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Closures of Ternary Relations

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Dedication

To my beloved father,

I dedicate this achievement to you with gratitude for your love, support, and endless sacrifices. May Allah bless you and keep you always my pride and strength. **To myself. . .**

After God, you were my strength in moments of weakness, my hope when the light dimmed. Today, I honor you with this achievement one that mirrors your perseverance alone.

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You, my answered prayer, my steadfast support when days grew difficult. I dedicate my success to you the fruit of your patience, your prayers, and your contentment.

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CONTENTS

GENERAL INTRODUCTION

Relationships between the elements of a same set or between the elements of different sets can be represented at a very basic level through the use of the mathematical construct called relation. Relations come in many flavours, such as binary or ternary, crisp or fuzzy, et cetera. Surprisingly, in contrast to binary relations, ternary and, more generally, n-ary relations, have received far less attention. However, In recent years, the interest in ternary relations is on the rise [11][12].

Ternary relations can display various interesting properties, such as reflexivity, symmetry, cyclicity and transitivity (see, e.g., [11]), some of which do not exist in the binary case (such as cyclicity) or come in a multitude of variations In the ternary case (such as transitivity). For instance, Pitcher and Smiley [25] introduced several types of four-point transitivity and five-point transitivity of betweenness relations. Additional types of four-point transitivity of ternary relations were proposed by Novák and Novotný [22]. Recently, Zedam et al. [32] characterized various properties of ternary relations using the notion of traces of a ternary relation.

Transitivity is by far the most interesting property of (crisp and fuzzy) binary relations. Thanks to the notion of composition of binary relations, the transitivity of a binary relation can conveniently be expressed as a relational inclusion. In addition to the role of relational compositions in the study and characterization of various properties of binary crisp or fuzzy relations [17][20], they also appear in many branches of mathematics, for instance in the study of fuzzy relational equations [13] and relation algebras [19].

By demonstrating how to create all possible compositions of ternary relations, this work aims to serve as a foundation for the study of ternary relations. Contrary to the binary case, two types of composition will emerge: we study each type, demonstrate connections between them. Since associativity is by far the most important property of the composition of binary relations, it should come as no surprise that our main focus in this work will be on the associativity of the compositions of ternary relations identified.

In case a binary relation R does not possess a desired property P, the question arises whether it is possible to find (if it exists) the smallest binary relation including R and possessing property P, which is called its P-closure. Such closures play an important role in many different mathematical areas, such as geometry [27] and logic [21].

The main aim of the present work is to derive results similar to those of Bandler and Kohout for the setting of ternary relations. For a given potential property P of ternary relations, we will lay bare necessary and sufficient conditions for the existence of a P-closure. Although we will discuss various properties, the main focus will be on the property of transitivity. Also, given the link between transitivity and relational composition we will study several new types of transitivity of a ternary relation based on the types of composition of ternary relations.

This work is structured as follows.

- In Chapter 1, we provide generalities on binary relations and ternary relations that we need throughout this work.
- In Chapter 2, we focus on the compositions of ternary relations. First, we recall several types of composition of ternary relations considering four points (considering five points). Also, we focus on the transitivity properties based on these compositions.
- In Chapter 3, we focus on the closures of ternary relations and apply the results to some basic properties of ternary relations.
- Finally, general conclusions and future research are drawn.

CHAPTER 1

GENERALITIES ON BINARY AND TERNARY RELATIONS

In this chapter, we recall the necessary basic properties and concepts of binary relations as well as of ternary relations that will be needed throughout this work.

1.1 Binary relations

In this section, we recall the basic definitions and properties of binary relations. Also, we recall the closures for these properties of binary relations.[2]

1.1.1 Definitions and properties

A binary relation R on a set X is a subset of X^2, i.e., it is a set of couples $(x, y) \in X^2$. Three special binary relations on X are the null relation $O_{X^2} = \emptyset$, the binary identity relation $I_{X^2} = \{(x, x) \mid x \in X\}$ and the universal binary relation X^2.

Two elements x and y of a set X equipped with a binary relation R are called comparable elements if it holds that $(x, y) \in R$ or $(y, x) \in R$, otherwise, they are called incomparable elements. A binary relation R_1 on a set X is said to be included in a binary relation R_2 on the same set X, denoted by $R_1 \subseteq R_2$, if, for any $x, y \in X$, $(x, y) \in R_1$ implies that $(x, y) \in R_2$. The intersection of two binary relations R_1 and R_2 on X is the binary relation $R_1 \cap R_2$ on X defined as $R_1 \cap R_2 = \{(x, y) \in X^2 \mid (x, y) \in R_1 \wedge (x, y) \in R_2\}$.

If $R_1 \cap R_2 = \emptyset$, then R_1 and R_2 are called disjoint relations. Also, the union of two binary relations R_1 and R_2 on X is the binary relation $R_1 \cup R_2$ on X defined as $R_1 \cup R_2 = \{(x, y) \in X^2 \mid (x, y) \in R_1 \vee (x, y) \in R_2\}$. The composition of two relations R_1 and R_2 on X is the relation $R_1 \circ R_2$ on X defined as:

$$R_1 \circ R_2 = \{(x, z) \in X^2 \mid (\exists y \in X)((x, y) \in R_1 \wedge (y, z) \in R_2)\}.$$

Let $\mathbb{N}^*$ be the set of non-zero natural numbers. The n-th power R^n of a binary relation R on X is recursively defined as:

$$R^1 = R \quad \text{and} \quad R^n = R^{n-1} \circ R,$$

for any $n \in \mathbb{N}^*$ and $n > 1$.

For a given binary relation R on a set X, we denote the *transpose* (*converse*) of R by R^t, i.e., for any $x, y \in X$, $(x, y) \in R^t$ means that $(y, x) \in R$. Also, we denote the *complement* of R by R^c, i.e., for any $x, y \in X$, $(x, y) \in R^c$ means that $(x, y) \notin R$. We denote the *dual* of R by R^d, i.e., for any $x, y \in X$, $(x, y) \in R^d$ means that $(y, x) \notin R$. A binary relation R on a set X is called:

(i) *reflexive*, if, for any $x \in X$, it holds that $(x, x) \in R$;

(ii) *irreflexive*, if, for any $x \in X$, it holds that $(x, x) \in R^c$;

(iii) *symmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R$ implies $(y, x) \in R$;

(iv) *asymmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R$ implies $(y, x) \in R^c$;

(v) *antisymmetric*, if, for any $x, y \in X$, it holds that $(x, y) \in R \wedge (y, x) \in R$ implies $x = y$;

(vi) *transitive*, if, for any $x, y, z \in X$, it holds that $(x, y) \in R \wedge (y, z) \in R$ implies $(x, z) \in R$;

(vii) *complete*, if, for any $x, y \in X$, it holds that $(x, y) \in R \vee (y, x) \in R$.

A preorder is binary relation over a set X which is reflexive and transitive. A partial order (order for short) is a binary relation on a set X that is reflexive, antisymmetric and transitive, a set with an order relation is called an ordered set (also called a poset). A total order is a partial order that satisfies the complete condition. A tolerance is a binary relation on a set X that is reflexive and symmetric. A binary relation on a set X is said to be an equivalence relation if is it reflexive, symmetric and transitive.

1.1.2 Closures of a binary relation

Where P is any property which a binary relation R on a set X may have or fail to have, the P-closure of R is defined to be the smallest relation S containing R and possessing P. Bandler and Kohout [12] have discussed the concept of closure of a given binary relation with respect to a property P as follows.

Definition 1.1. *If P is a property which a binary relation R on a set X may have or fail to have, then a binary relation S is the P-closure of R, written $S = P^{cl}(R)$, if and only if S satisfies all of*

(i) S has property P;

(ii) $R \subseteq S$;

(iii) If $R \subseteq T$ and T has property P, then $S \subseteq T$.

It is clear that a P-closure, if it exists, must be unique.

Corollary 1.1. *A binary relation R on a set X possesses property P if and only if $R = P^{cl}(R)$.*

For many properties P, a P-closure exists for some binary relations but not for others. Thus, all R that already possess P have a P-closure trivially (by the above corollary), but this guarantees nothing for other R. The interesting closures are those for properties where every R has a P-closure, because in these cases, and of course only in these, there is a P-closure operator on the entire set of binary relations on X. The following theorem states the conditions for this to occur.

Theorem 1.1. *A P-closure exists for all binary relations R on a set X if and only if*

(i) The universal relation X^2 possesses P;

(ii) The intersection of every (non-empty) family of binary relations, each of which possesses P, also possesses P.

Theorem 1.2. *If P and P' are properties for which closures exist (satisfying the conditions of Theorem 1.1), and if P^{cl} and P'^{cl} commute with each other, then $(P \wedge P')$ also satisfies the conditions of Theorem 1.1, and has a closure given by*

$$(P \wedge P')^{cl} = P^{cl}(P'^{cl}) = P'^{cl}(P^{cl}).$$

In [12], Bandler and Kohout gave a list of closures for the properties of a binary relation as follows.

Theorem 1.3. *Let R be a binary relation on a set X. It holds that:*

(i) *The reflexive closure of R is $Ref^{cl}(R) = R \cup I_{X^2}$;*

(ii) *The symmetric closure of R is $Sym^{cl}(R) = R \cup R^t$;*

(iii) *The transitive closure of R is $Tr^{cl}(R) = \bigcup_{n \geq 1} R^n$;*

(iv) *The tolerance closure of R is*

$$\begin{aligned} Tol^{cl}(R) &= Ref^{cl}(Sym^{cl}(R)) \\ &= Sym^{cl}(Ref^{cl}(R)); \end{aligned}$$

(v) *The preorder closure of R is*

$$\begin{aligned} Pre^{cl}(R) &= Ref^{cl}(Tr^{cl}(R)) \\ &= Tr^{cl}(Ref^{cl}(R)); \end{aligned}$$

(vi) *The equivalence closure of R is*

$$\begin{aligned} Equ^{cl}(R) &= Tr^{cl}(Tol^{cl}(R)) \\ &= Tr^{cl}(Sym^{cl}(Ref^{cl}(R))) \\ &= Tr^{cl}(Ref^{cl}(Sym^{cl}(R))) \\ &= Ref^{cl}(Tr^{cl}(Sym^{cl}(R))). \end{aligned}$$

The property of being complete fails to meet the second criterion of Theorem 1.1, while asymmetry and antisymmetry fail to meet even the first.

1.2 Ternary relations

In this section, we recall the basic definitions and properties of ternary relations.[2]

1.2.1 Definitions and examples

A ternary relation T on a set X is a subset of X^3, i.e., it is a set of triplets $(x, y, z) \in X^3$. Three special ternary relations on X are the null relation $O_{X^3} = \emptyset$, the ternary identity relation $I_{X^3} = \{(x, x, x) \mid x \in X\}$ and the universal ternary relation X^3.

Example 1.1. *Let T be the ternary relation on $\mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers) given as follows: $(a, b, c) \in T$ if $a \geq b \geq c$. Then, it is clear that $(15, 8, 3) \in T$, whereas $(3, 10, 2) \notin T$.*

Example 1.2. *Given any set X whose elements are arranged on a circle, one can define a ternary relation T on X, i.e., a subset of X^3, by stipulating that $(x, y, z) \in T$ holds If and only if the elements x, y and z are pairwise different and when going from x to z in a clockwise direction one passes through y. For example, if $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ represents the hours on a clock face, then $(8, 12, 4) \in T$ holds and $(12, 8, 4) \notin T$ does not hold.*

A ternary relation T_1 on a set X is said to be included in a ternary relation T_2 on the same set X, denoted by $T_1 \subseteq T_2$, if, for any $x, y, z \in X$, $(x, y, z) \in T_1$ Implies that $(x, y, z) \in T_2$. The intersection of two ternary relations T_1 and T_2 on X is the ternary relation $T_1 \cap T_2$ on X defined as $T_1 \cap T_2 = \{(x, y, z) \in X^3 \mid (x, y, z) \in T_1 \wedge (x, y, z) \in T_2\}$. If $T_1 \cap T_2 = \emptyset$, then T_1 and T_2 are called disjoint ternary relations. Also, the union of two ternary relations T_1 and T_2 on X Is the ternary relation $T_1 \cup T_2$ on X defined as $T_1 \cup T_2 = \{(x, y, z) \in X^3 \mid (x, y, z) \in T_1 \vee (x, y, z) \in T_2\}$.

Example 1.3. *(i) Let T_1 and T_2 be two ternary relations on $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ given by*

$$T_1 = \{(x_1, x_2, x_4), (x_1, x_3, x_2), (x_5, x_1, x_6)\},$$
$$T_2 = \{(x_1, x_2, x_4), (x_1, x_3, x_2), (x_5, x_1, x_6), (x_6, x_6, x_3)\}.$$

It is clear that $T_1 \subseteq T_2$.

(ii) Let T_1 and T_2 be two ternary relations on $X = \{x_1, x_2, x_3, x_4, x_5\}$ given by

$$T_1 = \{(x_1, x_2, x_4), (x_1, x_3, x_2), (x_5, x_1, x_4)\},$$
$$T_2 = \{(x_1, x_1, x_2), (x_1, x_2, x_4)\}.$$

One easily verifies that

$$T_1 \cap T_2 = \{(x_1, x_2, x_4)\};$$
$$T_1 \cup T_2 = \{(x_1, x_1, x_2), (x_1, x_2, x_4), (x_1, x_3, x_2), (x_5, x_1, x_4)\}.$$

For a given ternary relation T on a set X, we denote the *transpose* of T by T^t, i.e., for any $x, y, z \in X$, $(x, y, z) \in T^t$ means that $(z, y, x) \in T$. Also, we denote the *complement* of T by T^c, i.e., for any $x, y, z \in X$, $(x, y, z) \in T^c$ means that $(x, y, z) \notin T$. We denote the *dual* of T by T^d, i.e., for any $x, y, z \in X$, $(x, y, z) \in T^d$ means that $(z, y, x) \notin T$.[2]

1.2.2 Ternary relations obtained by permutation

Consider the two 2-permutations ρ_0 and ρ_1 mapping any 2-tuple (x, y) as follows:

$$\rho_0(x, y) = (x, y) \qquad \text{and} \qquad \rho_1(x, y) = (y, x).$$

For a binary relation R and a 2-permutation ρ, the binary relation R^ρ is defined as:

$$R^\rho = \{\, \rho(x, y) \in X^2 \mid (x, y) \in R \,\}.$$

Note that:

$$R^{\rho_0} = R \qquad \text{and} \qquad R^{\rho_1} = R^t,$$

Similarly, we consider the ternary relations obtained by permutation. The six 3-permutations are listed according to the lexicographical order, mapping any 3-tuple (x, y, z) as follows:

$$\sigma_0(x, y, z) = (x, y, z), \qquad \sigma_1(x, y, z) = (x, z, y), \qquad \sigma_2(x, y, z) = (y, x, z),$$
$$\sigma_3(x, y, z) = (y, z, x), \qquad \sigma_4(x, y, z) = (z, x, y), \qquad \sigma_5(x, y, z) = (z, y, x).$$

For a ternary relation T and a 3-permutation σ, the ternary relation T^σ on X is defined as [1]:

$$T^\sigma = \{\sigma(x,y,z) \in X^3 \mid (x,y,z) \in T\}.$$

Note that we can also write:

$$T^\sigma = \{(x,y,z) \in X^3 \mid \sigma^{-1}(x,y,z) \in T\},$$

With $\sigma_i^{-1} = \sigma_i$ for any $i \in \{0,1,2,5\}$ and $\sigma_3^{-1} = \sigma_4$. It is clear that $T^{\sigma_0} = T$ and $T^{\sigma_5} = T^t$.

Remark 1.1. *For any family of ternary relations $(T_j)_{j\in J}$ on a set X and a permutation σ_i, $i \in \{0,\dots,5\}$, the following equalities hold:*

$$\left(\bigcup_{j\in J} T_j\right)^{\sigma_i} = \bigcup_{j\in J} T_j^{\sigma_i} \quad \textit{and} \quad \left(\bigcap_{j\in J} T_j\right)^{\sigma_i} = \bigcap_{j\in J} T_j^{\sigma_i},$$

For any $i \in \{0,\dots,5\}$.

Definition 1.2. *Let T be a ternary relation on a set X.*

(i) The right-converse of T is the ternary relation T^{-1} on X defined as $T^{-1} = T^{\sigma_1}$;

(ii) The left-converse of T is the ternary relation $T^{\vdash}$ on X defined as $T^{\vdash} = T^{\sigma_2}$;

(iii) The right-rotation of T is the ternary relation T^+ on X defined as $T^+ = T^{\sigma_3}$;

(iv) The left-rotation of T is the ternary relation T^- on X defined as $T^- = T^{\sigma_4}$.

Proposition 1.1. *Let T be a ternary relation on a set X. The composition table for the considered permutations reads as follows: for given σ_i and σ_j, it lists $(T^{\sigma_i})^{\sigma_j}$.*

$\sigma_i \setminus \sigma_j$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
σ_0	T	T^{-1}	$T^{\dashv}$	T^+	T^-	T^t
σ_1	T^{-1}	T	T^-	T^t	$T^{\dashv}$	T^+
σ_2	$T^{\dashv}$	T^+	T	T^{-1}	T^t	T^-
σ_3	T^+	$T^{\dashv}$	T^t	T^-	T	T^{-1}
σ_4	T^-	T^t	T^{-1}	T	T^+	$T^{\dashv}$
σ_5	T^t	T^-	T^+	$T^{\dashv}$	T^{-1}	T

A ternary relation T on a set X is called:

(i) *reflexive, if, for any* $x \in X$, *It holds that* $(x, x, x) \in T$;

(ii) *strongly reflexive, if, for any* $x, y, z \in X$ *with card*$\{x, y, z\} \leq 2$, *it holds that* $(x, y, z) \in T$;

(iii) *irreflexive, if, for any* $x \in X$, *It holds that* $(x, x, x) \notin T$;

(iv) *strongly irreflexive, if, for any* $x, y, z \in X$ *with card*$\{x, y, z\} \leq 2$, *it holds that* $(x, y, z) \notin T$;

(v) *symmetric, if, for any* $x, y, z \in X$, *It holds that* $(x, y, z) \in T$ *implies* $(z, y, x) \in T$, *i.e.,* $T = T^t$;

(vi) *strongly symmetric, if* $T = T^{\sigma_i}$, *for any* $i \in \{1, \ldots, 5\}$;

(vii) *asymmetric, if, for any* $x, y, z \in X$ *such that card*$\{x, y, z\} \geq 2$, *it holds that* $(x, y, z) \in T$ *implies* $(z, y, x) \notin T$;

(viii) *strongly asymmetric, if, for any* $x, y, z \in X$ *such that card*$\{x, y, z\} \geq 2$, *it holds that* $(x, y, z) \in T$ *implies* $\sigma_i(x, y, z) \notin T$, *for any* $i \in \{1, \ldots, 5\}$;

(ix) *cyclic, if, for any* $x, y, z \in X$, *it holds that* $(x, y, z) \in T$ *implies* $(y, z, x) \in T$;

(x) *complete, if, for any* $x, y, z \in X$ *such that card*$\{x, y, z\} \geq 2$, *it holds that* $(x, y, z) \in T \vee (z, y, x) \in T$;

(xi) *strongly complete, if, for any* $x, y, z \in X$ *such that card*$\{x, y, z\} \geq 2$, *it holds that* $\sigma_i(x, y, z) \in T$, *for some* $i \in \{0, \ldots, 5\}$.

It is clear that a ternary relation T *on* X *Is called:*

(a) *left reflexive, if, for any* $x, y \in X$, *it holds that* $(x, y, y) \in T$;

(b) *middle reflexive, if, for any* $x, y \in X$, *it holds that* $(x, y, x) \in T$;

(c) *right reflexive, if, for any* $x, y \in X$, *it holds that* $(x, x, y) \in T$.

A cyclic order is a ternary relation that is asymmetric, transitive and cyclic. A ternary relation T *on a set* X *is a betweenness relation if it satisfies the following conditions:*

(i) *for any* $x, y, z \in X$, $(x, y, z) \in T$ *if and only if* $(z, y, x) \in T$;

(ii) for any $x, y, z \in X$, $(x, y, z) \in T$ *and* $(x, z, y) \in T$ *if and only If* $y = z$*;*

(iii) for any $x, y, z, u \in X$, $(x, y, u) \in T$ *and* $(x, u, z) \in T$ *implies* $(x, y, z) \in T$.

Also, a ternary relation T *is said to be a strict order-betweenness relation if, for any* $x, y, z \in X$*, It holds that* $(x, y, z) \in T$ *if and only if* $x < y < z$ *or* $z < y < x$*. An order-betweenness relation is a ternary relation* T *such that for any* $x, y, z \in X$*, It holds that* $(x, y, z) \in T$ *if and only If* $x \leq y \leq z$ *or* $z \leq y \leq x$.

For more details on ternary relations, we refer to [1, 6, 7, 5, 10, 11, 14, 18].

CHAPTER 2

COMPOSITIONS OF TERNARY RELATIONS

The main aim of this chapter is to study the compositions of (crisp) ternary relations. In this work, we discuss a systematic approach to the study of the composition of ternary relations from the point of view of the degrees of freedom available when linking a 3-tuple to two given 3-tuples. We will see a way of enumerating all possible four-point compositions (one degree of freedom) and five-point compositions (two degrees of freedom) of ternary relations [23], and establish a correspondence between them. Furthermore, we identify the associative compositions.

2.1 Compositions of relations

In contrast to binary relations, there is much less agreement in literature on the definition of composition of ternary relations. Indeed, various alternative definitions, each with its own motivation, have been proposed, for instance in [16]. In this section, we recall the definitions of the compositions of (binary and ternary) relations.

2.1.1 Composition of binary relations

In the theory of binary relations, a major role is played by the composition of relations, as it is the most important operation that allows to combine relations. In this subsection, an explanation of the usual notion of composition of two binary relations is given. The composition of two binary relations R and S on X is defined as follows [24]:

$$R \circ S = \Big\{(x,z) \in X^2 \;\Big|\; (\exists t \in X)\Big((x,t) \in R \wedge (t,z) \in S\Big)\Big\}.$$

One could imagine other definitions of the composition of binary relations:

$$R \circ_1 P = \{(x,z) \in X^2 \mid (\exists t \in X)((t,x) \in R \wedge (t,z) \in P)\};$$
$$R \circ_2 P = \{(x,z) \in X^2 \mid (\exists t \in X)((x,t) \in R \wedge (z,t) \in P)\};$$
$$R \circ_3 P = \{(x,z) \in X^2 \mid (\exists t \in X)((t,x) \in R \wedge (z,t) \in P)\};$$
$$R \circ_4 P = \{(x,z) \in X^2 \mid (\exists t \in X)((z,t) \in R \wedge (t,x) \in P)\};$$
$$R \circ_5 P = \{(x,z) \in X^2 \mid (\exists t \in X)((t,z) \in R \wedge (t,x) \in P)\};$$
$$R \circ_6 P = \{(x,z) \in X^2 \mid (\exists t \in X)((z,t) \in R \wedge (x,t) \in P)\};$$
$$R \circ_7 P = \{(x,z) \in X^2 \mid (\exists t \in X)((t,z) \in R \wedge (x,t) \in P)\}.$$

As all of these compositions link two 2-tuples while allowing for a degree of freedom, we refer to them as *3-point compositions*.

Any of the above 3-point compositions is determined by three 2-permutations ρ_i, ρ_j, and ρ_k, where $i,j,k \in \{0,1\}$, as explained next. First of all, let us fix two (possibly identical) 2-permutations ρ_i and ρ_j. If we say that a 2-tuple $(x,z) \in X^2$ belongs to some composition of two binary relations R and S on X if there exists an element $t \in X$ such that

$$\rho_i(x,t) \in R \qquad \text{and} \qquad \rho_j(t,z) \in S,$$

then this allows us to retrieve the first four compositions above. If, additionally, we allow to permute the 2-tuple (x,z), then we can also retrieve the last four compositions (after a proper renaming of variables). This view allows us to develop the following enumeration scheme. For any $q \in \{0,\ldots,7\}$, with $\circ_0 := \circ$ the basic composition, the composition $R \circ_q S$ can be written as:

$$R \circ_q S = \Big\{\rho_k(x,z) \in X^2 \;\Big|\; (\exists t \in X)\Big(\rho_i(x,t) \in R \wedge \rho_j(t,z) \in S\Big)\Big\}, \tag{1}$$

with $q = (kji)_2 = 4k+2j+i$ and $i, j, k \in \{0, 1\}$. Hence, the compositions $\circ_q$, $q = 1, \ldots, 7$, can be expressed in terms of the basic composition $\circ_0$ as follows:

$$R \circ_q S = \left(R^{\rho_i} \circ_0 S^{\rho_j}\right)^{\rho_k}.$$

Note that for a given 3-point composition $\circ_q$, with $q = (kji)_2$, there exists a 3-point composition $\circ_{q'}$ such that for any binary relations R and S on X, it holds that

$$R \circ_q S = S \circ_{q'} R,$$

with $q' = (\bar{k}\,\bar{i}\,\bar{j})_2,\ \bar{0} = 1,\ \bar{1} = 0.$

2.2 Compositions of ternary relations

In the literature, various compositions of two ternary relations S and T have been proposed, each with its own motivation. A first example is the composition $\circ_B$ (used in the definition of the transitivity of betweenness relations [25] defined as:

$$S \circ_B T = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in S \wedge (x, t, z) \in T)\}. \tag{2}$$

A 3-tuple belongs to this composition if there exist two 3-tuples that each share two components with the given tuple, of which exactly one in common, and one common degree of freedom t. We will refer to such a composition as a four-point composition. A second example is the composition $\circ_{c_1}$ (derived from the composition of a ternary relation with a binary relation in [32] defined as:

$$S \circ_{c_1} T = \{(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)((x, y, t) \in S \wedge (s, t, z) \in T)\}. \tag{3}$$

A 3-tuple belongs to this composition if there exist two 3-tuples of which one shares two components with the given tuple, while the other one contains the third component, complemented by two degrees of freedom s and t, of which one In common. We will refer to such a composition as a five-point composition. In the following, we will identify all 4-point compositions of ternary relations and study their properties, then, we will do the same for the five-point compositions [23].

2.2.1 Four-point compositions of ternary relations

The reasoning underlying the 3-point compositions of binary relations can be naturally extended to enumerate the four-point compositions of ternary relations. We start from the composition $\circ_B$ in Eq. 2 and use the 3-permutations to allow for a repositioning of the variables x, y, z and t, resulting in as many as 216 different four-point compositions.

Definition 2.1. *[23] Let $p \in \{0, \ldots, 215\}$ with $p = (kji)_6 = 36k + 6j + i$ and $i, j, k \in \{0, \ldots, 5\}$. For any ternary relations S and T on X, we define the composition $S\square_p^0 T$ as follows:*

$$S\square_p^0 T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(x, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\}\,.$$

It is indeed easy to see that the compositions $\square_p^0$, $p \in \{0, \ldots, 215\}$, are all different. However, not all four-point compositions of ternary relations can be obtained in this way. Indeed, one can also envisage compositions that match the following description. A 3-tuple belongs to such a composition if there exist two 3-tuples of which one shares two components with the given tuple, while the other contains the third component and a second occurrence of the corresponding element, again complemented with one common degree of freedom t.

Definition 2.2. *[23] Let $p \in \{0, \ldots, 215\}$ with $p = (kji)_6 = 36k + 6j + i$ and $i, j, k \in \{0, \ldots, 5\}$. For any ternary relations S and T on X, we define the compositions $S\square_p^1 T$ and $S\square_p^2 T$ as follows:*

$$S\square_p^1 T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(x, y, t) \in S \wedge \sigma_j(z, t, z) \in T)\}\,;$$
$$S\square_p^2 T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(y, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\}\,.$$

As the following proposition shows, we have only 54 different compositions of type $\square^1$ and as many of type $\square^2$.

Proposition 2.1. *Let $i, j, k \in \{0, \ldots, 5\}$. For any ternary relations S and T on X, it holds that*

$$S\square^1_{(kji)_6} T = S\square^1_{(\pi_1(k)j\pi_1(i))_6} T = S\square^1_{(k\pi_2(j)i)_6} T = S\square^1_{(\pi_1(k)\pi_2(j)\pi_1(i))_6} T$$

and

$$S\square^2_{(kji)_6}T = S\square^2_{(\pi_2(k)\pi_2(j)i)_6}T = S\square^2_{(kj\pi_1(i))_6}T = S\square^2_{(\pi_2(k)\pi_2(j)\pi_1(i))_6}T,$$

with π_1 *the permutation of* $\{0,\dots,5\}$ *given in Table 2.1*

u	0	1	2	3	4	5
$\pi_1(u)$	2	3	0	1	5	4

Table 2.1: The permutation π_1.

And π_2 *the permutation of* $\{0,\dots,5\}$ *given in Table 2.2*

u	0	1	2	3	4	5
$\pi_2(u)$	5	4	3	2	1	0

Table 2.2: The permutation π_2.

Note that if $u \in \{0,3,4\}$, then $\pi_1(u), \pi_2(u) \in \{1,2,5\}$. This implies that we can select the unique 54 compositions of type $\square^1$ and the 54 unique compositions of type $\square^2$ by restricting i,j to belong to $\{0,3,4\}$ (the same could be achieved by restricting i,j to belong to $\{1,2,5\}$). This corresponds to the compositions $\square^r_p$ ($r \in \{1,2\}$) with $p \bmod 36$ belonging to $\{0,3,4,18,21,22,24,27,28\}$.

The following examples illustrate the above definition of four-point compositions of ternary relations.

Example 2.1. *Consider* $p = 119$ *and two ternary relations* S *and* T *on* X*. In senary notation,* p *can be written as* $119 = 3*36+1*6+5 = (315)_6$*. Hence,*

$$\begin{aligned} T\square^0_{119}S &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_5(x,y,t) \in T \wedge \sigma_1(x,t,z) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (x,z,t) \in S)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (z,y,t) \in S)\}\,, \end{aligned}$$

$$\begin{aligned} T\square^1_{119}S &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_5(x,y,t) \in T \wedge \sigma_1(z,t,z) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (z,z,t) \in S)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (y,y,t) \in S)\}\,, \end{aligned}$$

$$
\begin{aligned}
T\square^2_{119}S &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_5(y,y,t) \in T \wedge \sigma_1(x,t,z) \in S)\} \\
&= \{(y,z,x) \in X^3 \mid (\exists t \in X)((t,y,y) \in T \wedge (x,z,t) \in S)\} \\
&= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,x) \in T \wedge (z,y,t) \in S)\} .
\end{aligned}
$$

Conversely, the following example shows how to find $r \in \{0,1,2\}$ and $p = (kji)_6 \in \{0,\dots,215\}$ for a given composition.

Example 2.2. *Consider two ternary relations S and T on X and the binary operation $*$ defined as follows:*

$$
S * T = \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (y,z,t) \in S)\} .
$$

The operation $*$ corresponds to a composition of the type $\square^0_p$, with $p \in \{0,\dots,215\}$. We express this composition as in Definition 2.1 by identifying the proper 3-permutations:

$$
\begin{aligned}
S * T &= \{(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_4(z,x,t) \in T \wedge \sigma_4(z,t,y) \in S)\} \\
&= \{(y,z,x) \in X^3 \mid (\exists t \in X)(\sigma_4(x,y,t) \in T \wedge \sigma_4(x,t,z) \in S)\} \\
&= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_4(x,y,t) \in T \wedge \sigma_4(x,t,z) \in S)\} .
\end{aligned}
$$

Hence, $i = 4, j = 4$ and $k = 3$. Thus, $p = (344)_6 = 136$ and $* = \square^0_{136}$.

Example 2.3. *Consider two ternary relations S and T on X and the binary operation $\star$ defined as follows:*

$$
S \star T = \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (t,x,x) \in S)\} .
$$

The operation $\star$ corresponds to a composition of the type $\square^1_p$, with $p \in \{0,\dots,215\}$. We express this composition as in Definition 2.2 by identifying the proper 3-permutations:

$$
\begin{aligned}
S \star T &= \{(z,y,x) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (t,z,z) \in S)\} \\
&= \{\sigma_5(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_4(x,y,t) \in T \wedge \sigma_3(z,t,z) \in S)\} .
\end{aligned}
$$

Hence, $i = 4, j = 3$ and $k = 5$. Thus, $p = (534)_6 = 202$ and $\star = \square^1_{202}$. According to Proposition 2.1, $\star$ can also be written as $\square^1_{161}$, $\square^1_{167}$ and $\square^1_{196}$.

The following proposition expresses all the four-point compositions in terms of the compositions $\Box_0^r, r \in \{0,1,2\}$, which we will refer to as the basic four-point compositions.

Proposition 2.2. *Let $p = (kji)_6 \in \{0,\ldots,215\}$ and $r \in \{0,1,2\}$. For any ternary relations S and T on X, the four-point composition $S\Box_p^r T$ can be written in terms of the composition $\Box_0^r$ as follows:*

$$S\Box_p^r T = (S^{\sigma_i^{-1}} \Box_0^r T^{\sigma_j^{-1}})^{\sigma_k}\,.$$

Proof. We give the proof for $r = 0$.

$$\begin{aligned} S\Box_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in S^{\sigma_i^{-1}} \wedge (x,t,z) \in T^{\sigma_j^{-1}})\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (x,y,z) \in S^{\sigma_i^{-1}} \Box_0^0 T^{\sigma_j^{-1}}\} \\ &= (S^{\sigma_i^{-1}} \Box_0^0 T^{\sigma_j^{-1}})^{\sigma_k}\,. \end{aligned}$$

□

Next, we study some basic properties of the four-point compositions of ternary relations. The following proposition identifies the right and left neutral elements of the basic four-point compositions. To that end, we introduce the following definition and lemma.

Definition 2.3. *We define the special ternary relations E, E_ℓ, E_m and E_r on X as follows:*
(i) $E = \{(x,x,x) \in X^3 \mid x \in X\}$;
(ii) $E_\ell = \{(x,x,y) \in X^3 \mid x,y \in X\}$;
(iii) $E_m = \{(x,y,x) \in X^3 \mid x,y \in X\}$;
(iv) $E_r = \{(x,y,y) \in X^3 \mid x,y \in X\}$.

Lemma 2.1. *For any ternary relation T on X, It holds that*
(i) $T\Box_0^0 E_r = T$;
(ii) $E_r \Box_0^0 T = T$.

Proof. We prove (i). Let $(x,y,z) \in T$, then with $(x,z,z) \in E_r$, it follows that $(x,y,z) \in T\Box_0^0 E_r$. Conversely, let $(x,y,z) \in T\Box_0^0 E_r$, then there exists $t \in X$ such that $(x,y,t) \in T$ and $(x,t,z) \in E_r$. The fact that $(x,t,z) \in E_r$ implies that $t = z$. Hence $(x,y,z) \in T$, and thus $T\Box_0^0 E_r \subseteq T$. □

Proposition 2.3. *Let $i,j,k \in \{0,\ldots,5\}$. For any ternary relation T on X, It holds that*

(i) $T\Box_{(kjk)_6}^0 E_{\zeta(j)} = T$;

(ii) $E_{\zeta(i)}\square^0_{(kki)_6}T = T$,

with $\zeta : \{0, \ldots, 5\} \to \{\ell, m, r\}$ *the mapping given in Table 2.3*

u	0	1	2	3	4	5
$\zeta(u)$	r	r	m	ℓ	m	ℓ

Table 2.3: The mapping ζ.

Proof. We prove (i). Due to Lemma 2.1, It holds for any $k \in \{0, \ldots, 5\}$ that

$$T^{\sigma_k^{-1}}\square^0_0 E_r = T^{\sigma_k^{-1}}\,.$$

Moreover, one can verify that $E_r = (E_{\zeta(j)})^{\sigma_j^{-1}}$ for any $j \in \{0, \ldots, 5\}$. Hence,

$$T^{\sigma_k^{-1}}\square^0_0(E_{\zeta(j)})^{\sigma_j^{-1}} = T^{\sigma_k^{-1}}\,,$$

And thus

$$\left(T^{\sigma_k^{-1}}\square^0_0(E_{\zeta(j)})^{\sigma_j^{-1}}\right)^{\sigma_k} = T\,.$$

Therefore, Proposition 2.2 ensures that

$$T\square^0_{(kjk)_6}E_{\zeta(j)} = T\,.$$

$\square$

Note that $E_{\zeta(k)}$ Is simultaneously the left and right neutral element of $\square^0_{(kkk)_6}$, and can hence be called the neutral element of $\square^0_{(kkk)_6}$.

Proposition 2.4. *Let* $p \in \{0, \ldots, 215\}$. *For any ternary relation* T *on* X*, It holds that*
(i) $T\square^1_p E = T$*;*
(ii) $E\square^2_p T = T$.

Proof. The proof is similar to that of Proposition 2.3. $\square$

Note that the compositions $\square^1_p$ and $\square^2_p$ do not have a neutral element.

2.2.2 Five-point compositions of ternary relations

In this subsection, we study the five-point compositions of ternary relations. In view of the notation used for four-point compositions, it is mnemonic notation $\pentagon$ for five-point compositions [23]. A four-point composition can be modified to obtain a five-point composition by introducing an additional degree of freedom, substituting one of the occurrences of the element appearing twice. More precisely, the four-point composition $\square_p^0$ in Definition 2.1 can be modified in two ways to obtain a five-point composition. The first one is obtained by replacing the element x in the first 3-tuple (x, y, t) by a new degree of freedom s as follows:

$$S \pentagon'_p T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists(s,t) \in X^2)(\sigma_i(s, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\}\,, \tag{4}$$

While the second one is obtained by replacing the element x in the second 3-tuple (x, t, z) as follows:

$$S \pentagon''_p T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists(s,t) \in X^2)(\sigma_i(x, y, t) \in S \wedge \sigma_j(s, t, z) \in T)\}\,. \tag{5}$$

This reasoning could give the impression that there are twice as many five-point compositions as there are four-point compositions. However, there are only 108 different compositions of type $\pentagon'_p$ and 108 different compositions of type $\pentagon''_p$. Indeed, each five-point composition is obtained in two different ways. For $p = (kji)_6 \in \{0, \ldots, 215\}$, it holds that $S \pentagon'_p T = S \pentagon'_q T$, with $q = (\pi_1(k) j \pi_1(i))_6$ and π_1 the permutation given in Table 2.1. Similarly, for $p = (kji)_6 \in \{0, \ldots, 215\}$, it holds that $S \pentagon''_p T = S \pentagon''_q T$, with $q = (\pi_2(k)\pi_2(j)i)_6$ and π_2 the permutation given in Table 2.2.

In the following definition, 216 different five-point compositions of ternary relations identified by using an elegant enumeration system.

Definition 2.4. *[23] Let $p \in \{0, \ldots, 215\}$ with $p = (kji)_6$. For any ternary relations S and T on X, we define the composition S*
pentagon$_p T$ as follows:

$$S \pentagon_p T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists(s,t) \in X^2)(\sigma_i(\hat{x}, y, t) \in S \wedge \sigma_j(\check{x}, t, z) \in T)\}\,, \tag{6}$$

With

$$\hat{x} = \begin{cases} x, & \text{if } k \in \{0,3,4\} \\ s, & \text{if } k \in \{1,2,5\} \end{cases}\,, \quad \check{x} = \begin{cases} s, & \text{if } k \in \{0,3,4\} \\ x, & \text{if } k \in \{1,2,5\} \end{cases}\,.$$

The same five-point compositions would have been obtained when starting from the four-point compositions $\square_p^1$ and $\square_p^2$. Indeed, by substituting one of the occurrences of the element z by s in the 54 compositions of the type $\square_p^1$, we obtain the 108 five-point compositions $\pentagon_p$ with $k \in \{0,3,4\}$. Similarly, starting from the compositions of the type $\square_p^2$, we obtain the other 108 five-point compositions $\pentagon_p$ with $k \in \{1,2,5\}$. Note that the five-point compositions $\pentagon_0, \pentagon_{12}, \pentagon_{18}, \pentagon_{84}, \pentagon_{86}$ and $\pentagon_{87}$ were already introduced in [3]. The following examples illustrate Definition 2.4.

Example 2.4. *Consider $p = 175$ and two ternary relations S and T on X. In senary notation, it holds that $p = (451)_6$. Since $k = 4$, we have*

$$\begin{aligned} S \pentagon_{175} T &= \{\sigma_4(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_1(\hat{x},y,t) \in S \wedge \sigma_5(\check{x},t,z) \in T)\} \\ &= \{\sigma_4(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_1(x,y,t) \in S \wedge \sigma_5(s,t,z) \in T)\} \\ &= \{(z,x,y) \in X^3 \mid (\exists (s,t) \in X^2)((x,t,y) \in S \wedge (z,t,s) \in T)\} \end{aligned}$$

Example 2.5. *Consider two ternary relations S and T on X and the binary operation $*$ defined as follows:*

$$S * T = \{(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)((s,t,z) \in S \wedge (y,t,x) \in T)\}.$$

The operation $*$ corresponds to a composition of the type $\pentagon_p$, with $p \in \{0,\dots,215\}$. We express this composition as in Definition 2.4 by identifying the proper 3-permutations. We distinguish two possibilities:

$$S * T = \{\sigma_1(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_1(s,y,t) \in S \wedge \sigma_5(x,t,z) \in T)\} \tag{2.1}$$

Or

$$S * T = \{\sigma_3(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_1(s,y,t) \in S \wedge \sigma_0(x,t,z) \in T)\}. \tag{2.2}$$

According to our enumeration scheme, it holds that $k \in \{1,2,5\}$. Therefore, Eq. 2.1 is the proper one, i.e., $i = 1$, $j = 5$ and $k = 1$. Hence $* = \pentagon_{67}$.

The following proposition expresses all the five-point compositions in terms of the compositions $\pentagon_0$ or $\pentagon_{210}$, which we will refer to as the basic five-point compositions.

Proposition 2.5. *Let $p = (kji)_6 \in \{0,\dots,215\}$. For any ternary relations S and T on X, the five-point composition $S \pentagon_p T$ can be written in terms of the compositions $\pentagon_0$ or*

$\pentagon_{210}$ *as follows:*

$$S \pentagon_p T = \begin{cases} \left(S^{\sigma_i^{-1}} \pentagon_0 T^{\sigma_j^{-1}}\right)^{\sigma_k} & , \text{if } k \in \{0,3,4\}, \\ \left(S^{\sigma_i^{-1}} \pentagon_{210} T^{\sigma_j^{-1}}\right)^{\sigma_k} & , \text{if } k \in \{1,2,5\}. \end{cases}$$

The following proposition shows some inclusion relationships between the four-point and five-point compositions.

Proposition 2.6. *Let $p = (kji)_6 \in \{0, \ldots, 215\}$. For any ternary relations S and T on X, it holds that*

(i) $S\square_p^0 T \subseteq S \pentagon_p T$;

(ii) $S\square_p^1 T \subseteq S \pentagon_p T$, *If* $k \in \{0,3,4\}$;

(iii) $S\square_p^2 T \subseteq S \pentagon_p T$, *If* $k \in \{1,2,5\}$.

***Proof*.** We give the proof of (i). If $k \in \{0,3,4\}$, then

$$\begin{aligned} S\square_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &\subseteq \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_i(x,y,t) \in S \wedge \sigma_j(s,t,z) \in T)\} \\ &= S \pentagon_p T. \end{aligned}$$

□

Similarly, if $k \in \{1,2,5\}$, then

$$\begin{aligned} S\square_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &\subseteq \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_i(s,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &= S \pentagon_p T. \end{aligned}$$

Next, we study some basic properties of the five-point compositions of ternary relations, some of them being similar to those of the four-point compositions, while others being different.

The following proposition identifies the right and left neutral elements of the five-point compositions of ternary relations.

Proposition 2.7. *Let $i, j, k \in \{0, \dots, 5\}$. For any ternary relation T on X, it holds that*

(i) $T \pentagon_{(kjk)_6} E_{\zeta(j)} = T$, *if* $k \in \{0, 3, 4\}$;

(ii) $E_{\zeta(i)} \pentagon_{(kki)_6} T = T$, *if* $k \in \{1, 2, 5\}$,

With ζ *as in Proposition 2.3.*

It is worth noting that in contrast to the four-point case, no five-point composition has a neutral element.

2.3 Associativity of compositions of ternary relations

The most desirable property of a relational composition undeniably is its associativity. In this section, we study the associativity of the four-point and five-point compositions of ternary relations [23].

2.3.1 Associativity of four-point compositions

The following proposition identifies the associative 4-point compositions.

Proposition 2.8. *The four-point composition* $\square_p^0$ *is associative if and only if* $p = (iii)_6 = 43i$, *with* $i \in \{0, \dots, 5\}$.

Proof. Let $p = (kji)_6$. For any three ternary relations R, S, T on X, we have

$$(R\square_p^0 S)\square_p^0 T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(x, y, t) \in R\square_p^0 S \wedge \sigma_j(x, t, z) \in T)\}.$$

Since the set of 3-permutations is a group, there exists $m \in \{0, \dots, 5\}$ such that $\sigma_i = \sigma_k \sigma_m$. Hence, we can write

$$\begin{aligned}
&(R\square_p^0 S)\square_p^0 T \\
&= \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_k(\sigma_m(x, y, t)) \in R\square_p^0 S \wedge \sigma_j(x, t, z) \in T)\} \\
&= \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_i(\sigma_m(x, y, s)) \in R \wedge \sigma_j(\sigma_m(x, s, t)) \in S \wedge \sigma_j(x, t, z) \in T)\}.
\end{aligned}$$

Using the same argument, there exists $n \in \{0, \dots, 5\}$ such that $\sigma_j = \sigma_k \sigma_n$, and we can write

$$
\begin{aligned}
&R\Box_p^0(S\Box_p^0 T)\\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists u \in X)(\sigma_i(x,y,u) \in R \wedge \sigma_j(x,u,z) \in S\Box_p^0 T)\}\\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists u \in X)(\sigma_i(x,y,u) \in R \wedge \sigma_k(\sigma_n(x,u,z)) \in S\Box_p^0 T)\}\\
&= \{\sigma_k(x,y,z) \in X^3 \mid (\exists (u,v) \in X^2)(\sigma_i(x,y,u) \in R \wedge \sigma_i(\sigma_n(x,u,v)) \in S \wedge \sigma_j(\sigma_n(x,v,z)) \in T)\}.
\end{aligned}
$$

It clearly holds that $(R\Box_p^0 S)\Box_p^0 T = R\Box_p^0(S\Box_p^0 T)$ if and only If $m = n = 0$, i.e., if and only If $i = j = k$. Therefore, $\Box_p^0$ is associative if and only If $i = j = k$. $\Box$

Remark 2.1. *For any $p \in \{0,\dots,215\}$, the compositions $\Box_p^1$ and $\Box_p^2$ are not associative. Indeed, consider the three ternary relations R, S, T on the set $X = \{a,b,c,d,e\}$ given as:*

$$
\begin{aligned}
R &= \{(a,b,c),(a,c,b),(b,a,c),(b,c,a),(c,a,b),(c,b,a)\};\\
S &= \{(d,c,d),(d,d,c),(c,d,d)\};\\
T &= \{(e,d,e),(e,e,d),(d,e,e)\}.
\end{aligned}
$$

One can verify that for any $p \in \{0,\dots,215\}$, there exists $i \in \{0,\dots,5\}$ such that

$$\sigma_i(a,b,e) \in (R\Box_p^1 S)\Box_p^1 T,$$

While $(R\Box_p^1 S)\Box_p^1 T = \emptyset$. The same holds for the compositions $\Box_p^2$ by considering the relations

$$
\begin{aligned}
R &= \{(a,a,b),(a,b,a),(b,a,a)\};\\
S &= \{(b,b,c),(b,c,b),(c,b,b)\};\\
T &= \{(c,d,e),(c,e,d),(d,c,e),(d,e,c),(e,c,d),(e,d,c)\}.
\end{aligned}
$$

In Table 5, we present the six associative four-point compositions denoted by the new notation $\diamond_i, i \in \{1,\dots,6\}$.

The 3-tuple (x, y, z) belongs to $S \diamond_i T$ if there exists an element $t \in X$ such that the other listed 3-tuples belong to S and T, respectively.

$\diamond_i$	$S \diamond_i T$	S	T
$\diamond_1 = \square_0^0$	(x, y, z)	(x, y, t)	(x, t, z)
$\diamond_2 = \square_{43}^0$	(x, y, z)	(x, t, z)	(x, y, t)
$\diamond_3 = \square_{86}^0$	(x, y, z)	(x, y, t)	(t, y, z)
$\diamond_4 = \square_{129}^0$	(x, y, z)	(x, t, z)	(t, y, z)
$\diamond_5 = \square_{172}^0$	(x, y, z)	(t, y, z)	(x, y, t)
$\diamond_6 = \square_{215}^0$	(x, y, z)	(t, y, z)	(x, t, z)

Table 2.4: The associative four-point compositions.

The following result is a particular case of Proposition 2.3. It identifies the neutral element of the associative four-point compositions.

Corollary 2.1. *(i) E_ℓ is the neutral element of $\diamond_1$ and $\diamond_2$;*

(ii) E_m is the neutral element of $\diamond_3$ and $\diamond_5$;

(iii) E_r is the neutral element of $\diamond_4$ and $\diamond_6$.

Remark 2.2. *It is clear that by using the commutativity of the conjunction, one proves that:*

$$T \diamond_1 S = S \diamond_2 T$$

$$T \diamond_3 S = S \diamond_5 T$$

$$T \diamond_4 S = S \diamond_6 T$$

2.3.2 Associativity of five-point compositions

Although there are more four-point compositions than five-point compositions, it turns out that there are more associative five-point compositions than associative four-point compositions. The following proposition identifies the associative five-point compositions [23].

Proposition 2.9. *Let $p = (kji)_6 \in \{0, \ldots, 215\}$. The composition $\pentagon_p$ is associative if and only if one of the following conditions holds:*

(i) $k = j = i$;

(ii) $k \in \{0, 3, 4\}$, $i = k$ and $j = \pi_4(k)$;

(iii) $k \in \{1, 2, 5\}$, $i = \pi_4(k)$ *and* $j = k$,

With π_4 *is the permutation of* $\{0, \ldots, 5\}$ *given in Table 2.5:*

u	0	1	2	3	4	5
$\pi_4(u)$	2	4	3	1	5	0

Table 2.5: The permutation π_4.

Proof. Let R, S, T be three ternary relations on X. We consider the case that $k \in \{0, 3, 4\}$, the other case being similar. For m as in Proposition 2.8, we have

$$\begin{aligned}(R \pentagon_p S) \pentagon_p T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_i(x,y,t) \in R \pentagon_p S \wedge \sigma_j(s,t,z) \in T)\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_k(\sigma_m(x,y,t)) \in R \pentagon_p S \wedge \sigma_j(s,t,z) \in T)\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t,s',t') \in X^4)(\sigma_i(\sigma_m(x,y,t')) \in R \\ &\quad \wedge \sigma_j(s_m(s',t',t)) \in S \wedge \sigma_j(s,t,z) \in T)\}.\end{aligned}$$

□

For n as in Proposition 2.8, we have

$$\begin{aligned}R \pentagon_p (S \pentagon_p T) = \{\sigma_k(x,y,z) \in X^3 \mid (\exists (u,u',v,v') \in X^4)(\sigma_i(x,y,u) \in R \\ \wedge \sigma_i(\sigma_n(u,v,u')) \in S \wedge \sigma_j(\sigma_n(v',u',z)) \in T)\}.\end{aligned}$$

By identification, $(R \pentagon_p S) \pentagon_p T = R \pentagon_p (S \pentagon_p T)$ if and only if $(m,n) = (0,0)$ or $(m,n) = (0,2)$, i.e., if and only If

$$K = j = i \quad \text{or} \quad i = k \text{ and } j = \pi_4(k).$$

The 3-tuple (x, y, z) belongs to $S \varpentagon_i T$ If there exists two elements $s, t \in X$ such that the other listed 3-tuples belong to S and T, respectively.

$\varpentagon_i$	$S \varpentagon_i T$	S	T
$\varpentagon_1 = \pentagon_0$	(x, y, z)	(x, y, t)	(s, t, z)
$\varpentagon_2 = \pentagon_{12}$	(x, y, z)	(x, y, t)	(t, s, z)
$\varpentagon_3 = \pentagon_{43}$	(x, y, z)	(s, t, z)	(x, y, t)
$\varpentagon_4 = \pentagon_{46}$	(x, y, z)	(t, s, z)	(x, y, t)
$\varpentagon_5 = \pentagon_{86}$	(x, y, z)	(x, s, t)	(t, y, z)
$\varpentagon_6 = \pentagon_{87}$	(x, y, z)	(x, t, s)	(t, y, z)
$\varpentagon_7 = \pentagon_{117}$	(x, y, z)	(x, t, z)	(s, y, t)
$\varpentagon_8 = \pentagon_{129}$	(x, y, z)	(x, t, z)	(t, y, s)
$\varpentagon_9 = \pentagon_{172}$	(x, y, z)	(t, y, z)	(x, s, t)
$\varpentagon_{10} = \pentagon_{178}$	(x, y, z)	(t, y, z)	(x, t, s)
$\varpentagon_{11} = \pentagon_{210}$	(x, y, z)	(s, y, t)	(x, t, z)
$\varpentagon_{12} = \pentagon_{215}$	(x, y, z)	(t, y, s)	(x, t, z)

Table 2.6: The associative five-point compositions.

CHAPTER 3

CLOSURES OF A TERNARY RELATION

In this chapter, we study the problem of closing a ternary relation with respect to various relational properties, with a focus on the many transitivity properties that have been proposed for ternary relations over the past years. In particular, we consider the transitivity properties corresponding to the associative four-point compositions and associative five-point compositions of ternary relations.

3.1 Closures of a ternary relation

For a property P which a ternary relation T on a set X may have or fail to have, the P-closure of T is defined to be the smallest relation S containing T and possessing P. In this section, we investigate the closures for various properties of a ternary relation.[2]

3.1.1 General results

In this subsection, all the results are obtained in the same way as Bandler and Kohout [12] did for binary relations.

Definition 3.1. *If P is a property which a ternary relation T on a set X may have or fail to have, then a ternary relation S is the P-closure of T, written $S = P^{cl}(T)$, If and only If S satisfies all of*

(i) S possesses property P;

(ii) $T \subseteq S$;

(iii) *If* $T \subseteq T'$ *and* T' *possesses property* P*, then* $S \subseteq T'$.

It is clear that a P*-closure, if it exists, must be unique.*

Corollary 3.1. *A ternary relation* T *on a set* X *possesses property* P *if and only if* $T = P^{cl}(T)$.

For many properties P, a P-closure exists for some ternary relations but not for others. The interesting properties P are those for which every ternary relation has a P-closure. The following theorem states the conditions for this to occur.

Theorem 3.1. *A* P*-closure exists for all ternary relations* T *on a set* X *if and only if*

(i) *The universal relation* X^3 *possesses* P;

(ii) *The intersection of every (non-empty) family of ternary relations, each of which possesses* P*, also possesses* P.

This characterization can be rephrased in the following corollary.

Corollary 3.2. *Let* T *be a ternary relation on a set* X*, then*

$$P^{cl}(T) = \bigcap\{S \mid T \subseteq S \wedge S \textit{ has property } P\}.$$

An example of a property P for whichh a P-closure does not exist in general is the completeness property. Indeed, this property is not preserved under intersection. Even more, there does not exist any non-complete ternary relation T that can be closed for completeness.

Theorem 3.2. *If* P *and* P' *are properties for which closures exist (satisfying the conditions of Theorem 3.1), and If* P^{cl} *and* P'^{cl} *commute with each other, then* $(P \wedge P')$ *also satisfies the conditions of Theorem 3.1, and has a closure given by*

$$(P \wedge P')^{cl} = P^{cl}(P'^{cl}) = P'^{cl}(P^{cl}).$$

3.1.2 Closures for some special properties of ternary relations

Bandler and Kohout [12] studied the closures for some special properties of binary relations, such as the symmetric closure, transitive closure, preorder closure and equivalence closure. In the following proposition, we follow the same direction and express the closures of a ternary relation for the reflexivity, symmetry, strong symmetry and cyclicity properties. Note that these properties are all preserved under intersection and that the universal relation X^3 possesses all of them. Hence, the closures for these properties always exist [2]. First, we need to give the following result that expresses the relationships between the properties of a ternary relation (symmetry and cyclicity) and those of its permutations. The proof is straightforward [2].

Proposition 3.1. *For a given ternary relation T, It holds that*

(i) T is symmetric if and only If $T = T^t$;

(ii) T is cyclic if and only if $T = T^+ = T^-$.

Proposition 3.2. *Let T be a ternary relation on a set X. It holds that:*

(i) The reflexive closure is $Ref^{cl}(T) = T \cup I_{X^3}$;

(ii) The symmetric closure is $Sym^{cl}(T) = T \cup T^t$;

(iii) The strongly symmetric closure is $Ssym^{cl}(T) = \bigcup_{i=0}^{5} T^{\sigma_i}$;

(iv) The cyclic closure is $Cyc^{cl}(T) = T \cup T^+ \cup T^-$.

Proof. Items (i) and (ii) are immediate.
(iii) To prove that $\mathrm{Ssym}^{\mathrm{cl}}(T) = \bigcup_{i=0}^{5} T^{\sigma_i}$, It suffices to prove that $\bigcup_{i=0}^{5} T^{\sigma_i}$ is the smallest strongly symmetric ternary relation containing T. First, it is clear that $T \subseteq \bigcup_{i=0}^{5} T^{\sigma_i}$. From Remark 1.1, it follows that $\bigcup_{i=0}^{5} T^{\sigma_i}$ is strongly symmetric. Next, if $T \subseteq S$ and S is strongly symmetric, then for any $i \in \{0, \ldots, 5\}$, it holds that $T^{\sigma_i} \subseteq S^{\sigma_i}$. Hence, $\bigcup_{i=0}^{5} T^{\sigma_i} \subseteq \bigcup_{i=0}^{5} S^{\sigma_i}$. Since S is strongly symmetric, it follows from Corollary 3.1 that $\mathrm{Ssym}^{\mathrm{cl}}(S) = S$. Thus, $\bigcup_{i=0}^{5} T^{\sigma_i} \subseteq S$. Therefore, $\mathrm{Ssym}^{\mathrm{cl}}(T) = \bigcup_{i=0}^{5} T^{\sigma_i}$.
(iv) We show that $\mathrm{Cyc}^{\mathrm{cl}}(T) = T \cup T^+ \cup T^-$. First, we need to prove that $T \cup T^+ \cup T^-$ is cyclic. In view of Proposition 3.1, It suffices to prove that $T \cup T^+ \cup T^- = (T \cup T^+ \cup T^-)^+ =$

$(T \cup T^+ \cup T^-)^-$. From Proposition 1.1 and Remark 1.1, it indeed follows that

$$\begin{aligned}(T \cup T^+ \cup T^-)^+ &= T^+ \cup (T^+)^+ \cup (T^-)^+ \\ &= T^+ \cup T^- \cup T \\ &= T \cup T^+ \cup T^-\end{aligned}$$

And

$$\begin{aligned}(T \cup T^+ \cup T^-)^- &= T^- \cup (T^+)^- \cup (T^-)^- \\ &= T^- \cup T \cup T^+ \\ &= T \cup T^+ \cup T^-.\end{aligned}$$

Furthermore, it is clear that $T \cup T^+ \cup T^-$ is the smallest cyclic ternary relation containing T. Thus, $\mathrm{Cyc}^{\mathrm{cl}}(T) = T \cup T^+ \cup T^-$. □

In the following, we focus on one of the most important types of closure, the transitive closures of a ternary relation. First, we discuss the transitivity properties of ternary relations.[2]

3.2 Transitivity properties of ternary relations

While the transitivity property of a binary relation is quite standard, it comes in a multitude of variations in the ternary case. Also, the fact that the transitivity property can conveniently be expressed as a relational inclusion using the notion of composition, leads us to give several notions of four-point (resp. five-point) transitivity properties. In this section, we discuss the four-point (resp. five-point) transitivity of a ternary relation from the point of view of four-point (resp. five-point) compositions of ternary relations[2]

3.2.1 Four-point transitivity properties

In this subsection, we discuss the four-point transitivity properties of ternary relations from the point of view of associative four-point compositions of ternary relations defined in [23].

Definition 3.2. *Let T be a ternary relation on a set X. For any $i \in \{1, \ldots, 6\}$, T is called $\diamond_i$-transitive if $T \diamond_i T \subseteq T$.*

The following proposition shows, for any $i \in \{1, \ldots, 6\}$, that the powers of a $\diamond_i$-transitive ternary relation are included in this ternary relation. The proof is straightforward.

Proposition 3.3. *Let T be a ternary relation on a set X. For any $i \in \{1, \ldots, 6\}$, T is $\diamond_i$-transitive if and only if $T^{n,\diamond_i} \subseteq T$, for any $n \in \mathbb{N}^*$.*

The following proposition shows, for any $i \in \{1, \ldots, 6\}$, the equivalence between the $\diamond_i$-transitivity of a ternary relation and that of its powers. The proof Is straightforward.

Proposition 3.4. *Let T be a ternary relation on a set X. For any $i \in \{1, \ldots, 6\}$, it holds that T is $\diamond_i$-transitive if and only if $T^{n,\diamond_i}$ is $\diamond_i$-transitive, for any $n \in \mathbb{N}^*$.*

The following proposition shows that, for any $i \in \{1, \ldots, 6\}$, $\diamond_i$-transitivity is preserved under intersection.

Proposition 3.5. *Let $(T_j)_{j \in J}$ be a family of ternary relations on a set X. For any $i \in \{1, \ldots, 6\}$, the $\diamond_i$-transitivity of $(T_j)_{j \in J}$ implies the $\diamond_i$-transitivity of $\bigcap_{j \in J} T_j$.*

Proof. To prove that $\bigcap_{j \in J} T_j$ is $\diamond_i$-transitive, it suffices to prove that $\bigcap_{j \in J} T_j \diamond_i \bigcap_{j \in J} T_j \subseteq \bigcap_{j \in J} T_j$. Suppose that $(T_j)_{j \in J}$ is $\diamond_i$-transitive, i.e., $T_j \diamond_i T_j \subseteq T_j$, for any $i \in \{1, \ldots, 6\}$. This implies that $\bigcap_{j \in J}(T_j \diamond_i T_j) \subseteq \bigcap_{j \in J} T_j$. It is clear that $\bigcap_{j \in J} T_j \diamond_i \bigcap_{j \in J} T_j \subseteq \bigcap_{j \in J}(T_j \diamond_i T_j)$, for any $i \in \{1, \ldots, 6\}$, and hence $\bigcap_{j \in J} T_j \diamond_i \bigcap_{j \in J} T_j \subseteq \bigcap_{j \in J} T_j$. Hence, $\bigcap_{j \in J} T_j$ is $\diamond_i$-transitive, for any $i \in \{1, \ldots, 6\}$. $\square$

3.2.2 Five-point transitivity properties

In this subsection, we discuss the five-point transitivity properties of ternary relations from the point of view of five-point compositions of ternary relations defined in [23].

Definition 3.3. *Let T be a ternary relation on a set X. For any $i \in \{1, \ldots, 12\}$, T is called $\pentagon_i$-transitive if $T \pentagon_i T \subseteq T$.*

As in the binary case, the following propositions show that the powers of a $\pentagon_i$-transitive ternary relation are included in this ternary relation. The proof is straightforward.

Proposition 3.6. *Let T be a ternary relation on a set X. For any $i \in \{1, \ldots, 12\}$, it holds that T is $\pentagon_i$-transitive if and only if $T^{n,\pentagon_i} \subseteq T$, for any $n \in \mathbb{N}^*$.*

The following proposition shows the equivalence between the $\pentagon_i$-transitivity of a ternary relation and that of its powers. The proof is straightforward.

Proposition 3.7. *Let T be a ternary relation on a set X. For any $i \in \{1, \ldots, 12\}$, it holds that T is $\pentagon_i$-transitive if and only if $T^{n,\pentagon_i}$ is $\pentagon_i$-transitive, for any $n \in \mathbb{N}^*$.*

The following proposition shows that for any $i \in \{1, \ldots, 12\}$, $\pentagon_i$-transitivity is preserved under intersection.

Proposition 3.8. *Let $(T_j)_{j \in J}$ be a family of ternary relations on a set X. For any $i \in \{1, \ldots, 12\}$, the $\pentagon_i$-transitivity of $(T_j)_{j \in J}$ implies the $\pentagon_i$-transitivity of $\bigcap_{j \in J} T_j$.*

Proof. To prove that $\bigcap_{j \in J} T_j$ is $\pentagon_i$-transitive, it suffices to prove that $\bigcap_{j \in J} T_j \pentagon_i \bigcap_{j \in J} T_j \subseteq \bigcap_{j \in J} T_j$. Suppose that $(T_j)_{j \in J}$ is $\pentagon_i$-transitive, i.e., $T_j \pentagon_i T_j \subseteq T_j$, for any $i \in \{1, \ldots, 12\}$. This implies that $\bigcap_{j \in J}(T_j \pentagon_i T_j) \subseteq \bigcap_{j \in J} T_j$. It is clear that $\bigcap_{j \in J} T_j \pentagon_i \bigcap_{j \in J} T_j \subseteq \bigcap_{j \in J}(T_j \pentagon_i T_j)$, for any $i \in \{1, \ldots, 12\}$, and hence $\bigcap_{j \in J} T_j \pentagon_i \bigcap_{j \in J} T_j \subseteq \bigcap_{j \in J} T_j$. Hence, $\bigcap_{j \in J} T_j$ Is $\pentagon_i$-transitive, for any $i \in \{1, \ldots, 12\}$. $\square$

3.3 Transitive closures

From a practical point of view, the important question often arises whether it is possible to find (if it exists), for a given type of transitivity, the smallest transitive ternary relation including a given ternary relation.

3.3.1 Four-point transitive closures

In this subsection, we characterize the four-point transitive closures of a ternary relation. Note that, for any $i \in \{1, \ldots, 6\}$, the $\diamond_i$-transitive closure always exist, as the universal relation X^3 is $\diamond_i$-transitive and intersection preserves $\diamond_i$-transitivity. Since $\diamond_i$-composition is associative, for any $i \in \{1, \ldots, 6\}$, the $\diamond_i$-transitive closure of a ternary relation can be written as the union of powers. First we give the following proposition, the proof of which is straightforward.

Proposition 3.9. *Let T and S be two ternary relations on a set X. If $T \subseteq S$, then it holds that $T^{n,\diamond_i} \subseteq S^{n,\diamond_i}$, for any $i \in \{1, \ldots, 6\}$.*

In the following result, we characterize the four-point transitive closures of a ternary relation.

Theorem 3.3. *Let T be a ternary relation on a set X. It holds that the $\diamond_i$-transitive closure of T Is $Tr^{cl}_{\diamond_i}(T) = \bigcup_{n\geq 1} T^{n,\diamond_i}$, for any $i \in \{1, \ldots, 6\}$.*

Proof. Let $i \in \{1, \ldots, 6\}$. First, we prove that $\bigcup_{n\geq 1} T^{n,\diamond_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$. Since $T \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$, it follows from Proposition 3.9 that $T^{n,\diamond_i} \subseteq (\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T))^{n,\diamond_i}$, for any $i \in \{1, \ldots, 6\}$. Since $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$ is $\diamond_i$-transitive, it follows from Proposition 3.3 that $(\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T))^{n,\diamond_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$, for any $i \in \{1, \ldots, 6\}$, and hence, $T^{n,\diamond_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T)$.
Conversely, to prove that $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T) \subseteq \bigcup_{n\geq 1} T^{n,\diamond_i}$, It suffices to prove that

$M = \bigcup_{n\geq 1} T^{n,\diamond_i}$ is $\diamond_i$-transitive. We only give the prove for $i = 1$, as the other cases can be proved analogously. Let $(x, y, z) \in M \diamond_1 M$, then there exist $t \in X$ such that $(x, y, t) \in M$ and $(x, t, z) \in M$. Hence, there exist $m_1, m_2 \in \mathbb{N}^*$ such that $(x, y, t) \in T^{m_1,\diamond_1}$ and $(x, t, z) \in T^{m_2,\diamond_1}$. It then follows that $(x, y, z) \in T^{m_1,\diamond_1} \diamond_1 T^{m_2,\diamond_1}$, which implies that $(x, y, z) \in T^{m_1+m_2,\diamond_1}$, and, hence, $(x, y, z) \in M$. Thus, M is $\diamond_1$-transitive. Since $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T)$ Is the smallest $\diamond_1$-transitive ternary relation including T, it follows that $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_1}(T) \subseteq \bigcup_{n\geq 1} T^{n,\diamond_1}$. Therefore, $\mathrm{Tr}^{\mathrm{cl}}_{\diamond_i}(T) = \bigcup_{n\geq 1} T^{n,\diamond_i}$, for any $i \in \{1, \ldots, 6\}$. $\square$

Remark 3.1. *It is clear that, for a ternary relation T under a non-associative four-point composition, the transitive closure cannot be expressed as the union of its powers, since the notion of powers of T is not well-defined.*

3.3.2 Five-point transitive closures

In this subsection, we characterize the five-point transitive closures of a ternary relation. Note that, for any $i \in \{0, \ldots, 215\}$, the $\pentagon_i$-transitive closure always exists, as the universal relation X^3 is $\pentagon_i$-transitive and intersection preserves $\pentagon_i$-transitivity. As in the binary case, the transitive closure of a ternary relation can be written as the union of powers. However, this only holds for the associative five-point compositions. First, we give the following proposition, the proof of which is straightforward.

Proposition 3.10. *Let T and S be two ternary relations on a set X. If $T \subseteq S$, then it holds that $T^{n,\pentagon_i} \subseteq S^{n,\pentagon_i}$, for any $i \in \{1, \ldots, 12\}$.*

In the following theorem, we characterize the five-point transitive closures of a ternary relation.

Theorem 3.4. *Let T be a ternary relation on a set X. It holds that the $\pentagon_i$-transitive closure of T is $Tr^{cl}_{\pentagon_i}(T) = \bigcup_{n\geq 1} T^{n,\pentagon_i}$, for any $i \in \{1,\ldots,12\}$.*

Proof. Let $i \in \{1,\ldots,12\}$. First, we prove that $\bigcup_{n\geq 1} T^{n,\pentagon_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\pentagon_i}(T)$. Since $T \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\pentagon_i}(T)$, it follows from Proposition 3.10 that $T^{n,\pentagon_i} \subseteq (\mathrm{Tr}^{\mathrm{cl}}_{\pentagon_i}(T))^{n,\pentagon_i}$. Since $\mathrm{Tr}^{\mathrm{cl}}_{\pentagon_i}(T)$ is $\pentagon_i$-transitive, it follows from Proposition 3.6 that $(\mathrm{Tr}^{\mathrm{cl}}_{\pentagon_i}(T))^{n,\pentagon_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\pentagon_i}(T)$, and hence, $T^{n,\pentagon_i} \subseteq \mathrm{Tr}^{\mathrm{cl}}_{\pentagon_i}(T)$.

Conversely, to prove that $\mathrm{Tr}^{\mathrm{cl}}_{\pentagon_i}(T) \subseteq \bigcup_{n\geq 1} T^{n,\pentagon_i}$, it suffices to prove that $S = \bigcup_{n\geq 1} T^{n,\pentagon_i}$ is $\pentagon_i$-transitive. We only give the proof for $i = 1$, as the other cases can be proved analogously. Let $(x, y, z) \in S \pentagon_1 S$, then there exist $t, s \in X$ such that $(x, y, t) \in S$ and $(s, t, z) \in S$. Hence, there exist $n_0, m_0 \in \mathbb{N}^*$ such that $(x, y, t) \in T^{n_0,\pentagon_1}$ and $(s, t, z) \in T^{m_0,\pentagon_1}$. It then follows that $(x, y, z) \in T^{n_0,\pentagon_1} \pentagon_1 T^{m_0,\pentagon_1}$, which implies that $(x, y, z) \in T^{n_0+m_0,\pentagon_1}$, and, hence, $(x, y, z) \in S$. Thus, S is $\pentagon_1$-transitive. Since $\mathrm{Tr}^{\mathrm{cl}}_{\pentagon_1}(T)$ is the smallest $\pentagon_1$-transitive ternary relation including T, It follows that $\mathrm{Tr}^{\mathrm{cl}}_{\pentagon_1}(T) \subseteq \bigcup_{n\geq 1} T^{n,\pentagon_1}$. Therefore, $\mathrm{Tr}^{cl}_{\pentagon_i}(T) = \bigcup_{n\geq 1} T^{n,\pentagon_i}$ for any $i \in \{1,\ldots,12\}$ □

Remark 3.2. *It is clear that, for a ternary relation T under a non-associative five-point composition, the transitive closure cannot be expressed as the union of its powers, since the notion of powers of T is not well-defined.*

GENERAL CONCLUSION

In this work, we have extended the composition of binary relations to the setting of ternary relations. First, we have recalled several types of four-point (resp. five-point) compositions of ternary relations, focusing on the associativity property. These compositions of ternary relations facilitated our study of the different notions of transitivity of a ternary relation, among others. Throughout this work, we have studied closures of ternary relations for some basic properties. More specifically, we have paid particular attention to the transitive closures of a ternary relation.

Given the importance of fuzzy relations, future efforts will be directed to the study of fuzzy ternary relations as well. We anticipate that it will be interesting to extend the closures of ternary relations to the fuzzy setting.

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•ملخص:

في هذا العمل، ندرس مشكلة إغلاق العلاقة الثلاثية بالنسبة إلى خصائص علائقية مختلفة، مع التركيز على الخصائص العديدة للتعدية التي تم اقتراحها للعلاقات الثلاثية خلال السنوات الماضية. وبصفة خاصة، ندرس خصائص التعدية المرتبطة بالتركيبات الترابطية ذات الأربع نقاط والتركيبات الترابطية ذات الخمس نقاط للعلاقات الثلاثية.

•Résumé :

Dans ce travail, nous étudions le problème de la fermeture d'une relation ternaire par rapport à diverses propriétés relationnelles, en mettant l'accent sur les nombreuses propriétés de transitivité qui ont été proposées pour les relations ternaires au cours des dernières années. En particulier, nous examinons les propriétés de transitivité correspondant aux compositions associatives à quatre points et aux compositions associatives à cinq points des relations ternaires.

•Abstract:

In this work, we study the problem of closing a ternary relation with respect to various relational properties, with a focus on the many transitivity properties that have been proposed for ternary relations over the past years. In particular, we consider the transi- tivity properties corresponding to the associative four-point compositions and associative five-point compositions of ternary relations.