

Exponential stability estimates for an axially travelling string damped at one end

Seyf Eddine Ghenimi & Abdelmouhcene Sengouga

To cite this article: Seyf Eddine Ghenimi & Abdelmouhcene Sengouga (2023): Exponential stability estimates for an axially travelling string damped at one end, *Applicable Analysis*, DOI: 10.1080/00036811.2023.2194307

To link to this article: <https://doi.org/10.1080/00036811.2023.2194307>



Published online: 27 Mar 2023.



Submit your article to this journal [↗](#)




View related articles [↗](#)



View Crossmark data [↗](#)



Exponential stability estimates for an axially travelling string damped at one end

Seyf Eddine Ghenimi and Abdelmouhcene Sengouga 

Laboratory of Functional Analysis and Geometry of Spaces, University of M'Sila, M'Sila, Algeria

ABSTRACT

We study the small vibrations of an axially travelling string with a dashpot damping at one end. The string is modelled by a wave equation in a time-dependent interval with two endpoints moving at a constant speed \mathbf{v} . For the undamped case, we obtain a conserved functional equivalent to the energy of the solution. We derive precise upper and lower exponentially decaying estimates for the energy with explicit constants. These estimates do not seem to be reported in the literature even for the non-travelling case $\mathbf{v} = 0$.

ARTICLE HISTORY

Received 22 February 2022
Accepted 15 March 2023

COMMUNICATED BY

KEYWORDS

Axially travelling strings; Fourier series; energy estimates; boundary stabilisation

AMS CLASSIFICATIONS

35L05; 35C10; 93D15

1. Introduction

We consider small transversal vibrations of a uniform string travelling with a constant speed \mathbf{v} between two pulleys (inlet and outlet) kept at a fixed distance L . The mechanical setting is sketched in Figure 1 where the inlet is fixed while the outlet is allowed to move transversely and attached to a damping device (a dashpot with damping factor η).

Many mechanical devices with axially moving continua, such as power transmission chains and belts, magnetic tapes, band saws and fibre winders, see for instance [1–5], are limited in their efficiency and utility due to unwanted vibrations. As a result, the stabilisation of axially moving systems is necessary to reduce or eliminate these vibrations and improve the overall performance and productivity of these mechanical systems.

The existing approaches in the literature describe the above problem in fixed space coordinates, see for instance [1, 3, 6–10]. Gaiko and van Horssen [11] considered a simplified mathematical model describing the small vibration of the string with a mass-spring-dashpot damping at the outlet. Under the restriction that the speed \mathbf{v} and the damping factor are small, i.e. $0 < \mathbf{v} \ll 1$ and $\eta \ll 1$, the authors in [11] obtained an asymptotic approximation for the solution of (1) using a multiple scale approach.

Denoting the displacement function by u , which depends on the time τ and the position s along the string, with only a dashpot damping at the outlet, their model can be stated as follows:

$$\begin{cases} u_{\tau\tau} + 2\mathbf{v}u_{s\tau} - (1 - \mathbf{v}^2)u_{ss} = 0, & \text{for } s \in (0, L) \text{ and } \tau > 0, \\ u(0, \tau) = 0, & \text{for } \tau > 0, \\ (1 - \mathbf{v}^2)u_s(L, \tau) + (\eta - \mathbf{v})u_{\tau}(L, \tau) = 0, & \text{for } \tau > 0, \\ u(s, 0) = u^0(s), \quad u_{\tau}(s, 0) = u^1(s), & \text{for } s \in (0, L), \end{cases} \quad (1)$$

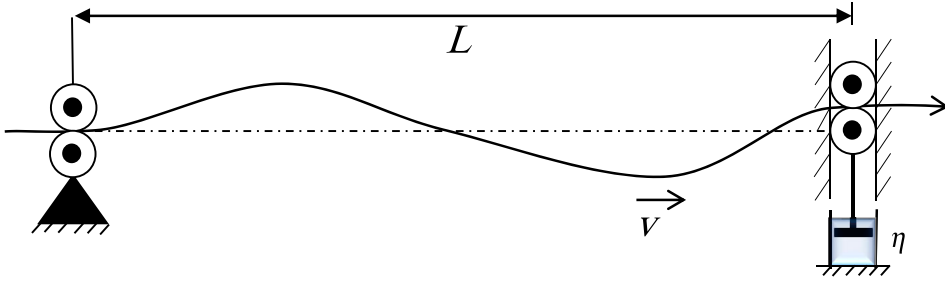


Figure 1. An axially travelling string with a dashpot at the outlet pulley.

where the subscripts τ and s stand for the derivatives in time and space variables respectively. The functions u^0 and u^1 represent the initial shape and the initial transverse speed of the string, respectively. See Appendix A for the derivation of the above governing equation and its boundary conditions.

In the present work, we do not consider the magnitudes of \mathbf{v} and η as small ones. We only assume that $\eta \geq 0$ and that the speed \mathbf{v} is strictly less than the speed of propagation of the wave (here normalised to $c = 1$), i.e.

$$0 \leq \mathbf{v} < 1. \tag{2}$$

If the speed \mathbf{v} approaches the critical speed $c = 1$, an instability will occur, as shown in [5, 10]. Another key difference with most of the existing works is that we consider the model in a moving space coordinates. We introduce the variables

$$s = L - x + \mathbf{v}t \quad \text{and} \quad \tau = t,$$

hence

$$x \in \mathbf{I}_t := (\mathbf{v}t, L + \mathbf{v}t), \text{ for } t \geq 0,$$

which is an interval travelling in the positive sense of the real axis (as in [4, 12]). It follows that

$$\partial_s = -\partial_x \quad \text{and} \quad \partial_\tau = \mathbf{v}\partial_x + \partial_t.$$

Rewriting Problem (1) in the new coordinates, we obtain the following (pure) wave equation with a damping at the moving boundary $x = \mathbf{v}t$,

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & \text{for } x \in \mathbf{I}_t \text{ and } t > 0, \\ (1 - \eta\mathbf{v})\phi_x(\mathbf{v}t, t) - (\eta - \mathbf{v})\phi_t(\mathbf{v}t, t) = 0, & \text{for } t > 0, \\ \phi(L + \mathbf{v}t, t) = 0, & \text{for } t > 0, \\ \phi(x, 0) = \phi^0(x), \phi_t(x, 0) = \phi^1(x), & \text{for } x \in \mathbf{I}_0, \end{cases} \tag{WP}$$

where $\phi^0 = u^0$ and $\phi^1 = u^1 - \mathbf{v}\phi_x^0$.

Let us denote

$$\gamma_{\mathbf{v}} := \frac{1 + \mathbf{v}}{1 - \mathbf{v}}, \quad \gamma_\eta := \frac{1 + \eta}{1 - \eta} \quad \text{and} \quad \omega_n := \begin{cases} \frac{(2n + 1)}{2}i\pi - \frac{1}{2} \ln \gamma_\eta, & \text{if } 0 \leq \eta < 1, \\ ni\pi - \frac{1}{2} \ln |\gamma_\eta|, & \text{if } \eta > 1. \end{cases}$$

Note that

- $\gamma_{\mathbf{v}} \geq 1$ for $0 \leq \mathbf{v} < 1$,
- $|\gamma_\eta| \geq 1$ and hence the real part of ω_n remains negative, for $\eta > 0$ with $\eta \neq 1$.

As a first result, we derive a closed form for the solution of (WP) given by the series formulas

$$\phi(x, t) = \sum_{n \in \mathbb{Z}} a_n \left(\gamma_\eta e^{\frac{1-\mathbf{v}}{L} \omega_n(t+x)} + e^{\frac{1+\mathbf{v}}{L} \omega_n(t-x)} \right), \quad \text{for } x \in \mathbf{I}_t \quad \text{and} \quad t \geq 0, \quad (3)$$

which is a sum of two waves travelling in opposite directions. The coefficients a_n are explicitly given in function of the initial data ϕ^0 and ϕ^1 , see Theorem 2.1.

Next, we demonstrate how the series formulas (3) can be used to achieve the following results:

- For the undamped case, i.e. $\eta = 0$, we show that the functional

$$\mathcal{E}_\mathbf{v}(t) = \frac{1}{2} \int_{\mathbf{I}_t} (\phi_t + \mathbf{v}\phi_x)^2 + (1 - \mathbf{v}^2) \phi_x^2 \, dx, \quad \text{for } t \geq 0, \quad (4)$$

depending on L, t, \mathbf{v} and the solution ϕ of (WP), is conserved in time. See Theorem 3.1. Note that under the assumption (2), the functional $\mathcal{E}_\mathbf{v}$ is positive-definite and we will call it the ‘energy’ of the solution ϕ .

- For the damped case $\eta > 0$ with $\eta \neq 1$, the (usual) energy

$$E_\mathbf{v}(t) = \frac{1}{2} \int_{\mathbf{I}_t} \phi_t^2(x, t) + \phi_x^2(x, t) \, dx, \quad \text{for } t \geq 0, \quad (5)$$

depending on L, t, \mathbf{v} and the solution of (WP), satisfies

$$\frac{1}{\gamma_\eta^2 \gamma_\mathbf{v}} E_\mathbf{v}(0) e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t} \leq E_\mathbf{v}(t) \leq \gamma_\eta^2 \gamma_\mathbf{v} E_\mathbf{v}(0) e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}, \quad \text{for } t \geq 0. \quad (6)$$

See Theorem 4.1 and its corollaries for sharper estimates. In particular, it follows that the energy does not vanish in finite time.

The upper and lower estimates given in (6), with explicit constants and the same exponential function in both estimates, are new to the best of our knowledge. The approach presented here relies on Fourier series and Parseval’s identity, and it does not involve semigroup theory as in [10, 13]. Even for a non-travelling string, i.e. when $\mathbf{v} = 0$, the precise estimate

$$\frac{1}{\gamma_\eta^2} E_0(0) e^{-\frac{1}{L} \ln|\gamma_\eta|t} \leq E_0(t) \leq \gamma_\eta^2 E_0(0) e^{-\frac{1}{L} \ln|\gamma_\eta|t}, \quad \text{for } t \geq 0, \quad (7)$$

seems to be not reported in the literature. The existing results assure only that

$$E_0(t) \leq K E_0(0) e^{-\frac{1}{L} \ln|\gamma_\eta|t},$$

for a (non-explicitly given) positive constant K , see for instance [14–17].

For the special case $\eta = 1$ and $0 \leq \mathbf{v} < 1$, the boundary condition at $x = \mathbf{v}t$ reads

$$\phi_x(\mathbf{v}t, t) - \phi_t(\mathbf{v}t, t) = 0.$$

This is a transparent condition, i.e. there is no reflection of waves from the boundary $x = \mathbf{v}t$ and consequently all the initial disturbances leave the interval $(\mathbf{v}t, L + \mathbf{v}t)$ at most after a time

$$T_\mathbf{v} := \frac{L}{1 + \mathbf{v}} + \frac{L}{1 - \mathbf{v}} = \frac{2L}{1 - \mathbf{v}^2}.$$

That is to say that the linear velocity feedback $\phi_t(\mathbf{v}t, t)$ steers the solution to the zero state in the finite time $T_\mathbf{v}$. See for instance [10] for the case $0 < \mathbf{v} < 1$ and [14, 15] for $\mathbf{v} = 0$. In the remaining of this paper, we assume that $\eta \neq 1$.

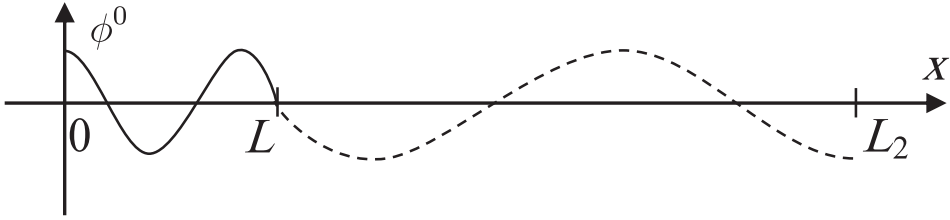


Figure 2. Example of the extension of an initial data ϕ^0 when $0 < \mathbf{v} < 1$.

After the present introduction, we derive the exact solution and the expression for the coefficients of the series formula (3). In Section 3, we show that the energy $\mathcal{E}_{\mathbf{v}}$ of the undamped equation is constant in time. In Section 4, we show the exponential stability when $\eta > 0$. The last section is devoted to some numerical examples.

2. Exact solution

To compute the solution of Problem (WP), given by (3), we must compute the coefficients $a_n, n \in \mathbb{Z}$. To this end, we need to know the functions ϕ^0 and ϕ^1 on an interval larger than $\mathbf{I}_0 = (0, L)$. As in [12, 18], we introduce

$$L_2 := \frac{2L}{1 - \mathbf{v}}$$

and extend ϕ to the interval $(L + \mathbf{v}t, L_2 + \mathbf{v}t)$ by setting

$$\tilde{\phi}(x, t) = \begin{cases} \phi(x, t), & x \in (\mathbf{v}t, L + \mathbf{v}t), \\ -\phi\left(\frac{1}{\gamma_{\mathbf{v}}}(\mathbf{v}t - x) + \frac{2L}{1 + \mathbf{v}} + \mathbf{v}t, t\right), & x \in (L + \mathbf{v}t, L_2 + \mathbf{v}t). \end{cases} \quad (8)$$

The obtained function is well defined since the first variable of ϕ remains in the interval $(\mathbf{v}t, L + \mathbf{v}t)$. In particular, the homogeneous boundary condition $\phi(L + \mathbf{v}t, t) = 0$ remains satisfied, for every $t \geq 0$.

Remark 2.1: Clearly, $0 < L \leq L_2/2$ for $0 \leq \mathbf{v} < 1$. If $\mathbf{v} = 0$, then $L_2 = 2L$ and the function $\tilde{\phi}$ on $(L, 2L)$ is an odd function on $(0, 2L)$, with respect to $x = L$. If $0 < \mathbf{v} < 1$, then ϕ is extended as an odd function with an extra dilatation on the added interval $(L + \mathbf{v}t, L_2 + \mathbf{v}t)$, see Figure 2.

Taking the derivative of (8) with respect to x , we obtain

$$\tilde{\phi}_x(x, t) = \begin{cases} \phi_x(x, t), & x \in (\mathbf{v}t, L + \mathbf{v}t), \\ \frac{1}{\gamma_{\mathbf{v}}}\phi_x\left(\frac{1}{\gamma_{\mathbf{v}}}(\mathbf{v}t - x) + \frac{2L}{1 + \mathbf{v}} + \mathbf{v}t, t\right), & x \in (L + \mathbf{v}t, L_2 + \mathbf{v}t). \end{cases} \quad (9)$$

On the other hand, the time derivative is extended as follows:

$$\tilde{\phi}_t(x, t) = \begin{cases} \phi_t(x, t), & x \in (\mathbf{v}t, L + \mathbf{v}t), \\ \frac{-1}{\gamma_{\mathbf{v}}}\phi_t\left(\frac{1}{\gamma_{\mathbf{v}}}(\mathbf{v}t - x) + \frac{2L}{1 + \mathbf{v}} + \mathbf{v}t, t\right), & x \in (L + \mathbf{v}t, L_2 + \mathbf{v}t). \end{cases} \quad (10)$$

Let us introduce the following family of Hilbert spaces:

$$\mathcal{H}_{L+\mathbf{v}t}(\mathbf{I}_t) := \{w \in H^1(\mathbf{I}_t), w(L + \mathbf{v}t) = 0\}, \quad \text{for } t \geq 0,$$

where $H^1(\mathbf{I}_t)$ is the Sobolev space defined on \mathbf{I}_t . We assume that the initial data satisfies

$$\phi^0 \in \mathcal{H}_L(\mathbf{I}_0), \quad \phi^1 \in L^2(\mathbf{I}_0). \tag{11}$$

Now, we are ready to state the following existence and uniqueness result for Problem (WP).

Theorem 2.1: *Let $T > 0$. Under the assumptions (2) and (11), the solution of Problem (WP)*

$$\phi \in C([0, T]; \mathcal{H}_{L+\mathbf{v}t}(\mathbf{I}_t)) \cap C^1([0, T]; L^2(\mathbf{I}_t)), \tag{12}$$

is given by the series (3) where the coefficients $a_n \in \mathbb{C}$ are computed by the following formula:

$$a_n = \frac{1}{4\gamma_\eta \omega_n} \int_0^{L_2} (\tilde{\phi}_x^0 + \tilde{\phi}^1) e^{-\frac{1-\mathbf{v}}{L} \omega_n x} dx, \quad \text{for } n \in \mathbb{Z}. \tag{13}$$

Moreover, we have

$$\sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 = \frac{L}{8(1-\mathbf{v})\gamma_\eta^2} \int_{\mathbf{v}t}^{L_2+\mathbf{v}t} e^{\frac{1-\mathbf{v}}{L} \ln|\gamma_\eta|(t+x)} (\tilde{\phi}_x + \tilde{\phi}_t)^2 dx < +\infty, \tag{14}$$

where $\tilde{\phi}_x^0$ and $\tilde{\phi}^1$ are extensions of the initial data ϕ^0 and ϕ^1 on the interval $(0, L_2)$ given above by (9) and (10) respectively.

Proof: The general solution of (WP) is given by D'Alembert's formula

$$\phi(x, t) = f(t+x) + g(t-x),$$

where f and g are arbitrary continuous functions. Let us check the boundary conditions. On one hand, at the left endpoint we have

$$(1-\eta\mathbf{v})f'((1+\mathbf{v})t) - g'((1-\mathbf{v})t) = (\eta-\mathbf{v})(f'((1+\mathbf{v})t) + g'((1-\mathbf{v})t)).$$

Setting $z = (1-\mathbf{v})t$, we obtain

$$(1-\eta\mathbf{v}-\eta+\mathbf{v})f'(\gamma_{\mathbf{v}}z) = (1-\eta\mathbf{v}+\eta-\mathbf{v})g'(z). \tag{15}$$

On the other hand, at the right endpoint, we infer that

$$f((1+\mathbf{v})t+L) + g((1-\mathbf{v})t-L) = 0.$$

Denoting $y = z-L$, we obtain

$$f(\gamma_{\mathbf{v}}y + \frac{2L}{1-\mathbf{v}}) = -g(y). \tag{16}$$

Then, taking the derivative with respect to y , we get

$$\gamma_{\mathbf{v}}f'(\gamma_{\mathbf{v}}y + \frac{2L}{1-\mathbf{v}}) = -g'(y). \tag{17}$$

Thanks to (15), (17) and taking $\xi = \gamma_{\mathbf{v}}y$, we deduce that f' satisfies

$$(1-\eta\mathbf{v}+\eta-\mathbf{v})\gamma_{\mathbf{v}}f'\left(\xi + \frac{2L}{1-\mathbf{v}}\right) = -(1-\eta\mathbf{v}-\eta+\mathbf{v})f'(\xi). \tag{18}$$

After few simplifications, this can be written as

$$f'\left(\xi + \frac{2L}{1-\mathbf{v}}\right) = -\frac{f'(\xi)}{\gamma_\eta}. \tag{19}$$

This formula suggests that $f'(\xi) = e^{\beta\xi}$, for some parameter β to be determined later. Substituting the form $e^{\beta\xi}$ in (19), we infer that $e^{\beta(\xi+\frac{2L}{1-\mathbf{v}})} = -e^{\beta\xi}/\gamma_\eta$. Then, it follows that:

- If $0 \leq \eta < 1$, then $\gamma_\eta \geq 1$ and we get

$$e^{\beta\left(\xi + \frac{2L}{1-\mathbf{v}}\right)} = -e^{-\ln \gamma_\eta} e^{\beta\xi}.$$

Solving this equation for β , we obtain a sequence of values $\beta_n = (1 - \mathbf{v})\omega_n/L$, $n \in \mathbb{Z}$, where

$$\omega_n = \frac{1}{2} (2n + 1) i\pi - \frac{1}{2} \ln \gamma_\eta.$$

- If $\eta > 1$, we have $\gamma_\eta < -1$ and we obtain another sequence of values $\beta_n = (1 - \mathbf{v})\omega_n/L$, $n \in \mathbb{Z}$, where this time

$$\omega_n = ni\pi - \frac{1}{2} \ln |\gamma_\eta|.$$

Note that in both cases we have $\ln |\gamma_\eta| \geq 1$ and the real part of ω_n is negative.

Due to the superposition principle, it follows that f' can be written as

$$f'(\xi) = \sum_{n \in \mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L} \omega_n \xi}, \quad c_n \in \mathbb{C},$$

where c_n are complex coefficients to be determined later. After integration, the function f can be written as

$$f(\xi) = c + \sum_{n \in \mathbb{Z}} \frac{Lc_n}{(1 - \mathbf{v}) \omega_n} e^{\frac{1-\mathbf{v}}{L} \omega_n \xi},$$

for some constant c . Using (16), we deduce that

$$g(\xi) = -c + \sum_{n \in \mathbb{Z}} \frac{Lc_n}{(1 - \mathbf{v}) \gamma_\eta \omega_n} e^{\frac{1+\mathbf{v}}{L} \omega_n \xi}, \quad (20)$$

where we have used the fact that $e^{2\omega_n} = -1/\gamma_\eta$ whether $0 \leq \eta < 1$ or $\eta > 1$.

Thanks to D'Alembert's formula, the solution of Problem (WP) is given by the series

$$\phi(x, t) = \sum_{n \in \mathbb{Z}} \frac{L}{(1 - \mathbf{v}) \gamma_\eta \omega_n} c_n \left(\gamma_\eta e^{\frac{1-\mathbf{v}}{L} \omega_n (t+x)} + e^{\frac{1+\mathbf{v}}{L} \omega_n (t-x)} \right). \quad (21)$$

To obtain (3), we set

$$a_n := \frac{L}{(1 - \mathbf{v}) \gamma_\eta \omega_n} c_n. \quad (22)$$

The coefficient c_n are determined as follows. Going back to (21), we infer that

$$\phi_x(x, t) = \sum_{n \in \mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L} \omega_n (t+x)} - \frac{\gamma_\mathbf{v}}{\gamma_\eta} e^{\frac{1+\mathbf{v}}{L} \omega_n (t-x)} \right), \quad (23)$$

$$\phi_t(x, t) = \sum_{n \in \mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L} \omega_n (t+x)} + \frac{\gamma_\mathbf{v}}{\gamma_\eta} e^{\frac{1+\mathbf{v}}{L} \omega_n (t-x)} \right), \quad (24)$$

for $x \in (\mathbf{v}t, L + \mathbf{v}t)$ and $t \geq 0$. It follows from (9) and (10) that the extensions $\tilde{\phi}_x$ and $\tilde{\phi}_t$ are given by

$$\tilde{\phi}_x(x, t) = \begin{cases} \sum_{n \in \mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L} \omega_n(t+x)} - \frac{\gamma_{\mathbf{v}}}{\gamma_{\eta}} e^{\frac{1+\mathbf{v}}{L} \omega_n(t-x)} \right), & \text{if } x \in (\mathbf{v}t, L + \mathbf{v}t), \\ \frac{1}{\gamma_{\mathbf{v}}} \sum_{n \in \mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L} \omega_n((1+\mathbf{v})t + \frac{\mathbf{v}t - x}{\gamma_{\mathbf{v}}} + \frac{2L}{1 + \mathbf{v}})} \right. \\ \left. - \frac{\gamma_{\mathbf{v}}}{\gamma_{\eta}} e^{\frac{1+\mathbf{v}}{L} \omega_n((1-\mathbf{v})t - \frac{\mathbf{v}t - x}{\gamma_{\mathbf{v}}} - \frac{2L}{1 + \mathbf{v}})} \right), & \text{if } x \in (L + \mathbf{v}t, L_2 + \mathbf{v}t), \end{cases} \quad (25)$$

$$\tilde{\phi}_t(x, t) = \begin{cases} \sum_{n \in \mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L} \omega_n(t+x)} + \frac{\gamma_{\mathbf{v}}}{\gamma_{\eta}} e^{\frac{1+\mathbf{v}}{L} \omega_n(t-x)} \right), & \text{if } x \in (\mathbf{v}t, L + \mathbf{v}t), \\ \frac{-1}{\gamma_{\mathbf{v}}} \sum_{n \in \mathbb{Z}} c_n \left(e^{\frac{1-\mathbf{v}}{L} \omega_n((1+\mathbf{v})t + \frac{\mathbf{v}t - x}{\gamma_{\mathbf{v}}} + \frac{2L}{1 + \mathbf{v}})} \right. \\ \left. + \frac{\gamma_{\mathbf{v}}}{\gamma_{\eta}} e^{\frac{1+\mathbf{v}}{L} \omega_n((1-\mathbf{v})t - \frac{\mathbf{v}t - x}{\gamma_{\mathbf{v}}} - \frac{2L}{1 + \mathbf{v}})} \right), & \text{if } x \in (L + \mathbf{v}t, L_2 + \mathbf{v}t). \end{cases} \quad (26)$$

Taking the sum of (25) and (26), we get

$$\tilde{\phi}_x + \tilde{\phi}_t = \begin{cases} 2 \sum_{n \in \mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L} \omega_n(t+x)}, & x \in (\mathbf{v}t, L + \mathbf{v}t), \\ \frac{-2}{\gamma_{\mathbf{v}}} \sum_{n \in \mathbb{Z}} c_n \frac{\gamma_{\mathbf{v}}}{\gamma_{\eta}} e^{-2\omega_n} e^{\frac{1-\mathbf{v}}{L} \omega_n(t+x)}, & x \in (L + \mathbf{v}t, L_2 + \mathbf{v}t). \end{cases}$$

Since $e^{-2\omega_n} = -\gamma_{\eta}$, then we have the unified expression

$$\tilde{\phi}_x + \tilde{\phi}_t = 2 \sum_{n \in \mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L} \omega_n(t+x)}, \quad \text{for } x \in (\mathbf{v}t, L_2 + \mathbf{v}t) \quad \text{and} \quad t \geq 0. \quad (27)$$

Using the definition of ω_n , we get

$$\tilde{\phi}_x + \tilde{\phi}_t = \begin{cases} 2e^{\frac{1-\mathbf{v}}{2L} (i\pi - \ln \gamma_{\eta})(t+x)} \sum_{n \in \mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L} n i \pi (t+x)}, & \text{if } 0 \leq \eta < 1, \\ 2e^{-\frac{1-\mathbf{v}}{2L} \ln |\gamma_{\eta}| (t+x)} \sum_{n \in \mathbb{Z}} c_n e^{\frac{1-\mathbf{v}}{L} n i \pi (t+x)}, & \text{if } \eta > 1. \end{cases} \quad (28)$$

Taking into account that $\{e^{\frac{n\pi i(1-\mathbf{v})}{L}(t+x)} / \sqrt{L_2}\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbf{v}t, L_2 + \mathbf{v}t)$, for every $t \geq 0$, we rewrite (27) as

$$\sum_{n \in \mathbb{Z}} c_n \frac{e^{\frac{1-\mathbf{v}}{L} n i \pi (t+x)}}{\sqrt{L_2}} = \begin{cases} \frac{1}{2\sqrt{L_2}} e^{-\frac{1-\mathbf{v}}{2L} (i\pi - \ln \gamma_{\eta})(t+x)} (\tilde{\phi}_x + \tilde{\phi}_t), & \text{if } 0 \leq \eta < 1, \\ \frac{1}{2\sqrt{L_2}} e^{\frac{1-\mathbf{v}}{2L} \ln |\gamma_{\eta}| (t+x)} (\tilde{\phi}_x + \tilde{\phi}_t), & \text{if } \eta > 1. \end{cases} \quad (29)$$

By consequence,

$$c_n = \begin{cases} \frac{1}{2L_2} \int_{\mathbf{v}t}^{L_2+\mathbf{v}t} e^{-\frac{1-\mathbf{v}}{2L} (i\pi - \ln \gamma_\eta)(t+x)} \left(\tilde{\phi}_x + \tilde{\phi}_t \right) e^{-\frac{n\pi i(1-\mathbf{v})}{L} (t+x)} dx, & \text{if } 0 \leq \eta < 1, \\ \frac{1}{2L_2} \int_{\mathbf{v}t}^{L_2+\mathbf{v}t} e^{\frac{1-\mathbf{v}}{2L} \ln |\gamma_\eta|(t+x)} \left(\tilde{\phi}_x + \tilde{\phi}_t \right) e^{-\frac{n\pi i(1-\mathbf{v})}{L} (t+x)} dx, & \text{if } \eta > 1, \end{cases}$$

for $n \in \mathbb{Z}$. Whether $0 \leq \eta < 1$ or $\eta > 1$, in both cases, we have

$$c_n = \frac{1}{2L_2} \int_{\mathbf{v}t}^{L_2+\mathbf{v}t} \left(\tilde{\phi}_x + \tilde{\phi}_t \right) e^{-\frac{(1-\mathbf{v})}{L} \omega_n(t+x)} dx, \quad \text{for } n \in \mathbb{Z}.$$

For $t = 0$, and tacking (22) into account, we obtain (13) as claimed.

Moreover, as a consequence of Parseval's equality, it comes that

$$\sum_{n \in \mathbb{Z}} |c_n|^2 = \begin{cases} \frac{1}{4L_2} \int_{\mathbf{v}t}^{L_2+\mathbf{v}t} \left| e^{-\frac{1-\mathbf{v}}{2L} (i\pi - \ln \gamma_\eta)(t+x)} \right|^2 \left(\tilde{\phi}_x + \tilde{\phi}_t \right)^2 dx, & \text{if } 0 \leq \eta < 1, \\ \frac{1}{4L_2} \int_{\mathbf{v}t}^{L_2+\mathbf{v}t} \left| e^{\frac{1-\mathbf{v}}{2L} \ln |\gamma_\eta|(t+x)} \right|^2 \left(\tilde{\phi}_x + \tilde{\phi}_t \right)^2 dx, & \text{if } \eta > 1. \end{cases}$$

Whether $0 \leq \eta < 1$ or $\eta > 1$, it follows that

$$\sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 = \frac{L^2}{\gamma_\eta^2 (1-\mathbf{v})^2} \sum_{n \in \mathbb{Z}} |c_n|^2 = \frac{L}{8\gamma_\eta^2 (1-\mathbf{v})} \int_{\mathbf{v}t}^{L_2+\mathbf{v}t} e^{\frac{1-\mathbf{v}}{L} \ln |\gamma_\eta|(t+x)} \left(\tilde{\phi}_x + \tilde{\phi}_t \right)^2 dx.$$

Thanks to (11), ϕ^0 and ϕ^1 belongs to $L^2(0, L_2)$. Thus the integral at the right-hand side for $t = 0$ is finite and

$$\sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 < +\infty.$$

Recalling that $|w_n|^2 = O(n^2)$, for large values of n , then

$$\sum_{n \in \mathbb{Z}} |na_n|^2 < +\infty. \quad (30)$$

Let $T > 0$ and $t \in [0, T]$. Due to the continuity of the exponential function, we get

$$\left| a_n \left(\gamma_\eta e^{\frac{1-\mathbf{v}}{L} \omega_n(t+x)} + e^{\frac{1+\mathbf{v}}{L} \omega_n(t-x)} \right) \right| \leq C_T |a_n|,$$

where C_T is a constant depending only on \mathbf{v}, η, L and T .

Going back to (23), (24) and due to (22), we can check that

$$\left| c_n \left(e^{\frac{1-\mathbf{v}}{L} \omega_n(t+x)} \pm \frac{\gamma_\mathbf{v}}{\gamma_\eta} e^{\frac{1+\mathbf{v}}{L} \omega_n(t-x)} \right) \right| \leq C'_T |na_n|,$$

for some constant C'_T .

Taking (30) into account, we infer that $\phi(x, t)$, $\phi_x(x, t)$ and $\phi_t(x, t)$ belong to $L^2(\mathbf{I}_t)$, for $t \geq 0$. In particular, $\phi(x, t) \in \mathcal{H}_{L+\mathbf{v}t}(\mathbf{I}_t)$, for $t \geq 0$. The continuity in time of ϕ and ϕ_t as functions of t with values in $\mathcal{H}_{L+\mathbf{v}t}(\mathbf{I}_t)$ and $L^2(\mathbf{I}_t)$, respectively, follows as they are the sums of uniformly converging series of continuous functions. This shows (12). \blacksquare

3. A conserved quantity for the string with no damper

For the undamped case, i.e. $\eta = 0$ in Problem (WP), we show that the energy \mathcal{E}_v given by (4) is conserved in time.

Theorem 3.1: *Under Assumptions (2) and (11), the solution of Problem (WP) satisfies*

$$\mathcal{E}_v(t) = \frac{\pi^2(1-v^2)}{2L} \sum_{n \in \mathbb{Z}} |(2n+1)a_n|^2, \quad \text{for } t \geq 0, \quad (31)$$

where the left-hand side is independent of t .

Proof: If $\eta = 0$, then $\omega_n = (2n+1)i\pi/2$ and the identity (14) becomes

$$\frac{1}{1-v} \int_{vt}^{L_2+vt} (\tilde{\phi}_x + \tilde{\phi}_t)^2 dx = \frac{2\pi^2}{L} \sum_{n \in \mathbb{Z}} |(2n+1)a_n|^2. \quad (32)$$

Using the extensions (9), (10) and considering the change of variable

$$x = \gamma_v(vt - \xi) + \frac{2L}{1-v} + vt,$$

in $(L+vt, L_2+vt)$, then we have

$$\frac{1}{1-v} \int_{L+vt}^{L_2+vt} (\tilde{\phi}_x(x, t) + \tilde{\phi}_t(x, t))^2 dx = \frac{1}{1+v} \int_{I_t} (\phi_x(\xi, t) - \phi_t(\xi, t))^2 d\xi.$$

Taking (32) into account, it comes that

$$\begin{aligned} \frac{1}{1-v} \int_{vt}^{L_2+vt} (\tilde{\phi}_x + \tilde{\phi}_t)^2 dx &= \frac{1}{1-v} \int_{I_t} (\phi_t + \phi_x)^2 dx + \frac{1}{1+v} \int_{I_t} (\phi_x - \phi_t)^2 dx \\ &= \frac{2\pi^2}{L} \sum_{n \in \mathbb{Z}} |(2n+1)a_n|^2. \end{aligned}$$

Expanding $(\phi_x \pm \phi_t)^2$ and collecting similar terms, we get

$$\frac{1}{1-v^2} \left(\int_{I_t} \phi_x^2 + \phi_t^2 + 2v\phi_x\phi_t dx \right) = \frac{\pi^2}{L} \sum_{n \in \mathbb{Z}} |(2n+1)a_n|^2, \quad \text{for } t \geq 0. \quad (33)$$

The left hand side is equal to $2\mathcal{E}_v(t)/(1-v^2)$ and (31) follows. ■

Remark 3.1: Using Leibnitz's rule for differentiation under the integral sign, we can check directly that $\frac{d}{dt}\mathcal{E}_v(t) = 0$, see Appendix B.

Remark 3.2: The energy expression $\mathcal{E}_v(t)$ is also shown to be conserved in time for the Dirichlet boundary conditions at both ends, see [12].

Let us now compare $\mathcal{E}_v(t)$ to the usual expression of energy $E_v(t)$ for the wave equation.

Corollary 3.2: Under Assumptions (2) and (11), the energy $E_v(t)$ of the solution of the undamped Problem (WP) satisfies

$$\frac{\mathcal{E}_v(0)}{1+v} \leq E_v(t) \leq \frac{\mathcal{E}_v(0)}{1-v}, \quad \text{for } t \geq 0 \tag{34}$$

and

$$\frac{1}{\gamma_v} E_v(0) \leq E_v(t) \leq \gamma_v E_v(0), \quad \text{for } t \geq 0. \tag{35}$$

Proof: We can write (33) as follows:

$$E_v(t) + v \int_{\mathbf{I}_t} \phi_x \phi_t \, dx = \mathcal{E}_v(t), \quad \text{for } t \geq 0. \tag{36}$$

Since $\pm \phi_x \phi_t \leq (\phi_x^2 + \phi_t^2)/2$, then it comes that

$$\mathcal{E}_v(t) \leq (1+v) E_v(t) \quad \text{and} \quad (1-v) E_v(t) \leq \mathcal{E}_v(t), \quad \text{for } t \geq 0. \tag{37}$$

This implies (34). Since (37) holds also for $t = 0$, then (35) follows by combining the two inequalities

$$\begin{aligned} (1-v) E_v(t) &\leq \mathcal{E}_v(t) = \mathcal{E}_v(0) \leq (1+v) E_v(0), \\ (1-v) E_v(0) &\leq \mathcal{E}_v(0) = \mathcal{E}_v(t) \leq (1+v) E_v(t), \end{aligned}$$

for $t \geq 0$. ■

Remark 3.3: The solution ϕ given by (3), with $\eta = 0$, satisfies the periodicity relation

$$\phi(x + vT_v, t + T_v) = -\phi(x, t), \quad t \geq 0. \tag{38}$$

It follows in particular that the energy E_v is a T_v -periodic function in time.

Remark 3.4: The equality in (34) holds at least if $\phi_t(x, t_0) = \pm \phi_x(x, t_0)$, for $x \in \mathbf{I}_{t_0}$ and some $t_0 \geq 0$. Indeed, we have

$$\mathcal{E}_v(t_0) = E_v(t_0) \pm v \int_{v t_0}^{L+v t_0} \phi_x(x, t_0) \phi_t(x, t_0) \, dx = (1 \pm v) E_v(t_0),$$

i.e. $E_v(t_0) = \mathcal{E}_v(t_0)/(1 \pm v)$. The + and - signs are used respectively.

Remark 3.5: Let $0 < v < 1$ and $\eta = 0$. Evaluating ϕ_x given by (23) at $x = vt$ and $x = L + vt$, we obtain

$$\begin{aligned} \phi_x(vt, t) &= \frac{-2v}{1-v} \sum_{n \in \mathbb{Z}} c_n e^{\frac{1-v^2}{2L}(2n+1)i\pi t}, \\ \phi_x(L + vt, t) &= \frac{2}{1-v} \sum_{n \in \mathbb{Z}} \left(c_n e^{\frac{1-v}{2}(2n+1)i\pi} \right) e^{\frac{1-v^2}{2L}(2n+1)i\pi t}, \end{aligned}$$

for $t \geq 0$. Then, by Parseval's identity and (31), we obtain

$$\frac{1}{\mathbf{v}^2} \int_0^{T_{\mathbf{v}}} \phi_x^2(\mathbf{v}t, t) dt = \int_0^{T_{\mathbf{v}}} \phi_x^2(L + \mathbf{v}t, t) dt = \frac{E_{\mathbf{v}}(0)}{(1 - \mathbf{v}^2)^2}.$$

Using (34) for $t = 0$, we get in particular

$$\frac{\mathbf{v}^2 E_{\mathbf{v}}(0)}{(1 + \mathbf{v})(1 - \mathbf{v}^2)} \leq \int_0^{T_{\mathbf{v}}} \phi_x^2(\mathbf{v}t, t) dt \quad \text{and} \quad \frac{E_{\mathbf{v}}(0)}{(1 + \mathbf{v})(1 - \mathbf{v}^2)} \leq \int_0^{T_{\mathbf{v}}} \phi_x^2(L + \mathbf{v}t, t) dt.$$

Each of these inequalities is called a boundary observability inequality. They ensure that if there is an initial disturbance of the string, i.e. $E_{\mathbf{v}}(0) > 0$, it will be detected by a sensor placed at an endpoint at most after a time $T \geq T_{\mathbf{v}} = 2L/(1 - \mathbf{v}^2)$. See [12, Section 4.1] and [18, Section 4.1] where the boundary observability of the Dirichlet case of the problem is well developed.

4. Exponential stability for the string with a boundary damper

In this section, we keep $0 \leq \mathbf{v} < 1$ and assume that $\eta > 0$ with $\eta \neq 1$.

Theorem 4.1: *Under Assumptions (2) and (11), the solution of Problem (WP) satisfies*

$$\begin{aligned} & \frac{1}{1 + \mathbf{v}} \int_{\mathbf{I}_t} e^{\frac{1+\mathbf{v}}{L} \ln|\gamma_{\eta}|(t-x)} (\phi_x - \phi_t)^2 dx \\ & + \frac{1}{\gamma_{\eta}^2 (1 - \mathbf{v})} \int_{\mathbf{I}_t} e^{\frac{1-\mathbf{v}}{L} \ln|\gamma_{\eta}|(t+x)} (\phi_t + \phi_x)^2 dx = \frac{8}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2, \end{aligned} \quad (39)$$

where the left-hand side is finite and independent of t . Moreover, it holds that

$$M_1 e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_{\eta}|t} \leq E_{\mathbf{v}}(t) \leq M_2 e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_{\eta}|t}, \quad \text{for } t \geq 0, \quad (40)$$

where

$$\begin{aligned} M_1 &:= \frac{2(1 - \mathbf{v})}{L} \min\{\gamma_{\mathbf{v}}, |\gamma_{\eta}|^{(1+\mathbf{v})}\} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2, \\ M_2 &:= \frac{2(1 - \mathbf{v})}{L} \max\{|\gamma_{\eta}|^{(1+\mathbf{v})}, \gamma_{\mathbf{v}}, \gamma_{\eta}^2\} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2. \end{aligned}$$

Proof: Let us split the integral in the identity (14) to the integrals

$$\int_{\mathbf{v}t}^{L_2+\mathbf{v}t} = \int_{\mathbf{I}_t} + \int_{L+\mathbf{v}t}^{L_2+\mathbf{v}t}, \quad (41)$$

then considering the change of variable $x = \gamma_{\mathbf{v}}(\mathbf{v}t - \xi) + \frac{2L}{1-\mathbf{v}} + \mathbf{v}t$ in $(L + \mathbf{v}t, L_2 + \mathbf{v}t)$, we obtain

$$\begin{aligned} & \frac{1}{(1 - \mathbf{v}) \gamma_{\eta}^2} \int_{L+\mathbf{v}t}^{L_2+\mathbf{v}t} e^{\frac{1-\mathbf{v}}{L} \ln|\gamma_{\eta}|(t+x)} \left(\tilde{\phi}_x(x, t) + \tilde{\phi}_t(x, t) \right)^2 dx \\ & = -\frac{1}{(1 - \mathbf{v}) \gamma_{\eta}^2} \int_{L+\mathbf{v}t}^{\mathbf{v}t} \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}^2} e^{\frac{1-\mathbf{v}}{L} \ln|\gamma_{\eta}|(t+\gamma_{\mathbf{v}}(\mathbf{v}t-\xi)+\mathbf{v}t)} e^{2 \ln|\gamma_{\eta}|} \left(\tilde{\phi}_x(\xi, t) - \tilde{\phi}_t(\xi, t) \right)^2 d\xi \end{aligned}$$

$$= \frac{1}{1 + \mathbf{v}} \int_{\mathbf{I}_t} e^{\frac{1+\mathbf{v}}{L} \ln|\gamma_\eta|(t-\xi)} (\phi_x(\xi, t) - \phi_t(\xi, t))^2 d\xi. \tag{42}$$

We used the definition of the extensions (9), (10) and the fact that $e^{2\ln|\gamma_\eta|} = \gamma_\eta^2$. Then, combining (41) and (42), we obtain (39).

Expanding $(\phi_x \pm \phi_t)^2$ and collecting similar terms, we get

$$\begin{aligned} & \int_{\mathbf{I}_t} \left(\frac{1}{1 + \mathbf{v}} e^{\frac{1+\mathbf{v}}{L} \ln|\gamma_\eta|(t-x)} + \frac{1}{\gamma_\eta^2(1 - \mathbf{v})} e^{\frac{1-\mathbf{v}}{L} \ln|\gamma_\eta|(t+x)} \right) (\phi_t^2 + \phi_x^2) dx \\ & - 2 \int_{\mathbf{I}_t} \left(\frac{1}{1 + \mathbf{v}} e^{\frac{1+\mathbf{v}}{L} \ln|\gamma_\eta|(t-x)} - \frac{1}{\gamma_\eta^2(1 - \mathbf{v})} e^{\frac{1-\mathbf{v}}{L} \ln|\gamma_\eta|(t+x)} \right) \phi_t \phi_x dx \\ & = \frac{8}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2. \end{aligned} \tag{43}$$

For $\mathbf{v}t \leq x \leq L + \mathbf{v}t$ and $t \geq 0$, let us denote

$$A(x, t) = \frac{1}{1 + \mathbf{v}} e^{\frac{1+\mathbf{v}}{L} \ln|\gamma_\eta|(t-x)} \quad \text{and} \quad B(x, t) = \frac{1}{\gamma_\eta^2(1 - \mathbf{v})} e^{\frac{1-\mathbf{v}}{L} \ln|\gamma_\eta|(t+x)}.$$

Then, we can rewrite (43) as

$$\int_{\mathbf{I}_t} (A + B) (\phi_t^2 + \phi_x^2) dx - 2 \int_{\mathbf{I}_t} (A - B) \phi_t \phi_x dx = \frac{8}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2.$$

Using the algebraic inequality

$$- |A - B| (\phi_t^2 + \phi_x^2) \leq \pm 2(A - B) \phi_t \phi_x \leq |A - B| (\phi_t^2 + \phi_x^2),$$

we get

$$\begin{aligned} & \int_{\mathbf{I}_t} ((A + B) - |A - B|) (\phi_t^2 + \phi_x^2) dx \leq \frac{2}{L} \sum_{n \in \mathbb{Z}} |2\omega_n a_n|^2 \\ & \leq \int_{\mathbf{I}_t} ((A + B) + |A - B|) (\phi_t^2 + \phi_x^2) dx. \end{aligned}$$

Knowing that

$$(a + b) - |a - b| = 2 \min \{a, b\} \quad \text{and} \quad (a + b) + |a - b| = 2 \max \{a, b\},$$

for $a, b \in \mathbb{R}$, then the precedent estimation reads

$$\int_{\mathbf{I}_t} \min \{A, B\} (\phi_t^2 + \phi_x^2) dx \leq \frac{4}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \leq \int_{\mathbf{I}_t} \max \{A, B\} (\phi_t^2 + \phi_x^2) dx.$$

Since $\ln|\gamma_\eta| \geq 0$ and $\mathbf{v}t \leq x \leq L + \mathbf{v}t$, we have

$$\frac{1}{1 + \mathbf{v}} e^{\frac{1+\mathbf{v}}{L} \ln|\gamma_\eta|(t-L-\mathbf{v}t)} \leq A(x, t) \leq \frac{1}{1 + \mathbf{v}} e^{\frac{1+\mathbf{v}}{L} \ln|\gamma_\eta|(t-\mathbf{v}t)},$$

i.e.

$$\frac{e^{-(1+\mathbf{v})\ln|\gamma_\eta|}}{1 + \mathbf{v}} e^{\frac{1-\mathbf{v}}{L} \ln|\gamma_\eta|t} \leq A(x, t) \leq \frac{1}{1 + \mathbf{v}} e^{\frac{1-\mathbf{v}}{L} \ln|\gamma_\eta|t}.$$

Similarly, we obtain

$$\frac{1}{\gamma_\eta^2 (1 - \mathbf{v})} e^{\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t} \leq B(x, t) \leq \frac{e^{-(1+\mathbf{v}) \ln|\gamma_\eta|t}}{1 - \mathbf{v}} e^{\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}.$$

It follows that

$$\begin{aligned} \min \left\{ \frac{1}{|\gamma_\eta|^{(1+\mathbf{v})} (1 + \mathbf{v})}, \frac{1}{\gamma_\eta^2 (1 - \mathbf{v})} \right\} E_\mathbf{v}(t) &\leq \frac{2e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \\ &\leq \max \left\{ \frac{1}{1 + \mathbf{v}}, \frac{1}{|\gamma_\eta|^{(1+\mathbf{v})} (1 - \mathbf{v})} \right\} E_\mathbf{v}(t), \end{aligned}$$

hence

$$\begin{aligned} \left(\frac{2}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \right) \min \left\{ 1 + \mathbf{v}, |\gamma_\eta|^{(1+\mathbf{v})} (1 - \mathbf{v}) \right\} e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t} &\leq E_\mathbf{v}(t) \\ &\leq \left(\frac{2}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \right) \max \left\{ |\gamma_\eta|^{(1+\mathbf{v})} (1 + \mathbf{v}), \gamma_\eta^2 (1 - \mathbf{v}) \right\} e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}. \end{aligned}$$

This shows (40) and the theorem follows. ■

Remark 4.1: If $\mathbf{v} = 0$ in (40), then we get the estimate

$$\left(\frac{2}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \right) e^{-\frac{1}{L} \ln|\gamma_\eta|t} \leq E_0(t) \leq \gamma_\eta^2 \left(\frac{2}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \right) e^{-\frac{1}{L} \ln|\gamma_\eta|t},$$

which is sharper than the estimate (7), stated in the introduction.

Remark 4.2: The solution ϕ given by (3), with $\eta \neq 1$, satisfies the periodicity relation

$$\phi(x + \mathbf{v}T_\mathbf{v}, t + T_\mathbf{v}) = -\phi(x, t) / \gamma_\eta,$$

hence

$$E_\mathbf{v}(t + T_\mathbf{v}) = E_\mathbf{v}(t) / \gamma_\eta^2, \quad \text{for } t \geq 0.$$

Remark 4.3: The constants in estimation (40) are (at least) asymptotically sharp in the sense that if $\eta \rightarrow 0$, we recover the estimation (34) with its sharp constants, see Remark 3.4.

The next corollary compares $E_\mathbf{v}(t)$ to the initial energy $E_\mathbf{v}(0)$.

Corollary 4.2: Under Assumptions (2) and (11), the energy of the solution of Problem (WP) satisfies

$$\begin{aligned} \frac{\min \left\{ \gamma_\mathbf{v}, |\gamma_\eta|^{(1+\mathbf{v})} \right\}}{\max \left\{ \gamma_\mathbf{v} |\gamma_\eta|^{(1+\mathbf{v})}, \gamma_\eta^2 \right\}} E_\mathbf{v}(0) e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t} &\leq E_\mathbf{v}(t) \\ &\leq \frac{\max \left\{ \gamma_\mathbf{v} |\gamma_\eta|^{(1+\mathbf{v})}, \gamma_\eta^2 \right\}}{\min \left\{ \gamma_\mathbf{v}, |\gamma_\eta|^{(1+\mathbf{v})} \right\}} E_\mathbf{v}(0) e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}, \quad \text{for } t \geq 0. \end{aligned} \tag{44}$$

Proof: Since (40) holds also for $t = 0$, then (44) follows by combining the two inequalities

$$\begin{aligned} \frac{e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}}{\max \left\{ |\gamma_\eta|^{(1+\mathbf{v})} (1+\mathbf{v}), \gamma_\eta^2 (1-\mathbf{v}) \right\}} E_{\mathbf{v}}(t) &\leq \frac{2}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \\ &\leq \frac{1}{\min \left\{ 1+\mathbf{v}, |\gamma_\eta|^{(1+\mathbf{v})} (1-\mathbf{v}) \right\}} E_{\mathbf{v}}(0) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\max \left\{ |\gamma_\eta|^{(1+\mathbf{v})} (1+\mathbf{v}), \gamma_\eta^2 (1-\mathbf{v}) \right\}} E_{\mathbf{v}}(0) &\leq \frac{2}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \\ &\leq \frac{e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}}{\min \left\{ 1+\mathbf{v}, |\gamma_\eta|^{(1+\mathbf{v})} (1-\mathbf{v}) \right\}} E_{\mathbf{v}}(t), \end{aligned}$$

for $t \geq 0$. ■

The next corollary gives more simple estimates, but less sharper than (40) and (44), for the energy $E_{\mathbf{v}}$.

Corollary 4.3: *Under Assumptions (2) and (11), the energy of the solution of Problem (WP) satisfies*

$$\begin{aligned} (1-\mathbf{v}) \left(\frac{2}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \right) e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t} &\leq E_{\mathbf{v}}(t) \\ &\leq \gamma_\eta^2 (1+\mathbf{v}) \left(\frac{2}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 \right) e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}, \quad \text{for } t \geq 0 \end{aligned} \quad (45)$$

and

$$\frac{1}{\gamma_\eta^2 \gamma_{\mathbf{v}}} E_{\mathbf{v}}(0) e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t} \leq E_{\mathbf{v}}(t) \leq \gamma_\eta^2 \gamma_{\mathbf{v}} E_{\mathbf{v}}(0) e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}, \quad \text{for } t \geq 0. \quad (46)$$

Proof: Since $0 \leq \mathbf{v} < 1$, then it suffices to simplify the constants in (40) and (44) using the fact that $1 \leq |\gamma_\eta| \leq |\gamma_\eta|^{(1+\mathbf{v})} < \gamma_\eta^2$. ■

5. Some numerical examples and final remarks

In this section, we will compute an approximate solution of Problem (WP), given by its series formula (3) for the first 20 frequencies, i.e.

$$\phi(x, t) \simeq \sum_{n=-20}^{n=20} a_n \left(\gamma_\eta e^{\frac{1-\mathbf{v}}{L} \omega_n(t+x)} + e^{\frac{1+\mathbf{v}}{L} \omega_n(t-x)} \right), \quad \text{for } x \in (\mathbf{v}t, \pi + \mathbf{v}t),$$

where the coefficient a_n are computed for the initial conditions

$$\phi^0(x) = (1 + \cos x)/10 \quad \text{and} \quad \phi^1(x) = 0,$$

hence $E_{\mathbf{v}}(0) \simeq 0.0079$. The values of \mathbf{v} and η will be chosen to emphasise how the solution and its energy depend on these parameters.

5.1. The undamped case $\eta = 0$

We plot the solution for $t = 0, \frac{T_v}{4}, \frac{T_v}{2}, \frac{3T_v}{4}, \dots, 2T_v$ and two different values of v . The energy $E_v(t)$ is plotted over two periods.

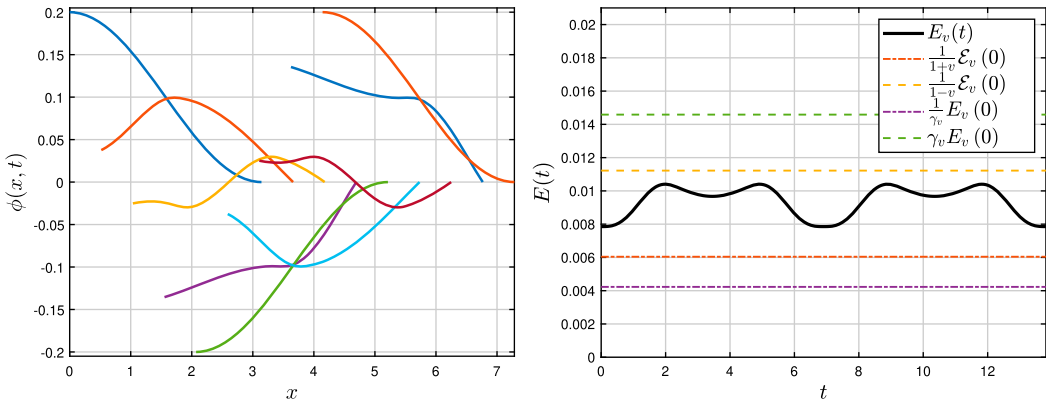


Figure 3. A string travelling at a speed $v = 0.3$, and its energy $E_v(t)$ plotted for $t \in [0, 2T_v]$, where $T_v \simeq 6.90$.

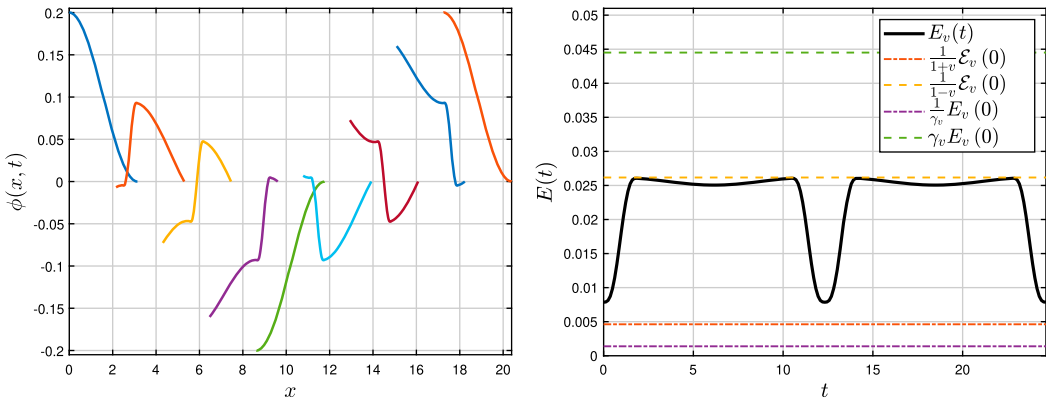


Figure 4. A string travelling at a speed $v = 0.7$, and its energy $E_v(t)$ plotted over two periods, where $T_v \simeq 12.32$.

5.2. The underdamped case $0 < \eta < 1$

We plot the solution for $t = 0, \frac{T_v}{4}, \frac{T_v}{2}, \frac{3T_v}{4}, \dots, 3T_v$, where $v = 0.5$, $T_v \simeq 8.38$ and two different values of η . The energy $E_v(t)$ is plotted for $t \in [0, 3T_v]$. The graphs show that the energy behave as predicted by estimations (40) and (46). In this case, the solution changes sign.

5.3. The overdamped case $\eta > 1$

We retain the speed value $v = 0.5$ and take two values of $\eta > 1$. The solution do not change sign this time.

Remark 5.1: Observe that $\gamma_{(1/\eta)} = -\gamma_\eta$. Thus $\sum_{n \in \mathbb{Z}} |\omega_n a_n|^2$ remains unchanged if we replace η by $1/\eta$ since the left-hand side of (14) remains unchanged. By consequence, the identity (39), M_1, M_2

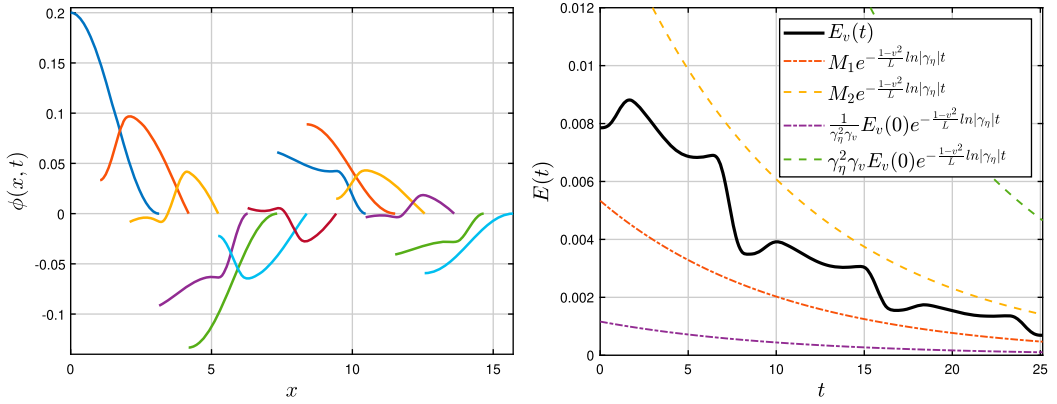


Figure 5. A string travelling at a speed $\mathbf{v} = 0.5$, and its energy $E_v(t)$, for a damping factor $\eta = 0.2$.

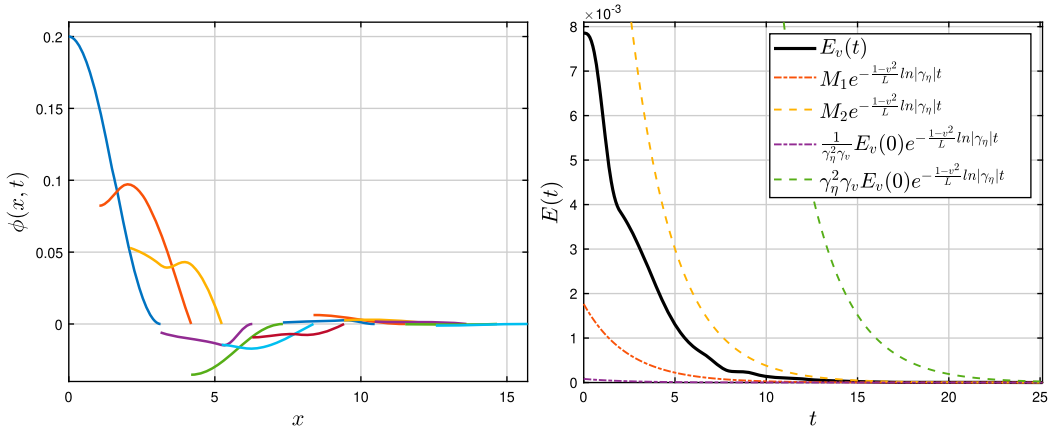


Figure 6. A string travelling at a speed $\mathbf{v} = 0.5$, and its energy $E_v(t)$, for a damping factor $\eta = 0.7$.

and the constants in the energy estimates of Section 4 remain the same for η and $1/\eta$. For instance, taking $\mathbf{v} = 0.5$, the upper and lower estimates for $E_v(t)$ with $\eta = 0.1$ are identical to those of Figure 8 for $\eta = 10$.

Remark 5.2: The case with a dashpot damping at the inlet pulley can be easily investigated by replacing \mathbf{v} by $-\mathbf{v}$ in the above sections. In Corollary 3.2 we have to change \mathbf{v} by $|\mathbf{v}|$, i.e.

$$\frac{\mathcal{E}_v(0)}{1 + |\mathbf{v}|} \leq E_v(t) \leq \frac{\mathcal{E}_v(0)}{1 - |\mathbf{v}|} \quad \text{and} \quad \frac{E_v(0)}{\gamma_{|\mathbf{v}|}} \leq E_v(t) \leq \gamma_{|\mathbf{v}|} E_v(0), \quad \text{for } t \geq 0.$$

More importantly, we still have the same exponential $e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}$ in the estimates of Section 4 when $\eta > 0$. The analogue of estimations (45) is

$$\begin{aligned} & \frac{2(1 - |\mathbf{v}|)}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t} \leq E_v(t) \\ & \leq \frac{2\gamma_\eta^2(1 + |\mathbf{v}|)}{L} \sum_{n \in \mathbb{Z}} |\omega_n a_n|^2 e^{-\frac{1-\mathbf{v}^2}{L} \ln|\gamma_\eta|t}, \quad \text{for } t \geq 0 \end{aligned}$$

and γ_η is replaced by $\gamma_{|\mathbf{v}|}$ in (46).

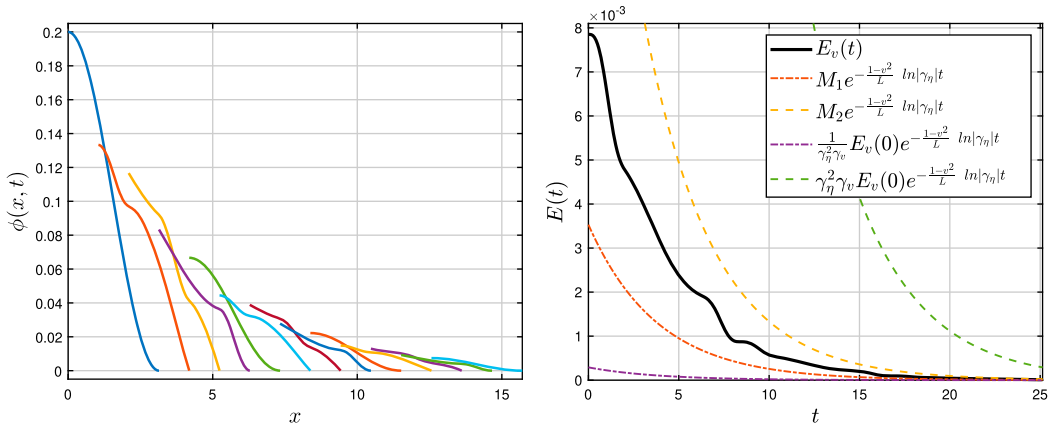


Figure 7. A string travelling at a speed $\mathbf{v} = 0.5$, and its energy $E_v(t)$, for a damping factor $\eta = 2$.

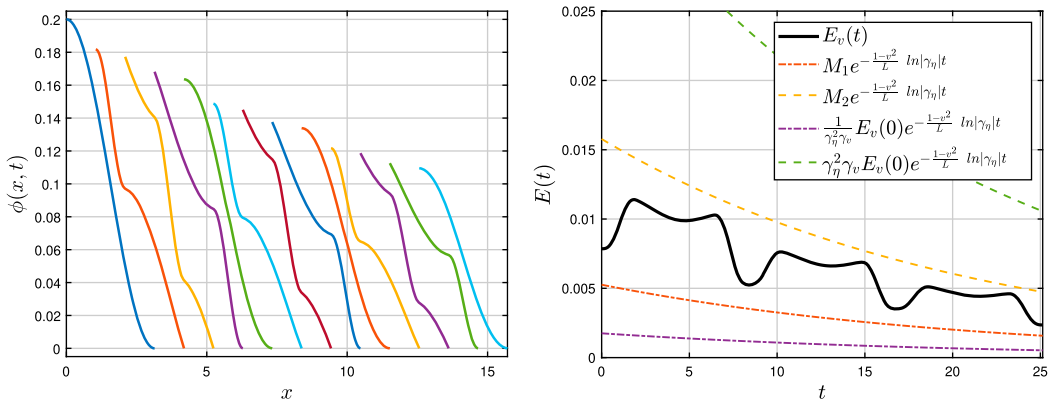


Figure 8. A string travelling at a speed $\mathbf{v} = 0.5$, and its energy $E_v(t)$, for damping factor $\eta = 10$.

Acknowledgments

The authors also thank the reviewers of this paper for their valuable suggestions on earlier version of this manuscript.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The authors have been supported by the General Direction of Scientific Research and Technological Development (Algerian Ministry of Higher Education and Scientific Research) PRFU # C00L03UN280120220010.

ORCID

Abdelmouhcene Sengouga  <http://orcid.org/0000-0003-3183-7973>

References

- [1] Marynowski K, Kapitaniak T. Dynamics of axially moving continua. *Int J Mech Sci.* 2014;81:26–41.
- [2] Hong KS, Pham PT. Control of axially moving systems: a review. *Int J Control Autom Syst.* 2019;17(12):2983–3008.
- [3] Chen LQ. Analysis and control of transverse vibrations of axially moving strings. *Appl Mech Rev.* 2005;58(2):91–116.

- [4] Miranker WL. The wave equation in a medium in motion. *IBM J Res Dev.* 1960;4(1):36–42.
- [5] Mote CD. A study of band saw vibrations. *J Franklin Inst.* 1965;279(6):430–444.
- [6] Tan CA, Ying S. Dynamic analysis of the axially moving string based on wave propagation. *ASME J Appl Mech.* 1997;64(2):394–400.
- [7] Chen EW, Ferguson NS. Analysis of energy dissipation in an elastic moving string with a viscous damper at one end. *J Sound Vib.* 2014;333(9):2556–2570.
- [8] Chen E, He Y, Zhang K, et al. A superposition method of reflected wave for moving string vibration with nonclassical boundary. *J Chin Inst Eng.* 2019;42(4):327–332.
- [9] Chen E, Yuan J, Ferguson N, et al. A wave solution for energy dissipation and exchange at nonclassical boundaries of a traveling string. *Mech Syst Signal Process.* 2021;150:107272.
- [10] Lee SY, Mote CD. Vibration control of an axially moving string by boundary control. *J Dyn Syst Meas Control.* 1996;118(1):66–74.
- [11] Gaiko NV, van Horssen WT. On the transverse, low frequency vibrations of a traveling string with boundary damping. *J Vib Acoust.* 2015;137(4):041004 (10p).
- [12] Ghenimi S, Sengouga A. Free vibrations of axially moving strings: energy estimates and boundary observability. 2022. <http://arxiv.org/abs/2201.01866>.
- [13] Fung RF, Wu JW, Wu SL. Exponential stabilization of an axially moving string by linear boundary feedback. *Automatica.* 1999;35(1):177–181.
- [14] Veselić K. On linear vibrational systems with one dimensional damping. *Appl Anal.* 1988;29(1-2):1–18.
- [15] Cox S, Zuazua E. The rate at which energy decays in a string damped at one end. *Indiana Univ Math J.* 1995;44(2):545–573.
- [16] Cherkaoui M. Estimation optimale du taux de décroissance de l'énergie pour une équation des ondes avec contrôle frontière. INRIA; 1994. (Research Report RR-2328). Available at <https://hal.inria.fr/inria-00074346>.
- [17] Quinn JP, Russell DL. Asymptotic stability and energy decay rates for solutions of hyperbolic equations with boundary damping. *Proc R Soc Edinb A Math.* 1977;77(1-2):97–127.
- [18] Sengouga A. Exact boundary observability and controllability of the wave equation in an interval with two moving endpoints. *Evol Equ Control Theory.* 2020;9(1):1–25.
- [19] McIver DB. Hamilton's principle for systems of changing mass. *J Eng Math.* 1973;7(3):249–261.
- [20] Cassel KW. Variational methods with applications in science and engineering. New York, USA: Cambridge University Press; 2013.

Appendix A. Derivation of the model (1)

For the sake of completeness, we now derive the mathematical model (1) for the vibration of a travelling string subject to a boundary damping, as described in Introduction. We refer to [11] for a model that considers more general boundary conditions.

We make the following assumptions:

- The vibrations of the string are transversal and small (compared to its length). This implies in particular that the slope u_s is also small.
- The string is uniform with a mass density ρ .
- The string travels with a constant speed \mathbf{v} between to two massless pulleys (inlet and outlet) kept at a fixed distance L .
- The string is perfectly flexible and the effects due to gravity are neglected. This means that the tension \mathbf{T} is constant.
- The inlet is not allowed to move transversely, hence we have the boundary condition

$$u(0, \tau) = 0, \quad \text{for } \tau \geq 0.$$

We apply the extended Hamilton's principle, over the finite time interval $\tau_1 \leq \tau \leq \tau_2$, in the following form:

$$\delta \int_{\tau_1}^{\tau_2} \mathcal{L} \, d\tau + \int_{\tau_1}^{\tau_2} \delta \mathcal{W} \, d\tau = 0, \quad (\text{A1})$$

(see [19] and [20, Chapters 4 and 5]). Here δ denotes the variation in a given function, \mathcal{L} is the Lagrangian and \mathcal{W} is the virtual work performed by nonconservative forces, i.e. the boundary damping force in the present case. The Lagrangian is

$$\mathcal{L} = E_k - E_p,$$

where E_k is the kinetic energy and E_p is the potential energy of the string. The first one is given by

$$E_k = \frac{1}{2} \int_0^L \rho \left(\frac{du}{d\tau} \right)^2 \, ds$$

where $\frac{du}{d\tau}(s, \tau) = u_\tau(s, \tau) + \mathbf{v}u_s(s, \tau)$ is the material velocity (also called the total derivative of u). The potential energy E_p is equal to the work done to deform the string from its rest position. Hence, the potential energy per unit length of a differential element is given by

$$dE_p = \mathbf{T} \cdot \text{elongation} = \mathbf{T} \left(\sqrt{(ds)^2 + (du)^2} - ds \right).$$

Since the slope of the string u_s is small, then

$$dE_p = \mathbf{T} \left(\sqrt{1 + (u_s)^2} - 1 \right) ds \approx \frac{\mathbf{T}}{2} u_s^2 ds,$$

hence the potential energy of the string is given by

$$E_p = \frac{\mathbf{T}}{2} \int_0^L u_s^2 ds$$

and therefore

$$\mathcal{L} = \frac{\rho}{2} \int_0^L (u_\tau + \mathbf{v}u_s)^2 ds - \frac{\mathbf{T}}{2} \int_0^L u_s^2 ds.$$

The damping force \mathbf{F}_η of the dashpot at $s = L$ is contrary to the sense of movement and thus the work done by the dashpot is

$$\mathcal{W} = -\mathbf{F}_\eta \cdot \text{displacement} = -\mathbf{F}_\eta u(L, \tau).$$

Now, we are led to search the critical points of the functional

$$I(u) := \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_0^L \rho (u_\tau + \mathbf{v}u_s)^2 - \mathbf{T}u_s^2 ds d\tau - \int_{\tau_1}^{\tau_2} \mathbf{F}_\eta u(L, \tau) d\tau. \quad (\text{A2})$$

When u is a critical point, we should have

$$\lim_{\theta \rightarrow 0} \frac{I(u + \theta v) - I(u)}{\theta} = 0, \quad (\text{A3})$$

where v is a smooth function such that

$$v(\cdot, \tau_1) = v(\cdot, \tau_2) = 0 \quad \text{and} \quad v(0, \cdot) = 0. \quad (\text{A4})$$

The function $u + \theta v$ is a small perturbation of u that does not affect the values of (the path) u at $\tau = \tau_1$ and $\tau = \tau_2$, as well as at $s = 0$. The limit (A3) equals

$$\int_{\tau_1}^{\tau_2} \int_0^L \rho (\mathbf{v}^2 u_s v_s + \mathbf{v}u_\tau v_s + \mathbf{v}u_s v_\tau + u_\tau v_\tau) - \mathbf{T}u_s v_s ds d\tau - \int_{\tau_1}^{\tau_2} \mathbf{F}_\eta v(L, \tau) d\tau = 0.$$

Integrating by parts, we infer that

$$\begin{aligned} & - \int_{\tau_1}^{\tau_2} \int_0^L \left\{ \rho (u_{\tau\tau} + 2\mathbf{v}u_{s\tau} + \mathbf{v}^2 u_{ss}) - \mathbf{T}u_{ss} \right\} v ds d\tau \\ & + \int_{\tau_1}^{\tau_2} \left\{ \rho \mathbf{v}u_\tau(L, \tau) + (\rho \mathbf{v}^2 - \mathbf{T}) u_s(L, \tau) - \mathbf{F}_\eta \right\} v(L, \tau) d\tau = 0. \end{aligned} \quad (\text{A5})$$

The other boundary terms vanishes due to (A4). Since v can be chosen arbitrary, and in particular its value at $s = L$, the first integral implies that

$$u_{\tau\tau} + 2\mathbf{v}u_{s\tau} + \mathbf{v}^2 u_{ss} - \frac{\mathbf{T}}{\rho} u_{ss} = 0, \quad \text{for } s \in (0, L) \quad \text{and} \quad \tau_1 \leq \tau \leq \tau_2.$$

The quantity $c = \sqrt{\mathbf{T}/\rho}$ is the wave speed in the string.

The second integral implies that

$$\rho \mathbf{v}u_\tau(L, \tau) + (\mathbf{T} - \rho \mathbf{v}^2) u_s(L, \tau) - \mathbf{F}_\eta = 0, \quad \text{for } \tau_1 \leq \tau \leq \tau_2.$$

Assuming that the damping force \mathbf{F}_η depends linearly on the velocity u_τ , i.e. $\mathbf{F}_\eta = \eta u_\tau$, we obtain the boundary condition

$$(\eta - \rho \mathbf{v}) u_\tau(L, \tau) + (\mathbf{T} - \rho \mathbf{v}^2) u_s(L, \tau) = 0, \quad \text{for } \tau_1 \leq \tau \leq \tau_2.$$

Rescaling the variables τ, \mathbf{v} and η as follows:

$$\tau \mapsto \tau \sqrt{\frac{\mathbf{T}}{\rho}}, \quad \mathbf{v} \mapsto \mathbf{v} \sqrt{\frac{\rho}{\mathbf{T}}} \quad \text{and} \quad \eta \mapsto \frac{\eta}{\sqrt{\rho \mathbf{T}}},$$

we obtain the model (1) with a normalised speed $c = 1$.

Appendix B. A second proof for the conservation of $\mathcal{E}_v(t)$

Let us check that $\frac{d}{dt}\mathcal{E}_v(t) = 0$ by using only the identities $\phi_{tt} = \phi_{xx}$, $\phi_x(\mathbf{v}t, t) + \mathbf{v}\phi_t(\mathbf{v}t, t) = 0$ and $\phi(L + \mathbf{v}t, t) = 0$ from (WP). First, when $\eta = 0$ the boundary condition at $x = \mathbf{v}t$ reads

$$\phi_x(\mathbf{v}t, t) + \mathbf{v}\phi_t(\mathbf{v}t, t) = 0. \quad (\text{A6})$$

At the other boundary, we have $\phi(L + \mathbf{v}t, t) = 0$ for $t \geq 0$. This means that

$$\frac{d}{dt}\phi(L + \mathbf{v}t, t) = \mathbf{v}\phi_x(L + \mathbf{v}t, t) + \phi_t(L + \mathbf{v}t, t) = 0. \quad (\text{A7})$$

It follows in particular that

$$\phi_x^2(\mathbf{v}t, t) = \mathbf{v}^2\phi_t^2(\mathbf{v}t, t) \quad \text{and} \quad \mathbf{v}^2\phi_x^2(L + \mathbf{v}t, t) = \phi_t^2(L + \mathbf{v}t, t).$$

Next, recall that

$$\mathcal{E}_v(t) = E_v(t) + \mathbf{v} \int_{\mathbf{I}_t} \phi_t \phi_x \, dx \quad (\text{A8})$$

and let us differentiate each term of the right hand side separately. On one hand, the identities (A6), (A7) and Leibniz's rule imply that

$$\frac{d}{dt}E_v(t) = \frac{\mathbf{v}}{2}(1 + \mathbf{v}^2)(\phi_x^2(L + \mathbf{v}t, t) - \phi_t^2(\mathbf{v}t, t)) + \int_{\mathbf{I}_t} (\phi_x \phi_{tx} + \phi_{tt} \phi_t) \, dx. \quad (\text{A9})$$

Taking into account that $\phi_{tt} = \phi_{xx}$, the last integral equals

$$\int_{\mathbf{I}_t} (\phi_x \phi_{tx} + \phi_{xx} \phi_t) \, dx = \int_{\mathbf{I}_t} (\phi_x \phi_t)_x \, dx = -\mathbf{v}(\phi_x^2(L + \mathbf{v}t, t) - \phi_t^2(\mathbf{v}t, t)),$$

hence

$$\frac{d}{dt}E_v(t) = -\frac{\mathbf{v}}{2}(1 - \mathbf{v}^2)(\phi_x^2(L + \mathbf{v}t, t) - \phi_t^2(\mathbf{v}t, t)). \quad (\text{A10})$$

On the other hand

$$\frac{d}{dt} \int_{\mathbf{I}_t} \phi_x \phi_t \, dx = -\mathbf{v}^2(\phi_x^2(L + \mathbf{v}t, t) - \phi_t^2(\mathbf{v}t, t)) + \int_{\mathbf{I}_t} \phi_{tx} \phi_t + \phi_x \phi_{tt} \, dx.$$

Again, using $\phi_{tt} = \phi_{xx}$ and integrating by parts, the last integral equals

$$\frac{1}{2} \int_{\mathbf{I}_t} (\phi_t^2 + \phi_x^2)_x \, dx = \frac{1}{2}(1 + \mathbf{v}^2)(\phi_x^2(L + \mathbf{v}t, t) - \phi_t^2(\mathbf{v}t, t)),$$

hence

$$\frac{d}{dt} \int_{\mathbf{I}_t} \phi_x \phi_t \, dx = \frac{1}{2}(1 - \mathbf{v}^2)(\phi_x^2(L + \mathbf{v}t, t) - \phi_t^2(\mathbf{v}t, t)). \quad (\text{A11})$$

Due to (A8), (A10) and (A11), we infer that $\frac{d}{dt}\mathcal{E}_v(t) = 0$ as claimed.