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Theme

On some regular properties of p-bounded variation spaces

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تصريح شرفي

خاص بالالتزام بقواعد النزاهة العلمية لإنجاز بحث

أنا الممضي(ة) أسفله

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و المكلف بإنجاز مذكرة ماستر تحت عنوان:

On some regular properties of p-bounded variation spaces

أصرح بشرفي أني التزم بمراعاة المعايير العلمية و المنهجية و معايير الأخلاقيات المهنية و النزاهة الأكاديمية المطلوبة في إنجاز البحث المذكور أعلاه.

التاريخ:

إمضاء المعني

إهـــداء ،

الحمد لله وكفى، الصلاة والسلام على الحبيب المصطفى وأهله ومن وفى، أمابعد، أهدي ثمرة جهدي إلى أعز وأغلى والدين في حياتي، اللذان أناريا دربي بنصائحهما وكانا بحرا صافيا، يجري بفيض الحب والبسمة، إلى من زينا حياتي بضياء البدر وشموع الفرح، إلى من منحاني القوة والعزيمة، لمواصلة الدرب، وكانا سببا في مواصلة دراستي، إلى من علماني الصبر والاجتهاد، إلى الغاليين على قلبي ،

أمي وأبي،

إلى زوجي العزيز،

إلى إبني حفظه الله عز وجل، إلى كل العائلة الكريمة،

وزملاء الدراسة متمنية لهم التوفيق،

إلى كل قسم الرياضيات، شعبة تحليل دالي دفعة 2022،

جامعة محمد بوضياف، بالمسيلة،

إلى كل الأشخاص الذين أحمل لهم المحبة والتقدير،

إلى كل من أحبهم قلبي ونسيهم قلمي،

زهـــــرة.

ملخص المذكرة باللغة الوطنية :

- في الفصل الأول، نقدم تعريفا لفضاء جبر بناخ للدوال الحقيقية المحدودية التغيير $\mathbf{BV}([a,b])$ المعرفة في المجالات الحقيقية المغلقة (تعريف حسب جوردان سنة 1881) مع شرح لخصائصها وبنيتها.

- في الفصل الثاني ندرس فضاء وينر للدوال p محدودية التغيير $\mathcal{V}_p(\mathbf{I})$ ومسألة تركيب المؤثرات SOP المطبقة على فضاء جديد $\mathbf{BV}_p^1$ للدوال الأصلية لتوزيعات $\mathcal{V}_p(\mathbf{I})$ (تعميم حديث لدوال وينر سنة 1937).

- في الفصل الثالث نقدم شرحا للفضاء الثنائي البعد للدوال محدودية التغيير $\mathbf{BV}(\sigma,\mathbb{C})$ (لآشتون و دوست سنة 2005)، المعرفة في المستوي المركب، مع شرح للتماثلات التقابلية في فضاء الدوال المستمرة مطلقا $\mathbf{AC}(\sigma,\mathbb{C})$ المتعلقة بمبرهنة بناخ - ستون (1932-1939).

الكلمات المفتاحية:

الدوال لمحدودية التغيير ذات الرتبة p ، مسألة تركيب المؤثرات، التباينات ، التحويلات التبولوجية ،
التشاكلات، التماثلات التقابلية.

Résumé:

- D'abord, dans le premier chapitre, on introduit l'espace $\mathbf{BV}([a,b])$ des fonctions à variation bornée de Jordan (introduites en 1881) définies sur les intervalles fermés de la droite réèlle qui constitue une structure d'Algebre de Banach.

- Ensuite, on présente l'espace de Wiener des fonctions à p-variation bornées $\mathcal{V}_p(\mathbf{I})$ et le problème de superposition des opérateurs (SOP), appliqué à un nouvel espace $\mathbf{BV}_p^1$ des primitives des distributions de $\mathcal{V}_p(\mathbf{I})$, expliquant la propriété de régularité du bornement. (Généralisation moderne des fonctions de Wiener de 1937).

- Enfin, on présente l'espace $\mathbf{BV}(\sigma,\mathbb{C})$, d'Ashton-Doust introduit en 2005 qui est un espace bi-dimensionnèl des fonctions à variation bornée, définies sur les compacts du plan complexe, éxpliquant l'isomorphisme entre $\mathbf{BV}(\sigma,\mathbb{C})$ et l'espace des fonctions absolument continues $\mathbf{AC}(\sigma,\mathbb{C})$ appliqué au théorème de Banach-Stone (1932-1939).

Mots clés :

Fonctions à p-variations bornées, Problème de superposition des opérateurs,
injections, homeomorphismes, isomorphismes.

Table of Contents

One - Dimensional Notations: $(\ n = 1)$

Notations	**Definitions**	**Designations**
$f:\ [a,b] \to \mathbb{F}$	$a \in \mathbb{R}$, $b \in \mathbb{R}$, $a < b$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$	The function f is defined on $[a,b] \subset \mathbb{R}$
$[a,b] = \bigcup\limits_{n} I_n$	$I_n \cap I_m = \begin{cases} I_n & \text{if} \quad n = m \\ \emptyset & \text{if} \quad n \neq m \end{cases}$	$I_n = [a_n, b_n]\,,\ \lvert I_n\rvert = \lvert b_n - a_n\rvert$ non overlapping sub-intervals (pairwise disjoint intervals)
$\mathcal{P}([a,b])$	$(t_i)_i \in \mathcal{P}([a,b]) \iff a = t_0 < t_1 < \cdots < t_m = b$	The Set of all partitions of $[a,b]$
$f \in \mathbf{C^0}\ [a,b]$	$(\forall \varepsilon > 0)\,(\forall x \in [a,b])\,(\exists \delta > 0)\,(\forall x' \in [a,b])$ $\lvert x - x'\rvert < \delta \implies \lvert f(x) - f(x')\rvert < \varepsilon$	The function f is continuous on $[a,b]$
$f \in UC\,[a,b]$	$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x \in [a,b])\,(\forall x' \in [a,b])$ $\lvert x - x'\rvert < \delta \implies \lvert f(x) - f(x')\rvert < \varepsilon$	f is uniformly continuous on $[a,b]$
$f \in AC\ [a,b]$	$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall n \in \mathbb{N},\ \sum\limits_{i=1}^{n} \lvert I_n\rvert < \delta \Rightarrow \sum\limits_{i=1}^{n} \lvert f(b_n) - f(a_n)\rvert < \varepsilon$	f is absolutely continuous
$f\ \nearrow$ on $[a,b]$	$\forall t_1, t_2 \in [a,b]\,,\ t_1 \leq t_2 \Rightarrow f(t_1) \leq f(t_2)$	The function f is increasing on $[a,b]$
$f\ \curvearrowright$ on $[a,b]$	$\forall \lambda \in [0,1]\,, \forall t_1, t_2 \in [a,b]\,,$ $f(\lambda . t_1 + (1-\lambda) . t_2) \geq \lambda . f(t_1) + (1-\lambda) . f(t_2)$	The function f is concave on $[a,b]$
$f\ \circlearrowleft$ on $[a,b]$	$\forall \lambda \in [0,1]\,, \forall t_1, t_2 \in [a,b]\,,$ $f(\lambda . t_1 + (1-\lambda) . t_2) \leq \lambda . f(t_1) + (1-\lambda) . f(t_2)$	The function f is convex on $[a,b]$
ϕ	$\phi : [0, \infty[\ \to [0, \infty[\,,\ \phi$ continuous, convex, $\phi(0) = 0$, $t > 0 \Rightarrow \phi(t) > 0,\quad \lim\limits_{t \to +\infty} \phi(t) = +\infty$	The Young function (or gauge function)
$\boldsymbol{\kappa}$	$\boldsymbol{\kappa} : [0,1] \to [0,1]\,,\ \boldsymbol{\kappa} \in C^0([0,1])\,,\ \boldsymbol{\kappa}(0) = 0,\ \boldsymbol{\kappa}(1) = 1,$ $\boldsymbol{\kappa} \nearrow$, $\boldsymbol{\kappa}$ concave, $\lim\limits_{t \to 0^+} \frac{\boldsymbol{\kappa}(t)}{t} = +\infty$	The distortion function $\boldsymbol{\kappa}$
$\Phi = \{\phi_n\}_n$	$\{\phi_n\}_n \nearrow,\ \{\phi_n\}_n \ \curvearrowright\,,\ \phi_n : [0, +\infty[\ \to [0, +\infty[\,,\ \phi_n(0) = 0$	Φ is a sequence of increasing convex functions

Multidimensional Notations: $(n \geq \ 2)$

Notations	**Definitions**	**Designations**
Y^X	$f \in Y^X \Leftrightarrow f : X \longrightarrow Y, \quad (X, Y \neq \varnothing)$	The family of functions from X to Y
$\Vert f\Vert_\infty < \infty$	$\Vert f\Vert_\infty = \sup\limits_{x \in X} \lvert f(x)\rvert$	f is a bounded function
$\mathcal{P}([t, u])$	Let $f \in \mathbb{R}^{[t,u]}$ and $\xi = \{t_i\}_{i=1}^k \in \mathcal{P}([t, u])$, $\Delta t_{i+1} = t_{i+1} - t_i$, $\Delta f(t_{i+1}) = f(t_{i+1}) - f(t_i)$	Partitions of the interval $[t, u] \subset \mathbb{R}$
I_a^b	$a = (a_1,\ b_1), b = (a_2,\ b_2) \in \mathbb{R}^2, \, , \, I_a^b = [a_1,\ a_2] \times [b_1, b_2]$,	The rectangle I_a^b of the plan $\mathbb{R}^2$
$\triangle_{ij} f$	If $f \in \mathbb{R}^{I_a^b}, \xi = \{t_i\}_{i=1}^{k_1} \subset [a_1,\ a_2]$ and $\eta = \{s_j\}_{j=1}^{k_2} \subset [b_1, b_2]$, then $\triangle_{10} f(t_{i+1},\ s_{j+1}) = f(t_{i+1},\ s_{j+1}) - f(t_i,\ s_{j+1})$, $\Delta_{01} f(t_{i+1},\ s_{j+1}) = f(t_{i+1},\ s_{j+1}) - f(t_{i+1},\ s_j)$, $\Delta_{11} f(t_{i+1},\ s_{j+1}) = f(t_i,\ s_j) - f(t_i, s_{j+1}) - f(t_{i+1},\ s_j) + f(t_{i+1},\ s_{j+1})$.	The difference operators $\triangle_{10}$, Δ_{01}, Δ_{11}
$(\alpha_1, ..., \alpha_d)$,	if $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^n$, then $\lvert\alpha\rvert = \alpha_1 + ... + \alpha_n$, $n \in \mathbb{N}^*$ if $a = (a_1, ..., a_d) \in \mathbb{R}^n$, $b = (b_1, ..., b_d) \in \mathbb{R}^n$, then $\lvert a\rvert^2 = a_1^2 + ... + a_n^2$ and $a^k = a_1^{k_1} \times ... \times a_n^{k_n}$. $a \leq b$ iff $a_i \leq b_i$ for all $i = 1, ..., n$,	Multi-indeces to an ordered -tuple $[a, b] = \{x \in \mathbb{R}^n,\ a \leq x \leq b\}$
$\partial_{x_k}^{\alpha_k} = \partial_k^{\alpha_k}$	$\partial_{x_k}^{\alpha_k} f = \frac{\partial^{\alpha_k} f}{\partial x_k}, \quad \partial^\alpha f = f^{(\alpha)} = \partial_1^{\alpha_1} ... \partial_n^{\alpha_n} f$, $k \in \mathbb{N}^*$, $n \in \mathbb{N}^*$.	Partial derivatives of a distribution $f : \mathbb{R}^n \to \mathbb{F}$
$E = \prod\limits_{i=1}^{i=n} [a_i, b_i]$	for $E = [a_1,\ b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$,	E is an n-dimensional parallelepiped.
1 or $\mathcal{X}$	$\mathbf{1}_E(t) = \begin{cases} 1 & \text{if} \quad t \in E \\ 0 & \text{if} \quad t \notin E \end{cases}$	Indicator (or characteristic) function of a set E
$f \in \mathcal{S}(\mathbb{R}^n)$	$\zeta_M(f) = \sup\limits_{\lvert\alpha\rvert \leq M} \sup\limits_{x \in \mathbb{R}^n} (1 + \lvert x\rvert)^M \lvert\partial^\alpha f(x)\rvert$ $\sim \sup\limits_{\lvert\alpha\rvert \leq M} \sup\limits_{x \in \mathbb{R}^n} \left\lvert x^M (D^\alpha f)(x)\right\rvert < \infty \quad (M \in \mathbb{N})$.	The Schwartz class of distributions $f : \mathbb{R}^n \to \mathbb{F}$
$f \in L^p(\Omega)$	$\Vert f\Vert_{L^p(\Omega)} = \Vert f\Vert_p = \left(\int_\Omega \lvert f(x)\rvert^p dx\right)^{\frac{1}{p}} < \infty$	Lebsgue space on a compact space $\Omega \subset \mathbb{R}^d$
$f \in W^{s,p}(\Omega)$	$\Vert u\Vert_{W^{s,p}(\Omega)} = \Vert u\Vert_{s,p} = \left(\sum\limits_{\lvert\alpha\rvert \leq R} \Vert u_\alpha\Vert_p^p\right)^{\frac{1}{p}} < \infty$	Sobolev space on a compact space $\Omega \subset \mathbb{R}^d$
$f * g = g * f$	$f * g(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy = \int_{\mathbb{R}^n} f(y) g(x - y) dy$	Convolution operation between distributions

Introduction

Bounded variation's function notion was first introduced in 1881, by the mathematician C. Jordan (1838 - 1922), characterizing them as the difference of two increasing functions, when he analyzed Dirichlet theorem's proof, on convergence of the Fourier series of bounded monotone functions.[1]
In 1924, N.Wiener (1894-1964), showed that the Fourier series of any real function of finite p-variation are converging almost everywhere.
In 1937, L. C. Young (1905 - 2000), developed an integration theory with respect to functions of finite ϕ -variation, such that $\phi = |\cdot|^p$, and showed that the Fourier series of such functions converges everywhere. This class of functions are actually used in the rectification of curves (Riemann-Stieltjes integrals of all continuous functions), the convergence of general multivariables Fourier series, geometric measure theory, variations calculus and mathematical physics. BV functions form an Algebra of discontinuous functions whose first derivatives exist almost everywhere, so they are used to define generalized solutions of nonlinear differential problems.

- In the first chapter, we introduce the classical Jordan's bounded variation functions space $BV([a,b])$, presenting the properties of its Algebra structure.

- In the second chapter, we study the structure of BV_p^1 space, of primitives of functions of Wiener's bounded p-variation and the superposition operator problem in this space, using the articles [BCS2] & [Mou].

- Finally, in the last chapter, we present a two-dimensional bounded variation space defined in [AD1] and an isomorphism theorem from [ASD], as an application to the famous theorem of Banach-Stone.

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[1] Throughout the whole year of 1874, Camille Jordan and Leopold Kronecker were discussing about the organization of the theory of bi-linear forms, opposing the Jordan's canonical reduction method to the Kronecker's invariant one, this polemic was finally resolved in 1967, by the theory of algorithms, in Bishop's book " Foundations of Constructive Analysis ", claiming that:
"Every mapping of a complete metric space into a metric space is sequentially continuous if and only if the limit of every nonzero positive Cauchy sequence of rational numbers is nonzero", which is a constructive form of the "Zermelo's Axiom choice".

Keywords :

Functions of bounded p-variation, boundedness property, **S**superposition **O**perators **P**roblem (SOP), embedding, homeomorphisms, isomorphisms.

Abstract:

- First, we introduce the Algebra of the Jordan's bounded variation functions space $BV\left([a,b]\right)$, defined on the closed intervals of the real line.

- After that, we present the Wiener space of $p-$bounded variation functions $\mathcal{V}_p(I)$ and the Superposition Operator Problem (**SOP**), applied to the new space BV_p^1 of primitives of $\mathcal{V}_p(I)$, explaining its regularity with respect to the boundedness property. .

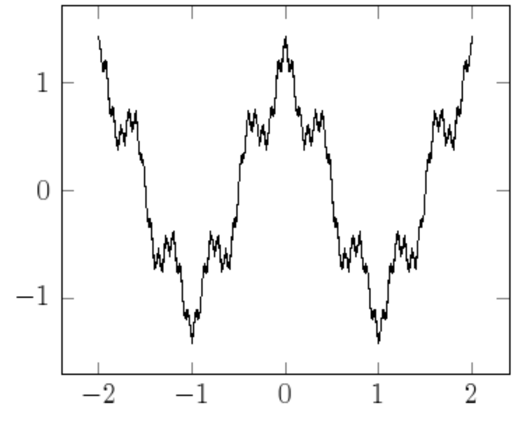
- Finally, we present the Ashton-Doust space $BV\left(\sigma,\mathbb{C}\right)$, of two-dimensional bounded variation functions, defined on the compacts of the complex plane, explaining the isomorphism between $BV\left(\sigma,\mathbb{C}\right)$ and the absolutely continuous functions space $AC\left(\sigma,\mathbb{C}\right)$ as an application to the Banach-Stone's Theorem

Chap 1: Jordan's real line $BV([a,b])$ space

In this chapter, we introduce the Algebra of the Jordan's bounded variation functions space, defined on the closed intervals of the real line.
Throughout this chapter, $[a, b]$ denotes the closed interval of reals between its lower born a, and its upper born b.

1.1 Preliminaries

Mathematicians proved that any continuous function from a compact space into a metric space is bounded. Lebesgue also proved that any BV function (Bounded Variation one) implies that it is almost everywhere differentiable. There are some functions, which are not BV, like the Weierstrass function (Continuous everywhere but differentiable nowhere, see Fig 1.1).



$$W_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

$$0 < a < 1,$$

$$ab > 1 + \frac{3\pi}{2}$$

$$W_{a,b} \notin BV$$

Figure 1.1:

For this reason we find some surprising facts about BV functions, like:

- Smooth functions are not dense on BV.
- Lipschits functions are dense on BV.
- Continuous BV functions $BV \cap C$ are dense on BV, although $C \not\subset BV$.

Definition 1.1.1.

Let S be a non-empty set of real numbers,

(a) The set S is bounded above $\Leftrightarrow \exists M \in \mathbb{R}, \forall x \in X, M \geq x$.

The number M is called an upper bound of S.

(b) The set S is bounded below $\Leftrightarrow m \in \mathbb{R}, \forall x \in S, m \leq x$.

The number m is called a lower bound of S.

The set S is bounded means it is bounded above and bounded below.

Proposition 1.1.1.

r is a bound for S $\Leftrightarrow \exists r > 0, \forall x \in S, \;\; |x| \leq r,$

Theorem 1.1.1.

Let S be a non-empty set of real numbers that is bounded above, and let b be an upper bound of S. Then the following assertions are equivalent:

(i) $b = \sup S$.

(ii) For all $\varepsilon > 0$ there exists $x \in S$ such that $|b - x| < \varepsilon$.

(iii) For all $\varepsilon > 0$ there exists $x \in S$ such that $x \in \,]b - \varepsilon, b]$

Definition 1.1.2.

A function $f : [a, b] \longrightarrow \mathbb{F}$ is said to be continuous at $x_0 \in [a, b]$ iff

$$\forall \varepsilon > 0, \; \exists \delta > 0, \; \forall x \in [a, b] \, , \; |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

- f is continuous, if it is continuous at every $x_0 \in [a, b]$.
- If δ is independent of x_0, then f is said to be a uniformly continuous.

Definition 1.1.3 (Classification of discontinuities).

Set $f : [a, b] \longrightarrow \mathbb{R}$, $x_0 \in [a, b]$, $f(x_0) \in \mathbb{R}$, such that,

$$f\left(x_0^-\right) = \lim_{t \overset{<}{\longrightarrow} x_0} f(t), \quad f\left(x_0^+\right) = \lim_{t \overset{<}{\longrightarrow} x_0} f(t), \quad f(x_0) \neq f\left(x_0^-\right) \quad \text{and} \quad f(x_0) \neq f\left(x_0^+\right).$$

Then, we define the eventual discontinuities of a function f on a point x_0 as,

1. Discontinuity of a first kind: If $f\left(x_0^-\right) \in \mathbb{R}$ and $f\left(x_0^+\right) \in \mathbb{R}$
    (a) If $f\left(x_0^-\right) = f\left(x_0^+\right)$, then it's a removable discontinuity.
    (b) If $f\left(x_0^-\right) \neq f\left(x_0^+\right)$, then it's a jump discontinuity.
2. Discontinuity of a second kind: If $f\left(x_0^-\right) \notin \mathbb{R}$ or $f\left(x_0^+\right) \notin \mathbb{R}$,
    (a) If $\lim\limits_{x \to x_0} f(x) \neq \infty$, then it's an oscillation Essential discontinuity.
    (b) If $\lim\limits_{x \to x_0} f(x) = \infty$, then it's an infinite discontinuity.

Theorem 1.1.2. *[Gor]*

If $f : [a,\ b] \to \mathbb{R}$ is an increasing function. Then f has one-sided limits at each point of $[a,\ b]$. These limits are given by

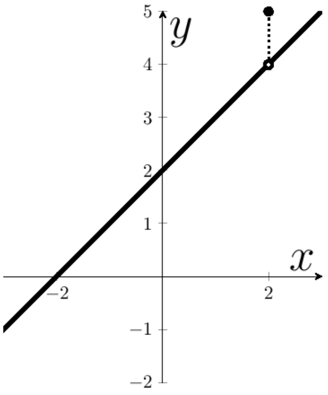
$$f(x^-) \;=\; \sup\{\ f(t) : \quad t \in [a,\ x[\quad \},\quad x \in]a,\ b]$$

$$f(x^+) \;=\; \inf\{\ f(t) : \quad t \in (]x,\ b] \quad \},\quad x \in [a,\ b[$$

such that $f(a) \leq f(a^+)$, $\qquad f(b) \geq f(b^-)$, *with*

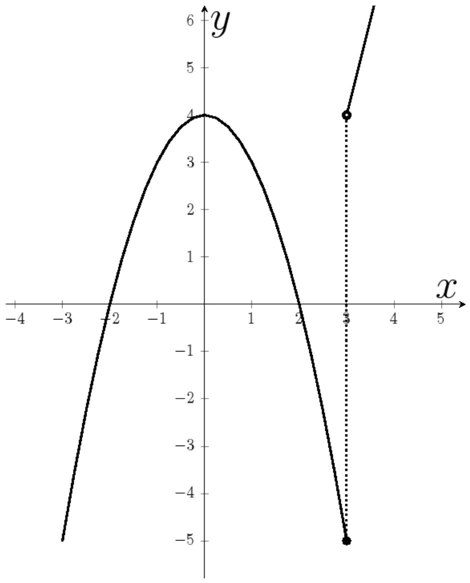
$$f(x^-) \leq f(x) \leq f\left(x^+\right),\ x \in [a,\ b]$$

<table>
<tr><td colspan="2">a) Discontinuity of a first kind: If $f\left(x_0^-\right) \in \mathbb{R}$ and $f\left(x_0^+\right) \in \mathbb{R}$</td></tr>
<tr>
<td>If $f\left(x_0^-\right) = f\left(x_0^+\right)$,

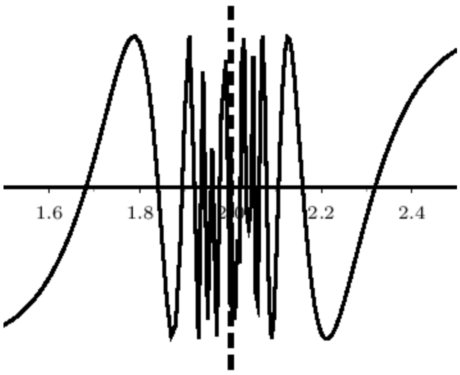


$x_0 = 2, \;\; f(x) = \dfrac{x^2-4}{x-2}$

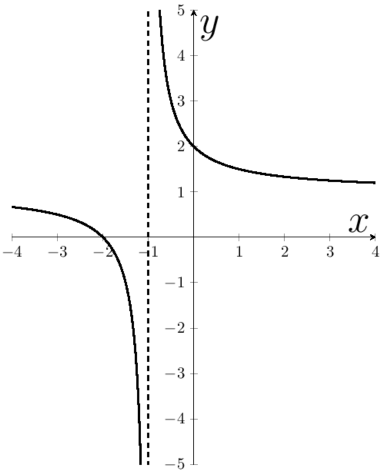
Removable discontinuity</td>
<td>If $f\left(x_0^-\right) \neq f\left(x_0^+\right)$,



$x_0 = 3, \;\; g(x) = \begin{cases} 4-x^2 & \text{if} \quad x \leq 3 \\ 4x-8 & \text{if} \quad x > 3 \end{cases}$

Jump discontinuity</td>
</tr>
<tr><td colspan="2">b) Discontinuity of a second kind: If $f\left(x_0^-\right) \notin \mathbb{R}$ or $f\left(x_0^+\right) \notin \mathbb{R}$,</td></tr>
<tr>
<td>

$x_0 = 0, \;\; h(x) = 2\sqrt{x}\sin(x)\cos(\ln x)$

Oscillation Essential discontinuity</td>
<td>

$x_0 = -1, \;\; k(x) = \dfrac{x+2}{x+1}$

Infinite discontinuity</td>
</tr>
</table>

Figure 1.2:

Summary 1.1.3 (Classical function spaces on the real line).

- The space $B([a,b])$ of all bounded functions $f:[a,b]\to\mathbb{R}$
- The space $C([a,b])$ of all continuous functions $f:[a,b]\to\mathbb{R}$ is defined with the norm
$$\|f\|_C=\|f\|_\infty=\max_{a\leq x\leq b}|f(x)|, \tag{1.1}$$
such that $(C([a,\ b\ \|\cdot\|_C)$ is a Banach space
- $C^1([a,b]):$ A function $f:[a,b]\to\mathbb{R}$ is called smooth function iff it is a continuously differentiable function.
- $R([a,b]):$ The space of all regular functions $f\in B([a,b])$, i.e. bounded functions which have, at most discontinuities of the first kind (The number of discontinuities is at most countable on $[a,\ b]$.)
- $S([a,\ b]):$ We say that f is a step function on $[a,b]$, if there exist finitely many points $a=t_0<t_1<\ldots<t_m=b$, such that f is constant on each open interval $]t_{j-1},\ t_j[$, for $j=1,2,\ldots,m$.
- $Lip([a,\ b]):$ A function $f:[a,\ b]\to\mathbb{R}$ is called Lipschitz continuous if there exists a constant $L>0$ such that
$$|f(x)-f(y)|\leq L|x-y| \qquad\qquad (a\leq x,\ y\leq b)$$
- $Lip_\alpha([a,\ b])$: A function $f:[a,b]\to\mathbb{R}$ is called Hölder continuous (or Lipschitz continuous for $0<\alpha\leq 1$, if there exists a constant $L>0$, such that
$$|f(x)-f(y)|\leq L|x-y|^\alpha \qquad (a\leq x,\ y\leq b)$$
- $L^p([a,\ b])$: A function $f:[a,b]\to\mathbb{R}$ is called Lebesgue function, iff
$$\int_a^b|f(x)|^p\,dx<\infty$$
- $H^1([a,b]):$ A function $f:[a,b]\to\mathbb{R}$ is called Sobolev function iff it is a continuous function on $[a,b]$ of the form: $f(x)=\displaystyle\int_a^x f'(t)\,dt,\quad x\in[a,b],\ f'\in L^2([a,b])$

1.2 Algebra of $BV([a,b])$

Definition 1.2.1.

The $\mathbb{F}$- sequence $(a_k)_{k\in\mathbb{N}^*}$ is said to be of bounded variation's sequence, if and only if it satisfies

$$\sum_{k=1}^{\infty}|a_{k+1}-a_k|<\infty \tag{1.2}$$

Proposition 1.2.1.

Every bounded variation's sequence $(a_n)_n$ satisfying the condition (1.2) is convergent.

Proof.

When, $m<n$ we form the telescoping sum

$$\sum_{k=m}^{n-1}(a_{k+1}-a_k)=a_n-a_m.$$

Using the triangular inequality we obtain:

$$|a_n-a_m|=\left|\sum_{k=m}^{n-1}(a_{k+1}-a_k)\right|\le\sum_{k=m}^{n-1}|a_{k+1}-a_k| \tag{1.3}$$

By (1.2) and using the Cauchy convergence definition we have from (1.3)

$$\forall\varepsilon>0,\exists N(\varepsilon)\in\mathbb{N},\forall n>m>N(\varepsilon)\,,\ |a_n-a_m|\le\sum_{k=m}^{n-1}|a_{k+1}-a_k|\le\sum_{k=1}^{n-1}|a_{k+1}-a_k|<\varepsilon$$

The bounded variation sequence $(a_n)_n$ is convergent into the complete metric space $\mathbb{R}$.

□ Q.E.D

Definition 1.2.2.

A partition of an interval $[a,b]$ is a set of points $\{x_0,x_1,...,x_n\}$ such that

$$a=x_0<x_1<x_2...<x_n=b.$$

Definition 1.2.3.

Let $f:[a,b]\rightarrow\mathbb{R}$ be a function and $[c,d]$ be any closed sub-interval of $[a,b]$. If the set

$$S=\left\{\sum_{i=1}^{n}\left|f\left(x_i\right)-f\left(x_{i-1}\right)\right|:\{x_i:1\leq i\leq n\}\text{ is a partition of }[c,d]\right\}$$

is bounded then the variation of f on $[c,d]$ is defined to be $V\left(f,[c,d]\right)=\sup S.$

If S is unbounded then the variation of f on $[c,d]$ is said to be ∞.

Theorem 1.2.1.

Let I be an interval. If $f:I\rightarrow\mathbb{R}$ is a monotone function on I. Then f has one-sided limits at each point of I.

Theorem 1.2.2.

If $f:[a,\ b]\rightarrow\mathbb{R}$ is monotone, then the set of discontinuities of f in $[a,\ b]$ is countable.

Theorem 1.2.3.

Let $f:[a,\ b]\rightarrow\mathbb{R}$ and let $c\in\,]a,\ b[$. If f is of bounded variation on $[a,\ c]$ and $[c,\ b]$. Then f is of bounded variation on $[a,\ b]$, such that:

$$V(f,\ [a,\ b])=V(f,\ [a,\ c])+V\left(f,\ [c,\ b]\right)$$

Definition 1.2.4.

A function $\varphi:\ [a,\ b]\rightarrow\mathbb{F}$, is said to be of bounded variation if there exists $M>0$, such that, for every partition $\mathcal{P}:a=x_0<x_1<\cdots<x_n=b$ of $[a,\ b]$,we have:

$$V_{a,b}^{\mathcal{P}}(\varphi)=\sum_{i=1}^{n}|\varphi(x_i)-\varphi(x_{i-1})|\leq M.$$

The quantity $V_{a,b}(\varphi)=\sup\limits_{\mathcal{P}\subset[a,b]}V_{a,b}^{\mathcal{P}}(\varphi)$ is called the total variation of φ, where the supremum is taken over all partitions $\mathcal{P}$ of $[a,\ b]$.

Example 1.2.1.

(1) If f is constant on $[a,b]$ then f is of bounded variation on $[a,b]$.

Consider the constant function $f(x) = k$ on $[a,b]$.

Notice that $\sum\limits_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ is zero for every partition of $[a,b]$.

Indeed: $V(f,[a,b]) = \sup\limits_{\mathcal{P}} \sum\limits_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sup\left(\sum\limits_{i=1}^{n} |k-k|\right) = 0 < \infty.$

(2) All increasing functions are of bounded variation, see Theorem 1.2.4

(3) Every characteristic function on $[a,\ b]$ is of bounded variation.

(4) As an example of a function which is not of bounded variation, consider the function

$\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} \sin(1/x)\ , & x > 0, \\ 1, & x = 0. \end{cases}$$

(5) A continuous function need not to be of bounded variation.

For example, consider the function $\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} x\sin(1/x)\ , & x > 0, \\ 0, & x = 0. \end{cases}$$

Then it can be shown that $\varphi \in C[0,1]$, but not of bounded variation.

However, if $\varphi : [a,\ b] \to \mathbb{F}$ is a Lipschitz function, then φ is of bounded variation.

Theorem 1.2.4.

If f is increasing on $[a,b]$, then f is of bounded variation, and the value of its total variation is $f(b) - f(a)$.

Proof.

Let $\{x_i : 1 \leq i \leq n\}$ be a partition of $[a,b]$,

$$x_i > x_{i-1} \quad \Rightarrow \quad f(x_i) > f(x_{i-1}), \quad \text{(f is increasing)}$$

$$\Rightarrow \quad f(x_i) - f(x_{i-1}) > 0 \quad \Rightarrow \quad |f(x_i) - f(x_{i-1})| = f(x_i) - f(x_{i-1}).$$

Hence, consider the telescopic sum:

$$\begin{aligned}\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| &= \sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))\\ &= f(x_2)-f(x_1)+\ldots+f(x_n)-f(x_{n-1})\\ &= f(x_n)-f(x_1)<f(x_n)<f(b)<\infty.\end{aligned}$$

Thus f is of bounded variation on $[a,b]$.

Because of the telescoping nature of this sum, it is the same for every partition $\mathcal{P}\subset[a,b]$.

Thus we see that,

$$V(f,[a,b])=\sup_{\mathcal{P}}\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|=f(b)-f(a).$$

☐ Q.E.D

Proposition 1.2.2.

The Dirichlet function f defined by

$$f(x)=\mathbb{1}_{\mathbb{Q}}=\begin{cases}0 & \textit{if} \quad x \textit{ is irrational}\\ 1 & \textit{if} \quad x \textit{ is rational}\end{cases}$$

is not of bounded variation on any interval.

Proof.

Let $[a,b]$ be a closed interval in $\mathbb{R}$.

By axiom's choice, we can always construct a partition $\mathcal{P}=\{x_0,x_1,...,x_n\}$ of $[a,b]$, such that $|f(x_i)-f(x_{i-1})|=1$, for all $n\in\mathbb{N}$, hence

$$V(f,[a,b])>\sum_{i=0}^{n}|f(x_i)-f(x_{i-1})|=n.$$

When n is arbitrarily large, $V(f,[a,b])$ diverges. ☐ Q.E.D

Theorem 1.2.5.

If $f:[a,\ b]\to\mathbb{R}$ is monotonically increasing, then it is differentiable $(\mathbf{a.e})$.

Definition 1.2.5.

If we denote by $\mathcal{L}[a,\ b]$ the set of all $\mathbb{F}$-valued functions defined on $[a,\ b]$, which are integrable with respect to $m(.)$, the Lebesgue measure, then for every $f \in \mathcal{L}[a,\ b]$ and $c \in \mathbb{F}$, we call the function $g : [a,\ b] \to \mathbb{F}$, defined by (1.4) an indefinite integral of f

$$g(x) = c + \int_a^x f dm,\ x \in [a,\ b], \tag{1.4}$$

Theorem 1.2.6. *[Nai1]*

Let $f \in \mathcal{L}[a,\ b]$ and $g : [a,\ b] \to \mathbb{F}$ be defined by: $g(x) = \displaystyle\int_a^x f dm,\ x \in [a,\ b]$.

Then g is continuous.

Theorem 1.2.7.

Let $\varphi : [a,\ b] \to \mathbb{R}$ be a monotonically increasing function.

Then φ is differential $(\mathbf{a}.\mathbf{e})$. *and φ' is non-negative, Lebesgue measurable, with*

$$\int_a^b \varphi' dm \leq \varphi(b) - \varphi(a)\ .$$

Theorem 1.2.8 (FTLI-1)**.**

Let $f \in \mathcal{L}[a,\ b]$ and $g : [a,\ b] \to \mathbb{F}$ be defined by: $g(x) = \displaystyle\int_a^x f dm,\ x \in [a,\ b]$.

Then g is differentiable $(\mathbf{a}.\mathbf{e})$.*, $g' \in \mathcal{L}[a,\ b]$ and $g' = f$* $(\mathbf{a}.\mathbf{e})$.

Definition 1.2.6.

A function $\varphi : [a,\ b] \to \mathbb{F}$ is said to be absolutely continuous iff

for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every family $\{I_i\ :\ i = 1,\ \ldots,\ n\}$ of non-overlapping subintervals of $[a,\ b]$,

$$\sum_{i=1}^{n} \ell(I_i) < \delta \implies \sum_{i=1}^{n} |\varphi(x_i) - \varphi(y_i)| < \varepsilon,$$

where x_i and y_i are the end points of I_i, $i = 1, \ldots, n.$, and $\ell(I_i)$ is the length of I_i.

Definition 1.2.7.

Let $\sigma \subseteq \mathbb{R}$. A family $\mathcal{I}$ of intervals is called a Vitali cover of E, if for every $x \in \sigma$ and for every $\varepsilon > 0$, there exists $I \in \mathcal{I}$ such that $x \in I$ and $\ell(I) < \varepsilon$.

Lemma 1.2.1 (Vitali covering lemma)**.**

Let $E \subseteq \mathbb{R}$ with $m^*(E) < \infty$. If $\mathcal{I}$ is a Vitali cover of E, then for every $\varepsilon > 0$, there exist pairwise disjoint intervals $I_1, \ldots, I_n$ in $\mathcal{I}$ such that $m^*(E \setminus \bigcup_{i=1}^n I_i) < \varepsilon$.

Proposition 1.2.3.

The following statements hold,

(i) If φ is a Lipschitz function, that is, if there exists $L > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq L|x-y| \quad for\ all \quad x, y \in [a,\ b],$$

then φ is absolutely continuous.

(ii) If φ is absolutely continuous, then φ is uniformly continuous.

Theorem 1.2.9.

Let $f \in \mathcal{L}[a,\ b]$ and $g : [a,\ b] \to \mathbb{F}$ be defined by

$$g(x) = \int_a^x f dm, \ x \in [a,\ b]$$

Then g is absolutely continuous.

Theorem 1.2.10 (fundamental theorem of Lebesgue integration)**.**

Suppose g : $[a,\ b] \to \mathbb{F}$ is absolutely continuous.

Then g is differentiable (**a**.**e**)., *$g' \in \mathcal{L}[a,\ b]$ and*

$$g(x) = g(a) + \int_a^x g' dm, \qquad \forall x \in [a,\ b].$$

Theorem 1.2.11.

Let $f \in \mathcal{L}[a,\ b]$ *and* $g : [a,\ b] \to \mathbb{F}$ *be defined by :* $g(x) = \displaystyle\int_a^x f dm,\ x \in [a,\ b]$.

Then g *is of bounded variation and* $V_{a,b}\,(g) = \displaystyle\int_a^b |f|\, dm$

Theorem 1.2.12.

Every absolutely continuous function on $[a,\ b]$ *is of bounded variation.*

Proposition 1.2.4.

Let $\varphi : [a,\ b] \to \mathbb{F}$ *be a function of bounded variation.*

Then for any $x, y, z \in [a,\ b]$ *with* $x < y < z$,

$$V_{x,y}(\varphi) + V_{y,z}(\varphi) = V_{x,z}(\varphi).$$

In particular, the function $\varphi_0 : [a,\ b] \to \mathbb{R}$ *defined by* $\varphi_0(x) = V_{a,x}(\varphi)$ *is monotonically increasing. Furthermore, if* φ *is real valued, then* $\varphi_0 - \varphi$ *is also a monotonically increasing.*

Proposition 1.2.5.

A function $\varphi : [a,\ b] \to \mathbb{R}$ *is of bounded variation if and only if there exist monotonically increasing functions* φ_1, φ_2 *on* $[a,\ b]$ *such that* $\varphi = \varphi_1 - \varphi_2$.

Remark 1.2.1.

We can choose φ_1, φ_2 positive functions with $\varphi = \varphi_1 - \varphi_2 + \varphi\,(a)$

Theorem 1.2.13.

Let $\varphi : [a,\ b] \to \mathbb{F}$ *be of bounded variation.*

Then φ *is differentiable* $(\mathbf{a}.\mathbf{e})$ *and* $\varphi' \in \mathcal{L}[a,\ b]$.

Theorem 1.2.14.

Suppose $\varphi : [a,\ b] \to \mathbb{R}$ *is absolutely continuous. Then* φ *is of bounded variation with*

$$V_{a,b}(\varphi) = \int_a^b |\varphi'| dm, \quad \text{and} \quad \varphi' \in \mathcal{L}[a,\ b]$$

Theorem 1.2.15 (FTLI-2).

A function $g : [a,\ b] \to \mathbb{F}$ *is an indefinite integral of an integrable function* f*:* $[a,\ b] \to \mathbb{F}$*, if and only if* g *is absolutely continuous, and in that case* $g' = f \quad (\mathbf{a}.\mathbf{e})$ *, and*

$$g(x) = g(a) + \int_a^x f dm, \qquad \forall x \in [a,\ b].$$

Theorem 1.2.16 (FTLI-3).

A function g *:* $[a,\ b] \to \mathbb{F}$ *is absolutely continuous if and only if there exists an integrable function* f *:* $[a,\ b] \to \mathbb{F}$ *such that* $g' = f \quad (\mathbf{a}.\mathbf{e})$ *, with:*

$$g(x) = g(a) + \int_a^x f dm, \qquad \forall x \in [a,\ b],$$

Theorem 1.2.17. *[Gor]*

Let f *and* g *be functions of bounded variation on* $[a,b]$ *and let* k *be a constant. Then*

(i) $|f|$ *is bounded on* $[a,b]$.

(ii) f *is of bounded variation on every closed subinterval of* $[a,b]$.

(iii) kf *is of bounded variation on* $[a,b]$.

(iv) $f+g$ *and* $f-g$ *are of bounded variation on* $[a,b]$.

(v) fg *is of bounded variation on* $[a,b]$.

(vi) If $1/g$ *is bounded on* $[a,b]$ *, then* $f\ /g$ *is of bounded variation on* $[a,b]$.

Definition 1.2.8.

Suppose that $J = [a,b]$. We shall denote by $\mathcal{P} = \mathcal{P}(J)$ the set of finite partition

$$\Lambda = \{\ a\ = \lambda_0 < \lambda_1 < \ .\ .\ .\ < \lambda_n = b\ \} \text{ of } J.$$

The variation of f over J is defined to be $\underset{J}{\text{var}} f = \sup\limits_{\Lambda \in \mathcal{P}} \sum\limits_{\Lambda} |f(\lambda_j) - f(\lambda_{j-1})|$.

If $\underset{J}{\text{var}} f < \infty$, then f is said to be of bounded variation over J.

$BV\,[a,b]$ is the space of all functions of bounded variation on $[a,b]$.

Remark 1.2.2.

- $BV\left(\mathbb{R}\right)$ is the Banach Algebra of functions on $\mathbb{R}$ which have finite total variation.

  We use the norm $$\|f\| = \sup_{t\in\mathbb{R}} |f(t)| + \underset{\mathbb{R}}{var} f, \quad (f \in BV(\mathbb{R}))$$

- $BV\left[a,b\right]$ becomes a Banach Algebra when given either one of the equivalent norms

$$\begin{aligned} \mathcal{N}_1\left(f,[a,b]\right) &= |f(b)| + \underset{[a,b]}{var} f \ , \\ \mathcal{N}_2\left(f,[a,b]\right) &= \sup_{t\in[a,b]} |f(t)| + \underset{[a,b]}{var} f \ , \quad (f \in BV\left[a,b\right]). \end{aligned}$$

- Nice properties of monotone functions (like Riemann integrability or diferentiability almost everywhere) carry over to functions of bounded variation.

  In particular, a function $f \in BV([a,\ b])$ has, at most, many countably points of discontinuity in $[a,\ b]$, all being of the frst kind (jumps) or removable.

Definition 1.2.9.

We denote by $BV^o([a,\ b])$ the subset of all functions $f \in BV([a,\ b])$ with $f(a) = 0$. Furthermore, we write $NBV([a,\ b])$ for the set of all functions $f \in BV^o([a,\ b])$ which are right continuous at any point $x_0 \in [a,\ b)$, i.e.

$$f(x_0^+) = \lim_{x \overset{>}{\longrightarrow} x_0} f(x) = f(x_0), \qquad (a \leq x_0 < b).$$

Functions in $NBV([a,\ b])$ will be called normalized .

Proposition 1.2.6.

Let $\{x_1,\ x_2,\ x_3,\ \ldots\} \subset [a,\ b]$, *be the set of discontinuities of a function* $f \in BV([a,\ b])$. *The jump function* $\sigma_f : [a,\ b] \to \mathbb{R}$ *of* f *defined by* $\sigma_f(a) = 0$ *and*

$$\sigma_f(t) = \sum_{t>x_k} (f(x_k^+) - f(x_k^-)) + (f(t) - f(t^-)), \qquad (a < t \leq b) \ ,$$

is veifying the inequality:

$$\sum_{k=1}^{n} (|f(x_k^+) - f(x_k)| + |f(x_k) - f(x_k^-)|) \leq V(f, [a,\ b]), \ \textit{for all } n \in \mathbb{N}.$$

Theorem 1.2.18.

Suppose that $f : [a,\ b] \to \mathbb{R}$ *is satisfying any one of the following:*

(i) f *is monotone on* $[a,\ b]$,

(ii) f *is piece-wise monotone function on* $[a,\ b]$,

(iii) f *is Lipschitz on* $[a,\ b]$,

(iv) f *is differentiable on* $[a,\ b]$ *such that* $f'(x)$ *is bounded on* $[a,\ b]$.

Then f *is of bounded variation on* $[a,\ b]$.

Theorem 1.2.19.

Suppose that $f : [a,\ b] \to \mathbb{R}$. *Then the following conditions are equivalent*

(i) f *is absolutely continuous,*

(ii) f *is differentiable almost everywhere on* $[a,\ b]$ *and*

$$f(x) = f(a) + \int_a^x f'(t)dt, \text{ for all } x \in [a,\ b]$$

(iii) For all $\varepsilon > 0$ *there exists a polynomial* p *such that* $\|f - p\|_{BV} < \varepsilon$.

Proposition 1.2.7. *[AD1]*

$AC[a,b]$ is a Banach Algebra with the following equivalent norm variation:

$$\|f\|_{AC[a,b]} \sim \underset{[a,b]}{var} f = \int_a^b |f'(t)|\, dt\ .$$

Hence the polynomials are not dense in $AC[a,b]$.

Lemma 1.2.2.

If $c \in [a,b]$, *then he norm*

$$\|f\|_{[a,b],c} = |f(c)| + \underset{[a,b]}{var} f,$$

is a norm on $BV[a,b]$ *which makes that space into a Banach space.*

Proposition 1.2.8.

$(BV[a,b],\ \|\cdot\|_{[a,b],\ a})$ *is a Banach Algebra*

Proof.

We prove the result for the norm

$$|||f|||_{[a,b]} = |f(a)| + \underset{[a,b]}{var} f$$

It is sufficient to work in $BV_0[a,b]$ of functions vanishing at a, to prove.

$$\underset{[a,b]}{var} f.g \leq (\underset{[a,b]}{var} f).(\underset{[a,b]}{var} g) \; .$$

Let $f, g \in BV_0[a,b]$, and fix $t \in \mathcal{P}[a,b]$. We split f in the following way:

$$f_k(t) = \begin{cases} 0 & \text{if} \quad t_0 \leq t \leq t_{k-1} \\ f(t) - f(t_{k-1}) & \text{if} \quad t_{k-1} \leq t \leq t_k \\ f(t_k) - f(t_{k-1}) & \text{if} \quad t \geq t_k \end{cases} , \qquad \text{for } k = 2, ..., m-1,$$

$$f_1(t) = \begin{cases} f(t) & 0 \leq t \leq t_1 \\ f(t_1) & t \geq t_1 \end{cases} \quad \cdots \quad f_m(t) = \begin{cases} 0 & 0 \leq t \leq t_{m-1} \\ f(t) - f(t_{m-1}) & t \geq t_{m-1} \end{cases}$$

Then

$$f = \sum_{k=1}^{m} f_k \Rightarrow, var(f) = \sum_{k=1}^{m} \underset{[a,b]}{var} f_k = \sum_{k=1}^{m} \underset{[t_{k-1},t_k]}{var} f_k,$$

and

$$f_k(t_j) = \begin{cases} 0 & j < k \\ f(t_k) - f(t_{k-1}) & j \geq k \end{cases} \quad .$$

Hence

$$\begin{aligned} \sum_{j=1}^{m} |f(t_j)\, g(t_j) - f(t_{j-1})\, g(t_{j-1})| = & \sum_{j=1}^{m} \left| \sum_{k=1}^{m} [f_k(t_j)\, g(t_j) - f_k(t_{j-1})\, g(t_{j-1})] \right| \\ \leq & \sum_{j=1}^{m} \sum_{k=1}^{m} |f_k(t_j)\, g(t_j) - f_k(t_{j-1})\, g(t_{j-1})| \end{aligned}$$

$$= \sum_{k=1}^{m} \left(\left| \left[f\left(t_k\right) - f\left(t_{k-1}\right) \right] g\left(t_k\right) \right| + \sum_{j=k+1}^{m} \left| \left[f\left(t_k\right) - f\left(t_{k-1}\right) \right] \left[g\left(t_j\right) - g\left(t_{j-1}\right) \right] \right| \right)$$

$$\leq \sum_{k=1}^{m} \left| f\left(t_k\right) - f\left(t_{k-1}\right) \right| \left(\left| g\left(t_k\right) - 0 \right| + \sum_{j=k+1}^{m} \left| g\left(t_j\right) - g\left(t_{j-1}\right) \right| \right)$$

$$\leq \sum_{k=1}^{m} \left| f\left(t_k\right) - f\left(t_{k-1}\right) \right| .\underset{[a,b]}{var}\ g \leq \left(\underset{[a,b]}{var}\ f \right) . \left(\underset{[a,b]}{var}\ g \right)$$

Since this is true for all $t \in \mathcal{P}\left[a,b\right]$, it follows that

$$\underset{[a,b]}{var}\left(f.g\right) \leq \left(\underset{[a,b]}{var}\ f\right).\left(\underset{[a,b]}{var}\ g\right)$$

for all $f, g \in BV_0\left[a,b\right]$, and so the theorem is proved. □ Q.E.D

Theorem 1.2.20. *[Dou]*

$\left(BV\left[a,b\right],\ \left\|\cdot\right\|_{[a,b],\ c}\right)$ *is a Banach Algebra for every* $c \in \left]a,b\right[$.

Proof.

$\left\|\cdot\right\|_{[a,b],\ c}$ is submultiplicative. Suppose that $[\alpha,\beta]$ is a compact and let

$$B_0 = \{\ f \in BV\left[\alpha,\ \beta\right] :\ f(\alpha) = 0\ \}.$$

First we proove that the norm $\left\|\cdot\right\|_{[\alpha,\beta],\alpha}$ is submultiplicative on B_0

Suppose that $f, g \in B_0$ and that $\Lambda = \{\ \alpha = \lambda_0 <\ \ldots\ < \lambda_n = \beta\ \} \in \mathcal{P}\left[\alpha,\ \beta\right]$.

Then $\displaystyle\sum_{j=1}^{n} \left| fg(\lambda_j) - fg(\lambda_{j-1}) \right| = \sum_{j=1}^{n} \left| f(\lambda_j)(g(\lambda_j) - g(\lambda_{j-1})) + g(\lambda_{j-1})(f(\lambda_j) - f(\lambda_{j-1})) \right|$

$$\leq \sum_{j=1}^{n} \left[\sum_{k=1}^{j} \left| f(\lambda_k) - f(\lambda_{k-1}) \right| \right] . \left| g(\lambda_j) - g(\lambda_{j-1}) \right| + \sum_{j=2}^{n} \left[\sum_{k=1}^{j-1} \left| g(\lambda_k) - g(\lambda_{k-1}) \right| \right] . \left| f(\lambda_j) - f(\lambda_{j-1}) \right|$$

$$\leq \sum_{j=1}^{n} \sum_{k=1}^{j} \left| f(\lambda_k) - f(\lambda_{k-1}) \right| . \left| g(\lambda_j) - g(\lambda_{j-1}) \right| + \sum_{j=1}^{n} \sum_{k=j+1}^{n} \left| f(\lambda_k) - f(\lambda_{k-1}) \right| . \left| g(\lambda_j) - g(\lambda_{j-1}) \right|$$

$$= \left[\sum_{k=1}^{n} |f(\lambda_k) - f(\lambda_{k-1})|\right] \left[\sum_{j=1}^{n} |g(\lambda_j) - g(\lambda_{j-1})|\right] \leq \left(\underset{[\alpha,\ \beta]}{var} f\right) . \left(\underset{[\alpha,\ \beta]}{var} g\right)$$

Thus $\|f.g\|_{[\alpha,\beta],\alpha} = \underset{[\alpha,\ \beta]}{var} (f.g) \leq \left(\underset{[\alpha,\ \beta]}{var} f\right) \cdot \left(\underset{[\alpha,\ \beta]}{var} g\right) = \|f\|_{[\alpha,\beta],\alpha} \cdot \|g\|_{[\alpha,\beta],\alpha}$

By symmetry, $\|.\|_{[\alpha,\beta],\beta}$ is sub-multiplicative on the subspace of $BV[\alpha,\ \beta]$ consisting of functions vanishing at β. Suppose then that $f, g \in BV[a,b]$.

We can write : $f = f_1 + f_2 + f_3$ and $g = g_1 + g_2 + g_3$

where f_1 and g_1 are constants, f_2 and g_2 vanish on $[c,b]$ and f_3 and g_3 vanish on $[a,c]$.

Hence $\|f.g\| = \|(f_1 + f_2 + f_3)(g_1 + g_2 + g_3)\| \leq \sum_{i,j=1}^{3} \|f_i.g_j\|$

The first part of the proof implies that $\|f_2.g_2\| \leq \|f_2\| . \|g_2\|$ and $\|f_3.g_3\| \leq \|f_3\| . \|g_3\|$.

Thus $\|f.g\| \leq \sum_{i,j=1}^{3} \|f_i\| . \|g_j\| = \left[\sum_{i=1}^{3} \|f_i\|\right] \left[. \sum_{j=1}^{3} \|g_j\|\right] = \|f\| . \|g\|$ □ Q.E.D

Proposition 1.2.9.

If $f : [a,b] \to \mathbb{R}$ *is a smooth function (continuously differentiable), then*

$$TV(f,[a,b]) = var(f,[a,b]) = V_{a,b}(f) = \int_a^b |f'(t)|\, dt.$$

Proposition 1.2.10. *[App]*

The strict inclusions in (1.5) hold, where the closure is taken in the norm (1.1).

$$\begin{aligned} S([a,\ b]) \ &\subsetneq\ BV([a,\ b]) \subsetneq R([a,\ b]) = \overline{S([a,b])} \\ \\ C^1\left([a,b]\right) \ &\subsetneq\ \overline{C^1\left([a,b]\right)} = H^1\left([a,b]\right) \subsetneq BV([a,\ b]) \end{aligned} \tag{1.5}$$

Theorem 1.2.21 (Theorem of Sierpiński)**.**

A function f belongs to $R([a,b])$ if and only if it can be represented as composition $f = g \circ \tau$, where $\tau : [a,b] \to [c,\ d]$ is strictly increasing and $g \in C([c,d])$

Definition 1.2.10.

The space $NBV[a,\ b]$ consists of all those $f \in BV[a,b]$ such that $f(a) = 0$ and f is continuous from right on $]a,\ b[$. Elements of $NBV[a,\ b]$ are called normalized functions of bounded variation on $[a,\ b]$.

Proposition 1.2.11.

$NBV[a,\ b]$ is a linear vector space and it becomes a normed space with the norm (1.6).

$$\phi \mapsto V(\phi) = \sup_{\mathcal{P}} \sum_{j=1}^{n} |\phi(t_j) - \phi(t_{j-1})|, \quad \phi \in NBV[a,b] \tag{1.6}$$

Theorem 1.2.22. *[Nai2]*

Let $C[a,\ b]$ be with $||.||_\infty$. For $\phi \in NBV[a,\ b]$, let

$$f_\phi(x) = \int_a^b x d\phi, \quad x \in C[a,\ b].$$

Then $f_\phi \in (C[a,\ b])'$ and the map $\phi \mapsto f_\phi$ is a linear isometry, from $NBV[a,\ b]$ onto the dual of $C[a,\ b]$.

Example 1.2.2.

If $[a,b] = [0,1]$, then the dual of $C[0,\ 1]$ is $BV[0,\ 1]$, which is isometric to $NBV[0,\ 1]$.

Chap 2: Wiener bounded p-variation space

In this chapter, we study the Wiener's Banach space $\mathcal{V}_p$ of bounded p-variation functions and their primitives, the Banach space BV_p of bounded p-variation distributions ($p \geq 1$). We also, study the Superposition Operator Problem (SOP) on Sobolev spaces $\mathbb{W}_p^m$ $(\mathbb{R}^n)$, $(n \geq 2)$ and on the space $BV_p^1(\mathbb{R})$ introduced by [BCS2] and [Mou].

Preliminaries

In general, all functions, distributions, etc. are defined on the Euclidean space $\mathbb{R}^n$.

- $\mathbb{Z}^n$ denotes the set of all lattice-points in $\mathbb{R}^n$
- Let A be a linear vector space. $\| a | A\|$ is said to be a quasi-norm if $\| a | A\||$ satisfies the usual conditions of a norm with exception of the triangle inequality, which is replaced by
$$\| a_1 + a_2 | A\| \leq c\,(\| a_1 | A\| + \| a_2 | A\|)$$
(here c does not depend on $a_1, a_2 \in A$).
- A quasi-normed space is said to be a quasi-Banach space if it is complete (i.e. any fundamental sequence in A with respect to $\| \cdot | A\|$ converges)
- By
$$\|a\|_1 \sim \|a\|_2 \,, \qquad a \in M$$
we indicate the existence of two constants $c_1 > 0$ and $c_2 > 0$ such that
$$c_1 \|a\|_1 \leq \|a\|_2 \leq c \|a\|_1$$
holds for all elements of M.
- If $\alpha = (\alpha_1, \ldots \alpha_n)$, then the partial derivative is defined by:
$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha n}} \text{ and } |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha_i \in \mathbb{N}_0, \quad i = 1, \ldots, n.$$

- Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\varphi \in S(\mathbb{R}^n)$. The topology is generated by the semi-norms

$$p_N(\varphi) = \sup_{x\in\mathbb{R}^n} (1+|x|)^N \sum_{|\alpha|\leq N} D^\alpha \varphi(x), \qquad N = 0,1,2,..., \;\; (N \in \mathbb{N}_0)$$

- Let $S'(\mathbb{R}^n)$ denote the set of all tempered distributions, i.e. the topological dual of $S(\mathbb{R}^n)$, equipped with the strong topology (if not otherwise stated). If $\varphi \in S(\mathbb{R}^n)$, then

$$\mathcal{F}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)\,dx, \qquad \xi \in \mathbb{R}^n, \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

($x\xi$ means the scalar product in $\mathbb{R}^n$) denotes the Fourier transform $\mathcal{F}\varphi$ of φ. The inverse transform $\mathcal{F}^{-1}\varphi$ of φ is given by

$$\mathcal{F}^{-1}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi}\varphi(x)\,dx, \qquad \xi \in \mathbb{R}^n, \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

- L_p is the space of the Lebesgue functions on $\mathbb{R}^n$ such that the norms (2.1) are finite:

$$\left.\begin{aligned} \|\, f|\, L_p\| &= \left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty, \qquad 0 < p < \infty \\ \|\, f|\, L_\infty\| &= \text{ess}\sup_{x\in\mathbb{R}^n} |f(x)| \end{aligned}\right\} \tag{2.1}$$

- $L_p^{\ell oc}$ is the space of the locally Lebesgue functions on $\mathbb{R}^n$ such that for all compact sets B the norm (2.2) is finite:

$$\int_B |f(x)|^p\,dx < \infty \tag{2.2}$$

- By C we denote the set of all complex-valued and uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup-norm.

$$\|f\|_\infty = \sup_{x\in\mathbb{R}^n} |f(x)|.$$

- If $m = 1, 2\ , \ .\ .\ .\ .\ ,$ we define the smooth regular functions:

$$C^m = \{f \in C : D^\alpha f \in C \ \text{ for all } \ |\alpha| \leq m\} \tag{2.3}$$

endowed with the norm

$$\|\, f |\, C^m\| = \sum_{|\alpha| \leq m} \|\, D^\alpha f |\, L_\infty\| \tag{2.4}$$

In (2.3) and (2.4) D^α means classical derivatives.

- Let $1 \leq p \leq \infty$ and $s \in \mathbb{N}$. Then we define the classic Sobolev space by

$$W_p^s = \left\{ f \in L_p : \quad \left\|\, f |\, W_p^s \right\| = \sum_{|\alpha| \leq s} \|\, D^\alpha f |\, L_p\| < \infty \right\}$$

- If s is a real number, then we put

$$s = [s]^- + \{s\}^+ \ \text{ with } \ [s]^- \text{ integer } \text{ and } 0 < \{s\}^+ \leq 1$$

- Slobodeckij spaces W_p^s. If $1 \leq p < \infty$, $0 < s \notin \mathbb{N}$, then

$$W_p^s = \left\{ f \in W_p^{[s]} : \quad \left\| f \left| W_p^s \right. \right\| = \left\| f \left| W_p^{[s]} \right. \right\| + \sum_{|\alpha| = [s]} \left(\iint \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n + \{s\} p}} dxdy \right)^{\frac{1}{p}} < \infty \right\}$$

- Bessel-potential spaces (or Sobolev spaces of fractional order) H_p^s.

  Let s be a real number and $1 < p < \infty$. Then

$$H_p^s = \left\{ f \in \mathcal{S}' : \quad \left\| f \left| H_p^s \right. \right\| = \left\| \mathcal{F}^{-1} \left(1 + |\xi|^2\right)^{\frac{s}{2}} \mathcal{F} f \right| L_p \right\| < \infty \right\}$$

Bounded p-variation functions and their primitives

2.2.1 The Banach space $\mathcal{V}_p$ of bounded p-variation functions

Definition 2.2.1.

Let $p \in [1, +\infty[$. Let I be an interval of $\mathbb{R}$. We consider the space $\mathcal{V}_p(I)$ of functions of bounded p-variation introduced by Wiener . A function $f : I \to \mathbb{R}$ belongs to $\mathcal{V}_p(I)$ if there exists $c > 0$ such that for all finite sequences $t_0 < t_1 < ... < t_N$ in I, we have:

$$\sum_{k=1}^{N} |f(t_k) - f(t_{k-1})|^p \leq c^p,$$

Remark 2.2.1.

- The infimum of such constants c is denoted by $\nu_p(f,\ I)$.
- We note that in the above definition, we could as well take $t_0 \leq t_1 \leq \cdots \leq t_N$.
- We use the abbreviations $\mathcal{V}_p = \mathcal{V}_p(\mathbb{R})$ and $\nu_p(f) = \nu_p(f,\ \mathbb{R})$.
- By considering a finite sequence with only two terms, we obtain

$$|f(x) - f(y)| \leq \nu_p(f,\ I) \ , \quad \text{for all} \quad x, y \in I.$$

Hence, every element of $\mathcal{V}_p(I)$ is a bounded function.

Proposition 2.2.1. *[BCS2]*

$\mathcal{V}_p(I)$ becomes a Banach space if endowed with the following norm

$$\|f\|_{\mathcal{V}_p(I)} = \sup_{x \in I} |f(x)| + \nu_p(f,\ I).$$

The right limit of $f \in \mathcal{V}_p(I)$ exists at each point of I which is not the right endpoint. Moreover,

$$\sum_{k=1}^{N} |f(t_k^+) - f(t_{k-1}^+)|^p \leq \nu_p(f,\ I)^p,$$

for all finite sequences $t_0 < t_1 < \cdots < t_N$ in I, such that t_N is not the right endpoint of I, and a corresponding property holds for left limits.

Definition 2.2.2.

If $\mathcal{V}_\infty(I)$ denotes the set of bounded functions of I to $\mathbb{R}$, having discontinuities only of the first kind, then $\mathcal{V}_\infty(I)$ is a Banach space for the norm:

$$\nu_\infty(f,\ I) = \sup_{x\in I} |f(x)|.$$

Definition 2.2.3.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function having discontinuities only of the first kind.

Then f is said to be normalized iff

$$f(x) = \frac{1}{2}(f(x^+) + f(x^-)) \ , \quad \text{for all } x \ \text{ in } \mathbb{R}.$$

Proposition 2.2.2.

Let $f : \mathbb{R} \to \mathbb{R}$ be a normalized function. If f vanishes almost everywhere, then f vanishes everywhere. Moreover, we have $f \in L^\infty(\mathbb{R})$ if and only if f is bounded.

If f is bounded, then

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$

Proposition 2.2.3.

Let $p \in [1,\ +\infty]$ and $f \in \mathcal{V}_p$. Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\tilde{f}(x) = \frac{1}{2}(f(x^+) + f(x^-))$$

for all $x \in \mathbb{R}$. Then the function $\tilde{f}$ is normalized, and belongs to $\mathcal{V}_p$, and satisfies the following inequality

$$\nu_p(\tilde{f}) \leq \nu_p(f) \ .$$

Proposition 2.2.4.

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Then:

- *There exists at most one normalized function $\tilde{f} : \mathbb{R} \to \mathbb{R}$, such that $f = \tilde{f} \quad (\mathbf{a.e})$.*
- *If $\tilde{f}$ exists and if $g : \mathbb{R} \to \mathbb{R}$ is a measurable function which equals to f almost everywhere, then $\tilde{g}$ exists and is equal to $\tilde{f}$.*

For further properties of functions of bounded p-variation, we refer to Bruneau [Bru].

2.2.2 Functions of bounded p-variation as distributions

Definition 2.2.4.

Let $p \in [1, +\infty]$. We denote by $\mathcal{BV}_p(\mathbb{R})$ the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that there exists a function $g \in \mathcal{V}_p$ which coincides with f almost everywhere, and we set

$$\varepsilon_p(f) = \inf\{\ \nu_p(g) : \quad g \in \mathcal{V}_p\ , \quad g = f \ \ (\mathbf{a.e}) \quad \} \qquad for \quad f \in \mathcal{BV}_p(\mathbb{R})\,.$$

Definition 2.2.5.

We denote by $BV_p(\mathbb{R})$ the quotient set of $\mathcal{BV}_p(\mathbb{R})$ modulo equality almost everywhere. If $h \in BV_p(\mathbb{R})$, then $\varepsilon_p(h) = \varepsilon_p(f)$, for any representative $f = h \ (\mathbf{a.e})$

Remark 2.2.2.

We denote by $BV(\mathbb{R})$ the set of all $f \in L^1_{\ell oc}(\mathbb{R})$ such that

$$\sup\left\{\int_{-\infty}^{\infty} f(x)\Phi'(x)dx : \quad \Phi \in \mathcal{D}(\mathbb{R})\,, \quad |\Phi(x)| \le 1 \quad \text{for all } x \in \mathbb{R}\right\} \tag{2.5}$$

is finite or, equivalently, such that the distributional derivative of f is a finite Borel measure. Then $BV_1(\mathbb{R}) = BV(\mathbb{R})$ and the number $\varepsilon_1(f)$ coincides with the expression (2.5) and with the total variation of the measure f'.

Proposition 2.2.5.

Let $p \in [1,\ +\infty]$*. If* $f \in BV_p(\mathbb{R})$*, then* f *has a unique normalized representative* $\tilde{f} \in \mathcal{V}_p$*. Moreover, we have* $\varepsilon_p(f) = \nu_p(\tilde{f})$ *.*

Remark 2.2.3.

$BV_p(\mathbb{R})$ is the Banach space of distributions, endowed with the following norm

$$\|f\|_{BV_p(\mathbb{R})} = \begin{cases} \varepsilon_p(f) + \|f\|_\infty = \nu_p(\tilde{f}) + \sup_{x\in\mathbb{R}} |\tilde{f}(x)| & \text{if} \qquad 1 \le p < \infty \\ \\ \sup\limits_{x\in\mathbb{R}} |\tilde{f}(x)| & \text{if} \qquad p = \infty \end{cases}$$

Definition 2.2.6.

Let $p \in [1,\ +\infty]$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ belongs to $BV_p^1(\mathbb{R})$ if f is Lipschitz continuous and if its distributional derivative belongs to $BV_p(\mathbb{R})$.

By classical properties of Lipschitz continuous functions and by Proposition 2.2.4, we derive the following proposition:

Proposition 2.2.6.

Let $p \in [1,\ +\infty]$*. Then the following statements hold.*

(i) If $h \in BV_p(\mathbb{R})$*, and* $\alpha \in \mathbb{R}$*, then the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by (2.6)*

$$f(x) = \alpha + \int_0^x h(t)dt, \qquad \forall x \in \mathbb{R} \tag{2.6}$$

belongs to $BV_p^1(\mathbb{R})$*.*

(ii) If $f \in BV_p^1(\mathbb{R})$*, then there exists an unique normalized function* $h \in \mathcal{V}_p$*, and a real number* α*, such that (2.6) holds. We endow* $BV_p^1(\mathbb{R})$ *with the norm*

$$\|f\|_{BV_p^1(\mathbb{R})} = |f(0)| + \|f'\|_{BV_p(\mathbb{R})} \qquad \textit{for all } f \in BV_p^1(\mathbb{R})$$

which renders $BV_p^1(\mathbb{R})$ *a Banach space.*

Composition Operator Problem

The origin of the non-linear Nemytskij operator applied to the theory of function spaces is linked to the partial differential equations. Here "function space" means a normed or quasi-normed space of functions or distributions defined on subsets of $\mathbb{R}^n$.

Definition 2.3.1.

1. Let X be some space of functions f: $[a,\ b] \to \mathbb{R}$, and let h: $\mathbb{R} \to \mathbb{R}$ be a fixed function. Under appropriate hypotheses on, h we may then define a nonlinear operator C_h on X by putting
$$C_h f(x) = h(f(x)) \qquad (a \leq x \leq b). \tag{2.7}$$
This operator is called the (autonomous) composition operator generated by h.

2. More generally if $h : [a,\ b] \times \mathbb{R} \to \mathbb{R}$ is a fixed function of two variables, one may also consider the (non-autonomous) operator
$$S_h f(x) = h(x,\ f(x)) \qquad (a \leq x \leq b). \tag{2.8}$$
In general, the behavior of the superposition operator (2.8) is far more complicated than that of the composition operator (2.7).

The most important problem related to the operator (2.7) is to find conditions on the function h both necessary and sufficient, under which the operator C_h generated by h maps a given function class X into itself.

Definition 2.3.2.

The composition operator problem (or COP for short), consists to find the set:

$$COP(X) = \{ \quad h : \quad C_h(X) \subseteq X \quad \} = \{ \quad h : \quad h \circ f \in X \ \text{ for all } \ f \in X \quad \} \tag{2.9}$$

The problem of determining the set COP(X) for given $X \subset \mathcal{F}([a,\ b], \mathbb{R})$.

For example, it is easy to see that COP(C) $= C$, which means that the operator (2.7) maps the space $C([a,\ b])$ into itself if and only if the corresponding function h is continuous.

Definition 2.3.3.

We say that X is COP-invariant if, whenever the operator (2.7) maps the space $X([a,\ b])$ into itself and is bounded in the norm of $X([a,\ b])$ for some interval $[a,\ b]$, it also maps the space $X([c,\ d])$ into itself and is bounded in the norm of $X([c,\ d])$.

Remark 2.3.1.

The usual way to prove COP-invariance is by considering the strictly increasing affine bijection:

$$P(t) = \frac{b-a}{d-c}(t-c) + a \qquad (c \le t \le d) \tag{2.10}$$

between $[c,\ d]$ and $[a,\ b]$ with inverse

$$P^{-1}(s) = \frac{d-c}{b-a}(s-a) + c \qquad (a \le s \le b) \tag{2.11}$$

Proposition 2.3.1. *[App]*

For $p \geq 1$ *and* $0 < \alpha < 1$, *we have*

$$COP(BV) \neq BV_{loc}(\mathbb{R}) \ , \qquad COP(WBV_p) \neq WBV_{p,loc}(\mathbb{R}) \ ,$$

$$COP(RBV_p) \neq RBV_{p,loc}(\mathbb{R}) \ , \qquad COP(AC) \neq AC_{loc}(\mathbb{R}) \ ,$$

$$COP(Lip_\alpha) \neq Lip_{\alpha,loc}(\mathbb{R}) \qquad (0 < \alpha < 1)$$

and

$$COP(BV) = COP(WBV_p) = COP(RBV_p) = COP(AC) = COP(Lip_\alpha) = Lip_{loc}(\mathbb{R}) \,.$$

Definition 2.3.4.

Let Ω be a domain in $\mathbb{R}^n$ and let $G(x, \xi)$ be a function defined for almost all $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$. We call G a Caratheodory function if and only if (a) and (b) hold

(a) For all $\xi \in \mathbb{R}^m$ the function $G(\cdot,\ \xi)$ is measurable on Ω.

(b) For almost all $x \in \Omega$ the function $G(x,\ \cdot)$ is continuous on $\mathbb{R}^n$

Proposition 2.3.2. *[RS]*

Let G be a Caratheodory function. Let $f_1, \dots , f_m$ be real-valued and measurable on Ω. Then the composite function $G\left(x, f_1(x), \dots, f_m(x)\right)$ is again a measurable function on Ω.

Definition 2.3.5.

Let Ω be a domain in $\mathbb{R}^n$ and $G(x,\ \xi) : \Omega \times \mathbb{R}^m \to \mathbb{C}$ be a Caratheodory function. Then

$$T_G\left(f_1,\ \dots,\ f_m\right)(x) = G(x,\ f_1(x),\ \dots,\ f_m(x))\ , \quad x \in \Omega$$

is called a Nemytskij operator.

Example 2.3.1.

- The nonlinear operator F

$$Fx(s) = f(s,\ x(s)) \tag{2.12}$$

associates to each function $x(s)$ on a compact domain of the metric space Ω the function $f(s,\ x(s))$ on Ω, is called a superposition operator.

- Apart from the operator (2.12), is the related composition operator

$$\Phi x(s) = x(\phi(s))$$

- The integral functional operator

$$\Phi x = \int_\Omega f(s,\ x(s))ds$$

is of fundamental importance in variational problems of nonlinear analysis.

Proposition 2.3.3. *[RS]*

Let $G : \mathbb{R}^m \longrightarrow \mathbb{R}$ *be a Caratheodory function. Let* $1 \leq r \leq p < n$. *Then*

$$T_G(f_1,\ \ldots,\ f_m) = G(f_1,\ \ldots,\ f_m) \tag{2.13}$$

maps $(\mathbb{W}_p^1)^m$ *into* W_r^1 *if and only if the following conditions hold:*

(i) $G(0) = 0$ *and* G *is locally Lipschitz continuous.*

(ii) The first order partial derivatives of G *satisfy the inequalities*

$$\frac{\partial G}{\partial \xi_i}(\xi)| \leq c(|\xi|^v + |\xi|^\mu) \qquad (\mathbf{a.e}). \quad i = 1,\ \ldots,\ m$$

where $$v = \frac{n(p-r)}{r(n-p)} \ \ \text{and} \ \ \mu = \frac{p-r}{r}$$

(iii) If $p > n$ $(\text{or } n = 1 \text{ and } p \geq 1)$ *and* $r = p$ *then* T_G *maps* $(\mathbb{W}_\rho^1)^m$ *into* W_r^1 *if and only if* (i) *holds. In either case, if* T_G *maps into then the mapping is bounded, too.*

Proposition 2.3.4 (Nemytskij operators in Sobolev spaces $W_p^1(\Omega)$)**.**

Suppose Ω *is a nontrivial bounded* $C^\infty-$ *domain. Let* $G : \mathbb{R}^m \to \mathbb{R}$ *be a Caratheodory function. Let* $1 \leq r \leq p < n$. *Let* T_G *defined as in (2.13), with the following conditions:*

(i) G *is locally Lipschitz continuous,*

(ii) There is a constant c *such that:* $\frac{\partial G}{\partial \xi_i}(\xi)| \leq c(1 + |\xi|^v) \qquad (\mathbf{a.e}).,\quad i = 1, \ldots, m,$

Then $\quad T_G\left[\mathbb{W}_p^1(\Omega))^m\right] \hookrightarrow W_r^1(\Omega) \iff (i) \ \ \text{and} \ \ (ii).$

Theorem 2.3.1.

.

(i) *Let* $G(0) = 0$. *Let* $1 \leq p < n$ *or* $p = n$ *and* $n \geq 2$. *Then*

$$T_G\left[W_p^1)^m\right] \hookrightarrow W_p^1 \iff G' \in L_\infty(\mathbb{R})$$

(ii) *Let* $G(0) = 0$. *Let* $p > n$ *or* $n = 1$ *and* $p \geq 1$. *Then*

$$T_G\left[(W_\rho^1(\Omega))^m\right] \hookrightarrow W_\rho^1(\Omega) \iff G' \in L_\infty^{loc}(\mathbb{R}).$$

Definition 2.3.6.

Let $A^s_{p,q}$ be either $F^s_{\rho,q}$ or $B^s_{p,q}$.

(a) We say $A^s_{\rho,q}$ is super-critical if $A^s_{p,q} \hookrightarrow L_\infty$.

(b) We say $A^s_{p,q}$ is sub-critical if $A^s_{p,q} \not\hookrightarrow L_\infty$.

Theorem 2.3.2.

Let $m \in \mathbb{N}$, $m \geq 2$, $1 \leq p < \infty$, $\mathbb{W}_p^m$ *be subcritical.*

Let $G : \mathbb{R} \to \mathbb{R}$ *be continuous and* $G(0) = 0$.

(i) *If* $m = n/p \geq 2$, *then*

$$T_G(\mathbb{W}_p^m) \hookrightarrow \mathbb{W}_p^m \iff G' \in W^{m-1}_{p,unif}(\mathbb{R}).$$

(ii) *Let* $1 < p < \infty$ *and* $2 \leq m < n/p$ *or* $p = 1$ *and* $3 \leq m < n$. *Then:*

$$T_G(\mathbb{W}_p^m) \hookrightarrow \mathbb{W}_p^m \iff G(t) = c.t, \quad t \in \mathbb{R} \ \ \textit{for some} \ \ c \in \mathbb{R}.$$

(iii) *If* $n \geq 3$, *then*

$$T_G(\mathbb{W}_1^2(\mathbb{R}^n)) \hookrightarrow \mathbb{W}_1^2(\mathbb{R}^n) \iff G'' \in L_1(\mathbb{R}).$$

Theorem 2.3.3.

Let $m \in \mathbb{N}$, $m \geq 2$, $1 \leq p < \infty$, $\mathbb{W}_p^m$ *be super-critical.*

Let $G : \mathbb{R} \to \mathbb{R}$ *be continuous and* $G(0) = 0$. *Then*

$$T_G(\mathbb{W}_p^m) \hookrightarrow \mathbb{W}_p^m \Longleftrightarrow G \in \mathbb{W}_p^{m,loc}(\mathbb{R}). \tag{2.14}$$

Theorem 2.3.4 (The main theorem on the COP in BV_p^1). *[BCS2]*

Let $1 \leq p < \infty$. *Then the following statements hold.*

(i) If $f, g \in BV_p^1(\mathbb{R})$, *then* $f \circ g \in BV_p^1(\mathbb{R})$ *and*

$$\|f \circ g\|_{BV_p^1(\mathbb{R})} \leq \|f\|_{BV_p^1(\mathbb{R})}(1 + 2^{1/p}\|g\|_{BV_p^1(\mathbb{R})})$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ *be a Borel measurable function. Then the operator* T_f *maps* $BV_p^1(\mathbb{R})$ *to itself if and only if* $f \in BV_p^1(\mathbb{R})$.

Remark 2.3.2.

- From the important theorem 2.3.4, we can deduce that **COP** $\left(BV_p^1(\mathbb{R})\right) = BV_p^1(\mathbb{R})$
- There is no common definition to $BV_p^1(\mathbb{R}^n)$, $(n \geq 2)$, in order to generalize the main Theorem 2.3.4. Nevertheless, we can generalize Wiener space by introducing the set $\mathcal{V}_p(\mathbb{R}^n)$ of the functions $g : \mathbb{R}^n \to \mathbb{R}$ such that

  $$\|g\|_{\mathcal{V}_p(\mathbb{R}^n)} = \sum_{j=1}^{n} (\int_{\mathbb{R}^{n-1}} \|g_{x'_j}\|^p_{BV_p^1(\mathbb{R})} dx_j'^{1/p} < +\infty,$$

  and so for all $f \in U_p^1(\mathbb{R})$ a class of of Lipschitz continuous functions satisfying $f(0) = 0$, the operator T_f takes $L_p(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$, where L_p is the Lebesgue space and $F^s_{p,q}$ is the Lizokin-Triebel space (Fore more detail about the subject see [Mou]).

$$f \in AC([a,b]), h \in Lip([c,d]) \Leftrightarrow h \circ f \in AC([a,b])$$

$$f \in BV([a,b]), h \in Lip([c,d]) \Leftrightarrow h \circ f \in BV([a,b])$$

$$g \in Lip([a,b]) \text{ and } \tau \text{monotone} \quad \Longrightarrow g \circ \tau \in BV([a,b])$$

$$g \in BV([a,b]) \text{ and } \tau \text{pseudomonotone} \quad \Longrightarrow g \circ \tau \in BV([a,b])$$

$$g \in C([a,b]) \text{ and } \tau \text{ strictly monotone} \quad \Longrightarrow g \circ \tau \in R([a,\ b])$$

$Diff:$	Continuously Differentiable	$UC:$	Uniformly Continuous
$LC:$	Lipschitz Continuous	$AC:$	Absolutely Continuous
$\alpha - HC:$	$\alpha-$Hölder Continuous	$C:$	Continuous
$BV:$	Bounded Variation	$Diff - a.e$	Differentiable almost everywhere
$B:$	Bounded	$R:$	Regular
C^1	Differentiable	C^n	n times derivable

$$\begin{array}{ccccc} L^p(\Omega) & \subset & L^p_{loc}(\Omega) & \subset & L^1_{loc}(\Omega) \\ \cup & & \cup & & \cup \\ W^{s,p}(\Omega) & \subset & W^{s,p}_{loc}(\Omega) & \subset & W^{1,p}_{loc}(\Omega) \\ \cup & & & & \\ W_0^{s,p}(\Omega) & & & & \end{array}$$

$$Diff \underset{\neq}{\subsetneq} LC \begin{array}{l} \nearrow \ \underset{\neq}{\subsetneq} HC \underset{\neq}{\subsetneq} UC \underset{\neq}{\subsetneq} C \\ \\ \searrow \ \underset{\neq}{\subsetneq} AC \underset{\neq}{\subsetneq} BV \underset{\neq}{\subsetneq} Diff - a.e. \end{array}$$

$$C_0^{(k)}(\Omega) \subset C^{(k)}(\Omega) \subset W^{1.1}(\Omega) \subset BV(\Omega) \subset L^1(\Omega)$$

$$\Omega = [0,1]^n \Rightarrow W^{1,1} = AC \cap L^1 \subset BV \subset L^1, \quad (n \geq 2)$$

$$\Omega = [0,1] \Rightarrow C^n \underset{\neq}{\subsetneq} C^1 \underset{\neq}{\subsetneq} Lip \underset{\neq}{\subsetneq} AC \underset{\neq}{\subsetneq} C \cap BV \underset{\neq}{\subsetneq} BV \underset{\neq}{\subsetneq} R \underset{\neq}{\subsetneq} B \cap L_\infty \underset{\neq}{\subsetneq} L_\infty \underset{\neq}{\subsetneq} L_p$$

$$I = [a,b] \subset \mathbb{R} \Longrightarrow Lip(I) \subset RBV_p(I) \subset BV(I) \subset R(I) \subset \ B(I)$$

$$\Longrightarrow Lip(I) \subset AC(I) \subset C(I) \subset \Lambda BV(I) \subset R(I) \subset \ B(I)$$

$$\Longrightarrow Lip(I) \subset AC(I) \subset BV(I) \subset \Lambda BV(I) \subset R(I) \subset \ B(I)$$

Figure 2.1:

Chap 3: Ashton-Dost plane $BV(\sigma, \mathbb{C})$ space

In this chapter, we present a two-dimensional bounded variation space defined by Ashton-Doust in [AD1] and an isomorphism theorem given by Al-shakarchi & Doust in [ASD], as an application to the famous Theorem of Banach-Stone.

Throughout this chapter, we denote by $\mathbb{F}$ the field $\mathbb{R}$ of the real numbers or the field $\mathbb{C}$ of the complex numbers. We denote by σ a nonempty compact subset of $\mathbb{C} \cong \mathbb{R}^2$, where $\cong$ denotes an isometric isomorphism, we shall denote an isomorphism by $\simeq$.

Linear Transformations.

Definition 3.1.1.

If (E_1, d_1) and (E_2, d_2) are two metric spaces, then

(a) A function $f : E_1 \longrightarrow E_2$ is said to be continuous at $u \in E_1$ iff

$$\forall \varepsilon > 0,\ \exists \delta > 0, \forall x \in E_1, \quad d_1(x, u) < \delta \Rightarrow d_2(f(x), f(u)) < \varepsilon$$

(b) f is continuous, if it is continuous at every $u \in E_1$.

(c) If δ is independent of u, then f is said to be a uniformly continuous.

(d) f is a homeomorphism iff f is bijective and continuous.

(e) A homeomorphism f is an isometry iff,

$$d_2(f(x), f(y)) = d_1(x, y), \qquad \forall x, y \in E_1.$$

Definition 3.1.2.

Two norms $\|\cdot\|_X$ and $\|\cdot\|'_X$ on a normed space X are called equivalent if there exist constants M, $m > 0$, such that,

$$m\,\|u\|_X \leq \|u\|'_X \leq M\,\|u\|_X \qquad (u \in X)\,.$$

Definition 3.1.3.
The normed space $(X,\ \|\cdot\|_X)$ is embedded into the normed space $(Y,\ \|\cdot\|_Y)$ iff

$$X \subseteq Y \quad \textit{and} \quad \forall u \in X,\ \ \|u\|_Y \leq c\|u\|_X \tag{3.1}$$

for some constant $c > 0$ independent of u.

In this case, we write $X \hookrightarrow Y$ and call c in (3.1) an embedding constant. $c(X,\ Y) = \sup\{\|u\|_Y\ :\ \|u\|_X \leq 1\}$, is the smallest possible embedding constant, and it is called the sharp embedding constant for $X \hookrightarrow Y$.

Remark 3.1.1. if $X \subseteq Y$ and by (3.1) , then $u_n \to u$ in X implies $u_n \to u$ in Y.

Definition 3.1.4.

A normed linear space $(X,\ \|\cdot\|_X)$ is called an Algebra if the product of two elements $u, v \in X$ also belongs to X and satisfies

$$\|uv\|_X \leq c\|u\|_X\|v\|_X \quad (u,\ v \in X),$$

for some constant $c > 0$ independent of f and g.

If $(X,\ \|\cdot\|_X)$ is complete, X is called a Banach Algebra.

We may take $c = 1$, in this case, we call $(X, \|.\|_X)$ a normalized Algebra.

Proposition 3.1.1.

The linear space $B([a,b])$ of all bounded functions $f : [a,b] \to \mathbb{R}$ is a normalized Banach Algebra with the norm

$$\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|. \tag{3.2}$$

Proposition 3.1.2. *[App]*

Let $(X,\ \|\cdot\|_X)$ be a Banach Algebra of functions $f:[a,b]\to\mathbb{R}$ which satisfies

$$X\hookrightarrow B([a,b]),\quad \textit{and}\quad \|fg\|_X\le\|f\|_\infty\|g\|_X+\|f\|_X\|g\|_\infty,\quad (f,\ g\in X)$$

where $\|\cdot\|_\infty$ is defined by (3.2). Then the space X equipped with the norm

$$||f||'_X=\|f\|_\infty+||f||_X\qquad (f\in X)$$

is a normalized Banach Algebra, i.e.

$$||fg||'_X\le||f||'_X\,||g||'_X\qquad (f,\ g\in X).$$

Moreover in case $X\hookrightarrow B([a,b])$ both norms $\|\cdot\|_X$ and $||\cdot||'_X$ are equivalent.

Definition 3.1.5.

Let V and W be vector spaces over the same field $\mathbb{F}$.

(a) A function $T:\ V\to W$ is said to be a linear transformation from V to W if it satisfies the following two conditions,

  (i) For all $x,y\in V$, $T(x+y)=T(x)+T(y)$. (Additivity)

  (ii) For all $x\in V$ and for all $\alpha\in\mathbb{F}$, $T(\alpha x)=\alpha T(x)$. (Homogeneity)

(b) A linear transformation from V to $\mathbb{F}$ is called a linear functional on V.

(c) A linear transformation from V to itself is called a linear operator on V.

Remark 3.1.2.

- The conditions (ai), (aii) in Definition 3.1.5 are equivalent to:

$$T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)\ \text{ for all } x,y\in V \text{ and for all }\ \alpha,\beta\in\mathbb{F}.$$

- Linear transformations are also called: linear maps, linear mappings, linear operators, homomorphisms.....etc
- A linear operator on a vector space is also called an endomorphism.

Example 3.1.1.

(1) For a given $t \in [a,\ b]$, let $f_t\ :\ C([a,\ b],\ \mathbb{F}) \rightarrow \mathbb{F}$ be defined by

$$f_t(x) = x(t)\ , \qquad for\ \ x \in C([a,\ b],\ \mathbb{F}).$$

Then f_t is a linear functional.

The functionals f_t for $t \in [a,\ b]$, are called evaluation functionals on $C[a,\ b]$.

(2) Given $t_1, \dots, t_n \in [a,\ b]$, $w_1, \dots w_n \in \mathbb{F}$, and $f\ :\ C([a,\ b],\ \mathbb{F}) \rightarrow \mathbb{F}$ by

$$f(x) = \sum_{i=1}^{n} x(t_i) w_i \quad for \quad x \in C([a,\ b],\ \mathbb{F}).$$

Then f is a linear functional called a quadrature formula.

(3) Let $f\ :\ C([a,\ b],\ \mathbb{F}) \rightarrow \mathbb{F}$ be defined by

$$f(x) = \int_a^b x(t)dt \quad for \quad x \in C([a,\ b],\ \mathbb{F}).$$

Then f is a linear functional.

(4) Define $T\ :\ C^1([a,\ b],\ \mathbb{R}) \rightarrow C([a,\ b],\ \mathbb{R})$ by

$Tx = x'\ \ for \quad x \in C^1([a,\ b],\ \mathbb{R})$, where x' denotes the derivative of x.

Then T is a linear transformation.

(5) Let $\alpha, \beta \in \mathbb{F}$. Define the function $T\ :\ C^1([a,\ b],\ \mathbb{R}) \rightarrow C([a,\ b],\ \mathbb{R})$ by

$$Tx = \alpha x + \beta x', \quad for \quad x \in C^1([a,\ b],\ \mathbb{R}).$$

Then T is a linear transformation.

(6) Let $T\ :\ C([a,\ b],\ \mathbb{R}) \rightarrow C([a,\ b],\ \mathbb{R})$ be defined by

$$(Tx)(s) = \int_a^s x(t)dt\ for\ x \in C([a,\ b],\ \mathbb{R}), s \in [a,\ b].$$

Then T is a linear transformation.

Definition 3.1.6.

Let be a topological vector space, we associate to each $a \in X$ and to each scalar $\lambda \neq 0$, the translation operator T_a and the multiplication operator M_λ by,

$$T_a(x) = a + x, \;\; M_\lambda(x) = \lambda x \qquad (x \in X) \; .$$

Proposition 3.1.3. *[Rud]*

T_a and M_λ are homeomorphisms from X onto X.

(Recall that a hemomorphism is a continuous mapping whose inverse is also continuous)

Theorem 3.1.1.

Let $T : V \rightarrow W$ be a linear transformation. Then T is bijective if and only if there exists a unique linear transformation $S : W \rightarrow V$, such that $ST = I_V$ and $TS = I_W$.

Definition 3.1.7.

A linear transformation $T : V \rightarrow W$ is said to be invertible if it is bijective.

In that case, the unique linear transformation $S : W \rightarrow V$ satisfying $ST = I_V$ and $TS = I_W$ is called the inverse of T, it is denoted by T^{-1}.

Definition 3.1.8.

A bijective linear transformation is called an isomorphism.

A vector space V is said to be isomorphic to a vector space W, written as $V \simeq W$, if there exists an isomorphism from V to W.

Theorem 3.1.2.

Let $T : V \rightarrow W$ be an isomorphism.

(i) If $\alpha \neq 0$ is a scalar, then αT is an isomorphism and $(\alpha T)^{-1} = (1/\alpha)T^{-1}$.

(ii) $T^{-1} : W \rightarrow V$ is an isomorphism and $(T^{-1})^{-1} = T$.

(iii) If $S : U \rightarrow V$ is an isomorhism, then so, for $TS : U \rightarrow W$ with, $(TS)^{-1} = S^{-1}T^{-1}$.

Remark 3.1.3.

- The composition of isomorphisms is an isomorphism.

  Isomorphism's relation is an equivalence one, on the collection of vector spaces.

- This relation is completely characterized by the dimensions of the spaces, in case the spaces are finite dimensional.

Theorem 3.1.3.

Let V and W be finite dimensional vector spaces over the same field $\mathbb{F}$. Then

$$V \simeq W \quad \textit{if and only if} \quad \dim(V) = \dim(W).$$

Definition 3.1.9.

An operator T on a finite dimensional inner product space V is called isometric iff

$$\|Tx\| = \|x\| \text{ for every } x \in V,$$

Remark 3.1.4.

- An isometric operator is also called an isometry.

- An isometry on a real inner product space need not have an eigenvalue.

Example 3.1.2.

The linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(a,\ b) = (b,\ -a)$ is an isometry, since

$$\|T\,(a,\ b\ )\| = \|(b,\ -a)\| = (b^2 + a^2)^{1/2} = \|(a,\ b)\|\,,$$

and it doesn't have any eigenvalue.

Isomorphisms between $BV(\sigma,\mathbb{C})$ and $AC(\sigma,\mathbb{C})$

In this section we present a definition of two variable bounded variation functions on $\mathbb{R}^2$, introduced by Ashton-Doust ([AD1]). The spaces related to this definition are used to explain the Banach-Stone and Gelfand-Kolmogorov theorems.

Definition 3.2.1.

The Banach Algebras A and B are said to be isomorphic, and we write $A \simeq B$, if there exists a linear and multiplicative bijection T: $A \to B$ such that T and T^{-1} are continuous. If $\|Tx\| = \|x\|$, for all $x \in A$, then they are isometric, i.e $A \cong B$.

Theorem 3.2.1 (Banach (1932), Stone (1937))**.**

Let K and L be normed compact spaces. with the norm

$$\|f\|_\infty = \sup\{|f(x), \quad x \in K\}.$$

Then $C(K) \cong C(L)$ if and only if K and L are homeomorphic.

Theorem 3.2.2 (Gelfand and Kolmogorov, 1939)**.**

Let $C(X)$ be the Algebra of continuous functions on a compact Hausdorff space X.

Then there is a one-to-one correspondence between the Algebra homomorphisms on $C(X)$ and the points of X, (All homomorphisms are the evaluation homomorphisms at $x \in X$).

Theorem 3.2.3 (Gelfand and Kolmogorov, modern version)**.**

If Ω_1 and Ω_2 are compact Hausdorff topological spaces, then (a), (b), (c) are equivalent.

- *(a) Ω_1 is homeomorphic to Ω_2.*
- *(b) $C(\Omega_1)$ and $C(\Omega_2)$ are linearly isometric as Banach spaces.*
- *(c) $C(\Omega_1)$ and $C(\Omega_2)$ are isomorphic as Algebras (or as C^*-Algebras).*

Definition 3.2.2.

Let σ be a nonempty compact subset of $\mathbb{R}$. Let $J=[a,\ b]$ be the smallest interval which contains σ. We say $\{s_i\}_{i=1}^n$ is a partition of σ if $s_1 \leq s_2 \leq \cdots \leq s_n$ and for all $i, s_i \in \sigma$. The set of partitions of σ is denoted by $\Lambda(\sigma)$.

Definition 3.2.3.

Let $S=\{s_i\}_{i=1}^n, T=\{t_i\}_{i=1}^m \in \Lambda(\sigma)$. The set T is said to be a *refinement* of S if $S \subset T$. Then $\Lambda(\sigma)$ is a lattice using refinement as a partial ordering.

Definition 3.2.4 ($BV(\sigma)$ for $\sigma \subset \mathbb{R}$ Compact)**.**

Let σ be a nonempty compact of $\mathbb{R}$. For $f : \sigma \mapsto \mathbb{C}$ we define the *variation* of f by

$$var(f,\ \sigma)=\sup_{\{s_i\}_{i=1}^n \in \Lambda(\sigma)} \sum_{i=1}^{n-1} |f(s_{i+1})-f(s_i)|.$$

Set

$$\|f\|_{BV(\sigma)}=\|f\|_1+\mathrm{var}(f,\ \sigma)\ .$$

The set of functions of bounded variation is

$$BV(\sigma)=\{f:\sigma\to\mathbb{C}: \quad \|f\|_{BV(\sigma)}<1\}.$$

Proposition 3.2.1.

Let $J=[a,b]$ be the smallest interval which contains σ. Then $BV(J) \hookrightarrow BV(\sigma)$.

Remark 3.2.1.

For many of the properties of $BV(\sigma)$, it is easier to embed $BV(\sigma)$ into $BV(J)$ and then use the classical theory. For $t \in J\backslash\sigma$, define

$$\alpha(t)=\sup\{x:[t,\ x]\subset J\backslash\sigma\ \} \quad \textit{and} \quad \beta(t)=\inf\{x\ :\ [x,\ t]\subset J\backslash\sigma\}$$

Definition 3.2.5.

Given $f : \sigma \to \mathbb{C}$, define the function $\varphi(f) : J \to \mathbb{C}$ by

$$\varphi(f)(t) = \begin{cases} f(t) & \text{if } t \in \sigma, \\ \left(\dfrac{f(\alpha(t)) - f(\beta(t))}{\alpha(t) - \beta(t)}\right)(t - \beta(t)) + f(\beta(t)) & \text{if } t \in J \backslash \sigma. \end{cases}$$

In other words, $\varphi(f)$ is defined so that it is linear on the gaps in σ.

Proposition 3.2.2.

Let $\sigma_1 \subset \sigma_2$ be compact subsets of $\mathbb{R}$ and let $f \in BV(\sigma_2)$. Then

$$\|f|\sigma_1\|_{BV(\sigma_1)} \leq \|f\|_{BV(\sigma_2)} \quad \textit{and} \quad f|\sigma_1 \in BV(\sigma_1).$$

Proposition 3.2.3.

Let $f \in BV(\sigma)$. Then $var(f,\ \sigma) = var(\varphi(f),\ J)$.

Proposition 3.2.4.

Let $f : \sigma \to \mathbb{C}$. Then $f \in BV(\sigma)$ if and only if $\varphi(f) \in BV(J)$.

Proposition 3.2.5.

The map φ: $BV(\sigma) \to BV(J)$ is a linear isometry.

Definition 3.2.6.

By a curve in the plane we shall mean an element of the set $\Gamma = C([0,1])$.

Remark 3.2.2.

Note that it will sometimes be important to distinguish between a curve (which includes its parameterization in $\mathbb{R}^2$) and its image in $\mathbb{C}$.

Notation 3.2.1.

- If $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1(1) = \gamma_2(0)$, let $\gamma_1 \circ \gamma_2 \in \Gamma$ be defined by:

$$(\gamma_1 \circ \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if} \quad 0 \leq t < \underline{1} \\ \gamma_2(2t-1) & \text{if} \quad \frac{1}{2} < t \leq 1. \end{cases}$$

- If $\gamma_1, \gamma_2 \in \Gamma$ and if there exists $h : [0,1] \to [0,1]$, where h is a continuous non-decreasing surjective function such that, $\gamma_1(t) = \gamma_2(h(t))$ for all $t \in [0,1]$, then we write $\gamma_1 \cong \gamma_2$.

Definition 3.2.7 (Variation in Two Variables)**.**

Let $f : \sigma \mapsto \mathbb{C}$ and let $\gamma \in \Gamma$, we denote the variation along the curve γ by

$$\mathbf{cvar}(f,\ \gamma,\ \sigma) = \mathbf{cvar}(f,\ \gamma) = \sup_{\{z_i\}_{i=1}^{n} \in \Lambda(\sigma,\gamma)} \sum_{i=1}^{n-1} |f(z_{i+1}) - f(z_i)|,$$

such that the value of the curve variation is $\mathbf{cvar}\ (f,\ \gamma) = var(\varphi(f \circ \gamma),\ [0,1])$.

Proposition 3.2.6. *[AD1]*

Let $\sigma_1 \subset \sigma \subset \mathbb{C}$ both be compact. Let $f, g : \sigma \to \mathbb{C}$ and let $k \in \mathbb{C}$.

Suppose $\gamma = \gamma_1 \circ \gamma_2 \in \Gamma$ with $\gamma_1(1) \in \sigma$. Then

(a) $\mathbf{cvar}\ (f+g,\ \gamma) \leq \mathbf{cvar}\ (f,\ \gamma) + \mathbf{cvar}(g,\ \gamma)$,

(b) $\mathbf{cvar}\ (fg,\ \gamma) \leq \|f\|_1\ \mathbf{cvar}\ (g,\ \gamma) + \|g\|_1\ \mathbf{cvar}\ (f,\ \gamma)$,

(c) $\mathbf{cvar}\ (kf,\ \gamma) = |k|\mathbf{cvar}(f,\ \gamma)$,

(d) $\mathbf{cvar}\ (f,\ \gamma) = \mathbf{cvar}(f,\ \gamma_1) + \mathbf{cvar}(f,\ \gamma_2)$,

(e) $\mathbf{cvar}\ (f,\ \gamma_1) \leq \mathbf{cvar}(f,\ \gamma)$,

(f) $\mathbf{cvar}\ (f,\ \gamma,\ \sigma_1) \leq \mathbf{cvar}(f,\ \gamma,\ \sigma)$.

Lemma 3.2.1.

Let $f:\sigma\to\mathbb{C}$. *Let* $\gamma_1,\gamma_2\in\Gamma$ *and suppose that* $\gamma_1\cong\gamma_2$.*Then*

$$\mathbf{cvar}(f,\ \gamma_1)=\mathbf{cvar}(f,\ \gamma_2).$$

Definition 3.2.8.

We assign to each curve γ a weight factor $\rho(\gamma)\in[0,1]$

Remark 3.2.3.

The weight factor is large for straight lines and low for very sinuous ones.

The two dimensional variation is then defined as the supremum of $\rho(\gamma)\mathrm{cvar}(f,\ \gamma)$ over all curves γ

Definition 3.2.9.

Let $f:\sigma\to\mathbb{C}$. Then the variation of f on σ is defined to be

$$var(f,\ \sigma)=\sup_{\gamma\in\Gamma}\rho(\gamma)\mathbf{cvar}(f,\ \gamma).$$

Here we take the convention that if $\gamma\in\Gamma$ is such that $\rho(\gamma)=0$, and if **cvar** $(f,\ \gamma)=\infty$, then $var(f,\ \sigma)=0$.

Proposition 3.2.7.

Let $\sigma_1\subset\sigma\subset\mathbb{C}$ *both be compacts. Let* $f,g:\sigma\to\mathbb{C},\ \ k\in\mathbb{C}$. *Then*

(a) $var(f+g,\ \sigma)\le var(f,\ \sigma)+var(g,\ \sigma)$.

(b) $var(fg,\ \sigma)\le\|f\|_1 var(g,\ \sigma)+\|g\|_1 var(f,\ \sigma)$.

(c) $var(kf,\ \sigma)=|k|var(f,\ \sigma)$.

(d) $var(f,\ \sigma_1)\le var(f,\ \sigma)$.

Definition 3.2.10 ($BV(\sigma)$ for $\sigma \subset \mathbb{C}$ Compact)**.**

For $f : \sigma \to \mathbb{C}$, set $\|f\|_{BV(\sigma)} = \|f\|_1 + var(f,\ \sigma)$. The functions of bounded variation with domain σ are defined to be

$$BV(\sigma) = \{f : \sigma \mapsto \mathbb{C} : \|f\|_{BV(\sigma)} < 1\}.$$

If f is continuous,we define the length of the curve f as its total variation.

Theorem 3.2.4.

$(BV(\sigma,\mathbb{C}),\ \|\cdot\|_{BV(\sigma)})$ is a Banach Algebra.

Theorem 3.2.5.

Suppose that σ_1 and σ_2 are nonempty compact subsets of the plane.

If $T : AC(\sigma_1) \to AC(\sigma_2)$ is an isomorphism then there exists a homeomorphism

$h : \sigma_1 \to \sigma_2$, such that $T(f) = f \circ h^{-1}$ for all $f \in AC(\sigma_1)$, furthermore T is continuous.

This theorem is not true for $BV(\sigma)$ spaces.

Example 3.2.1.

Let $\sigma_1 = \sigma_2 = [0,1]$ and define $h : \sigma_1 \to \sigma_2$,

$$h(x) = \begin{cases} \frac{1}{2} - x, & \text{if} \quad 0 \leq x \leq \frac{1}{2}, \\ x, & \text{if} \quad \frac{1}{2} < x \leq 1. \end{cases}$$

From the variation of $T_h(f)$ over any partition $\mathcal{P}$ of $[0,1]$, we have

$$V(\mathcal{P}, T_h(f)) \leq 2var(f,[0,1]).$$

Because of $T_h^{-1} = T_h$, we deduce

$$\frac{1}{2}\|f\|_{BV[0,1]} \leq \|T(f)\|_{BV[0,1]} \leq 2\|f\|_{BV[0,1]}.$$

Thus T_h is a Banach Algebra isomorphism from $BV(\sigma_1)$ to $BV(\sigma_2)$.

But the map h is not a homeomorphism in this case.

Theorem 3.2.6.

Suppose that σ_1 *and* σ_2 *are non-empty compact subsets of the plane.*

If $T : BV(\sigma_1) \to BV(\sigma_2)$ *is an isomorphism then there exists a bijection* $h : \sigma_1 \to \sigma_2$*, such that* $T(f) = f \circ h^{-1}$*, for all* $f \in BV(\sigma_1)$*.*

Notation 3.2.2.

Let $\mathbb{P}_2$ denote the space of complex polynomials in two real variables of the form

$$p(x,\ y) = \sum_{n,m} c_{nm} x^n y^m,$$

and let $\mathbb{P}_2(\sigma)$ denote the restrictions of elements of $\mathbb{P}_2$ to σ (now considered as a subset of $\mathbb{R}^2$). $\mathbb{P}_2(\sigma)$ is always a sub-algebra of $BV(\sigma)$.

Definition 3.2.11.

The set of absolutely continuous functions on σ, denoted $AC(\sigma)$, is the closure of $\mathbb{P}_2(\sigma)$ in $BV(\sigma)$, thus $AC(\sigma)$ always forms a closed sub-algebra of $BV(\sigma)$, since the BV norm agrees with the classical one if $\sigma \subseteq \mathbb{R}$.

Proposition 3.2.8.

If $AC(\sigma_1)$ *is isomorphic to* $AC(\sigma_2)$ *as Algebras, then the isomorphism must necessarily be bi-continuous, and so the spaces are isomorphic as Banach Algebras.*

Furthermore, if this is the case then σ_1*must be homeomorphic to* σ_2*.*

Proposition 3.2.9.

The $AC(\sigma)$ *spaces for the closed unit disk and a closed square in* $\mathbb{C}$ *are not isomorphic.*

Proposition 3.2.10.

The Banach Algebra of the $BV(\sigma)$ *spaces is invariant under homeomorphisms.*

Theorem 3.2.7.

If $h:\mathbb{C}\rightarrow\mathbb{C}$ *is an invertible affine transformation of the plane, then* $AC(h(\sigma))$ *is isometrically isomorphic to* $AC(\sigma)$*, via the Algebra isomorphism* $T(f)=f\circ h^{-1}$

Theorem 3.2.8.

We have the inclusion, Lip $(\sigma)\subseteq BV(\sigma)$*. Indeed,* $var(f,\sigma)\leq L(f,\sigma)C_\sigma$*, for* $f\in Lip(\sigma)$*.*

Theorem 3.2.9.

Suppose that σ *is a nonempty compact subset of the plane, and that h is locally piece-wise affine map. Then we have* $AC(\sigma)\simeq AC(h(\sigma))$ *and* $BV(\sigma)\simeq BV(h(\sigma))$*.*

Theorem 3.2.10.

Suppose that σ_1 *and* σ_2 *are non-empty compact subsets of the plane.*

If $T:BV(\sigma_1)\rightarrow BV(\sigma_2)$ *is an isomorphism then there exists a bijection* $h:\sigma_1\rightarrow\sigma_2$*, such that* $T(f)=f\circ h^{-1}$ *for all* $f\in BV(\sigma_1)$ *.*

Lemma 3.2.2.

Suppose that $h:\sigma_1\rightarrow\sigma_2$ *is a homeomorphism.*

If T_h *is an isomorphism from* $AC(\sigma_1)$ *to* $AC(\sigma_2)$*, then* $h\in AC(\sigma_1)$ *and* $h^{-1}\in AC(\sigma_2)$*.*

Example 3.2.2.

Let $\sigma_1=\sigma_2=[0,1]$, and $h:\sigma_1\rightarrow\sigma_2$ be an increasing continuous bijection which is not absolutely continuous, defined by, $h(x)=\dfrac{1}{2}\left(x+C(x)\right)$, where C is the Cantor function. Then T_h is an isomorphism from $BV(\sigma_1)$ to $BV(\sigma_2)$, but not an isomorphism from $AC(\sigma_1)$ to $AC(\sigma_2)$.

Theorem 3.2.11.

Suppose that $h:\sigma_1\rightarrow\sigma_2$ *is a homeomorphism such that* $h^{-1}\in AC(\sigma_2)$*.*

Then for any $p\in\mathbb{P}_2(\sigma_1)$*,* $T_h(p)\in AC(\sigma_2)$*.*

Theorem 3.2.12.

Suppose that M *is a projectable convex curve in* $\mathbb{R}^2$*, then* $AC(M) \simeq AC[0,1]$.

Theorem 3.2.13.

Suppose that $h : \sigma_1 \to \sigma_2$ *is a homeomorphism such that* $h^{-1} \in AC(\sigma_2)$*.*

Then for any $p \in \mathbb{P}_2(\sigma_1)$*,* $T_h(p) \in AC(\sigma_2)$*.*

Theorem 3.2.14 (Isomorphisms between $AC(\sigma)$ and $BV(\sigma)$ spaces)**.**

1. *Suppose that* $AC(\sigma_1) \simeq AC(\sigma_2)$*. Then* $BV(\sigma_1) \simeq BV(\sigma_2)$*.*
2. *If* $BV(\sigma_1) \simeq BV(\sigma_2)$ *and* $h : \sigma_1 \to \sigma_2$ *is a homeomorphism, then* T_h *is an isomorphism from* $AC(\sigma_1)$ *to* $AC(\sigma_2)$*, if and only if* h *and* h^{-1} *are absolutely continuous functions.*

Conclusion

- Ashton-Doust definition of bounded variation functions on the complex plane does match with the usual definition on the real line. The Isomorphisms theorems between $AC(\sigma, \mathbb{C})$ and $BV(\sigma, \mathbb{C})$ spaces are used to study the behavior of the well bounded operators in spectral theory and nonlinear functional analysis.
- The two propositions 3.2.9 and 3.2.10 explain the topological nature of $BV(\sigma, \mathbb{C})$. The theorem 3.2.14 explain easily the topological classic Theorems: Banach (1932) & Stone (1937) theorems in linear normed spaces and Gelfand-Kolmogorov's (1939) theorem in more general Hausdorff metric spaces.

Bounded variation Spaces (1881-2016)

$BV([a,b])$	$V(f,[a,b])=\sum_{i=1}^{n}\lvert f(t_i)-f(t_{i-1})\rvert$	The Jordan's Bounded Variation (1881)
$H([a_1,a_2]\times[b_1,b_2])$	$f\in BV_H \Leftrightarrow f(x,.)\in BV([a_1,a_2]),\, f(.,y)\in BV([b_1,b_2])$ $V(f,I_a^b)=\sup_{(\xi,\eta)}\sum_{i,j=1}^{n,m}\lvert\triangle_{11}f(t_{i+1},\ s_{j+1})<\infty$	The Hardy-Klaus variation (1903)
$A([a_1,a_2]\times[b_1,b_2])$	$V(f,I_a^b)=\sup_{(\xi,\eta)}\sum_{i}^{n}\lvert\triangle f(t_{i+1},\ s_{j+1})\rvert<\infty$	Arzelà and Hardy (1905)
$V([a_1,a_2]\times[b_1,b_2])$	$V(f,I_a^b)=\sup_{(\xi,\eta)}\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\lvert\triangle_{11}f(t_{i+1},\ s_{j+1})\rvert$	Vitali-Lebesgue-Fréchet-de la Vallée Poussin (1908)
$RBV_p([a,b])$	$V_p^R(f,[a,P;b])=\sum_{i=1}^{n}\frac{\lvert f(t_i)-f(t_{i-1})\rvert^p}{(t_i-t_{i-1})^{p-1}},\ p\geq 1$	The Riesz variation (1910)
$WBV_p([a,b])$	$V_p^W(f,[a,P;b])=\sum_{i=1}^{n}\lvert f(t_i)-f(t_{i-1})\rvert^p,\ \ p=2$	The Wiener variation (1924)
$T([a_1,a_2]\times[b_1,b_2])$	$V_T(f)=\int_{-\infty}^{\infty}TV(f(\cdot,y))\,dy+\int_{-\infty}^{\infty}TV(f(x,\cdot))\,dx$ $V_{TC}(f)=\inf\{V_T(g):g=f,\quad \lambda\text{-}(\mathbf{a.e}).\}\ .$	Tonelli and Cesari (1936)
$WBV_\phi([a,b])$	$Var_\phi^W(f,P;[a,b])=\sum_{i=1}^{n}\phi(\lvert f(t_i)-f(t_{i-1})\rvert)$ $\phi=\lvert\cdot\rvert^p,\ \ p\neq 2$	The Wiener-Young variation (1937)
$RBV_\phi([a,b])$	$Var_\phi^R(f,P;[a,b])=\sum_{i=1}^{n}\frac{\lvert f(t_i)-f(t_{i-1})\rvert^p}{(t_i-t_{i-1})^{p-1}}$	Riesz-Medvedev (1953)
$SBV(\Omega)$	$f\in BV(\Omega)$ and $\lVert Df\rVert_{L^\infty(\Omega)}<\infty,\ \Omega=[a_1,a_2]\times[b_1,b_2]$	De Giorgi and Fichera (1954)
$\kappa BV([0,1])$	$Var_\kappa(f,P;[0,1])=\frac{\sum_{i=1}^{n}\lvert f(t_i)-f(t_{i-1})\rvert}{\sum_{i=1}^{n}\kappa(t_i-t_{i-1})}$	The Korenblum variation (1975)
$\Phi BV([a,b])$	$\sum_{i=1}^{\infty}\Phi_k(\lvert f(b_k)-f(a_k)\rvert)$	Schramm (1985)
$\kappa BV_\Phi([a,b])$	$[a,b]=\bigcup_n I_n\Rightarrow \kappa V_\Phi(f\ )=\sup\frac{\sum_{i=1}^{n}\Phi_n(\lvert f(I_n)\rvert)}{\sum_{i=1}^{n}\kappa\left(\frac{\lvert I_n\rvert}{b-a}\right)}$	S. Ki Kim and J. Kim variation (1986)
$WBV_{p(\cdot)}([a,b])$	$V_{p(\cdot)}^W(f,P;[a,b])=\sum_{i=1}^{n}\lvert f(t_i)-f(t_{i-1})\rvert^{p(x_{i-1})}$	Castillo, Merentes, Rafeiro variation (2010)
$\kappa BV_\phi^R([a,b])$	$\kappa V_\phi^R(f;[a,b])=\sup_P\frac{\sum_{i=1}^{n}\lvert t_i-t_{i-1}\rvert\,\phi\left(\frac{\lvert f(t_i)-f(t_{i-1})\rvert}{\lvert t_i-t_{i-1}\rvert}\right)}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)}$	Castillo, Sanoja, Zea (2013) bounded $\kappa\phi$-variation in the sense of Riesz-Korenblu
$RBV_{p(\cdot)}([a,b])$	$V_{[a,b]}^{p(\cdot)}(f)=\sup_P\sum_{i=1}^{n}\frac{\lvert f(t_i)-f(t_{i-1})\rvert^{p(x_{i-1})}}{(t_i-t_{i-1})^{p(x_{i-1})-1}}$	Castillo, Mauricio, Rafeiro variation (2016) Variable exponent bounded variation spaces in the Riesz sense

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