal{H}\left(M_0h_1(t)h_0(t)\right). \tag{49}$$

Taking into account (47), thus, there exists a constant $c_2 > 0$ such that $\mathcal{F}_2(t) \ge c_2h_1(t)E(t)$. Let $\vartheta(t)$ satisfying (34) and (35). If

$$c_2h_1(t)E(t) \le 2\frac{\mathcal{H}_3(t)}{\vartheta(t)},$$

then

$$E(t) \le \frac{2}{c_2}\frac{\mathcal{H}_3(t)}{h_1(t)\vartheta(t)}. \tag{50}$$

Else if

$$c_2h_1(t)E(t) > 2\frac{\mathcal{H}_3(t)}{\vartheta(t)}.$$

According to $(h_1E)'(t) \le 0$, then we have, for $0 \le s \le t$

$$\mathcal{F}_2(s) \ge c_2h_1(s)E(s) > 2\frac{\mathcal{H}_3(t)}{\vartheta(t)}. \tag{51}$$

For $0 < \varepsilon_1 < 1$, using (9), $0 < \vartheta \le 1$ and the fact that $\mathcal{H}_2$ is convex, we obtain

$$\begin{aligned} &\mathcal{H}_2\left(\varepsilon_1\vartheta(s)\mathcal{F}_2(s) - \varepsilon_1\mathcal{H}_3(s)\right) \\ &= \mathcal{H}_2\left(\varepsilon_1\vartheta(s)\mathcal{F}_2(s) - \frac{\varepsilon_1\vartheta(s)\mathcal{H}_3(s)}{\vartheta(s)}\right) \\ &\le \varepsilon_1\vartheta(s)\mathcal{H}_2\left(\mathcal{F}_2(s) - \frac{\mathcal{H}_3(s)}{\vartheta(s)}\right) \\ &\le \varepsilon_1\vartheta(s)\left(\mathcal{F}_2(s) - \frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s) - \varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right) \\ &\le \varepsilon_1\vartheta(s)\mathcal{F}_2(s)\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s) - \varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right) \\ &\quad -\varepsilon_1\vartheta(s)\frac{\mathcal{H}_3(s)}{\vartheta(s)}\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s) - \varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right). \end{aligned}$$

Therefore, using (51) and the fact that $\mathcal{H}'$ is increasing, we have, for $0 \le s \le t$,

$$\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s) - \varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right) < \mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s)\right)$$

and

$$\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s) - \varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right) > \mathcal{H}'\left(\varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right).$$

These imply that

$$\begin{aligned} &\mathcal{H}_2\left(\varepsilon_1\vartheta(s)\mathcal{F}_2(s) - \varepsilon_1\mathcal{H}_3(s)\right) \\ &\le \varepsilon_1\vartheta(s)\mathcal{F}_2(s)\mathcal{H}'\left(\varepsilon_0\mathcal{F}_2(s)\right) \\ &\quad -\varepsilon_1\vartheta(s)\frac{\mathcal{H}_3(s)}{\vartheta(s)}\mathcal{H}'\left(\varepsilon_0\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right). \end{aligned} \tag{52}$$

Now, we let

$$\mathcal{F}_3(s) = \varepsilon_1\vartheta(s)\mathcal{F}_2(s) - \varepsilon_1\mathcal{H}_3(s). \tag{53}$$

By choosing ε_1 small enough so that $\mathcal{F}_3(0) \leq 1$. Then, using (53), we find, for $0 \leq s \leq t$,

$$\mathcal{H}_2\left(\mathcal{F}_3(s)\right) \leq \varepsilon_1 \vartheta(s) \mathcal{H}_2\left(\mathcal{F}_2(s)\right) - \varepsilon_1 \vartheta(s) \mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right). \tag{54}$$

As $\vartheta' \leq 0$, we have

$$\begin{aligned}\mathcal{F}_3'(s) &= \varepsilon_1 \vartheta'(s) \mathcal{F}_2(s) + \varepsilon_1 \vartheta(s) \mathcal{F}_2'(s) - \varepsilon_1 \mathcal{H}_3'(s) \\ &\leq \varepsilon_1 \vartheta(s) \mathcal{F}_2'(s) - \varepsilon_1 \mathcal{H}_3'(s).\end{aligned} \tag{55}$$

From (49), (55) and (54), we get, for $0 \leq s \leq t$,

$$\begin{aligned}\mathcal{F}_3'(s) \leq& -\alpha_1 \xi(s) \mathcal{H}_2(\mathcal{F}_3(s)) - \varepsilon_1 \alpha_1 \vartheta(s) \xi(s) \mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right) \\ &+ \varepsilon_1 \alpha_2 \vartheta(s) \xi(s) \mathcal{H}\left(M_0 h_1(s) h_0(s)\right) - \varepsilon_1 \mathcal{H}_3'(s).\end{aligned} \tag{56}$$

By the definition of $\mathcal{H}_1$ and $\mathcal{H}_3$, we have

$$\mathcal{H}_1(\mathcal{H}_3(s)) = \alpha_1 \int_0^s \xi(\varrho) d\varrho,$$

thus, for $0 \leq s \leq t$,

$$\mathcal{H}_3'(s) = -\alpha_1 \xi(s) \mathcal{H}_2(\mathcal{H}_3(s)). \tag{57}$$

Using (57), we obtain

$$\begin{aligned}&\varepsilon_1 \alpha_2 \vartheta(s) \xi(s) \mathcal{H}\left(M_0 h_1(s) h_0(s)\right) - \varepsilon_1 \mathcal{H}_3'(s) \\ &-\varepsilon_1 \alpha_1 \vartheta(s) \xi(s) \mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right) \\ =& \varepsilon_1 \alpha_2 \vartheta(s) \xi(s) \mathcal{H}\left(M_0 h_1(s) h_0(s)\right) - \varepsilon_1 \alpha_1 \vartheta(s) \xi(s) \mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right) \\ &+\varepsilon_1 \alpha_1 \xi(s) \mathcal{H}_2(\mathcal{H}_3(s)) \\ =& \varepsilon_1 \vartheta(s) \xi(s) \left(\alpha_2 \mathcal{H}\left(M_0 h_1(s) h_0(s)\right) - \alpha_1 \mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)\right. \\ &\left. + \alpha_1 \frac{\mathcal{H}_2(\mathcal{H}_3(s))}{\vartheta(s)}\right).\end{aligned}$$

So, by (35), we have

$$\begin{aligned}\varepsilon_1 \vartheta(s) \xi(s) &\left(\alpha_2 \mathcal{H}\left(M_0 h_1(s) h_0(s)\right) - \alpha_1 \mathcal{H}_2\left(\frac{\mathcal{H}_3(s)}{\vartheta(s)}\right)\right. \\ &\left. + \alpha_1 \frac{\mathcal{H}_2(\mathcal{H}_3(s))}{\vartheta(s)}\right) \leq 0.\end{aligned}$$

Consequently, (56) leads to

$$\mathcal{F}_3'(s) \leq -\alpha_1 \xi(s) \mathcal{H}_2(\mathcal{F}_3(s)).$$

Then, by the definition of $\mathcal{H}_1$ and $\mathcal{H}_2$, we get

$$\left[\mathcal{H}_1(\mathcal{F}_3(s))\right]' \geq \alpha_1 \xi(s). \tag{58}$$

By integrating (58) over $(0, t)$, we obtain

$$\mathcal{H}_1(\mathcal{F}_3(t)) \geq \alpha_1 \int_0^t \xi(s) \mathrm{d}s + \mathcal{H}_1(\mathcal{F}_3(0)),$$

Note that $\mathcal{H}_1$ is nonincreasing, $\mathcal{F}_3(0) \leq 1$ and $\mathcal{H}_1(1) = 0$. So,

$$\mathcal{F}_3(t) \leq \mathcal{H}_1^{-1}\left(\alpha_1 \int_0^t \xi(s) \mathrm{d}s\right) = \mathcal{H}_3(t).$$

Hence, by the definition of $\mathcal{F}_2$ and $\mathcal{F}_3$, we get

$$\mathcal{F}_2(t) \leq \frac{1+\varepsilon_1}{\varepsilon_1} \frac{\mathcal{H}_3(t)}{\vartheta(t)} \quad \text{and} \quad \mathcal{F}_1(t) \leq \frac{1+\varepsilon_1}{\varepsilon_1 c_1} \frac{\mathcal{H}_3(t)}{\vartheta(t) h_1(t)}.$$

Then, as $\mathcal{F}_1 \sim E$, there exists a constant $c_3 > 0$ such that $E(t) \leq c_3 \mathcal{F}_1$. Consequently, we deduce that

$$E(t) \leq \frac{c_3(1+\varepsilon_1)}{\varepsilon_1 c_1} \frac{\mathcal{H}_3(t)}{\vartheta(t) h_1(t)}.$$

From this estimate and (50), the estimate (36) is established with $C = \max\left\{\frac{2}{c_2}, \frac{c_3(1+\varepsilon_1)}{\varepsilon_1 c_1}\right\}$. ■

Now, we give an example to illustrate our result.

Example 1. [11] Let $g(t) = b(1+t)^{-q}$, where $q > 1$ and $0 < b < q-1$ so that **(A.1)** is satisfied, for any $\beta_0 \geq q$. In this case $\xi(t) = q b^{\frac{-1}{q}}$ and $\mathcal{H}(t) = t^{\frac{q+1}{q}}$. Then, there exist three positive constants $b_i; i = 1, \ldots, 3$ depending only on ε_0, α_1, q and b such that $\mathcal{H}_1(t) = b_1(t^{\frac{-1}{q}} - 1)$, $\mathcal{H}_2(t) = b_2 t^{\frac{q+1}{q}}$ and $\mathcal{H}_3(t) = (b_3 t + 1)^{-q}$.

If

$$m_0(1+t)^k \leq 1 + \|\nabla u_0(t)\|_2^2 \leq m_1(1+t)^k \tag{59}$$

where $0 < k < q-1$ and $m_0, m_1 > 0$, then, for some positive constants $b_i; i = 4, \ldots, 7$ depending only on q, b, m_0, m_1 and k, the estimates

$$b_4(1+t)^{1+k-q} \leq h_0(t) \leq b_5(1+t)^{1+k-q}, \tag{60}$$

$$\frac{c_2}{h_1(t)} \geq b_6 \left\{\begin{array}{ll} 1+\ln(1+t), & q-k=2; \\ 2, & q-k>2; \\ (1+t)^{k+2-q}, & 1<q-k<2. \end{array}\right. \tag{61}$$

$$\frac{c_2}{h_1(t)} \leq b_7 \left\{\begin{array}{ll} 1+\ln(1+t), & q-k=2; \\ 2, & q-k>2; \\ (1+t)^{k+2-q}, & 1<q-k<2. \end{array}\right. \tag{62}$$

Condition (35) is satisfied if

$$(1+t)^q h_0(t) h_1(t) \vartheta(t) \leq b_8 \left(1 - \vartheta^{\frac{1}{q}(t)}\right)^{\frac{q}{q+1}}, \tag{63}$$

where $b_8 > 0$. Choosing

$$\vartheta(t) = \lambda \left\{\begin{array}{ll} (1+t)^{-k-1}, & q-k \geq 2; \\ (1+t)^{1-q}, & 1<q-k<2; \end{array}\right. \tag{64}$$

so (34) is valid. Moreover, using (60) and (61), we see that (63) is satisfied if $0 < \lambda \leq 1$ is small enough in (64), and then (35) is satisfied. Therefore (36) and (62) imply that, for all $t \geq 0$

$$E(t) \leq C \left\{\begin{array}{ll} (1+\ln(1+t))^{1+k-q}, & q-k=2; \\ (1+t)^{1+k-q}, & q-k>2 \text{ or } 1<q-k<2. \end{array}\right. \tag{65}$$

This estimate gives $\lim_{t \to \infty} E(t) = 0$.

On the other hand, if $k = 0$ in (59), then (65) holds with $k = 0$.

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