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Theme

”Linearization of Nonlinear Operators in the Theory of
Summability”

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Dedication

I dedicate this work to:

***My self** — for having the patience and the courage to finish this journey.*

***To my dear parents** — your love and prayers have been my greatest support.*

***To my family and true friends** — thank you for always being there.*

With gratitude and love, Zahra

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List of Symbols

Symbol	Description
\mathbb{K}	The field of real or complex numbers (\mathbb{R} or \mathbb{C}).
\mathbb{R}_+	The field of non-negative real numbers.
p^*	The conjugate of the number p ($1 \leq p \leq \infty$), that is $\frac{1}{p} + \frac{1}{p^*} = 1$.
B_E	The closed unit ball of E .
E^*	The topological dual of E .
$L(E; F)$	The set of all linear operators from E to F .
$\mathcal{L}(E; F)$	The set of all continuous linear operators from E to F .
$\Pi_p(X, Y)$	The space of p -summing operators from X to Y .
$\ell_p(X)$	The space of strongly p -summable sequences in X , endowed with the norm $\ (x_n)\ _p$.
$\ell_p^w(X)$	The space of weakly p -summable sequences in X , endowed with the norm $\ (x_n)\ _p^w$.
$\text{Lip}(X, E)$	The space of Lipschitz maps from X to E .
$\text{Lip}(f)$	The Lipschitz constant of the map f .
$\text{Lip}_0(X, E)$	The space of Lipschitz maps $f : X \rightarrow E$ such that $f(0) = 0$.
$X^\#$	The Lipschitz dual space of X , i.e., $X^\# = \text{Lip}_0(X, \mathbb{K})$.
$\mathcal{F}(X)$	The Lipschitz-free space over X , the closed linear span of $\{\delta_x : x \in X\}$.
δ_x	The evaluation functional in $(X^\#)^*$, given by $\delta_x(f) = f(x)$.
$\delta(x, y)$	The evaluation functional in $\mathcal{F}(X)$, given by $\delta(x, y)(f) = f(x) - f(y)$.

$\Pi_p^{SL}(X, E)$	The space of strictly p -summing Lipschitz operators from X to E .
$D_p^L(X, E)$	The space of strongly Cohen p -summing Lipschitz operators from X to E .
$\text{Lip}_0^{\text{int}}(X, E)$	The space of integral Lipschitz operators from X to E .
$\mathcal{L}(X_1, \dots, X_m; Y)$	The space of continuous multilinear maps from $X_1 \times \dots \times X_m$ to Y .
$X_1 \otimes \dots \otimes X_m$	The algebraic tensor product of the spaces X_1, \dots, X_m .
$X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m$	The projective tensor product of the spaces X_1, \dots, X_m .
$D_p^m(X_1, \dots, X_m; Y)$	The space of strongly Cohen p -summing multilinear maps from $X_1 \times \dots \times X_m$ to Y .
$\text{Lip}_0(X, Y)$	The set of all Lipschitz operators between X and Y that vanish at 0.
$\Pi_p^L(X, Y)$	The class of all Lipschitz p -summing sublinear operators from X to Y ($1 \leq p \leq \infty$).
$D_p(X, Y)$	The set of all strongly p -summing operators from X to Y ($1 \leq p \leq \infty$).

Introduction

Linearization is an important tool in functional analysis, as it helps study nonlinear operators by linking them to linear ones. Since the theory of linear operators is rich with results and powerful mathematical tools, transforming certain classes of nonlinear operators into equivalent linear counterparts allows us to leverage these tools in their analysis. In this work, we focus on studying operators such as Lipschitz operators, multilinear operators, where the process of linearization provides a better understanding by transferring well-known properties of linear operators to these classes. This approach simplifies the study of nonlinear operators and expands the scope of mathematical applications in this field. This Master memory, written in the field of functional analysis, studies the relationship between nonlinear operators and their linearizations for various notions of summability. The content is organized into three chapters:

Chapter 1 recalls the background needed throughout this memory. We begin with the definition of Banach spaces, followed by linear operators and their basic properties. Next, we present the projective and injective tensor products. The chapter ends with the definitions of p -summing and strongly p -summing Cohen operators, along with Pietsch factorization theorems.

Chapter 2 focuses on Lipschitz mappings. We present the essential results and identification theorems. We then explore the relationship between a Lipschitz operator and its linearization for the classes of p -summing, strongly p -summing Cohen operators, and integral operators.

Chapter 3 is devoted to multilinear operators. We first present the main results of this class, then emphasize some identification properties. Finally, we study the relationship between a multilinear operator and its linearization for the same three operator ideals: p -summing, Cohen strongly p -summing, and integral operators.

Chapter 1: Generalities on linear operators of summing type

1.1 Introduction

In this chapter, we will present the definition of Banach and Hilbert spaces with some examples. Reflexive Banach spaces are also discussed in this part. Then, we will study bounded linear operators. We continue this study by giving the algebraic structure of a tensor product, providing the definition of the projective tensor product that we need later. For more details on the results of this chapter, please consult [DJT95],[DF93],[LT96], [Muj86], [Pie83] and [Zaa97].

1.2 Banach Spaces

Definition 1.1 (Banach Space) *A Banach space is a normed vector space X that is complete, i.e., every Cauchy sequence in X converges in X .*

Remark 1.1 *Every finite-dimensional normed space is a Banach space. Moreover, every closed subspace of a Banach space is a Banach space.*

Example 1.1 *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements in \mathbb{R} and $p \in [1, +\infty[$. We define the set ℓ_p by*

$$\ell_p = \left\{ (x_n)_{n \in \mathbb{N}} \subset \mathbb{R} : \left(\sum_{n \in \mathbb{N}} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

If $p = \infty$, we define ℓ_∞ by

$$\ell_\infty = \left\{ (x_n)_{n \in \mathbb{N}} \subset \mathbb{R} : \sup_{n \in \mathbb{N}} |x_n| < \infty \right\}.$$

Then, the space ℓ_p is a Banach space.

Definition 1.2 (Hilbert Space) A Hilbert space is a complete pre-Hilbert space with respect to the distance induced by the scalar product.

Theorem 1.1 (The Space ℓ_2) The space ℓ_2 is a Hilbert space with the scalar product

$$\langle x, y \rangle = \sum_{n \in \mathbb{N}} x_n y_n.$$

Proposition 1.1 (Comparison of Spaces) Let $1 \leq p \leq q \leq \infty$. We have

$$\ell_p \subset \ell_q.$$

1.3 Linear Operators

Definition 1.3 Let $u : X \rightarrow Y$ be a mapping between two Banach spaces. It is linear if

$$\forall x, y \in X, \forall \alpha, \beta \in \mathbb{R} : u(\alpha x + \beta y) = \alpha u(x) + \beta u(y).$$

We denote $L(X, Y)$ the set of linear mappings, equipped with the following algebraic operations:

1) $\forall x \in X : (u_1 + u_2)(x) = u_1(x) + u_2(x).$

2) $\forall x \in X, \forall \lambda \in \mathbb{R} : (\lambda u)(x) = \lambda u(x).$

Theorem 1.2 The linear mapping u is continuous (bounded) if there exists $C > 0$ such that

$$\forall x \in X : \|u(x)\| \leq C \|x\|.$$

We denote $\mathcal{L}(X, Y)$ the space of continuous linear mappings. We define an operator norm $\|\cdot\|$ on $\mathcal{L}(X; Y)$ by

$$\|u\| = \sup_{\|x\| \leq 1} \|u(x)\|,$$

We also have

$$\|u\| = \inf \{C : \text{satisfies the previous inequality}\}.$$

Then, the quantity $\|u\|$ defines a norm on $\mathcal{L}(X;Y)$. If Y is a Banach space, then $\mathcal{L}(X;Y)$ is also a Banach space.

Definition 1.4 (Topological Dual) Let X be a normed vector space. We call the topological dual, denoted X^* , the Banach space of continuous linear forms on X , i.e.,

$$X^* = \mathcal{L}(X, \mathbb{R}).$$

Note that the space X^* is always complete with respect to the operator norm.

Example 1.2 Let $1 < p < +\infty$. We have

$$(\ell_p)^* = \ell_{p^*},$$

where p^* is the conjugate of p , i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$.

1.4 Tensor Product

We start with the case of two spaces; the general case of n spaces is similar. Let X, Y be two Banach spaces. Let $x \in X$ and $y \in Y$. We define $x \otimes y$ as a linear form on the space of bilinear forms $\mathcal{L}(X \times Y)$ as follows:

$$\forall T \in \mathcal{L}(X \times Y) : x \otimes y(T) = \langle T, x \otimes y \rangle = T(x, y). \quad (1.1)$$

We denote $X \otimes Y$ the vector space generated by elements of the form $x \otimes y$ with $x \in X$ and $y \in Y$. The space $X \otimes Y$ is called the algebraic tensor product space of X and Y . We can represent this space as follows:

$$X \otimes Y = \left\{ \sum_{i=1}^n x_i \otimes y_i : n \in \mathbb{N}^*, x_i \in X, y_i \in Y \right\}.$$

From the definition of $X \otimes Y$, we have

$$X \otimes Y \subset \mathcal{L}(X \times Y)^*.$$

We will see some properties of the elements of $X \otimes Y$, starting with the relationship between the operations $+$ and \otimes .

Proposition 1.2 For all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$, we have:

- (i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$.
- (ii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$.
- (iii) $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y)$ with $\lambda \in \mathbb{K}$.
- (iv) $x \otimes 0 = 0 \otimes y = 0$.

proof We will only prove (i), as the others are immediate. Let $T \in \mathcal{L}(X \times Y)$, then

$$\begin{aligned} (x_1 + x_2) \otimes y(T) &= T(x_1 + x_2, y) \\ &= T(x_1, y) + T(x_2, y) \\ &= x_1 \otimes y(T) + x_2 \otimes y(T) \\ &= (x_1 \otimes y + x_2 \otimes y)(T), \end{aligned}$$

thus $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$. The following proposition explains what a zero element is in the space $X \otimes Y$.

Proposition 1.3 Let X, Y be two Banach spaces. For any

$$u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y,$$

the following statements are equivalent:

- (i) $u = 0$.
- (ii) $\sum_{i=1}^n x^*(x_i)y^*(y_i) = 0$ for all $x^* \in X^*, y^* \in Y^*$.
- (iii) $\sum_{i=1}^n x^*(x_i)y_i = 0$ for all $x^* \in X^*$.
- (iv) $\sum_{i=1}^n y^*(y_i)x_i = 0$ for all $y^* \in Y^*$.

proof (i) \Rightarrow (ii): Suppose $u = 0$. Then, for all $T \in \mathcal{L}(X \times Y)$, we have

$$u(T) = \sum_{i=1}^n T(x_i, y_i) = 0.$$

Let $x^* \in X^*, y^* \in Y^*$. Then the mapping

$$x^* \otimes y^* : X \times Y \rightarrow \mathbb{K}$$

defined by $x^* \otimes y^*(x, y) = x^*(x)y^*(y)$ is a linear form on $\mathcal{L}(X \times Y)$. That is,

$$u(x^* \otimes y^*) = \sum_{i=1}^n x^*(x_i)y^*(y_i) = 0.$$

(ii) \Rightarrow (iii): Suppose $\sum_{i=1}^n x^*(x_i)y^*(y_i) = 0$ for all $x^* \in X^*, y^* \in Y^*$. Let $x^* \in X^*$, then

$$y^* \left(\sum_{i=1}^n x^*(x_i)y_i \right) = 0 \text{ for all } y^* \in Y^*,$$

which implies $\sum_{i=1}^n x^*(x_i)y_i = 0$.

(iii) \Rightarrow (iv): Same argument.

(iv) \Rightarrow (i): Suppose $\sum_{i=1}^n y^*(y_i)x_i = 0$ for all $y^* \in Y^*$. Let $T \in \mathcal{L}(X \times Y)$. Let E, F be the vector subspaces generated by $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$, respectively. Let B be the restriction of T to $E \times F$. We can represent B as follows:

$$B(x, y) = \sum_{j=1}^m \varphi_j(x)\psi_j(y),$$

where $\varphi_j \in E^*$ and $\psi_j \in F^*$. By the Hahn-Banach theorem, we extend the linear forms φ_j and ψ_j to all of X and Y , respectively. Then,

$$\begin{aligned} u(T) &= \sum_{i=1}^n T(x_i, y_i) = \sum_{i=1}^n B(x_i, y_i) \\ &= \sum_{i=1}^n \sum_{j=1}^m \varphi_j(x_i)\psi_j(y_i) \\ &= \sum_{j=1}^m \psi_j \left(\sum_{i=1}^n \varphi_j(x_i)y_i \right) \\ &= \sum_{j=1}^m \psi_j(0) = 0. \end{aligned}$$

Thus, $u(T) = 0$ for all $T \in \mathcal{L}(X \times Y)$, so $u = 0$.

1.5 Projective Tensor Product

Let X_1, \dots, X_m be Banach spaces. We denote $X_1 \otimes \cdots \otimes X_m$ the algebraic tensor product of X_1, \dots, X_m

$$X_1 \otimes \cdots \otimes X_m = \left\{ \sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m : n \in \mathbb{N}^*, x_i^j \in X_j \right\}. \quad (1.2)$$

This is a vector space. We define the projective norm on $X_1 \otimes \cdots \otimes X_m$ by

$$\|v\|_\pi = \inf \left\{ \sum_{i=1}^n \prod_{j=1}^m \|x_i^j\| \right\},$$

where the infimum is taken over all possible representations of v of the form

$$v = \sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m.$$

We denote $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$ the projective tensor product of the spaces X_1, \dots, X_m , i.e., the completion of $X_1 \otimes \cdots \otimes X_m$ with respect to this norm. If $X_1 = \cdots = X_m = X$, we write simply $\widehat{\otimes}_\pi^m X$.

Proposition 1.4 *For all Banach spaces X_1, \dots, X_m, Y , we have the isometric identification*

$$\mathcal{L}(X_1, \dots, X_m; Y) = (X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m \widehat{\otimes}_\pi Y^*)^*.$$

Special case. *The dual of $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$ is identified with the space of bounded multilinear forms*

$$(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m)^* = \mathcal{L}(X_1, \dots, X_m).$$

1.6 Injective Tensor Product

Let X_1, \dots, X_m be Banach spaces. We denote $X_1 \otimes \cdots \otimes X_m$ the algebraic tensor product of X_1, \dots, X_m . We define the injective tensor norm on $X_1 \otimes \cdots \otimes X_m$ by

$$\|v\|_\xi = \sup_{x_j^* \in X_j^* (1 \leq j \leq m)} \left\{ \left| \sum_{i=1}^n x_1^*(x_i^1) \cdots x_m^*(x_i^m) \right| \right\}, \quad (1.3)$$

where the supremum is taken over all possible representations of v of the form

$$v = \sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m.$$

We denote $X_1 \widehat{\otimes}_{\xi} \cdots \widehat{\otimes}_{\xi} X_m$ the injective tensor product of the spaces X_1, \dots, X_m , i.e., the completion of $X_1 \otimes \cdots \otimes X_m$ with respect to this norm. If $X_1 = \cdots = X_m = X$, we write simply $\widehat{\otimes}_{\xi}^m X$.

1.7 p -Summing Linear Operators

Let X be a Banach space. Let $1 \leq p \leq \infty$. We denote $l_p^n(X)$ the Banach space of sequences $(x_i)_{1 \leq i \leq n}$ in X equipped with the norm

$$\|(x_i)_{1 \leq i \leq n}\|_{l_p^n(X)} = (\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}},$$

and $l_p^n \Omega(X)$ the Banach space of sequences $(x_i)_{1 \leq i \leq n}$ in X equipped with the norm

$$\|(x_i)\|_{l_p^n \Omega(X)} = \sup_{x^* \in B_{X^*}} (\sum_{i=1}^n |\langle x_i, x^* \rangle|^p)^{\frac{1}{p}},$$

where X^* is the topological dual of X . The closed unit ball of X is denoted B_X (if $p = \infty$, we take the supremum).

Definition 1.5 (Linear Case) *Let X, Y be two Banach spaces and $u \in \mathcal{B}(X; Y)$. We say that u is p -summing for $p \in [1, \infty[$, if there exists a constant $C > 0$ such that for all $x_1, \dots, x_n \in X$*

$$\left(\sum_{i=1}^n \|u(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |x^*(x_i)|^p \right)^{\frac{1}{p}}. \quad (1.4)$$

We denote $\Pi_p(X; Y)$ the Banach space of p -summing linear operators from X to Y equipped with the norm

$$\pi_p(u) = \inf\{C \text{ satisfying (1.4)}\}.$$

Remark 1.2 *The operator u is p -summing if it transforms every weakly p -summable sequence into a strongly p -summable sequence; if $p = \infty$, this is simply continuity.*

Example 1.3 *Let K be a compact space, μ a positive measure on K , and $1 \leq p < \infty$.*

(1) The multiplication operator defined by

$$\begin{aligned} u_\varphi : C(K) &\rightarrow L_p(\mu) \\ f &\mapsto f \cdot \varphi \end{aligned}$$

with $\varphi \in L_p(\mu)$, is p -summing and $\pi_p(u) = \|\varphi\|_p$.

(2) The canonical injection

$$\begin{aligned} J_p : C(K) &\rightarrow L_p(\mu) \\ f &\mapsto f \end{aligned}$$

is p -summing and $\pi_p(J_p) = \mu(K)^{\frac{1}{p}}$.

Theorem 1.3 (Pietsch Factorization [DJT95]) *Let $p \in [1, \infty[$ and $u : X \rightarrow Y$ a p -summing linear operator. Then, there exists a Radon probability measure λ on $(B_{X^*}, \sigma(X^*, X))$ such that*

$$\forall x \in X : \|u(x)\| \leq \pi_p(u) \left(\int_{B_{X^*}} |x^*(x)|^p d\lambda(x^*) \right)^{\frac{1}{p}}. \quad (1.5)$$

Conversely, if there exists a Radon probability measure λ on $(B_{X^}, \sigma(X^*, X))$ and $C > 0$ such that this formula is satisfied, then u is p -summing and $\pi_p(u) \leq C$.*

The actual Pietsch factorization is

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ i_X \downarrow & & \uparrow \tilde{u} \\ i_X(X) & \xrightarrow{k_p} & S \\ \cap & & \cap \\ C(K) & \xrightarrow{J_p} & L_p(\mu) \end{array}$$

where \tilde{u} is a bounded linear operator, k_p is the restriction of J_p , $K = B_{X^}$, and S is the closure of $k_p \circ i_X(X)$ in $L_p(\mu)$. In this case, $\pi_p(u) = \|\tilde{u}\|$.*

Remark 1.3 (Special Case) *If $p = 2$, the operator \tilde{u} can be extended to the whole $L_2(\mu)$, i.e., u factors through a Hilbert space. Indeed, let P be the projection of $L_2(\mu)$ onto S , so*

$$\bar{u} := \tilde{u} \circ P \text{ and } u = \bar{u} J_2 i_X : X \xrightarrow{J_2 i_X} L_2(\mu) \xrightarrow{\bar{u}} Y,$$

with the remark that $J_2 i_X$ is 2-summing.

1.8 Cohen Strongly p -Summing Operators

Let $u : X \rightarrow Y$ be a linear operator between Banach spaces. The operator u is strongly p -summing ($1 < p \leq \infty$) if there exists a positive constant C such that for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, and $y_1^*, \dots, y_n^* \in Y^*$, we have

$$\sum_{i=1}^n |\langle u(x_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}. \quad (1.6)$$

The smallest constant C , denoted $d_p(u)$, that satisfies the inequality (1.6), defines the strongly p -summing norm on the Banach space $\mathcal{D}_p(X; Y)$ of Cohen strongly p -summing operators from X to Y . For $p = 1$, the space $\mathcal{D}_1(X; Y)$ coincides with $\mathcal{B}(X; Y)$, the space of bounded operators from X to Y .

The Pietsch domination theorem for strongly p -summing linear operators is given as follows. For the proof, we directly apply the Pietsch theorem to the adjoint operators, as they are p^* -summing. See [Coh73] for more details.

Theorem 1.4 (Pietsch Factorization) *Let $p \in]1, \infty]$ and p^* its conjugate, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. Let $u : X \rightarrow Y$ be a strongly p -summing linear operator. Then, there exists a Radon probability measure μ on $(B_{Y^{**}}, \sigma(Y^{**}, Y^*))$ such that for all $x \in X$ and all $y^* \in Y^*$ we have*

$$|\langle u(x), y^* \rangle| \leq d_p(u) \|x\| \left(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}. \quad (1.7)$$

*Conversely, if there exists a Radon probability measure μ on $(B_{Y^{**}}, \sigma(Y^{**}, Y^*))$ and $C > 0$ such that the formula (1.7) is satisfied, then u is strongly p -summing and $d_p(u) \leq C$.*

Theorem 1.5 *Let $1 < p \leq \infty$. Let $u \in \mathcal{L}(X; Y)$ and u^* its adjoint. The following properties are equivalent:*

- (1) *The linear operator u is Cohen strongly p -summing.*
- (2) *The adjoint operator u^* is p^* -summing. Moreover,*

$$d_p(u) = \pi_{p^*}(u^*) = \inf\{C \text{ satisfying the condition}\}.$$

Corollary 1.1 *The following properties are equivalent:*

(1) *The space Y is finite-dimensional.*

(2) *For all Banach spaces X ,*

$$\mathcal{D}_p(X; Y) = \mathcal{L}(X; Y).$$

(3) *The identity $id_Y \in \mathcal{D}_p(Y; Y)$.*

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Chapter 2: Linearization of Lipschitz Operators

2.1 Lipschitz Mappings

Definition 2.1 Let X be a pointed metric space and E a Banach space. We say that $T : X \rightarrow E$ is Lipschitz if there exists a constant $C > 0$ such that:

$$\|f(x) - f(y)\| \leq Cd(x, y), \quad \forall x, y \in X \quad (2.1)$$

We denote by $Lip(X, E)$ the space of all Lipschitz mappings. For any function $f \in Lip(X, E)$, the Lipschitz constant is defined by:

$$Lip(f) = \inf\{C : C \text{ satisfies equation(2.1)}\}.$$

The elements of $Lip(X, E)$ are referred to as Lipschitz operators.

Theorem 2.1 The space $Lip(X, E)$ is a vector subspace.

Proof:

To show that $Lip(X, E)$ is a vector subspace, it suffices to verify the following three properties:

1. The zero function $f_0 : X \rightarrow E$, defined by $f_0(x) = 0$ for all $x \in X$, belongs to $Lip(X, E)$ because for all $x, y \in X$, we have:

$$\|f_0(x) - f_0(y)\| = \|0 - 0\| = 0 \leq Cd(x, y), \quad \text{for any } C > 0.$$

2. Let $f, g \in \text{Lip}(X, E)$. Then there exist constants $C_1, C_2 > 0$ such that:

$$\|f(x) - f(y)\| \leq C_1 d(x, y), \quad \|g(x) - g(y)\| \leq C_2 d(x, y), \quad \forall x, y \in X.$$

Then,

$$\|(f + g)(x) - (f + g)(y)\| = \|f(x) + g(x) - f(y) - g(y)\| \leq \|f(x) - f(y)\| + \|g(x) - g(y)\| \leq (C_1 + C_2) d(x, y).$$

Thus, $f + g \in \text{Lip}(X, E)$.

3. Let $f \in \text{Lip}(X, E)$ and $\lambda \in \mathbb{K}$. There exists $C > 0$ such that:

$$\|f(x) - f(y)\| \leq C d(x, y), \quad \forall x, y \in X.$$

Then,

$$\|(\lambda f)(x) - (\lambda f)(y)\| = |\lambda| \cdot \|f(x) - f(y)\| \leq |\lambda| C d(x, y).$$

Hence, $\lambda f \in \text{Lip}(X, E)$.

Therefore, $\text{Lip}(X, E)$ is a vector subspace.

Lemma 2.1 For any $f \in \text{Lip}(X, E)$, we have:

$$\text{Lip}(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

Proof

$$1: \text{Lip}(f) \leq \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} \right\}$$

1. From the definition of $\text{Lip}(f)$:

$$\text{Lip}(f) = \inf \{ C > 0 : \|f(x) - f(y)\| \leq C d(x, y), \quad \forall x, y \in X \}.$$

2. For all $x, y \in X$ such that $x \neq y$:

$$\frac{\|f(x) - f(y)\|}{d(x, y)} \leq C$$

3. Since $\text{Lip}(f)$ is the smallest constant C that satisfies this condition, it follows that:

$$\text{Lip}(f) \leq \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x,y)} : x \neq y \right\}.$$

2: $\text{Lip}(f) \geq \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x,y)} \right\}$ 1. From the definition of $\text{Lip}(f)$: For all $x, y \in X$:

$$\|f(x) - f(y)\| \leq \text{Lip}(f) \cdot d(x,y).$$

2. By dividing both sides of the inequality by $d(x,y) > 0$:

$$\frac{\|f(x) - f(y)\|}{d(x,y)} \leq \text{Lip}(f).$$

3. Taking the supremum:

$$\sup \left\{ \frac{\|f(x) - f(y)\|}{d(x,y)} : x \neq y \right\} \leq \text{Lip}(f).$$

Proposition 2.1 *The mapping $f \mapsto \text{Lip}(f)$ is a seminorm on $\text{Lip}(X, E)$.*

Proof:

To demonstrate that the application $f \mapsto \text{Lip}(f)$ is a seminorm on $\text{Lip}(X, E)$, the following properties must be verified:

1. Non-negativity: According to the definition of the seminorm:

$$\text{Lip}(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)}.$$

Since the norm $\|\cdot\|$ and the distance $d(x,y)$ are always positive:

$$\text{Lip}(f) \geq 0.$$

2. Homogeneity:

For any scalar $\alpha \in \mathbb{R}$:

$$\text{Lip}(\alpha f) = \sup_{x \neq y} \frac{\|\alpha f(x) - \alpha f(y)\|}{d(x,y)}.$$

By factoring out $|\alpha|$:

$$= |\alpha| \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x, y)} = |\alpha| \text{Lip}(f).$$

3. Triangle inequality:

For all $f, g \in \text{Lip}(X, E)$:

$$\text{Lip}(f + g) = \sup_{x \neq y} \frac{\|(f(x) + g(x)) - (f(y) + g(y))\|}{d(x, y)}.$$

Using the triangle inequality of the norm:

$$\leq \sup_{x \neq y} \frac{\|f(x) - f(y)\| + \|g(x) - g(y)\|}{d(x, y)}.$$

Separating the terms:

$$\leq \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x, y)} + \sup_{x \neq y} \frac{\|g(x) - g(y)\|}{d(x, y)}.$$

Thus:

$$\text{Lip}(f + g) \leq \text{Lip}(f) + \text{Lip}(g).$$

Since all conditions are satisfied, the application $f \mapsto \text{Lip}(f)$ is indeed a **seminorm** on $\text{Lip}(X, E)$.

• We denote by $\text{Lip}_0(X, E)$ the set of Lipschitz mappings satisfying $f(0) = 0$. If K is the field of real or complex numbers, the space $\text{Lip}_0(X, K)$, denoted by $X^\#$, is called the Lipschitz dual of X .

Proposition 2.2 *The pair $(\text{Lip}_0(X, E), \text{Lip}(\cdot))$ forms a Banach space.*

Proof:

We will demonstrate that $(\text{Lip}_0(X; E), \text{Lip}(\cdot))$ is a Banach space by verifying three fundamental properties: the vector space structure, the definition of a semi-norm, and completeness.

1. Vector space: Consider two functions $f, g \in \text{Lip}_0(X; E)$ and two scalars $\alpha, \beta \in \mathbb{K}$, where \mathbb{K} denotes the field of real or complex numbers. Define the following function:

$$h = \alpha f + \beta g.$$

Since $f(0) = 0$ and $g(0) = 0$, we obtain:

$$h(0) = \alpha f(0) + \beta g(0) = 0.$$

Moreover, for all $x, y \in X$, we have:

$$\|h(x) - h(y)\| \leq |\alpha| \|f(x) - f(y)\| + |\beta| \|g(x) - g(y)\|.$$

Thus, $h \in \text{Lip}_0(X; E)$, which shows that $\text{Lip}_0(X; E)$ is a vector space.

2. Semi-norm $\text{Lip}(\cdot)$: We have from previous **proposition 2.1** $\text{Lip}(f)$ is a seminorm on $\text{Lip}(X, E)$. Define the Lipschitz constant by:

$$\text{Lip}(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} : x \neq y \in X \right\}.$$

Let us verify the properties of a semi-norm:

- **Positivity:** $\text{Lip}(f) \geq 0$.
- **Homogeneity:** For any scalar $\alpha \in \mathbb{K}$:

$$\text{Lip}(\alpha f) = |\alpha| \cdot \text{Lip}(f).$$

- **Triangle inequality:** For all $f, g \in \text{Lip}_0(X; E)$:

$$\text{Lip}(f + g) \leq \text{Lip}(f) + \text{Lip}(g).$$

Thus, $\text{Lip}(\cdot)$ indeed defines a semi-norm on $\text{Lip}_0(X; E)$.

3. Completeness: Let (f_n) be a Cauchy sequence in $\text{Lip}_0(X; E)$ with respect to the semi-norm $\text{Lip}(\cdot)$. This implies that:

$$\text{Lip}(f_n - f_m) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.$$

For every $x \in X$, the sequence $(f_n(x))$ is Cauchy in the Banach space E , hence it converges to a limit $f(x)$. It remains to show that $f \in \text{Lip}_0(X; E)$ and that $\text{Lip}(f_n - f) \rightarrow 0$.

Using the definition of the Lipschitz constant and the uniform convergence on pairs (x, y) , we obtain:

$$\|f_n(x) - f(x)\| \leq \zeta d(x, y) \quad \text{for all } x, y \in X.$$

Thus, $\text{Lip}(f_n - f) \rightarrow 0$, which shows that $f \in \text{Lip}_0(X; E)$ and that the space is complete.

Conclusion: Since $\text{Lip}_0(X; E)$ is a vector space equipped with the semi-norm $\text{Lip}(\cdot)$ and is complete under this semi-norm, the pair $(\text{Lip}_0(X; E), \text{Lip}(\cdot))$ constitutes a Banach space.

2.2 Linearization of Lipschitz Mappings

For every $x \in X$, define the evaluation functional $\delta_x \in (X^\#)^*$ by

$$\delta_x(f) = f(x).$$

The closed linear span of $\{\delta_x : x \in X\}$ forms a predual of $X^\#$, known as the Lipschitz-free space over X , denoted $\mathcal{F}(X)$, as introduced by Godefroy and Kalton in [GK03]. In [Wea99], Weaver studied this predual, referring to it as the Arens–Eells space of X and denoting it (X) .

For $x, y \in X$ and $f \in X^\#$, we have

$$\delta_{(x,y)}(f) = \delta_x(f) - \delta_y(f) = f(x) - f(y).$$

Moreover,

$$\delta_{(x,y)} = \delta_{(x,0)} - \delta_{(y,0)}.$$

For $m \in \mathcal{F}(X)$, its norm is

$$\|m\| = \inf \left\{ \sum_{i=1}^n |\lambda_i| d(x_i, y_i) : m = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \right\}.$$

Proposition 2.3 *For any pointed metric space X , the free space $\mathcal{F}(X)$ equipped with the norm $\|m\|$ is a Banach space.*

Proof

- Proof that $\mathcal{F}(X)$ is a Banach space

1: Proving that $\mathcal{F}(X)$ is a normed space We define $\mathcal{F}(X)$ as the space consisting of all linear combinations of functions of the form:

$$\delta_{(x,y)}(f) = f(x) - f(y), \quad \forall f \in X^\#$$

Any element in $\mathcal{F}(X)$ is of the form:

$$m = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)}$$

The norm on $\mathcal{F}(X)$ is defined as:

$$\|m\| = \inf \left\{ \sum_{i=1}^n |\lambda_i| d(x_i, y_i) : m = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \right\}$$

,The norm satisfies the required properties:

- **Non-degeneracy:**

$$\|m\| = \inf \left\{ \sum_{i=1}^n |\lambda_i| d(x_i, y_i) : m = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \right\} = 0 \Rightarrow \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} = 0$$

$$d(x_i, y_i) > 0 \Rightarrow \lambda_i = 0.$$

$$\Rightarrow m = 0$$

- **Homogeneity:** For any real or complex number α , we have:

$$\|\alpha m\| = \inf \left\{ \sum_{i=1}^n |\alpha \lambda_i| d(x_i, y_i) \right\} = |\alpha| \inf \left\{ \sum_{i=1}^n |\lambda_i| d(x_i, y_i) \right\} = |\alpha| \|m\|.$$

- **Triangle inequality:** We Shall prove that $\|m_1 + m_2\| \leq \|m_1\| + \|m_2\|$.

$$\text{Let : } \|m\| = \inf \left\{ \sum_{i=1}^n |\lambda_i| d(x_i, y_i) : m = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \right\} \text{ then : } m = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)}$$

$$m_1 = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)}, \quad m_2 = \sum_{j=1}^m \mu_j \delta_{(a_j, b_j)}$$

where $\lambda_i, \mu_j \in \mathbb{R}$ or \mathbb{C} , and $x_i, y_i, a_j, b_j \in X$. Thus, their sum is:

$$m_1 + m_2 = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} + \sum_{j=1}^m \mu_j \delta_{(a_j, b_j)}$$

$$\|m_1\| = \inf \left\{ \sum_{i=1}^n |\lambda_i| d(x_i, y_i) \right\}$$

$$\|m_2\| = \inf \left\{ \sum_{j=1}^m |\mu_j| d(a_j, b_j) \right\}$$

$$\|m_1 + m_2\| = \inf \left\{ \sum_k |\alpha_k| d(u_k, v_k) : m_1 + m_2 = \sum_k \alpha_k \delta_{(u_k, v_k)} \right\}$$

$$\|m_1 + m_2\| \leq \sum_{i=1}^n |\lambda_i| d(x_i, y_i) + \sum_{j=1}^m |\mu_j| d(a_j, b_j)$$

$$\|m_1 + m_2\| \leq \|m_1\| + \|m_2\|$$

Thus, $\mathcal{F}(X)$ is a normed space.

2: Proving that $\mathcal{F}(X)$ is a Banach space

(a) Consider any Cauchy sequence (m_n) in $\mathcal{F}(X)$.

Since (m_n) is a Cauchy sequence, for every $\xi > 0$, there exists an integer N such that for all $n, m \geq N$,

$$\|m_n - m_m\| < \xi.$$

(b) Expanding the norm expression:

Each m_n can be written as:

$$m_n = \sum_{i=1}^{k_n} \lambda_i^{(n)} \delta_{(x_i, y_i)}.$$

and the norm satisfies:

$$\|m_n - m_m\| = \inf \left\{ \sum_{i=1}^k |\lambda_i^{(n)} - \lambda_i^{(m)}| d(x_i, y_i) \right\} < \xi.$$

(c) Finding the limit m of the sequence:

We define an element m as:

$$m = \sum_{i=1}^{\infty} \lambda_i \delta_{(x_i, y_i)},$$

where $\lambda_i = \lim_{n \rightarrow \infty} \lambda_i^{(n)}$.

By the definition of the norm, we obtain:

$$\|m_n - m\| = \inf \left\{ \sum_{i=1}^{\infty} |\lambda_i^{(n)} - \lambda_i| d(x_i, y_i) \right\}.$$

Since $\lambda_i^{(n)} \rightarrow \lambda_i$, we conclude:

$$\lim_{n \rightarrow \infty} \|m_n - m\| = 0.$$

Thus, the sequence (m_n) converges to m in $\mathcal{F}(X)$.

Conclusion Since every Cauchy sequence in $\mathcal{F}(X)$ converges to an element in $\mathcal{F}(X)$, the space $\mathcal{F}(X)$ is a Banach space.

Lemma: Let E, G be two Banach spaces, and let F be a dense vector subspace of E . Suppose that $T : F \rightarrow G$ is a continuous linear operator. Then, there exists a unique continuous linear operator $\tilde{T} : E \rightarrow G$ such that:

$$\tilde{T} = T \text{ on } F.$$

In other words, T admits a unique continuous linear extension defined on the whole space E .

Proof

Existence: Let $T : F \rightarrow G$ be a continuous linear operator. We will show that there exists a unique continuous linear operator $\hat{T} : E \rightarrow G$ extending T , i.e., such that:

$$\tilde{T}(x) = T(x), \quad \forall x \in F.$$

Since T is continuous on F , there exists a constant $C > 0$ such that:

$$\|T(x)\| \leq C\|x\|, \quad \forall x \in F.$$

Now, let $x \in E$. Since F is dense in E , there exists a sequence $(x_n) \subset F$ such that $x_n \rightarrow x$ in E .

By the continuity of T on F , the sequence $(T(x_n))$ is Cauchy in G :

$$\|T(x_m) - T(x_n)\| \leq C\|x_m - x_n\|.$$

Since G is complete, the sequence $(T(x_n))$ converges to an element $y \in G$. We then define $\tilde{T} : E \rightarrow G$ by:

$$\tilde{T}(x) = \lim_{n \rightarrow \infty} T(x_n).$$

This definition is well-posed because if (x'_n) is another sequence in F converging to x , then $(T(x'_n))$ must converge to the same y by the uniqueness of limits in G .

Linearity: For $x, y \in E$ and $\lambda \in \mathbb{K}$, let (x_n) and (y_n) be sequences in F such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Then:

$$T(x_n + \lambda y_n) = T(x_n) + \lambda T(y_n).$$

Taking the limit, we obtain:

$$\tilde{T}(x + \lambda y) = \tilde{T}(x) + \lambda \tilde{T}(y).$$

Thus, \tilde{T} is linear.

Continuity: Since T is bounded on F , we have:

$$\|\tilde{T}(x)\| = \lim_{n \rightarrow \infty} \|T(x_n)\| \leq C \lim_{n \rightarrow \infty} \|x_n\| \leq C\|x\|.$$

Thus, \tilde{T} is continuous and $\|\tilde{T}\| \leq C$.

Uniqueness: Suppose there exists another extension $S : E \rightarrow G$ such that $S(x) = T(x)$ for all $x \in F$. For $x \in E$, let $(x_n) \subset F$ be a sequence such that $x_n \rightarrow x$. By the continuity of S , we get:

$$S(x) = \lim_{n \rightarrow \infty} S(x_n) = \lim_{n \rightarrow \infty} T(x_n) = \hat{T}(x).$$

Thus, $S = \tilde{T}$, proving uniqueness.

Theorem 2.2 *For every Lip-linear operator $T \in Lip_0(X, E)$, there exists a unique bounded linear*

operator $\hat{T} : \mathcal{F}(X) \rightarrow E$ defined by:

$$\hat{T}(\delta_{(x,y)}) = T(x) - T(y), \quad \forall x, y \in X.$$

In other words, $T = \hat{T} \circ \delta_X$, ensuring that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ \delta_x \downarrow & \nearrow \hat{T} & \\ \mathcal{F}(X) & & \end{array}$$

In this case, we have:

$$\text{Lip}(T) = \|\hat{T}\|.$$

Proof

We aim to prove the existence of a unique bounded linear operator \hat{T} that satisfies the given condition.

.show that \hat{T} Linearity

Since $T \in \text{Lip}_0(X, E)$, we know that $T(0) = 0$ and that T is Lipschitz, meaning there exists a constant $C > 0$ such that:

$$\|T(x) - T(y)\| \leq Cd(x, y), \quad \forall x, y \in X.$$

We define \hat{T} by:

$$\hat{T} \left(\sum_{i=1}^n \lambda_i \phi_{(x_i, y_i)} \right) = \sum_{i=1}^n \lambda_i (T(x_i) - T(y_i)).$$

Since $\phi_{(x,y)}$ satisfies linearity, it follows that:

$$\hat{T}(m + m') = \hat{T}(m) + \hat{T}(m'), \quad \hat{T}(\alpha m) = \alpha \hat{T}(m).$$

Thus, \hat{T} is a linear operator.

. show that \hat{T} is Bounded

The norm in $\mathcal{F}(X)$ is given by:

$$\|m\| = \inf \left\{ \sum_{i=1}^n |\lambda_i| d(x_i, y_i) \text{ where } m = \sum_{i=1}^n \lambda_i \phi_{(x_i, y_i)} \right\}.$$

For any m with representation:

$$m = \sum_{i=1}^n \lambda_i \phi_{(x_i, y_i)},$$

applying \hat{T} gives:

$$\|\hat{T}(m)\| = \left\| \sum_{i=1}^n \lambda_i (T(x_i) - T(y_i)) \right\|.$$

Using the Lipschitz condition:

$$\|\hat{T}(m)\| \leq \sum_{i=1}^n |\lambda_i| \|T(x_i) - T(y_i)\| \leq C \sum_{i=1}^n |\lambda_i| d(x_i, y_i).$$

Taking the infimum over all representations, we obtain:

$$\|\hat{T}(m)\| \leq C \|m\|.$$

This proves that \hat{T} is bounded with $\|\hat{T}\| \leq C$.

. Uniqueness

Suppose there exist two linear operators \hat{T}_1, \hat{T}_2 satisfying the same conditions. Then, for all $x, y \in X$:

$$\hat{T}_1(\phi_{(x,y)}) = \hat{T}_2(\phi_{(x,y)}) = T(x) - T(y).$$

Since $\mathcal{F}(X)$ is the linear span of $\phi_{(x,y)}$, it follows that $\hat{T}_1 = \hat{T}_2$, proving uniqueness.

. show that $\|T\|_{\text{Lip}} = \|\hat{T}\|$

Since:

$$\|\hat{T}\| = \sup_{\|m\| \leq 1} \|\hat{T}(m)\|,$$

and using $\hat{T}(\phi_{(x,y)}) = T(x) - T(y)$, we get:

$$\|\hat{T}\| = \sup_{x \neq y} \frac{\|T(x) - T(y)\|}{d(x,y)} = \|T\|_{\text{Lip}}.$$

Conclusion We have successfully established the existence of a unique bounded linear operator \hat{T} such that $T = \hat{T} \circ \phi_X$. Moreover, we have shown that $\|\hat{T}\| = \|T\|_{\text{Lip}}$, proving that $\text{Lip}_0(X, E)$ is isometrically isomorphic to $\mathcal{L}(\mathcal{F}(X), E)$.

Proposition 2.4 *Let X be a pointed metric space and E a Banach space. We have the following isometric identification:*

$$\text{Lip}_0(X; E) = \mathcal{L}(\mathcal{F}(X), E).$$

proof

1. **Definition of the isomorphism:** For each $T \in \text{Lip}_0(X, E)$, we define the linear operator $\hat{T} : \mathcal{F}(X) \rightarrow E$ by:

$$\hat{T}(\delta_{(x,y)}) = T(x) - T(y), \quad \forall x, y \in X$$

where $\delta_{(x,y)}$ is the evaluation functional in $\mathcal{F}(X)$ such that $\delta_{(x,y)}(f) = f(x) - f(y)$.

2. **Linearity:** For any finite linear combination $m = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)} \in \mathcal{F}(X)$, we have:

$$\hat{T}(m) = \sum_{i=1}^n \lambda_i \hat{T}(\delta_{(x_i, y_i)}) = \sum_{i=1}^n \lambda_i (T(x_i) - T(y_i)).$$

This proves that \hat{T} is linear.

3. **Boundedness and isometry:** We compute the norm:

$$\|\hat{T}\| = \sup_{\|m\|_{\mathcal{F}(X)} \leq 1} \|\hat{T}(m)\|.$$

Since

$$\|\hat{T}(\delta_{(x,y)})\| = \|T(x) - T(y)\| \leq \text{Lip}(T)d(x,y),$$

it follows that

$$\|\hat{T}\| \leq \text{Lip}(T).$$

Similarly, for $m = \sum_{i=1}^n \lambda_i \delta_{(x_i, y_i)}$, we obtain:

$$\|\hat{T}(m)\| \leq \text{Lip}(T) \|m\|_{\mathcal{F}(X)}.$$

Hence, $\|\hat{T}\| = \text{Lip}(T)$, proving that \hat{T} is an isometric extension of T .

4. Bijection between the spaces: - Every bounded linear operator $S : \mathcal{F}(X) \rightarrow E$ defines a Lipschitz mapping $T : X \rightarrow E$ by setting $T(x) = S(\delta_x)$.

- From the definition of \hat{T} , we find that $S = \hat{T}$, proving the bijection between $\text{Lip}_0(X, E)$ and $(\mathcal{F}(X), E)$.

Conclusion: The space $\text{Lip}_0(X, E)$ is isometrically isomorphic to $(\mathcal{F}(X), E)$, i.e.,

$$\text{Lip}_0(X; E) = \mathcal{L}(\mathcal{F}(X), E).$$

2.3 Linearization of p -Summing Lipschitz Operators

Let $T : X \rightarrow E$ be a Lipschitz operator, and let $\hat{T} : \mathcal{F}(X) \rightarrow E$ be its linearized operator. In this section, we study the relationship between these operators concerning the notion of p -summing operators.

We start with the following result, which shows that if \hat{T} is p -summing, then T is also Lipschitz p -summing.

Theorem 2.3 *If $\hat{T} : \mathcal{F}(X) \rightarrow E$ is a p -summing operator, then $T : X \rightarrow E$ is Lipschitz p -summing.*

proof

Since \hat{T} is p -summing, it satisfies *Pietsch's Domination Theorem*, which guarantees the existence of a *Radon measure* μ on the unit ball $B_{X^\#}$ such that:

$$\|\hat{T}(m)\| \leq C \left(\int_{B_{X^\#}} |f(m)|^p d\mu(f) \right)^{\frac{1}{p}}, \quad \forall m \in \mathcal{F}(X)$$

Now, for $x_j, y_j \in X$, we define the following element in $F(X)$:

$$m = \sum_{j=1}^n \lambda_j (\delta_{x_j} - \delta_{y_j})$$

Since \tilde{T} satisfies the property above, we obtain:

$$\|\tilde{T}(m)\| \leq C \left(\int_{B_{X^\#}} \left| \sum_{j=1}^n \lambda_j (f(x_j) - f(y_j)) \right|^p d\mu(f) \right)^{\frac{1}{p}}$$

Since:

$$\tilde{T}(m) = \sum_{j=1}^n \lambda_j (T(x_j) - T(y_j))$$

this implies:

$$\left(\sum_{j=1}^n \|T(x_j) - T(y_j)\|^p \right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^\#}} \left(\sum_{j=1}^n |f(x_j) - f(y_j)|^p \right)^{\frac{1}{p}}$$

Thus, T is *Lipschitz p -summing*.

Remark 2.1 *The converse is not true, as illustrated by the following example:*

Consider the Lipschitz inclusion

$$\delta_X : \mathbb{R} \rightarrow \mathcal{F}(\mathbb{R}).$$

This is a Lipschitz operator since

$$\|\delta_X(x) - \delta_X(y)\| = |x - y|.$$

We show that the linearized operator of ϕ_X is given by

$$\hat{\delta}_X = \text{id}_{\mathcal{F}(\mathbb{R})}$$

.

Let $m \in \mathcal{F}(\mathbb{R})$ such that $m = \sum_{i=1}^n \lambda_i \delta(x_i, y_i)$,

then

$$\hat{\delta}_X(m) = \hat{\delta}_X \left(\sum_{i=1}^n \lambda_i \delta(x_i, y_i) \right) = \sum_{i=1}^n \lambda_i \hat{\delta}_X(\delta(x_i, y_i)) = \sum_{i=1}^n \lambda_i (\delta_X(x_i) - \delta_X(y_i)) = \sum_{i=1}^n \lambda_i \delta(x_i, y_i) = m.$$

Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\delta_x} & \mathcal{F}(\mathbb{R}) \\ \delta_x \downarrow & \nearrow id_{\mathcal{F}(\mathbb{R})} & \\ \mathcal{F}(\mathbb{R}) & & \end{array}$$

Now, the operator δ_X is Lipschitz p -summing because \mathbb{R} is finite-dimensional, but its linearized operator $id_{\mathcal{F}(\mathbb{R})}$ is not p -summing since $\mathcal{F}(\mathbb{R})$ is infinite-dimensional.

2.4 Strictly Lipschitz p -Summing

Definition 2.2 Let $1 \leq p \leq \infty$. A mapping $T : X \rightarrow E$ is said to be **strictly Lipschitz p -summing** if there exists a positive constant C such that for every $x_k, y_k \in X$ and $s_k^* \in E^*$ (where $1 \leq k \leq l$), we have:

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C d_p^L(u), \quad (2.2)$$

where $u = \sum_{k=1}^l \delta(x_k, y_k) \otimes s_k^*$. We denote by $\Pi_p^{SL}(X, E)$ the Banach space of all strictly C verifying (2.2). Lipschitz p -summing operators from X into E which its norm $\pi_p^{SL}(T)$ is the smallest constant C verifying (2.2). If we consider linear operators defined on Banach spaces, we have shown in [Theorem 2.4] that the three notions: p -summing, Lipschitz p -summing and strictly Lipschitz p -summing are coincide. The following characterization is the main result of this

Theorem 2.4 Let $1 \leq p \leq \infty$. Let X be a pointed metric space, and let E be a Banach space. Suppose $T : X \rightarrow E$ is a Lipschitz operator. The following properties are equivalent:

1. T is strictly Lipschitz p -summing.
2. \hat{T} is p -summing.

3. There exists a constant $C > 0$ and a Radon probability measure μ on the unit ball B_{X^*} such that for all sequences $(x^j)_{j=1}^n, (y^j)_{j=1}^n \subset X$ and $(\lambda^j)_{j=1}^n \subset \mathbb{K}$ (\mathbb{R} or \mathbb{C}) $n \in \mathbb{N}^*$, we have:

$$\left\| \sum_{j=1}^n \lambda^j (T(x^j) - T(y^j)) \right\| \leq C \left(\int_{B_{X^*}} \left| \sum_{j=1}^n \lambda^j (f(x^j) - f(y^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}}.$$

4. There exists a constant $C > 0$ such that for all sequences $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1} \subset X$ and $(\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K}$ (where $1 \leq j \leq n_2$, and $n_1, n_2 \in \mathbb{N}^*$), we have:

$$\left(\sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^*}} \left(\sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right\|^p \right)^{\frac{1}{p}}. \quad (2.3)$$

proof

(1) \Rightarrow (2): See Theorem 2.4 [DF93].

(2) \Rightarrow (3): We apply Pietsch's Domination Theorem for p -summing linear operators, then there is a Radon probability measure μ on B_{X^*} such that for any $m \in \mathcal{F}(X)$:

$$\|\widehat{T}(m)\| \leq C \left(\int_{B_{X^*}} |f(m)|^p d\mu(f) \right)^{\frac{1}{p}}.$$

Now, let $(x^j)_{j=1}^n, (y^j)_{j=1}^n \subset X$ and $(\lambda_j)_{j=1}^n \subset \mathbb{K}$. Define:

$$m = \sum_{j=1}^n \lambda^j \delta(x^j, y^j) \in \mathcal{F}(X).$$

then:

$$\left\| \widehat{T} \left(\sum_{j=1}^n \lambda^j \delta(x^j, y^j) \right) \right\| \leq C \left(\int_{B_{X^*}} \left| \sum_{j=1}^n \lambda^j f(\delta(x^j, y^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}}.$$

Thus, we obtain:

$$\left\| \sum_{j=1}^n \lambda^j (T(x^j) - T(y^j)) \right\| \leq C \left(\int_{B_{X^*}} \left| \sum_{j=1}^n \lambda^j (f(x^j) - f(y^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}}.$$

(3) \Rightarrow (4) : Let $(x_i^j)_{i=1}^{n_1}, (y_i^j)_{i=1}^{n_1} \subset X$ and $(\lambda_i^j)_{i=1}^{n_1} \subset \mathbb{K}$ (for $1 \leq j \leq n_2$). by (2.3) we have every $1 \leq i \leq n_1$:

$$\left\| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)) \right\| \leq C \left(\int_{B_{X^*}} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p d\mu(f) \right)^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)) \right\|^p &\leq C \int_{B_{X^*}} \sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p d\mu(f). \\ &\leq \sup_{B_{X^*}} \sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p d\mu(f). \end{aligned}$$

Thus:

$$\left(\sum_{i=1}^{n_1} \left\| \sum_{j=1}^{n_2} \lambda_i^j (T(x_i^j) - T(y_i^j)) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^*}} \left(\sum_{i=1}^{n_1} \left| \sum_{j=1}^{n_2} \lambda_i^j (f(x_i^j) - f(y_i^j)) \right|^p \right)^{\frac{1}{p}}.$$

(4) \Rightarrow (1) : let $u = \sum_{k=1}^l \phi(x_k, y_k) \otimes s_k^*$ where $s_k^* \in E^*$. Then, we define the set A_u as given in (2.1). Let: $m = \sum_{i=1}^{n_1} m_i \otimes e_i^*$ where $m_i = \sum_{j=1}^{n_2} \alpha_i^j \phi(x_i^j, y_i^j)$. Using equation (2.2), we get:

$$\begin{aligned} &\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \\ &= \left| \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \lambda_i^j \langle T(x_i^j) - T(y_i^j), e_i^* \rangle \right| \end{aligned}$$

by Hölder's inequality:

$$\begin{aligned} &\leq \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} |\lambda_i^j (T(x_i^j) - T(y_i^j))|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n_1} \|e_i^*\|^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq C \sup \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} |\lambda_i^j (f(x_i^j) - f(y_i^j))|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n_1} \|e_i^*\|^{p^*} \right)^{\frac{1}{p^*}} \end{aligned}$$

By Taking the infimum over all representations of $m \in A_u$, we obtain:

$$\left| \sum_{k=1}^l \langle T(x_k) - T(y_k), s_k^* \rangle \right| \leq C d_L^p(u)$$

Then, T is strictly Lipschitz p -summing.

2.5 Linearization of Lipschitz Cohen Strongly p -Summing Operators

A Lipschitz operator $T : X \rightarrow E$ is said to be *Lipschitz Cohen strongly p -summing* if there exists a constant $C > 0$ such that, for any $x_1, \dots, x_n, y_1, \dots, y_n \in X$ and any $y_1^*, \dots, y_n^* \in F^*$, we have:

$$\sum_{i=1}^n |\langle T(x_i) - T(y_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^n d(x_i, y_i)^p \right)^{\frac{1}{p}} \| (y_i^*) \|_{\ell_{p^*}^n, w}. \quad (2.4)$$

The class of Lipschitz Cohen strongly p -summing operators from X to E , denoted $D_p^L(X, E)$, is a Banach space with the norm $d_p^L(T)$, which is the smallest constant C satisfying the above inequality holds.

Theorem 2.5 *Let $1 \leq p \leq \infty$. Let X be a pointed metric space and E a Banach space. Let $T : X \rightarrow E$ be a Lipschitz operator. The following statements are equivalent:*

1. T is Lipschitz Cohen strongly p -summing.
2. \hat{T} is Cohen strongly p -summing.

Even more, $D_p^L(X, Y) = D_p(\mathcal{F}(X), Y)$. holds isometrically

Proof

First, suppose $T \in D_p^L(X, Y)$. Let $m \in \mathcal{F}(X)$ and $y^* \in Y^*$. Then, we have:

$$|\langle \hat{T}(m), y^* \rangle| \leq \sum_{i=1}^n |\lambda_i| \cdot |\langle T(x_i) - T(y_i), y^* \rangle|.$$

$$|\langle \hat{T}(m), y^* \rangle| \leq d_p^L(T) \sum_{i=1}^n |\lambda_i| \cdot d(x_i, x'_i) \|y^*\|_{\ell_{p^*}^p}(\mu).$$

Applying the Lipschitz Cohen strongly p -summing property of T , we obtain:

$$|\langle \hat{T}(m), y^* \rangle| \leq d_p^L(T) \|m\|_{\mathcal{F}(X)} \|y^*\|_{\ell_{p^*}^p}.$$

Since m is arbitrary, we conclude that \hat{T} satisfies Pietsch's domination theorem for Cohen strongly p -summing operators, so $\hat{T} \in D_p(\mathcal{F}(X), Y)$ and

$$d_p(\hat{T}) \leq d_p^L(T).$$

Conversely, suppose $\hat{T} \in D_p(\mathcal{F}(X), Y)$. For any $x, x' \in X$ and $y^* \in Y^*$, we apply Pietsch's domination theorem to \hat{T} , obtaining:

$$|\langle T(x) - T(x'), y^* \rangle| = |\langle \hat{T}(m_{(x,x')}), y^* \rangle| \leq d_p^L(\hat{T}) \|m\|_{F(X)} \|y^*\|_{\ell_{p^*}^p} \leq d_p(\hat{T}) \cdot d(x, x') \cdot \|y^*\|_{\ell_{p^*}^p}.$$

Thus, $T \in D_p^L(X, Y)$ and

$$d_p^L(T) \leq d_p(\hat{T}).$$

Corollary 2.1 *Let E, F be Banach spaces, and let $T : E \rightarrow F$ be a linear operator. The following statements are equivalent:*

1. T is Lipschitz Cohen strongly p -summing.
2. \hat{T} is Cohen strongly p -summing.

Proof

Let $T : E \rightarrow F$ be a linear operator. The corresponding linear operator \hat{T} is given by $\hat{T} = T \circ \beta_E$, where $\beta_E : \mathcal{F}(E) \rightarrow E$ is a linear quotient map satisfying

$$\beta_E \circ \delta_E = \text{id}_E, \quad \|\beta_E\| \leq 1.$$

$$\begin{array}{ccc} \mathcal{F}(E) & \xrightarrow{\beta_E} & E \xrightarrow{T} F \\ \delta_E \uparrow & & \\ E & & \end{array}$$

Since T is Cohen strongly p -summing, $\implies \hat{T} = T \circ \delta_E$ is also Cohen strongly p -summing, and hence T is Lipschitz Cohen strongly p -summing.

Conversely, if T is Lipschitz Cohen strongly p -summing, $\implies \hat{T} = T \circ \delta_E$ is Cohen strongly p -summing. This implies that $\hat{T} \circ \delta_E = T \circ \beta_E \circ \delta_E = T$ is also Lipschitz Cohen strongly p -summing.

2.6 Linearization of Integral Lipschitz Operators

The concept of integral linear operators was originally introduced in [DJT95]. The Lipschitz version has been introduced by in [DF93]. Let X be a pointed metric space and E a Banach space. A Lipschitz operator $T : X \times E \rightarrow F$ is said to be integral if there exists a constant $C > 0$ such that for any $x_1, \dots, x_n, y_1, \dots, y_n \in X$, $e_1^*, \dots, e_n^* \in E^*$, we have

$$\left| \sum_{i=1}^n \langle T(x_i) - T(y_i), e_i^* \rangle \right| \leq C \sup_{f \in B_{X^\#}} \left\| \sum_{i=1}^n (f(x_i) - f(y_i)) e_i^* \right\|_{E^*}. \quad (2.5)$$

The class of all integral Lip-Linear operators is denoted by $Lip_0^{int}(X \times E; F)$. For such operators, we define

$$\mathcal{J}_{int}^L(T) = \inf\{C : C \text{ satisfies (2.5)}\}.$$

It is straightforward to verify that the pair

$$(Lip_0^{int}(X \times E), \mathcal{J}_{int}^L(\cdot))$$

forms a Banach space. The importance of this new definition lies in its connection with the linearization operators, allowing the extension of certain classical results. The next theorem establishes this relationship.

Theorem 2.6 *Let X be a pointed metric space, and E a Banach space. The following properties are equivalent:*

1. *The Lipschitz operator T is integral.*

2. The linearized operator $\hat{T} : \mathcal{F}(X) \rightarrow E$ is integral.

Proof

(1) \Rightarrow (2): Suppose $T : X \rightarrow E$ is an integral Lipschitz operator. For any $u_j = \sum_{i=1}^n \lambda_i^j \delta(x_i^j, y_i^j) \in \mathcal{F}(X)$ and $e_j^* \in E^*$ (where $1 \leq j \leq m$), we have:

$$\begin{aligned} \left| \sum_{j=1}^m \langle \hat{T}(u_j), e_j^* \rangle \right| &= \left| \sum_{j=1}^m \sum_{i=1}^n \langle T(x_i^j) - T(y_i^j), \lambda_i^j e_j^* \rangle \right| \\ &\leq \mathcal{J}_{int}^L(T) \sup_{f \in B_{X^\#}} \left\| \sum_{j=1}^m \sum_{i=1}^n (f(x_i^j) - f(y_i^j)) \lambda_i^j e_j^* \right\|_{E^*} \\ &\leq \mathcal{J}_{int}^L(T) \sup_{f \in B_{X^\#}} \left\| \sum_{j=1}^m \sum_{i=1}^n \left(\lambda_i^j \delta(x_i^j, y_i^j) \right) e_j^* \right\|_{E^*} \\ &= \mathcal{J}_{int}^L(T) \sup_{f \in B_{X^\#}} \left\| \sum_{j=1}^m f(u_j) e_j^* \right\|_{E^*} \end{aligned}$$

Thus, \hat{T} is integral, with $\mathcal{J}(\hat{T}) \leq \mathcal{J}_{int}^L(T)$.

(2) \Rightarrow (1) : Suppose $\hat{T} : \mathcal{F}(X) \rightarrow E$ is integral. For any $(x_i)_{i=1}^n, (y_i)_{i=1}^n \in X$ and $e_1^*, \dots, e_n^* \in E^*$, we have:

$$\begin{aligned} \left| \sum_{i=1}^n \langle T(x_i) - T(y_i), e_i^* \rangle \right| &= \left| \sum_{i=1}^n \langle \hat{T}(\delta(x_i, y_i)), e_i^* \rangle \right| \\ &\leq \mathcal{J} \hat{T} \sup_{f \in B_{X^\#}} \left\| \sum_{i=1}^n (f(x_i) - f(y_i)) e_i^* \right\|_{E^*} \end{aligned}$$

Thus, T is integral and $\mathcal{J}_{int}(T) \leq \mathcal{J}(\hat{T})$.

Corollary 2.2 *Let X be a pointed metric space and E a Banach space. We have the following isometric identification:*

$$Lip_0^{int}(X, E) = \mathcal{I}(\mathcal{F}(X), E)$$

Proof

Define the mapping:

$$\Gamma : \text{Lip}_0^{\text{int}}(X, E) \rightarrow \mathcal{I}(\mathcal{F}(X), E)$$

by

$\text{varGamma}(T) = \hat{T}$. From the previous theorem, Γ is well-defined and isometric.

To show that Γ is surjective, let $\varphi \in \mathcal{I}(\mathcal{F}(X), E)$. Define:

$$T_\varphi(x) = \varphi(\delta_x)$$

Then T_φ is a Lipschitz operator. For any $(x_i)_{i=1}^n, (y_i)_{i=1}^n \in X$ and $(e_i^*)_{i=1}^n \in E^*$, we have:

$$\begin{aligned} &= \left| \sum_{i=1}^n \langle T_\varphi(x_i) - T_\varphi(y_i), e_i^* \rangle \right| = \left| \sum_{i=1}^n \langle \varphi(\delta(x_i)) - \varphi(\delta(y_i)), e_i^* \rangle \right| \\ &= \left| \sum_{i=1}^n \langle \varphi(\delta(x_i, y_i)), e_i^* \rangle \right| \\ &\leq \mathcal{I}(\varphi) \sup_{f \in B_{X^\#}} \left\| \sum_{i=1}^n (f(x_i) - f(y_i)) e_i^* \right\|_{E^*} \end{aligned}$$

Thus, T_φ is integral. Finally, for $u = \sum_{i=1}^n \alpha_i \delta(x_i, y_i)$, we have:

$$\hat{T}_\varphi(u) = \hat{T}_\varphi \left(\sum_{i=1}^n \alpha_i \delta(x_i, y_i) \right) = \sum_{i=1}^n \alpha_i (T_\varphi(x_i) - T_\varphi(y_i))$$

Substituting $T_\varphi(x_i) = \varphi(\delta_{x_i})$, we get:

$$\hat{T}_\varphi(u) = \sum_{i=1}^n \varphi(\alpha_i \delta_{x_i}) - \varphi(\alpha_i \delta_{y_i}) = \varphi(u)$$

Thus, $\hat{T}_\varphi = \varphi$, and by density, this equality extends to all of $\mathcal{F}(X)$. Therefore, we conclude that Γ is surjective

Chapter 3: Linearization of Multilinear Operators

In this chapter, we study the linearizations of multilinear operators.

3.1 Multilinear Mappings

Definition 3.1 Let X_1, \dots, X_m, Y be Banach spaces. A mapping $T : X_1 \times \dots \times X_m \rightarrow Y$ is called a *multilinear operator* (or *m-linear mapping*) if

$$T(x_1, \dots, \alpha x_j + \beta y_j, \dots, x_m) = \alpha T(x_1, \dots, x_j, \dots, x_m) + \beta T(x_1, \dots, y_j, \dots, x_m) \quad (3.1)$$

for every j ($1 \leq j \leq m$) and $x_j, y_j \in X_j, \alpha, \beta \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}). If $Y = \mathbb{K}$, then T is called a *multilinear form*.

We denote by $L(X_1, \dots, X_m; Y)$ the vector space of multilinear operators from $X_1 \times \dots \times X_m$ into Y . This space is equipped with the following norm:

$$\|T\| = \sup_{\substack{\|x_i\| \leq 1 \\ 1 \leq i \leq m}} \|T(x_1, \dots, x_m)\|.$$

Proposition 3.1 The mapping $T \mapsto \|T\|$ is a norm on $L(X_1, \dots, X_m; Y)$.

Proof

We will prove that the function $\|T\|$ defined by: $\|T\| = \sup_{\|x_i\| \leq 1, 1 \leq i \leq m} \|T(x_1, \dots, x_m)\|$ is a norm on $L(X_1, \dots, X_m; Y)$.

To show that $\|\cdot\|$ is a norm, we must verify the following three properties:

1. Positivity We need to show that:

$$\|T\| \geq 0, \quad \text{and} \quad \|T\| = 0 \Rightarrow T = 0.$$

From the definition of the norm, it is clear that $\|T\| \geq 0$.

If $\|T\| = 0$, then

$$\sup_{\|x_i\| \leq 1} \|T(x_1, \dots, x_m)\| = 0.$$

This implies that $T(x_1, \dots, x_m) = 0$ for all x_1, \dots, x_m , hence T is the zero operator.

2. Homogeneity We must prove that:

$$\|\lambda T\| = |\lambda| \|T\|, \quad \forall \lambda \in \mathbb{K}.$$

Using the definition of the norm,

$$\|\lambda T\| = \sup_{\|x_i\| \leq 1} \|\lambda T(x_1, \dots, x_m)\|.$$

By the homogeneity property in a Banach space,

$$\|\lambda T(x_1, \dots, x_m)\| = |\lambda| \|T(x_1, \dots, x_m)\|.$$

Taking the supremum on both sides, we obtain:

$$\sup_{\|x_i\| \leq 1} \|\lambda T(x_1, \dots, x_m)\| = |\lambda| \sup_{\|x_i\| \leq 1} \|T(x_1, \dots, x_m)\|,$$

thus:

$$\|\lambda T\| = |\lambda| \|T\|.$$

3. Triangle Inequality We must prove that:

$$\|T + S\| \leq \|T\| + \|S\|, \quad \forall T, S \in L(X_1, \dots, X_m; Y).$$

Using the definition of the norm,

$$\|T + S\| = \sup_{\|x_i\| \leq 1} \|T(x_1, \dots, x_m) + S(x_1, \dots, x_m)\|.$$

Applying the triangle inequality in a Banach space:

$$\|T(x_1, \dots, x_m) + S(x_1, \dots, x_m)\| \leq \|T(x_1, \dots, x_m)\| + \|S(x_1, \dots, x_m)\|.$$

Taking the supremum on both sides:

$$\sup_{\|x_i\| \leq 1} \|T(x_1, \dots, x_m) + S(x_1, \dots, x_m)\| \leq \sup_{\|x_i\| \leq 1} \|T(x_1, \dots, x_m)\| + \sup_{\|x_i\| \leq 1} \|S(x_1, \dots, x_m)\|.$$

Thus:

$$\|T + S\| \leq \|T\| + \|S\|.$$

Conclusion : Since the three properties of a norm are verified, we conclude that $\|\cdot\|$ is a norm on $L(X_1, \dots, X_m; Y)$.

The Banach space of multilinear operators from $X_1 \times \dots \times X_m$ to Y is denoted by $\mathcal{L}(X_1, \dots, X_m; Y)$.

3.2 Representation on a Tensor Product

Let X_1, \dots, X_m be Banach spaces. The algebraic tensor product of X_1, \dots, X_m is denoted by $X_1 \otimes \dots \otimes X_m$. The projective norm is defined by:

$$\|v\|_\pi = \inf \left\{ \sum_{i=1}^n \prod_{j=1}^m \|x_i^j\| \right\} \quad (3.2)$$

where the infimum is taken over all possible representations of v of the form:

$$v = \sum_{i=1}^n x_{1i} \otimes \dots \otimes x_{mi}.$$

The projective tensor product of the spaces X_1, \dots, X_m is denoted by $X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m$, which is the completion of $X_1 \otimes \dots \otimes X_m$ with respect to this norm. If $X_1 = \dots = X_m = X$, we simply write $\widehat{\otimes}_\pi^m X$. For each multilinear operator $T : X_1 \times \dots \times X_m \rightarrow Y$, one can associate a linear

operator, called the linearization of T , denoted $\tilde{T} : X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m \rightarrow Y$, defined by:

$$\tilde{T} \left(\sum_{i=1}^n x_i^1 \otimes \dots \otimes x_i^m \right) = \sum_{i=1}^n T(x_i^1, \dots, x_i^m). \quad (3.3)$$

It is well-defined because it does not depend on the chosen representation. Moreover, \tilde{T} is unique and satisfies $\|\tilde{T}\| = \|T\|$. Indeed, for $v = \sum_{i=1}^n x_i^1 \otimes \dots \otimes x_i^m$, we have:

$$\|\tilde{T}(v)\| = \left\| \sum_{i=1}^n T(x_i^1, \dots, x_i^m) \right\| \leq \|T\| \sum_{i=1}^n \|x_i^1\| \dots \|x_i^m\|,$$

which implies $\|\tilde{T}\| \leq \|T\| \|v\|_\pi$. Thus, \tilde{T} is bounded and $\|\tilde{T}\| \leq \|T\|$.

Conversely, we have $\|T\| \leq \|\tilde{T}\|$ since:

$$\|T(x^1, \dots, x^m)\| = \|\tilde{T}(x^1 \otimes \dots \otimes x^m)\| \leq \|\tilde{T}\| \|x^1\| \dots \|x^m\|.$$

Now, let \tilde{B} be another linearization of T . For all $(x^1, \dots, x^m) \in X_1 \otimes \dots \otimes X_m$, we have:

$$\tilde{B}(x^1 \otimes \dots \otimes x^m) = T(x^1, \dots, x^m) = \tilde{T}(x^1 \otimes \dots \otimes x^m).$$

By linearity, \tilde{B} and \tilde{T} coincide on $X_1 \otimes \dots \otimes X_m$ and, by density, they are identical on $X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m$.

Thus,

$$\mathcal{L}(X_1, \dots, X_m; Y) = B(X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m, Y), \quad (3.4)$$

because the application:

$$\Phi : \mathcal{L}(X_1, \dots, X_m; Y) \rightarrow B(X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m, Y), \quad (3.5)$$

$$T \mapsto \Phi(T) = \tilde{T}$$

is an isometric isomorphism. We will study in more detail the relationships between the properties of T and \tilde{T} . Special Case: The dual of $X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m$ is identified with the space of bounded multilinear forms:

$$(X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m)^* = \mathcal{L}(X_1, \dots, X_m). \quad (3.6)$$

This duality provides an alternative formula for the projective norm:

$$\|v\|_\pi = \sup\{|\langle v, T \rangle| : T \in B_{\mathcal{L}}(X_1, \dots, X_m)\}.$$

Remark 3.1 In the linear case, $B(X, Y^*)$ is identified with $(X \widehat{\otimes}_\pi Y)^*$. Here, according to (3.4), we have the isometric identification:

$$(X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X_1, \dots, X_m, Y^*). \quad (3.7)$$

Diagonal Operators The natural embeddings of X into $\underbrace{X \times \dots \times X}_m$ and into $\widehat{\otimes}_\pi^m X$, denoted by Δ_m and $\widehat{\delta}_m$, respectively, are defined by:

$$\Delta_m : X \rightarrow X \times \dots \times X,$$

$$x \mapsto (x, \dots, x),$$

$$\widehat{\delta}_m : X \rightarrow \widehat{\otimes}_\pi^m X,$$

$$x \mapsto x \otimes \dots \otimes x.$$

It is clear that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \Delta_m \swarrow & & \searrow \delta_m \\ \underbrace{X \times \dots \times X}_m & \xrightarrow{i_m} & \widehat{\otimes}_\pi^m X \end{array}$$

i.e $i_m \circ \Delta_m = \delta_m$ is the canonical multilinear operator from $X_1 \times \dots \times X_m$ to $X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m$, defined by:

$$i_m(x^1, \dots, x^m) = x^1 \otimes \dots \otimes x^m.$$

We also have the following commutative diagram:

$$\begin{array}{ccc} X_1 \times \dots \times X_m & \xrightarrow{T} & Y \\ \downarrow i_m & \nearrow \widehat{T} & \\ \widehat{\otimes}_\pi X_1 \dots \widehat{\otimes}_\pi X_m & & \end{array}$$

That is,

$$T = \tilde{T} \circ i_m. \quad (3.8)$$

3.3 Ideals of Multilinear Operators

Since the famous article by A. Pietsch titled "Ideals of Multilinear Functions" [Pie83], the success of the theory of linear ideals has had a considerable influence on the development...

This section likely continues discussing the impact of Pietsch's work on multilinear operators and ideals. Let me know if you need a full translation or explanation!

3.4 Linearization of strongly p -summing multilinear operators

Definition 3.2 Let X_1, \dots, X_m, Y be Banach spaces, and let $T : X_1 \times \dots \times X_m \rightarrow Y$ be a bounded m -linear operator, T is **factorable strongly p -summing** (for $1 \leq p < \infty$) if there exists a constant $C \geq 0$ such that, for any positive integers m_1, m_2 and any elements $(x_{ij}^k)_{1 \leq i \leq m_1, 1 \leq j \leq m_2} \subset X_k$ (for $k = 1, \dots, m$),

$$\left(\sum_{j=1}^{m_2} \left\| \sum_{i=1}^{m_1} T(x_{ij}^1, \dots, x_{ij}^m) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{\|\psi\| \leq 1} \left(\sum_{j=1}^{m_2} \left| \sum_{i=1}^{m_1} \psi(x_{ij}^1, \dots, x_{ij}^m) \right|^p \right)^{\frac{1}{p}},$$

where the supremum is taken over all bounded m -linear functionals $\psi : X_1 \times \dots \times X_m \rightarrow \mathbb{K}$ with norm $\|\psi\| \leq 1$. The norm of T in this space is

$$\pi_p^{f-st}(T) = \inf\{C \geq 0 \mid C \text{ satisfies the inequality above}\}.$$

Proposition 3.2 Let $n \geq 2$ be a natural number, X_1, \dots, X_n, Y Banach spaces, $T : X_1 \times \dots \times X_n \rightarrow Y$ a bounded n -linear operator and $1 \leq p < \infty$. Then: T is factorable strongly p -summing if and only if its linearization

$$\hat{T} : X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_n \rightarrow Y$$

is p -summing. Moreover,

$$\pi_p^{f-st}(T) = \pi_p(\hat{T}).$$

Proof

Let us suppose that T is factorable strongly p -summing. Let $(u_j)_{1 \leq j \leq m_2} \subset X_1 \otimes_\pi \cdots \otimes_\pi X_n$. Then there exists a natural number m_1 and $(x_{ij}^k)_{1 \leq i \leq m_1, 1 \leq j \leq m_2} \subset X_k (1 \leq k \leq n)$ such that $u_j = \sum_{i=1}^{m_1} x_{ij}^1 \otimes \cdots \otimes x_{ij}^n$ for all $1 \leq j \leq m_2$. (Use the relation $0 \otimes 0 \otimes \cdots \otimes 0 = 0$). Then, for all $1 \leq j \leq m_2$ we have

$$\hat{T}(u_j) = \sum_{i=1}^{m_1} \hat{T}(x_{ij}^1 \otimes \cdots \otimes x_{ij}^n) = \sum_{i=1}^{m_1} T(x_{ij}^1, \dots, x_{ij}^n)$$

and

$$\hat{\psi}(u_j) = \sum_{i=1}^{m_1} \hat{\psi}(x_{ij}^1 \otimes \cdots \otimes x_{ij}^n) = \sum_{i=1}^{m_1} \psi(x_{ij}^1, \dots, x_{ij}^n)$$

where $\psi : X_1 \times \cdots \times X_n \rightarrow \mathbb{K}$ is a bounded n -linear functional. Now since T is factorable strongly p -summing we have

$$\left(\sum_{j=1}^{m_2} \left\| \sum_{i=1}^{m_1} T(x_{ij}^1, \dots, x_{ij}^n) \right\|^p \right)^{\frac{1}{p}} \leq \pi_p^{f-st}(T) \sup_{\|\psi\| \leq 1} \left(\sum_{j=1}^{m_2} \left| \sum_{i=1}^{m_1} \psi(x_{ij}^1, \dots, x_{ij}^n) \right|^p \right)^{\frac{1}{p}}$$

or equivalently,

$$\left(\sum_{j=1}^{m_2} \|\hat{T}(u_j)\|^p \right)^{\frac{1}{p}} \leq \pi_p^{f-st}(T) \sup_{\|\tilde{\psi}\| \leq 1, \tilde{\psi} \in (X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n)^*} \left(\sum_{j=1}^{m_2} |\hat{\psi}(u_j)|^p \right)^{\frac{1}{p}}. \quad (3.9)$$

Since $X_1 \otimes_\pi \cdots \otimes_\pi X_n$ is (norm) dense in $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n$ we deduce from (10) that for every $(v_j)_{1 \leq j \leq m_2} \subset X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n$ the following relation holds

$$\left(\sum_{j=1}^{m_2} \|\hat{T}(v_j)\|^p \right)^{\frac{1}{p}} \leq \pi_p^{f-st}(T) \sup_{\|\tilde{\psi}\| \leq 1, \tilde{\psi} \in (X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n)^*} \left(\sum_{j=1}^{m_2} |\hat{\psi}(v_j)|^p \right)^{\frac{1}{p}}$$

that is $\hat{T} : X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n \rightarrow Y$ is p -summing and $\pi_p(\hat{T}) \leq \pi_p^{f-st}(T)$.

Conversely, let us suppose that $\hat{T} : X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n \rightarrow Y$ is p -summing. Let m_1, m_2 be natural numbers and $(x_{ij}^k)_{1 \leq i \leq m_1, 1 \leq j \leq m_2} \subset X_k (1 \leq k \leq n)$. Then, for each $1 \leq j \leq m_2$ the elements $u_j = \sum_{i=1}^{m_1} x_{ij}^1 \otimes \cdots \otimes x_{ij}^n \in X_1 \otimes_\pi \cdots \otimes_\pi X_n$ and $\hat{T}(u_j) = \sum_{i=1}^{m_1} T(x_{ij}^1, \dots, x_{ij}^n)$. Since

$\widehat{T} : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \rightarrow Y$ is p -summing we have

$$\left(\sum_{j=1}^{m_2} \|\widehat{T}(u_j)\|^p \right)^{\frac{1}{p}} \leq \pi_p(\widehat{T}) \sup_{\|\widehat{\psi}\| \leq 1, \widehat{\psi} \in (X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*} \left(\sum_{j=1}^{m_2} |\widehat{\psi}(u_j)|^p \right)^{\frac{1}{p}}$$

or equivalently,

$$\left(\sum_{j=1}^{m_2} \left\| \sum_{i=1}^{m_1} T(x_{ij}^1, \dots, x_{ij}^n) \right\|^p \right)^{\frac{1}{p}} \leq \pi_p(\widehat{T}) \sup_{\|\psi\| \leq 1} \left(\sum_{j=1}^{m_2} \left| \sum_{i=1}^{m_1} \psi(x_{ij}^1, \dots, x_{ij}^n) \right|^p \right)^{\frac{1}{p}}$$

which means that T is factorable strongly p -summing and $\pi_p^{f-st}(T) \leq \pi_p(\widehat{T})$.

Conclusion The operator T is factorable strongly p -summing if and only if its associated linear operator \widehat{T} is p -summing, with the norms satisfying $\pi_p^{f-st}(T) = \pi_p(\widehat{T})$.

3.5 Linearization of Cohen strongly p -summing multilinear operators

Definition 3.3 Let $1 \leq p \leq \infty$. An m -linear operator $T : X_1 \times \cdots \times X_m \rightarrow Y$ (where X_j and Y are arbitrary Banach spaces and $m \in \mathbb{N}$) is called Cohen strongly p -summing if and only if there exists a constant $C > 0$ such that for any elements $x_i^j \in X_j$ (for all $j = 1, \dots, m$) and any $y_1^*, y_2^*, \dots, y_n^* \in Y^*$, we have:

$$\left\| \left(\langle T(x_i^1, \dots, x_i^m), y_i^* \rangle \right) \right\|_{l_n^1} < C \left(\sum_{i=1}^n \prod_{j=1}^m \|x_i^j\|_{X_j}^p \right)^{1/p} \sup_{y \in B_Y} (\|y_i^*(y)\|)_{l_n^p} \quad (3.10)$$

*The class of all Cohen strongly p -summing m -linear operators from $X_1 \times \cdots \times X_m \rightarrow Y$ is denoted by $\mathcal{D}_p^m(X_1, \dots, X_m; Y)$, and it forms a Banach space with the norm $d_p^m(T)$, which is the smallest constant C such that the inequality (3.10) holds.

Proposition 3.3 Let m be a positive integer, X_1, \dots, X_m, Y be Banach spaces, $T : X_1 \times \cdots \times X_m \rightarrow Y$ be a bounded m -linear operator, and $1 \leq p < \infty$. Then, T is **Cohen strongly p -summing** if and only if its linearized operator $\widehat{T} : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m \rightarrow Y$ is Cohen strongly p -summing (in the linear sense). Moreover,

$$d_p^m(T) = d_p(\hat{T}),$$

where $d_p(\hat{T})$ is the Cohen strongly p -summing norm of \hat{T} .

Relation between T and \hat{T} for Cohen Strongly p -Summing Operators

To prove that $T : X_1 \times \cdots \times X_m \rightarrow Y$ is Cohen strongly p -summing if and only if its linearized operator $\hat{T} : X_1 \otimes_\pi \cdots \otimes_\pi X_m \rightarrow Y$ is Cohen p -summing, with $d_p^m(T) = d_p(\hat{T})$.

Let $\hat{T}(x_1 \otimes \cdots \otimes x_m) = T(x_1, \dots, x_m)$, and for any $u = \sum_{i=1}^k x_i^1 \otimes \cdots \otimes x_i^m \in X_1 \otimes_\pi \cdots \otimes_\pi X_m$,

$$\hat{T}(u) = \sum_{i=1}^k T(x_i^1, \dots, x_i^m).$$

T Cohen strongly p -summing $\implies \hat{T}$ Cohen p -summing.

Let $(u_j)_{1 \leq j \leq n} \subset X_1 \otimes_\pi \cdots \otimes_\pi X_m$, where

$$u_j = \sum_{i=1}^{k_j} x_{ij}^1 \otimes \cdots \otimes x_{ij}^m,$$

and let $(y_j^*)_{1 \leq j \leq n} \subset Y^*$. Define $v : \ell_p^n \rightarrow Y^*$ by $v(e_j) = y_j^*$, with

$$\|v\| = \sup_{y \in B_Y} \left(\sum_{j=1}^n |y_j^*(y)|^{p^*} \right)^{\frac{1}{p^*}},$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$.

Compute:

$$\langle \hat{T}(u_j), y_j^* \rangle = \left\langle \sum_{i=1}^{k_j} T(x_{ij}^1, \dots, x_{ij}^m), y_j^* \right\rangle = \sum_{i=1}^{k_j} \langle T(x_{ij}^1, \dots, x_{ij}^m), y_j^* \rangle.$$

To simplify, assume first that $k_j = 1$, i.e., $u_j = x_{1j}^1 \otimes \cdots \otimes x_{1j}^m$. Then:

$$\langle \hat{T}(u_j), y_j^* \rangle = \langle T(x_{1j}^1, \dots, x_{1j}^m), y_j^* \rangle.$$

Since T is Cohen strongly p -summing, by the equivalent form:

$$\sum_{j=1}^n \left| \langle T(x_{1j}^1, \dots, x_{1j}^m), y_j^* \rangle \right| \leq d_p^m(T) \left(\sum_{j=1}^n \prod_{l=1}^m \|x_{1j}^l\|_{X_l}^p \right)^{\frac{1}{p}} \|v\|.$$

Notice that:

$$\|u_j\|_{X_1 \otimes_{\pi} \dots \otimes_{\pi} X_m} = \|x_{1j}^1 \otimes \dots \otimes x_{1j}^m\| \leq \prod_{l=1}^m \|x_{1j}^l\|_{X_l}.$$

Thus:

$$\left(\sum_{j=1}^n \prod_{l=1}^m \|x_{1j}^l\|_{X_l}^p \right)^{\frac{1}{p}} \geq \left(\sum_{j=1}^n \|u_j\|_{X_1 \otimes_{\pi} \dots \otimes_{\pi} X_m}^p \right)^{\frac{1}{p}}.$$

Therefore:

$$\sum_{j=1}^n \left| \langle \hat{T}(u_j), y_j^* \rangle \right| \leq d_p^m(T) \left(\sum_{j=1}^n \|u_j\|_{X_1 \otimes_{\pi} \dots \otimes_{\pi} X_m}^p \right)^{\frac{1}{p}} \sup_{y \in B_Y} \left(\sum_{j=1}^n |y_j^*(y)|^{p^*} \right)^{\frac{1}{p^*}}.$$

This shows that \hat{T} is Cohen p -summing (in the linear context), with $d_p(\hat{T}) \leq d_p^m(T)$. For the general case ($k_j \geq 1$), we use the density of $X_1 \otimes_{\pi} \dots \otimes_{\pi} X_m$ in $X_1 \hat{\otimes}_{\pi} \dots \hat{\otimes}_{\pi} X_m$ to extend the inequality.

\hat{T} Cohen p -summing $\implies T$ Cohen strongly p -summing

Suppose \hat{T} is Cohen p -summing, i.e., there exists $C > 0$ such that for all $(u_j)_{1 \leq j \leq n} \subset X_1 \otimes_{\pi} \dots \otimes_{\pi} X_m$ and $(y_j^*)_{1 \leq j \leq n} \subset Y^*$:

$$\sum_{j=1}^n \left| \langle \hat{T}(u_j), y_j^* \rangle \right| \leq C \left(\sum_{j=1}^n \|u_j\|_{X_1 \otimes_{\pi} \dots \otimes_{\pi} X_m}^p \right)^{\frac{1}{p}} \sup_{y \in B_Y} \left(\sum_{j=1}^n |y_j^*(y)|^{p^*} \right)^{\frac{1}{p^*}}.$$

Take $u_j = x_j^1 \otimes \dots \otimes x_j^m$. Then:

$$\langle \hat{T}(u_j), y_j^* \rangle = \langle T(x_j^1, \dots, x_j^m), y_j^* \rangle,$$

and

$$\|u_j\|_{X_1 \otimes_{\pi} \dots \otimes_{\pi} X_m} \leq \prod_{l=1}^m \|x_j^l\|_{X_l}.$$

Applying the inequality:

$$\sum_{j=1}^n \left| \langle T(x_j^1, \dots, x_j^m), y_j^* \rangle \right| \leq C \left(\sum_{j=1}^n \|x_j^1 \otimes \dots \otimes x_j^m\|_{X_1 \otimes \pi \dots \otimes \pi X_m}^p \right)^{\frac{1}{p}} \sup_{y \in B_Y} \left(\sum_{j=1}^n |y_j^*(y)|^{p^*} \right)^{\frac{1}{p^*}}.$$

Since:

$$\left(\sum_{j=1}^n \|x_j^1 \otimes \dots \otimes x_j^m\|_{X_1 \otimes \pi \dots \otimes \pi X_m}^p \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^n \prod_{l=1}^m \|x_j^l\|_{X_l}^p \right)^{\frac{1}{p}},$$

we get:

$$\sum_{j=1}^n \left| \langle T(x_j^1, \dots, x_j^m), y_j^* \rangle \right| \leq C \left(\sum_{j=1}^n \prod_{l=1}^m \|x_j^l\|_{X_l}^p \right)^{\frac{1}{p}} \sup_{y \in B_Y} \left(\sum_{j=1}^n |y_j^*(y)|^{p^*} \right)^{\frac{1}{p^*}}.$$

T is Cohen strongly p -summing, with $d_p^m(T) \leq d_p(\hat{T})$.

Conclusion

we conclude that T is Cohen strongly p -summing if and only if \hat{T} is Cohen p -summing, with $d_p^m(T) = d_p(\hat{T})$.

3.6 Linearization of integral multilinear operators

Let X_1, \dots, X_m and Y be Banach spaces. In [Grothendieck1955, DefantFloret1993], the author introduced the concept of integral multilinear operators. A multilinear operator $T : X_1 \times \dots \times X_m \rightarrow Y$ is said to be integral (in the sense of Grothendieck) if there exists a constant $C > 0$ such that for any sequences $(x_i^j)_{i=1}^n \subset X_j$ and $(y_i^*)_{i=1}^n \subset Y^*$ ($1 \leq j \leq m$), we have

$$\left| \sum_{i=1}^n \langle T(x_1^i, \dots, x_m^i), y_i^* \rangle \right| \leq C \sup_{x_j^* \in B_{X_j^*} (1 \leq j \leq m)} \left\| \sum_{i=1}^n x_1^*(x_1^i) \dots x_m^*(x_m^i) y_i^* \right\|_{Y^*}. \quad (3.11)$$

The class of all integral bilinear operators is denoted by $\mathcal{L}_{int}^m(X_1 \times \dots \times X_m; Y)$, which forms a Banach space with the integral norm

$$\mathcal{I}_{int}^m(T) = \inf \{ C : C \text{ satisfies (3.11)} \}.$$

A linear version was originally introduced in [Grothendieck]. A linear operator $u : X \rightarrow Y$ is said to be integral (in the sense of Grothendieck) if its associated bilinear form $T_u : X \times Y^* \rightarrow \mathbb{R}$

is integral. We denote by $\mathcal{I}(X; Y)$ the Banach space of all integral linear operators and by $\mathcal{I}(\cdot)$ its norm. Furthermore, we have the identification

$$(X \widehat{\otimes}_{\xi} Y^*)^* = \mathcal{I}(X; Y).$$

Proposition 3.4 *For a linear operator $u : X \rightarrow Y$, the following statements are equivalent:*

- 1) $u \in \mathcal{I}(X; Y)$.
- 2) The bilinear form $T_u : X \times Y^* \rightarrow \mathbb{R}$ is integral.

In this case,

$$\mathcal{I}(u) = \mathcal{I}_{int}^2(T_u).$$

proof Let $u : X \rightarrow Y$ be a integral linear operator. Its corresponding bilinear form is defined as

$$\begin{aligned} T_u : X \times Y^* &\rightarrow \mathbb{R} \\ (x, y^*) &\mapsto y^*(u(x)) \end{aligned}$$

Let $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$. Then

$$\begin{aligned} \left| \sum_{i=1}^n \langle T_u(x_i, y_i^*), \lambda_i \rangle \right| &= \left| \sum_{i=1}^n \langle u(\lambda_i x_i), y_i^* \rangle \right| \\ &\leq \mathcal{I}(u) \sup \left\{ \left\| \sum_{i=1}^n \phi(x_1^i \otimes \dots \otimes x_m^i) y_i^* \right\|_{Y^*} : \phi \in B_{(X_1 \widehat{\otimes}_{\xi} \dots \widehat{\otimes}_{\xi} X_m)^*} \right\} \\ &\leq \mathcal{I}(u) \sup_{x^* \in B_{X^*}} \left\| \sum_{i=1}^n x^*(\lambda_i x_i) y_i^* \right\|_{Y^*} \\ &\leq \mathcal{I}(u) \sup_{x^* \in B_{X^*}, y^{**} \in B_{Y^{**}}} \left| \sum_{i=1}^n x^*(x_i) y^{**}(y_i^*) \lambda_i \right|. \end{aligned}$$

Then, T_u is integral and

$$\mathcal{I}_{int}^2(T_u) \leq \mathcal{I}(u).$$

Conversely, suppose that $T_u : X \times Y^* \rightarrow \mathbb{R}$ is integral. We show that $u : X \rightarrow Y$ is also integral.

Let $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$. We put $\lambda_i = 1$ for $i = 1, \dots, n$. We have

$$\begin{aligned}
\left| \sum_{i=1}^n \langle u(x_i), y_i^* \rangle \right| &= \left| \sum_{i=1}^n \langle T_u(x_i, y_i^*), \lambda_i \rangle \right| \\
&\leq \mathcal{I}_{int}^2(T_u) \sup_{x^* \in B_{X^*}, y^{**} \in B_{Y^{**}}} \left| \sum_{i=1}^n x^*(x_i) y^{**}(y_i^*) \lambda_i \right| \\
&\leq \mathcal{I}_{int}^2(T_u) \sup_{x^* \in B_{X^*}} \left\| \sum_{i=1}^n x^*(\lambda_i x_i) y_i^* \right\|_{Y^*} \\
&\leq \leq \mathcal{I}_{int}^2(T_u) \sup_{x^* \in B_{X^*}} \left\| \sum_{i=1}^n x^*(x_i) y_i^* \right\|_{Y^*}.
\end{aligned}$$

Then, u is integral and

$$\mathcal{I}(u) \leq \mathcal{I}_{int}^2(T_u).$$

The importance of the definition of itegral multilinear operators lies in its connection with the linearization operators, allowing the extension of certain classical results. The next theorem establishes this relationship.

Theorem 3.1 *Let X_1, \dots, X_m and Y be Banach spaces. The following properties are equivalent:*

- 1) *The multilinear operator T is integral.*
- 2) *The linearization $\widehat{T} : X_1 \widehat{\otimes}_{\xi} \dots \widehat{\otimes}_{\xi} X_m \longrightarrow Y$ is integral.*

proof 1) \implies 2) : Let $T : X_1 \times \dots \times X_m \longrightarrow Y$ be an integral multilinear operator. For $u_i = \sum_{j=1}^{m_i} x_{j,i}^i \otimes \dots \otimes x_{j,i}^i \in X_1 \widehat{\otimes}_{\xi} \dots \widehat{\otimes}_{\xi} X_m$ and $y_i^* \in Y^*$. We have

$$\begin{aligned}
\left| \sum_{i=1}^n \langle \widehat{T}(u_i), y_i^* \rangle \right| &= \left| \sum_{i=1}^n \left\langle \sum_{j=1}^{m_i} T(x_{j,i}^i, \dots, x_{j,i}^i), y_j^* \right\rangle \right| \\
&= \left| \sum_{i=1}^n \sum_{j=1}^{m_i} \langle T(x_{j,i}^i, \dots, x_{j,i}^i), y_j^* \rangle \right| \\
&\leq \mathcal{I}_{int}^m(T) \sup_{x_j^* \in B_{X_j^*} (1 \leq j \leq m)} \left\| \sum_{i=1}^n \sum_{j=1}^{m_i} x_1^*(x_{j,i}^i) \dots x_m^*(x_{j,i}^i) y_i^* \right\|_{Y^*} \\
&\leq \mathcal{I}_{int}^m(T) \sup_{x_j^* \in B_{X_j^*} (1 \leq j \leq m)} \left\| \sum_{i=1}^n (x_1^* \otimes \dots \otimes x_m^*)(u_i) y_i^* \right\|_{Y^*}.
\end{aligned}$$

Since

$$\sup_{x_j^* \in B_{X_j^*} (1 \leq j \leq m)} \left\| \sum_{i=1}^n (x_1^* \otimes \dots \otimes x_m^*) (u_i) y_i^* \right\|_{Y^*} = \sup \left\{ \left\| \sum_{i=1}^n \phi(u_i) y_i^* \right\|_{Y^*} : \phi \in B_{(X_1 \widehat{\otimes}_{\xi} \dots \widehat{\otimes}_{\xi} X_m)^*} \right\}$$

It follows that \widehat{T} is integral, with

$$\mathcal{I}(\widehat{T}) \leq \mathcal{I}_{int}^m(T).$$

2) \implies 1) : Suppose $\widehat{T} : X_1 \widehat{\otimes}_{\xi} \dots \widehat{\otimes}_{\xi} X_m \longrightarrow Y$ is integral. Let $(x_j^i)_{i=1}^n \subset X_j (1 \leq j \leq m)$ and $(y_i^*)_{i=1}^n \subset Y^*$. Then

$$\begin{aligned} \left| \sum_{i=1}^n \langle T(x_1^i, \dots, x_m^i), y_i^* \rangle \right| &= \left| \sum_{i=1}^n \langle \widehat{T}(x_1^i \otimes \dots \otimes x_m^i), y_i^* \rangle \right| \\ &\leq \mathcal{I}(\widehat{T}) \sup \left\{ \left\| \sum_{i=1}^n \phi(x_1^i \otimes \dots \otimes x_m^i) y_i^* \right\|_{Y^*} : \phi \in B_{(X_1 \widehat{\otimes}_{\xi} \dots \widehat{\otimes}_{\xi} X_m)^*} \right\} \\ &\leq \mathcal{I}(\widehat{T}) \sup_{x_j^* \in B_{X_j^*} (1 \leq j \leq m)} \left\| \sum_{i=1}^n x_1^*(x_1^i) \dots x_m^*(x_m^i) y_i^* \right\|_{Y^*}. \end{aligned}$$

Thus, T is integral and

$$\mathcal{I}_{int}^m(T) \leq \mathcal{I}(\widehat{T}).$$

Corollary 3.1 *Let X_1, \dots, X_m and Y be Banach spaces. We have the following isometric identification*

$$\mathcal{L}_{int}^m(X_1 \times \dots \times X_m; Y) = \mathcal{I}(X_1 \widehat{\otimes}_{\xi} \dots \widehat{\otimes}_{\xi} X_m; Y).$$

proof Define the map

$$\Gamma : \mathcal{L}_{int}^m(X_1 \times \dots \times X_m; Y) \rightarrow \mathcal{I}(X_1 \widehat{\otimes}_{\xi} \dots \widehat{\otimes}_{\xi} X_m; Y)$$

by $\Gamma(T) = \widehat{T}$. By the previous Theorem, Γ is well-defined and isometric. We will now show that Γ is surjective. Let $\varphi \in \mathcal{I}(X_1 \widehat{\otimes}_{\xi} \dots \widehat{\otimes}_{\xi} X_m; Y)$. Define

$$T_{\varphi}(x_1, \dots, x_m) = \varphi(x_1 \otimes \dots \otimes x_m).$$

Then, T_φ is multilinear operator. For any $(x_j^i)_{i=1}^n \subset X_j (1 \leq j \leq m)$ and $(y_i^*)_{i=1}^n \subset Y^*$, then

$$\left| \sum_{i=1}^n \langle T_\varphi(x_1^i, \dots, x_m^i), y_i^* \rangle \right| = \left| \sum_{i=1}^n \langle \varphi(x_1^i \otimes \dots \otimes x_m^i), y_i^* \rangle \right|.$$

Since φ is integral,

$$\left| \sum_{i=1}^n \langle T_\varphi(x_1^i, \dots, x_m^i), y_i^* \rangle \right| \leq \mathcal{I}(\varphi) \sup_{x_j^* \in B_{X_j^*} (1 \leq j \leq m)} \left\| \sum_{i=1}^n x_1^* (x_1^i) \dots x_m^* (x_m^i) y_i^* \right\|_{Y^*}.$$

Thus, T_φ is integral. Finally, for $u = \sum_{i=1}^n x_1^i \otimes \dots \otimes x_m^i$, we have

$$\widehat{T}_\varphi(u) = \widehat{T}_\varphi\left(\sum_{i=1}^n x_1^i \otimes \dots \otimes x_m^i\right) = \sum_{i=1}^n T_\varphi(x_1, \dots, x_m).$$

Substituting $T_\varphi(x_1, \dots, x_m) = \varphi(x_1 \otimes \dots \otimes x_m)$, we get

$$\widehat{T}_\varphi(u) = \sum_{i=1}^n \varphi(x_1 \otimes \dots \otimes x_m) = \varphi\left(\sum_{i=1}^n x_1 \otimes \dots \otimes x_m\right) = \varphi(u).$$

Then, $\widehat{T}_\varphi = \varphi$ on $X_1 \otimes_\xi \dots \otimes_\varepsilon X_m$. By density, this equality extends to $X_1 \widehat{\otimes}_\xi \dots \widehat{\otimes}_\xi X_m$. Therefore, we conclude that Γ is surjective.

By setting $Y = \mathbb{R}$ in (), we obtain

$$(X_1 \widehat{\otimes}_\xi \dots \widehat{\otimes}_\xi X_m)^* = \mathcal{L}_{int}^m(X_1 \times \dots \times X_m).$$

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Summary and Keywords

ملخص

تتناول هذه الأطروحة الخطية في التحليل الوظيفي، حيث تحول التطبيقات اللاخطية، مثل التطبيقات الليبشيتزية والمتعددة الخطية، إلى تطبيقات خطية. الهدف هو فهم خصائص هذه التطبيقات وإمكانية دراستها باستخدام خصائص إسومنج. **الكلمات المفتاحية:** فضاء آرينز-إيلز، العامل اللبشيتزي، العوامل بجمعية، الخطية.

Abstract

This thesis addresses linearization in functional analysis, where it transforms nonlinear mappings, such as Lipschitz and multilinear mappings, into linear mappings. The goal is to understand the properties of these mappings and the possibility of studying them using p-summing properties.

Keywords: Arens-Eells space, Lipschitz operator, p-summing operators, linearization.

Résumé

Cette thèse traite de la linéarisation en analyse fonctionnelle, où elle transforme les applications non linéaires, comme les applications lipschitziennes et multilinéaires, en applications linéaires. L'objectif est de comprendre les propriétés de ces applications et la possibilité de les étudier en utilisant les propriétés p-sommantes.

Mots-clés: Espace d'Arens-Eells, opérateur lipschitzien, opérateurs p-sommants, linéarisation.