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**Mohamed Boudiaf University of M'sila**
**Faculty of Mathematics and Informatics**
**Department of Mathematics**

# *Master memory*

**Field** : Mathematics and Informatics

**Branch** : Mathematics

**Option** : Algebra and Discrete Mathematics

## Theme

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# *On the fuzzy soft sets*

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*Presented by :*

***Ouafa Chettah*** ***Djoweyda Ferahtia***

**In front of the jury composed of :**

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| Soheyb Milles | MCA., | University of M'sila | **President**. |
| Lemnaouar Zedam | Prof., | University of M'sila | **Supervisor.** |
| Abdelaziz Amroune | Prof., | University of M'sila | **Examiner.** |

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# ❖شكر وعرفان

قال رسول الله صلى الله عليه وسلم

**من اصطنع إليكم معروفا فجازوه، فإن عجزتم عن مجازاته فادعوا له حتى يعلم أنكم شكرتم فإن الله شاكر يحب الشاكرين**

الحمد لله على فضله الكبير الذي أجازنا بالتوفيق لإتمام هذا العمل ويسعدنا أن نتقدم بالشكر الجزيل إلى أولئك الذين ساندونا طيلة فترة الدراسة ونقدم لهم هذه الكلمات: تتناثر الكلمات حبرا على صفائح الأوراق لكل من علمنا وأزال غيمة جهل مررنا بها برياح العلم الطيبة ولكل من أعاد رسم ملامحنا وتصحيح عثراتنا ، كما نبعث بتحية شكر وتقدير للأستاذ الفاضل المشرف **لمنور زدام** الذي قدم لنا كل الدعم والتوجيه، والذي كان صبره علينا وتشجيعه لنا أكبر عون وحافز لإتمام هذا العمل خاصة في ظل هذه الظروف الصعبة المصاحبة لجائحة كورونا فله منا جزيل الشكر والتقدير والامتنان. انه لمن دواعي السرور أيضا التقدم بالشكر الجزيل للدكتور **عمرون عبد العزيز** والدكتور **ميلاس صهيب** لقبولهما مناقشة مذكرة تخرجنا. كما نتقدم بالشكر الجزيل لكل من ساهم في إتمام هذا العمل.

❄ فراحتية جويدة ❄

❄ شتح وفاء ❄

# Contents

# Introduction

Molodtsov [2] has introduced the concept of soft sets that can be seen as a new mathematical theory for dealing with uncertainty. Molodtsov has applied this theory to several directions[2, 3, 4], and then formulated the notions of soft number, soft derivative, soft integral, etc. in [5]. The soft set theory has been applied to many different fields with great success. Maji et al. [19] have worked on the theoretical study of soft sets in detail, and[18] have presented an application of soft set in the decision making problem using the reduction of rough sets [23]. Chen et al. [6] have proposed parameterization reduction of soft sets, and then Kong et al.[22] have presented the normal parameterization reduction of soft sets.

In this memory, we present in brief, the concept of fuzzy sets introduced by zadeh [14] and the concept of fuzzy soft sets introduced by Molodtsov.

CHAPTER 1

# Generalities on crisp and fuzzy sets

In this chapter, we include some basic definitions, operations and properties of the crisp and fuzzy sets.

## 1.1 Crisp sets: Basic definitions and applications

- **Definition** [1]

Let $X$ be a set and $A$ be a subset of $X$ $(A \subseteq A) \cdot$ Then the function

$$X_A = \begin{cases} 1 \quad for \quad x \in A \\ 0 \quad for \quad x \notin A. \end{cases}$$

- **Properties of crisp sets operations**[8]

Let $X$ be the universal set, we define operations on $P(X)$ as follows. Let $A$ and $B$ be two sets (they are elements of $P(X)$). Then by $A^c$, called the complement of $A$, we mean the set of all elements in $X$ which are not members of $A$, or

$$A^c = \{a \in X \,/\, a \notin A\}.$$

The union $A \cup B$ of sets $A$ and $B$ is defined to be the set of all elements which are members of $A$ or $B$ or both, or

$$A \cup B = \{x \in U \,|\, x \in A \vee x \in B\}.$$

The intersection $A \cap B$ of sets $A$ and $B$ is defined to be the set of all elements which are members of both $A$ and $B$, in notation

$$A \cap B = \{x \in U \,|\, x \in A \wedge x \in B\}.$$

The fundamental properties of ${}^c$, $\cup$ , $\cap$ , which are similar to $\sim$ , $\vee$ , $\wedge$, respectively, are:

(1) Idempotency

$$A \cup A = A\, ,\; A \cap A = A$$

(2) Commutativity

$$A \cup B = B \cup A\, ,\; A \cap B = B \cap A$$

(3) Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

(4) Absorption

$$A \cup (A \cap B) = A\, ,\; A \cap (A \cup B) = A$$

(5) Distributivity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(6) Identity

$$A \cup \emptyset = A\,,\; A \cup X = X$$

$$A \cap \;X = A\,,\; A \cap \emptyset = \emptyset$$

(7) Laws of Contradiction

$$A \cap A^c = \emptyset$$

(8) Laws of Excluded Middle

$$A \cup A^c = X$$

(9) Involution

$$(A^c)^c = A$$

(10) De Morgan Laws

$$(A \wedge B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

- **Operation on crisp sets** [16]

| Operations | Let $A$, $B$ are two finite sets. Then |
|---|---|
| Union | $A \cup B = \{x \mid x \in A \vee x \in B\}$ |
| Intersection | $A \cap B = \{x \mid x \in A \wedge x \in B\}$ |
| Set Difference | $A - B = \{x \mid x \in A$ and $x \notin B\}$ |
| Complement | $A^c$ or $\overline{A} = \{x \mid x \notin A \wedge x \in U\}$ |

## 1.2 Fuzzy sets

The concept of a fuzzy set on a given universe introduced by L.zadeh [14] in 1965·

### 1.2.1 Definitions and examples

**Definition 1.1.** *[14] fuzzy set $A$ (fuzzy subset of $X$) is defined as a mapping*

$$A : X \to [0,1]$$

*where $A(x)$ is the membership degree of $x$ to the fuzzy set $A$. We denote by $F(x)$ the collection of all fuzzy subset of $x$.*

*Fuzzy sets are generalizations of the classical sets represented by their characteristic function $X_A : X \to \{0,1\}$. In our case $A(x) = 1$ means full membership of $x \in A$, while $A(x) = 0$ expresses non-membership, but in contrary to the classical case other membership degrees are allowed.*

*We identify a fuzzy set with its membership function. Other notations that can be used are the following $\mu_{A(x)} = A(x)$*

*Every crisp (classical) set is also a fuzzy set. We can define the membership of a crisp (classical) set $A \subseteq X$ as its characteristic function*

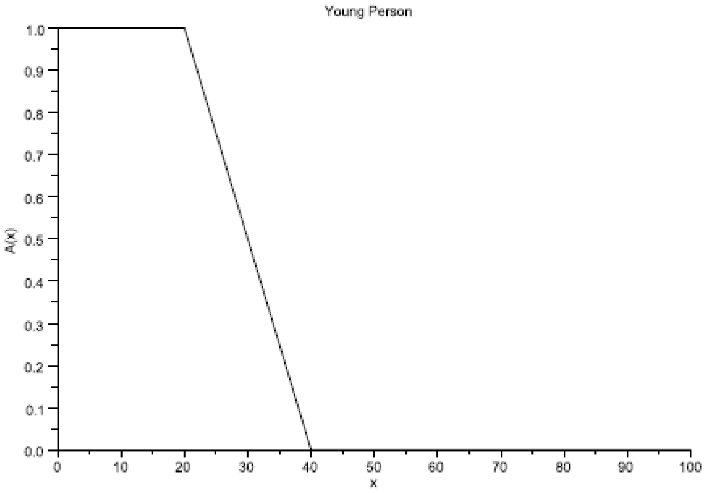

Figure 1.1: Example of a fuzzy set for modeling the expression young person

$$\mu_{A(x)} = \begin{cases} 1 \quad if \quad x \in A \\ 0 \qquad\quad otherwise. \end{cases}$$

*Fuzzy sets are able to model linguistic uncertainty and the following examples show how:*

**Example 1.1.** *In this example, we consider the expression "young" in the context "a young person" in order to exemplify how linguistic expression can be modeled using fuzzy sets.*

*The fuzzy set* $A : [0, 100] \rightarrow [0, 1]$,

$$A(x) = \begin{cases} 1 & if \quad 0 \le x \le 20 \\ \frac{40-x}{20} & if \quad 20 \le x \le 40; \\ 0 & if \quad otherwise \end{cases}$$

*is illustrated in Fig. 1.1*

**Example 1.2.** *Let us consider the fuzzy set* $A : \mathbb{R} \rightarrow [0, 1], A(x) = \frac{1}{1+x^2}$. *This fuzzy set can model* *the linguistic "real number near* $0$*" (see Fig 1.2 ).*

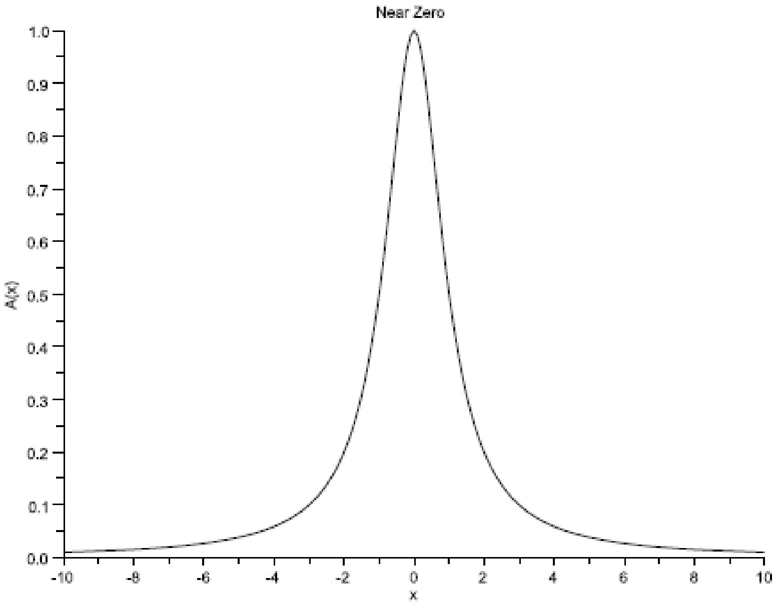

Figure 1.2: Fuzzy set that models a real number near 0

**Example 1.3.** *If given a crisp parameter which is known only through an expert's knowledge expressed in terms of estimates small, medium, and high can be modeled, e.g. by the fuzzy sets in Fig. 1.3*

**Example 1.4.** *Fuzzy sets can be used to express subjective perceptions in a mathematical form. Let* $X = [40, 100]$ *be the interval of temperatures for a room. Fuzzy sets* $A_1, A_2, ..., A_5$ *can be used to model the perception: cold, cool, just, right, warm, and hot (see Figure 1.4 )*

*cold:*

$$A_1(x) = \begin{cases} 1 & if \quad 40 \leq x < 50 \\ \frac{60-x}{10} & if \quad 50 \leq x < 60 \\ 0 & if \quad 60 \leq x \leq 100 \end{cases}$$

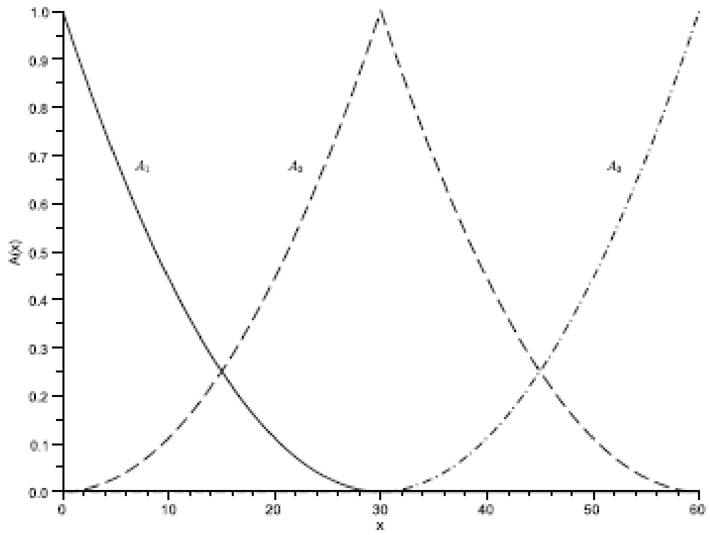

Figure 1.3: Fuzzy set that models a real number near 0

*cool:*

$$A_2(x) = \begin{cases} 0 & if \quad 40 \leq x < 50 \\ \frac{x-50}{10} & if \quad 50 \leq x < 60 \\ \frac{70-x}{10} & if \quad 60 \leq x < 70 \\ 0 & if \quad 70 \leq x \leq 100 \end{cases}$$

*hot:*

$$A_{(5)}(x) = \begin{cases} 0 & if \quad 40 \leq x < 80 \\ \frac{x-80}{10} & if \quad 80 \leq x < 90 \\ 1 & if \quad 90 \leq x \leq 100 \end{cases}$$

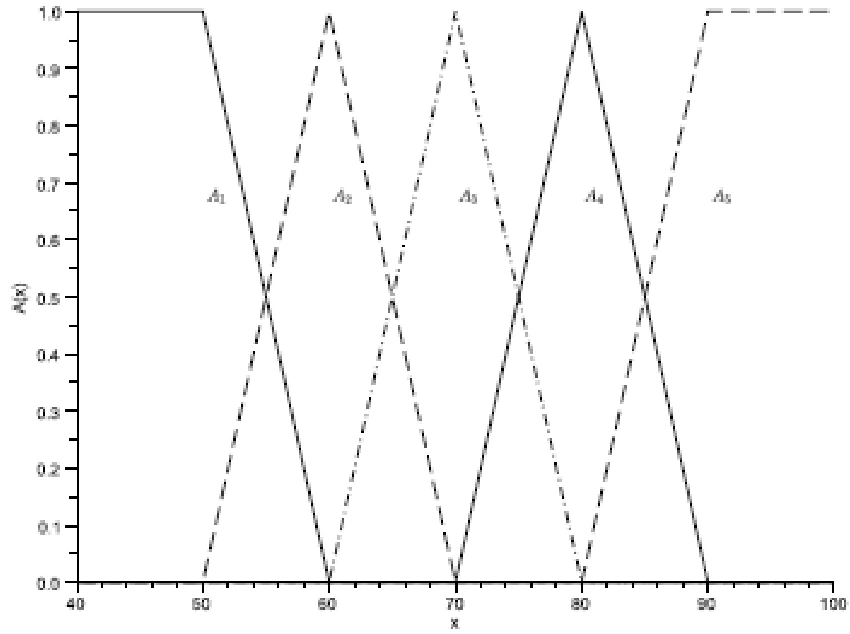

Figure 1.4: Fuzzy sets cold, cool, just Right, warm and hot used in a room temperature control example

**Definition 1.2.** *[1] Let* $A : X \to [0, 1]$ *be a fuzzy set. The level sets of* $A$ *are defined as the crisp sets*

$$A_\alpha = \{x \in X \mid A(x) \geq \alpha\},$$

$0 < \alpha \leq 1$.

$$A_1 = \{x \in X \mid A(x) \geq 1\},$$

*is called the core of the fuzzy set* $A$*, while*

$$suppA = \{x \in X \mid A(x) > 0\}$$

*is called the support of the fuzzy set.*

**Example 1.5.** *Let us consider the cool fuzzy set as in the previous example.*

$$A_2(x) = \begin{cases} 0 & if \quad 40 \leq x < 50 \\ \frac{x-50}{10} & if \quad 50 \leq x < 60 \\ 1 - \frac{x-60}{10} & if \quad 60 \leq x < 70 \\ 0 & if \quad 70 \leq x \leq 100 \end{cases}$$

*Its core is* $(A_2)_1 = \{60\}$*, the* $\frac{1}{2}$*-level set is* $(A_2)_{\frac{1}{2}} = [55, 65]$*, the* $\alpha$*-level set is* $(A_2)_\alpha = [50+10\alpha, 70-10\alpha]$*,* $0 < \alpha \leq 1$ *and the support is* $suppA_2 = (50, 70)$.

**Remark 1.1.** *If the universe of discourse is a finite set* $X = \{x_1, x_2, ..., x_n\}$ *then a fuzzy set* $A : X \to [0,1]$ *can be represented formally as*

$$A = \frac{A(x_1)}{x_1} + \frac{A(x_2)}{x_2} + ... + \frac{A(x_n)}{x_n}$$

**Example 1.6.** *Let us consider the expression "good grande in Mathematics".*

*This expression can be represented as a fuzzy set* $G : \{A, B, C, D, F\} \longrightarrow [0,1]$, $G = \frac{1}{A} + \frac{0.7}{B} + \frac{0.3}{C} + \frac{0}{D} + \frac{0}{F}$.

*The core of* $G$ *is* $G_1 = \{A\}$*, the support is* $suppG = \{A, B, C\}$ *and the* $\frac{1}{2}$*-level set is* $G_{\frac{1}{2}} = \{A, B\}$.

### 1.2.2 Set-theoretical operations on fuzzy sets

in this subsection we present the bassic set-theoretical operations on fuzzy sets.

- **Equality:** [8]

Two fuzzy sets A,B are said to be equal if $\mu_A(x) = \mu_B(x)$, $\forall x \in X$

Let $A = \{(x_1, 0.2), (x_2, 0.5)\}, B = \{(x_1, 0.3), (x_2, 0.5)\}, C = \{(x_1, 0.2), (x_2, 0.5)\}$ be three fuzzy sets

From the above example clearly $A$ and $C$ sets are equal sets bacause $\mu_A(x) = \mu_C(x), \forall x \in X$

Whereas A and B sets are not equal sets because $\mu_A(x) \neq \mu_C(x)$, $\forall x \in X$

- **Union and intersection:** [8]

The classical union ($\cup$) and intersection ($\cap$) of ordinary subsets of $X$ can be extended by the following formulas, proposed by Zadeh (1965):

$$\forall x \in X \, , \, \mu_{A\cup B}(x) = max(\mu_A(x), \mu_B(x))$$

$$\forall x \in X \, , \, \mu_{A\cap B}(x) = min(\mu_A(x), \mu_B(x))$$

where $\mu_{A\cup B}$ and $\mu_{A\cap B}$ are respectively the membership functions of $A \cup B$ and $A \cap B$.

These formulas give the usual union and intersection when the valuation set is reduced to $\{0, 1\}$.

Obviously, there are other extensions of $\cup$ and $\cap$ coinciding with the binary operators.

A justification of the choice of max and min was given by Bellman and Giertz (1973): *max* and *min* are the only operators $f$ and $g$ that meet the following requirements:

(i) The membership value of $x$ in a comounf fuzzy set depends on the membership value of $x$ in the elementary fuzzy sets that form it, but not on anything else:

$$\forall x \in X, \mu_{A\cup B}(x) = f(\mu_A(x), \mu_B(x))$$

$$\mu_{A\cap B}(x) = g(\mu_A(x), \mu_B(x))$$

(ii) $f$ and $g$ are commutative, associative, and mutually distributive operators.

(iii) $f$ and $g$ are continuous and nondecreasing with respect to each of their arguments. Intuitively, the membership of $x$ in $A \cup B$ or $A \cap B$ cannot decrease of $\mu_A(x)$ or $\mu_B(x)$ cannot induce a strong increase of $\mu_{A\cup B}(x)$ or $\mu_{A\cap B}(x)$.

(iv) $f(u,u)$ and $g(u,u)$ are strictly increasing. If $\mu_A(x_1) = \mu_B(x_1) \geq \mu_A(x_2) = \mu_B(x_2)$, then the membership of $x_1$ in $A \cup B$ or $A \cap B$ is certainly strictly greater than that of $x_2$.

(v) Membership in $A \cap B$ requires more, and membership in $A \cup B$ less, than the membership in one of $A$ or $B$:

$$\forall x \in X, \mu_{A\cup B}(x) \leq min(\mu_A(x), \mu_B(x))$$

$$\mu_{A\cap B}(x) \geq max(\mu_A(x), \mu_B(x))$$

Bellman and Giertz (1973):

(i) $\mu_{\overline{A}}(x)$ depend only on $\mu_A(x) : \mu_{\overline{A}}(x) = h(\mu_A(x))$.

(ii) $h(0) = 1$ and $h(1) = 0$, to recover the usual complementation when $A$ is an ordinary subset.

(iii) $h$ is continious and strictly monotonically decreasing, since membership in $\overline{A}$ should become smaller when membership in $A$ increases.

(iv) $h$ is involutive: $h(h(\mu_{\overline{A}}(x))) = \mu_A(x)$.

The above assumptions do not determine $h$ uniquely, not even if we require in addition $h(\frac{1}{2}) = \frac{1}{2}$. However, $h(u) = 1 - u$ if we introduce the following fifth requirement (Gaines, Reference from III.1, 1976b).

(v) $\forall x_1 \in X$, $\forall x_2 \in X$, if $\mu_A(x_1) + \mu_A(x_2) = 1$, then $\mu_{\overline{A}}(x_1) + \mu_{\overline{A}}(x_2) = 1$

(vi) Comlete membership in $A$ and in $B$ implies complete membership in $A \cap B$. Complete lack of membership in $A$ and in $B$ implies complete lack of membership in $A \cup B$:

$$g(1,1) = 1, f(0,0) = 0.$$

The above assumptions are consistent and sufficient to ensure the uniqueness af the choice of union and intersection operators.

Fung and Fu (1975) also found max and min to be the only possible operators. They use a slightly different set of assumptions. They Kept (i) and added the following:

(ii') $f$ and $g$ are commutative, associative, and mutually distributive operators.

(iii') $f$ and $g$ are nondecreasing.

(iv) $f$ and $g$ can be recursively extended to $m \geq 3$ arguments.

(v) $\forall x \in X, f(1, \mu_A(x)) = 1, g(0, \mu_A(x)) = 0.$

The interpretation of these axioms was given in the framework of group decision-making with a slightly more general valuation set (see IV.3.C).

**Example 1.7.** *Let* $A = \{(x_1, 0.4), (x_2, 0.3), (x_3, 1), (x_4, 0.5)\}$ *and* $B = \{(x_1, 0.6), (x_2, 0.8), (x_3, 0; 9), (x_4, 0.3)\}$ *are two fuzzy sets. Then*

$A \cup B = A = \{(x_1, 0.6), (x_2, 0.8), (x_3, 1), (x_4, 0.5)\}$

*Since,* $\mu_{A\cup B}(x) = max(\mu_A(x), \mu_B(x))$

$\forall \mu_{A\cup B}(x_1) = max(\mu_A(x_1), \mu_B(x_1)) = max(0.4, 0.6) = 0.6$ *respectively*

*Similarly* $\mu_{A\cup B}(x_2) = 0.8, \mu_{A\cup B}(x_3) = 1, \mu_{A\cup B}(x_4) = 0.5$

$A \cap B = A = \{(x_1, 0.4), (x_2, 0.3), (x_3, 0.9), (x_4, 0.3)\}$

*Since,* $\mu_{A\cap B}(x) = min(\mu_A(x), \mu_B(x))$

$\forall \mu_{A\cap B}(x_1) = min(\mu_A(x_1), \mu_B(x_1)) = min(0.4, 0.6) = 0.4$

*Similarly* $\mu_{A\cap B}(x_2) = 0.3, \mu_{A\cap B}(x_3) = 0.9, \mu_{A\cap B}(x_4) = 0.3$

- **Complement of a Fuzzy Set**[8]

The complement $\overline{A}$ of $A$ is defined by the membership function (Zadeh,1965)

$$\forall x \in X, \mu_{\overline{A}}(x) = 1 - \mu_A(x) ...(6)$$

The justification of (6) is more difficult than that of (4) and (5). Natural condition to impose on a complementation function $h$ were proposed by Instead of (v), Bellman and Giertz have proposed the following very strong condition:

(vi) $\forall x_1 \in X, \forall x_2 \in X, , \mu_A(x1) - \mu_A(x2) = \mu_{\overline{A}}(x2) - \mu_{\overline{A}}(x_1)$, which means that a certain

change in the membership value in A should have the same effect on the membership in $\overline{A}$. (i),(ii), and (vi) entail $h(u) = 1 - u$. However, there may be situation where (v) or (vi) may appear to be not really necessary assumptions. Sugeno defines the $\lambda$-complement $\overline{A}^{\lambda}$ of $A$

$$\mu_{\overline{A^{\lambda}}}(x) = (1 - \mu_A(x))/(1 + \lambda\mu_A(x)), \lambda \in ]-1, +\infty[.....(7)$$

$\lambda$-complementation satisfies (i), (ii), (iii), and (iv).

Lowen (1978) has developed a more general approach to the complementation of a fuzzy set in the framework of category theory.

When $A$ is an ordinary subset of $X$, the pair $(A, \overline{A})$ is a partition of X provided that $A \neq \varnothing$ and $A \neq X$. When $A$ is a fuzzy set $(\neq \varnothing, \neq X)$, the $(A_1, ..., A_n)$ of fuzzy sets $(\forall i, A_i \neq \varnothing$ and $A_i \neq X)$ such that

$$\forall x \in X, \Sigma_{i=1}^{m} \mu_{Ai}(x) = 1(orthogonality).....(8)$$

is still called a fuzzy partition of $X$ .

**Example 1.8.** $\overline{A} = \{(x_1, 0.6), (x_2, 0.7), (x_3, 0), (x_4, 0.5)\}$

*Since,* $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$

$\mu_{\overline{A}}(x_1) = 1 - \mu_A(x1) = 1 - 0.4 = 0.6$

*Similarly* $\mu_{\overline{A}}(x_2) = 0.7, \mu_{\overline{A}}(x_3) = 0, \mu_{\overline{A}}(x_4) = 0.5$ *respectively[1]*

### 1.2.3 Other operations on fuzzy set

**1 Product of two fuzzy sets:** [7]

$A.B(x) = \{(x_1, 0.24), (x_2, 0.24), (x_3, 0.9), (x_4, 0.15)\}$

Since, $\mu_{A.B}(x) = \mu_A(x) * \mu_B(x)$

$\mu_{A.B}(x_1) = \mu_A(x1) = \mu_A(x_1) * \mu_B(x_1) = (0.4) * (0.6) = 0.24$

Similarly $\mu_{A.B}(x_2) = 0.24, \mu_{A.B}(x_3) = 0.9, \mu_{A.B}(x_4) = 0.15$

**3 Product of fuzzy set with a crisp number:**[7]

Multuply a fuzzy set $A$ by a crisp number $'a'$, results in a new fuzzy set product $a.\ \overline{A}$ with member function

$\mu_{a.A}(x) = a.\mu_A(x)$

$A = \{(x_{1,0.2}), (x_{2,0.5}), (x_{3,0.14})\}$

Since $\mu_{a.A}(x) = a.\mu_A(x_1) = 0.2 * 0.2 = 0.04$

**4 Power of fuzzy set**[7]

The n power of fuzzy set $A$ is denoted by $A$ "whose membership function is given by

$\mu_A^n(x) = (\mu_A(x))^n$

$A = \{(x_1, 0.2), (x_3, 0.6)\}$

$n = 2$

$\mu_A^2(x) = (\mu_A(x))^2 = \{(x_1, 0.04), (x_2, 0.36)\}$

Since $\mu_A^2(x_1) = (\mu_A(x_1))^2 = (0.2)^2 = 0.04$

Similarly $\mu_A^2(x_2) = (\mu_A(x_2))^2 = (0.6)^2 = 0.36$

**5 Disjunctive sum:**[7]

$A \oplus B = (\overline{A} \cap B) \cup (A \cap \overline{B})$

Let $A = \{(x_1, 0.4), (x_2, 0.3), (x_3, 1), (x_4, 0.5)\}$ and $B = \{(x_1, 0.6), (x_2, 0.8), (x_3, 0.9), (x_4, 0.3)\}$ are two fuzzy sets

Then $\overline{A} = \{(x_1, 0.6), (x_2, 0.7), (x_3, 0), (x_4, 0.5)\}$

$(\overline{A} \cap B) = \{(x_1, 0.6), (x_2, 0.7), (x_3, 0), (x_4, 0.3)\}$

$\overline{B} = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0.1), (x_4, 0.5)\}$

$A \oplus B = (\overline{A} \cap B) \cup (A \cap \overline{B}) = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0), (x_4, 0.3)\}$

### 1.2.4 Properties of the set-theoritical operations on fuzzy sets

Let $A$, $B$ are two finite fuzzy sets. Then[1]

| Properties | Fuzzy set theory |
|---|---|
| Commutative | $A \cap B = B \cap A$ |
| Associativity | $(A \cap B) \cap C = A \cap (B \cap C)$ |
| Distributive | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ |
| Idempotence | $A \cup A = A$ , $A \cap A = A$ |
| Identity | $A \cup \varnothing = A$ , $A \cap U = A$, $A \cup U = U$ , $A \cap \varnothing = \varnothing$ |
| Transitivity | If $A \subseteq B, B \subseteq C, then A \subseteq C$ |
| Involution | $\overline{\overline{A}} = A$ |
| De Morgan's laws | $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ , $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ |
| Law Of Extended Middle | $A \cup \overline{A} \neq U$(this property does not hold good in fuzzy set theory) |
| Law Of contradiction | $A \cap \overline{A} = \varnothing$(this property does not hold good in fuzzy set theory) |

### 1.2.5 $\alpha$-Cut of fuzzy sets

- An $\alpha$-cuts set ($A_{\underline{\alpha}}$)

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\},$$

$$A_{\underline{\alpha}} = \{x \in X | \mu_A(x) > \alpha\}.$$

- The Core of a fuzzy set (Core($A$)), that is, the $\alpha$-cut set with $\alpha = 1$.

- The Support of a fuzzy set (Supp($A$)), that is, the strong $\alpha$-cut set with $\alpha = 0$.

- The Width of a fuzzy set, Width($A$) $= |x_2 - x_1|$, where $x_1$ and $x_2$ are crossover points of $A$ defined below.

- Crossover points of a fuzzy set

$$Crossover(A) = \{x \in X | \mu_A(x) = \frac{1}{2}\}.$$

- A fuzzy set is ”normal” if its core is not empty.
- A fuzzy set $A$ is ”convex” if and only if if and only if

$$\forall_{x_1,x_2 \in X} \forall_{\lambda \in [0,1]} \mu_A[\lambda x_1 + (1-\lambda)x_2] \geq min[\mu_A(x_1), \mu_A(x_2)].$$

- Two fuzzy sets $A$ and $B$ are equal if and only if

$$\forall_{x \in X} \mu_A(x) = \mu_B(x).$$

- A fuzzy sets $A$ is a subset of a fuzzy set $B$ if and only if

$$\forall_{x \in X} \mu_A(x) \leq \mu_B(x).$$

Core($A$), Supp($A$), Width($A$), and Crossover($A$) are illustrated in Fig 1.2.5. It is nothing that Core($A$) and Supp($A$) specify classic sets ($X_{Core(A)}(x)$) and $X_{Supp(A)}$.

Fig 1.2.5 denote their characteristic function.

**Theorem 1.1.** *Any fuzzy subset $A$ of the reference set $X$ is defined from its $\alpha$-cuts by:*

$$\forall x \in X \; \mu_A(x) = \sup_{0<\alpha\leq 1} \alpha.\chi_{A_\alpha}(x).$$

*$\chi_{A^\alpha}$ is the characteristic function of $A^\alpha$.*

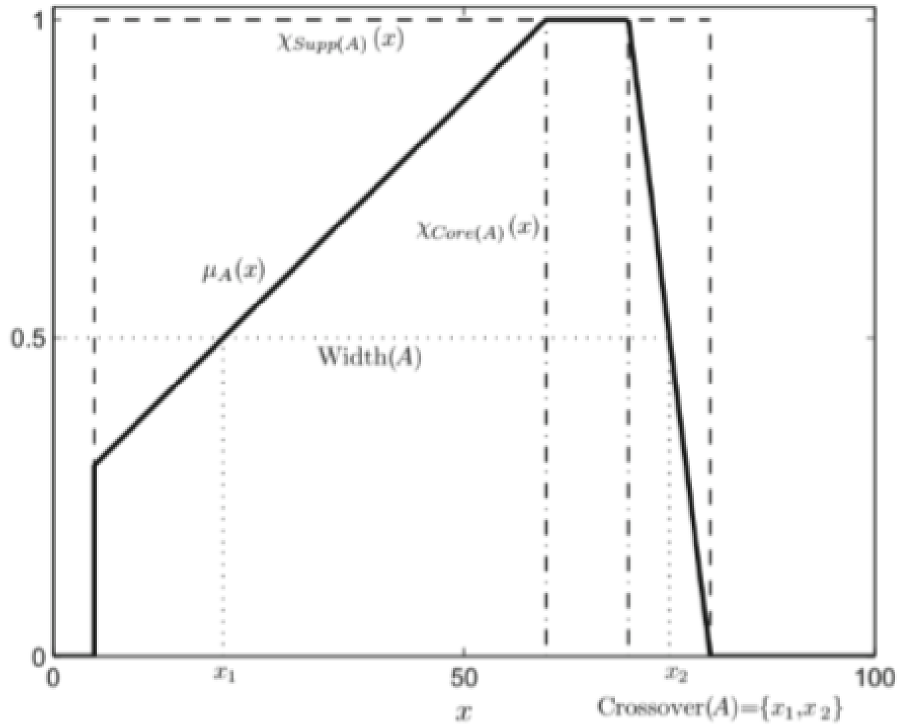

Figure 1.5: The illustration of the core, support, width, and crossover points of a fuzzy set

## 1.3 Further generalizations of fuzzy sets

### 1.3.1 L-fuzzy sets

In this subsection, we present an extension of the notion of a fuzzy set by an arbitrary replacing the real interval $[0, 1]$ by bounded lattice $L$. First, we recall the defintion of abounded lattice.

**Definition 1.3.** *[7] Let L be a set. An L-fuzzy set A is associated with a function $\mu_A$ from the universe X to L.if L has a given structure, such as lattice or group structure $\mathcal{P}_L(X)$, the set of L-fuzzy sets on X, will have this structure too, several structures are worth considering.*

*First, let L be a lattice. The intersection and the union of L-fuzzy sets can be induced in the following way:*

$$\forall x \in X, \mu_{A\cap B}(X) = \inf(\mu_A(X), \mu_B(X)),$$

$$\forall x \in X, \mu_{A\cup B}(X) = \sup(\mu_A(X), \mu_B(X)),$$

*Where* inf *and* sup *denote respectively the greatest lower bound and the least upper bound. Note that membership values of L-fuzzy sets cannot always be compared unless L is linearly ordered. Moreover, distributivity and complementation require a richer structure to be defined.*

*A Brouwerian lattice is a lattice* $L$ *such that* $\forall a \in L, \forall b \in L, \{x \in L \inf(a, x) \leqslant b\}$ *has a least upper bound, denoted* $a\alpha b.a\alpha b$ *is a relative pseudocomplement of a with respect to b.for example, a linearly ordered set having the greatest element* $\mathbb{I}$ *is a Brouwerian lattice.*

*A dual Brouwerian lattice is a lattice* $L$ *such that* $\forall a \in L, \forall b \in L, \{x \in L, \sup(a, x) \geqslant b\}$ *has a greatest lower bound, denoted* $a \in b$. *for instance,* $[0, 1]$ *is a complete Brouwerian lattice:*

$$a\,\alpha\,b = \begin{cases} 1 \quad if \quad a \leqslant b \\ b \quad if \quad b < a \end{cases} ;\; a \in b = \begin{cases} b \quad if \quad a < b \\ 0 \quad if \quad b \leqslant a \end{cases}$$

*the following theorem relates distributivity and Brouwerian lattices (Birkhoff,NF 1948 ):*

*A complete lattice is brouwerian iff inf is totally distrbutive over sup,i.e.,*

$$\forall I \subseteq L,\ \forall a \in L,\ \inf\left(a, \sup_{b\in I} b\right) = \sup_{b\in I} \inf(a, b).$$

*A complete lattice is dual Brouwerian* $\iff$ sup *is totally distributive over inf. Thus, if L has such properties,* $\bigcap$ *and* $\bigcup$ *are mutually distributive (De Luca and Termini, 1972a). moreover, a complete lattice L that is both Brouwerian and dual Brouwerian is boolean if and only if* $\forall a \in L, a\alpha\mathbb{O} = a \in \mathbb{I}$

.

*This property does not hold in* $L = [0, 1]$*.*

*In a boolean lattice,* $a \,\alpha\, b = \sup(\bar{a}, b)$*, where* $\bar{a}$ *is a boolean lattice.The compelent of a Bronwn* (1971) *studied L-fuzzy set A when L is a boolean lattice.The complement of an L-fuzzy sets A is then the* $\bar{A}$ *such that* $\mu_{\bar{A}}(x)$ *is the complement of* $\mu_A(x) \forall x$*. Borwn also gives some results about the convexity and the connectivity of L-fuzzy sets.*

*Negoita and Ralescu* (1975*b*) *considered other kids of structure for L, for instance, semigroupe and semiring structure.*

### 1.3.2 Intuitionistic fuzzy sets

In this subsection, we present the notion of intuitionistic fuzzy sets as an extension of fuzzy sets, by adding the non-membership degree in the definition of fuzzy sets. This notion was proposed by Atanassov [10] in 1983.

**Definition 1.4.** *[10] Let* $X$ *be a nonempty set. An intuitionistic fuzzy set (IFS, for short) A on* $X$ *is an object of the form* $A = \{\langle x, \mu_A(x), v_A(x)\rangle | x \in X\}$ *characterized by a membership function* $\mu_A : X \longrightarrow [0, 1]$ *and a non-membership function* $v_A : X \longrightarrow [0, 1]$ *which satisfy the condition:*

$$0 \leq \mu_A(x) + v_A(x) \leq 1,$$

*for any* $x \in X$*.*

*For any* $x \in X$ *the number* $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ *is called the hesitation degree or the intuitionistic index of* $x$ *to* $A$*.*

*The class of intuitionistic fuzzy sets on* $X$ *is denoted by IFS*$(X)$*.*

*Certainly, fuzzy sets are intuitionistic fuzzy sets by setting* $v_A(x) = 1 - \mu_A(x)$*.*

**Example 1.9.** *Let* $X$ *be the set of all countries with elective governments. Assume that we know* *for every country* $x \in X$ *the percentage of the electorate that has voted for the corresponding government. Denote it by* $M(x)$ *and let* $\mu(x) = \frac{M(x)}{100}$ *(degree of membership, validity, etc.). Let* $v(x) = 1 - \mu(x)$*. This number corresponds to the part of electorate who have not voted for the government. Using only the fuzzy set theory, we cannot consider this value in more detail. However, if we define* $v(x)$ *(degree of non-membership, non-validity, etc.) as the number of votes given to parties or persons outside the government, then we can show the part of electorate who have not voted at all or who have given bad voting-paper and the corresponding number will be* $\pi(x) = 1 - \mu(x) - v(x)$ *(degree of indeterminacy, uncertainty, etc.). Thus, we can construct the set* $\{\langle x, \mu(x), v(x)\rangle | x \in X\}$ *and obviously,*

$$0 \leq \mu(x) + v(x) \leq 1.$$

*For two intuitionistic fuzzy sets* $A$ *and* $B$ *on a set* $X$, *several operations are defined as follows (see, e.g., Atansssov [11, 12, 13], Biswas [20] and Gy [9] ). Here we will present only those which are related to the present work.*

*(i)* $A \subseteq B$ *if* $\mu_A(x) \leq \mu_B(x)$, *and* $v_B(x) \geq v_B(x)$, *for any* $x \in X$;

*(ii)* $A = B$ *if* $\mu_A(x) = \mu_B(x)$, *and* $v_A(x) = v_B(x)$, *for any* $x \in X$;

*(iii)* $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), v_A(x) \vee v_B(x)\rangle, | x \in X\}$;

*(iv)* $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), v_A(x) \wedge v_B(x)\rangle, | x \in X\}$;

*(v)* $\overline{A} = \{\langle x, v_A(x), \mu_A(x)\rangle | x \in X\}$;

*(vi)* $[A] = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\}$;

*(vii)* $\langle A\rangle = \{\langle x, 1 - v_A(x), v_A(x)\rangle | x \in X\}$;

*(viii)* $Ker(A) = \{x \in X | \mu_A(x) = 1 \wedge v_A(x) = 0\}$;

*(ix)* $Supp(A) = \{x \in X | \mu_A(x) > 0 \vee v_A(x) < 1\}$;

*In the sequel, we need the following definition of level sets (which is also often called* $(\alpha, \beta)$*-cuts) of an intuitionistic fuzzy set.*

**Definition 1.5.** *[?] Let* $A$ *be an intuitionistic fuzzy set on a set* $X$*. The* $(\alpha, \beta)$*-cut of* $A$ *is a crisp subset*

$$A_{\alpha,\beta} = \{x \in X | \mu_A(x) \geq \alpha \wedge v_A(x) \leq \beta\},$$

*where* $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1$*.*

Chapter 2

# Crisp and fuzzy soft set

In this chapter, we study the concept of (crisp)fuzzy soft sets and some related concepts.

## 2.1 Soft sets

In this section, we prsent the basic definitions of soft set theory.

### 2.1.1 Definitions and examples

To avoid difficulties, one must use adequate parameterization. Let $U$ be an initial universe set and let $E$ be a set of parameters.

**Definition 2.1.** *[17] A pair $(F, E)$ is called a soft set (over $U$) if and only if $F$ is a mapping of $E$ into the set of all subsets of the set $U$.*

*In other words, the soft set is a parametrized family of subsets of the set $U$. Every set $F(\epsilon)$, $\varepsilon \in E$, from this family may be considered as the set of $\varepsilon$-elements of the set $(F, E)$, or as the set of $\varepsilon$-approximation, let us consider the following examples.*

*(1) A soft set (F,E) describes the attractiveness of the houses which Mr. X is going to buy.*

*$U$ is the set of houses under consideration.*

*$E$ is the set of parameters. Each parameter is a word or a sentence.*

*$E$ = {expensive; beautiful; woonden; cheap; in the green surroundings; modern; in good repair; in bad repair}.*

*In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on.*

*It is worth noting that the sets $F(\varepsilon)$ may be arbitrary. Some of them may be empty, some may have a nonempty intersection.*

(2) *Zadeh's fuzzy set may be considered as a special case of the soft set. Let $A$ be a fuzzy set, and $\mu_A$ be the membership function of the fuzzy set $A$, that is $\mu_A$ is mapping of $U$ into $[0,1]$. Let us consider the family of $\alpha$-level sets fo function $\mu_A$*

$$F(\alpha) = \{x \in U | \mu_A(x) \geq \alpha\}, \alpha \in [0,1].$$

*If we know the family $F$, we can find the function $\mu_A(x)$ by means the following formulae:*

$$\mu_A(x) = \sup_{\alpha \in [0,1]_{x \in F(\alpha)}} \alpha.$$

*Thus, every zade'h fuzzy set $A$ may be considered as the soft set (F,[0,1]).*

(3) *Let $(X, \tau)$ be a topological space, that is, $X$ is a set and $\tau$ is a topology, in other words, $\tau$ is family of subsets of $X$, called the open sets of $X$.*

*Then tha family of open neighborhoods $T(x)$ of point $x$, where $T(x) = \{V \in \tau | x \in V\}$, may be considered as the soft set $(T(x), \tau)$.*

*The way of setting (or describing) any object in the soft set theory principally differs from the way in which we use classical mathematics.*

*In classical mathematics, we construct a mathematical model of an object and define the notion of the exact solution of this model. Usually, the mathematical model is too complicated and we cannot find the exact solution. So in the second step, we introduce the notion of the* *approximate solution and calculate that solution.*

*In the soft set theory, we have the opposite approach to this problem. The initial description of the object has an approximate nature, and we do not need to introduce the notion of the exact solution.*

*The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. We can use any parameterization we prefer: with the help of words and sentences, real number, function, mappings, and so on. It means that the problem of setting the membership function or any similar problem does not arise in the soft set theory.*

### 2.1.2 Operations on soft sets

[17]

Assume that we have a binary operation, denoted by $*$, for subsets of the set $U$. Let $(F, A)$ and $(G, B)$ be soft sets over $U$. Then, the operation $*$ for soft sets is defined in the following way:

$$(F, A) * (G, B) = (H, A \times B),$$

where $H(\alpha, \beta) = F(\alpha) * G(\beta)$, $\alpha \in A, \beta \in B$, and $A \times B$ is the Cartesian product of the sets $A$ and $B$.

This definition takes into account the individual nature of any soft set.

If we produce a lot of operations with soft sets, the result will be a soft set with a very wide set of parameters. Sometimes such expansion of the set of parameters may be useful. So, the intersection of the soft set from Example 1 with itself gives the soft set with a more detailed description. The resulting soft set points out the houses which are expensive and beautiful, modern and cheap, and so on.

In cases when such expansion of the set of parameters is not convenient, we may use a lot of cutting operations. Of course, the acceptance of these cutting operations depends on the special case and on the problem under consideration.

If we want to construct a general mathematical tool, we do not introduce a universal cutting operation for the set of parameters. If we look at the operations with fuzzy sets from the point of view of the soft set theory, we will figure out that all binary operations with fuzzy sets include the universal cutting operation. Let us consider, for example, the first version of the intersection of two fuzzy sets $A$ and $B$.

$$\mu_{A\cap B} = min\{\mu_A(x), \mu_B(x)\}.$$

To three fuzzy sets $A$, $B$ and $A\cap B$, three soft sets correspond to $(F_A, [0,1])$, $(F_B, [0,1])$, $(F_{A\cap B}, [0,1])$, where

$$F_A(\alpha) = \{x \in U|\mu_A(x) \geq \alpha\}, \alpha \in [0,1],$$

$$F_B(\alpha) = \{x \in U|\mu_B(x) \geq \alpha\}, \alpha \in [0,1],$$

$$F_{A\bigcap B}(\alpha) = \{x \in U|\mu_A(x) \geq \alpha, \mu_B(x) \geq \alpha\}, \alpha \in [0,1],$$

The intersection of soft sets $(F_A, [0,1])$ and $(F_B, [0,1])$ is denoted by $(H, [0,1]) \times [0,1]$. Then, we have

$$H(\alpha,\beta) = F_A(\alpha) \cap F_B(\beta) = \{x \in U|\mu_A(x) \geq \alpha, \mu_B(x) \geq \beta\}.$$

Comparing $H(\alpha,\beta)$ and $F_{A\bigcap B}(\alpha)$, we can see that, in this case, the cutting operation means the changing of the Cartesian product $[0,1] \times [0,1]$ to its diagonal.

The individual nature of a fuzzy set contradicts the universal cutting operation. It causes many difficulties in application areas of the theory.

### 2.1.3 Some applications of soft sets

**•Stability and Regularization** [17]

Let $(M, \rho)$ be a metric space, where $\rho$ is a metric. We shall call space $(M, \rho)$ a model space.

Let $U$ be a set and for every $m \in M$, we have a soft set (F(m),E) over $U$. Such a pair $(F, E)$ is said to be an s-function (soft function) and we shall use the notation $(F, E) : M \longrightarrow U$.

Now, we need to introduce the notion of "smoothness" for s-functions which is similar to "continuity" in the classical case. We understand smoothness as the proximity of two soft function values under the condition that their models are close, too.

To give a formal definition, we have to make more exact the notions of proximity for models $m, n$ and for s-function values $F(m, \alpha)$ and $F(n, \beta)$. To measure model proximity, we shall use the metric $\rho$, and for s-function values, we shall define the closeness as inclusion for sets $F(m, \alpha)$ and $F(n, \beta)$.

**Definition 2.2.** *[17] The s-function $(F, E)$ is said to be internally smooth from above on the pair $(m, \alpha)$ if and only if there exist such parameter $\beta \in E$ and positive number $\delta$ such that for every model $n \in M$, for which $\rho(m, n) \leq \delta$, the following inclusion hold:*

$$\emptyset \neq F(n, \beta) \subset F(m, \alpha).$$

**Definition 2.3.** *[17] The s-function $(F, E)$ is said to be internally smooth from below on the pair $(m, \alpha)$ if and only if there exist such parameter $\beta \in E$ and positive number $\delta$ such that for every* *model $n \in M$, for which $\rho(m,n) \leq \delta$, the following inclusion hold:*

$$\emptyset \neq F(m,\beta) \subset F(n,\alpha).$$

We say that s-function $(F,E)$ is internally from above (from below) on the subset $K$ of the set $M \times E$ if for every pair $(m,\alpha) \in K$, the s-function $(F,E)$ is internally smooth from above (from below) on the pair $(m,\alpha)$.

The words "from above" and "from below" in the Definition2.2 and 2.3 express the connection with the classical notions of continuity from above and continuity from below for point-set mapping. To prove it, let us consider the s-function $(G,P)$,

$$(G,P) : M \longrightarrow T,$$

Were $(M,\rho)$,$(T,\tau)$ are metric space, $P$ is the set of all positive numbers, and $G$ has the following from:

$$G(m,\alpha) = \{t \in T | r(t, g(m)) \leq \alpha\},$$

and $g$ is a point-set mapping from $M$ into the family of all subsets of the set $T$, $\alpha \in P$.

Let (G,P) be an internally smooth from above s-function on the set $m$. What does it mean for the point-set mapping $g$?

It is clear from Definition 2.3, for every positive number $\alpha$, there exists a positive number $\delta$, such that for every model $n \in M$, such that $r(m,n) < \delta$, the following inclusion holds:

$$g(n) \subset \{t \in T | r(t, g(m)) \leq \alpha\}.$$

It simply means that mapping $g$ is semicontinuous from above on the model $m$.

Now, let $(G, P)$ be an s-function internally smooth from below on the set $m \times p$. From Definition 2.3 it follows that for every positive number $\alpha$, there exists a positive number $\delta$ such that for every model $n \in M$, such that $r(m, n) < \delta$, the following inclusion,

$$g(m) \subset \{t \in T | r(t, g(n)) \leq \alpha\},$$

is valid.

**Definition 2.4.** *[17] The s-function $(F, A)$ internally approximates the s-function $(G, B)$ on the set $H$ if and only if for every pair $(m, \beta) \in H$ there exists parameter $\alpha \in A$ such that the inclusion*

$$\emptyset \neq F(m, \alpha) \subset G(m, \beta),$$

*is valid.*

**Definition 2.5.** *[17] The s-function $(F, E)$ is said to be internal regularization from above for the s-function $(G, B)$ on the set $H$ if and only if for every pair $(m, \beta) \in H$, there exist parameter $\alpha \in E$ and positive number $\delta$ for which for every model $n \in M$ for which $\rho(m, n) \leq \delta$, the inclusion*

$$\emptyset \neq F(n, \alpha) \subset G(m, \beta),$$

*is valid.*

**Definition 2.6.** *[17] The s-function $(F, E)$ is said to be internal regularization from below for the s-function $(G, B)$ on the set $H$ if and only if for every pair $(m, \beta) \in H$, there exist parameter*

*$\alpha \in E$ and positive number $\delta$ such that for every model $n \in M$ for which $\rho(m, n) \leq \delta$, the inclusion*

$$\emptyset \neq F(m, \alpha) \subset G(n, \beta),$$

*is valid.*

The notions of external approximation and regularization have asimilar structure but they are based on the inverse inclusion

$$F(m, \alpha) \supset G(m, \beta) \neq \emptyset.$$

- **The Game Theory and Operations Research**

Let us introduce some notations:

$n$ is the number of players,

$S_i$ is a set of strategies of player $i$,

$E_i$ is a set of parameters of player $i$,

$S = S_1 \times ... \times S_n$ is a set of situations,

$M(S)$ is a set of all subsets of the set S,

$(F_i, E_i) : M(S) \longrightarrow S, (F_i, E_i)$ is a soft choice function of player $i$.

IF $P \subset S$, that is, $P$ is a subset of admissible strategies, and $\epsilon$ is a parameter, $\epsilon \in E_i$, then $F_i(P, \epsilon)$ is a set of $\epsilon$-optimal situations for player $i$.

We will call such a game a soft game in a normal form with the following notation:

$$\langle (F_i, E_i), S_i, i = 1, ..., n \rangle.$$

For a soft game given in the normal form, the analogue of the Nash equilibrium is the following construction.

**Definition 2.7.** *[17] Situation $s \in S$ is called a situation of soft $\epsilon$-equilibrium, $\epsilon = (\epsilon_1, ..., \epsilon_n)$, $\epsilon_i \in E_i$, if and only if*

$$s \in F_i(s_1 \times ... \times s_{i-1} \times S_i \times s_{i+1} \times ... \times s_n, \epsilon_i),$$

*for every $i = 1, ..., n$.*

We denote the set of all situations or soft $\epsilon$-equilibrium by $N(\epsilon)$. Then, it is natural to call the soft set

$$(N, E_1 \times ... \times E_n),$$

a soft equilibrium

Now, we will suggest the soft guarantee concept.

Suppose, we have only one player and

$$X \text{ is the set of strategies of the player,}$$
$$E \text{ is the set of parameters of the player,}$$
$$S \text{ is the set of situations,}$$
$$(F, E) : M(S) \longrightarrow S \text{ is a soft choice function of the player.}$$

Let $\pi$ be a point-set mapping

$$\pi : X \longrightarrow M(S).$$

The player knows that if he chooses the strategy $x \in X$, then he will get one of the situations $s, s \in \pi(x)$.

We suggest to consider the soft set $(GarF, E)$, where

$$GarF(X, \epsilon) = \{x \in X | \pi(x) \subset F(\pi(x), \epsilon)\}, \pi(X) = \bigcup_{x \in X} \pi(x)$$

as a soft guarantee concept.

Let us apply the soft sets to the notion of Stackelberg solution. Suppose we have a soft game of two players in normal form

$$\langle (F_i, E_i), S_i, i = 1, 2 \rangle.$$

Player 1 makes the first move. He chooses a strategy $s_1 \in S_1$, and informs Player 2 about his choice. Since Player 2 knows the strategy $s_1$, he chooses his strategy $s_2 \in S_2$ so that $(s_1, s_2) \in F_2(s_1 \times S_2, \epsilon_2)$. We assume that Player 1 knows the value of the parameter $\epsilon_2 \in E_2$.

Player 1 considers uncertain the possible choices of Player 2. Applying the guarantee concept, described above, we come to the soft Stackelberg set $(St_1, E_{12})$ for Player 1, where

$$St_1(S_1 \times S_2, \epsilon_1, \epsilon_2) = \{s_1 \in S_1 | F_2(s_1 \times S_2, \epsilon_2) \subset F_1(S_1 \times S_2, \epsilon_1)\}.$$

•**Soft analysis** [17]

As it was mentioned above, a soft set is the collection of approximate descriptions of an object. The exact description is not necessary. If we want to keep this spirit of approximate descriptions, we should not base the soft set analysis on the classical notion of the limit.

We suggest the notion of ”soft limit” for real function, it is based on the following treatment: a number $A$ is a soft limit of the function $f$ at a point $a$, if from the fact that $x$ is close to $a$, it follows that $f(x)$ is close to $A$. To give a formal definition, we have to define exactly the notion of proximity. We will assume that for every point $x \in \mathbb{E}$, we have the set $\tau(x) \subset \mathbb{E}$ which is defined as a set of $\tau$-close points to the point $x$. Let $\alpha, \beta, \epsilon$ be positive numbers, too.

**Definition 2.8.** *[17] The upper $(\epsilon, \tau)$-softlimit of function $f$ at a point $x$ is the following set:*

$$\overline{Softlimit}[f, \epsilon, \tau](x) = \{v \in \mathbb{E} | f(y) \leq v + \epsilon, \forall y \in \tau(x)\}.$$

*The lower $(\epsilon, \tau)$-softlimit of function $f$ at a point $x$ is the following set:*

$$\overline{Softlimit}[f, \epsilon, \tau](x) = \{v \in \mathbb{E} | f(y) \geq v - \epsilon, \forall y \in \tau(x)\}.$$

*The set*

$$Softlimit[f, \alpha, \beta, \tau](x) = \{v \in \mathbb{E} | v - \alpha \leq f(y) \leq v + \beta, \forall y \in \tau(x)\},$$

*is called $(\alpha, \beta, \tau)$-softlimit of the functions $f$ at the point $x$.*

The collection of all these softlimits forms the notions of upper softlimit, lower softlimit, and softlimit of the functions, respectively.

The reader can find more details on softlimits in [2]. Now, we will construct the notion of ”soft approximator” which is the analogue of the classical differential. Let $\alpha$ and $\beta$ be positive numbers.

**Definition 2.9.** *[17] The set*

$$\overline{D}[f, \alpha, \beta, \tau](x) = \{v \in \mathbb{E} | f(y) \leq f(x) + (v + \alpha(x))(y - x) + \beta(x), \forall y \in \tau(x)\}$$

*is called an upper* $(\alpha,\beta,\tau)$*-approximator of function* $f$ *at the point* $x$*.*

*The set*

$$\underline{D}[f,\alpha,\beta,\tau](x) = \{v \in \mathbb{E} | f(y) \geq f(x) + (v - \alpha(x))(y - x) - \beta(x), \forall y \in \tau(x)\}$$

*is called a lower* $(\alpha,\beta,\tau)$*-approximator of function* $f$ *at the point* $x$*.*

The collection of upper and lower $(\alpha,\beta,\tau)$-approximators forms upper and lower soft approximators. Under the soft approximator $D$ we mean the intersection of upper and lower soft approximators

$$D[f,\alpha,\beta,\gamma,\delta,\tau](x) = \overline{D}[f,\alpha,\beta,\tau](x) \cap \underline{D}[f,\gamma,\delta,\tau](x).$$

The constructions used in Definition2.9 are well known in convex analysis.

Here, we present some of the simplest properties of the soft approximators

$$\overline{D}[-f,\alpha,\beta,\tau](x) = -\underline{D}[f,\alpha,\beta,\tau](x),$$

$$\overline{D}[kf,\alpha,\beta,\tau](x) = k\overline{D}[f,\alpha,\beta,\tau](x), k > 0,$$

$$\overline{D}[f,\alpha,\beta,\tau](x) + \overline{D}[g,\gamma,\delta,\tau](x) \subset \overline{D}[f+g,\alpha+\gamma,\beta+\delta,\tau](x),$$

$$D[f,\alpha,\beta,\gamma,\delta,\tau](x) + D[g,\alpha',\beta',\gamma',\delta',\tau](x)$$

$$\subset D[f+g,\alpha+\alpha',\beta+\beta',\gamma+\gamma',\delta+\delta',\tau](x),$$

$$D[f+g,\alpha+\alpha',\beta+\beta',\gamma+\gamma',\delta+\delta',\tau](x)$$

$$\subset D[f,\alpha,\beta,\gamma,\delta,\tau](x) + D[g,\alpha',\beta',\gamma',\delta',\tau](x) + [-X,X],$$

where

$$X = \alpha(x) + \gamma(x) + \alpha'(x) + \gamma'(x) + (\beta(x) + \delta(x) + \beta'(x) + \delta'(x))(\sup_{y\in\tau(x)} |y-x|)^{-1}.$$

Of course, these properties are not as simple as those for classical differential, but in contrast to the classical differential, the soft approximators are smooth soft functionals. Moreover, the soft approximator is the regularization from above and from below for the classical differential . Because of these properties of smoothness and regularization, the soft approximators are more convenient to deal with uncertain information and approximate calculation methods than the classical differentials.

We have introduced the soft analogue of differential, and it naturally raised up the question on the soft analogue of the integral. Two approaches can be used to construct the soft Integral.

The first one, called Riemann approach, is based on the integral sums.

Consider the interval $[a, b] \subset \mathbb{E}$. For simplicity, we will assume that $\tau(x) \subset x + \mathbb{E}^+$, that is, the $\tau$-close points of the point $x$ lay on the right side of $x$. We will consider the sequences of points where every pair of neighbor points are close.

Let us denote Pro$[x, \tau]$, the set of points which can be reached from the point x going only to the $\tau$-close points.

Here the formal definition is given,

$$Pro[x,\tau] = \bigcup_{k=0}^{\infty} \tau^k(x), \tau^0 = \{x\}, \tau^k(x) = \tau(\tau^{k-1}(x)).$$

For $b \in Pro[x, \tau]$, we introduce the set of admissible divisions for the interval [a, b],

$$dis[a, b, \tau] = \{\overline{x}(x_0, ..., x_n) | x_0 = a, x_n = b, x_{i+1} \in \tau(x_i)\}.$$

Note, that number $n$ depends on the division. Let $\epsilon$ be a noninegative real function.

**Definition 2.10.** *[17] The value*

$$\overline{I}R_a^b[f, \epsilon, \tau] = \sup_{\overline{x} \in dis[a,b,\tau]} \Sigma_{i=0}^{n(\overline{x})-1}\{f(x_i)(x_{i+1} - x_i) - \epsilon(x_i)\},$$

*is called the upper Riemann $(\epsilon, \tau)$-integral of the function $f$ between limits $a$ and $b$. The value*

$$\underline{I}R_a^b[f, \epsilon, \tau] = \inf_{\overline{x} \in dis[a,b,\tau]} \Sigma_{i=0}^{n(\overline{x})-1}\{f(x_i)(x_{i+1} - x_i) + \epsilon(x_i)\},$$

*is called the lower Riemann $(\epsilon, \tau)$-integral of the function $f$ between limits $a$ and $b$.*

The second approach of constructing a soft integral is based on the ideas of Perron. Denote

$$\tau_b(x) = \tau(x) \cap (-\infty, b].$$

We say that function $F$ is $(\epsilon, \tau)$-subfunction of the function $f$ between limits $a$ and $b$, if and only if

(1) $F$ is defined on the set $Pro[a, b, \tau]$,

(2) $F(a) = 0$,

(3) $f(x) \in \overline{D}[F, 0, \epsilon, \tau_b](x)$ for every $x \in Pro[a, b, \tau] \backslash \{b\}$.

We say that function $F$ is $(\epsilon,\tau)$-superfunction of the function $f$ between limits $a$ and $b$, if and only if

(1) $F$ is defined on the set $Pro[a,b,\tau]$,

(2) $F(a)=0$,

(3) $f(x)\in \underline{D}[F,0,\epsilon,\tau_b](x)$ for every $x\in Pro[a,b,\tau]\backslash\{b\}$.

**Definition 2.11.** *[17] We call the upper Perron $(\epsilon,\tau)$-integral of the function $f$ between limits $a$ and $b$ the following value:*

$$\overline{I}P_a^b[f,\epsilon,\tau]=inf F(b),$$

*where the infimum is considered with respect to all $(\epsilon,\tau)$-superfunctions $F$ for function $f$ between limits $a$ and $b$.*

*We call the lower Perron $(\epsilon,\tau)$-integral of the function $f$ between limits $a$ and $b$ the following value:*

$$\underline{I}P_a^b[f,\epsilon,\tau]=sup F(b),$$

*where the supremum is considered with respect to all $(\epsilon,\tau)$-subfunctions $F$ for function $f$ between limits $a$ and $b$.*

It is well known, that in the classical case, Riemann's approach and Perron's approach give us two different notions of the integral. For our case, these two approaches give identical results, that is

$$\overline{I}R_a^b[f,\epsilon,\tau]=\overline{I}P_a^b[f,\epsilon,\tau],$$

$$\underline{I}R_a^b[f,\epsilon,\tau]=\underline{I}P_a^b[f,\epsilon,\tau],$$

is called sample T-mean for function $f$.

## 2.2 Fuzzy soft sets

In this section, we have defined $fs$-sets and their operations. The parameter sets and the approximate functions are crisp. But in the $fs$-sets, while the parameters sets are crisp, the approximate functions are fuzzy subsets of $U$. From now on, we will use $\Gamma_A, \Gamma_B, \Gamma_C, ...,$ etc. for $fs$-sets and $\gamma_A, \gamma_B, \gamma_C, ...,$etc. for their fuzzy approximate functions, respectively.

### 2.2.1 Definitions and examples

**Definition 2.12.** *[15] An $fs$-set $\Gamma_A$ over $U$ is a set defined by a function $\gamma_A$ representing a mapping*

$$\gamma_A : E \longrightarrow F(U)$$

*such that $\gamma_A(x) = \emptyset$ if $x \notin A$.*

*Here, $\gamma_A$ is called a fuzzy approximate function of the $fs$-set $\Gamma_A$, and the value $\gamma_A(x)$ is a fuzzy set called $x$-element of the $fs$-set for all $x \in E$. Thus, an $fs$-set $\Gamma_A$ over $U$ can be represented by the set of ordered pairs*

$$\Gamma_A = \{(x, \gamma_A(x)) : x \in E, \gamma_A(x) \in F(U)\}.$$

*Note that from now on the set of all $fs$-sets over $U$ will be denoted by $FS(U)$.*

**Example 2.1.** *Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a universal set and $E = \{x_1, x_2, x_3, x_4\}$ be a set of parameters. If $A = \{x_1, x_2, x_4\} \subseteq E, \gamma_A(x_1) = \{0.9/u_2, 0.5/u_4\}, \gamma_A(x_2) = U,$ and $\gamma_A(x_4) =$*

$\{0.2/u_1, 0.4/u_3, 0.8/u_5\}$, *then the soft set* $F_A$ *is written by*

$$F_A = \{(x_1, \{0.9/u_2, 0.5/u_4\}), (x_2, U), (x_4, \{0.2/u_1, 0.4/u_3, 0.8/u_5\}\}).$$

**Definition 2.13.** *[15] Let* $\Gamma_A \in FS(U)$. *If* $\gamma_A(x) = \emptyset$ *for all* $x \in E$, *then* $\Gamma_A$ *is called an empty* $fs$*-set, denoted by* $\Gamma_\emptyset$.

**Universal fuzzy soft sets**

**Definition 2.14.** *[15] Let* $\Gamma_A \in FS(U)$. *If* $\gamma_A(x) = U$ *for all* $x \in A$, *then* $\Gamma_A$ *is called* $A$*-universal* $fs$*-set, denoted by* $\Gamma_{\overline{A}}$.

*If* $A = E$, *then the* $A$*-universal* $fs$*-set is called universal* $fs$*-set, denoted by* $\Gamma_{\overline{E}}$.

**Example 2.2.** *Assume that* $U = \{u_1, u_2, u_3, u_4, u_5\}$ *is a universal set and* $E = \{x_1, x_2, x_3, x_4\}$ *is a set of all parameters.*

*If* $A = \{x_2, x_3, x_4\}, \gamma_A(x_2) = \{0.5/u_2, 0.9/u_4\}, \gamma_A(x_3) = \emptyset$ *and* $\gamma_A(x_4) = U$, *then the* $fs$*-set* $\Gamma_A$ *is written by* $\Gamma_A = \{(x_2, \{0.5/u_2, 0.9/u_4\}), (x_4, U)\}$.

*If* $B = \{x_1, x_3\}$, *and* $\gamma_B(x_1) = \emptyset, \gamma_B(x_3) = \emptyset$, *then the* $fs$*-set* $\Gamma_B$ *is an empty* $fs$*-set, i.e.,* $\Gamma_B = \Gamma_\emptyset$.

*If* $C = \{x_1, x_2\}, \gamma_C(x_1) = U$, *and* $\gamma_C(x_2) = U$, *then the* $fs$*-set* $\Gamma_C$ *is a c-universal* $fs$*-set, i.e.,* $\Gamma_C = \Gamma_{\overline{C}}$.

*If D=E, and* $\gamma_D(x_i) = U$ *for all* $x_i \in E$, *where* $i = 1, 2, 3, 4$, *then the* $fs$*-set* $\Gamma_D$ *is a universal* $fs$*-set, i.e.,* $\Gamma_D = \Gamma_{\overline{E}}$.

### 2.2.2 Operations on fuzzy soft sets

• **Fuzzy soft subset:**[15]

Let $\Gamma_A, \Gamma_B \in FS(U)$. Then $\Gamma_A$ is an $fs$-subset of $\Gamma_B$, denoted by $\Gamma_A \subseteq \Gamma_B$. for all $x \in E$. if $x \in E, \Gamma_A(x) \subseteq \Gamma_B(x)$ for all $x \in E$.

**Remark 2.1.** *$\Gamma_A \subseteq \Gamma_B$ does not imply that every element of $\Gamma_A$ is an element of $\Gamma_B$ as in the definition of the classical subset.*

*For example, assume that $U = \{u_1, u_2, u_3, u_4\}$ is a universal set of objects and $E = \{x_1, x_2, x_3\}$ is a set of all the parameters. If $A = \{x_1\}, B = \{x_1, x_3\}, \Gamma_A = \{(x_1, \{\{0.2/u_2\}\})\}$ and $\Gamma_B = \{(x_1, \{0.9/u_2, 0.3/u_3, 0.5/u_4\}), (x_3, \{0.2/u_1, 0.7/u_5\})\}$, then for all $x \in E, \gamma_A(x) \subseteq \gamma_B(x)$ is valid. Hence, $\Gamma_A \subseteq \Gamma_B$. It is clear that $(x_1, \{0.2/u_2\}) \in \Gamma_A$, but $(x_1, \{0.2/u_2\}) \notin \Gamma_B$.*

**properties:** [15] Let $\Gamma_A, \Gamma_B \in FS(U)$. Then,

(1) $\Gamma_A \subseteq \Gamma_{\overline{E}}$

(2) $\Gamma_\emptyset \subseteq \Gamma_A$

(3) $\Gamma_A \subseteq \Gamma_A$

(4) $\Gamma_A \subseteq \Gamma_B$ and $\Gamma_B \subseteq \Gamma_C \Rightarrow \Gamma_A \subseteq \Gamma_C$

They can be proved easily by using the fuzzy approximate function of the $fs$-sets.

• **Equality of two fuzzy soft sets:**

Let $\Gamma_A, \Gamma_B \in FS(U)$. Then $\Gamma_A$ and $\Gamma_B$ are $fs$-equal, written as $\Gamma_A = \Gamma_B$ if and only if $\gamma_A(x) = \gamma_B(x)$ for all $x \in E$.

**properties:** [15] Let $\Gamma_A, \Gamma_B \in FS(U)$. Then,

(1) $\Gamma_A = \Gamma_B$ and $\Gamma_B = \Gamma_C \Rightarrow \Gamma_A = \Gamma_C$

(2) $\Gamma_A \subseteq \Gamma_B$ and $\Gamma_B \subseteq \Gamma_A \Leftrightarrow \Gamma_A = \Gamma_B$

The proofs are trivial.

- **Complement of the fuzzy soft sets:**[15]

Let $\Gamma_A \in FS(U)$. Then, the complement $\Gamma_A^{\overline{c}}$ of $\Gamma_A$ is an $fs$-set such that

$$\gamma_{A^{\overline{c}}}(x) = \gamma_A^c(x),$$

for all $x \in E$,

where $\gamma_A^c$ is the complement of the set $\gamma_A(x)$.

**Properties:** Let $\Gamma_A \in FS(U)$. Then,

(1) $(\Gamma_A^{\overline{c}})^{\overline{c}} = \Gamma_A$

(2) $\Gamma_\emptyset^{\overline{c}} = \Gamma_{\overline{E}}$

By using the fuzzy approximate functions of the $fs$-sets, the proofs are straightforward.

- **Union of two fuzzy soft sets:**[15]

Let $\Gamma_A, Gamma_B \in FS(U)$. Then the union of $\Gamma_A$ and $\Gamma_B$, denoted by $\Gamma_A \cup \Gamma_B$, is defined by its fuzzy approximate function

$$\gamma_{A\cup B}(x) = \gamma_A(x) \cup \gamma_B(x),$$

for all $x \in E$.

**Properties:** [15] Let $\Gamma_A, \Gamma_B, \Gamma_C \in FS(U)$. Then,

(1) $\Gamma_A \cup \Gamma_A = \Gamma_A$

(2) $\Gamma_A \cup \Gamma_\emptyset = \Gamma_A$

(3) $\Gamma_A \cup \Gamma_{\overline{E}} = \Gamma_{\overline{E}}$

(4) $\Gamma_A \cup \Gamma_B = \Gamma_B \cup \Gamma_A$

(5) $(\Gamma_A \cup \Gamma_B) \cup \Gamma_C = \Gamma_A \cup (\Gamma_B \cup \Gamma_C)$

The proofs can be easily obtained from definition 2.18.

• **Intersection of two fuzzy soft sets:**

Let $\Gamma_A, \Gamma_B \in FS(U)$. Then the intersection of $\Gamma_A$ and $\Gamma_B$, denoted by $\Gamma_A \cap \Gamma_B$, is defined by its fuzzy approximate function

$$\gamma_{A\cap B}(x) = \gamma_A(x) \cap \gamma_B(x),$$

for all $x \in E$.

**Properties:**[15] Let $\Gamma_A, \Gamma_B, \Gamma_C \in FS(U)$. Then,

(1) $\Gamma_A \cap \Gamma_A = \Gamma_A$

(2) $\Gamma_A \cap \Gamma_\emptyset = \Gamma_\emptyset$

(3) $\Gamma_A \cap \Gamma_{\overline{E}} = \Gamma_A$

(4) $\Gamma_A \cap \Gamma_B = \Gamma_B \cap \Gamma_A$

(5) $(\Gamma_A \cap \Gamma_B) \cap \Gamma_C = \Gamma_A \cap (\Gamma_B \cap \Gamma_C)$

The proofs can be easily obtained from Definition 2.19

**Remark 2.2.** *Let* $\Gamma_A \in FS(U)$.*If* $\Gamma_A \neq \Gamma_\emptyset$ *and* $\Gamma_A \neq \Gamma_{\overline{E}}$ *then* $\Gamma_A \cup \Gamma_A^{\overline{c}} \neq \Gamma_{\overline{E}}$ *and* $\Gamma_A \cap \Gamma_A^{\overline{c}} \neq \Gamma_\emptyset$.

**Properties:** [15] Let $\Gamma_A, \Gamma_B \in FS(U)$. Then, De Morgan's laws are valid as follows:

(1) $(\Gamma_A \cup \Gamma_B)^{\overline{c}} = \Gamma_A^{\overline{c}} \cap \Gamma_B^{\overline{c}}$

(2) $(\Gamma_A \cap \Gamma_B)^{\overline{c}} = \Gamma_A^{\overline{c}} \cup \Gamma_B^{\overline{c}}$.

The proofs can be obtained by using the respective approximate functions. For all $x \in E$,

$$\begin{aligned}
\gamma_{(A\cup B)^{\overline{c}}}(x) &= \gamma^c_{A\cup B}(x) \\
&= (\gamma_A(x) \cup \gamma_B(x))^c \\
&= (\gamma_A(x))^c \cap (\gamma_B(x))^c \\
&= \gamma^c_A(x) \cap \gamma^c_B(x) \\
&= \gamma_{A^{\overline{c}}}(x) \cap \gamma_{B^{\overline{c}}}(x)) \\
&= \gamma_{A^c \cap B^c}(x)
\end{aligned}$$

The proof of (2) is similar.

**Properties:** [15] Let $\Gamma_A, \Gamma_B, \Gamma_C \in FS(U)$. Then,

(1) $\Gamma_A \cup (\Gamma_B \cap \Gamma_C) = (\Gamma_A \cup \Gamma_B) \cap (\Gamma_A \cup \Gamma_C)$

(2) $\Gamma_A \cap (\Gamma_B \cup \Gamma_C) = (\Gamma_A \cap \Gamma_B) \cup (\Gamma_A \cap \Gamma_C)$

For all $x \in E$,

(1)

$$\begin{aligned}
\gamma_{A\cup(B\cap C)}(x) &= \gamma_A(x)\cup\gamma_{B\cap C}(x) \\
&= (\gamma_A(x)\cup(\gamma_B(x)\cap\gamma_C(x)) \\
&= (\gamma_A(x)\cup(\gamma_B(x))\cap(\gamma_A(x)\cup\gamma_C(x)) \\
&= \gamma_{A\cup B}(x)\cap\gamma_{A\cup C}(x) \\
&= \gamma_{(A\cup B)\cap(A\cup C)}(x).
\end{aligned}$$

Likewise, the proof of (2) can be made in a similar way.

We note that the binary $fs$-sets operations, $\cap$ and $\cup$ used in the subscripts of fuzzy approximate functions, are not classical set operations. They indicate that $\gamma_{A\cap B}$ and $\gamma_{A\cup B}$ are the fuzzy approximate function of $\Gamma_A\cap\Gamma_B$ and $\Gamma_A\cup\Gamma_B$, respectively.

### 2.2.3 Fuzzy soft aggregation operators

In this section, we define an $fs$-aggregation operator that produces an aggregate fuzzy set from an $fs$-set and its cardinal set. The approximate functions of an $fs$-set are fuzzy. An $fs$-aggregation operator on the fuzzy sets is an operation by which several approximate functions of an $fs$-set are combined to produce a single fuzzy set which is the aggregated fuzzy set of the $fs$-set. Once an aggregated fuzzy set has been arrived at, it may be necessary to choose the best single crisp alternative from this set.

**Definition 2.15.** *[15] Let $\Gamma_A\in FS(U)$. Assume that $U=\{u_1,u_2,...,u_m\}$, $E=\{x_1,x_2,...,x_n\}$ and $A\subseteq E$ then the $\Gamma_A$ can be presented by the following table,*

$$
\begin{array}{c|cccc}
\Gamma_A & x_1 & x_2 & \dots & x_n \\
\hline
u_1 & \mu_{\gamma_{A(x_1)}(u_1)} & \mu_{\gamma_{A(x_2)}(u_1)} & \cdots & \mu_{\gamma_{A(x_n)}(u_1)} \\
u_2 & \mu_{\gamma_{A(x_1)}(u_2)} & \mu_{\gamma_{A(x_2)}(u_2)} & \cdots & \mu_{\gamma_{A(x_n)}(u_2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
u_m & \mu_{\gamma_{A(x_1)}(u_m)} & \mu_{\gamma_{A(x_2)}(u_m)} & \cdots & \mu_{\gamma_{A(x_n)}(u_m)}
\end{array}
$$

*where* $\mu_{\gamma_{A(x_j)}}(u_i)$ *is the membership function of* $\gamma_A$.

*If* $a_{ij} = \mu_{\gamma_{A(x_j)}}(u_i)$, *for* $i = 1, 2, ..., m$ *and* $j = 1, 2, ..., n$, *then the* $fs$*-set* $\Gamma_A$ *is uniquely characterized by a matrix,*

$$
[a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
$$

*is called an* $m \times n$ $fs$*-matrix of the* $fs$*-set* $\Gamma_A$ *over* $U$.

**Definition 2.16.** *[15] Let* $\Gamma_A \in FS(U)$. *Then, the cardinal set of* $\Gamma_A$, *denoted by* $c\Gamma_A$ *and defined by*

$$
c\Gamma_A = \{\mu_{c\Gamma_A}(x)| : x \in E\},
$$

*is a fuzzy set over* $E$. *The membership function* $\mu_{c\Gamma_A}$ *of* $c\Gamma_A$ *is defined by*

$$
\mu_{c\Gamma_A} : E \longrightarrow [0, 1], \mu_{c\Gamma_{A(x)}} = \frac{|\gamma_A(x)|}{|U|}
$$

*where* $|U|$ *is the cardinality of universe* $U$, *and* $|\gamma_A(x)|$ *is the scalar cardinality of fuzzy set* $\gamma_A(x)$.

*Naote that the set of all cardinal sets of the $fs$-sets over $U$ will be denoted by $cFS(U)$. It is clear that $cFS(U) \subseteq F(E)$.*

**Definition 2.17.** *[15] Let $\Gamma_A \in FS(U)$ and $c\Gamma_A \in cFS(U)$. Assume that $E = \{x_1, x_2, ..., x_n\}$ and $A \subseteq E$ then $c\Gamma_A$ can be presented by the following table*

| $E$ | $x_1$ | $x_2$ | $...$ | $x_n$ |
|---|---|---|---|---|
| $\mu_{c\Gamma_A}$ | $\mu_{c\Gamma_{A(x1)}}$ | $\mu_{c\Gamma_{A(x2)}}$ | $\cdots$ | $\mu_{c\gamma_{A(x_n)}}$ |

*If $a_{ij} = \mu_{c\Gamma_{A(x_j)}}$ for $j = 1, 2, ..., n$, then the cardinal set $c\Gamma_A$ is uniquely characterized by a matrix,*

$$[a_{ij}]_{1\times n} = [a_{11}a_{12}...a_{1n}]$$

*which is called the cardinal matrix of the cardinal set $c\Gamma_A$ over $E$.*

**Definition 2.18.** *[15] Let $\Gamma_A \in FS(U)$ and $c\Gamma_A \in cFS(U)$. Then $fs$-aggregation operator, denoted by $FS_{agg}$, is defined by*

$$FS_{agg} : cFS(U) \times FS(U) \longrightarrow F(U), FS_{agg}(c\Gamma_A, \Gamma_A) = \Gamma_A^*$$

*where*

$$\Gamma_A^* = \{\mu_{\Gamma_A^*}(u)/u : u \in U\}$$

*is a fuzzy set over $U$. $\Gamma_A^*$ is called the aggregate fuzzy set of the $fs$-set $\Gamma_A$. The membership function $\Gamma_A^*$ of $\mu_{\Gamma_A^*}$ is defined as follows:*

$$\mu_{\Gamma_A^*} : U \longrightarrow [0,1], \mu_{\Gamma_A^*}(u) = \frac{1}{|E|}\sum_{x\in E} \mu_{c\Gamma_A}(x)\mu_{\gamma_{A(x)}}(u),$$

*where $|E|$ is the cardinality of $E$.*

**Definition 2.19.** *[15] Let $\Gamma_A \in FS(U)$ and $\Gamma_A^*$ be its aggregate fuzzy set. Assume that $U = \{u_1, u_2, ..., u_m\}$, then the $\Gamma_A^*$ can be presented by the following table*

| $\Gamma_A$ | $\mu_{\Gamma_A^*}$ |
|---|---|
| $u_1$ | $\mu_{\Gamma_A^*}(u_1)$ |
| $u_2$ | $\mu_{\Gamma_A^*}(u_2)$ |
| $\vdots$ | $\vdots$ |
| $u_m$ | $\mu_{\Gamma_A^*}(u_m)$ |

*If $a_{i1} = \mu_{\Gamma_{A^*(um)}}(u_i)$ for $i = 1, 2, ..., m$, then $\Gamma_A^*$ is uniquely characterized by the matrix,*

$$[a_{i1}]_{m\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

*which is called the aggregate matrix of $\Gamma_A^*$ over $U$.*

**Theorem 2.1.** *[15] Let $\Gamma_A \in FS(U)$ and $A \subseteq E$. If $M_{\Gamma_A}$, $M_{c\Gamma_A}$ and $M_{\Gamma_A^*}$are representation matrices of $\Gamma_A$, $c\Gamma_A$ and $\Gamma_A^*$, respectively, then*

$$|E| \times M_{\Gamma_A^*} = M_{\Gamma_A} \times M_{c\Gamma_A}^T$$

*where $M_{c\Gamma_A}^T$ is the transposition of $M_{c\Gamma_A}$ and $|E|$ is the cardinality of $E$.*

It is sufficient to consider $[a_{ij}]_{m\times 1} = [a_{ij}]_{m\times n} \times [a_{ij}]_{1\times n}^T$.

Theorem 2.2.3 is applicable to computing the aggregate fuzzy set of an $fs$-set.

### 2.2.4 Applications

[15]

Once an aggregate fuzzy set has been arrived at, it may be necessary to choose the best alternative from this set. Therefore, we can make a decision by the following algorithm

(Step 1) Construct an $fs$-set $\Gamma_A$ over $U$,

(Step 2) Find the cardinal set $c\Gamma_A$ of $\Gamma_A$,

(Step 3) Find the aggregate fuzzy set $\Gamma^*_A$ of $\Gamma_A$,

(Step 4) Find the best alternative from this set that has the largest membership grade by $max\mu_{\Gamma^*_A}(u)$.

**Example 2.3.** *Suppose a company wants to fill a position. There are eight candidates who form the set of alternatives,* $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$. *The hiring committee consider a set of parameters,* $E = \{x_1, x_2, x_3, x_4, x_5\}$. *For* $i = 1, 2, 3, 4, 5$, *the parameters* $x_i$ *stand for "experience", "computer knowledge", "young age","good speaking" and "friendly", respectively.*

*After a serious discussion each candidate is evaluated from the goals and constraint point of view of according to a chosen subset* $A = \{x_2, x_3, x_4\}$ *of* $E$. *Finally, the committee applies the following steps:*

*Step 1: The committee constructs an* $fs$*-set* $\Gamma_A$ *over* $U$*,*

$$\Gamma_A = \{(x_2, \{0.3/u_2, 0.5/u_3, 0.8/u_5, 0.7/u_7\}),$$

$$(x_3, \{0.4/u_1, 0.4/u_2, 0.9/u_3, 0.3/u_4\}),$$

$$(x_4, \{0.2/u_1, 0.5/u_2, 0.1/u_5, 0.7/u_7, 1/u_8\})\}$$

*Step 2: The cardinal is computed,*

$$c\Gamma_A = \{0.3/x_2, 0.25/x_3, 0.2/x_4\},$$

*Step 3: The aggregate fuzzy set is found by using Theorem 2.1 ,*

$$M_{\Gamma_A^*} = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0.4 & 0.2 & 0 \\ 0 & 0.3 & 0.4 & 0.5 & 0 \\ 0 & 0.5 & 0.9 & 0 & 0 \\ 0 & 0.1 & 0.3 & 0 & 0 \\ 0 & 0.8 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0.3 \\ 0.25 \\ 0.2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.028 \\ 0.058 \\ 0.075 \\ 0.021 \\ 0.052 \\ 0 \\ 0.070 \\ 0.040 \end{bmatrix}$$

*that means,*

$$\Gamma_A^* = \{0.028/u_1, 0.58/u_2, 0.075/u_3, 0.021/u_4, 0.052/u_5, 0/u_6, 0.070/u_7, 0.040/u_8\}),$$

*Step 4: Finally, the largest membership grade is chosen by*

$$max\mu_{\Gamma_A^*}(u) = 0.075$$

*which means that the candidate $u_3$ has the largest membership grade, hence he may be selected for the job.*

# Conclusion

In this work, we have presented the concept of (crisp) fuzzy soft sets and their basic connectives. To facilitate the comprehension of these concepts, we have started by recalling the basic concepts of fuzzy sets introduced by Zadeh.

In addition, we have presented as a motivation some applications of these concepts of (crisp) fuzzy soft sets.

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