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Thesis

On Lipschitz relations

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SAYAD. Sabrina

Dedication

I dedicate this thesis to all the people who have supported and encouraged me throughout my academic journey.

I would like to thank my parents in particular for their unconditional love and unwavering support. Your faith in me has allowed me to achieve my dreams and pursue my goals with determination.

I would also like to thank my brothers and to thank my sister: ***Ibtissam****, for their constant support and encouragement. Your presence in my life is a blessing that I will always cherish.*

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Contents

Introduction

A Lipschitz multivalued relation in Banach spaces is a mathematical concept that plays an important role in the study of functional analysis[5, 17]. Firstly, it is a generalization of the notion of a function, where instead of mapping a single point is maps to a set points.

More specifically, a Lipschitz multivalued relation is a set-valued mapping between Banach spaces, which satisfies a Lipschitz condition. This means that the Hausdorff distance between the sets of images is bounded by a constant multiple of the distance between the points of each set themselves.

The Lipschitz condition is an important property of multivalued relations, as it guarantees that the mapping is well-defined and continuous. This, in fact, allows for the development of powerful mathematical tools and techniques for studying properties of these relations.

More over, Lipschitz multivalued relations are an important and fascinating area of study in functional analysis, with numerous applications and implications for a wide range of fields and disciplines.

The objective of this paper is to study the theory of linear operators with multiple values or linear relations, which has been intensively developed in recent years. Linear relations were introduced in functional analysis by J.V. Neumann to consider the adjoint of operators used in applications to the theory of generalized equations, as well as to consider inverses of certain operators. A treatise on linear relations was written by R.W. Cross [5].

In the first chapter, we address classical definitions of mapping Lipschitz and linear operators, such as semi-continuous relations, continuity and norm of linear operators, classification of linear spaces, linear functions and conjugate spaces, Hahn-Banach extension and separation theorems, quotients, subspaces, and projections.

In the second chapter, we apply linear relations already seen in normed linear spaces. We present the difficulties of this study, particularly the algebra of linear relations, continuity and norm function for linear relations, open relations and minimal modules, linear selections, closed linear relations, adjoint of a linear relation, dimension and nullity theorems, deficiency and index, graphical operator and relative limit, as well as canonical factorization.

NOTATIONS

$d(.,.)$	distance application.	
(x,d)	metric space.	
$\mathbb{R}$	set of real numbers.	
$\mathbb{N}$	set of natural numbers.	
$Lip(.)$	Lipschitz function.	
$f \xrightarrow{c.s} f_n$	simple covergence.	
$\tilde{f}$	prolongation of f.	
$(X^{\#})$	dual space space.	
$B_{X^{\#}}$	unit ball.	
$B(X,Y)$	space of bounded linear operators from X to Y.	
$B(X)$	algebra of bounded linear operators on X.	
B_X	closed unit ball in metric space X.	lcl
Ker	kernel.	
dim	dimension.	
$LR(X,Y)$	set of linear relations from space X to space Y.	
I_E	identity application on set E.	
$\mathbb{K}$	any field.	
$(X, \|.\|_X)$	norm space.	
x_n	Suite.	
$\vert.\vert$	Complex module / absolute value.	
X/M	The quotient of space X by subspace M.	
$H(A,B)$	Haussdroff distance.	
$N(X)$	Closed and compact.	

Chapter 1

Preliminaries

1.1 Lipschtz mapping

Definition 1.1.1. *A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ between two metric spaces is called Lipschitz if there is a positive constant C such that*

$$\exists C > 0 : \forall x, y \in X, \quad d_Y(f(x), f(y)) \leqslant C d_X(x, y).$$

If $C = 1$, the map is called ***non expansive****.*

If $C < 1$, the map is called ***contraction****.*

For a Lipschitz map f, we define its Lipschitz constant by

$$\|f\|_{Lip} = Lip(f) := \sup_{x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} = \\ inf\ \{C : C \text{ verifying the above inequality}\}.$$

Let $(X, e_X, d_X), (Y, e_Y, d_Y)$ be pointed metric spaces.

We say a map $f : (X, e_X, d_X) \longrightarrow (Y, e_Y, d_Y)$ preserves distinguished point if $f(e_X) = e_Y$.

Definition 1.1.2. *Let $(X, d_X), (Y, d_Y)$ be two metric spaces.*

A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called bi-Lipschitz or quasi-isometry, if f is bijective (one-to-one = injective, and onto = surjective) and both f, f^{-1} are Lipschitz.

In this case X and Y are called

(1)- Lipschitz isomorphic or Lipschitz homeomorphic (Nigel Kalton)

or

(2)- Quasi-isometric (Nik Weaver).

A bi-Lipschitz function f is an isometry if

$$\forall x, y \in X, \quad d_Y(f(x), f(y)) = d_X(x, y).$$

Proposition 1.1.1. *Let X, Y and Z be metric spaces and let $f : (X, d_X) \longrightarrow (Y; d_Y), g : (Y, d_Y) \longrightarrow (Z, d_Z)$ be Lipschitz maps. Then $g \circ f : (X, d_X) \longrightarrow (Z, d_Z)$ is Lipschitz and*

$$Lip(g \circ f) \leqslant Lip(g) + Lip(f).$$

Proof. For $x, y \in X$, we have:

$$\begin{aligned} d_Z((g \circ f)(x), (g \circ f)(y)) &\leqslant Lip(g) d_Y(f(x), f(y)) \\ &\leqslant Lip(g) Lip(f) d_X(x, y) \end{aligned}$$

and this shows the proposition. □

Proposition 1.1.2. *Let (X, d) be metric space. For Lipschitz functions $f, g : (X, d) \longrightarrow \mathbb{R}$, and scalar $a \in \mathbb{R}$ the Lipschitz constant has the properties:*

a) $Lip(f + g) \leqslant Lip(f) + Lip(g)$.

b) $Lip(af) \quad = |a| Lip(f)$.

Proof. Let $x, y \in X$:

$$\begin{aligned} a)\quad d((f+g)(x),(f+g)(y)) &= |((f+g)(x)-(f+g)(y)| \\ &= |f(x)+g(x)-f(y)+g(y)| \\ &\leqslant |f(x)-f(y)|+|g(x)-g(y)| \\ &\leqslant Lip(f)d(x,y)+Lip(g)d(x,y) \\ &\leqslant (Lip(f)+Lip(g))d(x,y) \end{aligned}$$

$$\begin{aligned} b)\qquad Lip(af) &= \sup_{x\neq y}\frac{d(af(x),af(y))}{d(x,y)} \\ &= \sup_{x\neq y}\frac{|af(x)-af(y)|}{d(x,y)} \\ &= |a|\sup_{x\neq y}\frac{d|f(x),f(y)|}{d(x,y)} \\ &= |a|Lip(f). \end{aligned}$$

□

Theorem 1.1.1. *Let X_0, Y_0 be tow spaces and X, Y their complements. Let $f_0 : X_0 \longrightarrow Y_0$ be Lipschitz map then, f has a unique Lipschitz extension $f : X \longrightarrow Y$ such that*

$$Lip(f) = Lip(f_0).$$

Proof. Since Lipschitz functions are continuous and X_0 is dense in X, there is at most one Lipschitz extension. Consider x in $X \diagdown X_0$ and put

$$f(x) = \lim_{n\to+\infty} f_0(x_n)$$

where x is a Cauchy sequence in X_0 such that $x_n \longrightarrow x$. We have $Lip(f) = Lip(f_0)$. Indeed

$$\begin{aligned} d_Y(f(x),f(y)) &= d_Y(\lim_{n\to+\infty} f_0(x_n), \lim_{n\to+\infty} f_0(y_n)) \\ &= \lim_{n\to+\infty} d_Y(f_0(x_n), f_0(y_n)) \\ &\leqslant \lim_{n\to+\infty} Lip(f_0)d_X(x_n,y_n) \\ &\leqslant Lip(f_0)d_X(x,y) \\ &\leqslant Lip(f_0). \qquad (1.1) \end{aligned}$$

This implies that $Lip(f) \leqslant Lip(f_0)$ and we have in the first part

$$\begin{aligned} Lip(i_Y \circ f_0) &= \sup_{x\neq y} \frac{d_Y(i_Y \circ f_0(x), i_Y \circ f_0(y))}{d_X(x,y)} \\ &= \sup_{x\neq y} \frac{d_{Y_0}(f_0(x), f_0(y))}{d_X(x,y)} \\ &= Lip(f_0) \end{aligned}$$

and in the second part

$$\begin{aligned} Lip(i_Y \circ f_0) = Lip(f \circ i_X) &\leqslant Lip(f)Lip(i_X) \\ &\leqslant Lip(f) \cdot 1 \\ &\leqslant Lip(f). \end{aligned} \tag{1.2}$$

This implies that $Lip(f_0) \leqslant Lip(f)$ and this completes the proof. □

Example 1.1.1. *Let X be a metric space. Let $f : X \longrightarrow \mathbb{R}$ be a Lipschitz map. If $diam < \infty$ then $\|f\|_\infty$ is finite.*

$$diam(X) = \sup_{x,y\in X} d(x,y).$$

Let $x_0, x \in X$.

$$\begin{aligned} & |f(x) - f(x_0)| && \leqslant Lip(f)d(x,x_0) \\ \Rightarrow\ & |f(x)| - |f(x_0)| && \leqslant Lip(f)d(x,x_0) \\ \Rightarrow\ & |f(x)| && \leqslant |f(x_0)| + Lip(f)d(x,x_0) \\ \Rightarrow\ & \sup_{x\in X}|f(x)| && \leqslant \sup_{x\in X}(|f(x_0)| + Lip(f)d(x,x_0)) \\ & \|f\|_\infty && \leqslant |f(x_0)| + Lip(f)diam(X) < \infty. \end{aligned}$$

Proposition 1.1.3. *Let X, Y be two metric spaces f, f_n Lipschitz functions from X to Y, we suppose that $f_n \xrightarrow{c.s} f$ then:*

$$Lip(f) \leqslant \sup_{n\in\mathbb{N}} Lip(f_n)$$

$$f_n \xrightarrow{c.s} f \Longleftrightarrow \forall x \in X,\ \lim_{n\to+\infty} f_n(x) = f(x).$$

Proposition 1.1.4. *Let* X *be a metric space* $fg : X \longrightarrow \mathbb{R}$ *two Lipschitz operator then:*
a) $Lip(fg) \leqslant \|f\|_\infty Lip(g) + \|g\|_\infty Lip(f)$.

b) $Lip(\frac{1}{f}) \leqslant \frac{Lip(f)}{a^2} \quad$ *if* $|f(x)| \geqslant a > 0 \quad \forall x \in X$.

Remark 1. *If* fg *is Lipschitz so:* $diam(X) < \infty$.

Definition 1.1.3. ***(Contracting Lipschitz Function)***
Let I *interval* $\mathbb{R}$*, and* f *is function* $f : I \longrightarrow \mathbb{R}$ *is called contracting if it is Lipschitzian with a Lipschitz constant strictly less than.*

Remark 2. *By definition,*

(1) f *is contracting if and only if there exists* $k \in [1, 0[$ *such that for all* $x, y \in I$ *we have* $d(f(x) - f(y)) \leqslant kd(x - y)$.

(2) *A contracting function is Lipschitz, therefore continuous.*

(3) *If a contractive function admits a fixed point, it is unique.Indeed,* $f(a) = a$ *and* $f(b) = b$ *leads to* $|a - b| \leqslant k|a - b|$ *then to* $a = b$*, because* $0 \leqslant k < 1$.

(4) *If* $f : I \to I$ *is differentiable and if* $\sup |f'| = k < 1$ *then* f *is k-contractive by the mean value theorem.*

Example 1.1.2. *Let* f *is function*

$$\begin{array}{rcl} f : [-\pi, \pi] & \longrightarrow & [-1, 1] \\ x & \longmapsto & \sin(x). \end{array}$$

1.2 Linear Operators and Linear Relations

In this section we give some notions about linear operators and relations. Most of the Preliminaries in this section can be found in Cross [5].

Definition 1.2.1. *Let X, Y denote arbitrary nonempty sets. A* ***relation*** *T from X to Y is a map defined we have non-empty subset $\mathcal{D}(T)$ of X, called the* ***domain*** *of T, which takes values in $\mathcal{P}(Y)\setminus\emptyset$. We denote the class of relations of X in Y by $R(X,Y)$.*

Definition 1.2.2. ***(Graph)*** *For $T\in R(X,Y)$ we formally define its* ***graph*** *by $G(T)$, a subset $X\times Y$, as follows:*

$$G(T) := \{(x,y)\in X\times Y \mid x\in\mathcal{D}(T),\ y\in T(x)\}.$$

Remark 3. *If T sends the points of its domain to singletons, so T is said to be a one-to-one relation or a* ***function****.*

Definition 1.2.3. ***(Rank)*** *The* ***rank*** *$R(T)$ of T is defined as*

$$R(T) := \bigcup_{x\in\mathcal{D}(T)} Tx.$$

Remark 4. *If $R(T)=Y$, then we say T est* ***surjective****, and if $A\subset X$ then the image of A under T*

$$T(A) := \bigcup_{a\in A\cap\mathcal{D}(T)} Ta.$$

(**The inverse**) **The inverse** of the relation T^{-1} given by the graph

$$G\left(T^{-1}\right) := \{(y,x)\in Y\times X \mid (x,y)\in G(T)\}.$$

Proposition 1.2.1. *A relation is said to be* ***injective*** *if T^{-1} is single valued.*

Definition 1.2.4. ***(The inverse image)*** *if $B\subset Y$, then the* ***inverse image*** *of B under T is defined as the set*

$$T^{-1}(B) := \{x\in\mathcal{D}(T) \mid Tx\cap B\neq\emptyset\}.$$

Definition 1.2.5. *the* ***Direct image*** *of B under T defined as the set*

$$T^{+1}(B) :\ = \{x\in\mathcal{D}(T) \mid T(x)\subset B\}.$$

Definition 1.2.6. ***(Composition)*** *Let $S,T\in R(X,Z)$. The* ***composition*** *(or* ***product****) $ST\in R(X,Z)$ of T and S is defined by :*

$$ST(x) := S(Tx), x\in X.$$

Definition 1.2.7. *(**Restriction**) If $A \subset X$, then the **restriction** of T to A, denoted $T|_A$ is defined by :*

$$G(T|_A) := \{(x, y) \in G(T) \mid x \in A\} = G((x, y) \cap (x \times Y)).$$

Definition 1.2.8. *(**Extension**) Suppose $S \in R(X, Y)$. So R is said an **extension** from T if*

$$S|_{\mathcal{D}(T)} = T.$$

Notation 1. *The class of linear relations in $R(X, Y)$ will be denoted by $LR(X, Y)$ the set of linear relations from X into Y. If $X = Y$ we write $LR(X, X) = LR(X)$.*

Proposition 1.2.2. *[5] Let $T \in R(X, Y)$. So*

(a) *$T^{-1}y = \{x \in \mathcal{D}(T) : y \in Tx\}$ for $y \in R(T)$, and there for,*
$\mathcal{D}(T^{-1}) = R(T)$ and $R(T^{-1}) = \mathcal{D}(T)$.

(b) *If T is injective then $Tx_1 = Tx_2$ implies $x_1 = x_2$ for $x_1, x_2 \in \mathcal{D}(T)$.*

(c) *If T is unique, then $T^{-1}(B) = T^{+1}(B)$ for $B \subset Y$.*

(d) *For $B \in R(Y, Z)$, the domain and the graph of ST are given by*

$$\begin{aligned} \mathcal{D}(ST) &= \{x \in X : STx \neq \emptyset\} \\ &= \{x \in X : Tx \cap \mathcal{D}(S) \neq \emptyset\} \\ &= T^{-1}(\mathcal{D}(S)), \\ G(ST) &= \{(x, z) \in X \times Z \mid \exists y \in Y : (x, y) \in G(S)\}. \end{aligned}$$

(e) *For non-empty subsect A_1 and A_2 of X we have*

$$\begin{aligned} T(A_1 \cup A_2) &= T(A_1) \cup T(A_2), \\ T(A_1 \cap A_2) &= T(A_1) \cap T(A_2), \\ T(X \setminus A_1) &\supset R(T) \setminus T(A_1), \quad \textit{and} \\ A_1 \subset A_2 &\Rightarrow T(A_1) \subset T(A_2). \end{aligned}$$

Note that for X and Y vector spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A relation $T \in R(X, Y)$ is called a linear relation (or **multivalued linear operator**) if for all $x, z \in D(T)$ and nonzero scalars a we have the next definition.

Definition 1.2.9. *Let X and Y be vector spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $x_1, x_2 \in X$ and $\alpha \in \mathbb{K}$.*

*The **linear operator** $T : X \to Y$ is an application single valued from X into Y such that*

$$\begin{cases} T\left(x_1 + x_2\right) &= Tx_1 + Tx_2, \text{ and} \\ \alpha T x_1 &= T\left(\alpha x_1\right). \end{cases}$$

We denote the class of linear operator of the space X in Y by $L\left(X, Y\right)$.

*A **multivalued linear operator** (or **linear relation**) $T : X \to Y$ is a linear relation whose graph is vector subspace sous-espace of $X \times Y$. The class of linear relations from X in Y denoted by $LR\left(X, Y\right)$.*

Remark 5. *T is a linear relation if and only if T^{-1} linear relation.*

Corollary 1.2.1. *Let $T \in R\left(X, Y\right)$. So T is a linear relation if and only if*

$$\alpha T x_1 + \beta T x_2 = T\left(\alpha x_1 + \beta x_2\right)$$

for everyone $x_1, x_2 \in \mathcal{D}\left(T\right)$ and $\alpha, \beta \in \mathbb{K}^*$.

Proposition 1.2.3. *Let $T \in R(X, Y)$. The following properties are equivalent:*

(i) T is a linear relation.

(ii) $G(T)$ is a linear subspace of $X \times Y$.

(iii) T^{-1} is a linear relation.

(iv) $G(T^{-1})$ is a linear subspace of $X \times Y$.

Proof. $(i) \Rightarrow (ii)$ We have $(0, 0) \in G(T)$ because we have $\mathcal{D}(T)$ is a vector subspace therefore $0 \in \mathcal{D}(T)$ and for all $x, y \in \mathcal{D}(T)$ we have $0 \in 0Tx + 0Ty \subset T(0x + 0y) = T(0)$. be now $(x, y) \in G(T)$, $(a, b) \in G(T)$, $\alpha \in \mathbb{K}$, we have

$$(x, y) + \alpha(a, b) = (x + \alpha a, y + \alpha b).$$

On the one hand, we have $x + \alpha a \in \mathcal{D}(T)$ (because $x, a \in \mathcal{D}(T)$). And, if $\alpha \neq 0$ we have

$$T(x + \alpha a) = Tx + \alpha Ta.$$

Or $y \in Tx$ and $b \in Ta$, therefore

$$y + \alpha b \in Tx + \alpha Ta = T(x + \alpha a)$$

and hence $(x + \alpha a, y + \alpha b) \in G(T)$. If $\alpha = 0$ obvious.

$(ii) \Rightarrow (i)$ In the first part we show that $\mathcal{D}(T)$ is a vector subspace. Either

$$\begin{array}{rcl} P : X \times Y & \longrightarrow & X \\ (x, y) & \longmapsto & x \end{array}$$

we have $\mathcal{D}(T) = P(G(T))$ with $G(T)$ is a vector subspace and P linear. Therefore $\mathcal{D}(T)$ est un sous espace vectoriel. In the second part let $x, y \in \mathcal{D}(T)$, show that

$$T(x + y) = Tx + Ty.$$

Let $z \in T(x + y)$, then $(x + y, z) \in G(T)$ and we have $x, y \in \mathcal{D}(T)$ therefore $Tx \neq \emptyset$ and $Ty \neq \emptyset$. Let $a \in Tx, b \in Ty$, therefore $(x + y, a + b) \in G(T)$ and we have $(x + y, z) \in G(T)$, whence $(0, z - a - b) \in G(T)$. Therefore $z - a - b \in T(0)$ and hence $z \in a + b + T(0)$. Whence $z \in Tx + Ty + T(0)$ and we have $Ty + T(0) = Ty$ (bacause $(0, 0) \in G(T)$) therefore $0 \in T(0)$ and $Ty \subset Ty + T(0)$, let $z \in Ty + T(0)$ therefore

$$z = a + b, \quad a \in Ty,\ b \in T(0) \text{ and } (y, a),\ (0, b) \in G(T)$$

and consequently $(y, a + b) \in G(T)$, therefore $(y, z) \in G(T)$ and $z \in Ty$). Conversely let $z \in Tx + Ty$, so

$$z = a + b, \quad a \in Tx,\ b \in Ty$$

and hence $(x, a), (y, b) \in G(T)$. So,

$$(x + y, a + b) \in G(T) \text{ and } (x + y, z) \in G(T),$$

therefore $z \in T(x + y)$. Finally let $x \in \mathcal{D}(T), \alpha \in \mathbb{K}$ let's show that $T(\alpha x) = \alpha Tx$. We have

$$\begin{array}{rcl} z \in T(\alpha x) & \Longleftrightarrow & (\alpha x, z) \in G(T) \\ & \Longleftrightarrow & \alpha \left(x, \left(\dfrac{1}{\alpha} \right) z \right) \in G(T) \\ & \Longleftrightarrow & \left(x, \left(\dfrac{1}{\alpha} \right) z \right) \in G(T) \\ & \Longleftrightarrow & \left(\dfrac{1}{\alpha} \right) z \in Tx \\ & \Longleftrightarrow & z \in \alpha Tx, \end{array}$$

so T is linear.

$(iii) \Rightarrow (iv)$ It suffices to apply $(i) \Rightarrow (ii)$ to T^{-1}.

$(ii) \Rightarrow (iv)$ By symmetry. ☐

Corollary 1.2.2. *Let T be a linear relation. Then $T(0)$ and $T^{-1}(0)$ are linear subspaces.*

Corollary 1.2.3. *Let $T \in LR(X,Y)$ and let M a linear subspaces of X. So $T|_M \in LR(X,Y)$.*

Definition 1.2.10. *The subspaces $T^{-1}(0)$ is called the **null space** or **Kernel** de T and denoted $N(T)$.*

Proposition 1.2.4. *Let $T \in LR(X,Y)$.*

(a) *Let $x \in \mathcal{D}(T)$. We have the following equivalence:*

$$y \in Tx \Leftrightarrow Tx = y + T(0).$$

Especially,

$$0 \in Tx \Leftrightarrow Tx = T(0).$$

(b) *For $x_1, x_2 \in \mathcal{D}(T)$ We have the following equivalence:*

$$Tx_1 \cap Tx_2 \neq \emptyset \Leftrightarrow Tx_1 = Tx_2.$$

Corollary 1.2.4. *Let $T \in LR(X,Y)$, we have*

(a) $TT^{-1}(0) = T(0)$.

(b) $T^{-1}T(0) = T^{-1}(0)$.

Proof. We have T relation linear. So $T(0)$ and $T^{-1}(0)$ are vector subspaces, therefore $0 \in TT^{-1}(0)$. Or $T(0) = TT^{-1}(0)$. Substituting T^{-1} by T gives us the second equality. ☐

Corollary 1.2.5. *Let $T \in LR(X,Y)$, we have*

(a) *If $y \in R(T)$, so $TT^{-1}y = y + T(0)$.*

(b) *If* $x \in \mathcal{D}(T)$*, so* $TT^{-1}x = x + T^{-1}(0)$.

Corollary 1.2.6. *Suppose* $T, S \in LR(X, Y)$ *and* $G(S) \subset G(T)$*. Then* T *is an extension of* S *if and only if* $S(0) = T(0)$.

Proposition 1.2.5. *Let* $\alpha \in \mathbb{K}^*$*, and* $A, B \subset X$*,* $C \subset Y$*, let* $T \in LR(X, Y)$*, then*

(a) $T(\alpha A) = \alpha T(A)$.

(b) $T(A) + T(B) \subset T(A + B)$.

(c) *If* $A \subset \mathcal{D}(T)$ *or* $B \subset \mathcal{D}(T)$*, so* $T(A + B) = T(A) + T(B)$.

(d) *If* $A \subset \mathcal{D}(T)$ *or* $B \subset \mathcal{D}(T)$ *and* $A \cap B = \{0\}$*,*
so $T(A + B) = T(A) + T(B)$ *and* $T(A) \cap T(B) = T(0)$.

(e) $TT^{-1}C = C \cap R(T) + T(0)$.

(f) $T^{-1}T(A) = A \cap \mathcal{D}(T) + T^{-1}(0)$.

(g) $T^{-1}(0) \times \{0\} = G(T) \cap (X \times \{0\})$.

(h) $\{0\} \times T(0) = G(T) \cap (\{0\} \times Y)$.

(i) $X \times R(T) = G(T) + (X \times \{0\})$.

(j) $\mathcal{D}(T) \times Y = G(T) + (\{0\} \times Y)$.

The equality does not necessarily hold in (b) - we can consider the case $A = \{a\}, B = \{b\}$ *such that* $a + b \in \mathcal{D}(T)$ *while* $a \notin \mathcal{D}(T)$ *and* $b \notin \mathcal{D}(T)$.

Proof. Let $T \in LR(X, Y)$,

(a)

$$\begin{aligned} T(\alpha a) &= \cup\{(T(\alpha A)) : a \in A \cap D(T)\} \\ &= \cup\{\alpha(Ta) : a \in A \cap D(T)\} \\ &= \alpha \cup \{Ta : a \in A \cap D(T)\} \\ &= \alpha T(a). \end{aligned}$$

(c) Let $a \in A,\ b \in B \subset \mathcal{D}(T)$. If $a+b \notin \mathcal{D}(T)$,then $\emptyset = T(a+b) \subset Ta+Tb$ trivially. On the other hand if $a+b \in \mathcal{D}(T)$, then $a \in \mathcal{D}(T)$ and $T(a+b) = Ta+Tb \subset TA+TB$.

(d) Immédiate.

(e), (f) Follow immediately from Corollary (1.2.5).

(j), (g) Are simple consequences of the definitions.

□

1.3 Semi-Continuous Relations, Continuity and the Norm of Linear Relations

In this section, X and Y denote standard spaces.

Definition 1.3.1. *Let $\epsilon > 0$, and $M \subset X$. Then the sets $B(M,\epsilon), B_X, U(M,\epsilon), U_X$ and S_X are defined by:*

$$\begin{aligned} B(M,\epsilon) &:= \{x \in X \mid d(x,M) \leq \epsilon\}, \\ B_X &:= \{x \in X \mid d(x,0) \leq 1\}, \\ U(M,\epsilon) &:= \{x \in X \mid d(x,M) < \epsilon\}, \\ U_X &:= \{x \in X \mid d(x,0) < 1\}, \\ S_X &:= \{x \in X \mid d(x,0) = 1\}. \end{aligned}$$

Definition 1.3.2. *Asubet U of X is a set in the* ***vicinity*** *of point $x \in X$ if U contians an open set containing x.*

Definition 1.3.3. *A relation $T \in R(X,Y)$ is said* ***upper semi-continuous (u.s.c)*** *at the point $x \in \mathcal{D}(T)$ for any neighborhood U of $T(x)$ there exists $\epsilon > 0$ such that for any $z \in B(x,\epsilon)$ we have $T(z) \subset U$. T is said to be* ***upper semi-continuous*** *if it is upper semi-continuous to every x in its domain $\mathcal{D}(T)$.*

It follows from the pervious Definition that T is u.s.c at $x \in \mathcal{D}(T)$ if and only if the kernel of any open set is open.

Definition 1.3.4. *A relation $T \in R(X,Y)$ is said to be a* ***lower semi continuous (l.s.c)*** *at the point $x \in \mathcal{D}(T)$ if for all $y \in T(x)$ and for all sequence $x_n \subset \mathcal{D}(T)$ such that $x_n \to x$ there exists $y_n \in T(x_n)$ such that $y_n \to y$. T is called a* ***lower semi continuous*** *if it is lower semi-continuous to each x in its domain $\mathcal{D}(T)$.*

It follows that T is l.s.c à $x \in \mathcal{D}(T)$ if and only if the inverse image of any open set that intersects $T(x)$ is a neighborhood of x. Thus T is l.s.c if and ony if the inverse image of any open set is open.

Example 1.3.1. *We have*

(a) *The map $T_1 \in R(\mathbb{R},\mathbb{R})$ defined by:*

$$T_1(x) = \begin{cases} [-1,1] & \text{if} \quad x \neq 0 \\ \{0\} & \text{if} \quad x = 0 \end{cases}$$

is l.s.c at zero but not u.s.c at zero.

(b) *The map $T_2 \in R(\mathbb{R},\mathbb{R})$ defined by:*

$$T_2(x) = \begin{cases} \{0\} & \text{if} \quad x \neq 0 \\ [-1,1] & \text{if} \quad x = 0 \end{cases}$$

is u.s.c at zero but not l.s.c at zero.

Remark 6. *The definitions of upper and lower semi-continuity are equivalent for one-to-one maps. Moreover, it will known that the continuity of a (single valued) linear operator can characterize of the normed operator.*

Definition 1.3.5. *Let X and Y be normed spaces, and $T \in L(X,Y)$. The* ***norm*** *of T is defined as follows:*

$$\|T\| := \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Theorem 1.3.1. *Let $T \in L(X,Y)$. Then the following are equivalents:*

(i) *T is continuous at the point.*

(ii) *T is uniformly continuous over its domain.*

(iii) *There exists* $M \in \mathbb{R}$ *such that* $\|Tx\| \leq M\|x\|$ *for everything* x *in the domain of* T.

(iv) *The norm of* T *is finite* $\|T\| < \infty$.

Remark 7. *The definition of operator continuity between norm spaces can be extended linear reations. Furthermore, he can show that the property of lower semi-continuity of many-to-many operators is equivalent to the property of finite norm. For this reason we choose the notion of lower semi-continuity to serve for the defintion of the continuity of a many-sided linear operator. We provide definitions in the next chapter. The term "is continuous" also more convenient to use frequently, the more precise expression lower semi-continuity. Note that in the convex analysis literature, the map is said to be continuous if and only if it is both upper and lower semi-continuous.*

Notation 2. *Let* $B(X,Y)$ *denote the class of continuous one-to-one linear operators of the normed space* X *in* Y *and* $B(X)$ *indicates this class for the case* $X = Y$.

Definition 1.3.6. *let* $T \in LR(X,Y)$. *If* T *and the inverse mapp* T^{-1} *are single valued, continuous and every were defined, then* T *is said to be* ***isomorphism***. *Thus* T *is said to be* ***isometry*** *if* $\|Tx\| = \|x\|$.

Theorem 1.3.2. *Let* $T \in L(X,Y)$. *Then* T^{-1} *is continuous and single valued if and only if there exists* $m > 0$ *such that*

$$\|Tx\| \geq m\|x\|, \quad x \in D(T).$$

1.4 Propriets of Vector Spaces

This section serves to introduce the properties that are used in the following.

Definition 1.4.1. *A pair of normed spaces are said to be isomorphic (isometric) if there exists an isomorphism (isometry) wich maps one space onto the other.*
A norm space can be classified in terms of isomorphisms and isometries of the space into it self, or of its subset, into a subset of classical spaces. Characterization of space properties can be topological, for example, low compactness of the unite ball, or geomtric. Spaces are

also studied via their local properties, i.e, constructed from finite dimensional subspaces. More generally we have the following identifications.

Theorem 1.4.1. *If K and H compact metric spaces, then K is homemorphic to H si and only if $C(K)$ is isometric to $C(H)$.*

Theorem 1.4.2. *If K and H are uncountable compact metric spaces, then $C(K)$ is isomorphic to $C(H)$.*

Theorem (1.4.1) initially, was exended by M.H. Stone to compact Hausdorff spaces. However, $C(K)$ can be isomorphic to $C(H)$ without K being an uncountable compact metric space, it suffices to consider $K = [0, 1]$. On the other hand that K varies on countable compact metric spaces, there exists an uncountable number of classes for $C(K)$.

Theorem 1.4.3. *If X is ann dimensional normed space on sur $\mathbb{R}$ (or on $\mathbb{C}$), then X is isomorphic to $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$).*

Lemma 1.4.1. *If X is isomorphic to a Banach space, then X is also a space of Banach.*

Corollary 1.4.1. *If X is finite dimensional normed space, then X is complete.*

Corollary 1.4.2. *If B is a bounded closed set in normed space of dimension finite X, then B is compact.*

The downpour is also true, *i.e*, la property of the unit ball given in Corollary (1.4.2) characterizes finite dimensional vector spaces. Riesz's lemma is usully used to prove.

Theorem 1.4.4. *(* ***Riesz lemma****) Let M be subset of the normed space X, such that M is not dense. Then there exists a sequence $\{x_n\} \subset S_X$ such that $d(x_n, M) \to 1$.*

Theorem 1.4.5. *If X is a norm space such that B_X is totally bounded, then X is finite dimension.*

1.5 Linear functions and conjugate spaces

Definition 1.5.1. *Let X be a vectoriel topological vector space. The* ***algebraic conjugate*** *of X, noted $X^{\#}$, is the set of all functionals definied on X.i.e.*

$$X^{\#} := \{f : X \longrightarrow R \mid f(x+y) = f(x) + f(y);\ f(\alpha x) = \alpha f(x)$$

$$\text{for all} \quad x, y \in X \ \text{and}\ \alpha \in \mathbb{K}\}.$$

If X is a norm space, then the ***topological conjugate*** *of X denotes X', is the subset of $X^{\#}$ constitutes linear functionals that satisfies*

$$\|f\| := \sup_{x \in X} |f(x)| < \infty.$$

We generally denote x' an element in X', we denote X' simply as the conjugate or the adjoint of X, when there is no ambiguity. Note that X' is a Banach space with the norm given above.

Definition 1.5.2. *Let $x'_0 \in X'$. Sets*

$$U_{\epsilon, x_1, \ldots, x_n}(x'_0) := \{x' \in X' \mid |x'(x_i) - x'_0(x_i)| < \epsilon, 1 \leq i \leq n\},$$

$\epsilon > 0, \{x_1, \ldots, x_n\} \subset X, n \geq 1$ form neighborhood basis for $x'_0 \in X'$. The topology given by these sets a called the ***topology $*$-weak*** *on X', and it is denoted by $\sigma(X', X)$.*

Theorem 1.5.1. *The unit ball by $B_{X'}$ is $\sigma(X', X)$-compact.*

Definition 1.5.3. *The elements of the normed space X determine linear functionals on X' by the formula $x''(x') := x'(x)$ for $x \in X$. We say X* ***reflexive*** *if all the linear functionals determine on X' are determined in this way, i.e, if $X'' = X$ under this identification. The topology $\sigma(X'', X')$ on X, where X is considered a comme a subspace of X'', is called as the appelé comme la* ***weak - topology*** *on X.*

Theorem 1.5.2. *If X is a Banach, then X is reflexive if and only if X' is reflexive.*

Theorem 1.5.3. *If X is reflexive, then all closed subspaces are.*

Theorem 1.5.4. *X is reflexive if and only if the unit ball by is B_X est $\sigma\left(X'', X'\right)$ -compact.*

Example 1.5.1. *We have*

1. *C_0 is not reflexive since $C_0'' = l_\infty$.*
2. *$l_p, 1 < p < +\infty$ and $L_p, 1 < p < +\infty$ are reflexives.*
3. *l_0 is not reflexive, and so it follows that the spaces $L_1, C\left([0,1]\right)$, and L_∞ are not also reflexive since they have subspaces isomorphic to l_1.*

Theorem 1.5.5. *If X' is separable, then X is separable.*

Theorem 1.5.6. *If X is separable, then the topology $\sigma\left(X', X\right)$ of X' est metrizable.*

In general, for separable topologicalspaces, we use sequences in continuity and convergence. Sequences also suffice when the space is metrizable. However, when the weak topologies(infinite dimensional spaces) X and X', it is not generally the case that the topologies are metrizable.

Theorem 1.5.7. *Let X vector space over $\mathbb{R}$ or $\mathbb{C}$. Suppose that p is a real-valued function defined on X satisfies*

$$\begin{cases} p\left(x+y\right) & \leq & p\left(x\right) + p\left(y\right), \text{and} \\ p\left(\alpha x\right) & = & \left|\alpha\right| p\left(x\right). \end{cases}$$

If M is a subspace of X and f is a linear function defined on M that satisfies $\left|f\left(m\right)\right| \leq p\left(m\right)$ for $m \in M$, then there exists a linear function F that extends f for X and satisfies $\left|F\left(x\right)\right| \leq p\left(x\right)$ for $x \in X$.

Definition 1.5.4. *A subset K of a vector space X is said to be conveze if $\lambda x + (1-\lambda) y \in K$ whatever $x, y \in K$ and $\lambda \in [0,1]$.*

Theorem 1.5.8. *Let K be closed, under convex space of the norm space X. If $x \in X$ and $x \notin K$ then there exists $x' \neq 0$, $x' \in X'$ such that*

$$Re\ x'x \quad \geq \quad Re\ x'k, \quad k \in K.$$

1.6 Quotients, Subspaces and Projections

Definition 1.6.1. *Let M be a vector subspace of X, denoted by $[x]$ the set of all the elements equivalent to x under the relation of equivalent*

$$yRx \Leftrightarrow y - x \in M.$$

The quotient space X/M is defined by:

$$X/M := \{[x] \, | x \in X\}.$$

If M is the closed subspace of a normed space $(X, \|.\|_X)$, then X/M is a normed space of standard $\|.\|$ defined by:

$$\|[x]\| := d(x, M) = \inf_{m \in M} \|x - m\|.$$

Remark 8. *The fact that norm on X/M is well dfined results from the fact if yRx then*

$$\begin{aligned} \|[y]\| &= \inf_{m \in M} \|y - m\| \\ &= \inf_{m \in M} \|x - ((y - x) + m)\| \\ &= \inf_{m \in M} \|x - m\| \\ &= \|[x]\|. \end{aligned}$$

It thus follows that

$$\|[x]\| = \inf_{y \in [x]} \|y\|.$$

Definition 1.6.2. *The operator $Q_M^X : X \to X/M$ Defined by $Q_M^X x = [x]$ is called the natural quotient map with domain X nal space M.*

Theorem 1.6.1. *Let X be a Banach space. If M is a closed subspace of X then X/M is the Banach space.*

Proposition 1.6.1. *Let M be a closed subspace of X, and let $N \subset X$ be a subspace such that $M \subset N$. Then N is closed if and only if N/M is closed in X/M.*

Proof. Suppose que N/M is closed, and $\{x_n\}_{n\in\mathbb{N}} \subset N$ and a sequence converge to $x \in X$, then $\{[x_n]\}_{n\in\mathbb{N}} \subset N/M$, and

$$\begin{aligned} & \lim_{n\in\mathbb{N}} \inf\nolimits_{m\in M} \|(x_n - x) - m\| = 0 \\ \Rightarrow\ & \lim_{n\in\mathbb{N}} \|[x_n - x]\| = 0 \\ \Rightarrow\ & \lim_{n\in\mathbb{N}} \|[x_n] - [x]\| = 0. \end{aligned}$$

Since N/M is closed, $[x] \in N/M$ and $\exists y \in N$ such that $[y] = [x]$. Since $y - x \in M$ it follows that $x \in N$.

The inverse implication is similar. □

Proposition 1.6.2. *Let $T \in LR(X, Y)$, and let $M \subset X$. Then*

$$\dim R(T)/T(M) \leq \dim \mathcal{D}(T)/\mathcal{D}(T) \cap M \leq X/M.$$

In particular, if M is finitely codimensional of $R(T)$.

Proof. See Cross (5) □

Definition 1.6.3. *Let X a vector space, and let $M \subset X$. then, we define what is sometimes called the annilator $M^{\perp}$ of M by:*

$$M^{\perp} := \{x' \in X' \quad | \quad x'x = 0 \quad \forall x \in M\}.$$

Similarly, if $N \in X'$ then $N^{\top}$ is defined by:

$$N^{\top} := \{x \in X \quad | \quad x'x = 0 \quad \forall x' \in N\}.$$

Remark 9. *Let $M^{\perp}$ and $N^{\top}$ are closed subspaces of X' and X respectively. Moreover, $M^{\perp\top} = \overline{M}$, and $N^{\top\perp}$ is weak∗closure of N.*

Proposition 1.6.3. *Let M be a subspace of a normed space X. Then*

(a) *$X'/M^{\perp}$ is isometrically isomorphic to M' under the map U defined by :*

$$U[x'] := x'_M$$

where $[x'] \in X/M'$ and x'_M the restriction of x' to M.

(b) *If M is closed , then $(X/M)'$ is isometrically isomorphic to $M^{\perp}$ under the map V defined by par :*

$$(Vz')\,x := z'\,[x]\,,$$

where $z' \in (X/M)'$.

Definition 1.6.4. *Let M be a subspace of a vector space X, then a (one-to-one)* ***linear projection*** *from X onto M is a (one-to-one) linear operator which satisfies the condition $P^2 = P$.*
If M and N are subsets of a vector space X, then the sum $M + N$ disigns all

$$\{m + n \mid m \in M \text{ et } n \in N\}\,.$$

If M and N are vector subspaces of X wich satisfies $X = M + N$ and $M \cap N = \{0\}$, then N is called a ***complement*** *of M. If on the other hand, there is a continuous linear projection from X onto M, then N is called a* ***topological complement*** *of M. In this case write $X = M \oplus N$.*

Proposition 1.6.4. *Let M and N be linear subspaces of X. If N is a complement of M, then N is isomorphic to the quotient space X/M.*

Theorem 1.6.2. *Let M be a closed subspace of the Banach space X. there is continuous linear projection from X onto M if and only if there exists un a closed subspace N such that $X = M + N$ and $M \cap N = \{0\}$.*

Theorem 1.6.3. *If M a finite-dimensional subspace of the normed linear space, then M is topologically completed.*

Definition 1.6.5. *A closed subspace M of a Banach space X is said to be quasi-completions if there exists a closed subspace N such that $M \cap N = \{0\}$ and $M + N$ is dense in X.*

Theorem 1.6.4. *Every a closed subspace of a separable separable Banach space is quasi-complete.*

Chapter 2

Linear Relations between Normed Spaces

2.1 The Algebra of Linear Relations

In this section we define the operations of addition and scalar multiplication in $LR(X,Y)$.

Proposition 2.1.1. *Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$. Then $ST \in LR\,(X,Z)$.*

Proof. Let $x_1, x_2 \in \mathcal{D}(ST)$ and let α, β be nonzero scalars. Then

$$\begin{aligned} \alpha(STx_1) + \beta(STx_2) &= \alpha S(Tx_1) + \beta S(Tx_2) \\ &= S(\alpha Tx_1) + S(\beta Tx_2) \\ &= S(T(\alpha x_1)) + S(T(\beta x_2)) \\ &= ST\,(\alpha x_1 + \beta x_2)\,. \end{aligned}$$

Hence ST is linear, by (1.2.5). □

Definition 2.1.1. *Let $S, T \in LR(X,Y)$, be linear relations in $LR(X,Y)$ and let $\alpha \in \mathbb{K}$. **Addition** and **scalar multiplication** . The relations are defined as follows:*

$$\begin{cases} (S+T)x := Sx + Tx \quad x \in X \quad \textit{and} \\ (\alpha T)x := \alpha(Tx) \quad x \in X. \end{cases}$$

Remark 10. *The following proprietes for $R, S, T \in LR(X, Y)$ and $\alpha, \beta \in \mathbb{K}$ follow the definition:*

$$\begin{aligned}
\mathcal{D}(S+T) &= \mathcal{D}(S) \cap \mathcal{D}(T) \\
G(S+T) &= \{(x,y) \in X \times Y | y = s+t, s \in Tx, t \in Sx\} \\
S+T &= T+S \\
R+(S+T) &= (R+S)+T
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}(\alpha T) &= \mathcal{D}(T) \\
G(\alpha T) &= \{(x, \alpha y) \in X \times Y | (x,y) \in G(T)\} \\
&= \{(x,y) \in X \times Y | (x, \alpha^{-1} y) \in G(T)\} \\
&= \{(\alpha^{-1} x, \alpha y) \in X \times Y | (x,y) \in G(T)\} \\
\alpha(\beta T) &= (\alpha\beta)T.
\end{aligned}$$

It follows that $S+T$ et αT are linear relations, i.e, $LR(X,Y)$ is closed under addition and scalar multiplication.

Proposition 2.1.2. *Let $T, T_2 \in LR(X,Y)$ and $R, S \in LR(Y,Z)$. Then*

(a) $TT^{-1} = I_{R(T)} + (TT^{-1} - TT^{-1})$.

(b) $\alpha(ST) = (\alpha S)T = S(\alpha T), 0 \neq \alpha \in \mathbb{K}$.

(c) *Si $G(S) \subset G(R)$, alors $G(ST) \subset G(RT)$.*

(d) $G((R+S)T) \subset G(RT) + G(RT)$.

(e) *$S(T+T_2)$ is an extension of $ST + ST_2$, and we have equality if $\mathcal{D}(S)$ contains both $R(T)$ and $R(T_2)$.*

Proof. For the proof of this Proposition see [Cross (5)] □

Example 2.1.1. *Let $R \in LR(Y,Z)$ be single valued and nonzero, and let $S = -R$, let $G(T) = X \times Y$ where $Y \neq \{0\}$. Then*

$$(R+S)T(0) = (R-R)Y = \cup y \in \mathcal{D}(R)(R-R)y = \{0\}.$$

And

$$(RT+ST)(0) = (RT+RT)(0) = RT(0) - RT(0) = R(Y) - R(Y) = R(Y) \neq \{0\}.$$

This shows that $RT + ST$ *is not in general an extension of* $(R+S)T$, *(see Corollary* (1.2.6)*)*

2.2 Continuity and Normed Linear Relations

Let X, Y tow Banach spaces, $T \in LR(X,Y)$. Let Q_T be the canonical surjection:

$$\begin{array}{rcl} Q_T : Y & \longrightarrow & Y/\overline{T(0)} \\ x & \longmapsto & [x]. \end{array}$$

with $[x]$ it is the class of x. Q_T is a linear operator (is a linear relation).

We consider the linear relation :

$$\begin{array}{rcl} Q_T T : \mathcal{D}(T) & \longrightarrow & Y/\overline{T(0)} \\ x & \longmapsto & Q_T T x. \end{array}$$

Let U, V two parts of X.

$$d(U,V) = \inf\left\{\|y-z\| : y \in U, z \in V\right\}.$$

Definition 2.2.1. *If then* $U = \{a\}$, *then* $d(a,V) = \inf\left\{\|a-z\| : z \in V\right\}.$

Remark 11. (i) $Y/\overline{T(0)}$ *is equipped with a Banach space structure, with for* $\overline{y} \in Y/\overline{T(0)}$, *we have* $\|\overline{y}\| = d(y, \overline{T(0)}) = \inf\left\{\|y-\alpha\| : \alpha \in \overline{T(0)}\right\}.$
(ii) *For all* $y \in Y$ *on a* $d(y,T(0)) = d(y,\overline{T(0)})$. *In effect* $T(0) \subset \overline{T(0)}$, *so*

$$d(y,\overline{T(0)}) \leq d(y,T(0)).$$

Conversely we have on a $d(y,\overline{T(0)}) = \inf\left\{\|y-z\| : z \in \overline{T(0)}\right\}$. *If* $z \in \overline{T(0)}$, *then there exists* $z_n \in T(0)$ *such that* $z_n \underset{n\to+\infty}{\to} z$. *Or* $\|y-z\| = \lim\limits_{n\to\infty} \|y-z_n\|$ *and* $z_n \in T(0)$, *so*

$$d(y,T(0)) \leq \|y-z_n\|.$$

So,

$$d(y,T(0)) \leq \|y-z\| \quad \forall\ z \in \overline{T(0)}.$$

From where,

$$d(y,T(0)) \leq \inf\{\|y-z\| : z \in \overline{T(0)}\} = d(y,\overline{T(0)}).$$

Proposition 2.2.1. *QT is single valued.*

Proof. Let $x \in \mathcal{D}(T)$, and let $y_1, y_2 \in QTx$. Then $y_1 - y_2 \in QTx - QTx = QT(0) \subset Q\overline{T(0)} = 0$. Hence $y_1 = y_2$ ☐

Proposition 2.2.2. *Let $T \in LR(X,Y)$. Then*

$$N(T) \subset N(QT).$$

avec égalité si $T(0)$ is relativement closed in $R(T)$.

Proof. We apply Proposition (1.2.4):

$$\begin{aligned} N(T) = \{x \in X : Tx = T(0)\} &\subset \{x \in X : Tx \subset \overline{T(0)}\} \\ &= \{x \in X : QTx = 0\} = N(QT). \end{aligned}$$

Si $T(0)$ if relativly closed in $R(T)$, then the equality holds.

The following example shows that the equality generally does not hold in Proposition (2.2.2).

☐

Example 2.2.1. *Let X be a normed space of infinite dimension and let $f \in LR(X, \mathbb{K})$ be a discontinuous linear functional defined everywere. If $T = f^{-1}$, then $T(0)$ is a dence hyperplane dense in X and $Q_T = 0$. So*

$$N(T) = 0 \neq \mathbb{K} = N(QT).$$

Proposition 2.2.3. *Let $T \in LR(X,Y)$. If $R(T)$ is closed, then $R(QT)$ is closed. Conversely, if $R(QT)$ is closed and $\overline{T(0)} \subset R(T)$, then $R(QT)$ is closed.*

Proof. This is a special case of Proposition (1.6.1). ☐

Definition 2.2.2. *Let $T \in LR(X,Y)$. Then T is said to* ***continuous at a point*** *$x \in \mathcal{D}(T)$ if inverse image of all neighborhoods Tx is a neighborhood to x. T is said to be continuous if it is* ***continuous*** *at every point of its domain.*

for $x \in \mathcal{D}(T)$ we define $\|Tx\|$ by

$$\|Tx\| := \|QTx\|$$

and the quantity $\|T\|$, *wich is called the* ***norm*** *of* T, *is defined by*

$$\|T\| := \|QT\|.$$

We noted that $\|T\|$ *is not true a function "norm" since* $\|T\| = 0$ *does not imply* $T = 0$. *i.e,* $x \in T(0)$

Proposition 2.2.4. *Let* $T \in LR(X, Y)$.

(a) *For* $x \in \mathcal{D}(T)$

$$\begin{aligned} \|Tx\| &= d(y, T(0)) \text{ for all } y \in Tx \\ &= \inf_{y \in Tx} \|y\| \\ &= d(y + T(0), 0) \text{ for all } y \in Tx \\ &= d(Tx, 0) = d(Tx, T(0)). \end{aligned}$$

(b) $\|T\| = \sup_{x \in B_{\mathcal{D}(T)}} \|Tx\|$.

Proof. See Cross [5, theorem 2] □

Proposition 2.2.5. *On a*

(a) *For* $S, T \in LR(X, Y)$ *and* $x \in \mathcal{D}(S + T)$, *we have*

$$\|Sx + Tx\| \leq \|Sx\| + \|Tx\|.$$

If moreover $S(0) \in \overline{T(0)}$ *then*

$$\|Tx\| - \|Sx\| \leq \|Tx - Sx\|.$$

(b) *For* $\alpha \in \mathbb{K}$ *and* $x \in \mathcal{D}(T)$ *we have*

$$\|\alpha Tx\| = |\alpha| \|Tx\|.$$

Proof. **(a)** Let $s \in Sx$ and $t \in Tx$. Then $s + t \in Sx + Tx = (S + T)x$. So,

$$\begin{aligned} \|Sx + Tx\| &= d(s + t, (S + T)(0)) \\ &\leq d(s, S(0) + T(0)) + d(t, S(0) + T(0)) \\ &\leq d(s, S(0)) + d(t, T(0)) \\ &= \|Sx\| + \|Tx\|. \end{aligned}$$

If $S(0) \in \overline{T(0)}$, then by we just showed ce que nous venons de montrer,

$$\|Tx\| = \|Tx + Sx - Sx\| \leq \|Tx - Sx\| + \|Sx\|.$$

(b) We have $\|\alpha Tx\| = \|Q(\alpha T)(x)\| = \|\alpha QTx\| = |\alpha|\|QTx\| = |\alpha|\|Tx\|$.

□

The following exaple shows that it is not true in general that $\|Tx\| - \|Sx\| \leq \|Tx - Sx\|$ for linear ralations.

Example 2.2.2. *Let X be a nonzero norm space, let $G(S) = X \times X$ and let $T = I_X$. Then for $x \neq 0$ we have*

$$\|Tx - Sx\| = 0.$$

While

$$\|Tx\| - \|Sx\| = \|x\| \neq 0.$$

So $\|Tx\| - \|Sx\| \nleq \|Tx - Sx\|$.

Corollary 2.2.1. *Let $S, T \in LR(X, Y)$ and let $\alpha \in \mathbb{K}$. Then*

(a) $\|S + T\| \leq \|S\| + \|T\|$.

(b) *If $S(0) \in \overline{T(0)}$ then $\|T\| - \|S\| \leq \|T - S\|$.*

(c) $\|\alpha T\| = |\alpha|\|T\|$.

Proposition 2.2.6. *Let $T \in LR(X, Y)$ then.*

(a) *If $\|T\| < \infty \Leftrightarrow$ there exists $\lambda > 0$ such that*

$$TB_{\mathcal{D}(T)} \subset \lambda B_{R(T)} + T(0). \tag{2.1}$$

(b) *If $\|T\| < \infty$ then*

$$\|T\| = \inf_{\lambda > 0}\{\lambda | TB_{\mathcal{D}(T)} \subset \lambda B_{R(T)} + T(0)\}. \tag{2.2}$$

Proof. **(a)** suppse that $\|T\| < \infty$. We apply Proposition (2.2.4) : for $x \in B_{\mathcal{D}(T)}$ and $y \in Tx$ there exists $k \in T(0)$ such that given $\epsilon > 0$,

$$\|y - k\| < \|T\| + \epsilon,$$

i.e., $y - k \in (\|T\| + \epsilon)B_{R(T)}$. So

$$y \in (\|T\| + \epsilon)B_{R(T)} + T(0). \tag{2.3}$$

Conversely, suppose that (2.1) holds. Let $x \in B_{\mathcal{D}(T)}$ and choose $y \in Tx$. Then $y = \lambda y_1 + k$ were $\|y_1\| \leq 1$ and $k \in T(0)$. Thus $\|y - k\| \leq \lambda$, in particular, $d(y, T(0)) \leq \lambda$. It follows Proposition 2.2.7 that $\|T\| \leq \lambda < \infty$.

(b) Suppose that $\|T\| < \infty$. If $\|T\| = 0$, then $TB_{D(T)} \subset \overline{T(0)}$ and (2.2) holds. Suppose that $\|T\| > 0$. Then, it follows from (2.3) that

$$\|T\| \geq \inf_{\lambda > 0}\{\lambda | TB_{\mathcal{D}(T)} \subset \lambda B_{R(T)} + T(0)\}.$$

Let α a such that $0 < \alpha < \|T\|$, and choose $x \in B_{\mathcal{D}(T)}, y \in Tx$ such that

$$\alpha < d(y, T(0)) \tag{2.4}$$

If $TB_{\mathcal{D}(T)} \subset \alpha B_{R(T)} + T(0)$, then there exists $y_1 \in B_{R(T)}$, and $K \in T(0)$ such that $y = \alpha y_1 + k$. But then

$$\|y - k\| \leq \alpha.$$

Which contradicts (2.4). So,

$$\alpha < \inf_{\lambda > 0}\{\lambda | TB_{\mathcal{D}(T)} \subset \lambda B_{R(T)} + T(0)\}.$$

□

Proposition 2.2.7. *Let* $T \in LR(X, Y)$

(a) *T is continuous if only if $\|T\| < \infty$.*

(b) *If $\dim \mathcal{D}(T) < \infty$, then T is continuous.*

Proof. **(a)** Suppose that T is continuous. Since $T(0) + B_Y$ is a neighborhood to $T(0)$, then follows that $T^{-1}(T(0) + B_Y) = T^{-1}B_{R(T)}$ is a neighborhood to 0. So $\exists \lambda > 0$ such that.

$$\lambda B_{\mathcal{D}(T)} \subset T^{-1}B_{R(T)}.$$

Conversely,

$$\lambda T B_{\mathcal{D}(T)} \subset B_{R(T)} + T(0) \subset TT^{-1}B_{R(T)}.$$

By Proposition (2.2.6), this implies that $\|T\| < \infty$.

Conversely, suppose that $\|T\| < \infty$. Let $x \in \mathcal{D}(T)$, and let V be a non-trivial closed ball in $R(T)$ with center y where $y \in Tx$. Then $V_0 = V - \{y\} = \alpha B_{R(T)}$ for some $\alpha > 0$. Applying Proposition (2.2.6), there exists $\lambda > 0$ such that

$$TB_{\mathcal{D}(T)} \subset \lambda B_{R(T)} + T(0).$$

It then follows that

$$B_{\mathcal{D}(T)} + T^{-1}(0) \subset \lambda T^{-1}B_{R(T)} = \alpha^{-1}\lambda T^{-1}V_0.$$

Or, equivalent,

$$\lambda^{-1}\alpha B_{D(T)} + T^{-1}(0) \subset T^{-1}V_0 = T^{-1}(V - y).$$

As a result,

$$\lambda^{-1}\alpha B_{\mathcal{D}(T)} + T^{-1}y \subset T^{-1}V - T^{-1}y + T^{-1}y = T^{-1}V,$$

and $\lambda^{-1}\alpha B_{\mathcal{D}(T)} + T^{-1}y$ is a neighborhood of x in $\mathcal{D}(T)$. Suppose now that W let is a neighborhood of Tx, let $U \subset W$ be an open set containing $y \in Tx$, and let $V \subset U$ be non-trivial closed ball with center y. From what has already been shown, it follows that $T^{-1}W$ is a neighborhood of x. So T is continuous.

(b) if $\dim \mathcal{D}(T) < \infty$. Then QT is a single continuous operator, *i.e*, $\|QT\| < \infty$. Since $\|T\| = \|QT\|$, the result follows from (a).

□

2.3 Open Relations and Minimal Modulus

Definition 2.3.1. *A linear relation $T \in LR(X,Y)$ is said to be* ***open*** *if its inverse T^{-1} is a continuous linear relation.*

Definition 2.3.2. *Let $T \in LR(X,Y)$.*

$$\gamma(T) = \begin{cases} \infty \quad \text{if} \quad \mathcal{D}(T) \subset \overline{N(T)} \\ \inf\left\{ \dfrac{\|Tx\|}{d(x,N(T))} | x \in \mathcal{D}(T) \backslash \overline{N(T)} \right\} \text{ otherwise} \end{cases} \tag{2.5}$$

Proposition 2.3.1.

$$\gamma(T) = \sup\{\lambda : TB_{\mathcal{D}(T)} \supset \lambda B_{R(T)}\}. \tag{2.6}$$

Proof. see [Cross (5)] □

Proposition 2.3.2. *Let $T \in LR(X,Y)$.*

(i) *$Q_T T$ is closed and single valued.*

(ii) *$\|Tx\| = d(y, T(0))$ for all $x \in \mathcal{D}(T)$, $y \in Tx$.*

(iii) *$\|Tx\| = d(Tx, T(0)) = d(Tx, 0) (x \in \mathcal{D}(T))$.*

(iv) *$\|T\| = \sup\limits_{x \in B_X \cap \mathcal{D}(T)} \|Tx\|$.*

(v) *$\gamma(T) = \|T^{-1}\|^{-1}$.*

Proof. **(i)** We have $Q_T T(0) = \{0\}$ because $T(0) \subset \overline{T(0)}$.

(ii) Let $x \in \mathcal{D}(T), y \in Tx$. Therefore, $Tx = y + T(0)$ and hence

$$\|Tx\| = \|Q_T Tx\| = \|Q_T(y + T(0))\| = \|Q_T y\| = d(y, \overline{T(0)}) = d(y, T(0)).$$

(iii) We have $d(Tx, T(0)) = \inf\{d(y, T(0)) : y \in Tx\} = \inf\{\|Tx\| : y \in Tx\} = \|Tx\|$ (because from (ii), we have $\|Tx\| = d(y, T(0))$). Let us show the other equality, we have

$$d(Tx, T(0)) \leq d(0, Tx) \text{ because } 0 \in T(0).$$

For the other equality, we have

$$\{\|y-z\| : y \in Tx, z \in T(0)\} \subset \{\|z\| : z \in Tx\}$$

(car $y - z \in Tx$). So,

$$\inf\{\|z\| : z \in Tx\} \leq \inf\{\|y-z\| : y \in Tx, z \in T(0)\}.$$

Hence, $d(0, Tx) \leq d(Tx, T(0))$.

(iv) We have $\|T\| = \|Q_T T\| = \sup\limits_{x \in B_X \cap \mathcal{D}(T)} \|Q_T Tx\| = \sup\limits_{x \in B_X \cap \mathcal{D}(T)} \|Tx\|$.

(v) Let $\lambda > 0$, we have the following implications:

$$\begin{aligned} T^{-1} \supset \lambda B_{\mathcal{D}(T)} &\Rightarrow B_{R(T)} + T(0) \supset \lambda T B_{\mathcal{D}(T)} \\ &\Rightarrow T^{-1}(B_{R(T)} + T(0)) \supset \lambda B_{\mathcal{D}(T)} + T^{-1}(0) \\ &\Rightarrow T^{-1}B_{R(T)} + T^{-1}(0) \supset \lambda B_{\mathcal{D}(T)} + T^{-1}(0) \\ &\Rightarrow T^{-1}B_{R(T)} \supset \lambda B_{\mathcal{D}(T)}. \end{aligned}$$

As a result, $T^{-1}B_{R(T)} \supset \lambda B_{\mathcal{D}(T)} \Longleftrightarrow B_{R(T)} + T(0) \supset \lambda T B_{\mathcal{D}(T)}$. So, we have two case to be treated:

Case 1: $\|T\| < \infty$.

$$\begin{aligned} \|T\| &= \inf\{\lambda > 0 : \lambda^{-1} T B_{\mathcal{D}(T)} \subset B_{R(T)} + T(0)\} \\ &= (\sup\{\lambda T B_{\mathcal{D}(T)} \subset B_{R(T)} + T(0)\})^{-1} \\ &= (\sup\{T^{-1}B_{R(T)} \supset \lambda B_{\mathcal{D}(T)}\})^{-1} \\ &= (\gamma(T^{-1}))^{-1}. \end{aligned}$$

Case 2: $\|T\| = \infty$.

$$\begin{aligned} T B_{\mathcal{D}(T)} &\not\subset \lambda B_{R(T)} + T(0) \ \forall\ \lambda > 0 \\ \lambda T B_{\mathcal{D}(T)} &\not\subset B_{R(T)} + T(0) \ \forall\ \lambda > 0 \\ T^{-1}B_{R(T)} &\not\supset \lambda B_{\mathcal{D}(T)} \ \forall\ \lambda > 0. \end{aligned}$$

Consequently, we have $\gamma(T^{-1}) = 0$. So $\|T\| = \gamma(T^{-1})$.

in both cases we remplace T by T^{-1}, we find the result.

□

Proposition 2.3.3. *Let $T \in LR(X,Y)$.*

(a) *T is open if and only if $\gamma(T) > 0$.*

(b) *If $\dim R(T) < \infty$, then T is open.*

Proposition 2.3.4. *Let $T \in LR(X,Y)$. So*

$$\gamma(T) \leq \gamma(QT)$$

with equality if $T(0)$ is relativly closed in $R(T)$.

Proof.

$$\begin{aligned}\gamma(T) &= \sup\{\lambda : \|Tx\| \geq \lambda d(x, N(T)) \forall x \in D(T)\}\\ &\leq \sup\{\lambda : \|Tx\| \geq \lambda d(x, N(QT)) \forall x \in D(T)\}\\ &= \sup\{\lambda : \|QTx\| \geq \lambda d(x, N(T)) \forall x \in D(T)\}\\ &= \gamma(QT).\end{aligned}$$

By Proposition (2.2.2), the equality holds when $T(0)$ is closed in $R(T)$. Proposition (2.3.4) proof that T is open, so is QT. The converse is false and the equality must not contain. This is ullustrated in the the following example: ☐

Example 2.3.1. *Let X is an infinite-dimensional normed linear space, let f be a discontinuous linear functional with the domain $\mathcal{D}(T) = X$ and let $T := f^{-1}$ be a dense hyperplane and $N(T) = \{0\}$, while $N(QT) = \mathbb{K}$ and $\gamma(QT) = \infty$. However, since T is not open, $\gamma(T) = 0$.*

Corollary 2.3.1. *If $T \in LR(X,Y)$ is open and $N(T)$ is closed, then*

(a) *$N(T) = N(QT)$.*

(b) *$\gamma(T) = \gamma(QT)$.*

Proof. **(a)** Since $T^{-1}(0)$ is closed, $R(T) \cap \overline{T(0)} = T(0)$. The result follows from Proposition (2.2.2).

(b) As in (a), the result follows from Proposition (2.3.4).

☐

Proposition 2.3.5. *Let M be a non-empty subset of $R(T)$, and let $\gamma(T) < \infty$. So for $N \subset D(T)$ we have*

$$d(TN, M) \geq \gamma(T) d(N, T^{-1}M).$$

Proof. If $TN \cap M \neq \emptyset$, so $\emptyset \neq T^{-1}(TN \cap M) = (N + N(T)) \cap (T^{-1}M)$, therefore, $d(N, T^{-1}M) = 0$.

Suppose that $TN \cap M = \emptyset$, let $\epsilon > 0$, and choose $m \in M$ and $n \in N$ such that

$$d(TN, M) > d(Tn - m, 0) - \epsilon. \tag{2.7}$$

Now

$$\begin{aligned} d(Tn-m,0) &= d(Tn-m-T(0),0) = d(Tn, m+T(0)) \quad = \quad d(Tn, TT^{-1}m) \\ &= \inf_{h\in T^{-1}m} d(Tn, Th) = \inf_{h\in T^{-1}m} d(T(n-h),0) = \inf_{h\in T^{-1}m} \|T(n-h)\| \\ &\geq \gamma(T) \inf_{h\in T^{-1}m} d(n-h, T^{-1}(0)) = \gamma(T) \inf_{h\in T^{-1}m} d(n, h+T^{-1}(0)) \\ &= \gamma(T) d(n, T^{-1}m) \geq \gamma(T) d(n, T^{-1}M) \geq \gamma(T) d(N, T^{-1}M). \end{aligned}$$

Since ϵ was chosen arbitraily, it follows from (2.7) that

$$d(TN, M) \geq \gamma(T) d(N, T^{-1}M).$$

□

Proposition 2.3.6. *Let $T \in LR(X, Y)$ and $S \in LR(Y, Z)$. So*

$$\gamma(ST) \geq \gamma(S|_{R(T)})\gamma(T) \quad (\infty.0 \textit{ exclude}). \tag{2.8}$$

wich $\gamma(ST) = \infty$ when $\gamma(T) = \infty$ (even if $\gamma(S|_{R(T)}) = 0$). In other

$$S^{-1}(0) \subset R(T) \Rightarrow \gamma(ST) \geq \gamma(S)\gamma(T). \tag{2.9}$$

Remark 12. *If $\gamma(S) = \infty$ and $\gamma(T) = 0$, so inequality (2.8) may fail to hold considering $S := f$ and $T := f^{-1}$ where f is a discontinuous linear function an infinite dimensional space . So $\gamma(ST) = \gamma(ff^{-1}) = \gamma(I) = 1$ while $\gamma(S) = \infty$ (since $\overline{N(f)} = X$), and gamma(T) = $\|f\|^{-1} = 0$.*

Corollary 2.3.2. *Let* $T \in LR(X,Y)$ *and* $S \in LR(Y,Z)$. *So*

$$\|ST\| \leq \|S\| \|I_{d(S)}T\| \quad (\infty.0 \quad exclue). \tag{2.10}$$

with $\|ST\| = 0$ *when* $\|S\| = 0$ *(even if* $I_{d(T)} = \infty$*). In other,*

$$T(0) \subset \mathcal{D}(S) \Rightarrow \|ST\| \leq \|S\| \|T\|. \tag{2.11}$$

Proof. Applying Proposition (2.3.2), the inequality (2.10) follows from (2.8) of Proposition (2.3.6).
If $T(0) \subset \mathcal{D}(S)$, so for $x \in \mathcal{D}(ST)$ and $y \in Tx$ have that $Tx \subset \mathcal{D}(S) + T(0) = \mathcal{D}(S)$ (since $x \in T^{-1}(\mathcal{D}(S))$). Therefore $I_{d(S)}Tx = Tx \cap \mathcal{D}(S) = Tx$, and

$$\|ST\| \leq \|S\| \|T|_{\mathcal{D}(ST)}\|,$$

from which (2.11) follows. ☐

Proposition 2.3.7. *Suppose that* $T \in LR(X,Y)$, *and* $S \in LR(Y,Z)$ *is continuous with* $\mathcal{D}(S) \supset \overline{T(0)}$. *So*

$$Q_{ST}ST = Q_{ST}SQ_T^{-1}Q_TT.$$

2.4 Linear Selections

Definition 2.4.1. *A single valued linear operator A is called a* ***linear selection*** *(or* ***operator part*** *) an linear relation T if*

$$T = A + T - T. \tag{2.12}$$

If A is a selection of T then we have for all $x \in \mathcal{D}(T)$ *we have*

$$Tx = Ax + T(0). \tag{2.13}$$

It results from (2.13) *that* $R(T) = R(A) + T(0)$. *Hoever, this sum may not always be direct. The following result provides a method for constructing selections.*

Proposition 2.4.1. *If P is a single valued linear projection with domain $R(T)$ and kernel $T(0)$, then PT is a selection of T. Conversely, if A is a selection of T and $R(A) \cap T(0) = \{0\}$, then the single valued projection defined on $R(T)$ with the interval $R(A)$ and the kernel $T(0)$ satisfies $A = PT$.*

Proposition 2.4.2. *Let $T \in LR(X, Y)$.*

(a) *If T has a continuous selection A, then T is continuous and*

$$\|T\| \leq \|A\|.$$

(b) *If $T(0)$ is complete in $R(T)$, then T is continuous if and only if T has a continuous selection.*

2.5 Closed and Closable Linear Relations

Definition 2.5.1. *Let $T \in LR(X, Y)$. We call* ***closure*** *of T the relation defined by its graph $G(\overline{T}) = \overline{G(T)}$.*

Definition 2.5.2. *Let $T \in LR(X, Y)$. We say that T is* ***closed*** *if $G(T)$ is closed.*
In other words $T = \overline{T}$.

Notation 3. *we denote that set of linear relations on X by: $CR(X)$.*

Proposition 2.5.1. *Let $T \in LR(X, Y)$. we have $\overline{Q_T T} = Q_T \overline{T}$.*

Proof. " $\subset$ " Let $(x, \overline{y}) \in G(\overline{Q_T T}) = \overline{G(Q_T T)}$, therfore $\exists\ (x_n, \overline{y_n}) \in G(Q_T T) : (x_n, \overline{y_n}) \underset{n \to +\infty}{\to} (x, \overline{y})$. So,

$$\exists\ z_n : (x_n, z_n) \in G(T), (z_n, y_n) \in G(Q_T).$$

We have, $x_n \underset{n \to +\infty}{\to} x$ et $\overline{y_n} \underset{n \to +\infty}{\to} \overline{y}$, so

$$Q_T \overline{y_n} \underset{n \to +\infty}{\to} Q_T y.$$

We have also, $Q_T z_n = \overline{y_n} \underset{n \to +\infty}{\to} \overline{y}$. Therefore,

$$(x_n, Q_T z_n) \in G(Q_T T) \underset{n \to +\infty}{\to} (x, \overline{y}).$$

From where,

$$(x,\overline{y}) \in G(Q_T\overline{T}).$$

So, $\exists\ z : (x,z) \in G(\overline{T}) = \overline{G(T)}, (z,\overline{y}) \in G(Q_T)$. Therefore,

$$Q_T z_n \underset{n\to+\infty}{\to} \overline{y} = Q_T\overline{y}.$$

From where,

$$Q_T(z_n - \overline{y}) \underset{n\to+\infty}{\to} 0$$

and consequently, $d(z_n - \overline{y}, T(0)) \underset{n\to+\infty}{\to} 0$. So, $\exists\ \alpha_n \in T(0) : z_n - \overline{y} - \alpha_n \underset{n\to+\infty}{\to} 0$. Therefore,

$$z_n - \alpha_n \underset{n\to+\infty}{\to} y.$$

As a result, $(x_n, z_n - \alpha_n) \in G(T)$. We stretch n towards infinity we get, $(x,y) \in \overline{G(T)} = G(\overline{T})$. Therefore, we have $(x,y) \in G(\overline{T})$, $(y,\overline{y}) \in G(Q_T)$, so

$$(x,\overline{y}) \in G(Q_T\overline{T}).$$

" $\supset$ " Let $(x,\overline{y}) \in G(Q_T\overline{T})$, so

$$\exists\ z : (x,z) \in G(\overline{T}) = \overline{G(T)}, (z,\overline{y}) \in G(Q_T).$$

So,

$$\exists\ (x_n, z_n) \in G(T) : (x_n, z_n) \underset{n\to+\infty}{\to} (x,z).$$

Therefore, $(x_n, Q_T z_n) \in G(Q_T T)$ and $(x_n, Q_T z_n) \underset{n\to+\infty}{\to} (x, Q_T z) = (x, \overline{y})$. From where, $(x,\overline{y}) \in \overline{G(Q_T T)} = G(\overline{Q_T T})$. □

Proposition 2.5.2. *The following properties are equivalent:*

(i) *T is closed.*

(ii) *Q_T closed and $T(0)$ is closed.*

Proof. **"(i)⇒(ii)"** On the one hand we have if T is closed, so $G(T)$ is closed and we have $(\{0\} \times Y)$ is closed. Therefore, $\{0\} \times T(0) = G(T) \cap (\{0\} \times Y)$ is closed. from where, $T(0)$ is closed. On the other hand $\overline{Q_T T} = Q_T\overline{T} = Q_T T$ (because T is closed). Therefore, $Q_T T$ is closed.

"(ii)⇒(i)" To show that T is closed it suffices to show that $G(T)$ is closed. Let

$$(x_n, y_n) \in G(T) : (x_n, y_n) \underset{n\to+\infty}{\to} (x, y),$$

therefore we have $y_n \in Tx_n$, from where

$$Q_T y_n \in Q_T T x_n.$$

Therefore, $(x_n, Q_T y_n) \in G(Q_T T)$ and we have

$$(x_n, Q_T y_n) \underset{n\to+\infty}{\to} (x, Q_T y).$$

But $Q_T T$ est closed, therefore $(x, Q_T y) \in G(Q_T T)$. So,

$$\exists \;\; z \;: (x, z) \in G(T), \;(z, Q_T y) \in G(Q_T).$$

Therefore, $Q_T z = Q_T y$ which gives that $-z + y \in N(Q_T) = \overline{T(0)} = T(0)$ (because $T(0)$ is closed). So,

$$y \in z + T(0) \in Tx + T(0) = Tx \text{ (because } z \in Tx).$$

Therefore, $(x, y) \in G(T)$.

□

Proposition 2.5.3. *Let T be a closed linear relation and $F \subset Y$ of finite dimension. So, $Q_F T$ is closed.*

Definition 2.5.3. *We say that a linear relation T is closable if $\overline{T}$ is an extension of T, i.e.,*

$$Tx = \overline{T}x \;\; \forall \; x \in \mathcal{D}(T).$$

2.6 Adjoint of a Linear Relation

Definition 2.6.1. *Let X, Y be two Banach spaces, $D \subset X \times Y$. We denote by X' : the topological dual space of X, and $D^{\perp}$ the orthogonal of D defined by,*

$$D^{\perp} = \{(a, b) \in X' \times Y' : ax + by = 0 \;\; \forall \; (x, y) \in D\}.$$

Definition 2.6.2. *Let $T \in LR(X,Y)$. We callthe* ***adjoint*** *or* ***conjugate*** *of T the linear relation $T^{'} \in LR(Y^{'}, X^{'})$ defined by:*

$$G(T') := G(-T^{-1})^{\perp} \subset Y' \times X'.$$

or

$$[(y,x),(y',x')] := [x,x'] + [y,y'] = x'x + y'y.$$

Remark 13. *We note tha the terms adjoint and conjugate are used interchangeably replaceable everywhere.*
If $(y',x') \in G(T')$ then $y'y = x'x$ for all $y \in Tx,\ x \in \mathcal{D}(T)$, i.e $x' \in T'y' \Leftrightarrow x'x = y'Tx$ for all $x \in \mathcal{D}(T)$, i.e,

$$x'|_{D(T)} = y'T.$$

If T is densely defined, then $y'T$, which is single valued,has a unique extension of X, making T' single valued. Thus, we can make the following assertions.

Proposition 2.6.1. *$T' \in LR(Y',X')$ is a closed relation with*

$$\mathcal{D}(T') = \{y' \in Y' \mid y'T \text{ is continuous and single valued}\}$$

and $T'y'x = y'Tx \in \mathbb{K}$ for $x \in \mathcal{D}(T)$ and $y' \in \mathcal{D}(T')$.

Proposition 2.6.2. *Let $T \in LR(X,Y)$. So*

(a) $(\overline{T})' = T'$.

(b) $(T')^{-1} = (T^{-1})'$.

(c) $(\lambda T)' = \lambda T'$.

Proof. It suffices to verify (c), let $\lambda \in \mathbb{K}, \lambda \neq 0$. So

$$\begin{aligned} G((\lambda T)') &= \{(y',x') | y'y = x'x \text{ pour } (x,y) \in G(\lambda T)\} \\ &= \{(y',\lambda x') | y'(\lambda y) = (\lambda x')x \text{ pour } (x,y) \in G(T)\} \\ &= G(\lambda T'). \end{aligned}$$

□

Proposition 2.6.3. *[5] Let $T \in LR(X, Y)$. So*

(a) $N(T') = R(T)^{\perp}$.

(b) $T'(0) = \mathcal{D}(T)^{\perp}$.

(c) $N(\overline{T}) = R(T')^{\top}$.

(d) $\overline{T}(0) = \mathcal{D}(T')^{\top}$.

Proposition 2.6.4. *Let $S, T \in LR(X, Y)$. So*

(a) $G(S' + T') \subset G((S+T)')$.

(b) *$(S+T)'$ is an extension of $S' + T'$ if and only if $(\mathcal{D}(S) \cap \mathcal{D}(T))^{\perp} = \mathcal{D}(S)^{\perp} + \mathcal{D}(T)^{\perp}$.*

(c) *If $\mathcal{D}(T) \subset \mathcal{D}(S)$ and S is continuous, so $s' + T' = (S+T)'$.*

Proposition 2.6.5. *Let $T \in LR(X, Y)$ and $S \in LR(Y, Z)$. So*

(a) $G(T'S') \subset G((ST)')$.

(b) *If*

$$\begin{aligned} &(1) \quad R(T') = X' \quad \textit{and} \quad \mathcal{D}(S) \subset R(T) \\ \textit{or} \quad &(2) \quad \mathcal{D}(S') = Z' \quad \textit{and} \quad R(T) \subset \mathcal{D}(S) \end{aligned}$$

so $(ST)' = T'S'$.

Notation 4. *Let E be a subspace of a norm linear space X. Let we noted J_E^X the* ***natural injection*** *map from E in X, i.e., for $x \in E$, $J_E^X x = x \in X$.*

Proposition 2.6.6. *Let E be subspace of X. So*

(a) $(J_E^X)' = Q_{E^{\perp}}^{X'}$.

(b) *If E is closed, so $(Q_E^X)' = J_{E^{\perp}}^{X'}$.*

Proof. **(a)** In applying the Proposition (1.6.3), $Q_{E^{\perp}}^{X'} : X' \to E'$ with

$$(Q_{E^{\perp}}^{X'} x')(e) = x'e \tag{2.14}$$

for $x' \in X'$ and $e \in E$.

Likewise $(J_E^X)' : X' \to E'$ and

$$((J_E^X)'x')(e) = x'(J_E^X)e = x'e \tag{2.15}$$

for $x' \in X'$ and $e \in E$. Equality follows by combining (2.14) and (2.15).

(b) Applying proposition (1.6.3), $(Q_E^X)' : E^{\perp} \to X'$ with

$$((Q_E^X)'e')(x) = e'(Q_E^X)x = e'x \tag{2.16}$$

for $x \in X$ and $e' \in E^{\perp}$.

Similarly $J_{E^{\perp}}^{X'} : E^{\perp} \to X'$ with

$$(J_{E^{\perp}}^{X'}e')(x) = e'x \tag{2.17}$$

pour $x \in X$ e $e' \in E^{\perp}$. Equality follows by combining (2.16) and (2.17).

$\square$

Proposition 2.6.7. *Let $T \in LR(X,Y)$. So*

(a) $(Q_T T)' = T' J_{T(0)^{\perp}}^{Y'}$.

(b) $(T J_{D(T)})' = Q_{T'} T'$.

(c) $(Q_T T J_{D(T)})' = Q_{T'} T' J_{T(0)^{\perp}}$.

Proof. These equalities follow from the direct of Propositions (2.6.5) and (2.6.6). $\square$

Corollary 2.6.1. *Let $T \in LR(X,Y)$, so $\mathcal{D}(T') = \mathcal{D}((QT)')$. Moreover, $T'y' = (QT)'y'$ for $y \in \mathcal{D}(T')$.*

Proposition 2.6.8. *Let $T \in LR(X,Y)$. So*

$$\|T'\| \leq \|T\|.$$

Proposition 2.6.9. *Let $T \in LR(X,Y)$. So*

$$\gamma(T') \geq \gamma(T).$$

Proof. This follows from Proposition (2.6.8) combined with Proposition (2.3.2). ☐

Proposition 2.6.10. *Let $T \in LR(X, Y)$. So*

(a) *If T is continuous, so $\mathcal{D}(T') = T(0)^{\perp}$.*

(b) *If T is open, so $R(T') = N(T)^{\perp}$.*

(c) *If T est continuous, so $\|T'\| = \|T\|$.*

(d) *If T is open, so $\gamma(T') = \gamma(T) > 0$.*

Proof. It suffices to show that (a) and (c) hold.

(a) Suppose T is continuous. Then by Proposition (2.6.8), $(QTJ)'$ is continuous, and by Proposition (2.6.1), its domain is the entire space *i.e.*, $T(0)^{\perp}$. Thus,by Proposition (2.6.7), the desired equality is true.

(c)

$$\begin{aligned}
\|T'\| &= \sup_{y' \in B_{\mathcal{D}(T')}} \|(QTJ)'y'\| \\
&= \sup_{y' \in B_{\mathcal{D}(T')}} \sup_{x \in B_{\mathcal{D}(T)}} \|y'(QTJ)x\| \\
&= \sup_{y' \in B_{T(0)^{\perp}}} \sup_{x \in B_{\mathcal{D}(T)}} \|y'(QTJ)x\| \\
&= \sup_{x \in B_{\mathcal{D}(T)}} \|(QTJ)x\| \\
&= \|QTJ\| = \|T\|.
\end{aligned}$$

☐

Chapter 3

Lipschitz multi-valued mapping between Banach spaces

3.1 Hausdroff's distance

If (X, d) is metric space, then

1. $CB\left(X\right) = \left\{C \text{ is a nonempty closed and bounded sub set of } X\right\}.$

2. $2^{X} = \left\{C \text{ is a nonempty compact subset of } X\right\}.$

3. $N\left(\epsilon, C\right) = \left\{x \in X/d(x, c) < \epsilon \text{ for some } c \in C \text{ and for all } \epsilon > 0 \text{ and } C \in CB\left(X\right)\right\},$

4. $H\left(A, B\right) = \inf\left\{\epsilon/A \subset N\left(\epsilon, B\right) \text{ and } B \subset N\left(\epsilon, A\right)\right\} \text{ if } A, B \in CB\left(X\right).$

The function H is a metric for $CB(X)$ called the Hausdorff metric. We note that the metric H actually depends on the metric for X and that two equivalent metrics for X may not generate equivalent Hausdorff metrics for $CB(X)$ (see[7, p.131]). We shall not notate this dependency except where confusion may arise. It will be understood, unless otherwise stated, that the symbol H stands for the Hausdorff metric obtained from a fixed

Definition 3.1.1. *Let X, Y be two metric spaces. A multivalued map $T : (X, d_X) \longrightarrow CB\left(Y\right)$ is called multivalued Lipschitz mapping (abbreviated m.v.l.m.) if there is a positive* *constant C such that*

$$\forall x, y \in X, \quad H(T(x), T(y)) \leq C d_X(x, y). \text{ For all } x, y \in X. \tag{3.1}$$

If $C = 1$, the map is called nonexpansive (and contraction if $C < 1$). For a multivalued Lipschitz map T, we define its Lipschitz constant by

$$\mathrm{Lip}\,(T) := \sup_{x \neq y} \frac{H(T(x), T(y))}{d_X(x, y)} = \inf \{C : C \text{ verifying } 3.1\}.$$

Note that, every m.v.l.m. is continuous.

Definition 3.1.2. *Let be A, B tow subsets of a normed vector space X. We call excess of A and B, the continuity*

$$e\,(A, B) = \sup_{a \in A} d\,(a, B).$$

With $e\,(\varnothing, B) = 0$ if $B \neq \varnothing$ and $e\,(A, \varnothing) = +\infty$ for all A.
Equivalently, we have

$$e\,(A, B) = \inf \{\epsilon \geq 0 / A \subset B + \epsilon B_X\},$$

such that B_X denotes the unit ball of X. It is general $e\,(A, B) \neq e\,(B, A)$.

3.2 Multi-valued Lipschitz relations

Definition 3.2.1. *Multi-Valued Lipschitz relations are a generalization of Lipschitz continuity where the function is allowed to take multiple values at each point in its domain. In other words, Multi-Valued Lipschitz relations are a generalization of Lipschitz continuity that allows for more complex functions.*

Example 3.2.1. *Let X be a metric space and let $T : X \to 2^X$ be a multivalued map. We say that T is a multivalued Lipschitz relation if there exists a constant $K > 0$ such that for all $x, y \in X$,*

$d(T(x), T(y)) \leq K d(x, y)$,

where d is the metric on X.

Definition 3.2.2. *(**Multi-Valued Lipschitz maps**) We say that a multi-valued map $T : X \longrightarrow 2^Y$ is Lipschitz relative to a non-empty subset D of dom T,if it is closed-valued on D and if there exists a constant $K \geqslant 0$ such that*

$$H(T(x), T(y)) \leqslant Kd(x, y), \forall x, y \in D \tag{3.2}$$

Equivalently,

$$T(x) \subset T(y) + Kd(x, y)IB_Y, \forall x, y \in D \tag{3.3}$$

Example 3.2.2. *Here are some examples of Multi-Valued Lipschitz relations:*

- *The set-valued function defined by $T(x) = \{0\}$ if $x < 0$ and $T(x) = \{0, 1\}$ if $x \geq 0$ is a Multi-Valued Lipschitz relation with Lipschitz constant 1.*
- *The set-valued function defined by $T(x) = \{0\}$ if $x < 0$ and $T(x) = \{1\}$ if $x \geq 0$ is a Multi-Valued Lipschitz relation with Lipschitz constant 0.*
- *The set-valued function defined by $T(x) = \{x\}$ is a Multi-Valued Lipschitz relation with Lipschitz constant 1.*

Definition 3.2.3. *Let (X, d) be a complete metric space and T be a multi-valued map on X such that $T(x)$ is a nonempty closed bounded subset of X for any $x \in X$. If there exists $C \in (0, 1)$ such that*

$$H(T(x), T(y)) \leqslant Cd(x, y), \quad \forall x, y \in X,$$

then T has a fixed point in X.

3.3 Fixed point for multi-valued contraction mapping

In this section we studied Fixed point for multi-valued contraction mapping. in this section can be found in Nadler [16, 17]

Definition 3.3.1. *A multi-valued mapping $T : X \longrightarrow N(X)$ is said to be upper semi-continuous, if for any $x \in X$ and a neighborhood $V \supset T(x)$, there is a neighborhood U of x such that for any $y \in U$, we have $T(y) \subset V$.*

Definition 3.3.2. *An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \longrightarrow N(X)$, if $x \in T(x)$.*

Theorem 3.3.1. ***(Fixed point theorems)*** *[17] Let (X, d) be a generalized complete metric space and let $x \in X$. If $T : X \to CL(X)$ is an (ε, λ)-uniformly locally contractive multi-valued mapping, then the following alternative holds: either*

(1.1) *$d(x_{i+1}, x_i) \geq \varepsilon$ for each $i = 1, 2, \ldots$ for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of T at x_0, or*

(1.2) *there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of T at x_0 such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of T.*

Proof. Suppose **(1.1)** does not hold. Then there is a choice (*) of $x_l \in T(x_0), x_2 \in T(x_1), \ldots,$ and $x_N \in T(x_{N-1})$ such that

$$d(x_{N-1}, x_N) < \varepsilon$$

for some fixed integer $N \geqq 1$. This implies

$$H(T(x_{N-l}), T(x_N)) \leqq \lambda \cdot d(x_{N-1}, x_N) < \lambda \cdot \varepsilon.$$

Therefore, since $x_N \in T(x_{N-1})$, there exists $x_{N+1} \in T(x_N)$ such that

$$d(x_N, x_{N+1}) < \lambda \cdot \varepsilon. \text{ Now } H(T(x_N), T(x_{N+1})) \leqq \lambda \cdot d(x_N, x_{N+1}) < \lambda^2 \cdot \varepsilon$$

and hence, since $x_{N+1} \in F(x_N)$, there exists $x_{N+2} \in T(x_{N+1})$ such that

$$d(x_{N+1}, x_{N+2}) < \lambda^2 \cdot \varepsilon.$$

Continuing in this fashion we produce a sequence $\{x_i\}_{i=1}^{\infty}$ of points of X such that $x_{N+i+1} \in T(x_{N+i})$ and

$$d(x_{N+i}, x_{N+i+1}) < \lambda^{i+1} \cdot \varepsilon \text{ for all } i \geqq 1.$$

It follows that the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence which, by the completeness of (X, d), converges to some point $p \in X$. Hence, the sequence $\{T(x_i)\}_{i=1}^{\infty}$ converges to $T(p)$ and, since $x_{i+1} \in T(x_i)$ for all i and $T(p)$ is closed, $p \in T(p)$. This proves T has a fixed point x^*. Furthermore, the sequence $\{x_i\}_{i=1}^{\infty}$ satisfies the conditions in **(1.2)** of the alternative.

Example 3.3.1. *Let T is function*

$$\begin{aligned} T : X &\to 2^X \\ x &\to [-|x|, |x|] \end{aligned}$$

$T(1) = [-1, 1]$

$T(2) = [-2, 2]$

$T(-2) = [-2, 2]$

$T(0) = 0$

T is multi-valued Lipschitz relation

Theorem 3.3.2. *Let (X, d) be a complete metric space, $T : X \longrightarrow C(X)$ be a multi-valued mapping. If there exists a constant $C \in (0, 1)$ such that for any $x \in X$ there is $y \in I_b^x$ satisfying*

$$d(y, T(y)) \leqslant C d(x, y),$$

then T has a fixed point in X provided $C < b$ and f is lower semi-continuous.

Proof. For the proof see □

Remark 14. *For a positive constant $b(b \in (0, 1))$, define the set $I_b^x \subset X$ as*

$$I_b^x = \{y \in T(x) \mid b d(x, y) \leqslant d(x, T(x))\}.$$

The following theorem is the main result of this section.

Example 3.3.2. *The following example shows that Theorem (3.3.2) is an extension of **Nadler's** fixed point theorem.*
Let $X = \{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\} \bigcup \{0, 1\}, d(x, y) = |x - y|$, for $x, y \in X$; then X is complete metric space. Define mapping $T : X \longrightarrow C(X)$ as

$$T(x) = \begin{cases} \{\frac{1}{2^{n+1}}, 1\}, & x = \frac{1}{2^n}, n = 0, 1, 2, \ldots, \\ \{0, \frac{1}{2}\}, & x = 0 \end{cases}$$

*. Obviously, T is not a contractive mapping in **Nadler's** meaning, in fact,*

$$H\left(T\left(\frac{1}{2^n}\right), T(0)\right) = \frac{1}{2} \geqslant \frac{1}{2^n} = \left|\frac{1}{2^n} - 0\right| = d\left(\frac{1}{2^n}, 0\right), \quad n = 1, 2, \ldots$$

On the other hand, it is easy to compute

$$f(x) = d(x, T(x)) = \begin{cases} \dfrac{1}{2^{n+1}}, & x = \dfrac{1}{2^n},\ n = 1, 2, \ldots, \\ 0, & x = 0, 1, \end{cases}$$

. hence f is continuous. Furthermore, there exists $y \in I^x_{0,7}$ for any $x \in X$ such that

$$d(y, T(y)) \leqslant \frac{1}{2} d(x, y).$$

Then the existence of a fixed point follows from Theorem (3.3.2) *.*
Hence Theorem (3.3.2) *is an extension of* ***Nadler's theorem****.*

□

Corollary 3.3.1. *Let (X, d) be a generalized complete metric space and let $x_0 \in X$. If $T : X \longrightarrow CL(X)$ is a m.v.c.m., then the following alternative holds: either*

1. *For each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of T at $x_0, d(x_{i-1}, x_i) = \infty$ for each $i = 1, 2, \ldots,$ or*
2. *There exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of T at x_0 such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of T.*

Proof. For the proof see [17]. □

Corollary 3.3.2. *[17] Let (X, d) be a complete ε-chainable generalized metric space and let $x_0 \in X$. If $T : X \longrightarrow CL(X)$ is an (ε, λ)-uniformly locally contractive multi-valued mapping, then the following alternative holds: either*

(1) *for each iterative sequence $\{x_i\}_{i=1}^{\infty}$ of T at $x_0, d(x_{i-1}, x_i) = \infty$ for each $i = 1, 2, \ldots,$ or*

(2) *there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ of T at x_0 such that $\{x_i\}_{i=1}^{\infty}$ converges to a fixed point of T.*

Proof. For the proof see [6]. □

Corollary 3.3.3. *[6] Let* (X, d) *be a complete metric space and let* $x_0 \in X$. *If* $T : X \longrightarrow CL(X)$ *is a m.v.c.m., then there exists an iterative sequence* $\{x_i\}_{i=1}^{\infty}$ *of* T *at* x_0 *such that* $\{x_i\}_{i=1}^{\infty}$ *converges to a fixed point of* T*.The following corollary is an immediate consequence of Corollary* (3.3.2) *above.It is asubstantial extension of theorem 6 of Nadler [17] which states that an* (ε, λ)*-uniformly locally contractive multi-valued mapping* T *on a complete* ε*-chainable metric space has a fixed point if each point image* $T(x)$ *is nonempty and compact.*

Theorem 3.3.3. *Let* (X, d) *be a complete* ε*-chainable (respectively, well-chained) metric space, let* A *be a nonempty subset of* X*, and let* $f : A \longrightarrow X$ *be an* (ε, λ)*-uniformly locally expansive (continuous) mapping of onto* X *. If* T^{-1} *is closed in* X *for each* $x \in X$ *and* $T^{-1} : X \longrightarrow CL(X)$ *is* ε*-nonexpansive (respectively, uniformly* ε*-continuous), then* T *has a fixed point.To see that some metric type of restriction even stronger than uniform continuity must be placed on* T^{-1} *, the reader is referred to example 3 of [17]. Theorem 2 above makes theorem 8 of [17] superfluous (see the remark at the end of section 3 of [17]).*

Conclusion

In this paper we have learned a lot of notions of mathematics namely the multivalued linear relations in Banach spaces, in the second chapter we have studied the theory of multivalued linear relations succinctly and give a lot of acknowledgements and the proprieties about this theory in the last chapter we have introducing the Lipschitz relations which is more complicated and finding the solutions of the Lipschitz inequality as the standard one in single mappings, by The Hausdorff metric, also fixed point theorem in multivalued case which have a lot of applications and we give also some examples.

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ملخص:

في هذه المذكرة، قمنا بدراسة نظريةالدوال الليبشتزيةالمتعددة القيم على الفضاءات الناظمية وكذا فضاءات باناخ. وأيضا تطرقناإلى نظرية النقطة الصامدة لهذه الدوال مع إعطاء أمثلة.

الكلمات المفتاحية: (العلاقات الخطية، مسافة هاوسدروف، علاقات ليبشز متعددة القيم، نظرية النقطة الصامدة)

Abstract:

In this note, we have studied the theory of Multivalued Lipschitz relations on normed spaces and Banach spaces. And also we have touched on the fixed point theory of these functions with giving some examples.

Keywords: (Linear Relations, Hausdroff's distance, Multi-valued Lipschitz relations, Fixed point theorem.)

Resumé:

Dans cette note, nous avons étudié la théorie des Relations Lipschitzienne Multivoques sur les espaces normés et les espaces de Banach. Et nous avons également abordé la théorie du point fixe de ces fonctions en donnant quelques exemples.

Mots-clés: (Les relations linéaires, la distance de Hausdorff, les relations Lipschitzienne Multivoques, le théorème du point fixe.)