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كـلية التكنـــولــوجيـا
نيابة العمادة لما بعد التدرج والبحث العلمي والعلاقات الخارجية

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شهادة ادارية

المصادقة على تقارير خبرة للموافقة على مطبوعة بيداغوجية

بعد الإطلاع على تقارير لجنة الخبراء للموافقة على المطبوعة البيداغوجية للأستاذ : **حرايز توفيق– أستاذ محاضر قسم ب** ، بالقاعدة المشتركة بكلية التكنولوجيا بجامعة محمد بوضياف بالمسيلة والتي كانت كلها ايجابية ، تمّ تقرير التّالي:

1–المصادقة على تقارير لجنة الخبراء للموافقة المطبوعة البيداغوجية والمعنونة بـــ:

Analyse 1
Engineering cycle

2– تمّ تشكيل لجنة الخبراء المكونة من السّادة الآتية أسماؤهم:

–**حميدي خالد**، أستاذ محاضر أ ، جامعة محمد بوضياف– المسيلة

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وتمت الموافقة بالاجماع على هذه المطبوعة.

رئيس المجلس العلمي للكلية

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Lecture Notes on Analysis 1

Domain: Engineering

Field: Electrical Eng., Mechanical Eng., Civil Eng.

Level: First Year (L1)

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Lecture Notes on Analysis 1 for First-Year Engineering disciplines Students

Electrical Eng., Mechanical Eng., and Civil Eng.

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Abstract

These lecture notes are specifically designed for first-year undergraduate students pursuing degrees in Engineering, Technology, Computer Science, or Natural Sciences. The material covers foundational concepts essential for Engineering disciplines, with a focus on building both theoretical understanding and practical problem-solving skills.

These notes are primarily intended for:

- Freshmen in Computer Science and Information Technology
- First-year Engineering and Technology students
- Natural Sciences undergraduates

Contents

Introduction

Mathematics forms the backbone of engineering, providing the language and tools to model, analyze, and solve real world problems. These lecture notes are designed to equip first year engineering students whether in electrical, mechanical, or civil disciplines with the foundational mathematical concepts critical to your academic and professional journey. Over the course of three chapters, we will explore the real number system, sequences, and real functions, each of which serves as a cornerstone for advanced topics in calculus, physics, and engineering design. By mastering these principles, you will develop the analytical rigor necessary to tackle complex systems, optimize solutions, and innovate in your field.

Our exploration begins with the real number system, the bedrock of all quantitative reasoning in engineering. Here, you will delve into the properties of real numbers, their completeness, and their role in representing continuous quantities such as time, distance, and force. Understanding the structure of real numbers is essential for interpreting measurements, designing tolerances, and ensuring precision in computational models. This chapter will also introduce you to inequalities, absolute values, and intervals tools that underpin error analysis and stability criteria in engineering systems. These concepts are not merely abstract; they are the framework for translating physical realities into solvable mathematical problems.

From there, we transition to sequences, which bridge discrete and continuous mathematics. Sequences model phenomena that evolve step-by-step, such as digital signals in electrical circuits or iterative approximations in mechanical simulations. You will study convergence, limits, and the behavior of infinite series skills vital for analyzing algorithms, stability in control systems, and harmonic responses. Finally, the course culminates with real functions, where you will learn to describe relationships between variables, such as voltage current dependencies or stress strain curves. Through continuity, limits, and function classification, you will gain the intuition to model dynamic systems, predict outcomes, and validate designs. Together, these chapters lay a robust mathematical foundation, empowering you to think critically and creatively as you progress into specialized engineering disciplines.

Chapter 1

Real numbers System

The set of real numbers $\mathbb{R}$, forms a foundational continuum in mathematics, seamlessly integrating algebraic structure, order, and completeness. Central to its study are bounded subsets, which confine elements within fixed numerical limits, enabling precise analysis of convergence, extrema, and topological behavior. This chapter explores the interplay between the real number line's intrinsic properties—such as the Least Upper Bound Axiom—and the constraints imposed by boundedness. We examine how bounded intervals, sequences, and functions serve as critical tools for modeling phenomena in calculus, optimization, and beyond, while highlighting pivotal results like the Bolzano-Weierstrass theorem that bridge finite limitations with infinite possibilities.

1.1 Real Numbers

i. Natural Numbers

The set of natural numbers is denoted by $\mathbb{N} = \{0, 1, 2, \ldots\}$. In this set, we define the two operations of addition and multiplication as follows:

$$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (x, y) \mapsto x + y,$$

$$\times : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (x, y) \mapsto x \times y.$$

Subtraction of x and y is defined if there exists a number $a \in \mathbb{N}$ such that $x = y + a$, and we denote $a = x - y$.

Division of x by y ($y \neq 0$) is defined if there exists a number $a \in \mathbb{N}$ such that $y \times a = x$, and we denote $a = \frac{x}{y}$.

Note that subtraction and division may not always be defined in $\mathbb{N}$, so the set of natural numbers is not sufficient for these operations. This necessitates extending this set.

ii. Set of Integers

From $\mathbb{N}$, we construct the set of integers, denoted by $\mathbb{Z} = \{\ldots, -2, -1, 0, +1, +2, \ldots\}$. In this set, the equation $a + x = y$ always has a solution. The elements $(-x)$ for $x \in \mathbb{N}$ are the additive inverses of x, and they satisfy:

$$\forall x \in \mathbb{Z}, \quad x + (-x) = (-x) + x = 0.$$

The set $(\mathbb{Z}, +, \times)$ is a commutative unital ring. However, this set is not sufficient for solving the equation $a \times y = x$ for $y \neq 0$.

iii. Set of Rational Numbers

From $\mathbb{Z}$, we construct the set of rational numbers, denoted by:

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \right\} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N} \right\}.$$

Example 1.1.

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \dots$$

For simplicity, we choose p and q to be coprime.

The set $\mathbb{Q}$ equipped with addition and multiplication forms a commutative unital field. However, $\mathbb{Q}$ is not sufficient to solve equations like $x^2 - 2 = 0$, whose solution $x = \sqrt{2}$ is not in $\mathbb{Q}$. Indeed, assume by contradiction that $\sqrt{2} \in \mathbb{Q}$. Then:

$$\sqrt{2} = \frac{p}{q} \quad \text{(with } p \text{ and } q \text{ coprime).}$$

$$\Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \Rightarrow p \text{ is even} \Rightarrow \exists m \in \mathbb{Z}, p = 2m \Rightarrow p^2 = 4m^2 = 2q^2.$$

$$\Rightarrow q^2 = 2m^2 \Rightarrow q \text{ is even, contradicting the coprimality of } p \text{ and } q.$$

Thus, $\sqrt{2} \notin \mathbb{Q}$.

Decimal Expansion of a Rational Number

Let $\frac{p}{q} \in \mathbb{Q}$. By performing the Euclidean division of p by q, we obtain:

$$\frac{p}{q} = \alpha_0.\alpha_1\alpha_2\alpha_3\dots$$

If there exists $k \in \mathbb{N}$ such that $\alpha_k = 0$, then we obtain a finite decimal expansion:

$$\frac{p}{q} = \alpha_0.\alpha_1\alpha_2\dots\alpha_n.$$

Example 1.2.

$$\frac{1}{8} = 0.125.$$

If the decimal expansion of a rational number $\frac{P}{q}$ is infinite, i.e., $\forall n \in \mathbb{N}$, $\alpha_n \neq 0$, then:

$$\frac{P}{q} = \alpha_0.\alpha_1\alpha_2\ldots\alpha_n\ldots$$

where $\alpha_i \in \{0,1,2,3,4,5,6,7,8,9\}$.

Any finite decimal expansion can be written as:

$$\frac{P}{q} = \alpha_0.\alpha_1\ldots\alpha_n 000\ldots = \alpha_0.\alpha_1\ldots\alpha_n(0) = \alpha_0.\alpha_1\ldots\alpha_{n-1}(\alpha_n{-}1)999\ldots = \alpha_0.\alpha_1\ldots\alpha_{n-1}(\alpha_n{-}1)(9).$$

In this way, we find that:

$$\frac{P}{q} = \alpha_0.\alpha_1\ldots\alpha_n\beta_1\beta_2\ldots\beta_m\beta_1\beta_2\ldots\beta_m\ldots$$

Example 1.3.

$$1030 \div 3300 = 0.3121212\ldots$$

Thus, we have demonstrated the following theorem:

Theorem 1.4. *There exists a bijective correspondence between the set of rational numbers and the set of infinite periodic decimal expansions with a period different from 9.*

Example 1.5. *The number* $1.202200220002\ldots$ *is not rational, indicating that there are numbers beyond the rationals.*

Set of Real Numbers

Definition 1.6. *An irrational number is any infinite non-periodic decimal expansion.*

Definition 1.7. *the set of real numbers, denoted by* $\mathbb{R}$*, is the set consisting of rational and irrational numbers.*

It can be shown that the relation $x \leq y \iff x - y \geq 0$ is a total order on $\mathbb{R}$.

Axiomatic Definition of Real Numbers:

Axiom 1.8. *The set of real numbers* $\mathbb{R}$*, equipped with the two binary operations of addition* $(+)$ *and multiplication* $(\cdot)$*, and a total order relation* $(\leq)$*, satisfies the following axioms:*
1. $\forall x, y \in \mathbb{R}, x + y = y + x$ *(commutativity of addition).*
2. $\forall x, y, z \in \mathbb{R}, x + (y + z) = (x + y) + z$ *(associativity of addition).*
3. $\exists 0 \in \mathbb{R}$*, called the additive identity, such that* $0 + x = x + 0 = x, \forall x \in \mathbb{R}$.
4. $\forall x \in \mathbb{R}, \exists(-x) \in \mathbb{R}$*, called the additive inverse, such that* $x + (-x) = (-x) + x = 0$.
5. $\forall x, y \in \mathbb{R}, x \cdot y = y \cdot x$ *(commutativity of multiplication).*
6. $\forall x, y, z \in \mathbb{R}, x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *(associativity of multiplication).*
7. $\exists 1 \in \mathbb{R}$*, called the multiplicative identity, such that* $1 \cdot x = x \cdot 1 = x, \forall x \in \mathbb{R}$.
8. $\forall x \in \mathbb{R}, x \neq 0, \exists x^{-1} \in \mathbb{R}$*, called the multiplicative inverse, such that* $x \cdot x^{-1} = 1$.
9. $\forall x, y, z \in \mathbb{R}, x \cdot (y + z) = x \cdot y + x \cdot z$ *(distributivity).*
10. $\forall x \in \mathbb{R}, x \leq x$ *(reflexivity).*

11. $\forall x, y \in \mathbb{R}, (x \leq y \wedge y \leq x) \Rightarrow x = y$ *(antisymmetry).*
12. $\forall x, y, z \in \mathbb{R}, (x \leq y \wedge y \leq z) \Rightarrow x \leq z$ *(transitivity).*
13. $\forall x, y \in \mathbb{R}$, *either* $x \leq y$ *or* $y \leq x$ *(total order).*
14. $x \leq y \Rightarrow x + z \leq y + z, \forall z \in \mathbb{R}$.
15. $\forall x, y, z \in \mathbb{R}, (x \leq y \wedge z \geq 0) \Rightarrow x \cdot z \leq y \cdot z$.
16. If $X, Y \subseteq \mathbb{R}$ *are such that* $\forall x \in X, \forall y \in Y, x \leq y$*, then* $\exists c \in \mathbb{R}, x \leq c \leq y$ *(axiom of continuity or Dedekind completeness).*

Remark 1.9. *Axioms 1 to 15 hold in* $\mathbb{Q}$*, but the set of rational numbers does not satisfy the axiom of continuity.*

Supremum and Infimum of a Subset of $\mathbb{R}$

Let X be a subset of $\mathbb{R}$.

Definition 1.10. *We say that the set* X *has a maximum (resp. minimum) if there exists* $x_0 \in X$ *such that:*

$$\forall x \in X, \quad x \leq x_0 \quad (\textit{resp. } x \geq x_0).$$

The set X *is said to be bounded above (resp. bounded below) if it has a maximum (resp. minimum), and we write:*

$$(X \textit{ is bounded above}) \iff \exists M \in \mathbb{R}, \quad \forall x \in X, \quad x \leq M.$$

The number M *is called an upper bound of* X*.*

$$(X \textit{ is bounded below}) \iff \exists m \in \mathbb{R}, \quad \forall x \in X, \quad x \geq m.$$

The number m *is called a lower bound of* X*.*

If a set X *is bounded above (resp. bounded below), then it has infinitely many upper bounds (resp. lower bounds).*

Supremum and Infimum

Definition 1.11. *The least upper bound (resp. greatest lower bound) of a set* X *is called the supremum (resp. infimum) of* X*, denoted by* $\sup X$ *(resp.* $\inf X$*), and we write:*

$$\sup X = M \iff \begin{cases} 1.\, M \textit{ is an upper bound of } X, \\ 2.\, M \textit{ is the least upper bound.} \end{cases} \iff \begin{cases} 1.\, \forall x \in X,\, x \leq M, \\ 2.\, \forall \varepsilon > 0,\, M - \varepsilon \textit{ is not an upper bound.} \end{cases}$$

$$\Rightarrow \begin{cases} 1.\, \forall x \in X,\, x \leq M, \\ 2.\, \forall \varepsilon > 0,\, \exists x_0 \in X,\, x_0 > M - \varepsilon. \end{cases}$$

$$\inf X = m \iff \begin{cases} 1.\, m \textit{ is a lower bound of } X, \\ 2.\, m \textit{ is the greatest lower bound.} \end{cases} \iff \begin{cases} 1.\, \forall x \in X,\, x \geq m, \\ 2.\, \forall \varepsilon > 0,\, m + \varepsilon \textit{ is not a lower bound.} \end{cases}$$

$$\Rightarrow \begin{cases} 1.\, \forall x \in X,\, x \geq m, \\ 2.\, \forall \varepsilon > 0,\, m + \varepsilon \textit{ is not a lower bound.} \end{cases}$$

Remark 1.12. *In general,* $\sup X$ *and* $\inf X$ *do not necessarily belong to the set* X *itself. If they do,* $\sup X$ *is called the greatest element, denoted by* $\max X$*, and* $\inf X$ *is called the least element, denoted by* $\min X$*. Otherwise, we say that* $\max X$ *and* $\min X$ *do not exist.*

Example 1.13. *Let* $X =]-1,2]$*. Determine* $\sup X$*,* $\inf X$*,* $\max X$*, and* $\min X$ *if they exist. For* $\sup X$*: It is clear that* $\forall x \in X$, $x \leq 2$*, so* $M = 2$ *is an upper bound of* X*. Since* $M = 2 \in X$*, we conclude that* $\sup X = \max X = 2$*. For* $\inf X$*: It is clear that* $\forall x \in X$, $x \geq -1$*, so* $m = -1$ *is a lower bound of* X*. We show that* $m = -1$ *is the greatest lower bound. Indeed, for any* $\varepsilon > 0$*, we seek* $x_0 \in X$ *such that:*

$$-1 \leq x_0 < m + \varepsilon = \varepsilon - 1.$$

Two cases arise: 1. If $\varepsilon > 1$*, then* $\varepsilon - 1 > 0$*, and we can take* $x_0 = \frac{\varepsilon - 1}{2}$*, which satisfies* $-1 \leq x_0 < \varepsilon - 1$*. 2. If* $\varepsilon < 1$*, then* $\varepsilon - 1 < 0$*, and we can take* $x_0 = -1 + \frac{\varepsilon}{2}$*, which satisfies* $-1 \leq x_0 < -1 + \varepsilon < 0$*.*

The existence of x_0 *shows that* $\inf X = -1$*. Since* $-1 \notin X$*, the least element* $\min X$ *does not exist.*

Theorem (Bolzano-Weierstrass)

Theorem 1.14. *1. The supremum and infimum of a subset of* $\mathbb{R}$*, if they exist, are unique 2. Every bounded above (resp. bounded below) set has a supremum (resp. infimum).*

Proof. 1. Let M_1 and M_2 be two suprema of a set X. Then:

$$M_1 \text{ is a supremum and } M_2 \text{ is an upper bound} \Rightarrow M_1 \leq M_2 \quad (1)$$

$$M_2 \text{ is a supremum and } M_1 \text{ is an upper bound} \Rightarrow M_2 \leq M_1 \quad (2)$$

From (1) and (2), we conclude $M_1 = M_2$.

2. Let X be a bounded above subset of $\mathbb{R}$. Let Y be the set of upper bounds of X. Then:

$$\forall x \in X, \forall y \in Y, \quad x \leq y \Rightarrow \exists c \in \mathbb{R}, \quad x \leq c \leq y.$$

We have $\exists c \in \mathbb{R}, \forall x \in X, x \leq c$, so c is an upper bound of X.

$$c \leq y, \forall y \in Y \Rightarrow c \text{ is the least upper bound of } X \Rightarrow c = \sup X.$$

□

Archimedean Axiom

The Archimedean property states:

Axiom 1.15.

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}, \text{ such that } x < n.$$

Proof. Assume the contrary, i.e., there exists $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, x \geq n$. This implies:

$$\exists x \in \mathbb{R}, \forall n \in \mathbb{N}, n \leq x \Rightarrow \mathbb{N} \text{ is bounded above by } x \in \mathbb{R}.$$

$$\Rightarrow \exists \sup \mathbb{N} = M.$$

By definition of the supremum:

$$\sup \mathbb{N} = M \iff \begin{cases} 1.\, \forall n \in \mathbb{N}, n \leq M, \\ 2.\, \forall \epsilon > 0, \exists n_0 \in \mathbb{N}, n_0 > M - \epsilon. \end{cases}$$

Let $\epsilon = 1$. Then $n_0 > M - 1 \Rightarrow M < n_0 + 1$. Thus:

$$n_0 \leq M < n_0 + 1 \quad \text{and} \quad (n_0 + 1) \in \mathbb{N}.$$

This contradicts the definition of the supremum, proving the Archimedean property. □

Consequences of Archimedean Property

The Archimedean property can be written as:

$$\forall h > 0, \forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, \quad nh > x.$$

A direct consequence is:

Corollary 1.16.

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, \quad n \leq x < n + 1.$$

Integer Part of a Real Number

Definition 1.17. *The integer part of $x \in \mathbb{R}$ is the integer k satisfying $k \leq x < k + 1$. It is denoted by $E(x)$ or $[x]$, i.e.:*

$$E(x) \leq x < E(x) + 1 \quad \text{or} \quad E(x) - 1 < x \leq E(x).$$

Alternatively, $E(x)$ is defined as the greatest integer less than or equal to x.

Example 1.18.

$$E(2.35) = 2; \quad E(0.01) = 0; \quad E(-2.35) = -3.$$

Proposition 1.19. *1. $\forall x \in \mathbb{Z}, \quad E(x) = x$. 2. $x = E(x) + \alpha$, where $0 \leq \alpha < 1$.*

Absolute Value of a Real Number

Definition 1.20. *The absolute value of a real number x is defined as:*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It is clear that the absolute value satisfies:

$$|x| \leq \alpha \iff -\alpha \leq x \leq +\alpha$$

Intervals in $\mathbb{R}$

Definition 1.21. *Let I be a subset of $\mathbb{R}$. I is an interval $\Leftrightarrow$ for all $x \in I$, $y \in I$, if $x < y$ then $\forall z \in \mathbb{R} : (x < z < y \implies z \in I)$.*

- Open interval: $]a, b[= \{x \in \mathbb{R} \mid a < x < b\}$
- Closed interval: $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
- Half-open interval: $[a, b[= \{x \in \mathbb{R} \mid a \leq x < b\}$

Remark 1.22. *$\mathbb{R}$ is both an open and closed interval, denoted $]-\infty, +\infty[$.*

Proof by Induction

To prove a proposition $P(n)$ depending on a natural number n, it suffices to:

1. Prove the validity of $P(n_0)$ for $n = n_0$, with $n_0 \geq 0$.
2. Assuming that the proposition $P(n)$ is true, prove $P(n+1)$. Then the proposition $P(n)$ is true for all $n \in \mathbb{N}$.

Countable Sets

Definition 1.23. *Two subsets A and B of $\mathbb{R}$ are equipotent if there exists a bijection f from A to B, i.e.,*

$$\forall b \in B, \exists! a \in A, b = f(a)$$

Definition 1.24. *A subset A of $\mathbb{R}$ is called countable if it is equipotent to the set of natural numbers $\mathbb{N}$. In this case, the set A can be written as $A = \{x_1, x_2, x_3, \ldots, x_m, x_{m+1}, \ldots\}$.*

1.2 Complex Numbers

Introduction

Note that $\mathbb{R}$ is not sufficient to solve equations of the form $x^2 + \alpha = 0$ for $\alpha > 0$. This necessitates extending this set.

We equip $\mathbb{R}^2$ with two internal composition laws, addition $(+)$ and multiplication $(\cdot)$, defined by:

Complex Numbers

Definition and Operations

For all $(a, b), (a', b') \in \mathbb{R}^2$,

$$(a, b) + (a', b') = (a + a', b + b') \quad \text{and} \quad (a, b) \cdot (a', b') = (aa' - bb', ab' + ba')$$

We verify that $(\mathbb{R}^2, +, \cdot)$ forms a commutative unitary field. If we denote $z = (a, b)$, then:

1. $0 = (0, 0)$ is the identity element for addition.
2. $-z = (-a, -b)$ is the additive inverse of $z = (a, b)$.
3. $1 = (1, 0)$ is the identity element for multiplication.
4. $z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ is the multiplicative inverse of z.

This field is called the field of complex numbers, denoted $\mathbb{C}$.
Consider the map $f : \mathbb{R} \to \mathbb{C}$:

$$x \mapsto (x, 0)$$

Then f is a field isomorphism:

1. f is bijective.
2. $f(x + x') = (x + x', 0) = (x, 0) + (x', 0) = f(x) + f(x')$.
3. $f(x \cdot x') = (x \cdot x', 0) = (x, 0) \cdot (x', 0) = f(x) \cdot f(x')$.

Thus, $\mathbb{R}$ can be identified with a subfield of $\mathbb{C}$ consisting of elements $(a, 0)$ with $a \in \mathbb{R}$. Through this isomorphism, the complex number $(a, 0)$ is simply denoted a.

Imaginary Unit

Let $i = (0, 1)$. Then:

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1$$

Any complex number $z = (a, b)$ can be written as:

$$z = (a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1) \cdot (b, 0) = a + ib, \quad a, b \in \mathbb{R}$$

Properties of Complex Numbers

For a complex number $z = a + ib$:

1. $a = \mathrm{Re}(z)$ (real part) and $b = \mathrm{Im}(z)$ (imaginary part).
2. If $b = 0$, z is real. If $a = 0$, z is purely imaginary.
3. The conjugate of z is $\overline{z} = a - ib$.
4. $\overline{z + z'} = \overline{z} + \overline{z'}$ and $\overline{z \cdot z'} = \overline{z} \cdot \overline{z'}$.
5. The modulus of z is $|z| = \sqrt{a^2 + b^2}$.
6. $|z \cdot z'| = |z| \cdot |z'|$ and $|z + z'| \leq |z| + |z'|$.

Geometric Representation

Each complex number $z = a + ib$ corresponds to a point $M = (a, b)$ in the plane $\mathbb{R}^2$, or equivalently to the vector $\overrightarrow{OM}$. The point M is called the image of the complex number z, and z is called the affix of the point M. **Complex Plane** The plane $0xy$ containing the images of complex numbers $z = (a, b)$ is called the complex plane, where $0x$ is the real axis and $0y$ is the imaginary axis.

Trigonometric Form (Polar or Geometric Form)

The image M of the complex number $z = a + ib$ can also be determined by the angle θ measured from $0x$ to $0M$ and the number r, which is the length of the vector $\overrightarrow{OM}$.

For every complex number z, there exists a pair (r, θ) in $\mathbb{R}_+ \times \mathbb{R}$ such that:

$$z = r(\cos\theta + i\sin\theta)$$

1. For $z \neq 0$, we have:
$$r = |z|, \quad \cos\theta = \frac{\Re z}{|z|}, \quad \sin\theta = \frac{\Im z}{|z|}$$
The angle θ is defined up to multiples of 2π.
2. For $z = 0$, we have $r = 0$ and θ is arbitrary.

The number θ is called the argument of z, and there are infinitely many values for θ. When $\theta \in [0, 2\pi]$, we write $\arg z$.

Proposition 1.25. *If z_1 and z_2 are two complex numbers with $\theta = \arg z$ and $\theta' = \arg z'$, then:*

1. $z = z' \Leftrightarrow |z| = |z'|$ *and* $\theta = \theta' + 2\pi k$ *for some integer* k.
2. $z \cdot z' = |z| \cdot |z'|(\cos(\theta + \theta') + i\sin(\theta + \theta'))$.
3. *If* $z' \neq 0$,
$$\frac{z}{z'} = \frac{|z|}{|z'|}(\cos(\theta - \theta') + i\sin(\theta - \theta')).$$

Complex Exponential

For $z = a + ib$, we define:

$$e^z = e^{a+ib} = e^a \cdot e^{ib}, \quad \text{where} \quad e^{ib} = \cos b + i\sin b.$$

Thus, any complex number can be written as:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

This definition preserves the properties of the real exponential function.

De Moivre's Formula

For any real number θ and any integer $n \in \mathbb{N}$, we have:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof. 1. For $n = 1$, the formula holds:

$$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta.$$

2. Assume the formula holds for n:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Then for $n+1$:

$$(\cos\theta+i\sin\theta)^{n+1} = (\cos\theta+i\sin\theta)^n(\cos\theta+i\sin\theta) = (\cos n\theta+i\sin n\theta)(\cos\theta+i\sin\theta).$$

Expanding the right-hand side:

$$\cos n\theta\cos\theta - \sin n\theta\sin\theta + i(\sin\theta\cos n\theta + \cos\theta\sin n\theta).$$

By identification:

$$r^n = \rho \quad \text{and} \quad n\theta = x + 2\pi k \Rightarrow r = \sqrt[n]{\rho} \quad \text{and} \quad \theta = \frac{x+2\pi k}{n}$$

Thus, we obtain:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right); \quad k = 0,1,2,\ldots,n-1$$

□

Example 1.26. *Calculate* $\sqrt[3]{1-i}$

$$\begin{aligned}\sqrt[3]{1-i} &= \sqrt[3]{\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)} = \sqrt[3]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\\ &= \sqrt[3]{2}\left(\cos\frac{\frac{3\pi}{4}+2k\pi}{3} + i\sin\frac{\frac{3\pi}{4}+2k\pi}{3}\right); \quad k = 0,1,2\\ &= \sqrt[3]{2}\left(\cos\frac{(3+8k)\pi}{12} + i\sin\frac{(3+8k)\pi}{12}\right); \quad k = 0,1,2\end{aligned}$$

For $k=0$*:*

$$w_0 = \sqrt[3]{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt[3]{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

For $k=1$*:*

$$w_1 = \sqrt[3]{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

For $k=2$*:*

$$w_2 = \sqrt[3]{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$

1.3 Inverse of a Complex Number

The inverse of a complex number $z = (a,b)$ is given by:

$$z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

This inverse is the symmetric of z with respect to the operation $\langle . \rangle$.

1.4 Field of Complex Numbers

The field thus defined is called the field of complex numbers, denoted by L. Consider the application $f : \mathbb{R} \to L$ defined by:

$$x \to (x, 0)$$

Then f is a field isomorphism. Indeed:

1. f is bijective (obvious!).
2. $f(x + x') = (x + x', 0) = (x, 0) + (x', 0) = f(x) + f(x')$.
3. $f(x \cdot x') = (x \cdot x', 0) = (x, 0) \cdot (x', 0) = f(x) \cdot f(x')$.

Therefore, we can identify $\mathbb{R}$ with a subfield of L consisting of elements $(a, 0)$ with $a \in \mathbb{R}$. Using this isomorphism, the complex number $(a, 0)$ will simply be denoted by a. We set $i = (0, 1)$, then we have:

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1$$

Consequently, any complex number $z = (a, b)$ can be written as:

$$z = (a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1)(b, 0) = a + ib, \quad a, b \in \mathbb{R}$$

Hence, the usual notation for a complex number, and we have the following relations:

1. $z = a + ib$, where $a = \mathrm{Re}(z)$ (real part of z) and $b = \mathrm{Im}(z)$ (imaginary part of z).
2. If $b = 0$, then z is a real number. If $a = 0$, then z is said to be purely imaginary.
3. If $z = a + ib$, then the conjugate of z is the complex number $\bar{z} = a - ib$.
4. $\overline{z + z'} = \bar{z} + \bar{z}'$.
5. The modulus of z is the positive real number $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$.
6. $|z \cdot z'| = |z| \cdot |z'|$ and $|z + z'| \leq |z| + |z'|$.

1.5 Representation of a Complex Number

1.5.1 Image of a Complex Number

To each complex number $z = a + ib$, we associate in the Oxy plane the point M with abscissa x and ordinate y ($M = (a, b)$), or the vector $\overrightarrow{OM}$. The point M is called the image of the complex number z, and z is said to be the affix of the point M.

The Oxy plane containing the images of the complex numbers $z = (a, b)$ is called the complex plane, where Ox is the real axis and Oy is the imaginary axis.

1.5.2 Trigonometric Form (Polar or Geometric Form)

The image M of the complex number $z = a + ib$ can also be determined by the measure θ of the angle $(\Omega x, \Omega M)$ and by the number r which measures the length of the vector $\overrightarrow{OM}$. For every complex number z, there exists a pair (r, θ) in $\mathbb{R}_+ \times \mathbb{R}$ such that:

$$z = r(\cos\theta + i\sin\theta)$$

1. For $z \neq 0$ we have: $r = |z|$, $\cos\theta = \frac{\mathrm{Re} z}{|z|}$, $\sin\theta = \frac{\mathrm{Im} z}{|z|}$, and the number θ is defined up to 2π.

2. For $z = 0$ we have $r = 0$ and θ arbitrary. The number θ is called the argument of z and there are infinitely many, and when $\theta \in [0, 2\pi]$ we write $\arg z$.

Proposition 1.27. *If z_1 and z_2 are two complex numbers such that: $\theta = \arg z$ and $\theta' = \arg z'$ then we have:*

1. $z = z' \Leftrightarrow |z| = |z'|$ and $\theta = \theta' + 2\pi k$
2. $z \cdot z' = |z| \cdot |z'|(\cos(\theta + \theta') + i\sin(\theta + \theta'))$
3. If $z' \neq 0$, $\frac{z}{z'} = \left|\frac{z}{z'}\right| (\cos(\theta - \theta') + i\sin(\theta - \theta'))$

1.5.3 Complex Exponential:

When $z = a + ib$ we define $e^z = e^{a+ib} = e^a \cdot e^{ib}$ with $e^{ib} = \cos b + i\sin b$, thus any complex number can be written as:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

this definition preserves the properties of the real exponential.

1.5.4 De Moivre's Formula:

Proposition 1.28. *For any real number θ and any integer n in $\mathbb{N}$ we have:*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof. 1. $n = 1$, $P(1) : (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$, which is true. 2. Assume $P(n) : (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ then we have:

$$(\cos\theta + i\sin\theta)^{n+1} = (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta)$$

$$= (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta)$$

$$= \cos n\theta\cos\theta - \sin n\theta\sin\theta + i(\sin\theta\cos n\theta + \cos\theta\sin n\theta)$$

$$= \cos(n+1)\theta + i\sin(n+1)\theta$$

hence $P(n + 1)$ is true. □

1.5.5 Application of De Moivre's Formula – n-th Roots of a Complex Number:

Definition 1.29. *We call the n-th root of z a complex number* $w = r(\cos\theta + i\sin\theta)$ *such that:* $w^n = z$ *and we write* $w = \sqrt[n]{z}$.

We have

$$w^n = z \Leftrightarrow (r(\cos\theta + i\sin\theta))^n = \rho(\cos x + i\sin x)$$
$$\Leftrightarrow r^n(\cos n\theta + i\sin n\theta) = \rho(\cos x + i\sin x)$$

by identification:

$$r^n = \rho \quad \text{and} \quad n\theta = x + 2\pi k \Rightarrow r = \sqrt[n]{\rho} \quad \text{and} \quad \theta = \frac{x + 2\pi k}{n}$$

then we obtain:

$$\sqrt[n]{r(\cos\theta + i\sin\theta)} = \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right), \quad k = 0, 1, 2, ..., n-1$$

Example 1.30. *Calculate* $\sqrt[3]{1-i}$

$$\sqrt[3]{1-i} = \sqrt[3]{\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)} = \sqrt[3]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= \sqrt[3]{2}\left(\cos\frac{\frac{3\pi}{4} + 2k\pi}{3} + i\sin\frac{\frac{3\pi}{4} + 2k\pi}{3}\right); \quad k = 0, 1, 2$$

$$= \sqrt[3]{2}\left(\cos\frac{(3+8k)\pi}{12} + i\sin\frac{(3+8k)\pi}{12}\right); \quad k = 0, 1, 2$$

For $k = 0$*:*

$$w_0 = \sqrt[3]{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt[3]{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

For $k = 1$*:*

$$w_1 = \sqrt[3]{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

For $k = 2$*:*

$$w_2 = \sqrt[3]{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$

Chapter 2

Numerical Sequences

Sequences, ordered lists of numbers or elements defined by a specific rule, form a cornerstone of mathematical analysis and its applications. This chapter explores foundational concepts, including types such as arithmetic, geometric, and recursive sequences, as well as the critical study of convergence where terms approach a finite limit. We examine pivotal theorems like the Monotone Convergence Theorem and the Cauchy Criterion, which establish rigorous conditions for convergence. These ideas not only underpin calculus and differential equations but also drive advancements in numerical methods and data science, highlighting their indispensable role in both theoretical and applied mathematics.

2.1 Definitions and properties

Let X be a subset of $\mathbb{R}$.

Definition 2.1. *A numerical sequence is any function* $f : \mathbb{N} \to X$ *where* $n \mapsto f(n) = x_n$.

Any numerical sequence can be written as $x_0, x_1, x_2, \ldots, x_n$, *where* x_n *is called the general term.*

Example 2.2. *The sequence with the general term* $x_n = \frac{(-1)^n}{n}$ *is written as:*

$$-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \ldots, \frac{(-1)^n}{n}$$

2.1.1 Operations on Sequences

Let (x_n) and (y_n) be two numerical sequences:

1. The sequences (x_n) and (y_n) are equal if $x_n = y_n$ for all $n \in \mathbb{N}$.
2. The sum of (x_n) and (y_n) is the sequence (z_n) such that $z_n = x_n + y_n$.
3. The product of (x_n) and (y_n) is the sequence (w_n) such that $w_n = x_n \cdot y_n$.

2.2 Bounded Sequences

Definition 2.3. *A sequence u_n is said to be:*

- ***Bounded above*** *if there exists $M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, u_n \leq M$.*
- ***Bounded below*** *if there exists $m \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, u_n \geq m$.*
- ***Bounded*** *if it is both bounded above and below, i.e., $\exists m, M \in \mathbb{R}, \forall n \in \mathbb{N}, m \leq u_n \leq M$ or equivalently $\exists c > 0, \forall n \in \mathbb{N}, |u_n| \leq c$.*

Additionally:

$$sup\ u_n = M \iff \begin{cases} i, & \forall n \geq 0, u_n \leq M \\ ii, & \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, M - \varepsilon < u_{n_0} \leq M \end{cases}$$

$$inf\ u_n = m \iff \begin{cases} i, & \forall n \geq 0, u_n \geq m \\ ii, & \forall \varepsilon > 0, \exists n_1 \in \mathbb{N}, m \leq u_{n_1} < m + \varepsilon \end{cases}$$

2.2.1 Limit of a Numerical Sequence

Definition 2.4. *The number $l \in \mathbb{R}$ is called the limit of the sequence u_n as n tends to $+\infty$ if for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_0$, $|u_n - l| < \varepsilon$. We write:*

$$\lim_{n\to\infty} u_n = l \iff (\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon), \forall n : (n \geq n_0 \Rightarrow |u_n - l| < \varepsilon))$$

Example 2.5. *Show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. Let $\varepsilon > 0$, we have:*

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon \iff \left|\frac{n-n-1}{1+n}\right| < \varepsilon \Rightarrow \frac{1}{1+n} < \varepsilon \Rightarrow 1 < \varepsilon + n\varepsilon \Rightarrow n > \frac{1}{\varepsilon} - 1$$

It suffices to take $n_0 = E\left(\frac{1}{\varepsilon} - 1\right) + 1$, and thus:

$$\forall \varepsilon > 0, \exists n_0 = E\left(\frac{1}{\varepsilon} - 1\right) + 1 \in \mathbb{N}, \forall n : n \geq n_0 \Rightarrow \left|\frac{n}{n+1} - 1\right| < \varepsilon \iff \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = 1$$

Infinite limits

Definition 2.6.

$$\lim_{n\to\infty} u_n = +\infty \iff (\forall A > 0, \exists n_0 = n_0(A), \forall n : n \geq n_0 \Rightarrow u_n > A)$$

$$\lim_{n\to\infty} u_n = -\infty \iff (\forall A > 0, \exists n_0 = n_0(A), \forall n : n \geq n_0 \Rightarrow u_n < -A)$$

Definition 2.7.

1. *A sequence (u_n) is called **infinitesimal** if $\lim_{n\to\infty} u_n = 0$.*
2. *A sequence (u_n) is called **infinitely large** if $\lim_{n\to\infty} u_n = +\infty$.*

Properties 2.8.

1. *The sum (resp. product) of a finite number of infinitesimal sequences is an infinitesimal sequence.*
2. *The product of a number and an infinitesimal sequence is an infinitesimal sequence.*

2.3 Convergent Sequences

Definition 2.9. *1. A sequence (u_n) is called **convergent** if there exists $l \in \mathbb{R}$ such that $\lim_{n\to\infty} u_n = l$.*

*2. A sequence (u_n) is called **divergent** if its limit is infinite or does not exist.*

2.3.1 Uniqueness of the Limit

Theorem 2.10. *The limit of any convergent sequence is unique.*

Proof. Let (u_n) be a convergent sequence with $\lim_{n\to\infty} u_n = l_1$ and $\lim_{n\to\infty} u_n = l_2$. Then:

$$\Big(\lim_{n\to\infty} u_n = l_1\Big) \iff \Big(\forall\varepsilon > 0, \exists n_1 = n_1(\varepsilon), \forall n : \Big(n \geq n_1 \implies |u_n - l_1| < \frac{\varepsilon}{2}\Big)\Big) \quad (1)$$

$$\Big(\lim_{n\to\infty} u_n = l_2\Big) \iff \Big(\forall\varepsilon > 0, \exists n_2 = n_2(\varepsilon), \forall n : \Big(n \geq n_2 \implies |u_n - l_2| < \frac{\varepsilon}{2}\Big)\Big) \quad (2)$$

Let $n_0 = \max(n_1, n_2)$. Then both (1) and (2) hold for $n \geq n_0$. This gives:

$$|l_1 - l_2| = |(l_1 - u_n) + (u_n - l_2)| \leq |l_1 - u_n| + |u_n - l_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, for $n \geq n_0$:

$$\forall\varepsilon > 0, |l_1 - l_2| < \varepsilon \implies l_1 - l_2 = 0 \implies l_1 = l_2 \quad \text{Q.E.D.}$$

□

Example 2.11. *The sequences $u_n = (-1)^n$, $v_n = \sin n$, and $w_n = \cos n$ are not convergent because they do not have unique limits.*

2.3.2 Boundedness of Convergent Sequences

Theorem 2.12. *Every convergent sequence is bounded.*

Proof. Let (u_n) be a convergent sequence with $\lim_{n\to\infty} u_n = l$. Then:

$$\Big(\lim_{n\to\infty} u_n = l\Big) \iff (\forall\varepsilon > 0, \exists n_0 = n_0(\varepsilon), \forall n : (n \geq n_0 \implies |u_n - l| < \varepsilon))$$

$$\iff \forall n, n \geq n_0, -\varepsilon < u_n - l < +\varepsilon$$

$$\iff \forall n, n \geq n_0, l - \varepsilon < u_n < l + \varepsilon$$

Thus, for $n \geq n_0$, all terms of (u_n) lie within the interval $[l - \varepsilon, l + \varepsilon]$. Outside this interval, there are only finitely many terms $u_0, u_1, \ldots, u_{n_0-1}$.

Let:

$$M = \max(u_0, u_1, \ldots, u_{n_0-1}, l + \varepsilon) \quad \text{and} \quad m = \min(u_0, u_1, \ldots, u_{n_0-1}, l - \varepsilon)$$

Then:

$$\forall n \geq 0, m \leq u_n \leq M \implies (u_n) \text{ is bounded.}$$

□

Remark 2.13. *The converse of this theorem is not true.*

Example 2.14. *The sequence $u_n = (-1)^n$ is bounded but not convergent because its limit is not unique.*

2.3.3 Sequences of the Same Nature

Two sequences are of the same nature if the convergence of one implies the convergence of the other, and similarly for divergence.

Proposition 2.15. *Passage to the Limit in Inequalities Let u_n and v_n be two convergent sequences with $\lim_{n\to\infty} u_n = l_1$ and $\lim_{n\to\infty} v_n = l_2$.*

If $l_1 < l_2$, then from a certain rank onwards, $u_n < v_n$.

Proof. Let l be a number such that $l_1 < l < l_2$.

$$\lim_{n\to\infty} u_n = l_1 \iff \forall \varepsilon > 0, \exists n_1 = n_1(\varepsilon), \quad \forall n : n \geq n_1 \implies |u_n - l_1| < \varepsilon$$

For $\varepsilon = l - l_1$, we obtain:

$$n \geq n_1 \implies -\varepsilon < u_n - l_1 < +\varepsilon \implies -(l - l_1) < u_n - l_1 < l - l_1 \implies u_n < l$$

Similarly,

$$\lim_{n\to\infty} v_n = l_2 \iff \forall \varepsilon > 0, \exists n_2 = n_2(\varepsilon), \quad \forall n : n \geq n_2 \implies |v_n - l_2| < \varepsilon$$

For $\varepsilon = l_2 - l$, we obtain:

$$n \geq n_2 \implies -\varepsilon < v_n - l_2 < +\varepsilon \implies -(l_2 - l) < v_n - l_2 < l_2 - l \implies v_n > l$$

Let $n_0 = \max(n_1, n_2)$. Then for $n \geq n_0$:

$$u_n < l \quad \text{and} \quad l < v_n \quad \text{thus} \quad n \geq n_0 \implies u_n < v_n.$$

□

Corollary 2.16.

1. *If $u_n < v_n$ from a certain rank onwards, then $l_1 = \lim_{n\to\infty} u_n \leq l_2 = \lim_{n\to\infty} v_n$.*
2. *If $u_n < a$ from a certain rank onwards, then $l_1 \leq a$. In particular, if $a = 0$, we deduce that any positive sequence cannot have a negative limit.*

Proof. By contradiction; if $l_1 > l_2$, then there exists n_0 such that for $n \geq n_0$, $u_n > v_n$, which is a contradiction. □

2.3.4 Theorem of Three Sequences

Theorem 2.17. *Let* (u_n), (v_n), *and* (w_n) *be three sequences such that from a certain rank onwards,* $v_n \leq u_n \leq w_n$. *If* $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = l$, *then* $\lim_{n\to\infty} u_n = l$.

Proof.

$$\lim_{n\to\infty} v_n = l \iff (\forall \varepsilon > 0, \exists n_1 = n_1(\varepsilon), \quad \forall n : (n \geq n_1 \implies |v_n - l| < \varepsilon)) \quad (1)$$

$$\lim_{n\to\infty} w_n = l \iff (\forall \varepsilon > 0, \exists n_2 = n_2(\varepsilon), \quad \forall n : (n \geq n_2 \implies |w_n - l| < \varepsilon)) \quad (2)$$

Let $n_0 = \max(n_1, n_2)$. Then for $n \geq n_0$, both (1) and (2) hold. That is:

$$n \geq n_0 \implies l - \varepsilon < v_n < l + \varepsilon \quad \text{and} \quad l - \varepsilon < w_n < l + \varepsilon$$

Since $v_n \leq u_n \leq w_n$, it follows that:

$$n \geq n_0 \implies l - \varepsilon < u_n < l + \varepsilon \implies \lim_{n\to\infty} u_n = l$$

□

Theorem 2.18. *Let* (u_n), (v_n), *and* (w_n) *be three sequences such that from a certain rank onwards,* $v_n \leq u_n \leq w_n$.

If $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = l$, *then* $\lim_{n\to\infty} u_n = l$.

Proof.

$$l - \varepsilon < v_n \leq u_n \leq w_n < l + \varepsilon$$

$$\Rightarrow l - \varepsilon < u_n < l + \varepsilon \iff \lim_{n\to\infty} u_n = l$$

□

Example 2.19. *Calculate* $\lim_{n\to\infty} \frac{\sin n}{n}$.

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \Rightarrow 0 = \lim_{n\to\infty} \frac{-1}{n} \leq \lim_{n\to\infty} \frac{\sin n}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{\sin n}{n} = 0$$

2.3.5 Arithmetic Operations on Limits

Theorem 2.20. *Let* (u_n) *and* (v_n) *be two convergent sequences. Then the sequences* $(u_n + v_n)$, $(u_n - v_n)$, $(u_n \cdot v_n)$, (λu_n), *and* $\left(\frac{u_n}{v_n}\right)$ *(with* $v_n \neq 0$*) are also convergent, and:*

1. $\lim_{n\to\infty}(u_n \pm v_n) = \lim_{n\to\infty} u_n \pm \lim_{n\to\infty} v_n$
2. $\lim_{n\to\infty}(u_n \cdot v_n) = \lim_{n\to\infty} u_n \cdot \lim_{n\to\infty} v_n$
3. $\lim_{n\to\infty}(\lambda u_n) = \lambda \cdot \lim_{n\to\infty} u_n$
4. $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\lim_{n\to\infty} u_n}{\lim_{n\to\infty} v_n}$ *if* $\lim_{n\to\infty} v_n \neq 0$
5. $\lim_{n\to\infty} |u_n| = |\lim_{n\to\infty} u_n|$

To prove this theorem, we first demonstrate the following lemma:

Lemma 2.21. *For* $\lim_{n\to\infty} u_n = l$*, it is necessary and sufficient that* $u_n = l + \alpha_n$ *where* $\lim_{n\to\infty} \alpha_n = 0$.

Proof. **Necessary Condition:** Assume $\lim_{n\to\infty} u_n = l$ and show that $u_n = l + \alpha_n$ with $\lim_{n\to\infty} \alpha_n = 0$.

$$\lim_{n\to\infty} u_n = l \iff \forall\varepsilon > 0, \exists n_0 = n_0(\varepsilon), \forall n : n \geq n_0 \Rightarrow |u_n - l| < \varepsilon \quad (1)$$

Let $u_n - l = \alpha_n$. Then:

$$(1) \Rightarrow \forall n; \quad (n \geq n_0 \Rightarrow |\alpha_n| < \varepsilon) \iff \lim_{n\to\infty} \alpha_n = 0$$

Thus, $u_n = l + \alpha_n$ and $\lim_{n\to\infty} \alpha_n = 0$.

Sufficient Condition:

Assume $u_n = l + \alpha_n$ with $\lim_{n\to\infty} \alpha_n = 0$ and show that $\lim_{n\to\infty} u_n = l$.

$$\lim_{n\to\infty} \alpha_n = 0 \iff \forall\varepsilon > 0, \exists n_0 = n_0(\varepsilon), \forall n : n \geq n_0 \Rightarrow |\alpha_n| < \varepsilon \Rightarrow n \geq n_0; |u_n - l| < \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} u_n = l$$

□

Proof. Proof of the Theorem We want to prove:

$$\lim_{n\to\infty} (u_n v_n) = \lim_{n\to\infty} u_n \cdot \lim_{n\to\infty} v_n$$

Given:

$$\lim_{n\to\infty} u_n = l_1 \iff u_n = l_1 + \alpha_n \text{ where } \lim_{n\to\infty} \alpha_n = 0$$
$$\lim_{n\to\infty} v_n = l_2 \iff v_n = l_2 + \beta_n \text{ where } \lim_{n\to\infty} \beta_n = 0$$

Then:

$$u_n v_n = (l_1 + \alpha_n)(l_2 + \beta_n) = l_1 l_2 + l_1 \beta_n + \alpha_n l_2 + \alpha_n \beta_n$$

where $\lim_{n\to\infty}(l_1 \beta_n + \alpha_n l_2 + \alpha_n \beta_n) = 0$.

Thus:

$$\lim_{n\to\infty} (u_n v_n) = l_1 l_2 = \lim_{n\to\infty} u_n \cdot \lim_{n\to\infty} v_n$$

□

Example 2.22. *Calculate:*

$$\lim_{n\to\infty} \frac{7n^2 + 15n + 13}{4n^2 + 10n + 23} \left(\frac{\infty}{\infty}\right)$$

$$\lim_{n\to\infty} \frac{7n^2 + 15n + 13}{4n^2 + 10n + 23} = \lim_{n\to\infty} \frac{n^2\left(7 + \frac{15}{n} + \frac{13}{n^2}\right)}{n^2\left(4 + \frac{10}{n} + \frac{23}{n^2}\right)} = \lim_{n\to\infty} \frac{7 + \frac{15}{n} + \frac{13}{n^2}}{4 + \frac{10}{n} + \frac{23}{n^2}} = \frac{7}{4}$$

2.3.6 Convergence Criteria for Monotone Sequences

Definition 2.23. *A sequence (u_n) is called:*

- ***Increasing*** *if* $\forall n \geq 1$, $u_n \leq u_{n+1}$.
- ***Decreasing*** *if* $\forall n \geq 1$, $u_n \geq u_{n+1}$.

Increasing and decreasing sequences are called ***monotone****.*

Theorem 2.24. *Monotonically Convergence Theorem MCT Every increasing sequence that is bounded above (resp. decreasing sequence that is bounded below) is convergent. Moreover:*

$$\lim_{n\to\infty} u_n = \sup u_n \quad (\textit{resp.} \lim_{n\to\infty} u_n = \inf u_n)$$

Proof. Let (u_n) be an increasing sequence bounded above. Then $\exists \sup u_n = M_0$.

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, u_{n_0} > M_0 - \epsilon$$

Since (u_n) is increasing:

$$\forall n \geq n_0, M_0 - \epsilon < u_n \leq M_0 < M_0 + \epsilon$$

Thus:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |u_n - M_0| < \epsilon \iff \lim_{n\to\infty} u_n = M_0 = \sup u_n$$

□

Notable Limits

1. $$\lim_{n\to\infty} \sqrt[n]{a} = 1 \quad (a > 0)$$

2. $$\lim_{n\to\infty} \sqrt[n]{n^p} = 1 \quad (\forall p \in \mathbb{N})$$

3. $$\lim_{n\to\infty} q^n = \begin{cases} 0 & \text{if } |q| < 1 \\ 1 & \text{if } q = 1 \\ \text{does not exist if } q = -1 & \\ \infty & \text{if } |q| > 1 \end{cases}$$

4. $$\lim_{n\to\infty} \frac{n^k}{a^n} = 0 \quad (k \in \mathbb{N}, a > 1)$$

5. $$\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a \in \mathbb{R})$$

6.

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

7.

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e, \quad (2 < e < 3)$$

Proof. **1.** $\lim_{n\to\infty} \sqrt[n]{a} = 1$ **for** $a > 0$

Let $a > 1$. Then:

$$\sqrt[n]{a} = 1+\alpha_n, \quad \alpha_n > 0 \iff a = (1+\alpha_n)^n = 1+n\alpha_n+\frac{n(n-1)}{2}\alpha_n^2+\cdots+C_n^n\alpha_n^n > 1+n\alpha_n$$

$$\iff a-1 > n\alpha_n \Rightarrow 0 < \alpha_n < \frac{a-1}{n} \Rightarrow \lim_{n\to\infty} \alpha_n = 0$$

This shows that:

$$\lim_{n\to\infty} \sqrt[n]{a} = 1$$

For $0 < a < 1$, let $a = \frac{1}{b}$ where $b > 1$. Then:

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{b}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{b}} = \frac{1}{1} = 1$$

3. $\lim_{n\to\infty} q^n$ **for** $|q| < 1$

Let $|q| < 1$. We want to show:

$$\lim_{n\to\infty} q^n = 0 \iff \forall\epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n : n \geq n_0 \implies |q|^n < \epsilon$$

$$\implies n\log|q| < \log\epsilon \implies n > \frac{\log\epsilon}{\log|q|}$$

It suffices to take:

$$n_0 = E\left(\frac{\log\epsilon}{\log|q|}\right) + 1$$

If $q = 1$, then $q^n = 1$ and $\lim_{n\to\infty} q^n = 1$.
If $q = -1$, then $\lim_{n\to\infty}(-1)^n$ does not exist.
If $|q| > 1$, let $|q| = \frac{1}{p}$ where $0 < p < 1$. Then:

$$\lim_{n\to\infty} |q|^n = \lim_{n\to\infty} \frac{1}{p^n} = \frac{1}{\lim_{n\to\infty} p^n} = \frac{1}{0} = \infty$$

Limit of $\left(1+\frac{1}{n}\right)^n$ We want to prove that:

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e, \quad 2 < e < 3$$

Proof that $u_n = \left(1 + \frac{1}{n}\right)^n$ is Bounded Consider the sequence $u_n = \left(1 + \frac{1}{n}\right)^n$. Expanding u_n using the binomial theorem:

$$u_n = 1 + C_1^1 \cdot \frac{1}{n} + C_2^2 \cdot \frac{1}{n^2} + C_3^3 \cdot \frac{1}{n^3} + \cdots + C_n^n \cdot \frac{1}{n^n}$$

$$= 2 + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \cdots + \frac{n(n-1)(n-2)\dots(n-(n-1))}{n!} \cdot \frac{1}{n^n}$$

$$= 2+\frac{1}{2}\left(1-\frac{1}{n}\right)+\frac{1}{3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-(n-1)}{n}\right)$$

This shows that $u_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} > u_n$, so u_n is increasing.

It is clear that $u_n > 2$. Now, we show that $u_n < 3$ for all n:

$$1 - \frac{1}{n} < 1, \quad 1 - \frac{2}{n} < 1, \quad \dots, \quad 1 - \frac{n-1}{n} < 1$$

Thus:

$$u_n < 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \quad (\text{since } n! > 2^{n-1})$$

$$< 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots = 2 + \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 3$$

Therefore, $2 < u_n < 3$. Since u_n is increasing and bounded, it converges. Euler proposed that:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e, \quad 2 < e < 3$$

Generalization Let $n = -m$. Then as $n \to -\infty$, $m \to +\infty$. Thus:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{m\to+\infty} \left(1 - \frac{1}{m}\right)^{-m} = \lim_{m\to+\infty} \left(\frac{m-1}{m}\right)^{-m} = \lim_{m\to+\infty} \left(\frac{m}{m-1}\right)^m$$

$$= \lim_{m\to+\infty} \left(1 + \frac{1}{m-1}\right)^m = \lim_{m\to+\infty} \left(1 + \frac{1}{m-1}\right)^{m-1} \left(1 + \frac{1}{m-1}\right) = e \cdot 1 = e$$

In general, if $\lim_{n\to\infty} p_n = \pm\infty$, then:

$$\lim_{n\to\infty} \left(1 + \frac{1}{p_n}\right)^{p_n} = e$$

More generally, if:

$$\lim_{n\to\infty} p_n = \pm\infty \quad \text{and} \quad \lim_{n\to\infty} q_n = l, \quad \text{then} \quad \lim_{n\to\infty} \left(1 + \frac{1}{p_n}\right)^{p_n q_n} = e^l$$

□

Example 2.25. *Calculate:*

$$\lim_{n\to\infty}\left(\frac{3n^2+5n+1}{3n^2+3n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{3n^2+5n+1}{3n^2+3n+1}-1\right)^n = \lim_{n\to\infty}\left(1+\frac{2n}{3n^2+3n+1}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\frac{3n^2+3n+1}{2n}}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{3n^2+3n+1}{2n}}\right)^{\frac{3n^2+3n+1}{2n}\cdot\frac{2n}{3n^2+3n+1}\cdot n}$$

$$= e^{\lim_{n\to\infty}\frac{2n^2}{3n^2+3n+1}} = e^{\frac{2}{3}}$$

2.3.7 Convergence of Adjacent Sequences

Definition 2.26. *Two sequences (u_n) and (v_n) are called **adjacent** if one is increasing, the other is decreasing, and $\lim_{n\to\infty}(u_n - v_n) = 0$.*

Theorem 2.27. *Adjacent sequences converge to the same limit.*

Proof. Assume (u_n) is increasing and (v_n) is decreasing, with $\lim_{n\to\infty}(u_n - v_n) = 0$. We need to show that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$.

First, show that $(v_n - u_n)$ is decreasing:

$$(v_{n+1} - u_{n+1}) - (v_n - u_n) = (v_{n+1} - v_n) - (u_{n+1} - u_n) < 0$$

Thus, $(v_n - u_n)$ is decreasing.

Moreover:

$$\lim_{n\to\infty}(v_n - u_n) = 0 \Rightarrow \inf(v_n - u_n) = 0 \Rightarrow v_n - u_n \geq 0 \quad \forall n \Rightarrow v_n \geq u_n \quad \forall n.$$

Since (u_n) is increasing and bounded above by v_1, and (v_n) is decreasing and bounded below by u_1, both sequences are convergent. Therefore:

$$0 = \lim_{n\to\infty}(v_n - u_n) = \lim_{n\to\infty} v_n - \lim_{n\to\infty} u_n \Rightarrow \lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n$$

□

2.3.8 Application: Theorem of Nested Intervals

Theorem 2.28. *Let $I_n = [u_n, v_n]$ be a sequence of intervals satisfying:*

1. *$I_{n+1} \subset I_n$ for all $n \geq 1$.*
2. *$\lim_{n\to\infty}(v_n - u_n) = 0$.*

Then there exists a unique point c belonging to all intervals I_n, i.e., $\bigcap_{n=1}^{\infty} I_n = \{c\}$.

Proof. Since (u_n) is increasing and (v_n) is decreasing, and $\lim_{n\to\infty}(v_n - u_n) = 0$, we have:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = c$$

Thus, $u_n \leq c \leq v_n$ for all $n \geq 1$.

To show that c is unique, assume by contradiction that there exists $c' \neq c$ such that $\bigcap_{n=1}^{\infty} I_n = \{c'\}$. This leads to a contradiction, proving the uniqueness of c. □

2.3.9 Cauchy Sequence and Cauchy Criterion for Convergence

Definition 2.29. *A sequence (u_n) is called a* ***Cauchy sequence*** *if:*

$$\forall \varepsilon > 0, \quad \exists n_0 = n_0(\varepsilon), \quad \forall p, q; \quad (p \geq n_0, q \geq n_0 \Rightarrow |u_p - u_q| < \varepsilon)$$

Remark 2.30. *A sequence (u_n) is* ***not*** *a Cauchy sequence if:*

$$\exists \varepsilon > 0, \quad \forall n_0 = n_0(\varepsilon), \quad \exists p, q; \quad (p \geq n_0, q \geq n_0 \wedge |u_p - u_q| \geq \varepsilon)$$

Theorem 2.31. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume (u_n) converges. We need to show that (u_n) is a Cauchy sequence.

$$(u_n) \text{ converges } \Rightarrow \lim_{n\to\infty} u_n = l \iff \forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon), \forall p; p \geq n_0 \Rightarrow |u_p - l| < \frac{\varepsilon}{2}$$

For $q \geq n_0$, we have:

$$|u_q - l| < \frac{\varepsilon}{2}$$

Thus:

$$|u_p - u_q| = |(u_p - l) + (l - u_q)| \leq |u_p - l| + |u_q - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, (u_n) is a Cauchy sequence. □

Example 2.32. *Consider the sequence:*

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

We study its nature.
Let p, q be two integers with $p \geq q$. Then:

$$|u_p - u_q| = \left|\frac{1}{q+1} + \frac{1}{q+2} + \cdots + \frac{1}{p}\right|$$

$$\frac{1}{q+1} + \frac{1}{q+2} + \cdots + \frac{1}{p} > \frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p} = \frac{p-q}{p}$$

For $p = 2q$, we have:

$$|u_p - u_q| = |u_{2q} - u_q| > \frac{2q - q}{2q} = \frac{1}{2}$$

Thus:

$$\exists \varepsilon = \frac{1}{2}, \forall n_0 \in \mathbb{N}, \exists p, q(p = 2q), \text{ such that } p > n_0, q > n_0 \text{ and } |u_p - u_q| \geq \varepsilon$$

Therefore, the sequence (u_n) is not a Cauchy sequence and hence not convergent.

2.3.10 Subsequences, Extracted Sequences

Definition 2.33. *Let (u_n) be a sequence, and let $n \to n_k$ be a strictly increasing sequence of natural numbers. Then the sequence (u_{n_k}) is called a* ***subsequence*** *or* ***extracted sequence*** *of (u_n).*

Example 2.34. *Consider the sequence defined by $u_n = \frac{1}{n}$ for $n \geq 1$:*

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$$

The following are subsequences of (u_n):

$$u_{2k} : \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots, \frac{1}{2k}$$

$$u_{2^k} : \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{2^k}$$

$$u_{2k+1} : 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots, \frac{1}{2k-1}$$

$$u_{3k} : \frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \dots, \frac{1}{3k}$$

Theorem 2.35. *Every subsequence of a convergent sequence converges to the same limit.*

Proof. Assume (u_n) converges to l:

$$\lim_{n\to\infty} u_n = l \iff \forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon), \forall n : n \geq n_0 \implies |u_n - l| < \varepsilon$$

Let (u_{n_k}) be a subsequence of (u_n). We need to show that $\lim_{k\to\infty} u_{n_k} = l$.

Since $\lim_{k\to\infty} n_k = \infty$, for any n_0, there exists $q \in \mathbb{N}$ such that for all $k \geq q$, $n_k > n_0$. Thus:

$$n_k > n_0 \implies |u_{n_k} - l| < \varepsilon \implies \lim_{k\to\infty} u_{n_k} = l$$

□

2.3.11 Recurrent Sequences

Definition 2.36. *A sequence (u_n) is called* ***recurrent*** *if:*

1. *u_1 is given.*
2. *There exists a function $f : X \to \mathbb{R}$ (where $X \subset \mathbb{R}$) such that $f(X) \subset X$ and $u_{n+1} = f(u_n)$.*

We write:

$$u_n = \begin{cases} u_1 & \textit{given} \\ u_{n+1} = f(u_n) \end{cases}$$

Proposition 2.37. *Monotonicity for Recurrent Sequences*

1. *If the function f is increasing, then:*
   - $f(u_1) - u_1 \geq 0$ *implies that the sequence* (u_n) *is increasing.*
   - $f(u_1) - u_1 \leq 0$ *implies that the sequence* (u_n) *is decreasing.*
2. *If f is decreasing, then the two subsequences* u_{2n} *and* u_{2n+1} *are monotonic in opposite directions.*

Proof. Let f be an increasing function, and assume $f(u_1) - u_1 \geq 0$. Then:

$$f(u_1) - u_1 \geq 0 \Rightarrow u_2 - u_1 \geq 0 \Rightarrow u_2 \geq u_1$$

Since f is increasing:

$$f(u_2) - f(u_1) = u_3 - u_2 \geq 0 \Rightarrow u_3 \geq u_2$$

Assume $u_n \geq u_{n-1}$. We show that $u_{n+1} \geq u_n$:

$$u_n \geq u_{n-1} \Rightarrow f(u_n) \geq f(u_{n-1}) \Rightarrow u_{n+1} \geq u_n$$

Thus, the sequence (u_n) is increasing. □

Theorem 2.38. *If* (u_n) *is a recurrent sequence defined by a continuous function* f *on* X*, and* (u_n) *converges to* l*, then* l *is a solution to the equation* $f(l) = l$*.*

Proof. This follows directly from the definition of a continuous function. □

Example 2.39. *Study the nature of the recurrent sequence defined by:*

$$\begin{cases} u_1 = 0 \\ u_{n+1} = \sqrt{6 + u_n}; \quad n \geq 2 \end{cases}$$

We have:

$$u_1 = 0, \quad u_2 = \sqrt{6} \Rightarrow u_1 < u_2$$

Assume $u_{n-1} < u_n$*. We show that* $u_n < u_{n+1}$*:*

$$u_{n+1} - u_n = \sqrt{6 + u_n} - \sqrt{6 + u_{n-1}} = \frac{u_n - u_{n-1}}{\sqrt{6 + u_n} + \sqrt{6 + u_{n-1}}} > 0$$

Thus, (u_n) *is increasing. Moreover, assume* $u_n < 3$*. Then:*

$$u_{n+1} - 3 = \sqrt{6 + u_n} - 3 = \frac{u_n - 3}{3 + \sqrt{6 + u_n}} < 0 \Rightarrow u_{n+1} < 3$$

Since (u_n) *is increasing and bounded above, it converges. Let* l *be its limit. By the previous theorem,* l *satisfies:*

$$f(l) = l \Leftrightarrow l = \sqrt{6 + l} \Leftrightarrow l^2 - l - 6 = 0 \Rightarrow \begin{cases} l = 3 \\ l = -2 \ \textit{(excluded since } l > 0) \Rightarrow l = 3 \end{cases}$$

2.3.12 Limit Superior and Limit Inferior

Definition 2.40. *A number x_0 is called a **limit point** (or **adherent value**) of the sequence (u_n) if there exists a subsequence (u_{n_k}) of (u_n) that converges to x_0.*
The set of all limit points of (u_n) is denoted by $Ad(u_n)$.
*The **limit superior** (resp. **limit inferior**) of a sequence (u_n) is the supremum (resp. infimum) of the set of adherent values $Ad(u_n)$. We write:*

$$\limsup_{n\to\infty} u_n = \sup Ad(u_n), \quad \liminf_{n\to\infty} u_n = \inf Ad(u_n)$$

Example 2.41. *Consider the sequence $(u_n) = (-1)^n$. We have:*

$$u_{2k} \xrightarrow{k\to\infty} 1 \quad \textit{and} \quad u_{2k+1} \xrightarrow{k\to\infty} -1$$

Thus, -1 and $+1$ are the adherent values of the sequence (u_n), and they are the only ones. Therefore:

$$Ad(u_n) = \{-1, +1\}$$

$$\limsup_{n\to\infty} u_n = +1, \quad \liminf_{n\to\infty} u_n = -1$$

Theorem 2.42. *For a sequence (u_n) to be convergent, it is necessary and sufficient that:*

$$\limsup_{n\to\infty} u_n = \liminf_{n\to\infty} u_n$$

Proof. If $\limsup_{n\to\infty} u_n = \liminf_{n\to\infty} u_n = l$, then all subsequences of (u_n) converge to l, implying that (u_n) itself converges to l. Conversely, if (u_n) converges to l, then all subsequences converge to l, so $\limsup_{n\to\infty} u_n = \liminf_{n\to\infty} u_n = l$. □

Chapter 3

Real Functions

Real-valued functions form the cornerstone of mathematical analysis, bridging abstract theory with practical applications across science, engineering, and economics. Defined as mappings from a subset of the real numbers to another, these functions assign a unique output $f(x)$ to each input (x) in their domain. From simple linear relationships like $f(x) = 2x+1$ to complex transcendental expressions, they encapsulate phenomena such as growth, motion, and optimization. This chapter explores their fundamental properties (domain, range, continuity, and differentiability), while examining graphical interpretations and algebraic manipulations. By analyzing these functions, we unlock tools to model real-world systems, solve equations, and understand the behavior of dynamic processes through the lens of calculus and beyond.

3.1 Definitions and properties

Definition 3.1. *A **real-valued function** defined on a domain X is any mapping f that assigns to each point x in X a unique element y in $\mathbb{R}$. We write:*

$$f : X \to \mathbb{R}$$

$$x \mapsto y = f(x)$$

Here, X is the domain of definition of f, and the set of values of f (or the image of f) is:

$$f(X) = Im\ f = \{y \in \mathbb{R} \mid \exists x \in X; y = f(x)\}$$

Definition 3.2. *Graph of a Function The **graph** of a function f is the set of points $M(x, y)$ where $x \in X$ and $y = f(x)$. We write:*

$$G_f = \{M(x, y) \mid x \in X \text{ and } y = f(x)\}$$

Operations on Real-Valued Functions

Let $f, g : X \to \mathbb{R}$.

Equality and Inequality

1. f is equal to g, written $f = g$, if:
$$f(x) = g(x) \quad \forall x \in X$$
2. f is less than or equal to g, written $f \leq g$, if:
$$f(x) \leq g(x) \quad \forall x \in X$$
3. f is greater than or equal to g, written $f \geq g$, if:
$$f(x) \geq g(x) \quad \forall x \in X$$

Arithmetic Operations

- Sum: $(f+g)(x) = f(x) + g(x)$, $\forall x \in X$
- Difference: $(f-g)(x) = f(x) - g(x)$, $\forall x \in X$
- Product: $(f \cdot g)(x) = f(x)g(x)$, $\forall x \in X$
- Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $\forall x \in X$, $g(x) \neq 0$

Composition of Functions

Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ such that $f(X) \subset Y$. The **composition** of f and g, denoted $g \circ f$, is defined on X by:
$$(g \circ f)(x) = g(f(x)); \quad \forall x \in X$$

Example 3.3. *Let* $f(x) = \cos x$ *and* $g(x) = x^2$*, where* $x \in \mathbb{R}$*. Then:*
$$(f \circ g)(x) = \cos(g(x)) = \cos(x^2)$$
$$(g \circ f)(x) = g(f(x)) = (f(x))^2 = \cos^2 x$$
Clearly, $g \circ f \neq f \circ g$*.*

General Properties of Functions

Even and Odd Functions

A set $X \subset \mathbb{R}$ is called **symmetric with respect to the origin** if:
$$\forall x \in X \Rightarrow -x \in X$$

Definition 3.4. *A function* f *defined on a symmetric set* X *is:*

1. ***Even*** *if:*
$$\forall x \in X, f(-x) = f(x)$$
2. ***Odd*** *if:*
$$\forall x \in X, f(-x) = -f(x)$$

Periodicity

Definition 3.5. *Let $f : X \to \mathbb{R}$. The function f is called* ***periodic*** *if there exists $\alpha \in \mathbb{R}_+$ such that:*

1. $x + \alpha \in X$
2. $f(x + \alpha) = f(x), \forall x \in X$

It is evident that:

$$f(x + k\alpha) = f(x)$$

Definition 3.6. *The* ***period*** *of f is the smallest positive number T such that:*

$$f(x + T) = f(x)$$

Monotonicity

Definition 3.7. *Let $f : X \to \mathbb{R}$. The function f is called:*

1. ***Increasing*** *if:*
$$\forall x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$
2. ***Strictly increasing*** *if:*
$$\forall x_1, x_2 \in X, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$
3. ***Decreasing*** *if:*
$$\forall x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$
4. ***Strictly decreasing*** *if:*
$$\forall x_1, x_2 \in X, x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Bounded Functions

Definition 3.8. *A function f is called:*

1. ***Bounded above*** *on X if there exists $M \in \mathbb{R}$ such that:*
$$f(x) \leq M, \forall x \in X$$
2. ***Bounded below*** *on X if there exists $m \in \mathbb{R}$ such that:*
$$f(x) \geq m, \forall x \in X$$
3. ***Bounded*** *on X if it is both bounded above and below, i.e., there exist $m, M \in \mathbb{R}$ such that:*
$$m \leq f(x) \leq M, \forall x \in X$$
or equivalently, there exists $c \in \mathbb{R}_+$ such that:
$$|f(x)| \leq c, \forall x \in X$$
4. ***Unbounded*** *if for every $c \in \mathbb{R}_+$, there exists $x' \in X$ such that:*
$$|f(x')| > c$$

Supremum and Infimum of Functions

Definition 3.9. *The* ***supremum*** *(resp.* ***infimum****) of f on X is the smallest upper bound (resp. largest lower bound) of f, denoted by:*

$$\sup f \quad (\textit{resp. } \inf f)$$

For a function $f : X \to \mathbb{R}$:

$$M = \sup_{x\in X} f(x) \iff \begin{cases} 1/\forall x \in X, f(x) \leq M \\ 2/\forall \varepsilon > 0, \exists x_0 \in X, f(x) > M - \varepsilon \end{cases}$$

$$m = \inf_{x\in X} f(x) \iff \begin{cases} 1/\forall x \in X, f(x) \geq m \\ 2/\forall \varepsilon > 0, \exists x_1 \in X, f(x) < m + \varepsilon \end{cases}$$

Theorem 3.10. *Every function that is bounded above (resp. below) admits a supremum (resp. infimum).*

Maximum and Minimum of a Function

Let $f : X \to \mathbb{R}$.

Definition 3.11. *The function f is said to have a* ***maximum*** *(resp.* ***minimum****) at a point $x_0 \in X$ if:*

$$\forall x \in X, f(x) \leq f(x_0) \quad (\textit{resp. } f(x) \geq f(x_0))$$

Inverse Functions

Definition 3.12. *Let $f : X \to \mathbb{R}$. The function f is called:*

1. ***Injective*** *if:*

$$\forall x_1, x_2 \in X, (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)) \quad \textit{or} \quad (x_1 = x_2 \Rightarrow f(x_1) = f(x_2))$$

2. ***Surjective*** *if:*

$$\forall y \in \mathbb{R}, \exists x \in X, y = f(x)$$

3. ***Bijective*** *if it is both injective and surjective.*

Definition 3.13. *A function $f : X \to Y$ is called* ***invertible*** *if there exists a function $g : Y \to X$ such that:*

$$(g \circ f)(x) = x \quad \textit{and} \quad (f \circ g)(y) = y$$

The function g is called the ***inverse*** *of f and is denoted by $g = f^{-1}$. We have:*

$$y = f(x) \iff x = f^{-1}(y)$$

Properties 3.14. *Let $f : X \to Y$ be invertible (bijective). Then:*

1. *The inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.*
2. *If f is odd (resp. even), then f^{-1} is also odd (resp. even).*
3. *If f is strictly monotone, then f^{-1} is also strictly monotone.*

Graph of an Inverse Function

Proposition 3.15. *Let G_f be the graph of an invertible function f, and let $G_{f^{-1}} = \{(y, f^{-1}(y)) \mid y \in Y\}$ be the graph of f^{-1}. In the Cartesian coordinate system Oxy, we have:*

$$(y,x) \in G_{f^{-1}} \iff x = f^{-1}(y), y \in Y \iff y = f(x), x \in X \iff (x,y) \in G_f$$

Thus, $G_{f^{-1}}$ is symmetric to G_f with respect to the first bisector $y = x$.

Elementary Functions

The following functions are called **elementary functions**:

1. **Constant functions**: $f(x) = c, \forall x \in X$

Power Functions

The **power function** is defined as:

$$f(x) = x^{\alpha}, \quad \alpha \in \mathbb{R}$$

The domain D_f of the power function depends on the value of α:

$$\begin{cases} \alpha = n : & D_f = \mathbb{R} \\ \alpha = -n : & D_f = \mathbb{R}^* \\ \alpha = \frac{p}{q} : & f(x) = x^{\frac{p}{q}} = \sqrt[q]{x^p}; \\ D_f = \begin{cases} \mathbb{R}, & \text{if } q \text{ is odd} \\ \mathbb{R}_+, & \text{if } q \text{ is even} \end{cases} \end{cases}$$

Exponential Functions

The **exponential function** with base a ($a > 0$, $a \neq 1$) is defined as:

$$f(x) = a^x, \quad D_f = \mathbb{R}, \quad \text{Im } f = \mathbb{R}_+$$

The function f is:

- **Increasing** if $a > 1$
- **Decreasing** if $0 < a < 1$

Additionally, $f(0) = 1$.

Logarithmic Functions

The **logarithmic function** with base a ($a > 0$, $a \neq 1$) is defined as:

$$y = \log_a x \iff x = a^y, \quad D_f = \mathbb{R}^*, \quad \operatorname{Im} f = \mathbb{R}$$

The logarithmic function satisfies:

$$x = 1 \Rightarrow y = 0$$

The function $y = \log_a x$ is:

- **Increasing** if $a > 1$
- **Decreasing** if $0 < a < 1$

The graph of $y = \log_a x$ is symmetric to the graph of $y = a^x$ with respect to the first bisector $y = x$.

Trigonometric Functions

The **trigonometric functions** include:

- Sine (sin)
- Cosine (cos)
- Tangent (tan)
- Cotangent (cot)

Inverse Trigonometric Functions

The **inverse trigonometric functions** include:

- Arcsine (arcsin)
- Arccosine (arccos)
- Arctangent (arctan)

Trigonometric Functions

Sine Function

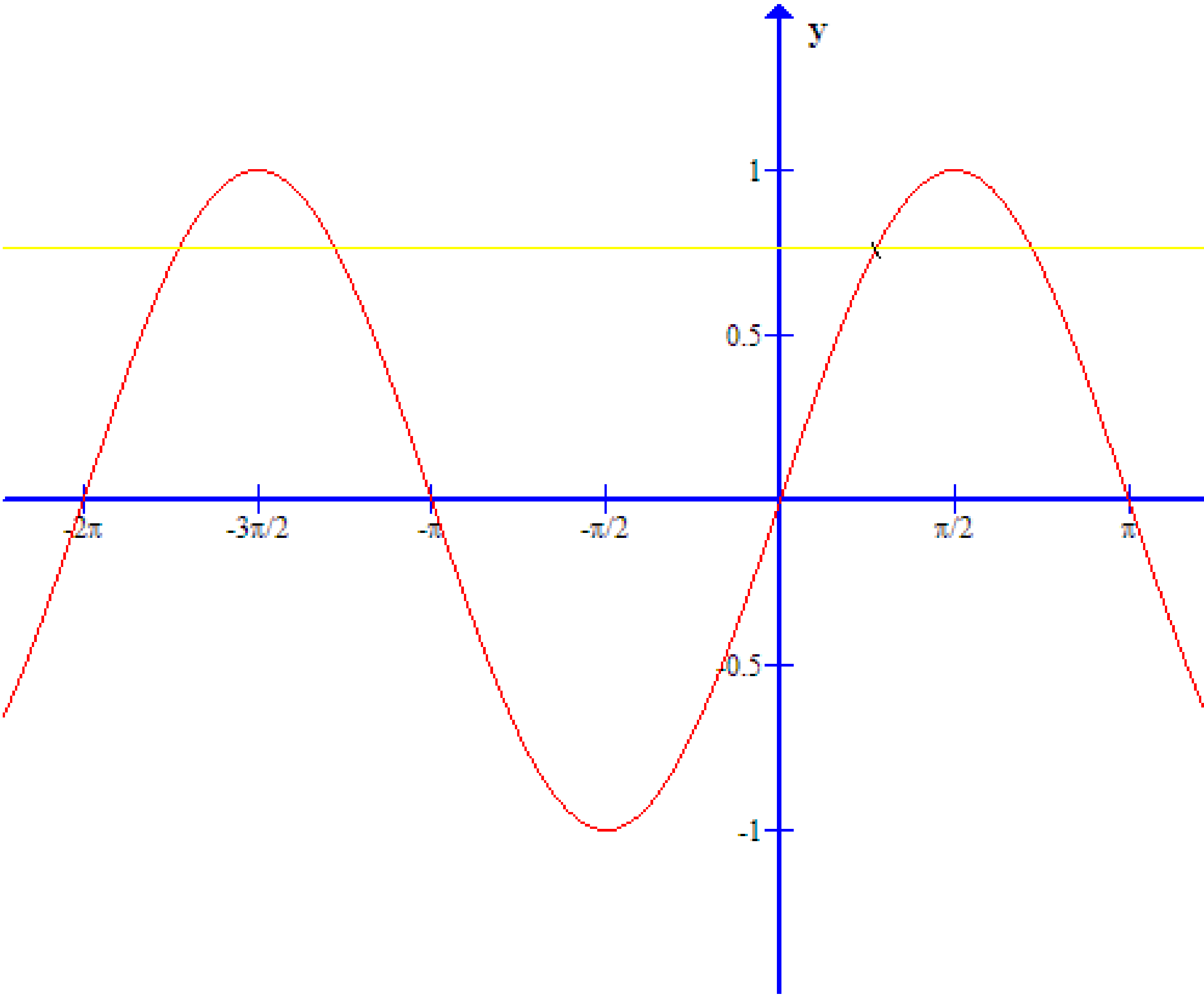
The **sine function** is defined as:

$$y = \sin x$$

Properties 3.16. 1. *Domain:* $D_f = \mathbb{R}$

2. *The sine function is* ***odd*** *and* ***periodic*** *with period* 2π*.*

3. $|\sin x| \leq 1$
4. $\sin x = 0 \iff x = k\pi$, *where* $k \in \mathbb{Z}$
5. *In the interval* $[0, 2\pi]$, *the sine function is:*
   - ***Increasing*** *on* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
   - ***Decreasing*** *on* $\left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$



Cosine Function

The **cosine function** is defined as:

$$y = \cos x$$

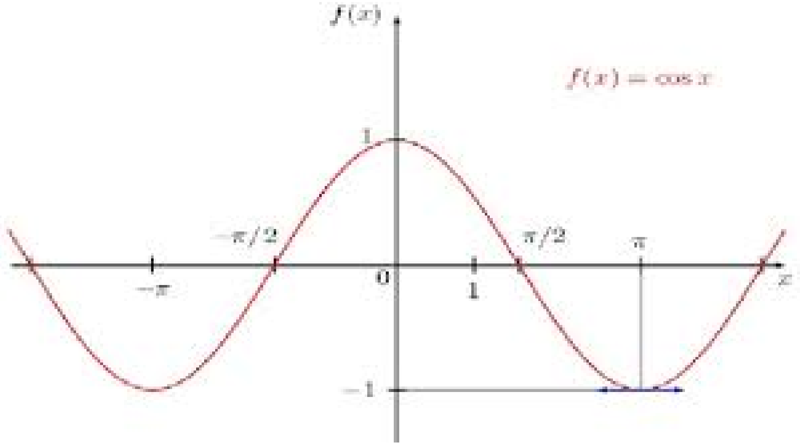
Properties 3.17. *1. Domain:* $D_f = \mathbb{R}$

2. *The cosine function is* ***even*** *and* ***periodic*** *with period* 2π.

3. $|\cos x| \leq 1$

4. $\cos x = 0 \iff x = \frac{\pi}{2} + k\pi$, *where* $k \in \mathbb{Z}$

5. *The cosine function is:*
   - ***Increasing*** *on* $\left[-\frac{\pi}{2}, 0\right] \cup \left[\pi, \frac{3\pi}{2}\right]$
   - ***Decreasing*** *on* $[0, \pi]$



Tangent Function

The **tangent function** is defined as:

$$y = \tan x = \frac{\sin x}{\cos x}$$

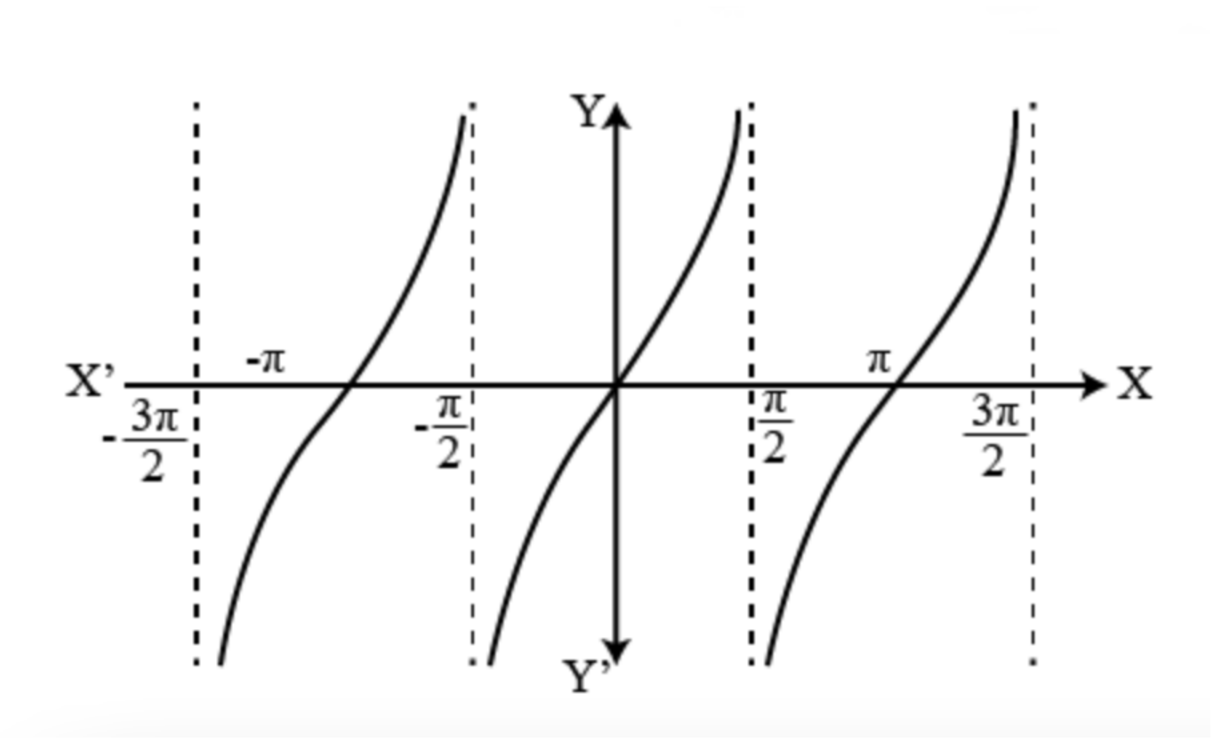
Properties 3.18. 1. *Domain:* $D_f = \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$

2. $\tan(-x) = -\tan x$, *so the tangent function is* ***odd***.

3. *The tangent function is* ***periodic*** *with period* $T = \pi$.

4. *The tangent function is* ***increasing*** *on its domain.*

5. $\tan x = 0 \iff x = k\pi$, *where* $k \in \mathbb{Z}$



Cotangent Function

The **cotangent function** is defined as:

$$y = \cot x = \frac{\cos x}{\sin x}$$

Properties 3.19. *1. Domain:* $D_f = \mathbb{R} - \{k\pi, k \in \mathbb{Z}\}$

2. $\cot x = 0 \iff x = \frac{\pi}{2} + k\pi$, *where* $k \in \mathbb{Z}$

3. The cotangent function is ***odd*** *and* ***decreasing*** *on its domain.*

4. The cotangent function is ***periodic*** *with period* $T = \pi$.

Inverse Trigonometric Functions

Arcsine Function

The **arcsine function** is the inverse of the sine function restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The sine function:

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$$

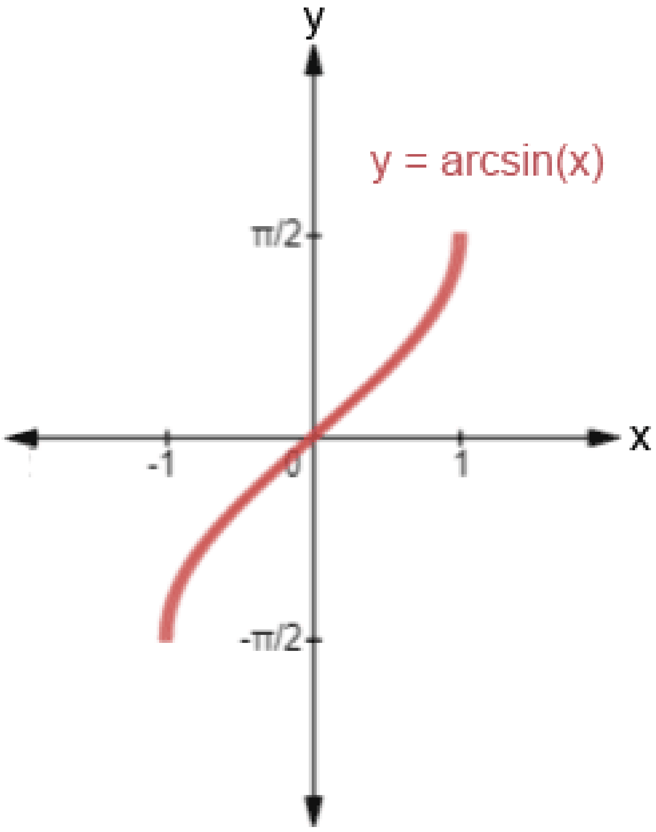
is continuous and monotonic, hence bijective and invertible. Its inverse is the arcsine function:

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$y = \sin x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \iff x = \arcsin y, \quad y \in [-1, 1]$$

Properties 3.20. *Let* $y = \arcsin x$*, where* $x \in [-1, 1]$*. Then:*

1. $\arcsin x = 0 \iff x = 0$
2. *The arcsine function is* ***odd****:*
$$\arcsin(-x) = -\arcsin x, \quad \forall x \in [-1, 1]$$
3. *The arcsine function is* ***increasing*** *for all* $x \in [-1, 1]$*.*
4. $\sin(\arcsin x) = x, \quad \forall x \in [-1, 1]$
5. $\arcsin(\sin x) = x, \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
6. $\cos(\arcsin x) = \sqrt{1 - x^2}, \quad \forall x \in [-1, 1]$



Example 3.21. *Solve the equation:*

$$2 \sin x = 1 \iff \sin x = \frac{1}{2} \Rightarrow x = (-1)^k \frac{\pi}{6} + k\pi, \quad k \in \mathbb{Z}$$

Arccosine Function

The **arccosine function** is the inverse of the cosine function restricted to the interval $[0, \pi]$. The cosine function:

$$\cos : [0, \pi] \to [-1, 1]$$

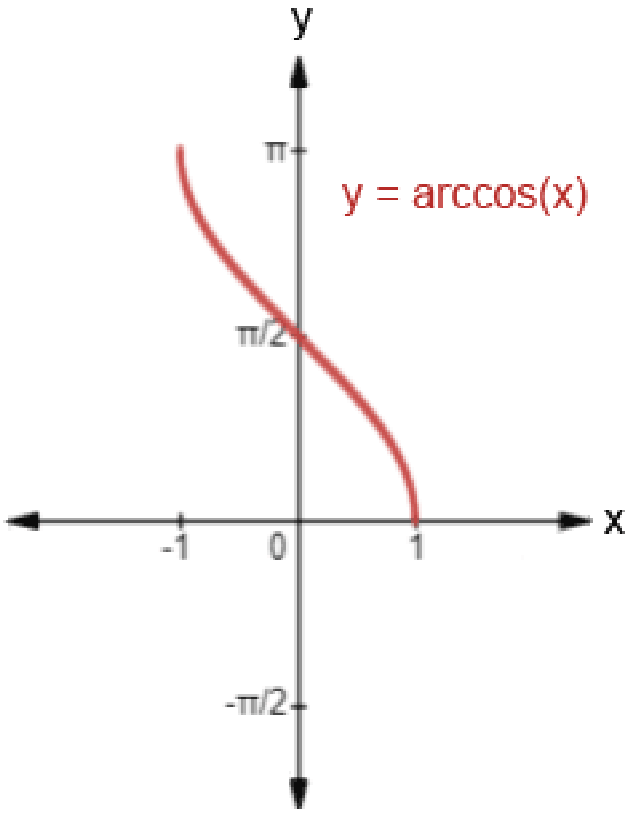
is continuous and decreasing, hence bijective and invertible. Its inverse is the arccosine function:

$$\arccos : [-1,1] \to [0,\pi]$$

$$y = \cos x, \quad x \in [0,\pi] \iff x = \arccos y, \quad y \in [-1,1]$$

Properties 3.22. *Let* $y = \arccos x$*, where* $x \in [-1,1]$*. Then:*

1. $\arccos x = 0 \iff x = 1$
2. $y = \arccos x$ *is decreasing for all* $x \in [-1,1]$
3. $\cos(\arccos x) = x$*, for all* $x \in [-1,1]$
4. $\arccos(\cos x) = x$*, for all* $x \in [0,\pi]$
5. $\sin(\arccos x) = \sqrt{1-x^2}$
6. $\arcsin x + \arccos x = \dfrac{\pi}{2}$*, for all* $x \in [-1,1]$
7. $y = \cos x,\ x \in \mathbb{R} \iff x = \pm\arccos y + 2k\pi,\ k \in \mathbb{Z}$



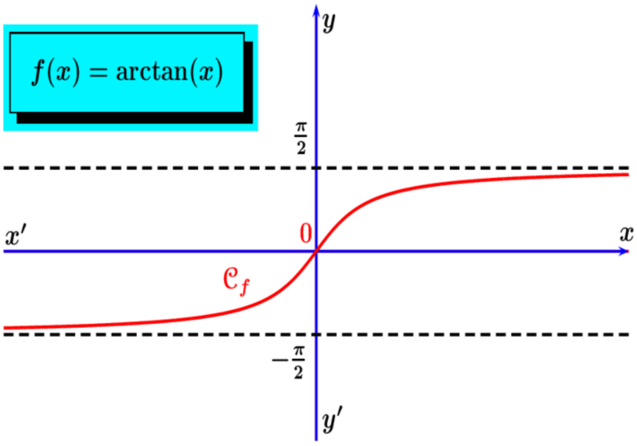
arctangente function

The function $\tan\colon \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\longrightarrow \mathbb{R}$ is one to one then it's invertible, its inverse is the function arctan*gente*, noted arctan, and defined by

$$\begin{aligned}&\arctan\colon \mathbb{R} \longrightarrow \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\\ &y=\tan x,\ x\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\iff x=\arctan y,\ y\in\mathbb{R}\end{aligned}$$

Proposition 3.23. *if* $y=\arctan x$, $x\in\mathbb{R}$ *then*

1. $\arctan x = 0 \iff x=0$
2. $\arctan(-x)=\arctan(x)$, *for all* $x\in\mathbb{R}$
3. arctan *is increasing function for all* $x\in\mathbb{R}$
4. $\tan(\arctan x)=x$, *for all for all* $x\in\mathbb{R}$, *and* $\arctan(\tan x)=x$, *for all* $x\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$
5. $\sin(\arctan x)=\dfrac{x}{\sqrt{1+x^2}}$, $x\in\mathbb{R}$ *and* $\tan(\arcsin x)=\dfrac{x}{\sqrt{1-x^2}}$, $x\in\left]-1,1\right[$
6. $y=\tan x$, *for* $x\in\mathbb{R}\backslash\left\{\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\right\} \iff x=\arctan y+k\pi$, $k\in\mathbb{Z}$



arccotangente function

The function $\cot\colon \left]0,\pi\right[\longrightarrow \mathbb{R}$ is one to one (bijection) then it's invertible, it's inverse is arccotangente function, noted by arccot, then we have

$$\begin{aligned}&\operatorname{arccot}\colon \mathbb{R} \longrightarrow \left]0,\pi\right[\\ &y=\cot x,\ x\in\left]0,\pi\right[\iff x=\operatorname{arccot} y,\ y\in\mathbb{R}\end{aligned}$$

Proposition 3.24. *Let* $y=\operatorname{arccot} x$, $x\in\mathbb{R}$, *then*

1. *The arccotangent function is* ***decreasing*** *for all* $x\in\mathbb{R}$.

2. $\text{arccot}(-x) = \pi - \text{arccot}\ x$, *for all* $x \in \mathbb{R}$.
3. $\cot(\text{arccot}\ x) = x$, *for all* $x \in \mathbb{R}$, *and* $\text{arccot}(\cot x) = x$, *for all* $x \in [0, \pi]$.
4. $\arctan x + \text{arccot}\ x = \frac{\pi}{2}$, *for all* $x \in \mathbb{R}$.

Elementary Functions

An **elementary function** is any real-valued function obtained from basic functions using a finite number of arithmetic operations and function compositions.

Example 3.25.

$$y = \frac{x^2 + \sin 3x + \arcsin x}{e^x + \log x}$$

Rational Functions

A **rational function** is the ratio of two polynomials:

$$y = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}{b_0 + b_1x + b_2x^2 + \ldots + b_nx^n}$$

where the domain $D_f = \{x \mid Q(x) \neq 0\}$.

Irrational Functions

An **irrational function** is any function $y = f(x)$ where $f(x)$ is composed of arithmetic operations on x and non-integer powers.

Example 3.26.

$$y = f(x) = \sqrt[3]{1 - x^2}$$

Hyperbolic Functions

The **hyperbolic functions** are defined as follows:

1. **Hyperbolic sine**:

$$y = \sinh x = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbb{R}$$

2. **Hyperbolic cosine**:

$$y = \cosh x = \frac{e^x + e^{-x}}{2}, \quad x \in \mathbb{R}$$

3. **Hyperbolic tangent**:

$$y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{2e^x - 1}{2e^x + 1}, \quad x \in \mathbb{R}$$

4. **Hyperbolic cotangent**:

$$y = \coth x = \frac{\cosh x}{\sinh x} = \frac{2e^x + 1}{2e^x - 1}, \quad x \in \mathbb{R}$$

Hyperbolic Functions

Hyperbolic Sine (sinh)

The **hyperbolic sine** function $\sinh x$ has the following properties:

1. $\sinh x = 0 \iff x = 0$
2. $\sinh x$ is **odd**.
3. $\sinh x$ is **strictly increasing**.
4. $\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \sinh x_2 \cosh x_1$

Hyperbolic Cosine (cosh)

The **hyperbolic cosine** function $\cosh x$ has the following properties:

1. $\cosh x \neq 0$ for all $x \in \mathbb{R}$.
2. $\cosh x$ is **even**.
3. $\cosh x$ is **increasing** on $\mathbb{R}_+$ and **decreasing** on $\mathbb{R}_-$.
4. $\cosh^2 x - \sinh^2 x = 1$
5. $\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2$

Hyperbolic Tangent (tanh)

The **hyperbolic tangent** function $\tanh x$ has the following properties:

1. $\tanh x = 0 \iff x = 0$
2. $\tanh x$ is **odd**.
3. $\tanh x$ is **strictly increasing**.

Hyperbolic Cotangent (coth)

The **hyperbolic cotangent** function $\coth x$ has the following properties:

1. $\coth x \neq 0$ for all $x \in \mathbb{R}^*$.
2. $\coth x$ is **odd**.
3. $\coth x$ is **strictly decreasing**.

Inverse Hyperbolic Functions

Inverse Hyperbolic Sine (argsinh)

The **inverse hyperbolic sine** function, denoted argsinh, is the inverse of the hyperbolic sine function sinh. The hyperbolic sine function:

$$\sinh : \mathbb{R} \to \mathbb{R}$$

is bijective, hence invertible. Its inverse is:

$$\text{argsinh} : \mathbb{R} \to \mathbb{R}$$

$$y = \text{argsinh } x, \quad x \in \mathbb{R} \iff x = \sinh y, \quad y \in \mathbb{R}$$

Properties 3.27. *:*

1. *$y = \text{argsinh } x$ is* ***odd****.*
2. *$y = \text{argsinh } x$ is* ***strictly increasing****.*
3. *$\text{argsinh } x = 0 \iff x = 0$*

Inverse Hyperbolic Cosine (argcosh)

The **inverse hyperbolic cosine** function, denoted argcosh, is the inverse of the hyperbolic cosine function cosh. The hyperbolic cosine function:

$$\cosh : [0, +\infty) \to [1, +\infty)$$

is bijective, hence invertible. Its inverse is:

$$\text{argcosh} : [1, +\infty) \to [0, +\infty)$$

$$y = \cosh x, \quad x \in [0, +\infty) \iff x = \text{argcosh } y, \quad y \in [1, +\infty)$$

Properties 3.28. *:*

1. *$\text{argcosh } x = 0 \iff x = 1$*
2. *$y = \text{argcosh } x$ is* ***strictly increasing*** *on its domain.*

:

Inverse Hyperbolic Tangent (argtanh)

The **inverse hyperbolic tangent** function, denoted argtanh, is the inverse of the hyperbolic tangent function tanh. The hyperbolic tangent function:

$$\tanh : \mathbb{R} \to [-1, 1]$$

is bijective, hence invertible. Its inverse is:

$$\text{argtanh} : [-1, 1] \to \mathbb{R}$$

$$y = \tanh x, \quad x \in \mathbb{R} \iff x = \text{argtanh } y, \quad y \in [-1, 1]$$

Properties 3.29. *:*

1. *argtanh* $x = 0 \iff x = 0$
2. $y =$ *argtanh* x *is* ***odd*** *and* ***strictly increasing*** *on its domain.*

:

Inverse Hyperbolic Cotangent

The **inverse hyperbolic cotangent** function, denoted argcoth, is the inverse of the hyperbolic cotangent function coth. The hyperbolic cotangent function:

$$\coth : \mathbb{R}^* \to (-\infty, -1) \cup (1, +\infty)$$

is bijective, hence invertible. Its inverse is:

$$\text{argcoth} : (-\infty, -1) \cup (1, +\infty) \to \mathbb{R}^*$$

$$y = \coth x, \quad x \in \mathbb{R}^* \iff x = \text{argcoth } y, \quad |y| > 1$$

Properties 3.30. *:*

1. $y =$ *argcoth* x *is* ***odd****.*
2. $y =$ *argcoth* x *is* ***strictly decreasing*** *on its domain.*

:

Theorem 3.31. *The inverse hyperbolic functions can be expressed as logarithms:*

1. *argsinh* $x = \log(x + \sqrt{x^2 + 1})$*, for all* $x \in \mathbb{R}$
2. *argcosh* $x = \log(x + \sqrt{x^2 - 1})$*, for all* $x \geq 1$
3. *argtanh* $x = \frac{1}{2} \log \frac{1+x}{1-x}$*, for all* $x \in (-1, 1)$

4. argcoth $x = \frac{1}{2}\log\frac{1+x}{1-x}$, *for all* $|x| > 1$

Proof. For argsinh x:

$$y = \operatorname{argsinh} x \iff x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{e^{2y}-1}{e^y} \iff e^{2y} - 2xe^y - 1 = 0$$

$$\iff e^y = x \pm \sqrt{x^2+1} \Rightarrow e^y = x + \sqrt{x^2+1} \iff y = \log(x + \sqrt{x^2+1})$$

Similarly, the expressions for argcosh x, argtanh x, and argcoth x can be derived. □

Limits of Functions

Definition 3.32. *Definition of Limit Let* f *be defined in a neighborhood of* x_0*, except possibly at* x_0*. The number* l *is called the limit of* f *as* x *approaches* x_0*, written:*

$$l = \lim_{x\to x_0} f(x)$$

if:

$$\forall\varepsilon > 0, \exists\delta = \delta(\varepsilon) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon)$$

Example 3.33. *Prove that:*

$$\lim_{x\to 0}\sin x = 0$$

Indeed:

$$|\sin x - 0| = |\sin x| \le |x|$$

Let $\delta(\varepsilon) = \varepsilon$*. Then:*

$$\forall\varepsilon > 0, \exists\delta(\varepsilon) > 0, \text{ such that } |x| < \delta(\varepsilon) \Rightarrow |\sin x - 0| = |x| < \varepsilon$$

Definition 3.34. *Alternative Definition The number* l *is called the limit of* f *as* x *approaches* x_0 *if for every sequence* (x_n) *in* $V^0(x_0)$ *(a punctured neighborhood of* x_0*) converging to* x_0*, the sequence* $y_n = f(x_n)$ *converges to* l*. We write:*

$$\forall x_n \in V^0(x_0)$$

$$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = l$$

Remark 3.35. *According to definition 2, if there exist two sequences* (u_n), (v_n) *converging to* x_0 *such that*

$$\lim_{n\to\infty} f(u_n) \neq \lim_{n\to\infty} f(v_n)$$

then the limit of f *does not exist at* x_0*.*

Example 3.36. *Study the limit of* $y = \sin\frac{\pi}{x}$ *as* $x \to 0$*. Let* $u_n = \frac{1}{n} \implies 0$ *and* $v_n = \frac{1}{\frac{1}{2}+2n} \implies 0$*. Then we have:*

$$f(u_n) = \sin n\pi = 0$$

and

$$f(v_n) = \sin\left(\frac{\pi}{2} + 2n\right) = 1$$

Hence, we conclude that the limit of f *does not exist.*

Extension of the Limit:

1. One-Sided Limits at a Point:

Let $V \subset \mathbb{R}$ be a set containing the interval $]a, x_0[$ (or $]x_0, b[$).

Definition 3.37. *The number $l \in \mathbb{R}$ is said to be the right-hand limit (resp. left-hand limit) of f at x_0 if:*

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) \in \mathbb{R}_+^*, \quad \forall x \in V, \quad (0 < x_0 - x < \delta) \Rightarrow |f(x) - l| < \varepsilon$$

Respectively:

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) \in \mathbb{R}_+^*, \quad \forall x \in V, \quad (0 < x - x_0 < \delta) \Rightarrow |f(x) - l| < \varepsilon$$

We denote: Right-hand limit:

$$l = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0 + 0)$$

Left-hand limit:

$$l = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} f(x) = f(x_0 - 0)$$

Theorem 3.38.

$$\lim_{n \to x_0} f(x) = l \Leftrightarrow f(x + 0) = f(x - 0)$$

2. Limit at Infinity:

$$\lim_{x \to +\infty} f(x) = l \Leftrightarrow (\forall \varepsilon > 0, \exists A > 0, \forall x \in V(+\infty), (x > A \Rightarrow |f(x) - l| < \varepsilon))$$

And

$$\lim_{x \to -\infty} f(x) = l \Leftrightarrow (\forall \varepsilon > 0, \exists A > 0, \forall x \in V(-\infty), (x < -A \Rightarrow |f(x) - l| < \varepsilon))$$

3. Infinite Limit:

Definition 3.39.

$$\lim_{x \to x_0} f(x) = +\infty \Leftrightarrow (\forall A > 0, \exists \delta = \delta(A) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow f(x) > A))$$

$\lim_{x \to x_0} f(x) = -\infty$ *if and only if:*

$$\forall A > 0, \exists \delta = \delta(A) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow f(x) < -A)$$

2. $\lim_{x \to +\infty} f(x) = +\infty$ *if and only if:*

$$\forall A > 0, \exists B > 0, \forall x \in V(+\infty), (x > B \Rightarrow f(x) > A)$$

Theorem 3.40. *Uniqueness of Limits If a function f has a limit at x_0, then the limit is unique.*

Proof. Assume f has two limits l_1 and l_2 at x_0. Then:

$$\lim_{x\to x_0} f(x) = l_1 \iff \forall\epsilon > 0, \exists\delta_1 = \delta_1(\epsilon) > 0, \forall x \in V(x_0), (0 < |x-x_0| < \delta_1 \Rightarrow |f(x)-l_1| < \frac{\epsilon}{2})$$

$$\lim_{x\to x_0} f(x) = l_2 \iff \forall\epsilon > 0, \exists\delta_2 = \delta_2(\epsilon) > 0, \forall x \in V(x_0), (0 < |x-x_0| < \delta_2 \Rightarrow |f(x)-l_2| < \frac{\epsilon}{2})$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for $0 < |x - x_0| < \delta$, both conditions are satisfied, and:

$$|l_1 - l_2| = |(l_1 - f(x)) + (f(x) - l_2)| \leq |f(x) - l_1| + |f(x) - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, $l_1 = l_2$. □

Local Properties

Theorem 3.41. *If* $\lim_{x\to x_0} f(x) = l$*, then there exists a neighborhood of* x_0 *in which* f *is bounded. That is:*

$$\exists V(x_0) \text{ such that } \forall x \in V(x_0), |f(x)| \leq M$$

Passage to the Limit in Inequalities

Theorem 3.42. *Let* f *and* g *be two functions defined in a neighborhood of* x_0 *such that:*

$$\lim_{x\to x_0} f(x) = l_1, \quad \lim_{x\to x_0} g(x) = l_2, \quad \text{and} \quad l_1 < l_2$$

Then there exists a punctured neighborhood of x_0 *in which* $f(x) \leq g(x)$*.*

Proof.

$$\lim_{x\to x_0} f(x) = l_1 \iff \forall\epsilon > 0, \exists\delta_1 = \delta_1(\epsilon) > 0, \forall x \in V^0(x_0), (0 < |x-x_0| < \delta_1 \Rightarrow |f(x)-l_1| < \epsilon)$$

$$\lim_{x\to x_0} g(x) = l_2 \iff \forall\epsilon > 0, \exists\delta_2 = \delta_2(\epsilon) > 0, \forall x \in V(x_0), (0 < |x-x_0| < \delta_2 \Rightarrow |f(x)-l_2| < \epsilon)$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for $\forall x \in V(x_0), 0 < |x - x_0| < \delta$, both conditions are satisfied. Moreover:

$$|f(x) - l_1| < \epsilon \iff l_1 - \epsilon < f(x) < l_1 + \epsilon$$

Let $l_1 < l < l_2$. Given $\varepsilon = l - l_1 > 0 \Rightarrow l_1 - (l - l_1) < f(x) < l_1 + (l - l_1) \Rightarrow \forall x \in V(x_0), \quad 0 < |x - x_0| < \delta, \quad f(x) < l \ldots (3)$

$$|g(x) - l_2| < \varepsilon \Leftrightarrow l_2 - \varepsilon < g(x) < l_2 + \varepsilon$$

Given $\varepsilon = l_2 - l > 0 \Rightarrow l_2 - (l_2 - l) < g(x) < l_2 + (l_2 - l) \Rightarrow \forall x \in V(x_0), \quad 0 < |x - x_0| < \delta, \quad g(x) > l \ldots (4)$

From (3) and (4), we deduce $\forall x \in V(x_0), \quad 0 < |x - x_0| < \delta, \quad f(x) < g(x)$ □

Corollary 3.43. *Let f be a function defined in a neighborhood of x_0 such that $\lim_{x\to x_0} f(x) = l$ and $f(x) > a$*
then $l \geq a$

Theorem 3.44. ***Intermediate Function Criterion***
Let f, g, h be three functions defined in a punctured neighborhood of x_0 such that

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in V^0(x_0)$$

If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l$ then $\lim_{x\to x_0} g(x) = l$

Proof.

$$\lim_{x\to x_0} f(x) = l \Leftrightarrow \text{for every sequence } x_n \text{ converging to } x_0, \quad \lim_{n\to\infty} f(x_n) = l$$

$$\lim_{x\to x_0} h(x) = l \Leftrightarrow \text{for every sequence } x_n \text{ converging to } x_0, \quad \lim_{n\to\infty} h(x_n) = l$$

Hence $f(x_n) \leq g(x_n) \leq h(x_n), \quad \forall (x_n) \in V^0(x_0)$ From the theorem of three sequences, we deduce that $\lim_{x_n\to\infty} g(x_n) = l$ Which is equivalent to saying $\lim_{x\to x_0} g(x) = l$ □

Example 3.45. *Study the limit of $f(x) = x\sin\frac{1}{x}; \quad x_0 = 0$*

We have $\forall x \in \mathbb{R} - \{0\}, \quad -1 \leq \sin\frac{1}{x} \leq +1 \Leftrightarrow \begin{cases} -x \leq x\sin\frac{1}{x} \leq x & \textit{if } x > 0 \\ -x \geq x\sin\frac{1}{x} \geq x & \textit{if } x < 0 \end{cases}$

Taking the limit, we obtain

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

Operations on Limits

Proposition 3.46. *Let f, g be two functions defined in a neighborhood of x_0 such that $\lim_{x\to x_0} f(x) = l_1$,*

$$\lim_{x\to x_0} g(x) = l_2 \text{ and then we have:}$$

1/ $\lim_{x\to x_0}(f(x) \pm g(x)) = l_1 \pm l_2$
2/ $\lim_{x\to x_0}(f(x)g(x)) = l_1 l_2$
3/ $\lim_{x\to x_0}(Af(x)) = \lambda l_1$
4/

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}$$

if

$$\lim_{x\to x_0} g(x) \neq 0$$

5/

$$\lim_{x\to x_0} |f(x)| = |l_1|$$

Proof. Let's prove (4) for example:

$$\lim_{x\to x_0} f(x) = l_1 \iff \text{for every sequence } x_n \text{ converging to } x_0, \quad \lim_{n\to+\infty} f(x_n) = l_1$$

$$\lim_{x\to x_0} g(x) = l_2 \iff \text{for every sequence } x_n \text{ converging to } x_0, \quad \lim_{n\to+\infty} g(x_n) = l_2$$

By the theorem of the limit of the ratio of two sequences, we find

$$\lim_{n\to+\infty} \frac{f(x_n)}{g(x_n)} = \frac{\lim_{n\to+\infty} f(x_n)}{\lim_{n\to+\infty} g(x_n)} = \frac{l_1}{l_2}$$

□

Example 3.47. *Calculate*

$$\lim_{n\to+\infty}(\sqrt{x^2+3x-4}-x) = \lim_{n\to+\infty} \frac{3x-4}{\sqrt{x^2+3x-4}+x} = \lim_{n\to+\infty} \frac{x(3-\frac{4}{x})}{\sqrt{1+\frac{3}{x}-\frac{4}{x^2}}+1} = \frac{3}{2}$$

Infinitely Large and Infinitely Small Functions

Definition 3.48. *Let f be a function defined in a punctured neighborhood of x_0, then:*

If

$$\lim_{x\to x_0} f(x) = 0$$

, then f is said to be infinitely small.

If

$$\lim_{x\to x_0} f(x) = +\infty$$

, then f is said to be infinitely large.

Remark 3.49. *If f is an infinitely small function at x_0 and if*

$$\lim_{x\to x_0} f(x) \neq 0$$

in

$$V^0(x_0)$$

, then the function

$$g(x) = \frac{1}{f(x)}$$

is infinitely large.

Remarkable Limits:

1. Let's prove that

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

The function

$$f(x)=\frac{\sin x}{x}$$

is an even function on

$$\mathbb{R}^*$$

, so it suffices to calculate the limit at

$$0^+$$

. (To be seen!)

2.

$$\lim_{x\to 0}\cos x=1$$

indeed:

$$0<|1-\cos x|=\left|2\sin^2\frac{x}{2}\right|=2\sin^2\frac{x}{2}\Rightarrow\lim_{x\to 0}\cos x=1$$

3. a)

$$\lim_{x\to 0}\frac{\tan x}{x}=1;\quad b)\lim_{x\to 0}\frac{\arcsin x}{x}=1;\quad c)\lim_{x\to 0}\frac{\arctan x}{x}=1$$

For (b), let

$$\arcsin x=y\iff\sin y=x;\quad x\to 0\implies y\to 0$$

, so we have:

$$\lim_{x\to 0}\frac{\arcsin x}{x}=\lim_{y\to 0}\frac{\sin y}{y}=1$$

4.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)=\frac{1}{2}$$

5.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$

Let $x\to+\infty$, let $E(x)=n$, then:

$$n\leq x<n+1\Rightarrow\frac{1}{n+1}<\frac{1}{x}\leq\frac{1}{n}\Rightarrow\frac{1}{n+1}+1<1+\frac{1}{x}\leq 1+\frac{1}{n}\Rightarrow\left(1+\frac{1}{n+1}\right)^{n+1}<\left(1+\frac{1}{x}\right)^x\leq\Big(1+$$

$$\Rightarrow\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e$$

This formula is also true when $x\to-\infty$

Generalization:

Let's compute

$$\lim_{x\to x_0}(f(x))^{g(x)}$$

, $f(x) > 0$, then we have the following cases:
i. If

$$\lim_{x\to x_0} f(x) = l_1$$

and

$$\lim_{x\to x_0} g(x) = l_2$$

, then

$$\lim_{x\to x_0} (f(x))^{g(x)} = (l_1)^l$$

ii. If

$$\lim_{x\to x_0} f(x) = 1$$

and

$$\lim_{x\to x_0} g(x) = \infty$$

, then

$$\lim_{x\to x_0} (f(x))^{g(x)} = e^{\lim_{x\to x_0}(f(x)-1)g(x)}$$

Example 3.50. *1.*

$$\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} = e^{\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}} = e^{\lim_{x\to 0}\frac{\sin x}{x}} = e$$

2.

$$\lim_{x\to+\infty}\left(\frac{2x+5}{x+3}\right)^{x+3} = 2^1 = 2$$

because

$$\lim_{x\to+\infty}\frac{2x+5}{x+3} = 2$$

and

$$\lim_{x\to+\infty}\frac{x+3}{x-5} = 1$$

3.

$$\lim_{x\to+\infty}\frac{a^0+a^1x+a^2x^2+\cdots+a^nx^n}{b^0+b^1x+b^2x^2+\cdots+b^mx^m} = \lim_{x\to+\infty}\frac{a^nx^n}{b^mx^m} = \begin{cases} \frac{a}{b} & \text{if } n=m \\ \infty & \text{if } n>m \\ 0 & \text{if } n<m \end{cases}$$

Change of Variables in Limit Calculation:

Proposition 3.51. *Let $y = f(x)$ be defined in a neighborhood $V^0(x_0)$, $z = g(y)$ be defined in $W^0(y_0)$ satisfying the relations:*

$$f(x) \neq y_0, \quad \forall x \in V^0(x_0), \quad \lim_{x\to x_0} f(x) = y_0 \quad \& \quad \lim_{y\to y_0} g(y) = l$$

Then

$$\lim_{x\to x_0} g(f(x)) = l$$

Proof.

$$\left(\lim_{y\to y_0} g(x) = l\right) \Leftrightarrow (\forall \epsilon > 0, \exists \delta_1 > 0, \forall y \in V(y_0), \quad (0 < |y - y_0| < \delta_1 \Rightarrow |g(x) - l| < \epsilon)$$

$$\left(\lim_{x\to x_0} f(x) = y_0\right) \Leftrightarrow (\text{for } \delta_1 > 0, \exists \delta_2 > 0, \forall x \in V(x_0), \quad (0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - y_0| < \delta_1)$$

Then we obtain:

$$\forall x \in V(x_0), \quad (0 < |x - x_0| < \delta_2 \Rightarrow |y - y_0| < \delta_1 \Rightarrow |g(x) - l| < \epsilon)$$

That is:

$$\forall \epsilon > 0, \exists \delta_2 > 0, \forall x \in V(x_0), \quad (0 < |x - x_0| < \delta_2 \Rightarrow |g(f(x)) - l| < \epsilon) \Rightarrow \lim_{x\to x_0} g(f(x)) = l$$

□

Example 3.52. *Calculate*

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{1}{\cos x} - \tan x\right)$$

Let

$$x - \frac{\pi}{2} = y \Leftrightarrow x = y + \frac{\pi}{2}, \quad x \to \frac{\pi}{2} \Leftrightarrow y \to 0$$

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{1}{\cos x} - \tan x\right) = \lim_{y\to 0} \left(\frac{1}{\cos\left(\frac{\pi}{2} + y\right)} - \tan\left(\frac{\pi}{2} + y\right)\right) = \lim_{y\to 0} \left(\frac{1}{-\sin y} + \cot y\right)$$

$$= \lim_{y\to 0} \left(\frac{\cos y}{\sin y} - \frac{1}{\sin y}\right) = \lim_{y\to 0} \left(\frac{\cos y - 1}{\sin y}\right) = \lim_{y\to 0} \left(\frac{2\sin^2 \frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}\right) = \lim_{y\to 0} \tan\frac{y}{2} = 0$$

Limit of a Monotone Function:

Theorem 3.53. *Let f be an increasing (respectively decreasing) function on the interval $I = (a, b)$.*

If f is bounded above (resp. bounded below) on the interval I, then the limit of f exists, and moreover:

$$\lim_{x\to b^-} f(x) = \sup_{(a,b)} f(x) \quad \textit{if} \quad f \quad \textit{is bounded above.}$$

$$\lim_{x\to a^+} f(x) = \inf_{(a,b)} f(x) \quad \textit{if} \quad f \quad \textit{is bounded below.}$$

Proof. f is bounded above on $I = (a, b) \Rightarrow$

$$\exists M_0 = \sup_{(a,b)} f(x) \Leftrightarrow \begin{cases} 1. \quad \forall x \in (a, b),\ f(x) \le M_0 \\ 2. \quad \forall \epsilon > 0,\ \exists x' \in (a, b),\ f(x) > M_0 - \epsilon \end{cases}$$

f is increasing on $I = (a, b) \Rightarrow$

$$(x' < x \Rightarrow f(x') < f(x))$$

Let

$$b - x' = \delta \quad \text{then} \quad x' = b - \delta \quad \text{and :}$$

$$x' < x < b \Rightarrow (b - \delta < x < b) \Rightarrow M_0 - \epsilon < f(x') \leq f(x) < M_0 < M_0 + \epsilon$$

In this way:

$$b - \delta < x < b \Rightarrow M_0 - \epsilon < f(x) < M_0 + \epsilon \Leftrightarrow |f(x) - M_0| < \epsilon \Leftrightarrow \lim_{x \to b^-} f(x) = M_0 = \sup_{(a,b)} f(x)$$

□

Cauchy Criterion:

Let f be a function defined in a punctured neighborhood of x_0.

Definition 3.54. *We say that f satisfies the Cauchy criterion at the point x_0 if:*

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \forall x', x'' \in V^0(x_0)$$

$$(0 < |x' - x_0| < \delta \quad \& \quad 0 < |x'' - x_0| < \delta \Rightarrow |f(x') - f(x'')| < \epsilon)$$

Theorem 3.55. *For the limit of a function f to exist at x_0, it is necessary and sufficient that f satisfies the Cauchy criterion at x_0.*

Comparison of Functions – Landau Notation:

Let f, g be two functions defined in a neighborhood of x_0, except possibly at x (x_0 can be equal to $+\infty$).

Definition 3.56. *Negligible Functions:*

We say that f is negligible compared to g in the neighborhood of x_0 if there exists a function $h(x)$ that is infinitely small at x_0 such that:

$$f(x) = h(x)g(x), \quad \forall x \in V^0(x_0)$$

We write:

$$f = o(g)(x \to x_0) \quad (\textit{Landau notation})$$

Read as: f equals little-o of g in the neighborhood of x_0.

Properties 3.57. *1/ If $g(x) \neq 0$, $\forall x \in V^0(x_0)$, then:*

$$f = o(g)(x \to x_0) \iff \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

2/ $f = l + o(l)(x \to x_0) \iff \lim_{x \to x_0} f(x) = l$. In particular, $f = o(l)(x \to x_0) \iff \lim_{x \to x_0} f(x) = 0$.

Example 3.58.

$$\sin^2 x = o(x), \quad x \to 0$$

Indeed:

$$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \sin x \right) = 1 \cdot 0 = 0$$

Definition 3.59. *Dominant Functions: We say that the function f is dominated by the function g at x_0 if there exists $k > 0$ such that:*

$$|f(x)| \leq k|g(x)|, \quad \forall x \in V^0(x_0)$$

And we write: $f = O(g)(x \to x_0)$ (Landau notation).
Read as: f equals big-O of g in the neighborhood of x_0.

Properties 3.60. *1/ If $g(x) \neq 0$, $\forall x \in V^0(x_0)$, then:*

$$f = O(g)(x \to x_0) \iff \exists k > 0, \quad \left| \frac{f(x)}{g(x)} \right| \leq k$$

2/ $f = O(l)(x \to x_0) \iff \exists k > 0, \quad |f(x)| \leq k$.

Example 3.61.

$$2x \sin x = O(x^2), \quad x \to 0$$

Definition 3.62. *Equivalent Functions: We say that f is equivalent to g in the neighborhood of x_0 if there exists a function h such that:*

$$\lim_{x\to x_0} h(x) = 1 \quad \text{and} \quad f(x) = h(x)g(x), \quad \forall x \in V^0(x_0)$$

We write $f \sim g$ (as $x \to x_0$).
Read as: f is equivalent to g in the neighborhood of x_0.

Properties 3.63. *1/ If $g(x) \neq 0$, $\forall x \in V^0(x_0)$, then:*

$$f \sim g \iff \lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$$

2/ If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = l \neq 0$, then $f \sim g$.
3/ If $\lim_{x\to x_0} f(x) = l$ and $f \sim g$, then $\lim_{x\to x_0} g(x) = l$.
4/ $f \sim g \iff f(x) = g(x)(1 + \alpha(x))$, where $\lim_{x\to x_0} \alpha(x) = 0$.

Theorem 3.64.

$$f \sim g \iff f = g + o(g), \quad x \to x_0$$

Theorem 3.65. *Let $f \sim f_1$ and $g \sim g_1$ as $x \to x_0$, then:*

$$f \cdot g \sim f_1 \cdot g_1 \quad \text{and} \quad \frac{f}{g} \sim \frac{f_1}{g_1} \quad \text{with} \quad (g \neq 0; \quad g_1 \neq 0)$$

Proof.

$$f \sim f_1 \iff f(x) = f_1(x)h_1(x) \quad \text{and} \quad \lim_{x\to x_0} h_1(x) = 1$$

$$g \sim g_1 \iff g(x) = g_1(x)h_2(x) \quad \text{and} \quad \lim_{x\to x_0} h_2(x) = 1$$

$$f(x)g(x) = h_1(x)h_2(x)f_1(x)g_1(x) = h(x)f_1(x)g_1(x) \quad \text{and} \quad \lim_{x\to x_0} h(x) = 1$$

This shows that $f \cdot g \sim f_1 \cdot g_1$. □

Remark 3.66.

$$f \sim f_1 \quad \textit{and} \quad g \sim g_1 \quad \textit{does not imply} \quad (f \pm g) \sim f_1 \pm g_1$$

Example 3.67.

$$\cos x \sim 1 + x^3 \quad \textit{and} \quad -1 \sim -1 \quad \textit{But} \quad \cos x - 1 \sim x^3 \quad \textit{because}$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x^3} = \lim_{x\to 0} \frac{-2\sin^2 \frac{x}{2}}{x^3} = -\frac{2}{4}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 \frac{1}{x} = \infty$$

Proposition 3.68. *Composition of Equivalences: Let $t = \varphi(x)$ be defined in $V^0(x_0)$.*
Let $f(t), g(t)$ be defined in $V^0(a)$.
If $f(t) \sim g(t)$ and $\varphi(x) \to a$ with $\varphi(x) \neq 0$, then $f(\varphi(x)) \sim g(\varphi(x))$.

Proof.

$$f \sim g \iff f(t) = g(t)h(t) \quad \text{and} \quad \lim_{t\to a} h(t) = 1$$

$$\Rightarrow f(\varphi(x)) = h(\varphi(x))g(\varphi(x)) \quad \text{and} \quad \lim_{x\to x_0} h(\varphi(x)) = \lim_{t\to a} h(t) = 1$$

Let $h(\varphi(x)) = h_1(x)$, then $\lim_{x\to x_0} h_1(x) = 1$ and we obtain:

$$f(\varphi(x)) = h_1(x)g(\varphi(x)) \iff f(\varphi(x)) \sim g(\varphi(x))$$

□

Remark 3.69. *In general, $f \sim g$ does not imply $\varphi(f) \sim \varphi(g)$.*

Example 3.70.

$$f(x) = e^{x^2+x}, \quad g(x) = e^{x^2}; \quad \varphi(x) = \ln(x), \quad x_0 = 0$$

We have $e^{x^2+x} \sim e^{x^2}$.

$$\varphi(f) = \ln(e^{x^2+x}) = x^2 + x; \quad \varphi(g) = \ln(e^{x^2}) = x^2.$$

$$\lim_{x\to 0} \frac{\varphi(f)}{\varphi(g)} = \lim_{x\to 0} \frac{x^2 + x}{x^2} = \infty$$

Thus,

$$\varphi(f) \neq \varphi(g)$$

Table of Equivalences for Elementary Functions: The notion of equivalence is very useful in calculating limits. For this purpose, we provide a table of equivalences for some commonly used elementary functions in the neighborhood of $x_0 = 0$: 1. $\ln(1+x) \sim x \iff \ln(1+x) = x + o(x)$ 2. $e^x - 1 \sim x \iff e^x - 1 = x + o(x)$ 3. $\sin x \sim x \iff \sin x = x + o(x)$ 4. $\cos x - 1 \sim -\frac{x^2}{2} \iff \cos x - 1 = -\frac{x^2}{2} + o(x^2)$ 5. $\tan x \sim x \iff \tan x = x + o(x)$ 6. $(1+x)^r - 1 \sim rx \iff (1+x)^r - 1 = rx + o(x), \forall r \in \mathbb{Q}^+$ 7. $\arcsin x \sim x \iff \arcsin x = x + o(x)$ 8. $\arctan x \sim x \iff \arctan x = x + o(x)$ 9. $\sinh x \sim x \iff \sinh x = x + o(x)$ 10. $\cot x \sim \frac{1}{x} \iff \cot x = \frac{1}{x} + o\left(\frac{1}{x}\right)$

Scales of Comparison for Infinitesimal Functions – Principal Parts:

Let g be an infinitesimal function in the neighborhood of x_0 with $g(x) \neq 0$.

Definition 3.71. *1. f is said to be an infinitesimal of higher order with respect to g at x_0 if g is negligible compared to f, i.e.,*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \iff f = o(g)(x \to x_0)$$

2. f is said to be an infinitesimal of lower order with respect to g at x_0 if f is negligible compared to g, i.e.,

$$\lim_{x \to x_0} \frac{g(x)}{f(x)} = 0 \iff g = o(f)(x \to x_0)$$

3. f is said to be an infinitesimal of order k with respect to g at x_0 if

$$\lim_{x \to x_0} \frac{f(x)}{g^k(x)} = l \neq 0$$

i.e.,

$$\lim_{x \to x_0} \frac{f(x)}{l \cdot g^k(x)} = 1 \iff f \sim l \cdot g^k \iff f(x) = l \cdot g^k(x) + o(g^k(x))$$

In this case, the function $l \cdot g^k(x)$ is called the principal part of f at x_0, and we write:

$$p.p.f(x) = l \cdot g^k(x)$$

Scales of Comparison for Infinitely Large Functions:

Let g be an infinitely large function at x_0.

Definition 3.72. *1. f is said to be infinitely large of higher order with respect to g at x_0 if g is negligible compared to f, i.e.,*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \infty \iff g = o(f)(x \to x_0)$$

2. f is said to be infinitely large of lower order with respect to g at x_0 if f is negligible compared to g, i.e.,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \iff f = o(g)(x \to x_0)$$

3. f is said to be infinitely large of order k with respect to g at x_0 if

$$\lim_{x\to x_0}\frac{f(x)}{g^k(x)}=l\neq 0$$

i.e.,

$$\lim_{x\to x_0}\frac{f(x)}{l\cdot g^k(x)}=1 \iff f\sim l\cdot g^k \iff f(x)=l\cdot g^k(x)+o(g^k(x))$$

Example 3.73. *Determine the principal part of the function $f(x)=1-\cos x$ in the neighborhood of $x_0=0$ of the form $\alpha\cdot x^\beta$. Indeed:*

$$\lim_{x\to 0}\frac{1-\cos x}{\alpha\cdot x^\beta}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\alpha\cdot x^\beta}=\lim_{x\to 0}\frac{2\cdot\frac{x^2}{4}}{\alpha\cdot x^\beta}=\frac{1}{2\alpha}\lim_{x\to 0}x^{2-\beta}=1 \quad \textit{iff} \quad \begin{cases}\beta=2\\ 2\alpha=1\end{cases} \Rightarrow \begin{cases}\beta=2\\ \alpha=\frac{1}{2}\end{cases}$$

Thus,

$$p.p.(1-\cos x)=\frac{1}{2}x^2$$

Examples of Limit Calculations Using Equivalences and Little "o":

1. Calculate

$$\lim_{x\to 2}\frac{\sin(x^2-4)}{x^2-5x+6}$$

We have

$$\sin x\sim x\Rightarrow \sin u(x)\sim u(x) \quad \text{when} \quad \lim_{x\to x_0}u(x)=0 \quad \text{and thus:}$$

$$\lim_{x\to 2}\frac{\sin(x^2-4)}{x^2-5x+6}=\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x-3)}=-4$$

2. Calculate

$$\lim_{x\to 0}\frac{e^x-\sqrt[3]{1+x}}{2\arctan x-\arcsin x}$$

$$\sqrt[3]{1+x}=1+\frac{1}{3}x+o(x) \quad \text{and} \quad o(x)\pm o(x)=o(x) \quad \text{and} \quad e^x-1=x+o(x)$$

$$\arctan x=x+o(x) \quad \text{and} \quad \arcsin x=x+o(x)$$

Thus:

$$\lim_{x\to 0}\frac{e^x-\sqrt[3]{1+x}}{2\arctan x-\arcsin x}=\lim_{x\to 0}\frac{1+(e^x-1)-\sqrt[3]{1+x}}{2\arctan x-\arcsin x}=\lim_{x\to 0}\frac{1+(x+o(x))-(1+\frac{1}{3}x+o(x))}{2(x+o(x))-(x+o(x))}$$

$$=\lim_{x\to 0}\frac{\frac{2}{3}x+o(x)}{x+o(x)}=\lim_{x\to 0}\frac{x\left(\frac{2}{3}+o(1)\right)}{x(1+o(1))}=\frac{\frac{2}{3}+\lim_{x\to 0}o(1)}{1+\lim_{x\to 0}o(1)}=\frac{2}{3}$$

Upper and Lower Limits:

Definition 3.74. *Let f be a function defined in $V^0(x_0)$. The following values $\lim_{\delta\to 0}\sup f(x)$ and $\liminf_{\delta\to 0} f(x)$, for $0 < |x - x_0| < \delta$, are called the upper limit and lower limit of f at x_0, respectively. They are denoted by:*

$$\limsup_{x\to x_0} f(x) \quad \textit{and} \quad \liminf_{x\to x_0} f(x)$$

Theorem 3.75.

$$\lim_{x\to x_0} f(x) = l \iff \limsup_{x\to x_0} f(x) = \liminf_{x\to x_0} f(x) = l$$

Definition 3.76. *The number $\lambda \in \mathbb{R}$ is called a partial limit of f at x_0 (or a cluster value) if there exists a sequence (x_n) different from x_0 such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = \lambda$.*

We denote by $\Lambda(x_0)$ the set of cluster values of f at x_0.

Then:

$$\limsup_{x\to x_0} f(x) = \sup \Lambda(x_0) \quad \textit{and} \quad \liminf_{x\to x_0} f(x) = \inf \Lambda(x_0)$$

Example 3.77. *Determine $\limsup_{x\to 0} f(x)$ and $\liminf_{x\to 0} f(x)$ where $f(x) = \cos\frac{1}{x}$ and $x_0 = 0$.*

For $x_n = \frac{1}{2\pi n} \to 0$:

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \cos(2\pi n) = 1 \quad \textit{since} \quad |\cos x| \leq 1$$

For $x'_n = \frac{1}{\pi+2\pi n} \to 0$:

$$\lim_{n\to\infty} f(x'_n) = \lim_{n\to\infty} \cos(\pi + 2\pi n) = -1$$

Thus:

$$\Lambda(0) = \{-1, +1\} \implies \limsup_{x\to 0} f(x) = 1 \quad \textit{and} \quad \liminf_{x\to 0} f(x) = -1$$

Continuous Functions

Definitions and Properties

Definition 3.78. *We say that the function f is continuous at $x_0 \in \mathbb{R}$ if: 1. f is defined in a neighborhood of x_0. 2.*

$$\lim_{x\to x_0} f(x) = f(x_0)$$

In other words:

$(f$ *continuous at* $x_0) \Leftrightarrow (\forall\epsilon > 0, \exists\delta = \delta(\epsilon) > 0, \forall x \in V^0(x_0), (|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \epsilon))$

Definition 3.79. *Increment of the Function: Let f be a function defined in a neighborhood of x_0. The value $x - x_0 = \Delta x$ is called the increment of the argument x at the point x_0. The value $f(x) - f(x_0) = \Delta f(x)$ is called the increment of the function $f(x)$ at the point x_0.*

Definition 3.80. *Using Sequences: Let f be a function defined in a neighborhood V of x_0. We say that f is continuous at x_0 if:*

$$\forall (u_n) \subset V;\ \lim_{n\to\infty} u_n = x_0 \Rightarrow \lim_{n\to\infty} f(u_n) = f(x_0)$$

Continuity from the Right – Continuity from the Left:

Definition 3.81. *i. f is said to be continuous from the right at $x_0 \in \mathbb{R}$ if: 1. f is defined on $[x_0, b]$. 2.*

$$\lim_{x\to x_0^+} f(x) = f(x_0 + 0) = f(x_0)$$

ii. f is said to be continuous from the left at $x_0 \in \mathbb{R}$ if: 1. f is defined on $[a, x_0]$. 2.

$$\lim_{x\to x_0^-} f(x) = f(x_0 - 0) = f(x_0)$$

Theorem 3.82. *For f to be continuous at x_0, it is necessary and sufficient that:*

$$f(x_0 + 0) = f(x_0 - 0) = f(x_0)$$

Theorem 3.83. *Continuity on a Set: f is continuous on an interval I if it is continuous at every point of this interval.*

Discontinuity – Classification of Points of Discontinuity:

Definition 3.84. *The function f is said to be discontinuous at a point x_0 if it is not continuous at x_0, i.e.: 1. f is not defined at x_0. 2. The limit of f at x_0 exists but is different from $f(x_0)$. 3. The limit of f at x_0 does not exist.*

For the first case: If f is not defined at x_0 but

$$\exists \lim_{x\to x_0} f(x) = l$$

, then by setting $f(x_0) = l$, we make f continuous at x_0 (i.e., we have extended f by continuity at x_0). The point x_0 is called a removable discontinuity in this case.

Example 3.85. *The function*

$$f(x) = \frac{\sin x}{x}$$

is not defined at $x_0 = 0$, but

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

. Thus, by setting $f(0) = 1$, we obtain a function continuous at $x_0 = 0$:

$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

For the second case:

$$\exists f(x_0+0), \quad \exists f(x_0-0), \quad \text{but} \quad f(x_0+0) \neq f(x_0-0)$$

The point x_0 is called a discontinuity of the first kind.

Example 3.86.

$$f(x) = E(x), \quad x_0 = m; \quad m \in \mathbb{Z}$$

$$\lim_{x\to m^+} f(x) = \lim_{x\to m^+} E(x) = m \quad \textit{and} \quad \lim_{x\to m^-} f(x) = \lim_{x\to m^-} E(x) = m-1$$

Thus:

$$f(m+0) \neq f(m-0)$$

For the third case: If one of the limits $f(x_0+0)$ or $f(x_0-0)$ does not exist or is infinite, the point x_0 is called a discontinuity of the second kind.

Example 3.87.

$$f(x) = \frac{1}{x-2}; \quad x_0 = 2, \quad \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{1}{x-2} = \pm\infty$$

Discontinuity Points of Monotone Functions

Theorem 3.88. *Every monotone function on an interval (a,b) can only have discontinuities of the first kind.*

Proof. Let f be an increasing function on (a,b) and let $x_0 \in (a,b)$. For every point $x \leq x_0$, we have $f(x) \leq f(x_0)$. By the theorem on the limit of a monotone function, it follows that:

$$f(x_0-0) = \lim_{x\to x_0^-} f(x) \leq f(x_0)$$

Thus: - If $f(x_0-0) = f(x_0)$, then f is continuous from the left at x_0. - If $f(x_0-0) < f(x_0)$, then x_0 is a discontinuity of the first kind. □

3.1.1 Operations on Continuous Functions

Proposition 3.89. *Let f, g be two functions continuous at a point x_0. Then:*

$$f \pm g; \quad fg; \quad Af(\lambda \in \mathbb{R}); \quad \frac{f}{g}(g(x_0) \neq 0) \quad \textit{and} \quad |f|$$

are functions continuous at x_0.

Proof. f is continuous at $x_0 \iff \lim_{x\to x_0} f(x) = f(x_0)$. g is continuous at $x_0 \iff \lim_{x\to x_0} g(x) = g(x_0)$. Let $\varphi(x) = f(x) + g(x)$, then:

$$\lim_{x\to x_0} \varphi(x) = \lim_{x\to x_0} (f(x)+g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = f(x_0) + g(x_0) = \varphi(x_0)$$

□

Continuity of Composite Functions:

Proposition 3.90. *Let $y = f(x)$ be continuous at x_0, and $z = g(y)$ be continuous at $y_0 = f(x_0)$. Then the composite function $(g \circ f)(x) = g(f(x))$ is continuous at x_0.*

Proof. Let (x_n) be an arbitrary sequence in the domain of $y = f(x)$ such that:

$$\lim_{n\to\infty} x_n = x_0$$

Since f is continuous at x_0 and g is continuous at $y_0 = f(x_0)$, we have:

$$\lim_{n\to\infty}(g \circ f)(x_n) = \lim_{n\to\infty} g(f(x_n)) = g\left(\lim_{n\to\infty} f(x_n)\right) = g(f(x_0)) = (g \circ f)(x_0)$$

This shows that $(g \circ f)$ is continuous at x_0. □

Proposition 3.91. *Continuity of Monotone Functions: If f is strictly monotone on an interval $I = (a, b)$ and the image $f(I) = J$ is an interval, then f is continuous on I.*

**

Proof. We prove by contradiction. Assume that f has a discontinuity at x_0 in $I = (a, b)$. x_0 can only be a discontinuity of the first kind (since f is monotone). For every point $x < x_0$, we have $f(x) < f(x_0 - 0) < f(x_0)$ (assuming f is increasing).

$$\begin{array}{c} f(x_0) \\ \underline{\;f(x_0-0)\;} \end{array}$$

Consider the interval $(f(x_0 - 0), f(x_0))$. For every $y \in (f(x_0 - 0), f(x_0))$, there is no point $x \in (a, b)$ such that $y = f(x)$. This contradicts the hypothesis of the theorem that $f(I) = J$ is an interval. . □

Continuity of Elementary Functions:

Theorem 3.92. *All elementary functions are continuous in their respective domains..*

Example 3.93. *1. $y = \sin x$, $x_0 \in \mathbb{R}$, arbitrary.*

$$|\sin x - \sin x_0| = \left|2\sin\frac{x - x_0}{2}\cos\frac{x - x_0}{2}\right| \le 2\left|\frac{x - x_0}{2}\right| = |x - x_0|$$

since $|\sin\alpha| \le |\alpha|$ and $\cos\alpha \le 1$.

Choose $\delta(\epsilon) = \epsilon$. Then $|x - x_0| < \delta = \epsilon$ implies $|\sin x - \sin x_0| < \epsilon$. Thus, $y = \sin x$ is continuous at x_0.

2. $y = \cos x = \sin\left(\frac{\pi}{2} - x\right)$, $y = \tan x = \frac{\sin x}{\cos x}$, and $y = \cot x = \frac{\cos x}{\sin x}$ are continuous because they are composite functions.

3. $y = a^x$ $(a > 0,\ a \neq 1)$ is continuous because it is monotone, etc.

Properties of Continuous Functions

Theorem 3.94. *Bolzano-Cauchy: Let f be a function defined and continuous on a closed interval $[a, b]$ such that $f(a)f(b) < 0$ (i.e., $f(a)$ and $f(b)$ have opposite signs). Then there exists at least one point $c \in [a, b]$ such that $f(c) = 0$.*

Theorem 3.95. *Bolzano-Cauchy Intermediate Value Theorem: Let f be a function defined and continuous on an interval I of $\mathbb{R}$, and let $a, b \in I$ $(a < b)$. Then for every point α between $f(a)$ and $f(b)$, there exists a number c in $[a, b]$ such that $f(c) = \alpha$.*

Proof. Let $g(x) = f(x) - \alpha$. Then:

$$g(a) = f(a) - \alpha < 0$$

$$g(b) = f(b) - \alpha > 0$$

$$\Rightarrow g(a)g(b) < 0$$

By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $g(c) = 0$, which implies $f(c) = \alpha$. □

Corollary 3.96. *The image of an interval under a continuous function is an interval.*

Theorem 3.97. *First Weierstrass Theorem: Every function defined and continuous on a closed interval $[a, b]$ is bounded.*

Proof. By contradiction, assume that f is not bounded on $[a, b]$. Then there exists a sequence (x_n) in $[a, b]$ such that $f(x_n) > n$. Since (x_n) is a bounded sequence, we can extract a convergent subsequence $x_{n_k} \to x_0$. Then:

$$f(x_{n_k}) \to f(x_0)$$

This implies that $f(x_{n_k})$ is bounded, which contradicts the assumption $f(x_{n_k}) > n_k$. □

Theorem 3.98. *Second Weierstrass Theorem: Let f be a function defined and continuous on a closed interval $[a, b]$. Then f attains its supremum and infimum.*

Proof. Let $M = \sup_{[a,b]} f(x)$. We need to show that there exists $x_i \in [a, b]$ such that $f(x_i) = M$. Assume, for contradiction, that $f(x) < M$ for all $x \in [a, b]$. Consider the function:

$$\varphi(x) = \frac{1}{M - f(x)}$$

Since $\varphi(x)$ is continuous on $[a, b]$, it is bounded. Thus, there exists $c > 0$ such that:

$$\varphi(x) < c \Rightarrow \frac{1}{M - f(x)} < c \Rightarrow M - f(x) > \frac{1}{c} \Rightarrow f(x) < M - \frac{1}{c}$$

This contradicts the fact that M is the least upper bound of f. □

Theorem 3.99. *Inverse Function Theorem: Let f be a function defined and continuous on $I = [a, b]$, strictly increasing (resp. decreasing). Then f has an inverse function defined on the interval $J = f(I)$, which is strictly increasing (resp. decreasing). Moreover:*

$$f([a, b]) = [f(a), f(b)] \quad \textit{if } f \textit{ is increasing, and } f([a, b]) = [f(b), f(a)] \quad \textit{if } f \textit{ is decreasing.}$$

3.1.2 Uniform Continuity:

Definition 3.100. *A function f defined on an interval I is said to be uniformly continuous on I if:*

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) > 0; \quad \forall x, x' \in I; \quad (|x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon)$$

Remark 3.101. *It is clear that uniform continuity on an interval I implies simple continuity at every point x_0 of I, by taking x' instead of x_0 in the definition.*

The converse is generally not true, but the following theorem holds:

Theorem 3.102. *(Cantor):** Every function defined and continuous on a closed interval $[a, b]$ is uniformly continuous on this interval.*

Proof. We prove by contradiction. Assume that f is not uniformly continuous on $[a, b]$, i.e.:

$$\exists \varepsilon > 0, \quad \forall \delta = \delta(\varepsilon) > 0; \quad \exists x, x' \in I; \quad (|x - x'| < \delta \wedge |f(x) - f(x')| \geq \varepsilon)$$

Consider a sequence (δ_n) converging to 0. Then:

$$\exists \varepsilon > 0, \quad \forall \delta_n > 0; \quad \exists x_n, x'_n \in I; \quad (|x_n - x'_n| < \delta_n \wedge |f(x_n) - f(x'_n)| \geq \varepsilon)$$

The sequences x_n, x'_n are bounded, and from any bounded sequence, we can extract a convergent subsequence (Bolzano-Weierstrass theorem). Thus, there exist $(x_{n_k}) \subset (x_n)$ and $(x'_{n_k}) \subset (x'_n)$ such that:

$$|x_{n_k} - x'_{n_k}| < \delta_{n_k} \implies \lim_{k \to \infty} |x_{n_k} - x'_{n_k}| = 0$$

Therefore:

$$\lim_{k \to \infty} x_{n_k} = \lim_{k \to \infty} x'_{n_k} = x_0$$

By the continuity of f on $[a, b]$, we obtain:

$$\lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(x'_{n_k}) = f(x_0) \implies \lim_{k \to \infty} (f(x_{n_k}) - f(x'_{n_k})) = 0$$

This contradicts the fact that $|f(x_n) - f(x'_{n_k})| \geq \varepsilon$ for all $n = 1, 2, \ldots$. □

Example 3.103. *The function $f(x) = \frac{3}{x^2 - 25}$ is uniformly continuous on* $[1, 3]$.

3.2 Differentiable Functions

3.2.1 Derivatives at a Point:

Let f be a function defined in a neighborhood $V(x_0)$ of the point x_0 in $\mathbb{R}$.

Definition 3.104. *We say that f is differentiable at x_0 if the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. This limit is called the derivative of f at x_0 and is denoted by $f'(x_0)$.

Alternative Notations: - Let $x - x_0 = \Delta x = h$. Then:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Example 3.105. *Find the derivative of $y = x^3$ at a point x_0 in $\mathbb{R}$:*

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^3 - x_0^3}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x^2 + xx_0 + x_0^2)}{x - x_0} = x_0^2 + x_0^2 + x_0^2 = 3x_0^2$$

3.2.2 Differentiability at a Point:

Definition 3.106. *We say that a function f is differentiable at x_0 if: 1. f is defined in a neighborhood of x_0. 2. There exists a number A in $\mathbb{R}$ and an infinitesimal function $\alpha(x)$ such that the increment Δf of f corresponding to the increment Δx of x can be written as:*

$$\Delta f(x) = f(x + \Delta x) - f(x) = A\Delta x + \alpha(\Delta x)\Delta x; \quad \lim_{\Delta x \to 0} \alpha(\Delta x) = 0 \quad (*)$$

The following theorem holds:

Theorem 3.107. *For a function to be differentiable at a point x_0, it is necessary and sufficient that it be differentiable at that point.*

Proof. **Necessary Condition:** If f is differentiable at x in $\mathbb{R}$, then (*) holds, and thus:

$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} (A + \alpha(\Delta x)) = A \Rightarrow f'(x) = A$$

Sufficient Condition: If f is differentiable at x in $\mathbb{R}$, then f is differentiable at that point, which follows from the definition. □

Theorem 3.108. *Every function differentiable at a point x_0 is continuous at that point.*

Proof.

$$f \text{ differentiable at } x_0 \Rightarrow \exists \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$\Leftrightarrow \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \alpha(x - x_0) \quad \text{where} \quad \lim_{x \to x_0} \alpha(x - x_0) = 0$$

$$\Rightarrow f(x) - f(x_0) = f'(x_0)(x - x_0) + \alpha(x - x_0)(x - x_0)$$

$$\Rightarrow f(x) = f(x_0) + f'(x_0)(x - x_0) + \alpha(x - x_0)(x - x_0)$$

$$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

□

Remark 3.109. *The converse is not true.*

Example 3.110. *The function $y = |x|$, $x_0 = 0$ is continuous at $x_0 = 0$, but:*

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Thus, it is not differentiable at x_0.

3.2.3 Left and Right Derivatives:

Definition 3.111. *We say that f is differentiable at x_0 from the right (resp. from the left) if: 1. f is defined on $[x_0, b]$ (resp. $[a, x_0]$).*
2. $\exists \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = f'_+(x_0)$ $\left(\text{resp.}\ \ \exists \lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = f'_-(x_0)\right)$

Theorem 3.112. *For a function f to be differentiable at a point x_0, it is necessary and sufficient that it be differentiable at x_0 from the left and from the right and that $f'_+(x_0) = f'_-(x_0)$.*

3.2.4 Differential:

Let f be a function differentiable at x_0. Then:

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \alpha(x - x_0)(x - x_0)$$

Let $x - x_0 = \Delta x \Rightarrow x = x_0 + \Delta x$:

$$f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0) = \alpha(\Delta x)\Delta x \Rightarrow \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

Thus:

$$\Delta f(x_0) = f'(x_0)\Delta x + \alpha(\Delta x)\Delta x$$

Definition 3.113. *The expression $f'(x)\Delta x$ is called the differential of f at x and is denoted by $df(x) = f'(x)\Delta x$.*

In particular, for $f(x) = x$, we have $f'(x) = 1$, so $dx = 1 \cdot \Delta x = \Delta x$. Therefore:

$$df(x) = f'(x)\Delta x \Rightarrow f'(x) = \frac{df(x)}{\Delta x}$$

3.2.5 Application of the Differential to Approximate Calculations:

Find an approximate value of $\sqrt{1.0005}$. If f is differentiable at x_0 in D_f, then:

$$\Delta f(x) = df(x) + \alpha(\Delta x)\Delta x \Rightarrow \Delta f(x) \approx df(x) \Rightarrow f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x$$

$$x - x_0 = \Delta x \Leftrightarrow x = x_0 + \Delta x$$

If $x_0 = 0$, then $f(x) = f(0) + f'(0)x$. In our example: $f(x) = (1 + x)^{\alpha} \Longrightarrow f'(x) = \alpha\,(1 + x)^{\alpha - 1}$

$f(x) = \sqrt{1 + x} = (1 + x)^{\frac{1}{2}} \Longrightarrow f'(x) = \frac{1}{2}(1 + x)^{-\frac{1}{2}}$

$f(0) = 1$, $f'(0) = \frac{1}{2} \Longrightarrow f(x) = 1 + \frac{1}{2}x$

Then $\sqrt{1.0005} = \sqrt{1 + 0.0005} = 1 + \frac{1}{2}(0.0005) = 1.00025$

Geometric interpretation of the derivative

let f a derivable function at x_0 and $f'(x_0) \neq 0$ and lat C the graph of the function f, $M(x_0, y_0)$ a point of C that is $f(x_0) = y_0$ and $M_0\left(x_0 + \triangle x,\ y_0 + \triangle y\right)$ a variable point on C that is, $f\left(x_0 + \triangle x\right) = y_0 + \triangle y$. i.e MM_0 is a secant of the curve C. When $\triangle x \longrightarrow 0$, $M \longrightarrow M_0$ and the line (MM_0) take the position of the tangent; that is, the derivative of y with respect to x is the slope of the line

$$\begin{aligned}\lim_{x \longrightarrow x_0} \frac{\triangle y}{\triangle x} &= \lim_{x \longrightarrow x_0} \tan\beta \Longrightarrow \lim_{x \longrightarrow x_0} \frac{f\left(x_0 + \triangle x\right) - f(x_0)}{x - x_0} \\ &= \lim_{\beta \longrightarrow \alpha} \tan\beta = \tan\alpha \Longrightarrow f'(x_0) = \tan\alpha.\end{aligned}$$

That is, the tangent line to the graph of $y = f(x)$ at $x = a$ is the line with equation $y = f\prime(a)(x - a) + f(a)$

Hence a tangent line to the graph of a function f is a line through a point on the graph of f whose slope is equal to the slope of the graph at that point.

Arithmetic operation on the derivatives

Let f and g be two derivable functions at x_0 then

$f \pm g$, $f.g$, $\lambda f\ (\lambda \in \mathbb{R})$, $\frac{f}{g}$ (si $g(x) \neq 0$ on $V\left(x_0\right)$ the neighborhood of x_0) are derivable at x_0 and we have

1. $(f \pm g)' = f' \pm \pm g'$
2. $(f.g)' = f'.g + f.g'$
3. $(\lambda f)' = \lambda f'$
4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Let's show the first statement

$$\begin{aligned}(f+g)'(x_0) &= \lim_{x \longrightarrow x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} \\ &= \lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0) + g(x) - g(x_0)}{x - x_0} = f'(x_0) + g'(x_0)\end{aligned}$$

Derivative of a Composite Function:

Proposition 3.114. *If $y = f(x)$ is differentiable at x_0 and $z = g(y)$ is differentiable at $y_0 = f(x_0)$, then the composite function $g \circ f(x) = g(f(x))$ is differentiable at x_0, and we have:*

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0)$$

Proof.

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{y\to y_0}\frac{g(y)-g(y_0)}{y-y_0}\cdot\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = g'(y_0)f'(x_0)$$

This holds because $y=f(x)$ is differentiable at x_0 and thus continuous at that point. ☐

3.2.6 Derivative of an Inverse Function:

Theorem 3.115. *If f is bijective in a neighborhood of x_0 and $f'(x_0)$ exists and is non-zero, then f^{-1} is differentiable at $y_0=f(x_0)$, and moreover:*

$$\left(f^{-1}\right)'(y_0)=\frac{1}{f'(x_0)}=\frac{1}{f'(f^{-1}(y_0))}$$

Proof.

$$\left(f^{-1}\right)'(y_0)=\lim_{y\to y_0}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}=\lim_{y\to y_0}\frac{1}{\frac{y-y_0}{f^{-1}(y)-f^{-1}(y_0)}}=\lim_{y\to y_0}\frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}=\frac{1}{f'(x_0)}$$

☐

Theorem 3.116. *The following elementary functions are differentiable in the indicated domains, and we have: 1. $y=constant$, $y'=0$, $\forall x\in\mathbb{R}$ 2. $y=x^a$, $y'=ax^{a-1}$, $x>0$ 3. $y=a^x$, $y'=a^x\ln a$, $\forall x\in\mathbb{R}$, $(a>0$ and $a\neq 1)$ 4. $y=\log_a x$, $y'=\frac{1}{x}\log_a e$, $x>0$, $(a>0$ and $a\neq 1)$ 5. $y=\sin x$, $y'=\cos x$, $\forall x\in\mathbb{R}$ 6. $y=\cos x$, $y'=-\sin x$, $\forall x\in\mathbb{R}$ 7. $y=\tan x$, $y'=\frac{1}{\cos^2 x}$, $x\neq\frac{\pi}{2}+k\pi$ 8. $y=\cot x$, $y'=-\frac{1}{\sin^2 x}$, $x\neq k\pi$ 9. $y=\arcsin x$, $y'=\frac{1}{\sqrt{1-x^2}}$, $x\in[-1,+1]$ 10. $y=\arccos x$, $y'=-\frac{1}{\sqrt{1-x^2}}$, $x\in[-1,+1]$ 11. $y=\arctan x$, $y'=\frac{1}{1+x^2}$, $x\in\mathbb{R}$ 12. $y=arccot\ x$, $y'=\frac{-1}{1+x^2}$, $x\in\mathbb{R}$*

13. $y=\sinh x$, $y'=\cosh x$, $x\in\mathbb{R}$
14. $y=\cosh x$, $y'=\sinh x$, $x\in\mathbb{R}$
15. $y=\tanh x$, $y'=\frac{1}{\cosh^2 x}$, $x\in\mathbb{R}$
16. $y=\coth x$, $y'=\frac{-1}{\sinh^2 x}$, $x\neq 0$
17. $y=argsinh\ x$, $y'=\frac{1}{\sqrt{1+x^2}}$, $x\in\mathbb{R}$
18. $y=argcosh\ x$, $y'=\frac{1}{\sqrt{x^2-1}}$, $x>1$
19. $y=argtanh\ x$, $y'=\frac{1}{1-x^2}$, $x\in[-1,+1]$
20. $y=argcoth\ x$, $y'=\frac{-1}{1-x^2}$, $|x|>1$

Proof. 3. $(a^x)'=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=a^x\lim_{h\to 0}\frac{a^h-1}{h}=a^x\ln a$

4. $(\log_a x)'=\frac{1}{(a^y)'}=\frac{1}{a^y\ln a}=\frac{1}{x\ln a};\quad (y=\log_a x\Leftrightarrow x=a^y)$

5. $(\cos x)'=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}=-\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}=-\sin x$

9. $(\arcsin x)'=\frac{1}{(\sin y)'}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}};\quad (y=\arcsin x\Leftrightarrow x=\sin y)$ ☐

Example 3.117. *Calculate the derivative of the composite function $y=\ln^4\left(\arctan\frac{x}{x^2+1}\right)$.*

$$y'=4\left(\arctan\frac{x}{x^4+1}\right)\ln^3\left(\arctan\frac{x}{x^4+1}\right)=4\left(\frac{x}{x^4+1}\right)\ln^3\left(\arctan\frac{x}{x^4+1}\right)$$

Remark 3.118. *If f is differentiable at x_0 and $f(x_0) \neq 0$, then the function $y = \log|f(x)|$ is differentiable at x_0, and we have:*

$$(\log|f(x)|)' = \frac{f'(x)}{f(x)}$$

3.2.7 Higher-Order Derivatives:

Let f be a differentiable function on an interval I. Then f' is called the first-order derivative of f.

Definition 3.119. *If the derivative $y' = f'(x)$ is differentiable on I, then its derivative is called the second-order derivative of f, denoted by f'', and we have:*

$$f''(x) = (f'(x))'$$

Similarly, we define the n-th order derivative of f, denoted by $f^{(n)}(x)$, as:

$$f^{(n)}(x) = (f^{(n-1)}(x))'$$

Definition 3.120. *The function f is said to be of class $C^{(n)}(I)$ if the n-th derivative of f exists and is continuous on I.*

Example 3.121.

$$\begin{aligned} y &= \sin x,\ x \in \mathbb{R},\ y^{(n)} = ? \\ y' &= \cos x = \sin\left(x + \frac{\pi}{2}\right) \\ y'' &= -\sin x = \sin(x+\pi) = \sin\left(x + 2\cdot\frac{\pi}{2}\right) \\ &\vdots \\ y^{(n)} &= \sin\left(x + n\cdot\frac{\pi}{2}\right) \quad \textit{(proof by induction)} \end{aligned}$$

3.2.8 Leibniz Formula:

Proposition 3.122. *Let f and g be two functions n times differentiable at x. Then:*

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)$$

Proof. by induction For $n = 1$:

$$(f\cdot g)'(x) = \sum_{k=0}^{1} \binom{1}{k} f^{(1-k)}(x) g^{(k)}(x) = \binom{1}{0} f'(x)g(x) + \binom{1}{1} f(x)g'(x) = f'(x)g(x) + f(x)g'(x)$$

Thus, $P(1)$ is true. Assume that $P(n)$ is true and prove $P(n+1)$:

$$(f\cdot g)^{(n+1)} = \left[\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)\right]' = \sum_{k=0}^{n} \binom{n}{k} \left[f^{(n-k+1)}(x) g^{(k)}(x) + f^{(n-k)}(x) g^{(k+1)}(x)\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k+1)}(x) g^{(k)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k+1)}(x)$$

$$= \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)}(x) g^{(k)}(x) + \binom{n}{0} f^{(n+1)}(x) g(x) + \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)}(x) g^{(k)}(x) + \binom{n}{n} f(x) g^{(n+1)}(x)$$

$$= \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) f^{(n-k+1)}(x) g^{(k)}(x) + f^{(n+1)}(x) g(x) + f(x) g^{(n+1)}(x)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)}(x) g^{(k)}(x)$$

Hence, the proposition $P(n+1)$ holds. □

Example 3.123. *Calculate* $(x^3 \cos 4x)^{(4)}$.

$$(x^3 \cos 4x)^{(4)} = \sum_{k=0}^{4} \binom{4}{k} (\cos 4x)^{(4-k)} (x^3)^{(k)}$$

$$= \binom{4}{0} (\cos 4x)^{(4)} (x^3) + \binom{4}{1} (\cos 4x)^{(3)} (x^3)' + \binom{4}{2} (\cos 4x)^{(2)} (x^3)^{(2)} + \binom{4}{3} (\cos 4x)' (x^3)^{(3)} + \binom{4}{4} (\cos 4x)(x^3$$

$$= 4^4 x^3 \cos 4x - \binom{4}{1} 4^3 \cdot 3x^2 \sin 4x + \binom{4}{2} 4^2 \cdot 6x \cos 4x - \binom{4}{3} \cdot 24 \cdot \sin 4x + \binom{4}{4} \cos 4x \cdot 0$$

3.2.9 Higher-Order Differentials:

Let f be a differentiable function on a set X. Then it is differentiable on X, and we have:

$$df(x) = f'(x)dx$$

This is the first-order differential of f.

If f' is differentiable on X, then $df(x)$ is differentiable on X, and we have:

$$d^2 f(x) = d(df(x)) = d(f'(x)dx) = (f''(x)dx)dx = f''(x)dx^2$$

This is the second-order differential of f.

Similarly, if f is n times differentiable on X, then it is n times differentiable on X, and we have:

$$d^n f(x) = f^{(n)}(x) dx^n$$

3.2.10 Fundamental Theorems on Differentiable Functions:

Rolle's Theorem:

Theorem 3.124. *Rolle's Theorem: Let f be a function satisfying:*

1. *Defined and continuous on the closed interval $[a, b]$*
2. *Differentiable on the open interval (a, b)*
3. *$f(a) = f(b)$ Then there exists a point c in (a, b) such that $f'(c) = 0$.*

**

Proof. Since f attains its maximum and minimum on $[a, b]$ (at least one of these extrema is in $[a, b]$), let:

$$\sup_{[a,b]} f(x) = f(c), \quad c \in [a, b]$$

This gives:

$$\forall \Delta x > 0 \quad (\text{or } \Delta x < 0); \quad f(c + \Delta x) \leq f(c)$$

$$\Rightarrow \begin{cases} \frac{f(c+\Delta x)-f(c)}{\Delta x} \geq 0 & \text{if } \Delta x < 0 \\ \frac{f(c+\Delta x)-f(c)}{\Delta x} \leq 0 & \text{if } \Delta x > 0 \end{cases}$$

Taking the limit as $\Delta x \to 0$:

$$\begin{cases} \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} \geq 0 \implies f'(c) \geq 0 \\ \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} \leq 0 \implies f'(c) \leq 0 \\ \Rightarrow f'(c) = 0 \end{cases}$$

□

Geometric Interpretation:

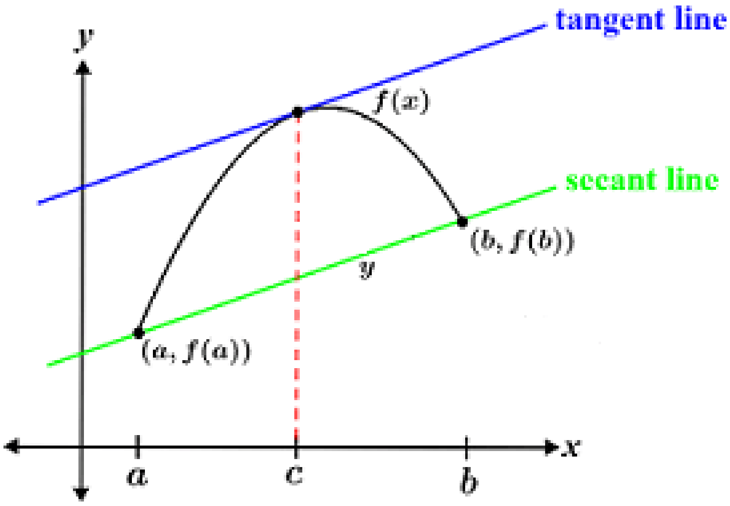
Remark 3.125. *All the conditions of Rolle's Theorem are necessary.*

Lagrange's Theorem (Mean Value Theorem):

Theorem 3.126. *Let f be a function satisfying: 1. Defined and continuous on $[a, b]$ 2. Differentiable on (a, b)*

Then there exists a point c in (a, b) such that:

$$f(b) - f(a) = (b - a)f'(c)$$



Proof. Consider the function:

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

The function $g(x)$ is: 1. Continuous on $[a, b]$ because it is composed of continuous functions. 2. Differentiable on (a, b). 3. $g(a) = 0$ and $g(b) = 0$. By Rolle's Theorem, there exists a point c in (a, b) such that $g'(c) = 0$. Now:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Thus:

$$g'(c) = 0 \Rightarrow f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \Rightarrow f(b) - f(a) = (b - a)f'(c)$$

□

Cauchy's Theorem:

Theorem 3.127. *Let f and g be two functions satisfying: 1. Defined and continuous on $[a, b]$ 2. Differentiable on (a, b) 3. $g'(x) \neq 0$ for all $x \in (a, b)$ Then there exists c in (a, b) such that:*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof. Consider the function:

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$$

It is clear that the function F satisfies the conditions of Rolle's Theorem, and thus there exists a point c in (a, b) such that $F'(c) = 0$. Now:

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(x)$$

$$F'(c) = 0 \implies f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(c) = 0 \implies \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

□

3.2.11 Derivative of an Implicit Function:

Let $y = f(x)$ be an implicit function given by the equation $F(x,y) = 0$. That is, $F(x, f(x)) = 0$ for all $x \in (a,b)$.

Derivative of a Function Given in Parametric Form:

Let $y = f(x)$ such that

$$\begin{cases} x = \varphi(t) \\ y = \phi(t) \end{cases} \qquad t_0 \leq t \leq T.$$

Assume that φ and ϕ are differentiable on (t_0, T) and that $t = \mathcal{G}(x)$ (where $\mathcal{G}$ is the inverse of φ). Then $y = \phi(t) = \phi(\mathcal{G}(x))$. Thus, by the chain rule, we obtain:

$$y'_x = y'_t \cdot t'_x = \frac{y'_t}{x'_t} \implies y'_x = \frac{y'_t}{x'_t}$$

Example 3.128. *Let*

$$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases} \qquad 0 \leq t \leq \frac{\pi}{2} \quad \textit{then:}$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{a\cos t}{-a\sin t} = -\cot t; \quad t \neq k\pi$$

Equation of the Tangent and Normal to a Curve:

Let $y = f(x)$, $x \in I$, with the curve C and let $M_0(x_0, y_0) \in C$. We want to write the equations of the tangent and normal to the curve C at the point $M_0(x_0, y_0)$. The equation of the tangent passing through the point $M_0(x_0, y_0)$ is given by:

$$y - y_0 = k(x - x_0), \quad y_0 = f(x_0) \quad \text{and} \quad k = \tan\alpha = f'(x_0)$$

Thus:

$$y_T = f'(x_0)(x - x_0) + f(x_0)$$

Equation of Normal

It is of the form $y_N = k_1(x - x_0) + y_0$.

$$MT \perp NA \Rightarrow k_1 = -\frac{1}{k}$$

Thus:

$$y_N = -\frac{1}{k}(x - x_0) + f(x_0)$$

3.2.12 Limit of the Ratio of Two Infinitesimals

First L'Hôpital's Rule:

Theorem 3.129. *Let f and g be two functions defined and differentiable in a punctured neighborhood $V^0(x_0)$ satisfying the following conditions: 1.*

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$$

2.

$$g'(x) \neq 0 \quad \textit{on} \quad V^0(x_0)$$

If there exists:

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = I \quad \textit{then} \quad \exists \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} \quad \textit{and we have:} \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = I$$

Proof. Condition 1 implies that f and g can be extended to x_0 and that $f(x_0) = g(x_0) = 0$.

Let x be an arbitrary point in $V^0(x_0)$. Then f and g satisfy the conditions of Cauchy's Theorem on the interval $[x_0, x]$, i.e.:

$$\exists c \in [x_0, x];$$

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \Rightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(c)}{g'(c)} = I$$

□

Remark 3.130. *L'Hôpital's Rule is valid if:*

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} \quad \textit{exists, because sometimes} \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} \quad \textit{exists}$$

while:

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} \quad \textit{does not exist.}$$

Example 3.131.

$$f(x) = x^2 \cos\frac{1}{x}, \quad g(x) = x; \quad x \to 0$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^2 \cos\frac{1}{x}}{x} = \lim_{x\to 0} x \cos\frac{1}{x} = 0$$

However:

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{2x\cos\frac{1}{x} + \sin\frac{1}{x}}{1} \quad \textit{does not exist!}$$

Remark 3.132. *L'Hôpital's Rule remains valid when $x \to \infty$ because:*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t\to 0} \frac{f'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}{g'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)} = \lim_{t\to 0} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Remark 3.133. *If $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \frac{0}{0}$ and the functions $f'(x)$ and $g'(x)$ satisfy the conditions of L'Hôpital's Rule, then we can apply L'Hôpital's Rule again.*

Limit of the Ratio of Two Infinitely Large Functions: Second L'Hôpital's Rule:

Theorem 3.134. *Let f and g be two functions defined and differentiable in a punctured neighborhood $V^0(x_0)$ satisfying the following conditions: 1. $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \infty$ 2. $g'(x) \neq 0$ on $V^0(x_0)$ If there exists $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l$, then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and we have:*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = l$$

Remark 3.135. *Remarks 1-3 remain valid for Theorem 2.*

Example 3.136. *Calculate $\lim_{x\to\infty} \frac{x^n}{e^x}$.*

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0$$

Other Indeterminate Forms:

$\alpha 0, \omega, \gamma, \omega + \infty - \infty, \omega, \gamma, \omega^0, \gamma, \omega^0$. 1. If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = \infty$, then:

$$\lim_{x\to x_0} [f(x)g(x)] = \lim_{x\to x_0} \frac{f(x)}{\frac{1}{g(x)}} = \left(\frac{0}{0}\right)$$

2. If $\lim_{x\to x_0} f(x) = \infty$ and $\lim_{x\to x_0} g(x) = \infty$, then:

$$\lim_{x\to x_0} [f(x) - g(x)] = \lim_{x\to x_0} \frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{g(x)}} = \left(\frac{0}{0}\right)$$

3. If $\lim_{x\to x_0} f(x) = 1$ and $\lim_{x\to x_0} g(x) = \infty$, then by setting $y = (f(x))^{g(x)}$, we obtain:

$$\ln y(x) = g(x) \ln f(x)$$

$$\lim_{x\to x_0} (\ln y(x)) = \lim_{x\to x_0} g(x) \ln f(x) = l \quad \text{(of the form } \infty \cdot 0)$$

$$\ln\left(\lim_{x\to x_0} y(x)\right) = l \Rightarrow \lim_{x\to x_0} y(x) = e^l$$

4. If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = \infty$, then by setting $y = (f(x))^{g(x)}$, we obtain a limit of the form $(\infty \cdot 0)$, as mentioned in the first case. 5. If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$, then by setting $y = (f(x))^{g(x)}$, we obtain a limit of the form 1.

3.2.13 Criterion for Monotonicity of a Real Function:

Theorem 3.137. *Let f be a function from $I \to \mathbb{R}$ differentiable on I. Then the following equivalences hold: 1. $f' \geq 0$ on $I \iff f$ is increasing on I.*

2. $f' \leq 0$ on $I \iff f$ is decreasing on I.

Proof. ($\Rightarrow$) Suppose f is increasing on I and prove that $f' \geq 0$. Indeed, let x_0 be an arbitrary point in I. Then for all $h \in \mathbb{R}^*$, we have:

$$\frac{f(x_0+h) - f(x_0)}{h} \geq 0 \Rightarrow \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \geq 0 \Rightarrow f'(x_0) \geq 0$$

($\Leftarrow$) Suppose $f' \geq 0$ for all $x \in I$ and prove that f is increasing.
Let x_1, x_2 be two arbitrary points in I such that $x_1 < x_2$.
By the Mean Value Theorem on $[x_1, x_2]$, there exists a point $c \in [x_1, x_2]$ such that:

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$$

$$f'(c) \geq 0 \quad \text{and} \quad (x_2-x_1) \geq 0 \Rightarrow f'(c)(x_2-x_1) \geq 0 \Rightarrow f(x_2)-f(x_1) \geq 0 \Rightarrow f(x_2) \geq f(x_1)$$

Thus, f is increasing on I. ☐

Example 3.138. *Show that* $\tan x > x + \frac{x^3}{3}$ *for all* $x \in \left[0, \frac{\pi}{2}\right]$.
Let $y = \tan x - \left(x + \frac{x^3}{3}\right)$.

$$y(0) = 0$$

$$y'(x) = \frac{1}{\cos^2 x} - (1+x^2) = 1+\tan^2 x - 1 - x^2 = \tan^2 x - x^2 > 0 \quad \forall x \in \left[0, \frac{\pi}{2}\right] \Rightarrow y(x) \textit{ is increasing.}$$

Thus:

$$\tan x - \left(x + \frac{x^3}{3}\right) > 0 \iff \tan x > x + \frac{x^3}{3}$$

3.2.14 Extrema:

Let f be a function defined on I, and $x_0 \in I$.

Definition 3.139. *We say that f has a local maximum (resp. local minimum) at the point x_0 if there exists a neighborhood V of x_0 such that: For all $x \in V$, $f(x) \leq f(x_0)$ [resp. $f(x) \geq f(x_0)$]. "max f" and "min f" are called the extrema of the function f.*

Maximum local **Minimum local**

x_0 x_1

Definition 3.140. *We say that f has an absolute maximum (resp. absolute minimum) at the point x_0 of I if*

$$\forall x \in I, f(x) \leq f(x_0) \quad \textit{[resp. } f(x) \geq f(x_0)\textit{]}$$

Necessary Condition for an Extremum:

Theorem 3.141. *If f has an extremum at x_0 and is differentiable at this point, then* $f'(x_0) = 0$.

Proof. Analogous to the proof of Rolle's Theorem. ☐

Remark 3.142. *This theorem is not sufficient for the existence of an extremum.*

Example 3.143. *The function $y = x^3$ does not have an extremum at $x_0 = 0$ even though* $y'(0) = 0$.

Remark 3.144. *A function f may have an extremum at a point x_0 without being differentiable at that point.*

Example 3.145. *The function $y = |x|$ is not differentiable at $x_0 = 0$, but it has a minimum at this point. The points where f' is zero or does not exist are called critical points of f.*

First Sufficient Condition:

Proposition 3.146. *Let f be defined and continuous in a neighborhood V of the point x_0, and differentiable in this neighborhood except possibly at x_0. If the derivative f' changes sign from left to right of the point x_0, then f has an extremum at this point. Moreover:*

$$\begin{cases} f' > 0 \quad \textit{if } x < x_0 \quad \textit{and } f' < 0 \quad \textit{if } x > x_0 \Rightarrow x_0 \quad \textit{is a maximum point of } f \\ f' < 0 \quad \textit{if } x < x_0 \quad \textit{and } f' > 0 \quad \textit{if } x > x_0 \Rightarrow x_0 \quad \textit{is a minimum point of } f \end{cases}$$

Proof. Assume that $f' > 0$ if $x < x_0$ and $f' < 0$ if $x > x_0$, and show that x_0 is a maximum of f. By the Mean Value Theorem, there exists a point c such that:

$$f(x) - f(x_0) = (x - x_0)f'(c)$$

If $x < x_0$, $f'(c) > 0$ and $(x - x_0) < 0$, then $f'(c)(x - x_0) < 0 \Rightarrow f(x) < f(x_0)$.
If $x > x_0$, $f'(c) < 0$ and $(x - x_0) > 0$, then $f'(c)(x - x_0) < 0 \Rightarrow f(x) < f(x_0)$.
Thus, x_0 is a maximum point of f. □

Example 3.147. *Determine the extrema of $y = 3x^2 - 10x + 5$. We have:*

$$y'(x) = 0 \Leftrightarrow 6x - 10 = 0 \Leftrightarrow x = \frac{5}{3}$$

And:

$$y' : \quad \begin{array}{ccc} - & | & + \\ \frac{5}{3} & \frac{5}{3} & \end{array}$$

Second Sufficient Condition:

Proposition 3.148. *Let f be a function twice differentiable at the point x_0 with $f'(x_0) = 0$. Then:*

- If $f''(x_0) < 0$, then x_0 is a maximum of f. - If $f''(x_0) > 0$, then x_0 is a minimum of f.

Proof. Assume that $f'(x_0) = 0$ and $f''(x_0) < 0$, and prove that x_0 is a maximum point. Indeed:

$$f''(x_0) < 0 \Rightarrow (f')'(x_0) < 0 \Rightarrow f \text{ is decreasing in the neighborhood of } x_0.$$

Moreover, $f'(x_0) = 0$, so:

$$f' > 0 \quad \text{if} \quad x < x_0 \quad \Rightarrow \quad x_0 \text{ is a maximum point of } f.$$

$$f' < 0 \quad \text{if} \quad x > x_0.$$

□

Remark 3.149. *If $f'(x_0) = 0$ and $f''(x_0) = 0$, we cannot conclude anything.*

Third Sufficient Condition:

Proposition 3.150. *Let f be a function n times differentiable at $x = x_0$. If $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$, then:*

1. When n is an even number, the function f has an extremum at x_0, and moreover: - a) $f^{(n)}(x_0) < 0 \Rightarrow x_0$ is a maximum. - b) $f^{(n)}(x_0) > 0 \Rightarrow x_0$ is a minimum.

2. When n is an odd number, the function f does not have an extremum.

Convexity – Concavity

Definition 3.151. *The function $f : I \to \mathbb{R}$ with the representative curve is said to be convex (resp. concave) on the interval I if any chord passing through arbitrary points of C lies above the arc AB (resp. below AB) of this curve.*

Definition 3.152. *The function $f : I \to \mathbb{R}$ with the representative curve is said to be convex (resp. concave) on the interval I if:*

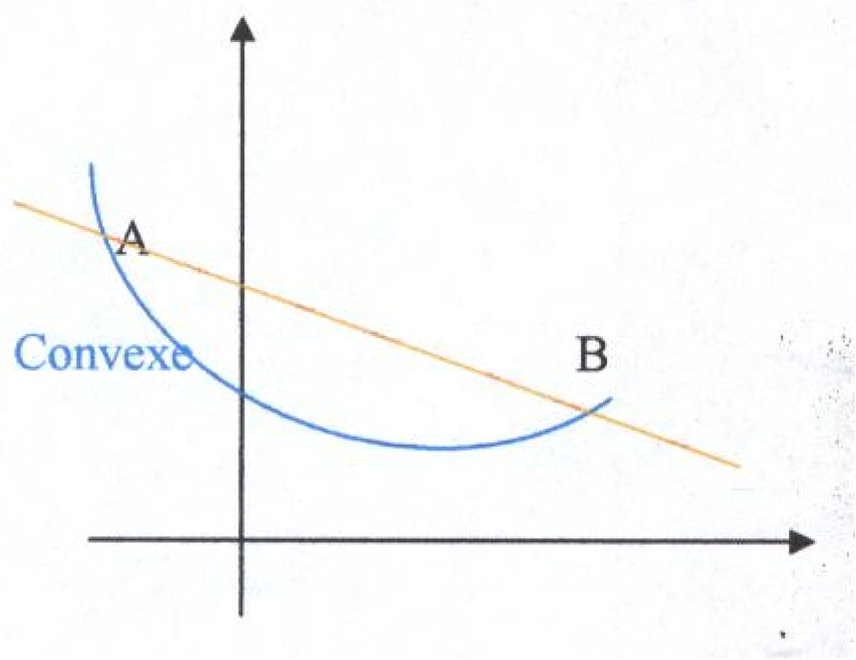
$$\forall x_1, x_2 \in I, \quad \forall q_1 > 0, \quad q_2 > 0 \quad \textit{with} \quad q_1 + q_2 = 1 \quad \textit{we have}$$

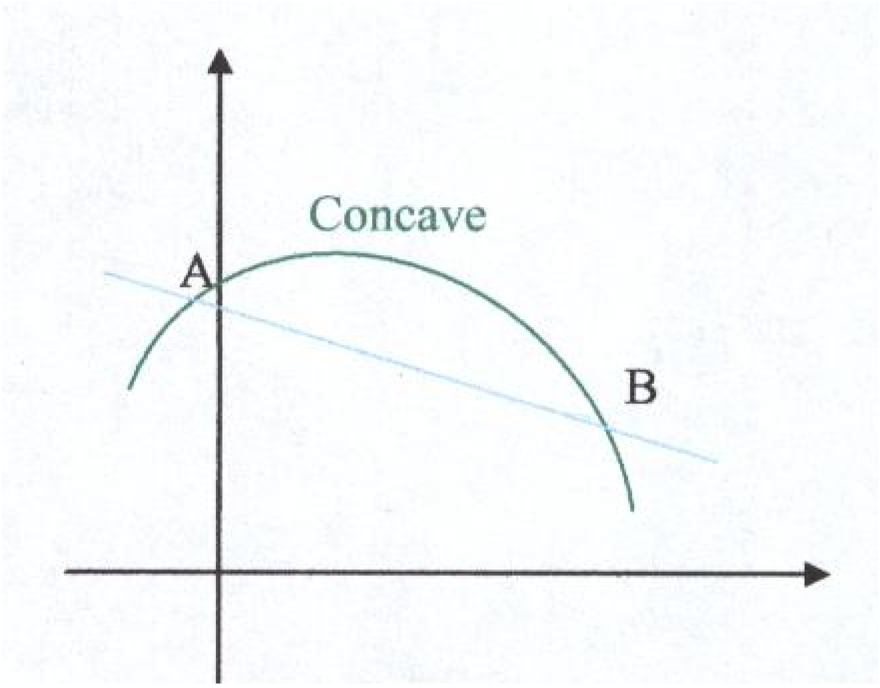
$$f(q_1x_1 + q_2x_2) \leq q_1f(x_1) + q_2f(x_2)\,[\textit{resp.} \cdot f(q_1x_1 + q_2x_2) \geq q_1f(x_1) + q_2f(x_2)]$$

Definition 3.153. *The function $f : I \to \mathbb{R}$ with the representative curve is said to be convex (resp. concave) on the interval I if any tangent to the curve C at the point with abscissa x_0 of I lies above the curve (resp. below the curve).*

Theorem 3.154. *If $f : I \to \mathbb{R}$ is twice differentiable on I, then the following equivalences hold:*

1. *f is convex on $I \Leftrightarrow f'' \geq 0$ on I*
2. *f is concave on $I \Leftrightarrow f'' \leq 0$ on I*





Proof. Let $f : I \to \mathbb{R}$ be twice differentiable on I and x_0 an arbitrary point of I. The equation of the tangent to the curve at the point x_0 is given by $y_T = f'(x_0)(x-x_0)+f(x_0)$. The equation of the curve is $y = f(x)$. We have:

$y-y_T = f(x)-f(x_0)-f'(x_0)(x-x_0)$ By Lagrange's Theorem: $f'(c)(x-x_0)-f'(x_0)(x-x_0), \quad (x < c <$

$= [f'(c)-f'(x_0)](x-x_0)$ By Lagrange's Theorem: $f''(c)(c-x_0)(x-x_0), \quad (c < c < x_0)$

The sign of $y - y_T$ is the same as that of $f''(c)$ because $(c - x_0)(x - x_0) > 0$. □

3.2.15 Points of Inflection

Definition 3.155. *Let $f : I \to \mathbb{R}$ with the curve C.*

The point $M_0(x_0, y_0)$ is called a point of inflection of the curve C at x_0 if the curve C is convex on one side of x_0 and concave on the other side.

Necessary Condition: If f is twice differentiable at x_0 and f' has a point of inflection at x_0, then $f''(x_0) = 0$.

Sufficient Condition: Let f be twice differentiable in a neighborhood of the point x_0, except possibly at x_0. If f'' changes sign from left to right of the point x_0, then f has a point of inflection at x_0.

3.2.16 Asymptotes:

Let $f : I \to \mathbb{R}$ with the curve C.

Definition 3.156. *The line $x = x_0$ is called a vertical asymptote of the curve C if:*

$$\lim_{x \to x_0} f(x) = \infty$$

Example 3.157. *The line $x = 0$ is an asymptote of the curve $y = \log x$.*

Definition 3.158. *The line $y = y_0$ is called a horizontal asymptote of the curve C if:*

$$\lim_{x \to \infty} f(x) = y_0$$

Example 3.159. *The line $y = \frac{\pi}{2}$ is an asymptote of the curve $y = \arctan x$.*

Definition 3.160. *The line $y = kx + b$ is called an oblique asymptote of the curve C if there exists a function $h = h(x)$ such that $f(x) = kx + b + h(x)$ and $\lim_{x \to \infty} h(x) = 0$. We have:*

$$f(x) = kx + b + h(x) \Rightarrow k = \frac{f(x)}{x} - \frac{b}{x} - \frac{h(x)}{x}$$

Taking the limit as x tends to infinity, we obtain:

$$k = \lim_{x \to \infty} \frac{f(x)}{x} \quad \textit{and} \quad b = \lim_{x \to \infty} [f(x) - kx]$$

General Scheme for Plotting a Curve:

The study of functions generally involves determining:

1. The domain of definition of the function.
2. The points of discontinuity of the function.
3. The intervals of increase and decrease of the function.
4. The points of extrema as well as the maximum and minimum values of the function.
5. The intervals of convexity and concavity as well as the points of inflection.
6. The graphical asymptotes of the function.

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