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*Quelques caractérisations de certains espaces
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Notation

We will use the following notations throughout the work :

- \mathbb{R}^n is the n -dimensional real Euclidian space.
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the collection of all natural numbers.
- \mathbb{Z} is the collection of all integer numbers.
- \mathbb{C} is the collection of all complex numbers.
- \mathbb{R} is the collection of all real numbers.
- If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands the (Lebesgue)measure of $|E|$.
- For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in \mathbb{R}^n with center x and radius r .
- $\overline{B}(x, r)$ is the closure of the open ball $B(x, r)$
- For any $u > 0, k \in \mathbb{Z}$ we set $C(u) = \{x \in \mathbb{R}^n : u/2 < |x| \leq u\}$ and $C_k = C(2^k)$.
- $\alpha = (\alpha_1 \cdots \alpha_n) \in \mathbb{N}_0^n$ stands for some multi-index, whose length is denoted by $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}.$$

- The Euclidean scalar product of $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is given by

$$x \cdot y = x_1 y_1 + \dots + x_n y_n.$$

- The expression $f \lesssim g$ means that $f < cg$ for some independent constant c (and non-negative functions f and g).
- $f \approx g$ means $f \lesssim g \lesssim f$.
- The notation $X \hookrightarrow Y$ stands for continuous embeddings from X to Y , where X and Y are quasi-normed spaces.
- $f * g = \int_{\mathbb{R}^n} f(x-y)g(y)dy$ is the convolution of functions f and g .
- $L^1_{loc}(\mathbb{R}^n)$ is the collection of all locally integrable functions on \mathbb{R}^n .
- $L_p(\mathbb{R}^n)$ for $0 < p \leq \infty$ stands for the Lebesgue spaces on \mathbb{R}^n normed by (quasi-normed

for $p < 1$)

$$\|f\|_{L_p(\mathbb{R}^n)} = \|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty, \quad 0 < p < \infty$$

and

$$\|f\|_{L_\infty(\mathbb{R}^n)} = \|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$

- If $1 \leq p \leq \infty$, then p' is the conjugate exponent of p given by $\frac{1}{p} + \frac{1}{p'} = 1$
- c_1, c_2, c_3, \dots positive constants, their values may depend on certain parameters and some auxiliary functions, and change from one line to another.
- By $\operatorname{supp} f$ we denote the support of function f .
- $\mathcal{D}(\mathbb{R}^n)$ is the space of functions with continuous derivatives of all orders and compact support.
- $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on \mathbb{R}^n . The topology in the complete locally convex space $\mathcal{S}(\mathbb{R}^n)$ is generated by the norms

$$p_N(\varphi) = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \sum_{|\alpha| \leq N} |D^\alpha \varphi(x)|, \quad N = 1, 2, 3, \dots$$

- The dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of temperate distributions.
- The Fourier transform is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} \exp^{-ix\xi} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

and its inverse Fourier transform by :

$$\mathcal{F}^{-1}f(\xi) = \check{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp^{ix\xi} f(x) dx, \quad x \in \mathbb{R}^n.$$

- f is spectrum in $B(0, r)$ and $r > 0$, $\operatorname{supp} \hat{f} \subset B(0, r)$.
- By $\ell^q, 0 < q \leq \infty$, we denote the space of all (complex) sequences equipped with the quasi-norm

$$\|\{a_k\}_{k \in \mathbb{Z}}\|_{\ell^q} = \left(\sum_{k=-\infty}^{\infty} |a_k|^q \right)^{\frac{1}{q}}.$$

(with the usual modification if $q = \infty$).

- **"i.e."** stands simply for "in other words".
- **"a.e."** stands simply for "almost every where".
- χ_E is the characteristic function of $E \subset \mathbb{R}^n$.
- χ_k , for $k \in \mathbb{Z}$, denote the characteristic function of the set C_k .
- $\eta_{v,m} = 2^{nv}(1 + 2^v|x|)^{-m}$, for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$.

Introduction

Function spaces have been a central topic in modern analysis, and are now of increasing applications in many fields of mathematics especially harmonic analysis and partial differential equations. The most known general scales of function spaces are the scales of Besov spaces and Triebel-Lizorkin spaces and it is known that they cover many well-known classical function spaces such as Hölder-Zygmund spaces, Hardy spaces and Sobolev spaces. For more details one can refer to Triebel's books [7], [8] and [9].

It is well known that Herz spaces play an important role in Harmonic Analysis. After they have been introduced in [4], the theory of these spaces had a remarkable development in part due to its usefulness in applications.

The purpose of this thesis is to present some properties of Herz-type Besov spaces $\dot{K}_q^{\alpha,p}B_\beta^s$. These function spaces introduced earlier in the papers of J. Xu and D. Yang [13], [14] and [15].

The thesis is organised as follows. In Chapter 1 we fix some notations and recall some basics facts on function spaces. In Chapter 2, several basic properties of Herz-type Besov spaces such as the characterization by ball means of differences. In Chapter 3 we present the theorem of an equivalent norm in Herz-Besov spaces based on differences.

Chapitre 1

Function spaces

We present the Fourier analytical definitions of Herz-type Besov spaces $\dot{K}_q^{\alpha,p} B_\beta^s$ and recall their basic properties.

1.1 Definitions and basic properties

Definition 1.1.1 *Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}$ is defined by*

$$\dot{K}_q^{\alpha,p} := \{f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_q^p \right)^{\frac{1}{p}} < \infty,$$

with the usual modifications made :

(i) *If $p = \infty$, then*

$$\|f\|_{\dot{K}_q^{\alpha,\infty}} = \sup_{k \in \mathbb{Z}} 2^{k\alpha} \|f\chi_k\|_q,$$

(ii) *If $p = q = \infty$, then*

$$\|f\|_{\dot{K}_\infty^{\alpha,\infty}} = \sup_{k \in \mathbb{Z}} 2^{k\alpha} \|f\chi_k\|_\infty.$$

Remark 1.1.2 (i) The spaces $\dot{K}_q^{\alpha,p}$ are quasi-Banach spaces and if $\min(p, q) \geq 1$, then $\dot{K}_q^{\alpha,p}$ are Banach spaces.

(ii) If $\alpha = 0$ and $0 < p = q \leq \infty$, then $\dot{K}_p^{0,p}$ coincides with the Lebesgue spaces $L^p(\mathbb{R}^n)$.

(iii) If $0 < p_1 \leq p_2 \leq \infty$, then we may derive the embedding

$$\dot{K}_q^{\alpha,p_1} \hookrightarrow \dot{K}_q^{\alpha,p_2}.$$

(iii) Let $0 < p, q \leq \infty, r > 0, \alpha > -\frac{n}{q}, j \in \mathbb{N}_0$ and $m > n$ large enough. From [2] there exists $c > 0$ such that for all $f \in \dot{K}_q^{\alpha,p}$ we have

$$\|(\eta_{j,m} * |f|^q)^{1/q}\|_{\dot{K}_q^{\alpha,p}} \leq c \|f\|_{\dot{K}_q^{\alpha,p}}, \quad (1.1)$$

where $c > 0$ is independent of $j \in \mathbb{N}_0$.

Definition 1.1.3 Let ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ such that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq \frac{3}{2}. \end{cases}$$

we put $\varphi_0(x) = \psi(x)$, $\varphi_1(x) = \psi(\frac{x}{2}) - \psi(x)$ and

$$\varphi_j(x) = \varphi_1(2^{-j+1}x) \quad \text{for } j = 2, 3, \dots$$

Then we have

- $\text{supp}\varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| \leq 3 \cdot 2^{j-1}\}$,
- $\varphi_j(x) = 1$ for $\{3 \cdot 2^{j-2} \leq |x| \leq 2^j\}$
- $\psi(x) + \sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$.
- The system of functions $\{\varphi_j\}_{j \in \mathbb{N}_0}$ is called a smooth dyadic resolution of unity.
- We define the convolution operators Λ_j and Q_k by the following :

$$\Lambda_j f = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} f), \quad j \in \mathbb{N},$$

$$Q_k f = \mathcal{F}^{-1}(\psi(2^{-k} \cdot) \mathcal{F} f), \quad k \in \mathbb{N}_0,$$

and we set $Q_0 = \Lambda_0$

- The Littlewood-Paley decomposition

$$f = \sum_{j=0}^{\infty} \Lambda_j f.$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

Definition 1.1.4 Let $\alpha, s \in \mathbb{R}$ and $0 < p, q, \beta \leq \infty$. The Herz-type Besov space $\dot{K}_q^{\alpha,p} B_\beta^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

- If $\beta < \infty$, then

$$\|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s} = \left(\sum_{j=0}^{\infty} 2^{js\beta} \|\Lambda_j f\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{1/\beta} < \infty,$$

- If $\beta = \infty$, then

$$\|f\|_{\dot{K}_q^{\alpha,p} B_\infty^s} = \sup_{j \in \mathbb{N}_0} 2^{js} \|\Lambda_j f\|_{\dot{K}_q^{\alpha,p}}.$$

Théorème 1.1.5 let $s \in \mathbb{R}, 0 < p, q, \beta \leq \infty$ and $\alpha > -\frac{n}{q}$.

(i) the spaces $\dot{K}_q^{\alpha,p} B_\beta^s$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j \in \mathbb{N}_0}$ appearing in their definitions (in the sens of equivalent quasi-norms)

(ii) Both $\dot{K}_q^{\alpha,p} B_\beta^s$ are quasi-Banach spaces and if $p, q, \beta \geq 1$, then $\dot{K}_q^{\alpha,p} B_\beta^s$ are Banach spaces.

Definition 1.1.6 Let $s \in \mathbb{R}$ and $0 < p, \beta \leq \infty$. The Besov space $B_{p,\beta}^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,\beta}^s} = \left(\sum_{j=0}^{\infty} 2^{js\beta} \|\Lambda_j f\|_p^\beta \right)^{1/\beta} < \infty,$$

Remark 1.1.7 (i) With $p = q = \infty, s > 0$, one recovers Hölder-Zygmund spaces $C^s = B_{\infty, \infty}^s$

(ii) For $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leq \infty$.

$$\dot{K}_p^{0,p} B_\beta^s = B_{p,\beta}^s.$$

1.2 Embeddings

Théorème 1.2.1 Let $s \in \mathbb{R}, 0 < p, q \leq \infty$ and $\alpha > -\frac{n}{q}$.

(i) If $0 < \beta_1 \leq \beta_2 \leq \infty$, then

$$\dot{K}_q^{\alpha,p} B_{\beta_1}^s \hookrightarrow \dot{K}_q^{\alpha,p} B_{\beta_2}^s$$

(ii) If $0 < \beta_1, \beta_2 \leq \infty$, and $\epsilon > 0$, then

$$\dot{K}_q^{\alpha,p} B_{\beta_1}^{s+\epsilon} \hookrightarrow \dot{K}_q^{\alpha,p} B_{\beta_2}^s$$

(iii) If $0 < p_1 \leq p_2 \leq \infty$, then

$$\dot{K}_q^{\alpha,p_1} B_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p_2} B_\beta^s.$$

Théorème 1.2.2 (i) Let $s \in \mathbb{R}, 0 < p, q, \beta \leq \infty$, and $\alpha > -n/q$. The Herz-type Besov space $\dot{K}_q^{\alpha,p} B_\beta^s$ is a quasi-Banach space. Furthermore,

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \dot{K}_q^{\alpha,p} B_\beta^s \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

(ii) If $s \in \mathbb{R}, 0 < p, q, \beta < \infty$, and $\alpha > -n/q$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $\dot{K}_q^{\alpha,p} B_\beta^s$.

Théorème 1.2.3 Let $\alpha > 0, 0 < s, p, q \leq \infty$ and $0 < \beta \leq \infty$.

Let us define

$$\sigma_q = n \left(\frac{1}{\min(1, q)} - 1 \right) \text{ and } \bar{q} = \max(1, q)$$

(i) If $s > \sigma_q + \alpha$, then

$$\dot{K}_q^{\alpha,p} B_\beta^s \hookrightarrow L^{\bar{q}}(\mathbb{R}^n).$$

(ii) Let $0 < p, q < \infty$ and $0 < \alpha < n/p - n/q$. If $s > n/q - n + \alpha$ and $0 < q \leq 1$ or $s \geq \alpha$ and $1 < q < \infty$, then

$$\dot{K}_q^{\alpha,p} F_\beta^s \hookrightarrow L^{\bar{q}}(\mathbb{R}^n).$$

Let C_u be the space of all bounded uniformly continuous functions on \mathbb{R}^n equipped with the sup norm. Concerning embeddings into C_u , we have the following result.

Théorème 1.2.4 Let $\alpha \geq 0$ and $0 < p, q \leq \infty$. Then

$$\dot{K}_q^{\alpha,\theta} B_1^{\alpha+n/q} \hookrightarrow C_u.$$

where

$$\theta = \begin{cases} \infty & \text{if } \alpha = 0 \\ p & \text{if } \alpha > 0. \end{cases}$$

The proof of these results are given in [2].

Chapitre 2

Characterization in terms of ball means of differences

In this chapter we present a equivalent norm of Herz-type Besov spaces in terms of ball means of differences.

2.1 Necessary tools

Various important results have been proved in the space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ under some

The proof of some basic tools is based on the following result, see Tang and Yang [6].assumptions on α, p and q . The conditions $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$, $1 < q < \infty$ and $0 < p \leq \infty$ is crucial in the study of boundness of classical operators in $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ spaces. This fact was first realised by Li and Yang [5] with the proof of the boundedness of the maximal function ;

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad L_{loc}^1.$$

Lemma 2.1.1 *Let $1 < q < \infty$ and $0 < p < \infty$. If f is a locally integrable function on \mathbb{R}^n and $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$, then*

$$\|f\|_{\dot{K}_q^{\alpha,p}} \leq c \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Notice that if $\theta \in L^1$ and $f \in L^1_{loc}$, then

$$|(t^{-n}\theta(\cdot/t)) * f| \leq \|\theta\|_1 \mathcal{M}(f)(x), \quad t > 0, x \in \mathbb{R}^n.$$

We need the following lemma, which is a discrete convolution inequality.

Lemma 2.1.2 *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ be a sequence of positive real numbers and denote*

$$\delta_k = \sum_{j=-\infty}^k a^{k-j} \varepsilon_j$$

and

$$\eta_k = \sum_{j=k}^{\infty} a^{j-k} \varepsilon_j, k \in \mathbb{Z}.$$

Then there exists constant $c > 0$ only depending on a and q such that

$$\left(\sum_{k=-\infty}^{\infty} \delta_k^q \right)^{1/q} + \left(\sum_{k=-\infty}^{\infty} \eta_k^q \right)^{1/q} \leq c \left(\sum_{k=-\infty}^{\infty} \varepsilon_k^q \right)^{1/q}.$$

Lemma 2.1.3 *Let r, R, N , let $m > n$, and let $\theta, \omega \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp } \mathcal{F}\omega \subset \overline{B(0, 1)}$. Then there exists $C = C(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$, we have*

$$|\theta_R * \omega_N * g(x)| \leq C \max \left(1, \left(\frac{N}{R} \right)^m \right) (\eta_{N,m} * |\omega_N * g|^r(x))^{1/r}, \quad x \in \mathbb{R}^n,$$

where $\theta_R(\cdot) = R^n \theta(R\cdot)$, $\omega_N(\cdot) = N^n \omega(N\cdot)$ and $\eta_{N,m}(\cdot) = N^n (1 + N|\cdot|)^{-m}$,

The proof is given in [3].

Remark 2.1.4 *For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we denote Peetre maximal function,*

$$\Lambda_k^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Lambda_k f(y)|}{(1 + 2^k |x - y|)^a}.$$

2.2 Main result

We present a fundamental characterization of the above spaces, which plays an essential role in the characterization of Herz-Besov spaces in terms of ball means of differences, see [17].

Théorème 2.2.1 *Let $s \in \mathbb{R}, 0 < p, q \leq \infty, 0 < \beta \leq \infty, \alpha > -n/q$. Let $a > n/q$. Then*

$$\|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}^* = \left(\sum_{j=0}^{\infty} 2^{js\beta} \|\Lambda_k^{*,a} f\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{1/\beta},$$

is an equivalent quasi-norm in $\dot{K}_q^{\alpha,p} B_\beta^s$.

Proof

We divide the proof in two steps.

Step1. It is easy to see that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}^* < \infty$ and any $x \in \mathbb{R}^n$ we have

$$|\varphi_k * f(x)| \leq \varphi_k^{*,a} f(x), \quad k \in \mathbb{N}_0.$$

Step2. We will prove in this step that there is a constant $C > 0$ such that for every $f \in \dot{K}_q^{\alpha,p} B_\beta^s$.

$$\|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}^* \leq C \|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}. \quad (2.1)$$

We choose $t > 0$ such that $a > \frac{n}{t} > \frac{n}{p}$. By lemma 2.1.3 the estimates

$$|\varphi_k * f(y)| \leq C_2 (\eta_{k,m} * |\varphi_k * f|^t(y))^{1/t} \quad (2.2)$$

are true for any $y \in \mathbb{R}^n, k \in \mathbb{N}_0$ and any $m > 0$.

Now divide both sides of (2.2) by $(1 + 2^k |x - y|)^a$, in the right-hand side we use the inequality

$$(1 + 2^k |x - y|)^{-a} \leq (1 + 2^k |x - z|)^{-a} (1 + 2^k |y - z|)^a, \quad x, y, z \in \mathbb{R}^n,$$

In the left-hand side take the supremum over $y \in \mathbb{R}^n$ and get for all $f \in \dot{K}_q^{\alpha,p} F_\beta^s$, any $x \in \mathbb{R}^n$ any $v \in \mathbb{N}_0$ and any $m > at$

$$(\varphi_k^{*,a} f(x))^t \leq C_2 \eta_{k,at} * |\varphi_k * f|^t(x) \leq (\mathcal{M}_t(\varphi_k * f)(x))^t,$$

where $C_2 > 0$ is independent of x, k and f .

An application of lemma 2.1.1 gives that the left hand side of (2.1) is bounded by

$$\begin{aligned} & \left(\sum_{k=0}^{\infty} 2^{ks\beta} \|\mathcal{M}_t(\varphi_k * f)\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{1/\beta} \\ & \lesssim \left(\sum_{k=0}^{\infty} 2^{ks\beta} \|\varphi_k * f\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{1/\beta} \\ & \lesssim \|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}. \end{aligned}$$

This finishes the proof . ■

Definition 2.2.2 *Let f be an arbitrary function on \mathbb{R}^n and $x, h \in \mathbb{R}^n$. Then*

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^{M+1} f(x) = \Delta_h(\Delta_h^M f)(x), \quad M \in \mathbb{N}.$$

These are well-known differences of functions which play an important role in the theory of function spaces. Using mathematical induction one can show the explicit formula

$$\Delta_h^M f(x) = \sum_{j=0}^M (-1)^j C_j^M f(x + (M-j)h),$$

where C_j^M are binomial coefficients. By ball means of differences we mean the quantity

$$d_t^M f(x) = t^{-n} \int_{|h| \leq t} |\Delta_h^M f(x)| dh = \int_B |\Delta_{th}^M f(x)| dh,$$

where $B = \{y \in \mathbb{R}^n : |h| \leq 1\}$ is the unit ball of \mathbb{R}^n , $t > 0$ is a real number and M is a natural number.

The following theorem play a central role in our work.

Théorème 2.2.3 *Let $1 < p, q < \infty, 1 \leq \beta \leq \infty, -\frac{n}{q} < \alpha < n(1 - \frac{1}{q}), f \in \dot{K}_q^{\alpha,p} B_\beta^s$ and $M \in \mathbb{N} \setminus \{0\}$. Assume that*

$$0 < s < M.$$

Then

$$\|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}^* = \|f\|_{\dot{K}_q^{\alpha,p}} + \left(\int_0^1 t^{-s\beta} \|d_t^M f\|_{\dot{K}_q^{\alpha,p}}^\beta \frac{dt}{t} \right)^{\frac{1}{\beta}}.$$

is an equivalent norm in $\dot{K}_q^{\alpha,p} B_\beta^s$.

Proof. We split it into three steps.

Step 1. Let $f \in \dot{K}_q^{\alpha,p} B_\beta^s$. since $s > 0$, we see that

$$\|f\|_{\dot{K}_q^{\alpha,p}} \leq \sum_{j=0}^{\infty} \|\Lambda_j f\|_{\dot{K}_q^{\alpha,p}} \leq c \|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}.$$

Step 2. We put

$$H = \left(\int_0^1 t^{-s\beta} \|d_t^M f\|_{\dot{K}_q^{\alpha,p}}^\beta \frac{dt}{t} \right)^{1/\beta}.$$

After a change of variable $t = 2^{-y}$, we get

$$H = (\log 2)^{1/\beta} \left(\int_1^{+\infty} 2^{ys\beta} \|d_{2^{-y}}^M f\|_{\dot{K}_q^{\alpha,p}}^\beta dy \right)^{1/\beta}.$$

Then

$$H \leq c \left(\sum_{k=0}^{\infty} 2^{ks\beta} \|d_{2^{-k}}^M f\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{1/\beta}.$$

Let $\{\varphi_j\}_j$ be a smooth dyadic resolution of unity. Obviously we need to estimate

$$\left\{ 2^{ks} \sum_{j=0}^k \|d_{2^{-k}}^M (\Lambda_j f)\|_{\dot{K}_q^{\alpha,p}} \right\}_k \quad (2.3)$$

and

$$\left\{ 2^{ks} \sum_{j=k+1}^{\infty} \|d_{2^{-k}}^M (\Lambda_j f)\|_{\dot{K}_q^{\alpha,p}} \right\}_k, \quad (2.4)$$

in ℓ^β -norm. Using the same arguments of [7] we get

$$|\Delta_h^M \Lambda_j f(x)| \leq c 2^{(j-k)M} \Lambda_k^{*,a} f(x),$$

If $0 \leq j \leq k$, $x \in \mathbb{R}^n$ and $|h| \leq 2^{-k}$, with $c > 0$ independent of j, k, h and x , which yields that

$$|d_{2^{-k}}^M(\Lambda_j f)(x)| \leq c 2^{(j-k)M} \Lambda_k^{*,a} f(x),$$

If $0 \leq j \leq k$, $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$. Since $s < M$, (2.3) in ℓ^β -norm does not exceed

$$\left(\sum_{j=0}^{\infty} 2^{js\beta} \|\Lambda_j^{*,a} f\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{\frac{1}{\beta}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s},$$

by Theorem 2.2.1. Let $j > k$. Recalling the definition of $d_{2^{-k}}^M(\Lambda_j f)$ and observe that

$$\int_B |\Lambda_j f(x + (M-i)2^{-k}h)| dh \leq c \mathcal{M}(\Lambda_j f)(x),$$

If $j, k \in \mathbb{N}_0$, $i \in \{0, \dots, M\}$, $|h| \leq 2^{-k}$ and $x \in \mathbb{R}^n$. Therefore,

$$d_{2^{-k}}^M(\Lambda_j f)(x) \leq c \mathcal{M}(\Lambda_j f)(x),$$

for any $j > k$ and $x \in \mathbb{R}^n$. Hence

$$2^{ks} \sum_{j=k+1}^{\infty} \|d_{2^{-k}}^M(\Lambda_j f)\|_{\dot{K}_q^{\alpha,p}} \leq c 2^{ks} \sum_{j=k+1}^{\infty} \|\mathcal{M}(\Lambda_j f)\|_{\dot{K}_q^{\alpha,p}}.$$

Using Lemma 2.1.2, we obtain that (2.4) in ℓ^β -norm can be estimated from above by

$$\left(\sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{M}(\Lambda_j f)\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{\frac{1}{\beta}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s},$$

where we used Lemma 2.1.1.

Step 3. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq \frac{3}{2}$, in addition radiallysymmetric. We make use of an observation made by Nikol'skij [10] (see also [11]). We put

$$\psi(x) := (-1)^{M+1} \sum_{i=0}^{M-1} (-1)^i C_i^M \Psi(x(M-i)).$$

The function ψ satisfies $\psi(x) = 1$ for $|x| \leq 1/M$ and $\psi(x) = 0$ for $|x| \geq \frac{3}{2}$. Then, taking $\varphi_0(x) = \psi(x)$, $\varphi_1(x) = \psi(x/2) - \psi(x)$ and $\varphi_j(x) = \varphi_1(2^{-j+1}x)$ for $j = 2, 3, \dots$, we obtain that $\{\varphi_j\}$ is a smooth dyadic resolution of unity. This yields that

$$\left(\sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1}\varphi_j * f\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{1/\beta}$$

is a norm equivalent in $\dot{K}_q^{\alpha,p} B_\beta^s$. Let us prove that the last expression is bounded by

$$C \|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}^*$$

obviously from (1.1)

$$\|\mathcal{F}^{-1}\varphi_0 * f\|_{\dot{K}_q^{\alpha,p}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p}} \lesssim \|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}^*.$$

Moreover, it holds for $x \in \mathbb{R}^n$ and $j = 1, 2, \dots$

$$\mathcal{F}^{-1}\varphi_j * f(x) := (-1)^{M+1} \int_{\mathbb{R}^n} \Delta_{2^{-j}y}^M f(x) \tilde{\Psi}(y) dy,$$

with $\tilde{\Psi}(\cdot) = \mathcal{F}^{-1}\tilde{\Psi}(\cdot) - 2^{-n}\mathcal{F}^{-1}\tilde{\Psi}(\cdot/2)$. Now, for $j \in \mathbb{N}$ we write

$$\begin{aligned} \int_{\mathbb{R}^n} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy &= \int_{|y| \leq 1} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy \\ &+ \int_{|y| > 1} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy. \end{aligned}$$

obviously we need only to estimate the second term. We have

$$\begin{aligned} \int_{|y| > 1} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy &\leq \sum_{k=0}^{\infty} \int_{\overline{C}_k} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy \\ &\leq c \sum_{k=0}^{\infty} 2^{nj-Nk} \int_{\overline{C}_{k-j}} |\Delta_h^M f(x)| dh, \end{aligned} \quad (2.5)$$

where $\overline{C}_i = \{x \in \mathbb{R}^n : 2^i < |x| \leq 2^{i+1}\}$, $i \in \mathbb{N}_0$ and $N > 0$ is our disposal, and we have used the properties of the function $\tilde{\Psi}$,

$$|\tilde{\Psi}| \leq c(1 + |y|)^{-N}$$

for any $y \in \mathbb{R}^n$ and any $N > n$. Now the right-hand side of (2.5) in $\ell_\beta^s(\dot{K}_q^{\alpha,p})$ -norm is bounded by

$$c \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_{j=0}^{\infty} 2^{-(s+n)j\beta} (\omega_{j,k})^\beta \right)^{1/\beta}, \quad (2.6)$$

where

$$\omega_{j,k} = \left\| \int_{\bar{C}_{k-j}} |\Delta_h^M f(\cdot)| dh \right\|_{\dot{K}_q^{\alpha,p}}, \quad j, k \in \mathbb{N}_0.$$

The sum(2.6) can be rewritten as

$$\begin{aligned} & c \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_{j=0}^{k+1} \dots + \sum_{j=k+2}^{\infty} \dots \right)^{1/\beta} \\ & \leq c \sum_{k=0}^{\infty} 2^{-Nk} \left(\left(\sum_{j=0}^{k+1} \dots \right)^{1/\beta} + \left(\sum_{j=k+2}^{\infty} \dots \right)^{1/\beta} \right) \\ & = c \sum_{k=0}^{\infty} 2^{-Nk} (S_k^1 + S_k^2). \end{aligned}$$

Clearly

$$(S_k^1)^\beta \leq c 2^{kn\beta} \|\mathcal{M}(f)\|_{\dot{K}_q^{\alpha,p}}^\beta \sum_{j=0}^{k+1} 2^{sj\beta} \leq c 2^{(s+n)k\beta} \|f\|_{\dot{K}_q^{\alpha,p}}^\beta,$$

where $c > 0$ is independent of k . Now let us estimate S_k^2 . We obtain

$$\begin{aligned} (S_k^2)^\beta & \lesssim 2^{(n+s)k\beta} \sum_{v=1}^{\infty} 2^{(s+n)v\beta} \left\| \int_{|h| \leq 2^{-v}} |\Delta_h^M f| dh \right\|_{\dot{K}_q^{\alpha,p}}^\beta \\ & \lesssim 2^{(n+s)k\beta} \sum_{v=1}^{\infty} 2^{sv\beta} \|d_{2^{-v}}^M(f)\|_{\dot{K}_q^{\alpha,p}}^\beta \\ & \lesssim 2^{(n+s)k\beta} \int_0^1 t^{-s\beta} \|d_t^M(f)\|_{\dot{K}_q^{\alpha,p}}^\beta \frac{dt}{t}. \end{aligned}$$

Taking N large enough such that $N > s + n$, we estimate the right-hand side of (2.5) in $\ell_\beta^s(\dot{K}_q^{\alpha,p})$ -norm by

$$c \|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s}^*.$$

This finishes the proof. \blacksquare

Chapitre 3

Characterization by differences

In this chapter we obtain the characterization of the above spaces by differences.

3.1 Key Lemmas

To prove our result, we need this technical lemmas. The following lemma is the $\dot{K}_q^{\alpha,p}$ -version of the Plancherel-Polya-Nikolskij inequality.

Lemma 3.1.1 *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < s, \tau, q, r \leq \infty$. We suppose that $\alpha_1 + \frac{n}{s} > 0, 0 < q \leq s \leq \infty$ and $\alpha_2 \geq \alpha_1$. Then there exist a positive constant $c > 0$ independent of R such that for all $f \in K_q^{\alpha_2,p} \cap \mathcal{S}'(\mathbb{R}^n)$ with $\text{supp } \mathcal{F}f \subset \{\xi : |\xi| \leq R\}$, we have*

$$\left\| f \mid \dot{K}_s^{\alpha_1,r} \right\| \leq c R^{\frac{n}{q} - \frac{n}{s} + \alpha_2 - \alpha_1} \left\| f \mid \dot{K}_q^{\alpha_2,\delta} \right\|,$$

where

$$\delta = \begin{cases} r, & \text{if } \alpha_2 = \alpha_1, \\ \tau, & \text{if } \alpha_2 > \alpha_1. \end{cases}$$

For the proof see [2].

Lemma 3.1.2 *Let $\alpha \in \mathbb{R}, h \in \mathbb{R}^n, 0 < p, q \leq \infty$ and $\alpha > -\frac{n}{p}$. Then there exists a constant $c > 0$ independent of R and h such that for all $g \in K_p^\alpha, q \cap \mathcal{S}'(\mathbb{R}^n)$ with $\text{supp } \mathcal{F}g \subset \overline{B}(0, R)$,*

we have

$$\left\| g(\cdot + h) \mid \dot{K}_p^{\alpha, q} \right\| \leq cA \left\| g \mid \dot{K}_p^{\alpha, q} \right\|,$$

where

$$A = \begin{cases} \left(\frac{1}{R|h|} \right)^{-|\alpha|}, & \text{if } \alpha \neq 0, \\ 1 + \log R|h|, & \text{if } \alpha = 0, \end{cases}$$

If $R \geq \frac{2}{|h|} > 0$ and

$$A = 1,$$

otherwise.

Proof. We will present the proof in two cases.

- $R \geq \frac{2}{|h|} > 0$.

Assume that $-\frac{n}{p} < \alpha \leq 0$. We have

$$\begin{aligned} \left\| g(\cdot + h) \right\|_{\dot{K}_p^{\alpha, q}}^q &= \sum_{k=-\infty}^{\infty} 2^{k\alpha q} \left\| g(\cdot + h) \chi_k \right\|_p^q \\ &= \sum_{2^k < \frac{|h|}{2}} \cdots + \sum_{\frac{|h|}{2} \leq 2^k < 2|h|} \cdots + \sum_{2^k \geq 2|h|} \cdots \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Estimation of I_1 . We write

$$I_1 = \sum_{2^k < \frac{|h|}{2}} \cdots = \sum_{2^k < \frac{1}{R}} \cdots + \sum_{\frac{1}{R} \leq 2^k < \frac{|h|}{2}} \cdots.$$

Using Lemma 3.1.1 to estimate the first sum by

$$c \left\| g \right\|_{\dot{K}_p^{\alpha, q}} \sum_{2^k < \frac{1}{R}} (2^k R)^{\left(\frac{n}{p} + \alpha\right)q} \lesssim \left\| g \right\|_{\dot{K}_p^{\alpha, q}},$$

because of $\frac{n}{p} + \alpha > 0$. The second sum clearly is bounded by

$$\begin{aligned} \sum_{\frac{1}{R} \leq 2^k < \frac{|h|}{2}} 2^{k\alpha q} \|g\chi_{\widetilde{C(|h|)}}\|_p^q &\lesssim \|g\|_{\dot{K}_p^{\alpha, \infty}}^q \sum_{\frac{1}{R} \leq 2^k < \frac{|h|}{2}} 2^{k\alpha q} |h|^{-\alpha q} \\ &= c \|g\|_{\dot{K}_p^{\alpha, \infty}}^q \left(\frac{1}{R|h|}\right)^\alpha \sum_{\frac{1}{R} \leq 2^k < \frac{|h|}{2}} (2^k R)^{\alpha q} \\ &\lesssim \left(\frac{1}{R|h|}\right)^\alpha \|g\|_{\dot{K}_p^{\alpha, \infty}}^q \end{aligned}$$

where $\widetilde{C(|h|)} = \{x : \frac{|h|}{2} \leq |x| < \frac{3|h|}{2}\}$ with $\alpha \neq 0$. Clearly

$$\sum_{\frac{1}{R} \leq 2^k < \frac{|h|}{2}} \|g\chi_{\widetilde{C(|h|)}}\|_p^q \leq \|g\|_{\dot{K}_p^{0, \infty}}^q (1 + \log R|h|).$$

Estimation of I_2 . We have

$$I_2 \lesssim \sum_{\frac{|h|}{2} \leq 2^k < 2|h|} 2^{k\alpha p} \|g\chi_{B(0, 3|h|)}\|_q^p.$$

We have

$$\begin{aligned} \|g\chi_{B(0, 3|h|)}\|_p &= \sum_{i \in \mathbb{Z}, 2^i < 3|h|} \|g\chi_i\|_p \leq \|g\|_{\dot{K}_p^{\alpha, q}} \left(\sum_{i \in \mathbb{Z}, 2^i < 3|h|} 2^{-i\alpha q'} \right)^{1/q'} \\ &\lesssim |h|^{-\alpha} \|g\|_{\dot{K}_p^{\alpha, q}}, \end{aligned}$$

but if $-\frac{n}{p} < \alpha < 0$ and $1 \leq q \leq \infty$. If $-\frac{n}{p} < \alpha < 0$ and $0 < q < 1$, then

$$\|g\chi_{B(0, 3|h|)}\|_p^q \leq \sum_{i \in \mathbb{Z}, 2^i < 3|h|} \|g\chi_i\|_p^q \lesssim |h|^{-\alpha} \|g\|_{\dot{K}_p^{\alpha, q}}.$$

Now if $\alpha = 0$, we obtain

$$\begin{aligned} I_2 &\lesssim \sum_{k \in \mathbb{Z}, \frac{|h|}{2} \leq 2^k < 2|h|} \|g\chi_{B(0, 3|h|)}\|_p^q \lesssim \|g\|_{\dot{K}_p^{\alpha, q}}^q \sum_{k \in \mathbb{Z}, \frac{|h|}{2} \leq 2^k < 2|h|} 1 \\ &\lesssim \|g\|_{\dot{K}_p^{0, q}}^q. \end{aligned}$$

Estimation of I_3 . Clearly

$$I_2 \leq \sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|g\chi_{\tilde{C}_k}\|_p^q \leq \|g\|_{\dot{K}_p^{\alpha, q}}^q$$

where $\tilde{C}_k = \{x : 2^{k-1} \leq |x| < 2^{k+1}\}$

Now assume $\alpha > 0$. First let $1 \leq p, q \leq \infty$. We have

$$|g(x+h)| \lesssim \eta_{R,N} * |g|(x+h)$$

By duality

$$\|g(\cdot + h)\|_{\dot{K}_p^{\alpha,q}} \leq \sup \int_{\mathbb{R}^n} \eta_{R,N} * |g|(x+h) |\omega(x)| dx,$$

where the supremum is taking over all functions $\omega \in \dot{K}_{p'}^{-\alpha,q'}$ with

$$\|\omega\|_{\dot{K}_{p'}^{-\alpha,q'}} \leq 1,$$

where p' and q' are the conjugate exponent of p and q respectively. we have

$$\begin{aligned} \int_{\mathbb{R}^n} \eta_{R,N} * |g|(x+h) |g(x)| dx &= \int_{\mathbb{R}^n} \eta_{R,N} * |g|(h-y) |f(y)| dy \\ &\leq \|g\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} \|\eta_{R,N} * |\omega|(h-\cdot)\|_{\dot{K}_{p'}^{-\alpha,q'}(\mathbb{R}^n)} \end{aligned}$$

by Hölder's inequality. From Step 1, because of $-\alpha < 0$, we easily get

$$\|\eta_{R,N} * |\omega|(h-\cdot)\|_{\dot{K}_{p'}^{-\alpha,q'}} \lesssim A \|\eta_{R,N} * |\omega|\|_{\dot{K}_{p'}^{-\alpha,q'}} \lesssim A \|\omega\|_{\dot{K}_{q'}^{-\alpha,p'}}.$$

• $R < \frac{2}{|h|}$. We have

$$\begin{aligned} \|g(\cdot + h)\|_{\dot{K}_p^{\alpha,q}}^q &= \sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|g(\cdot + h)\chi_k\|_p^q \\ &= \sum_{2^k \leq \frac{8}{R}} \cdots + \sum_{2^k > \frac{8}{R}} \cdots \\ &= I_1 + I_2. \end{aligned}$$

We have

$$I_1 \leq c \|g\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} \sum_{2^k \leq \frac{8}{R}} (2^k R)^{\left(\frac{n}{p} + \alpha\right)q} \lesssim \|g\|_{\dot{K}_p^{\alpha,q}},$$

because of $\frac{n}{p} + \alpha > 0$. Observe that

$$\frac{3|x|}{4} \leq |x+h| \leq 3|x|$$

if $2^k > \frac{8}{R}$. The second sum clearly is bounded by

$$\sum_{2^k > \frac{8}{R}} 2^{k\alpha q} \|f \chi_{\widetilde{C}_k}\|_p^q \lesssim \|f\|_{\dot{K}_p^{\alpha,q}}^q,$$

where $\widetilde{C}_k = \{x : 32^{k-2} \leq |x| < 32^k\}$.

3.2 Main result

We have the following results :

Théorème 3.2.1 *Let $1 < p, q < \infty, 1 \leq \beta \leq \infty, -\frac{n}{p} < \alpha < n(1 - \frac{1}{p}), f \in \dot{K}_p^{\alpha,q} B_\beta^s$ and $M \in \mathbb{N} \setminus \{0\}$. Assume that*

$$|\alpha| < s < M.$$

Then

$$\|f\|_{\dot{K}_p^{\alpha,q} B_\beta^s}^* = \|f\|_{\dot{K}_p^{\alpha,q}} + \left(\int_0^1 t^{-s\beta} \sup_{|h| \leq t} \|\Delta_h^M f\|_{\dot{K}_p^{\alpha,q}}^\beta \frac{dt}{t} \right)^{1/\beta}$$

is an equivalent norm in $\dot{K}_p^{\alpha,q} B_\beta^s$.

Proof. We split it into three steps.

Step 1. Let $f \in \dot{K}_p^{\alpha,q} B_\beta^s$. Since $s > 0$, we see that

$$\|f\|_{\dot{K}_p^{\alpha,q}} \leq \sum_{j=0}^{\infty} \|\varphi_j * f\|_{\dot{K}_p^{\alpha,q}} \leq c \|f\|_{\dot{K}_p^{\alpha,q} B_\beta^s}.$$

Step 2. We put

$$H = \int_0^1 t^{-s\beta} \sup_{|h| \leq t} \|\Delta_h^M f\|_{\dot{K}_p^{\alpha,q}}^\beta \frac{dt}{t}.$$

Obviously

$$H \lesssim \sum_{k=0}^{\infty} 2^{ks\beta} \sup_{|h| \leq 2^{-k}} \|\Delta_h^M f\|_{\dot{K}_p^{\alpha,q}}^\beta.$$

Let $\psi, \psi_0 \in \mathcal{S}(\mathbb{R}^n)$ be two functions such that $\mathcal{F}\psi = 1$ and $\mathcal{F}\psi_0 = 1$ on $\text{supp}\varphi_1$ and $\text{supp}\Psi$ respectively. Using the mean value theorem we obtain for any $x \in \mathbb{R}^n, j \in \mathbb{N}_0$ and $|h| \leq 2^{-k}$

$$\begin{aligned} |\Delta_h^1(\varphi_j * f)(x)| &= |\Delta_h^1(\psi_j * \varphi_j * f)(x)| \\ &\leq 2^{-k} \sup_{|x-y| \leq c2^{-k}} \sum_{|\beta|=1} |D^\beta \psi_j * \varphi_j * f(y)|, \end{aligned}$$

with some positive constant c , independent of j and k , and $\psi_j(\cdot) = 2^{(j-1)n}\psi(2^{j-1}\cdot)$ for $j = 1, 2, \dots$ by induction on M , We show that

$$|\Delta_h^M(\varphi_j * f)(x)| \leq 2^{-kM} \sup_{|x-y| \leq c2^{-k}} \sum_{|\beta|=M} |D^\beta \psi_j * \varphi_j * f(y)|.$$

We see that if $|\beta| = M$ and $a > 0$

$$\begin{aligned} |D^\beta \psi_j * \varphi_j * f(y)| &= 2^{(j-1)n} \left| \int_{\mathbb{R}^n} D^\beta \psi(2^{(j-1)}(y-z)) * \varphi_j * f(z) dz \right| \\ &\leq 2^{(j-1)(n+M)} \int_{\mathbb{R}^n} |(D^\beta \psi)(2^{(j-1)}(y-z))| |\varphi_j * f(z)| dz. \end{aligned}$$

The right hand side may be estimated as follows :

$$\begin{aligned} &c2^{j(n+M)} \varphi_j^{*,a} f(y) \int_{\mathbb{R}^n} |(D^\alpha \psi)(2^{j-1}(y-z))| (1 + 2^j|y-z|)^a dz \\ &\leq c2^{jM} \varphi_j^{*,a} f(y). \end{aligned}$$

Then we obtain for any $x \in \mathbb{R}^n, |h| \leq 2^{-k}$ and any $0 \leq j \leq k$

$$\begin{aligned} |\Delta_h^M(\varphi_j * f)(x)| &\leq c2^{(j-k)M} \sup_{|x-y| \leq c2^{-k}} \varphi_j^{*,a} f(y) \\ &\leq c2^{(j-k)M} (1 + 2^{j-k})^a \sup_{|x-y| \leq c2^{-k}} \frac{\varphi_j^{*,a} f(y)}{(1 + 2^j|x-y|)^a} \\ &\leq c2^{(j-k)M} \varphi_j^{*,a} f(x). \end{aligned}$$

Recalling the definition of $\Delta_h^M(\varphi_j * f)$,

$$|\Delta_h^M(\varphi_j * f)(x)| \leq \sum_{m=0}^M C_m^M |\varphi_j * f(x + (M - m)h)|.$$

We obtain for any $j > k$ and any $|h| \leq 2^{-k}$

$$\|\Delta_h^M(\varphi_j * f)\|_{\dot{K}_p^{\alpha,q}} \lesssim A(j, k) \|\varphi_j * f\|_{\dot{K}_p^{\alpha,q}},$$

Where

$$A(j, k) = \begin{cases} 2^{(j-k)|\alpha|}, & \text{if } \alpha \neq 0, \\ j - k, & \text{if } \alpha = 0. \end{cases}$$

Take the ℓ^β -norm and we obtain the desired estimate.

Step 3. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq \frac{3}{2}$, in addition radialsymmetric. We make use of an observation made by Nikol'skij [10] (see also [11]). We put

$$\psi(x) := (-1)^{M+1} \sum_{i=0}^{M-1} (-1)^i C_i^M \Psi(x(M-i)).$$

The function ψ satisfies $\psi(x) = 1$ for $|x| \leq 1/M$ and $\psi(x) = 0$ for $|x| \geq 3/2$. Then, taking $\varphi_0(x) = \psi(x)$, $\varphi_1(x) = \psi(x/2) - \psi(x)$ and $\varphi_j(x) = \varphi_1(2^{-j+1}x)$ for $j = 2, 3, \dots$, we obtain that $\{\varphi_j\}$ is a smooth dyadic resolution of unity. This yields that

$$\left(\sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1}\varphi_j * f\|_{\dot{K}_p^{\alpha,q}}^\beta \right)^{1/\beta}$$

is a quasi-norm equivalent in $\dot{K}_p^{\alpha,q} B_\beta^s$. Let us prove that the last expression is bounded by

$$C \|f\|_{\dot{K}_p^{\alpha,q} B_\beta^s}^*.$$

Obviously from (1.1)

$$\|\mathcal{F}^{-1}\varphi_0 * f\|_{\dot{K}_p^{\alpha,q}} \lesssim \|f\|_{\dot{K}_p^{\alpha,q}} \lesssim \|f\|_{\dot{K}_p^{\alpha,q} B_\beta^s}^*.$$

Moreover, it holds for $x \in \mathbb{R}^n$ and $j = 1, 2, \dots$

$$\mathcal{F}^{-1}\varphi_j * f(x) := (-1)^{M+1} \int_{\mathbb{R}^n} \Delta_{2^{-j}y}^M f(x) \tilde{\Psi}(y) dy,$$

with $\tilde{\Psi}(\cdot) = \mathcal{F}^{-1}\Psi(\cdot) - 2^{-n}\mathcal{F}^{-1}\Psi(\cdot/2)$. Now, for $j \in \mathbb{N}$ we write

$$\int_{\mathbb{R}^n} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy = \int_{|y| \leq 1} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy + \int_{|y| > 1} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy.$$

Obviously we need only to estimate the second term. We have

$$\begin{aligned} \int_{|y| > 1} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy &\leq \sum_{k=0}^{\infty} \int_{\bar{C}_k} |\Delta_{2^{-j}y}^M f(x)| |\tilde{\Psi}(y)| dy \\ &\leq c \sum_{k=0}^{\infty} 2^{nj-Nk} \int_{\bar{C}_{k-j}} \frac{|\Delta_h^M f(x)|}{(1+2^j|h|)^L} dh, \end{aligned} \quad (3.1)$$

where $\bar{C}_i = \{x \in \mathbb{R}^n : 2^i < |x| \leq 2^{i+1}\}$, $i \in \mathbb{N}_0$ and $N > 0$ is at our disposal, and we have used the properties of the function $\tilde{\Psi}$,

$$|\tilde{\Psi}(y)| \leq c(1+|y|)^{-N-L}$$

for any $y \in \mathbb{R}^n$ and any $N, L > n$. Now the right-hand side of (3.1) in $\ell_\beta^s(\dot{K}_p^{\alpha,q})$ -norm is bounded by

$$c \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_{j=0}^{\infty} 2^{(s+n)j\beta} (\omega_{j,k})^\beta \right)^{1/\beta}, \quad (3.2)$$

where

$$\omega_{j,k} = \left\| \int_{\bar{C}_{k-j}} \frac{|\Delta_h^M f(\cdot)|}{(1+2^j|h|)^L} dh \right\|_{\dot{K}_p^{\alpha,q}}, \quad j, k \in \mathbb{N}_0,$$

which can be estimated by

$$\begin{aligned} c \sum_{m=0}^M C_m^M \left\| \int_{\mathbb{R}^n} \frac{|f(\cdot + (M-m)h)|}{(1+2^j|h|)^L} dh \right\|_{\dot{K}_p^{\alpha,q}} &\lesssim 2^{-jn} (\|f\|_{\dot{K}_p^{\alpha,q}} + \|\eta_{j,L} * f\|_{\dot{K}_p^{\alpha,q}}) \\ &\lesssim 2^{-jn} \|f\|_{\dot{K}_p^{\alpha,q}}, \end{aligned}$$

where we have used (1.1) and L large enough. The sum (3.2) can be rewritten as

$$\begin{aligned}
& c \sum_{k=0}^{\infty} 2^{-Nk} \left(\sum_{j=0}^{k+1} \cdots + \sum_{j=k+2}^{\infty} \cdots \right)^{1/\beta} \\
& \leq c \sum_{k=0}^{\infty} 2^{-Nk} \left(\left(\sum_{j=0}^{k+1} \cdots \right)^{1/\beta} + \left(\sum_{j=k+2}^{\infty} \cdots \right)^{1/\beta} \right) \\
& = c \sum_{k=0}^{\infty} 2^{-Nk} (S_k^1 + S_k^2).
\end{aligned}$$

Clearly

$$(S_k^1)^\beta \leq c \|f\|_{\dot{K}_p^{\alpha,q}} \sum_{j=0}^{k+1} 2^{sj} \leq c 2^{sk} \|f\|_{\dot{K}_p^{\alpha,q}},$$

where $c > 0$ is independent of k . Now let us estimate S_k^2 . We obtain

$$\begin{aligned}
(S_k^2)^\beta & \lesssim 2^{(n+s)k\beta} \sum_{v=1}^{\infty} 2^{sv\beta} \sup_{|h| \leq 2^{-v}} \|\Delta_h^M f\|_{\dot{K}_p^{\alpha,q}}^\beta \\
& \lesssim 2^{(n+s)k\beta} \left(\int_0^1 t^{-s\beta} \sup_{|h| \leq t} \|\Delta_h^M f\|_{\dot{K}_p^{\alpha,q}}^\beta \frac{dt}{t} \right)^{1/q}.
\end{aligned}$$

Taking N large enough such that $N > s + n$, we estimate the right-hand side of (3.1) in $\ell_\beta^s(\dot{K}_p^{\alpha,q})$ -norm by

$$c \|f\|_{\dot{K}_p^{\alpha,q} B_\beta^s}.$$

This finishes the proof.

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