PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
UNIVERSITY MOHAMED BOUDIAF OF M'SILA
Faculty of Mathematics and Informatics
Department of Mathematics

Order Num : ....

Memory

Presented to obtain a Master

Specialty
Mathematics

Option
Functional analysis

By
BABECH HOUDA

Title

On the ideal of two- Lipschitz operators

Defended on .. / .. /2022 before the jury composed of :

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Academic year : 2021/2022

Acknowledgments

We begin by thanking God who has given us strength and perseverance
to
reach our goals.
I send my strongest express thanks and deepest gratitude to my
supervisor
the
professor **Hamidi khaled** for his assistance and his patience during the
period of preparing this memory.
My sincere thanks to the president of the jury professor **Addelaziz
Hellal** for accepting the chairmanship of this committee and for
taking care of my work.
My special thanks to the professor **Tallab Abdelhamid** of participating
in the supervisor.
My thanks to the professor **Maazouz Ahmed** to agree to examine this
memory.
I also extend my thanks to all members of the Faculty of Mathematics
and
Informatics of the university Mohamed Boudaif M'sila.
I thank all members of the family for their help and constant support.

Dedication

It is with great pleasure that I dedicate
this modest work:
To the dearest person in my life, my
mother.
To the one who made me a woman, my
father.
To my dear brothers, for their support and
encouragement.
As well as to all my teachers, my friends,
and all my family for their
support throughout my academic career.
Thank you for always being there for me.

Contents

Introduction

The theory of operator ideal was introduced by Pietsch in the linear case and is nowadays well established [30]. In 2016, Achour et al .[4] introduced the notion of Lipschitz operator ideal in the same spirit of linear operator ideals.

In 2009, Dubai et,all. to the. [14] defined a two-Lipschitz operator between the Cartesian product of two pointed metric spaces and a Banach space as being the one that is Lipschitz in each of the coordinates, making a natural allusion to the operators bilinear. There Sánchez-Pérez, in [36], presented a more suitable definition for the two-Lipschitz operator at real values (called the bi-form Lipschitz) to which it is directly associated with a continuous bilinear form. In [22, 21], the authors extended the definition of Sánchez-Pérez for two-Lipschitz operators valued in a Banach space will arbitrate and, on occasion, present some examples and show that these operators are intimately related to continuous bilinear operators between Banach spaces.

The thesis consists of three chapters. In the preliminaries (Chapter 1) we establish the notation of the thesis. We introduce basic notions in Banach space theory and we recall the main definitions and properties of the theory of Bilinear operator and Lipschitz operator that we will use later. Also, we recall the most important results for the Bilinear operator ideal, Lipschitz operator ideals. The stated results have their statements duly referenced, based primarily on [9, 10, 24, 25, 33]

For the construction of the second chapter, the main reference used was the article [22] and the thesis [21, 16]. We will start the chapter with the definition of the two-Lipschitz operator and show that the space formed by these operators is Banach with an appropriate norm and also that each two-Lipschitz operator is uniquely related to a continuous bilinear map. Following the same steps of the ideals of linear operators (see [30]), multilinear operators (see [18, 31]) and Lipschitz (see [4]), we will discuss the ideal definition of two-Lipschitz operators and we will exhibit a method, called the composition method, for constructing an ideal of two-Lipschitz operators from an ideal of linear operators. This new ideal of two-Lipschitz operators is called the ideal of composition of two-Lipschitz operators and we will see that it inherits some properties from the ideal of linear operators that generates it.

In the third and final chapter, we will present news two examples of operator ideals two-Lipschitz, the first is the two-Lipschitz Cohen p-nuclear operators and the second is the Factorable strongly p-nuclear two-Lipschitz operators in order to apply the technique of composition methods for this examples.

Chapter 1

Preliminaries

We will write $\mathbb{K}$ for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For $1 \leq p \leq \infty$, by p^* we denote the conjugate of p, that is $\frac{1}{p} + \frac{1}{p^*} = 1$. Along this thesis the letters E, F and G denote Banach space. Given a Banach space E, B_E is its closed unit ball and $S_E$is unit sphere of E. By $\mathcal{L}(E, F)$ we denote the Banach space of all continuous linear operators between E and F with the norm.

$$\|T\| = \sup_{x \in B_X} \|T(x)\|.$$

1.1 Bilinear operator ideals

1.1.1 Bilinear operators

Definition 1.1. *Let $E_j\,(j = 1, 2)\,, F$ be a normed spaces over $\mathbb{K}$, (either $\mathbb{K}$ or $\mathbb{C}$). A mapping $T : E_1 \times E_2 \longrightarrow Y$ is called bilinear (or b-linear) if the mapping*

$$\begin{array}{ccc} T_j & : & E_j \longrightarrow F \\ x^j & \longrightarrow & T\left(E^1, E^2\right), \end{array}$$

are linear for each set of fixed $x^k \in E_k, k \neq j$, i.e.

$T(\lambda x^1 + x^2, y) = \lambda T(x^1, y) + T(x^2, y)$,

$T(x, \lambda y^1 + y^2) = \lambda T(x, y^1) + T(x, y^2)$

for all $\lambda \in \mathbb{K}$ and $x^j, y^j \in E_j\,(j = 1, 2)$ The vector space of such mappings is denoted by $\mathcal{L}(E_1, E_2; F)$. If $F = \mathbb{K}$, We write $\mathcal{L}(E_1, E_2)$.

Remark 1.2. *The set S of all vectors in F of the form $T(x^1, x^2)\,, x^j \in E_j\,(j = 1, 2)$ is not in general a vector subspace of F (see [20, section 1.1]).*

Definition 1.3. *An b-linear mapping $T : E_1 \times E_2 \longrightarrow F$ is continuous if it is continuous as a function between two normed spaces.*

As a consequence of this definition, similar to the linear case, we have a result that gives the characterization of the continuous b-linear mapping.

Theorem 1.4. *Let E_1, E_2 be normed spaces. For $T \in \mathcal{L}(E_1, E_2; F)$ the following assertions are equivalent*

i) *T is continuous.*

ii) *T is continuous in $(0, 0)$.*

ii) *There is a constant $C \geq 0$ with*

$$\left\|T\left(x^1, x^2\right)\right\| \leq C \left\|x^1\right\| \left\|x^2\right\|, \tag{1.1}$$

for all $x^j \in E_j \, (j = 1, 2)$.

iv) $\|T\| = \sup\limits_{\|x^j\| \leq 1, j=1,2} \|T(x^1, x^2)\| \leq \infty.$

We will write $\mathcal{L}(E_1, E_2; f)$ for the vector space of all continuous b-linear mapping. If $Y = \mathbb{K}$, we write $\mathcal{L}(E_1, e_2)$.
It is easy to see that

$$\|T\| = \inf \{C \geq 1, \text{verifying the in equality } 1.1\}.$$

Defines a norm on $\mathcal{L}(E_1, E_2; F)$ which is complete norm when $\|.\|_F$ is completed.
For the general theory of bilinear mappings we refer to([13], or [27]).

1.1.2 Tensor products of Banach spaces

The 2-fold tensor product $E_1 \otimes E_2$ of the vector spaces E_1, E_2 can be constructed from the elements of the space $\mathcal{L}(E_1, E_2)^*$. For $x^j \in E_j \; (j = 1, 2)$. we define the linear mapping

$$x^1 \otimes x^2 : \mathcal{L}(E_1, E_2) \longrightarrow \mathbb{K},$$

by

$$x^1 \otimes x^2 (\phi) := \phi\left(x^1, x^2\right),$$

for each b-linear form ϕ on $E_1 \times E_2$. The functional $x^1 \otimes x^2$ is called an elementary tensor

Definition 1.5. *The subspace of $\mathcal{L}(E_1, E_2)^*$ spanned by the collection of elementary tensors*

$$\left\{x^1 \otimes x^2, x^j \in E_j\,(j = 1, 2)\right\},$$

is called the 2-fold tensor product of E_1, E_2 and will be denoted by $E_1 \otimes E_2$.
The elements of this space are called tensors. So a typical tensor $u \in E_1 \otimes E_2$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes x_i^2 \tag{1.2}$$

where $(\lambda_i)_{i=1}^n \subset \mathbb{K}$, $\left(x_i^j\right)_{i=1}^n \subset E_j (j = 1, 2)$ and $n \in \mathbb{N}$ is arbitrary. Note that the tensor u can always be rewritten, using the properties of the elementary tensors, in the form

$$u = \sum_{i=1}^{n} x_i^1 \otimes x_i^2.$$

The projective norm on $E \otimes F$, the tensor product of E and F, is defined by

$$\pi(u) = \inf \sum_{i=1}^{n} \|x_i\| \|y_i\|, \tag{1.3}$$

where the infimum is taken over all possible representations of $u \in E \otimes F$ of the form $\sum\limits_{i=1}^{n} x_i \otimes y_i$.
The tensor product $E \otimes F$ endowed with the projective norm π is denoted by $E \otimes_\pi F$ and its completion by $E \widehat{\otimes}_\pi F$. Which call the projective tensor product.

We now recall the linearization of continuous bilinear mappings. Consider the canonical continuous bilinear mapping

$$\sigma_2 : E \times F \longrightarrow E \hat{\otimes}_\pi F,$$

defined by

$$\sigma_2(x, y) = x \otimes y.$$

Theorem 1.6.
Let E, F and G be Banach spaces. For every continuous bilinear mapping $T : E \times F \longrightarrow G$ there exists a unique continuous linear operator $T_L : E \widehat{\otimes}_\pi F \longrightarrow G$ satisfying

$$T_L \circ \sigma_2 = T,$$

i.e.,

$$T_L(x \otimes y) = T(x, y),$$

for every $(x, y) \in E \times F$. That is, the following diagram commutes

$$\begin{array}{ccc} E \times F & \xrightarrow{\quad T \quad} & G \\ & {\scriptstyle \sigma_2} \searrow \quad \nearrow {\scriptstyle T_L} & \\ & E \hat{\otimes}_\pi F & \end{array} \tag{1.4}$$

Furthermore, $\|T_L\| = \|T\|$.

The previous theorem gives the canonical identification

$$\mathcal{L}(E,F;G)=\mathcal{L}(E\widehat{\otimes}_\pi F,G).$$

For more details refer to [32] and [12].

1.1.3 Ideals of bilinear mapping

Let E_1,E_2,F be Banach spaces. Consider non-zero $\varphi^j\in E_j^*(j=1,2)$, and $y\in F$. Define the bilinear mapping

$$\varphi^1\otimes\varphi^2\otimes y:E_1\times E_2\longrightarrow F,$$

by

$$\varphi^1\otimes\varphi^2\otimes y(x^1,x^2):=\varphi^1(x^1)\varphi^2(x^2)y. \tag{1.5}$$

It is clear that $\varphi^1\otimes\varphi^2\otimes y\in\mathcal{L}(E_1,E_2;F)$ the vector space of continuous bilinear operators and

$$\left\|\varphi^1\otimes\varphi^2\otimes y\right\|=\left\|\varphi^1\right\|\left\|\varphi^2\right\|\|y\|.$$

We denote by $\mathcal{L}_f(E_1,E_2;F)$, the vector space of finite type bilinear operators, which is generated by the mappings of the form (1.5). All elements T of this space a finite representation of the form

$$T=\sum_{i=1}^{n}\lambda_i\varphi_i^1\otimes\varphi_i^2\otimes y_i,$$

where $(\lambda_i)_{i=1}^n\subset\mathbb{K}$, $(\varphi_i^j)_{i=1}^n\subset E_j^*(j=1,2)$ and $(y_i)_{i=1}^n\subset F$.

Definition 1.7.
An ideal of bilinear mappings (or bilinear ideal) is a subclass $\mathcal{M}$ of all continuous bilinear mappings between Banach spaces such that for all Banach spaces E_1,E_2 and F, the components

$$\mathcal{M}(E_1,E_2;F):=\mathcal{L}(E_1,E_2;F)\cap\mathcal{M}$$

satisfy:

(i) $\mathcal{M}(E_1,E_2;F)$ is a vector subspace of $\mathcal{L}(E_1,E_2;F)$ which contains the bilinear mappings of finite type.

(ii) The ideal property: If $T\in\mathcal{M}(G_1,G_2;H)$, $u_j\in\mathcal{L}(E_j,G_j)$ for $j=1,2$ and $v\in\mathcal{L}(H,F)$, then $v\circ T\circ(u_1,u_2)$ is in $\mathcal{M}(E_1,E_2;F)$.

If $\|.\|_{\mathcal{M}}:\mathcal{M}\longrightarrow\mathbb{R}^+$ satisfies

(i') $(\mathcal{M}(E_1,E_2;F),\|\cdot\|_{\mathcal{M}})$ is a normed (Banach) space for all Banach spaces E_1,E_2,F.

(ii') The bilinear form $T^2 : \mathbb{K}^2 \longrightarrow \mathbb{K}$ *given by* $T^2(x^1, x^2) = x^1 x^2$ *satisfies* $\|T^2\|_{\mathcal{M}} = 1$.

(iii') If $T \in \mathcal{M}(G_1, G_2; H), u_j \in \mathcal{L}(E_j, G_j)$ *for* $j = 1, 2$ *and* $v \in \mathcal{L}(H, F)$*, then*

$$\|v \circ T \circ (u_1, u_2)\|_{\mathcal{M}} \leq \|v\| \|T\|_{\mathcal{M}} \|u_1\| \|u_2\|,$$

we say that $(\mathcal{M}, \|.\|_{\mathcal{M}})$ *is a normed (Banach) bilinear ideal.*

Of course the Banach spaces considered in this definition are all over the same fixed scalar field.
The bilinear ideal $\mathcal{M}$ is said to be closed if each $\mathcal{M}(E_1, E_2; F)$ is a closed subspace of $\mathcal{L}(E_1, E_2; F)$ for the sup norm.
Note that $\mathcal{L}$, the class of all continuous bilinear mappings between arbitrary Banach spaces, is the largest bilinear ideal.

Proposition 1.8. *[35]*
Let $\mathcal{M}$ *be normed bilinear ideal and* E_1, E_2, F *be Banach spaces.*

1- $\|T\| \leq \|T\|_{\mathcal{M}}$ *for all* $T \in \mathcal{M}(E_1, E_2; F)$.

2- $\|\varphi^1 \otimes \varphi^2 \otimes y\|_{\mathcal{M}} = \|\varphi^1\| \|\varphi^2\| \|y\|$ *for any* $\varphi^j \in E_j^*(j = 1, 2)$ *and* $y \in F$.

1.2 Lipschitz operator ideals

1.2.1 Lipschitz functions

The notion of metric space was formalized by Maurice Fréchet in [17]. He was among the first who used the word space. Recall that a metric or distance on a non empty set X is a function

$$d : X \times X \to \mathbb{R}_+$$

with the following properties:

(i) (Positivity) For all x, $y \in X$, $d(x, y) \geq 0$ with equality if and only if $x = y$.

(ii) (Symmetry) For all x, $y \in X$,$d(x, y) = d(y, x)$.

(iii) (Triangle inequality) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

The set X equipped with the distance d is called a metric space. Along this thesis, X and Y denote pointed metric spaces. The natural morphisms between metric spaces are the Lipschitz functions. A map $T : X \to Y$ between two metric spaces is called Lipschitz if there is a positive constant C such that

$$\forall x, y \in X, d(T(x), T(y)) \leq Cd(x, y). \tag{1.6}$$

If $C = 1$, the map is called nonexpansive (and contraction if $C \leq 1$).
For a Lipschitz map $T : X \to Y$ its Lipschitz constant is given by

$$\begin{aligned} Lip(T) &= \sup\left\{\tfrac{d(T(x),T(y))}{d(x,y)}, x \neq y\right\} \\ &= \inf\{C : C \text{ verifying the equality } 1.6\}. \end{aligned}$$

A pointed metric space X is a metric space with a base point in X, that is, a designated special point, which we will always denote by 0. We will consider a normed space E over $\mathbb{K}$ as a pointed metric space with the distance defined by its norm and the zero vector as the base point. Given two pointed metric spaces X and Y, we denote by $Lip_0(X, Y)$ the set of all base-point preserving Lipschitz maps from X to Y. If E is a Banach space, $Lip(X, E)$ is a Banach space under the Lipschitz norm given by

$$Lip(T) = \left\{\frac{\left\|T(x) - T(x')\right\|}{d(x, y)}, x \neq y\right\}.$$

For $E = \mathbb{K}$, we designate $Lip_0(X, \mathbb{K}) = Lip_0(X) = X^{\#}$. The Banach space $X^{\#}$ of Lipschitz functions is called also Lipschitz dual it has been used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case.

Now we are going to present some concepts about the space of molecules, the reader can for more details see [37] or [6]

1.2.2 The Arens-Eells spaces

Definition 1.9. *Let X be a metric space. A molecule on X is a scalar valued function m on X with finite support that satisfies $\sum_{x\in X} m(x) = 0$. We denote by $\mathcal{M}(X)$ the linear space of all molecules on X. For x, $x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \mathcal{X}_{\{x\}} - \mathcal{X}_{\{x'\}}$, where $\mathcal{X}_A$ is the characteristic function of the set A. For $m \in \mathcal{M}(X)$ we can write*

$$m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}$$

for some e suitable scalars λ_j, and we write

$$\|m\|_{\mathcal{M}(X)} = \inf\left\{\sum_{j=1}^{n} |\lambda_j| \, d(x_j, x'_j), m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}\right\},$$

where the infimum is taken over all representations of the molecule m. Denote by Æ(X) the completion of the normed space $\left(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)}\right)$. This space was first introduced by Arens and Eells [6] in 1956. The terminology Arens-Eells space Æ$(X)$is due to Weaver [37].A different notation was used in [19] by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X)$.

Proposition 1.10. *Let X be a pointed metric space*

(i) *The map $\delta_X : X \longrightarrow$Æ$(X)$ defined by*

$$\delta_X(x) = m_{x0}$$

*is an isometric embedding of X into*Æ(X) *(see [37, thereme 2.2.4])*

(ii) *The map $Q_X : X^{\#} \longrightarrow$Æ$(X)^*$defined by*

$$Q_X(f) = f_L, \text{where } f_L(m) = \sum_{x\in X} f(x)m(x),$$

Theorem 1.11. *[23, Lemma 3.1]*
Let $T \in Lip_0(X,Y)$, there exists a unique linear operator $\widehat{T} \in \mathcal{L}($« (X), « $(Y))$ such that

$$\widehat{T} \circ \delta_X = \delta_Y \circ T, \tag{1.7}$$

that is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Y} \\ «(X) & \xrightarrow[\widehat{T}]{} & «(Y) \end{array} \tag{1.8}$$

commutes. Furthermore, $\left\|\widehat{T}\right\| = Lip(T)$.

Theorem 1.12. *[37, Theorem 2.2.4 (b)]*
Let $T \in Lip_0(X,E)$, there is a unique bounded linear map T_L : «$(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$ that is, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ & {\scriptstyle \delta_X}\searrow \quad \nearrow{\scriptstyle T_L} & \\ & «(X) & \end{array} \tag{1.9}$$

Furthermore $\|T_L\| = Lip(T)$.

The operator T_L is referred to as the linearization of T. The correspondence $T \longleftrightarrow T_L$ establishes an isomorphism between the vector spaces $Lip_0(X,E)$ and $\mathcal{L}($« (X), $E)$.
In particular, the spaces $X^{\#}$ and «$(X)^*$ are isometrically isomorphic via the linearization $R(f) := f_L$, where $f_L(m) = \sum\limits_{x\in X} f(x)m(x)$ (see [37, Theorem 2.2.2]).

1.2.3 Ideals of Lipschitz mappings

The notion of Lipschitz operator ideal was introduced by Achour, Rueda, Sánchez-Pérez and Yahi [4]. This can be seen as an extension of the linear operator ideal.

Definition 1.13. *[4, Definition 2.1]*
A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components

$$\mathcal{I}_{Lip}(X,E) := Lip_0(X,E) \cap \mathcal{I}_{Lip}$$

satisfy:

(i) $\mathcal{I}_{Lip}(X,E)$ *is a vector subspace of* $Lip_0(X,E)$.

(ii) $eg \in \mathcal{I}_{Lip}(X,E)$ *for* $e \in E$ *and* $g \in X^{\#}$.

(iii) The ideal property: if $S \in Lip_0(X,Y)$, $T \in \mathcal{I}_{Lip}(Y,F)$ *and* $u \in \mathcal{L}(F,E)$, *then the composition* $u \circ T \circ S$ *is in* $\mathcal{I}_{Lip}(X,E)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a normed (Banach) Lipschitz operator ideal if there is a function $\|.\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0,+\infty[$ that satisfies

(i') For every pointed metric space X and every Banach space E, the pair $(\mathcal{I}_{Lip}(X,E), \|.\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X,E)$.

(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.

(iii') If $S \in Lip_0(X,Y)$, $T \in \mathcal{I}_{Lip}(Y,F)$ *and* $u \in \mathcal{L}(F,E)$, *then*

$$\|u \circ T \circ S\|_{\mathcal{I}_{Lip}} \leq Lip(S) \, \|T\|_{\mathcal{I}_{Lip}} \, \|u\| \, .$$

Following [4, Definition 3.1], there is a way to construct a (Banach) Lipschitz operator ideal from a (Banach) linear operator ideal, called composition method. Let $\mathcal{I}$ be a (Banach) linear operator ideal. A Lipschitz mapping $T \in Lip_0(X,E)$ belongs to the composition Lipschitz operator ideal $\mathcal{I} \circ Lip_0$ if there exists a Banach space F, a Lipschitz operator $S \in Lip_0(X,F)$ and a linear operator $u \in \mathcal{I}(F,E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a Banach operator ideal we write

$$\|T\|_{\mathcal{I} \circ Lip_0} = \inf \|u\|_{\mathcal{I}} \, Lip(S),$$

where the infimum is taken over all u and S as above.
In [4], the authors establish a criterion to decide whenever a Lipschitz operator ideal is of composition or not.

Proposition 1.14. *[4, Proposition 3.2]*
Let X be a pointed metric space, E a Banach space and $\mathcal{I}$ an operator ideal. A Lipschitz operator $T \in Lip_0(X, E)$ belongs to $\mathcal{I} \circ Lip_0(X, E)$ if and only if its linearization T_L belongs to $\mathcal{I}(«(X), E)$.
Furthermore, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a Banach operator ideal then $(\mathcal{I} \circ Lip_0, \|\cdot\|_{\mathcal{I} \circ Lip_0})$ is Banach Lipschitz operator ideal with

$$\|T\|_{\mathcal{I} \circ Lip_0} = \|T_L\|_{\mathcal{I}}.$$

Chapter 2

Two-Lipschitz operators ideals

2.1 Bi-linearization of two-Lipschitz operators

Definition 2.1. *[?, Section 3.4]*
Let (X, d_X) *and* (Y, d_Y) *be pointed metric spaces and let* E *be a Banach space, we say that a map* $T : X \times Y \longrightarrow E$ *is a two-Lipschitz operator if there is a constant* $C > 0$ *such that for each* $x, x' \in X$ *and* $y, y' \in Y$,

$$\|T(x,y) - T(x,y') - T(x',y) + T(x',y')\| \leq C.d_X(x,x')\, d_Y(y,y')\,. \tag{2.1}$$

By $BLip_0(X,Y;E)$ *we denote the set of all two-Lipschitz operators from* $X \times Y$ *to* E *such that*

$$T(x,0) = T(0,y) = 0, \tag{2.2}$$

for all $x \in X$ *and* $y \in Y$. *For* $T \in BLip_0(X,Y;E)$ *we set*

$$BLip(T) = \inf C = \sup_{x \neq x', y \neq y'} \frac{\|T(x,y) - T(x,y') - T(x',y) + T(x',y')\|}{d_X(x,x')\, d_Y(y,y')}. \tag{2.3}$$

The next result shows that the expression(2.3) can be used to define a norm in space $BLip_0(X,Y;E)$

Proposition 2.2. *The application*

$$\begin{aligned} BLip(.) \quad &: \quad B_0Lip(X,Y;E) \longrightarrow \mathbb{R} \\ T \quad &\longrightarrow \quad BLip(T) \end{aligned}$$

define a norm in space $B_0Lip(X,Y;E)$.

Proof. we give $T \in B_0Lip(X,Y;E)$,follow this $BLip(T) \geq 0$.It is clear that $BLip(0) = 0$.On the other side,impose that $BLip(0) = 0$,in(2.3)follow that

$$0 = \sup_{x \neq 0, y \neq 0} \frac{\|T(x,y) - T(x,0) - T(0,y) + T(0,0)\|}{d_X(x,0) d_Y(y,0)} = \sup_{x \neq 0, y \neq 0} \frac{\|T(x,y)\|}{d_X(x,0) d_Y(y,0)}.$$

with that ,for any $x \in X$ and $y \in Y$,we get

$$\|T(x,y)\| = 0 \Rightarrow T(x,y) = 0,$$

i.e, $T = 0$.Now ,if $\alpha \in \mathbb{K}$ we have

$$\begin{aligned}
BLip\,(\alpha T) &= \sup_{x\neq 0, y\neq 0} \frac{\|(\alpha T)(x,y) - (\alpha T)(x,y') - (\alpha T)(x',y) + (\alpha T)(x',y')\|}{d_X(x,x')d_Y(y,y')} \\
&= \sup_{x\neq 0, y\neq 0} \frac{\|\alpha.T(x,y) - \alpha.T(x,y') - \alpha.T(x',y) + \alpha.T(x',y')\|}{d_X(x,x')d_Y(y,y')} \\
&= \sup_{x\neq 0, y\neq 0} \frac{\|(\alpha(T(x,y) - T(x,y') - T(x',y) + T(x',y'))\|}{d_X(x,x')d_Y(y,y')} \\
&= \sup_{x\neq 0, y\neq 0} \frac{|\alpha|\,\|T(x,y) - T(x,y') - T(x',y) + T(x',y')\|}{d_X(x,x')d_Y(y,y')} \\
&= |\alpha| \sup_{x\neq 0, y\neq 0} \frac{\|T(x,y) - T(x,y') - T(x',y) + T(x',y')\|}{d_X(x,x')d_Y(y,y')} \\
&= |\alpha|\, BLip\,(T)
\end{aligned}$$

Finally ,to prove the triangle inequality, considering $S \in B_0Lip\,(X,Y;E)$ and points $x, x' \in X$ and $y, y' \in Y$,with $x \neq x'$and $y \neq y'$,we have

$$\begin{aligned}
& \|(T+S)(x,y) - (T+S)(x,y') - (T+S)(x',y) + (T+S)(x',y')\| \\
= \; & \|T(x,y) + S(x,y) - T(x,y') + S(x,y') - T(x',y) + S(x',y) + T(x',y') + S(x',y')\| \\
\leq \; & \|T\,(x,y) - T\,(x,y') - T\,(x',y) + T\,(x',y')\| + \|S\,(x,y) - S\,(x,y') - S\,(x',y) + S\,(x',y')\| \\
\leq \; & BLip(T)d_X(x,x')d_Y(y,y') + BLip(S)d_X(x,x')d_Y(y,y') \\
= \; & (BLip(T) + BLip(S))d_X(x,x')d_Y(y,y'),
\end{aligned}$$

implying in $(BLip(T+S) \leq BLip(T) + BLip(S)$.Therefore ,$BLip(.)$ is a norm in $BLip(X,Y;E)$. □

For the mapping $T : X \times Y \longrightarrow E$, consider $A_y : X \longrightarrow E$ and $A_x : Y \longrightarrow E$ such that $A_y(x) = T(x,y)$ for every fixed $y \in Y$ and $A_x(y) = T(x,y)$ for every fixed $x \in X$. According to Dubei et al. in [14], T is said to be two-Lipschitz if A_x is Lipschitz for every fixed $x \in X$ and A_y is Lipschitz for every fixed $y \in Y$.

The following proposition, give the relation between the operator $x \longrightarrow A_x$ or $y \longrightarrow A_y$ and Definition 2.1. The prove is in [22, 21, 16].

Proposition 2.3. *[22]*
For a mapping $T : X \times Y \longrightarrow E$, the following statements are equivalent.

i) $T \in BLip_0(X,Y;E)$.

ii) $A_x \in Lip_0(Y, E)$ *for every fixed* $x \in X$ *and* $G : x \longrightarrow A_x$ *belongs to* $Lip_0(X, Lip_0(Y, E))$.

iii) $A_y \in Lip_0(X, E)$ *for every fixed* $y \in Y$ *and* $H : y \longrightarrow A_y$ *belongs to* $Lip_0(Y, Lip_0(X, E))$.

In the example below,we will show that bilinear operators between spaces of Banach are also 2-lipschitz operators.

Example 2.4.

Let X, Y *and* E *Banach spaces,Then every bilinear operator* $T : X \times Y \to E$ *belongs to* $BLip_0(X, Y; E)$ *and* $BLip(T) = \|T\|$ *.For any* $x, x \in X$ *and* $y, y \in Y$*,we have*

$$\begin{aligned} \|T(x, y) - T(x, y') - T(x', y) + T(x', y')\| &= \|T(x - x', y - y')\| \\ &\leq \|T\| \|x - x'\| \|y - y'\| \end{aligned}$$

and

$$T(x, 0) = T(0, y) = 0.$$

So,$T \in BLip_0(X, Y; E)$ *how* $BLip(T) \leq \|T\|$ *.To show the other inequality,it is enough to observe that*

$$\begin{aligned} BLip(T) &= \sup_{x \neq x', y \neq y'} \frac{\|T(x, y) - T(x, y') - T(x', y) + T(x', y')\|}{d(x, x')d(y, y')} \\ &\geq \sup_{x \neq 0, y \neq 0} \frac{\|T(x, y) - T(x, 0) - T(0, y) + T(0, 0)\|}{d(x, 0)d(y, 0)} \\ &= \sup_{x \neq 0, y \neq 0} \frac{\|T(x, y)\|}{\|x\| \|y\|} = \|T\| . \end{aligned}$$

We find ,$BLip(T) = \|T\|$ *.*

Proposition 2.5.

Let X, Y, Z, W *be pointed metric spaces and let* E, F *be Banach spaces. If* $f \in Lip_0(Z, X)$*,* $g \in Lip_0(W, Y)$*,* $T \in BLip_0(X, Y; E)$ *and* $u \in \mathcal{L}(E, F)$ *then*

$$u \circ T \circ (f, g) \in BLip_0(Z, W; F)$$

where $(f, g) : Z \times W \longrightarrow X \times Y$*,is set to* $(f, g)(z, w) = (f(z),g(w))$*. Moreover,*

$$BLip(u \circ T \circ (f, g)) \leq \|u\| BLip(T)Lip(f)Lip(g). \tag{2.4}$$

Proof. First, for each $x \in X$ and $y \in Y$, note that

$$u \circ T \circ (f, g)(x, y) = (u \circ T)(f, g)(x, y) = (u \circ T)(f(x), g(y)) = u(T(f(x), g(y)))$$

So,$u \circ T \circ (f, g)$ satisfied (2.2),then

$$u \circ T \circ (f, g)(x, 0) = u(T(f(x), 0)) = u(0) = 0$$

and

$$u \circ T \circ (f, g)(0, y) = u(T(f(0), y)) = u(0) = 0$$

Now, we prove $u \circ T \circ (f, g)$ is 2-lipschitz.u is linear,$T \in BLip_0(X, Y; F)$ and f and g we have

$$\begin{aligned}
&\|u(T(f(x), g(y))) - u(T(f(x), g(y'))) - u(T(f(x'), g(y))) + u(T(f(x'), g(y')))\| \\
&= \|u(T(f(x), g(y)) - T(f(x), g(y')) - T(f(x'), g(y)) + T(f(x'), g(y')))\| \\
&\leq \|u\| \, \|T(f(x), g(y)) - T(f(x), g(y')) - T(f(x'), g(y)) + T(f(x'), g(y'))\| \\
&\leq \|u\| \, BLip(T) d(f(x), f(x')) d(g(y), g(y')) \\
&\leq \|u\| \, BLip(T) Lip(f) d(x, x') Lip(g) d(y, y') \\
&= \|u\| \, BLip(T) Lip(f) Lip(g) d(x, x') d(y, y')
\end{aligned}$$

it is clear that for all $x, x' \in X$ and $y, y' \in Y$. Then, $u \circ T \circ (f, g) \in BLip_0(Z, W; F)$ and $BLip(u \circ T \circ (f, g)) \leq \|u\| \, BLip\,(T)\, Lip\,(f)\, Lip\,(g)\,.$ □

The next theorem and its proof are similar to the Lipschitz case (see [37, Proposition 1.6.2]).

Theorem 2.6. *[22]*
$BLip_0\,(X, Y; E)$ is a Banach space under the norm $BLip\,(\cdot)$ defined by (2.3).

In what follows, let (X, d_X) and (Y, d_Y) be metric spaces and consider the product metric space $X \times Y$ equipped with the metric

$$d\,((x, y)\,, (x', y')) = d_X\,(x, x') + d_Y\,(y, y')\,,$$

for all $x, x' \in X$ and $y, y' \in Y$.

Also if E, F be Banach spaces, the product Banach space $E \times F$ is equipped with the norm

$$\|(x, y)\|_{E \times F} = \|x\|_E + \|y\|_F\,,$$

for all $x \in E$ and $y \in F$.

Proposition 2.7.
Let X, Y pointed metric spaces: The mapping $(\delta_X, \delta_Y) : X \times Y \longrightarrow Æ(X) \times Æ(Y)$ defined by

$$(\delta_X, \delta_Y)(x, y) = (\delta_X(x), \delta_Y(y)) = (m_{x0}, m_{y0}),$$

isometrically embeds $X \times Y$ in $Æ(X) \times Æ(Y)$.

For all two-Lipschitz operator $T : X \times Y \longrightarrow E$, we define a bilinear mapping $T_B : \mathcal{M}(X) \times \mathcal{M}(Y) \longrightarrow E$ by

$T_B\left(\sum_{i=1}^{n} \alpha_i m_{xx'}, \sum_{j=1}^{m} \beta_i m_{yy'}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j (T(x_i, y_j) - T(x_i, y'_j) - T(x'_i, y_j) + T(x'_i, y'_j)$

In particular, we have

$$T_B(m_{xx'}, m_{yy'}) = T(x, y) - T(x, y') - T(x', y) + T(x', y'), \tag{2.5}$$

Also the equality $T = T_B \circ (\delta_X, \delta_Y)$, then

$$\begin{aligned} T_B \circ (\delta_X, \delta_Y)(x, y) &= T_B((\delta_X, \delta_Y)(x, y)) \\ &= T_B(\delta_X(x), \delta_Y(y)) \\ &= T_B(m_{x0}, m_{y0}) \\ &= T(x, y) - T(x, 0) - T(0, y) + T(0, 0) \\ &= T(x, y) \end{aligned}$$

for all $x, x' \in X$ and $y, y' \in Y$. Thus the two-Lipschitz operator T is associated with the bilinear mapping T_B to the space $Æ(X) \times Æ(Y)$.

Theorem 2.8. *[22, Theorem 2.6]*
For every two-Lipschitz operator $T \in BLip_0(X, Y; E)$ there exists a unique continuous bilinear mapping $\tilde{T}_B : Æ(X) \times Æ(Y) \longrightarrow E$ satisfying (2.5) and

$$T = \tilde{T}_B \circ (\delta_X, \delta_Y) : X \times Y \overset{(\delta_X, \delta_Y)}{\longrightarrow} Æ(X) \times Æ(Y) \overset{T_B}{\longrightarrow} E.$$

Furthermore $BLip(T) = \left\|\tilde{T}_B\right\|$. The continuous bilinear mapping $\tilde{T}_B$ is called bi-linearization of the two-Lipschitz operator T.

From now on,we will denote the bi-linearization $\tilde{T}_B$ by T_B, the bilinear operator T_B admits a linearization :

$$(T_B)_L : Æ(X) \widehat{\otimes}_\pi Æ(Y) \longrightarrow E$$

satisfies

$$T = T_B \circ (\delta_X, \delta_Y) = (T_B)_L \circ \sigma_2 \circ (\delta_X, \delta_Y), \tag{2.6}$$

where $\|T_B\| = \|(T_B)_L\|$. From this it follows that

$$(T_B)_L(m_{x0}, m_{y0}) = T_B(m_{x0} \otimes m_{y0}) = T(x, y).$$

In addition we have

$$BLip(T) = \|T_B\| = \|(T_B)_L\|.$$

Definition 2.9.

The linear operator $(T_B)_L$ is referred to as the linearization of the two-Lipschitz operator T. For the simplification, write T_L instead of $(T_B)_L$.

Remark 2.10.

The map associate $T \to T_L$ is linear ,$\alpha \in \mathbb{K}$, $T, S \in BLip_0(X, Y; E)$ and the linearization T_L, S_L and $(\alpha T + S)_L$, be there

$$\begin{aligned}(\alpha T + S)_L (m_{x0} \otimes m_{y0}) &= (\alpha T + S)(x, y) = \alpha T(x, y) + S(x, y) \\ &= \alpha T_L (m_{x0} \otimes m_{y0}) + S_L(m_{x0} \otimes m_{y0})\end{aligned}$$

We will now present a simple example of 2-lipschitz operators that will be based on used in the next sections.

Example 2.11.

Let X and Y be pointed metric spaces and consider the application

$$\sigma_2 \circ (\delta_X, \delta_Y) : X \times Y \longrightarrow Æ(X) \widehat{\otimes}_\pi Æ(Y)$$

$$(x, y) \longrightarrow \sigma_2 \circ (\delta_X, \delta_Y)(x, y) = m_{x0} \otimes m_{y0}.$$

We have

$$\sigma_2 \circ (\delta_X, \delta_Y) \in BLip_0(X, Y; Æ(X) \widehat{\otimes}_\pi Æ(Y)) \text{ and } BLip(\sigma_2 \circ (\delta_X, \delta_Y)) = 1.$$

For any $x, x' \in X$ and $y, y' \in Y$, we have

$$\begin{aligned}&\|\sigma_2 \circ (\delta_X, \delta_Y)(x, y) - \sigma_2 \circ (\delta_X, \delta_Y)(x, y') - \sigma_2 \circ (\delta_X, \delta_Y)(x', y) + \sigma_2 \circ (\delta_X, \delta_Y)(x', y')\| \\ =\ & \|\sigma_2(\delta_X(x), \delta_Y(y)) - \sigma_2(\delta_X(x), \delta_Y(y')) - \sigma_2(\delta_X(x'), \delta_Y(y)) + \sigma_2(\delta_X(x'), \delta_Y(y'))\| \\ \leq\ & BLip(\sigma_2) \|\delta_X(x) - \delta_X(x')\| \|\delta_Y(y) - \delta_Y(y')\| \\ =\ & BLip(\sigma_2) d(x, x') d(y, y')\end{aligned}$$

after that

$$\sigma_2 \circ (\delta_X, \delta_Y)(x, 0) = m_{x0} \otimes m_{00} = 0 \text{ and } \sigma_2 \circ (\delta_X, \delta_Y)(0, y) = m_{00} \otimes m_{y0} = 0.$$

thus,$\sigma_2 \circ (\delta_X, \delta_Y) \in \sigma_2 \circ (\delta_X, \delta_Y) \in BLip_0(X, Y; Æ(X) \widehat{\otimes}_\pi Æ(Y))$.*finally,* $BLip(\sigma_2 \circ$

$(\delta_X, \delta_Y)) = 1$, *then*

$$\begin{aligned} BLip(\sigma_2 \circ (\delta_X, \delta_Y)) &= \sup_{x\neq x', y\neq y'} \frac{\|\sigma_2 \circ (\delta_X, \delta_Y)(x,y)\sigma_2 - (\delta_X, \delta_Y)(x,y')\sigma_2 - (\delta_X, \delta_Y)(x',y)\sigma_2 + (\delta}{d(x,x')d(y,y')} \\ &= \sup_{x\neq x', y\neq y'} \frac{\|m_{x0} \otimes m_{y0} - m_{x0} \otimes m_{y'0} - m_{x'0} \otimes m_{y0} + m_{x'0} \otimes m_{y'0}\|}{d(x,x')d(y,y')} \\ &= \sup_{x\neq x', y\neq y'} \frac{\|m_{x0} - m_{x'0} \otimes m_{y0} - m_{y'0}\|}{\|m_{xx'}\| \, \|m_{yy'}\|} \\ &= \sup_{x\neq x', y\neq y'} \frac{\|m_{xx'} \otimes m_{yy'}\|}{\|m_{xx'}\| \, \|m_{yy'}\|} \\ &= \sup_{x\neq x', y\neq y'} \frac{\|m_{xx'}\| \, \|m_{yy'}\|}{\|m_{xx'}\| \, \|m_{yy'}\|} = 1 \end{aligned}$$

Next we give a simple but crucial example of a two-Lipschitz operator. Let X, Y be pointed metric spaces and let E be Banach space.

Example 2.12.

Consider non-zero Lipschitz functions $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$. Define the mapping $f \cdot g \cdot e : X \times Y \longrightarrow E$ by

$$f \cdot g \cdot e(x,y) = f(x)g(y)e. \tag{2.7}$$

Then, an easy computation shows that this mapping is two-Lipschitz and

$$BLip(f \cdot g \cdot e) = Lip(f)Lip(g)\|e\|. \tag{2.8}$$

Definition 2.13.

We denote by $BLip_{0\mathcal{F}}(X,Y;E)$, the vector subspace of all two-Lipschitz operators generated by the mappings of the special form (2.7). All elements T of this space are called of finite type. So, any $T \in BLip_{0\mathcal{F}}(X,Y;E)$ admits a finite representation of the form

$$T = \sum_{i=1}^{n} f_i \cdot g_i \cdot e_i$$

where $(f_i)_{i=1}^n \subset X^{\#}$, $(g_i)_{i=1}^n \subset Y^{\#}$ and $(e_i)_{i=1}^n \subset E$.

2.2 Two-Lipschitz operator ideals

We will follow the spirit of the definitions of multilinear operator ideals ([37] or [35]) and Lipschitz operator ideals [4], for defining the concept of two-Lipschitz operator ideals.

Definition 2.14.

A two-Lipschitz operator ideal between pointed metric spaces and Banach spaces, $\mathcal{I}_{BLip}$, *is a subclass of $BLip_0$ such that for every pointed metric spaces X, Y and every Banach space E the components*

$$\mathcal{I}_{BLip}(X, Y; E) := BLip_0(X, Y; E) \cap \mathcal{I}_{BLip}$$

satisfy:

(i) *$\mathcal{I}_{BLip}(X, Y; E)$ is a vector subspace of $BLip_0(X, Y; E)$.*

(ii) *For any $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$, the map $f \cdot g \cdot e$ belongs to $\mathcal{I}_{BLip}(X, Y; E)$.*

(iii) *The ideal property: if $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in \mathcal{I}_{BLip}(X, Y; E)$ and $u \in \mathcal{L}(E, F)$, then the composition $u \circ T \circ (f, g)$ is in $\mathcal{I}_{BLip}(Z, W; F)$.*

A two-Lipschitz operator ideal $\mathcal{I}_{BLip}$ is a normed (Banach) two-Lipschitz operator ideal if there is $\| \cdot \|_{\mathcal{I}_{BLip}} : \mathcal{I}_{BLip} \longrightarrow [0, +\infty[$ that satisfies

(i') *For every pointed metric spaces X, Y and every Banach space E , the pair $(\mathcal{I}_{BLip}(X, Y; E), \| \cdot \|_{\mathcal{I}_{BLip}})$ is a normed (Banach) space and $BLip(T) \leq \|T\|_{\mathcal{I}_{BLip}}$ for all $T \in \mathcal{I}_{BLip}(X, Y; E)$.*

(ii') $\|Id_{\mathbb{K}^2} : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K} : Id_{\mathbb{K}^2} (\alpha, \beta) = \alpha\beta\|_{\mathcal{I}_{BLip}} = 1.$

(iii') *If $f \in Lip_0(Z, X)$, $g \in Lip_0(W, Y)$, $T \in \mathcal{I}_{BLip}(X, Y; E)$ and $u \in \mathcal{L}(E, F)$, the inequality $\|u \circ T \circ (f, g)\|_{\mathcal{I}_{BLip}} \leq \|u\| \, \|T\|_{\mathcal{I}_{BLip}} Lip(f) Lip(g)$ holds.*

Of course the Banach spaces considered in this definition are all over the same fixed scalar field.

The two-Lipschitz operator ideal $\mathcal{I}_{BLip}$ is said to be closed if each $\mathcal{I}_{BLip}(X, Y; E)$ is a closed subspace of $BLip(X, Y; E)$ with the norm $BLip(\cdot)$.

Proposition 2.15.
Let $\mathcal{I}_{BLip}$ be a normed two-Lipschitz operator ideal, X, Y be pointed metric spaces and E be Banach space. Then

$$\|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}} = \|e\| \, Lip(f) Lip(g) = BLip(f.g.e)$$

for any $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$.

Proof. Let $f \in X^{\#}$, $g \in Y^{\#}$ and $e \in E$. We can write $f \cdot g \cdot e$ in the following way

$$f \cdot g \cdot e = id_{\mathbb{K}}.e \circ id_{\mathbb{K}^2} \circ (f, g)$$

because,for any $x \in X$ and $y \in Y$,we have

$$\begin{aligned} id_{\mathbb{K}}.e \circ id_{\mathbb{K}^2} \circ (f,g)(x,y) &= id_{\mathbb{K}}.e \circ id_{\mathbb{K}^2}((f,g)(x,y)) \\ &= id_{\mathbb{K}}.e \circ id_{\mathbb{K}^2}(f(x),g(y)) \\ &= id_{\mathbb{K}}.e(id_{\mathbb{K}^2}(f(x),g(y))) \\ &= id_{\mathbb{K}}.e(f(x),g(y)) \\ &= e.f(x).g(y) \\ &= f(x).g(y).e \\ &= f.g.e(x,y). \end{aligned}$$

By (i'),(ii'), (iii') and (2.8), we obtain directly

$$\begin{aligned} \|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}} &= \|id_{\mathbb{K}}.e \circ id_{\mathbb{K}^2} \circ (f,g)\|_{\mathcal{I}_{BLip}} \\ &\leq \|id_{\mathbb{K}}.e\| \, \|id_{\mathbb{K}^2}\|_{\mathcal{I}_{BLip}} Lip(f)Lip(g) \\ &= \|e\| \, \|id_{\mathbb{K}}\| \, Lip(f)Lip(g) \\ &= \|e\| \, Lip(f)Lip(g) = BLip(f \cdot g \cdot e) \\ &\leq \|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}}, \end{aligned}$$

this gives, $\|f \cdot g \cdot e\|_{\mathcal{I}_{BLip}} = \|e\| \, Lip(f)Lip(g) = BLip(f \cdot g \cdot e)$. □

Remark 2.16.
By the above definition, the class $BLip_{0\mathcal{F}}$ is the smallest two-Lipschitz operator ideal and the class of all two-Lipschitz operators between arbitrary pointed metric spaces and Banach spaces, is the largest two-Lipschitz operator ideal.

The next definition given the composition method for the two-Lipschitz operators

Definition 2.17. *[22, 21]*
Let $\mathcal{I}$ be an operator ideal. A two-Lipschitz operator $T \in BLip_0(X,Y;E)$ belongs to the composition two-Lipschitz operator ideal $\mathcal{I} \circ BLip_0$, in this case we write $T \in \mathcal{I} \circ BLip_0(X,Y;E)$, if there is a Banach space F, a two-Lipschitz operator $S \in BLip_0(X,Y;F)$ and a linear operator $u \in \mathcal{I}(F,E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal we write

$$\|T\|_{\mathcal{I}\circ BLip_0} = \inf \|u\|_{\mathcal{I}} \, BLip(S),$$

where the infimum is taken over all u, S as above.

Theorem 2.18. *[22, Theorem 3.5]*
Let $\mathcal{I}$ be an operator ideal. A two-Lipschitz operator $T \in BLip_0(X,Y;E)$ belongs to

$\mathcal{I} \circ BLip_0(X, Y; E)$ if and only if its linearization T_L belongs to $\mathcal{I}(« (X) \widehat{\otimes}_\pi « (Y), E)$. Furthermore, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal, then

$$\|T\|_{\mathcal{I}\circ BLip_0} = \|T_L\|_{\mathcal{I}}, \tag{2.9}$$

and we have the isometric identification

$$(\mathcal{I} \circ BLip_0(X, Y; E), \|\cdot\|_{\mathcal{I}\circ BLip_0}) = \Big(\mathcal{I}(« (X) \widehat{\otimes}_\pi « (Y), E), \|\cdot\|_{\mathcal{I}}\Big). \tag{2.10}$$

Proposition 2.19.

If $\mathcal{I}$ is a (normed, closed, Banach) operator ideal then, $\mathcal{I} \circ BLip_0$ is a (respectively normed, closed, Banach) two-Lipschitz operator ideal.

Chapter 3

Some examples of two-Lipschitz operators ideals

In this chapter we give some examples of two-Lipschitz operator ideal, the ideals of p-neclear two-Lipschitz operators and, we establish a natural relation between two-Lipschitz and bilinear maps and show that the two-Lipschitz factorable p-neclear operators.

3.1 Two-Lipschitz Cohen p-nuclear operators

The class of p-nuclear operators introduced by Cohen [11] and generalized to Cohen (p,q)-nuclear operators by Apiola [5]. We start by recalling the linear case.

Definition 3.1.
A linear operator T between Banach spaces E and F, is Cohen p-nuclear $(1 < p < \infty)$ if there is a positive constant C such that for all $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subset E$ and $(a_i^*)_{i=1}^n \subset F$ we have:

$$\left| \sum_{i=1}^{n} \langle T(x_i), a_i^* \rangle \right| \leq C \sup_{x^* \in B_{E^*}} \|x^*(x_i)\|_{l_p^n} \sup_{a \in B_F} \|(a_i^*(a))_{i=1}^n\|_{l_{p^*}^n}$$

The smallest constant C, denoted by $\eta_p(T)$, such that the above inequality holds, is called the Cohen p-nuclear norm on the space $\mathcal{N}_p(E,F)$ of all Cohen p-nuclear operator from E into F, which is a Banach space. For $p = 1$ and $p = \infty$ we have $\mathcal{N}_1(X,Y) = \Pi_1(X,Y)$ and $\mathcal{N}_\infty(X,Y) = \mathcal{D}_\infty(X,Y)$ (for $1 < p \leq \infty$, $\mathcal{D}_p(E,F)$ is the Banach space of all strongly p-summing linear operators, see [11]).

The definition of Cohen p-nuclear m-linear operators is due to D. Achour and A. Ahlem, (see [1]) in order to generalize the concept of p-nuclear linear operators.

Definition 3.2.
An bi-linear operator $T : E_1 \times E_2 \to F$ is Cohen p-nuclear $(1 < p < \infty)$ if there is a constant

$C>0$ such that for any $n\in\mathbb{N}, x_1,...,x_n\in E_1, y_{1,},...,y_n\in E_2$ and and $a_1^*,...,a_n^*\in F^*$, we have

$$\left|\sum_{i=1}^{n}\langle T(x_i,y_i),a_i^*\rangle\right|\leq C\left(\sup_{\substack{\varphi_1\in B_{E_1^*}\\ \varphi_2\in B_{E_2^*}}}\sum_{i=1}^{n}|\varphi_1(x_i)|^p|\varphi_2(y_i)|^p\right)^{\frac{1}{p}}\sup_{a\in B_F}\|(a_i^*(a))_{i=1}^n\|_{l_{p^*}^n}. \tag{3.1}$$

The vector space of these mappings is indicated by $\mathcal{N}_p^2(E_1,E_2;F)$, and the smallest constant C such that (3.1) holda by $\eta_p^2(T)$.

For $p=\infty$, (3.1) becomes

$$\left|\sum_{i=1}^{n}\langle T(x_i,y_i),a_i^*\rangle\right|\leq C\left(\sup_{1\leq i\leq n}\|x_i\|_{E_1}\|y_i\|_{E_2}\right)\sup_{a\in B_F}\|(a_i^*(a))_{i=1}^n\|_{l_1^n}$$

It is clear that every $T\in\mathcal{N}_p^2(E_1,E_2;F)$ is continuous and $\|T\|\leq\eta_p^2(T)$.

The next Lipschitz generalization of the concept of Cohen p-nuclear linear operators was introduced by T. Hamid and L. Mezrag [26].

Definition 3.3.

A Lipschitz operator $T:X\to E$ is Cohen Lipschitz p-nuclear $(1<p<\infty)$ if there is a positive constant C such that for any $n\in\mathbb{N}, (x_i)_{1\leq i\leq n}, (x_i^{'})_{1\leq i\leq n}\subset X, (a_i^*)_{i=1}^n\subset E^*$ and $(\lambda_i)_{1\leq i\leq n}\subset\mathbb{R}_+$,we have

$$\left|\sum_{i=1}^{n}\lambda_i\langle T(x_i)-T(x_i'),a_i^*\rangle\right|\leq C\sup_{f\in B_{X^\#}}\left(\sum_{i=1}^{n}|\lambda_i(f(x_i)-f(x_i'))|^p\right)^{\frac{1}{p}}\sup_{a\in B_E}\left(\sum_{i=1}^{n}|\langle a_i^*,a\rangle|^{p^*}\right)^{\frac{1}{p^*}}$$

The smallest constant C above, denoted by $\eta_p^L(T)$, is called the Cohen Lipschitz p-nuclear norm, it makes the space $\mathcal{N}_p^L(X,E)$ of all Cohen Lipschitz p-nuclear operators from X into E a Banach space.

Theorem 3.4. *[26, Theorem 3.3.] If $T\in\mathcal{N}_p^L(X,E)$, there exist Radon probability measures μ_1 on $B_{X^\#}$ and μ_2 on $B_{E^{**}}$, such that for all x,x' in X and a in E , the following*

$$\begin{aligned}&|\langle T(x)-T(x'),a^*\rangle|\\&\leq C\left(\int_{B_{X^\#}}|f(x)-f(x')|^p d\mu_1\right)^{\frac{1}{p}}\left(\int_{B_{E^{**}}}|a^{**}(a^*)|^{p^*}d\lambda(a^{**})\right)^{\frac{1}{p^*}}\end{aligned} \tag{3.2}$$

In this case

$$\eta_p^L(T)=\inf\{C>0: \textit{for all } C \textit{ verifying the inequality } 3.2.\}$$

Now we extended the definition of the class of Cohen p-nuclear bi-linear operators to the case of two-Lipschitz, for with the resulting vector space of two-Lipschitz Cohen p-nuclear operator is a Banach two-Lipschitz operator ideals.

Definition 3.5.
Let $1 < p < \infty$. An two-Lipschitz operator $T : X \times Y \to E$ is said to be two-Lipschitz Cohen p-nuclear operator if there is a constant $C > 0$ such that for any $(x_i)_{i=1}^n, (x_i')_{i=1}^n \subset X$, $(y_i)_{i=1}^n, (y_i')_{i=1}^n \subset Y$ and $(a_i^*)_{i=1}^n \subset E^*$ we have

$$\left|\sum_{i=1}^n \langle T(x_i, y_i) - T(x_i, y_i') - T(x_i', y_i) + T(x_i', y_i'), a_i^*\rangle\right| \leq C \sup_{\substack{f\in B_{X^\#} \\ g\in B_{Y^\#}}} \left(\sum_{i=1}^n |f(x_i) - f(x_i')|^p |g(y_i) - g(y_i')|^p\right)^{\frac{1}{p}} \sup_{a\in B_E} \|(a_i^*(a))_{i=1}^n\|_{l_{p^*}^n}. \tag{3.3}$$

We denote this class of two-Lipschitz operators by $\mathcal{N}_p^{BL}(X, Y; E)$, In this case, we define

$$\|T\|_{\mathcal{N}_p^{BL}} = \inf\{C : \textit{satisfying } (3.3)\}.$$

For $p = \infty$, (3.3) becomes:

$$\left|\sum_{i=1}^n \langle T(x_i, y_i) - T(x_i, \acute{y}_i) - T(\acute{x}_i, y_i) + T(\acute{x}_i, \acute{y}_i), a_i^*\rangle\right| \leq C \sup_{1\leq i\leq n} d(x_i, x_i') d(y_i, y_i') \sup_{a\in B_E} \|(a_i^*(a))_{i=1}^n\|_{l_1^n}.$$

It is clear that every $T \in \mathcal{N}_p^{BL}(X, Y; E)$ is two-Lipschitz and $BLip(T) \leq n_p^{BL}(T)$.

Now,we characterize this type of operators by giving the Pietsch Domination Theorem.

Recently, a general version of the Pietsch Domination Theorem was proved in [29], which is an improved version of a similar result in [7] (see also [28]).

Theorem 3.6. *Let $1 < p < \infty$. A two-Lipschitz mapping $T : X \times Y \to E$ is Cohen p-nuclear if and only if there are a regular palpability measures μ_1 on $C(B_{X^\#}), \mu_2$ on $C(B_{Y^\#})$ and λ on $C(B_{E^{**}})$, such that for any $x, x' \in X, y, y' \in Y$ and $a^* \in E^*$ we have*

$$\begin{aligned} &|\langle T(x, y) - T(x, \acute{y}) - T(\acute{x}, y) + T(\acute{x}, \acute{y}), a^*\rangle| \\ &\leq C \left(\int_{B_{X^\#}} |f(x) - f(x')|^p d\mu_1\right)^{\frac{1}{p}} \left(\int_{B_{Y^\#}} |g(y) - g(y')|^p d\mu_2\right)^{\frac{1}{p}} \\ &\left(\int_{B_{E^{**}}} |a^{**}(a^*)|^{p^*} d\lambda(a^{**})\right)^{\frac{1}{p^*}} \end{aligned} \tag{3.4}$$

Moreover, in this case $\|T\|_{\mathcal{N}_p^{BL}} = \inf\{C : \textit{satisfying } (3.4)\}$

Example 3.7. *Let $1 < p < \infty$ and $L : X \to E$ be a Lipschitz p-summing mapping and $S : Y \to E$ be a Lipschitz Cohen p-nuclear mapping. The mapping*

$$T : X \times Y \to E, \qquad T(x, y) = L(x) S(y)$$

is two-Lipschitz Cohen p-summing operator with $\|T\|_{\mathcal{N}_p^{BL}} \leq \pi_p^L(L)\,\eta_p^L(S)$. *Indeed, for any* $x, x' \in X, y, y' \in Y$ *and* $a^* \subset E^*$ *we have*

$$\begin{aligned}
&|\langle T(x,y) - T(x,\acute{y}) - T(\acute{x},y) + T(\acute{x},\acute{y}), a^*\rangle| \\
&= |\langle L(x)\,S(y) - L(x)\,S(\acute{y}) - L(\acute{x})\,S(y) + L(\acute{x})\,S(\acute{y}), a^*\rangle| \\
&= \|L(x) - L(\acute{x})\|\,|\langle S(y) - S(\acute{y}), a^*\rangle|
\end{aligned}$$

By Theorem 1 *in [15] and Theorem 3.4, there exist Radon probability measures* μ_1 *on* $B_{X^\#}, \mu_2$ *on* $B_{Y^\#}$ *and* λ *on* $B_{E^{**}}$ *, such that for all* x, x' *in* X y, y' *in* Y *and* a^* *in* E^* *we have*

$$\begin{aligned}
&\|L(x) - L(\acute{x})\|\,|\langle S(y) - S(\acute{y}), a^*\rangle| \\
&\leq \pi_p^L(L)\,\eta_p^L(S)\left(\int_{B_{X^\#}} |f(x) - f(x')|^p\,d\mu_1\right)^{\frac{1}{p}}\left(\int_{B_{Y^\#}} |g(y) - g(y')|^p\,d\mu_2\right)^{\frac{1}{p}} \\
&\quad\left(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*}\,d\lambda(a^{**})\right)^{\frac{1}{p^*}}
\end{aligned}$$

it follows that $T \in \mathcal{N}_p^{BL}(X,Y;E)$ *and* $\|T\|_{\mathcal{N}_p^{BL}} \leq \pi_p^L(L)\,\eta_p^L(S)$.

We don't know if two-Lipschitz Cohen p-nuclear implies Cohen p-nuclear bilinear operator whenever the mapping T is bilinear. the converse is of course.

Theorem 3.8.

If $T : X \times Y \to E$ *is a bilinear Cohen p-nuclear between Banach spaces* X, Y *and* E, *then* T *is two-Lipschtz Cohen p-nuclear. Furthermore,* $\|T\|_{\mathcal{N}_p^{BL}} \leq \|T\|_{\mathcal{N}_p^2}$.

Proof. Suppose that $T \in \mathcal{D}_p^2(X,Y;E)$. By [1, Theorem 2.5 (iii)] There exist Radon probability measures $\mu_j \in C(B_{X_j^*})(1 \leq j \leq 2)$ and $\lambda \in C(B_{E^{**}})$, such that for all $x, x' \in X$, $y, y' \in Y$ and $a^* \in E^*$, we have

$$\begin{aligned}
&|\langle T(x,y) - T(x',y) - T(x,y') + T(x',y'), e^*\rangle| \\
&= |\langle T(x-x', y-y'), e^*\rangle| \\
&\leq \|T\|_{\mathcal{D}_p^2}\left(\int_{B_{X^*}} |\varphi_1(x-x')|^p\,d\mu_1\right)^{\frac{1}{p}}\left(\int_{B_{Y^*}} |\varphi_2(y-y')|^p\,d\mu_2\right)^{\frac{1}{p}} \\
&\left(\int_{B_{E^{**}}} |\langle a^*, a^{**}\rangle|^{p^*}\,d\mu(\phi)\right)^{\frac{1}{p^*}} = *
\end{aligned}$$

Let $\tilde{\mu}_1$, $\tilde{\mu}_2$ an extension of μ_1, μ_2 respectively in $B_{X^\#}$ and $B_{Y^\#}$, Since «$(Z)^*$ and $Z^\#$ ($Z = X$ or Y) are isometrically isomorphic via the linearization, for all $h \in Z^\#$ there is $\varphi \in$ «$(Z)^*$ such that

$$\varphi(m_{zz'}) = h_L(m_{zz'}) = h(z) - h(z'),$$

for all $z, z' \in Z$, we obtain

$$* = \|T\|_{\mathcal{D}_p^2} \left(\int_{B_{X\#}} |f(x) - f(x')|^p \, d\mu_1 \right)^{\frac{1}{p}} \left(\int_{B_{Y\#}} |g(y) - g(y')|^p \, d\mu_2 \right)^{\frac{1}{p}} \left(\int_{B_{E^{**}}} |\langle a^*, a^{**} \rangle|^{p^*} d\mu(\phi) \right)^{\frac{1}{p^*}}$$

Therefore, $T \in \mathcal{N}_p^{BL}(X, Y; E)$ and $\|T\|_{\mathcal{N}_p^{BL}} \leq \|T\|_{\mathcal{N}_p^2}$ □

Theorem 3.9.

Let X, Y be pointed metric spaces and E be a Banach space. For $1 < p < \infty$, we have $T : X \times Y \to E$ is two-Lipschitz Cohen p-nuclear mapping if and only if its bi-linearization T_B is Cohen p-nuclear from Æ$(X) \times$Æ(Y) to E. In this case

$$\|T\|_{\mathcal{N}_p^{BL}} = \eta_p^2(T_B). \tag{3.5}$$

Proof. Suppose that $T \in \mathcal{N}_p^{BL}(X, Y; E)$. Let $(m_j^1)_{j=1}^s \subset \mathcal{M}(X)$ and $(m_j^2)_{j=1}^s \subset \mathcal{M}(Y)$, with $m_j^1 = \sum_{i=1}^n \alpha_i^j m_{x_i^j x_i'^j}$ and $m_j^2 = \sum_{k=1}^r \beta_k^j m_{y_k^j y_k'^j}$ then, we have

$$\begin{aligned}
&\sum_{j=1}^s \left| \left\langle T_B(m_j^1, m_j^2), a_j^* \right\rangle \right| \\
&= \sum_{j=1}^s \left| \left\langle T_B(\sum_{i=1}^n \alpha_i^j m_{x_i^j x_i'^j}, \sum_{k=1}^r \beta_k^j m_{y_k^j y_k'^j}), a_j^* \right\rangle \right| \\
&= \sum_{j=1}^s \left| \left\langle \sum_{i=1}^n \alpha_i^j \sum_{k=1}^r \beta_k^j T_B \left(m_{x_i^j x_i'^j}, m_{y_k^j y_k'^j} \right), a_j^* \right\rangle \right| \\
&= \sum_{j=1}^s \left| \left\langle \sum_{i=1}^n \alpha_i^j \sum_{k=1}^r \beta_k^j \left(T(x_i^j, y_k^j) - T(x_i^j, y_k'^j) - T(x_i'^j, y_k^j) + T(x_i'^j, y_k'^j) \right), a_j^* \right\rangle \right| \\
&\leq \|T\|_{\mathcal{N}_p^{BL}} \sup_{\substack{f \in B_{X\#} \\ g \in B_{Y\#}}} \left(\sum_{j=1}^s \left| \sum_{i=1}^n \alpha_i^j \left(f(x_i^j) - f(x_i'^j) \right) \right|^p \left| \sum_{k=1}^r \beta_i^j \left(g(y_k^j) - g(y_k'^j) \right) \right|^p \right)^{\frac{1}{p}} \\
&\sup_{a \in B_E} \|(a_i^*(a))_{i=1}^n\|_{l_{p^*}^n} = (*)
\end{aligned}$$

Since «$(Z)^*$ and $Z^\#$ $(Z = X$ or $Y)$ are isometrically isomorphic via the linearization, for all $h \in Z^\#$ there is $\varphi \in$ «$(Z)^*$ such that

$$\varphi(m_{zz'}) = h_L(m_{zz'}) = h(z) - h(z'),$$

for all $z, z' \in Z$, we obtain

$$\begin{aligned}
(*) &= \|T\|_{\mathcal{N}_p^{BL}} \sup_{\substack{\|\varphi_1\| \leq 1 \\ \|\varphi_2\| \leq 1}} \left(\sum_{j=1}^s \left| \sum_{i=1}^n \alpha_i^j \varphi_1(m_{x_i^j x_i'^j}) \right|^p \left| \sum_{k=1}^r \beta_i^j \varphi_2(m_{y_k^j y_k'^j}) \right|^p \right)^{\frac{1}{p}} \sup_{a \in B_E} \|(a_i^*(a))_{i=1}^n\|_{l_{p^*}^n} \\
&= \|T\|_{\mathcal{N}_p^{BL}} \sup_{\substack{\|\varphi_1\| \leq 1 \\ \|\varphi_2\| \leq 1}} \left(\sum_{j=1}^s \left| \varphi_1(m_j^1) \right|^p \left| \varphi_2(m_j^2) \right|^p \right)^{\frac{1}{p}} \sup_{a \in B_E} \|(a_i^*(a))_{i=1}^n\|_{l_{p^*}^n}
\end{aligned}$$

Therefore, $T_B \in \mathcal{N}_p^2($« (X), « $(Y); E)$ and $\eta_p^2(T) \leq \|T\|_{\mathcal{N}_p^{BL}}$.

Conversely, suppose that $T_B \in \mathcal{N}_p^2(« (X), « (Y); E)$. Let $x_i^j, x_i'^j \in X$, $y_i^j, y_i'^j \in Y$, $\lambda_i^j \in \mathbb{K}$, $(1 \leq i \leq n, 1 \leq j \leq s)$, we have

$$\sum_{j=1}^{s} \left| \left\langle \sum_{i=1}^{n} \lambda_i^j \left(T(x_i^j, y_i^j) - T(x_i^j, y_i'^j) - T(x_i'^j, y_i^j) + T(x_i'^j, y_i'^j) \right), a_j^* \right\rangle \right|$$
$$= \sum_{j=1}^{s} \left| \left\langle T_B \left(\sum_{i=1}^{n} \lambda_i^j m_{x_i^j x_i'^j}, m_{y_i^j y_i'^j} \right), a_j^* \right\rangle \right|$$
$$= \sum_{j=1}^{s} \left| \left\langle T_B \left(\sum_{i=1}^{n} \lambda_i^j m_{x_i^j x_i'^j}, m_{y_i^j y_i'^j} \right), a_j^* \right\rangle \right|$$
$$\leq \eta_p^2 (T_B) \sup_{\substack{\|\varphi_1\| \leq 1 \\ \|\varphi_2\| \leq 1}} \left(\sum_{j=1}^{s} \left| \varphi_1(m_j^1) \right|^p \left| \varphi_2(m_j^2) \right|^p \right)^{\frac{1}{p}} \sup_{a \in B_E} \left\| \left(a_j^* (a) \right)_{j=1}^{s} \right\|_{l_{p^*}^s}$$
$$= \eta_p^2 (T_B) \sup_{\substack{f \in B_{X^\#} \\ g \in B_{Y^\#}}} \left(\sum_{j=1}^{s} \left| f(x_j) - f(x_j') \right|^p \left| g(y_j) - g(y_j') \right|^p \right)^{\frac{1}{p}} \sup_{a \in B_E} \left\| \left(a_j^* (a) \right)_{j=1}^{s} \right\|_{l_{p^*}^s}.$$

Which means that $T \in \mathcal{N}_p^{BL}(X, Y; E)$ and $\|T\|_{\mathcal{N}_p^{BL}} \leq \eta_p^2 (T_B)$. □

Corollary 3.10. *The class $\left(\mathcal{N}_p^{BL}, \|.\|_{\mathcal{N}_p^{BL}} \right)$ is a Banach ideal of two-Lipschitz operators.*

Proof. The proof is based and the following statements that we get directly from the uniqueness of the bi-linearization maps,

1) $(\alpha S + T)_B = \alpha S_B + T_B$,for all $S, T \in \mathcal{N}_p^{BL}$ and $\alpha \in \mathbb{K}$.

2) $(f.g.e)_B = f_L.g_L.e$,for any $f \in X^\#, g \in Y^\#$ and $e \in E$,where $f_L.g_L.e \in \mathcal{L}\left(Æ(X), Æ(Y); E\right)$ is defined by $f_L.g_L.e(m, m') = f_L(m).g_L(m').e$.

3) $(u \circ T \circ (f, g)) = u \circ T_B \left(\hat{f}, \hat{g} \right)$ for all $f \in Lip_0 (Z, X), g \in Lip_0 (W, Y), T \in \mathcal{N}_p^{BL} (X, Y, E)$

□

3.2 Two-Lipschitz factorable p-nuclear operators

The class of factorable strongly p-nuclear multilinear mappings were introduced in [2] by D. Achour et all .

Definition 3.11. *Let $1 \leq p \leq \infty$,A bounded b-linear operator $T : E_1 \times E_2 \to F$ is factorable strongly p-nuclear, if there exists $C \geq 0$ such that for all natural numbers s, n, all $\left(x_i^j \right)_{1 \leq i \leq n} \in E_1$, $\left(y_i^j \right)_{1 \leq j \leq s} \in E_2$ $(j = 1, 2)$,and for all $\left(a_j^* \right)_{1 \leq j \leq s} \in F^*$, the following holds*

$$\sum_{j=1}^{s} \left| \sum_{i=1}^{n} \left\langle T \left(x_i^j, y_i^j \right), a_j^* \right\rangle \right| \leq C \sup_{\substack{\varphi \in \mathcal{L}(E_1, E_2) \\ \|\varphi\| \leq 1}} \left(\sum_{j=1}^{s} \left\| \sum_{i=1}^{n} \varphi \left(x_i^j, y_i^j \right) \right\|^p \right)^{\frac{1}{p}} \left\| \left(a_j^* \right)_j \right\|_{p^*, w}. \quad (3.6)$$

The space of all factorabel strongly p-nuclear operators is denoted by $\mathcal{L}^{2,f,s}_{p,N}(E_1, E_2; F)$ *and endowed with the norm* $\eta^{2,f,s}_{p,N}(T)$ *where* $\eta^{2,f,s}_{p,N}(T)$ *is given by the infimum of all costantes* $C \geq 0$ *that satisfy the inequality* (3.6).

Note that if T *factorable strongly p-nuclear operators then :*

- *Making* $n = 1$ *we have*

$$\sum_{j=1}^{s} \left| \left\langle T\left(x_1^j, y_1^j\right), e_j^* \right\rangle \right| \leq \sup_{\substack{\varphi \in \mathcal{L}(E_1,E_2) \\ \|\varphi\| \leq 1}} \left(\sum_{j=1}^{s} \left\| \varphi\left(x_1^j, y_1^j\right) \right\|^p \right)^{\frac{1}{p}} \left\| \left(a_j^*\right)_j \right\|_{p^*,w}.$$

  When T *satisfies just this condition, we get* T *is strongly p-nuclear. The corresponding class is denoted by* $\mathcal{L}^{2,s}_{p,N}(E_1, E_2; F)$ *and endowed with the norm* $\eta^{2,s}_{p.N}(T)$, *where* $\eta^{2,s}_{p.N}(T)$ *is given by the infimum of all constants* $C \geq 0$ *that satisfy the above inequality. Therefore,* $\mathcal{L}^{2,f,s}_{p,N}(E_1, E_2; F) \subset \mathcal{L}^{2,s}_{p,N}(E_1, E_2; F)$ *continuously.*

- *In particular if* $T \in \mathcal{L}(E, F)$ *we have* $\mathcal{L}^{s}_{p,N}(E, F) = \mathcal{L}^{f,s}_{p,N}(E, F) = \mathcal{N}_p(E, F)$ *where* $\mathcal{N}_p(E, F)$ *is the class of all Cohen p-nuclear operators from* E *to* F. *For more details refer to [11].*

- *Factorable strongly* ∞*-nuclear multilinear operators coincide with all Cohen strongly* ∞*-summing multilinear operators (see [3]).*

Definition 3.12. *Let* $1 \leq p \leq \infty$. *A two-Lipschitz operator* $T : X \times Y \to E$ *is called factorable strongly p-nuclear, if there exists* $C \geq 0$ *such that for any, all* $x_i^j, x_i'^j \in X, y_i^j, y_i'^j \in Y, a_j^* \in E^*$ $(1 \leq i \leq n, 1 \leq j \leq s)$, *and all positive integers* n, s *we have*

$$\begin{aligned} &\sum_{j=1}^{s} \left| \sum_{i=1}^{n} \left\langle T\left(x_i^j, y_i^j\right) - T\left(x_i^j, y_i'^j\right) - T\left(x_i'^j, y_i^j\right) + T\left(x_i'^j, y_i'^j\right), a_j^* \right\rangle \right| \\ &\leq C \sup_{\substack{\varphi \in B_{BLip_0(X,Y)} \\ BLip(\varphi) \leq 1}} \left(\sum_{j=1}^{s} \left\| \sum_{i=1}^{n} \varphi\left(x_i^j, y_i^j\right) - \varphi\left(x_i^j, y_i'^j\right) - \varphi\left(x_i'^j, y_i^j\right) + \varphi\left(x_i'^j, y_i'^j\right) \right\|^p \right)^{\frac{1}{p}} \left\| \left(a_j^*\right)_j \right\|_{p^*,w}. \end{aligned} \tag{3.7}$$

For $p = \infty$ *we have*

$$\begin{aligned} &\sum_{j=1}^{s} \left| \sum_{i=1}^{n} T\left(x_i^j, y_i^j\right) - T\left(x_i^j, y_i'^j\right) - T\left(x_i'^j, y_i^j\right) + T\left(x_i'^j, y_i'^j\right) \right| \\ &\leq C \sup_{\varphi \in B_{BLip(X,Y)}} \sup_{1 \leq j \leq s} \left| \sum_{i=1}^{n} \varphi\left(x_i^j, y_i^j\right) - \varphi\left(x_i^j, y_i'^j\right) - \varphi\left(x_i'^j, y_i^j\right) + \varphi\left(x_i'^j, y_i'^j\right) \right| \left\| \left(a_j^*\right)_j \right\|_{1,w}. \end{aligned} \tag{3.8}$$

The set of oll factorable strongly p-nuclear two-Lipschitz operators is denoted by $BL^{f,s}_{p,N}(X, Y; E)$, *and endowed with the norm* $\|T\|_{BL^{f,s}_{p,N}} = \inf\{C\}$ *where the infimum is taken over all constants* $C \geq 0$ *that satisfy the inequality* (3.7) *and* (3.8).

Remark 3.13.

Note that if T is two-Lipschitz factorable p-nuclear, then taking $n = 1$, we have

$$\sum_{j=1}^{s}\left|\left\langle T(x_j,y_j) - T(x_j,\acute{y}_j) - T(\acute{x}_j,y_j) + T(\acute{x}_j,\acute{y}_j), a_j^*\right\rangle\right| \\ \leq C \sup_{\substack{\varphi\in B_{BLip_0(X,Y)}\\ BLip(\varphi)\leq 1}}\left(\sum_{i=1}^{s}\left|\varphi(x_j,y_j) - \varphi\left(x_j,y_j'\right) - \varphi\left(x_j',y_j\right) + \varphi\left(x_j',y_j'\right)\right|^p\right)^{\frac{1}{p}}\left\|\left(a_j^*\right)_j\right\|_{p^*,w}. \tag{3.9}$$

When T satisfies just this condition then it is called strongly p-nuclear. The corresponding class of two-Lipschitz operators is denoted by $BL_{p,N}^{s}(X,Y;E)$.

In this case, we define $\|T\|_{BL_{p,N}^s} = \inf\{C : \text{satisfying } (3.9)\}$.

Therrfore $BL_{p,N}^{f,s}(X,Y;E) \subset BL_{p,N}^{s}(X,Y;E)$ and $\|T\|_{BL_{p,N}^s} \leq \|T\|_{BL_{p,N}^{f,s}}$.

Now, we study the connection between a mapping belonging to $BL_{p,N}^{f,s}(X,Y;E)$ and its bi-linearization.

Theorem 3.14. *Let X, Y be pointed metric spaces and E be a Banach space. For $1 \leq p \leq \infty$, we have $T : X\times Y \to E$ is factorable strongly p-nuclear two-Lipschitz mapping if and only if its bi-linearization T_B:Æ$(X)\times$Æ$(Y) \to E$ is strongly p-nuclear. In this case*

$$\|T\|_{BL_{p,N}^{f,s}} = \eta_{p,N}^{2,s}(T_B). \tag{3.10}$$

Proof. Suppose that $T \in BL_{p,N}^{f,s}(X,Y;E)$. Let $(m_j^1)_{j=1}^s \subset \mathcal{M}(X)$ and $(m_j^2)_{j=1}^s \subset \mathcal{M}(Y)$, with $m_j^1 = \sum_{i=1}^n \alpha_i^j m_{x_i^j x_i'^j}$ and $m_j^2 = \sum_{k=1}^r \beta_k^j m_{y_k^j y_k'^j}$ then, we have

$$\begin{aligned}
&\sum_{j=1}^{s}\left|\left\langle T_B(m_j^1,m_j^2), a_j^*\right\rangle\right| \\
&= \sum_{j=1}^{s}\left|\left\langle T_B\Big(\sum_{i=1}^{n}\alpha_i^j m_{x_i^j x_i'^j}, \sum_{k=1}^{r}\beta_k^j m_{y_k^j y_k'^j}\Big), a_j^*\right\rangle\right| \\
&= \sum_{j=1}^{s}\left|\left\langle \sum_{i=1}^{n}\alpha_i^j \sum_{k=1}^{r}\beta_k^j T_B\left(m_{x_i^j x_{l,i}'^j}, m_{y_k^j y_k'^j}\right), a_j^*\right\rangle\right| \\
&= \sum_{j=1}^{s}\left|\left\langle \sum_{i=1}^{n}\alpha_i^j \sum_{k=1}^{r}\beta_k^j \left(T(x_i^j,y_k^j) - T(x_i^j,y_k'^j) - T(x_i'^j,y_k^j) + T(x_i'^j,y_k'^j)\right), a_j^*\right\rangle\right| \\
&\leq \|T\|_{BL_{p,N}^{f,s}} \sup_{\substack{\varphi\in BLip_0(X,Y)\\ BLip(\varphi)\leq 1}}\left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\alpha_i^{l,j}\sum_{k=1}^{r}\beta_k^j \varphi(x_i^j,y_k^{l,j}) - \varphi(x_i^j,y_k'^j) - \varphi(x_i'^j,y_k^{l,j}) + \varphi(x_i'^j,y_k'^j)\right|^p\right)^{\frac{1}{p}} \\
&\left\|\left(a_j^*\right)_j\right\|_{p^*,w} = (*)
\end{aligned}$$

By Theorem 2.8, $\varphi \in BLip_0(X,Y)$ then there exist a function $\varphi_B \in \mathcal{L}$(Æ$(X)$, Æ$(Y)$) via the bilinearization, such that

$$\varphi_B(m_{x,x'}, m_{yy'}) = \varphi(x,y) - \varphi(x,y') - \varphi(x',y) + \varphi(x',y'),$$

and $\|\varphi_B\| = BLip(\varphi) \leq 1$. Moreover we have

$$(*) = \|T\|_{BL^{f,s}_{P,N}} \sup_{\substack{\|\varphi_1\|\leq 1\\ \|\varphi_2\|\leq 1}} \left(\sum_{j=1}^{s}\left(\sum_{i=1}^{n}\alpha_i^j \sum_{k=1}^{r}\beta_k^j \left|\varphi_B(m_{x_i^j x'^j_{l,i}}, m_{y_k^j y'^j_k}\right|^p\right)^p\right)^{\frac{1}{p}} \left\|\left(a_j^*\right)_j\right\|_{p^*,w}$$

$$= \|T\|_{BL^{f,s}_{P,N}} \sup_{\|\varphi_B\|\leq 1} \left(\sum_{j=1}^{s}\left|\varphi_B\left(\sum_{i=1}^{n}\alpha_i^j m_{x_i^j x'^j_i}, \sum_{k=1}^{r}\beta_k^j m_{y_k^j y'^j_k}\right)\right|^p\right)^{\frac{1}{p}} \left\|\left(a_j^*\right)_j\right\|_{p^*,w}$$

$$= \|T\|_{BL^{f,s}_{P,N}} \sup_{\substack{\|\varphi_1\|\leq 1\\ \|\varphi_2\|\leq 1}} \left(\sum_{j=1}^{s}\left(\left|\varphi_B\left(m_j^1, m_j^2\right)\right|^p\right)^p\right)^{\frac{1}{p}} \left\|\left(a_j^*\right)_j\right\|_{p^*,w}$$

Therefore, $T_B \in \mathcal{L}^{2,s}_{p,N}(« (X), « (Y); E)$ and $\eta^{2,s}_{p,N}(T_B) \leq \|T\|_{BL^{f,s}_{P,N}}$.

Conversely, suppose that $T_B \in \mathcal{L}^{2,s}_{p,N}(Æ(X), Æ(Y); E)$. Let $x_i^j, x'^j_i \in X$, $y_i^j, y'^j_i \in Y$, $(1 \leq i \leq n, 1 \leq j \leq s)$, we have

$$\sum_{j=1}^{s}\left|\left\langle \sum_{i=1}^{n}\lambda_i^j\left(T(x_i^j, y_i^j) - T(x_i^j, y'^j_i) - T(x'^j_i, y_i^j) + T(x'^j_i, y'^j_i)\right), a_j^*\right\rangle\right|$$
$$= \sum_{j=1}^{s}\left|\left\langle T_B\left(\sum_{i=1}^{n}\lambda_i^j m^1_{x_i^j x'^j_i}, m^2_{y_i^j y'^j_i}\right), a_j^*\right\rangle\right|$$
$$\leq \eta^{2,s}_{p,N}(T_B) \sup_{\|\varphi\|\leq 1}\left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\varphi(m^1_{x_i^j x'^j_i}, m^2_{y_i^j y'^j_i})\right|^p\right)^{\frac{1}{p}} \left\|\left(a_j^*\right)_j\right\|_{p^*,w}$$
$$= \eta^{2,s}_{p,N}(T_B) \sup_{\substack{\phi\in BLip_0(X,Y)\\ BLIp(\phi)\leq 1}} \left(\sum_{j=1}^{s}\left|\sum_{i=1}^{n}\phi(x_i^j, y_i^j) - \phi(x_i^j, y'^j_i) - \phi(x'^j_i, y_i^j) + \phi(x'^j_i, y'^j_i)\right|^p\right)^{\frac{1}{p}} \left\|\left(a_j^*\right)_j\right\|_{p^*,w}.$$

Which means that $T \in BL^{f,s}_{p,N}(X, Y; E)$ and $\|T\|_{BL^{f,s}_{p,N}} \leq \eta^{2,s}_{p,N}(T_B)$. □

The proof of the following proposition is a consequence of Proposition 3.12 in [3] and the previous theorem.

Proposition 3.15.

Let X, Y be pointed metric spaces and E be a Banach space. For $1 \leq p \leq \infty$, we have $T \in BL^{f,s}_{p,N}(X, Y; E)$ if and only if its linearization $T_L \in \mathcal{N}_p(Æ(X)\widehat{\otimes}_\pi Æ(Y), E)$. In this case $\|T\|_{BL^{f,s}_{p,N}} = \eta_p(T_L)$.

As a consequence, we obtain the following corollary which is a straightforward consequence of the preceding proposition, Theorem 2.18 and Proposition 2.19.

Corollary 3.16.

The class $BL^{f,s}_{p,N}$ is the Banach two-Lipschitz operator ideal generated by the composition method from the operator ideal $\mathcal{N}_p$, i.e.,

$$BL^{f,s}_{p,N}(X, Y; E) = \mathcal{N}_p \circ BLip_0(X, Y; E),$$

for all pointed metric spaces X, Y and Banach space E.

Remark 3.17.

If we consider $T \in Lip_0(X, E)$, and $n = 1$ we obtain a characterization of Cohen Lipschitz p-nuclear operators that was introduced by T. Hamid and L. Mezrage in [26].

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Abstract:

In this memory we have studied and analyzed the article entitled "Two-Lipschitz operator ideals" . We also give a new example with called Cohen p-nuclear and factorable strongly p-nuclear two-Lipschitz operator's ideals

Keywords: Lipschitz operators, bilinear operators, tensor product, Arens-Eells space, linear operator ideals, multi-linear operator ideals, Lipschitz operator ideals, p-nuclear operators, Cohen p-nuclear operators .

Résumé:

Dans ce mémoire nous avons étudié et analysé l'article intitulé "Two-Lipschitz operator ideals" Nous donnons également un nouvel exemple avec des idéaux appelés Cohen p-nucléaires et facturables fortement p-nucléaires des opérateurs deux-Lipchitziens

Mots clés : Opérateurs Lipchitziens, opérateurs bilinéaires, produit tensoriel, espace de Arens-Eells, idéaux d'opérateurs linéaires, idéaux d'opérateurs multilinéaires, idéaux d'opérateurs Lipchitziens l'opérateur p- nucléaire, Les opérateurs Cohen p-nucléaire.

ملخص:

قمنا بدراسة وتحليل المقال المعنون "Two-Lipschitz operator ideals" كما قدمنا امثلة جديدة جديدة تسمى كوهان p-نكليار, p-نكليار التفكيكي لمثاليات ثنائي-ليبشيتز

الكلمات المفتاحية: المؤثرات الليبشيتزية, المؤثرات المتعددة الخطية Arens-Eells , مثاليات المؤثرات الخطية ,المتعددة الخطية والليبشيتزية, المؤثرات P-نكليار, المؤثرات كوهان p-نكليار