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تحت عنوان : ELECTRONICS

وفق البرنامج المقترح لطلبة السنة الثانية هندسة ميكانيكية

.

رئيس اللجنة العلمية للقسم
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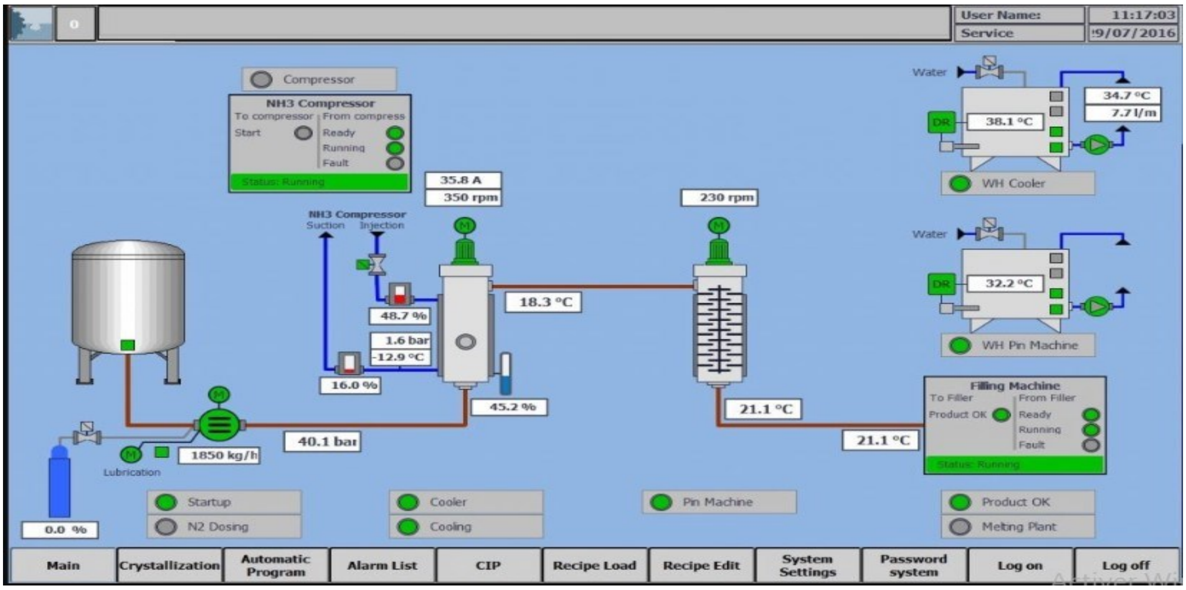
DEPARTMENT OF MECHANICAL ENGINEERING

ELECTRONICS

2nd year Mechanical Engineering



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FOREWORD

This course handout for the Electronics module is intended for second-year Engineering Cycle students in the Mechanical Engineering program and fully complies with the official academic curriculum. It has been prepared as a structured and coherent teaching material designed to support students in acquiring the fundamental concepts related to the study of control and servo systems.

The main objective of this module is to enable students to master the theoretical and methodological foundations of control and regulation systems. This technical discipline occupies a central position in modern engineering, as it is essential for the analysis, modeling, stability assessment, and performance improvement of dynamic systems. Through this course, students will gain a solid understanding of the principles underlying industrial control systems and many contemporary technological applications.

More specifically, this handout aims to:

- introduce students to the main control and command components, such as sensors, actuators, and controllers;
- explain the fundamental principles of control and regulation, in both open-loop and closed-loop systems;
- present methods for the analysis of control systems, based on mathematical modeling tools, particularly the Laplace transform and transfer functions;
- link theoretical concepts to practical engineering applications drawn from mechanical, electrical, and thermal systems.

This course material has been developed following a progressive and pedagogical approach, emphasizing clarity, logical organization, and scientific rigor. Special attention has been given to highlighting the essential concepts and key ideas of the module, presented in a simplified manner that is well suited to the academic level of second-year engineering students.

Consequently, this document serves as a lecture reference and a fundamental learning resource for the Electronics and Control Systems module, supporting students in building a strong theoretical foundation necessary for advanced engineering studies and future professional practice.

CHAPTER 01 : TERMINOLOGY OF CONTROL SYSTEMS

1 INTRODUCTION

Today, automatic control systems play a major role in almost every area of daily life. They are present in household appliances (washing machines, refrigerators…), vending machines, elevators, transportation systems, robotics, aeronautics, econometrics, and many other fields.

In industry, automation has become essential. A well-known example is the introduction of robots in automobile assembly lines, where they replaced certain manual operations. Automated systems are also used in environments that are too dangerous for humans, such as nuclear facilities.

The term automatic control, often described as the science of automation, refers to systems capable of performing a set of operations without human intervention. In many situations, the purpose is to replace human activity for economic, safety, or precision reasons.

An automatic system reproduces or replaces human functions such as effort, observation, memory, and decision-making using technological elements (controllers, programmable logic devices, sensors, electronic circuits…). This allows:

- relieving humans from repetitive, exhausting, or dangerous tasks;
- improving production efficiency and product quality;
- increasing the accuracy and reliability of operations;
- enhancing the safety of people and equipment.

Automation aims to control various quantities coming from industrial processes. These quantities can be:

- **electrical**: voltage, current, power;
- **mechanical**: speed, force, torque, position;
- **hydraulic**: pressure, flow rate, liquid level;
- **thermal**: temperature, gradient;
- **other physical properties**: humidity, viscosity, etc.

Examples of measurable quantities that may need to be controlled include:

- the output voltage or current of a source,

- the rotational speed of a motor,
- the temperature of a room.

2. HISTORICAL BACKGROUND

Automatic control has existed since ancient times. To compensate for physical limitations, humans quickly sought ways to control the natural sources of energy available in their environment, such as wind and water. Early mechanisms—including hydraulic regulators, water clocks, and simple mechanical governors—can already be seen as primitive forms of automation.

Modern automatic control began to take shape in the early 20th century, driven by the foundational work of engineers and mathematicians such as **Harold Black**, **Nathaniel Nichols**, **Hendrik Bode**, and **Harry Nyquist**. Their contributions laid the theoretical basis for feedback, stability analysis, frequency response, and control system design.

Throughout the 20th century, both the theoretical framework and practical applications of control and regulation expanded dramatically. The rise of modern electronics and computer science enabled the design and implementation of increasingly sophisticated automated systems at much lower cost.

Today, automatic control permeates virtually every aspect of daily life. Automated systems can be found everywhere ranging from children's toys to advanced military aircraft highlighting the essential role of automation in contemporary technology and society.

3. GENERAL CONCEPTS ABOUT SYSTEMS

A *system* is defined as an isolated physical entity governed by the laws of physics and characterized by specific measurable quantities. In a single-variable system, we are primarily interested in the relationship between:

- an input variable, denoted $e(t)$, which represents an external action applied to the system, and
- an output variable, denoted $s(t)$, which characterizes the internal state or observable response of the system.

Applying the physical laws that govern the system allows us to establish a functional relationship between *e(t)* and *s(t)*. Any other external quantities that influence the system thereby modifying the relationship between input and output—are known as disturbances or parasitic inputs.

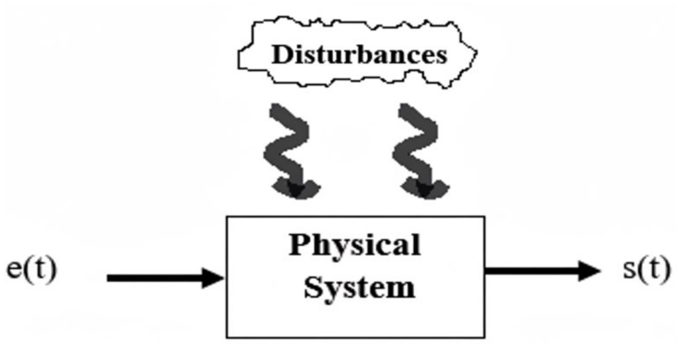


Figure 1-1 – Diagram of a System

The relationship between the input and the output is symbolically represented as in Figure 1-1, in which the physical system is modeled as a block (or dipole) that transforms the input into an output.

In practical terms, the external action *e(t)* corresponds to the application of a certain form of energy, characterized by two physical components depending on the domain:

- electrical signals: voltage and current
- mechanical signals: velocity and force
- pneumatic or hydraulic signals: pressure and flow rate
- thermal signals: temperature and heat flux

4. CONTROL SYSTEMS

A control system is an assembly of interconnected physical components arranged so that they can:

- control or regulate their own behavior, or
- control, direct, or regulate another system.

The idea of "control" is intuitive and can be illustrated with simple examples.

Examples:

- When a crank is connected to a shaft through a gear train, the position of the crank controls the position of the mechanical gripper.
- When the field current of a separately excited electric motor is adjusted using a switch, the motor's operating regime is controlled by the value chosen at the input.

In these examples, the term *control* indicates that there exists a precise functional relationship between:

- the input (crank angle, potentiometer setting), and
- the output (gripper position, motor operating state).

A control system therefore transforms an input command into a desired, predictable output behavior.

5. CONTINUOUS SYSTEMS

A physical system is said to be continuous when all its characteristic quantities are continuous in nature. This means that:

- the information representing these variables exists at every instant.
- the variables can take any value within a given range (not just discrete values).

The evolution of such variables over time forms a continuous-time signal, in the mathematical sense. By contrast, discrete systems operate with signals that are sampled, quantized, logical, or sequential, either in time or in amplitude.

Thus, continuous systems are defined in opposition to discrete-time and discrete-amplitude systems.

6. LINEAR SYSTEMS

A physical system is considered linear if the relationship between its input variables and its output variables can be described by a linear differential equation.

A linear system exhibits two fundamental properties:

❖ **Superposition :** The response to a sum of inputs equals the sum of the responses to each input individually.

- **Homogeneity (scaling) :** If the input is multiplied by a constant factor, the output is multiplied by the same factor.

Linear systems are essential in control theory because they allow the use of powerful mathematical tools such as Laplace transforms, transfer functions, and frequency analysis.

7- NATURE OF INPUT AND OUTPUT SIGNALS

In an automatic or controlled system, signals define how information circulates from the operator to the system and back. They can be classified according to their role**,** origin, and physical nature.

7-1. Input Signals

Input signals are all the information provided to the system. They include:

a) Setpoint (Reference Input)

- The desired value of the system output.
- Defined by the operator or an external system.
- Usually constant or slowly varying.

Example: Desired speed of a motor, target temperature.

b) Disturbances

- Unwanted signals that affect the system.
- Can be internal or external.
- Generally not controllable.

Example: Load variation, outside temperature change.

c) Control Signal

- Generated by the controller.
- Drives the actuator to act on the system.

Example: Voltage applied to a motor, PWM signal to a heater.

7-2. Output Signals

Output signals represent the actual behavior of the system.

a) Measured Output

- The real physical variable produced by the system.
- Measured by a sensor.

Example: Actual speed, real temperature.

b) Feedback Signal

- The processed form of the output used for comparison with the setpoint.
- Can be amplified, filtered, or converted.

Example: Voltage proportional to motor speed (tachometer).

8. PHYSICAL NATURE OF SIGNALS

Signals may also be classified based on how they are represented:

a) Analog Signals

- Continuous in time and amplitude.

Example: 0–10 V voltage signal, 4–20 mA current loop.

b) Digital Signals

- Discrete states (0/1 or coded levels).

Example: Logic signals, PWM, microcontroller data.

c) Continuous vs. Discrete Signals

- **Continuous:** Vary at all instants (e.g., raw sensor voltage).
- **Discrete:** Defined at specific sampling times (e.g., sampled data).

Simple Examples

Heating System (Thermostat)

- **Input:** Desired temperature.
- **Output:** Actual room temperature.

- **Disturbance:** Window opening, external weather.
- **Control Signal:** ON/OFF heater command.

Speed-Controlled Motor

- **Input:** Desired speed.
- **Output:** Actual measured speed.
- **Control Signal:** Motor drive voltage or current.

9. OPEN-LOOP AND CLOSED-LOOP SYSTEMS

Regardless of the nature of the system to be controlled (electrical, mechanical, hydraulic, etc.), control architectures can generally be classified into two major families:

- Open-loop control structures.
- Feedback-based control structures, also called closed-loop control systems.

These two approaches differ fundamentally in how they treat the relationship between the input (command) and the output (system response).

9.1 Open-Loop Control

In an open-loop configuration, the control action is applied directly to the system without any form of monitoring or measurement of the output. In other words,the control signal is completely independent of the actual output.

There is no feedback path, which means that the system does not verify whether the desired objective has been achieved.

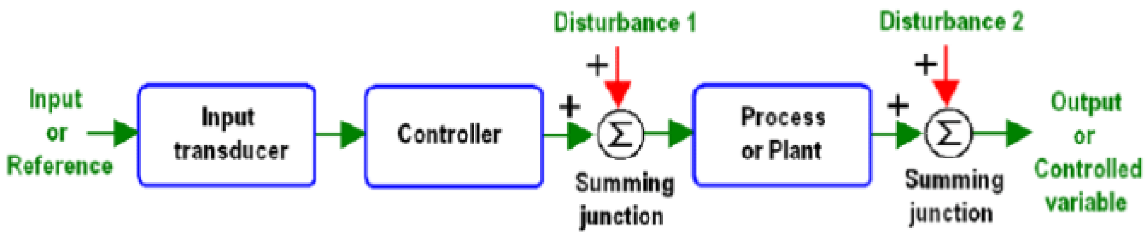


Figure 1-2 – Open-loop control

Examples of Open-Loop Systems

Washing machine:A classical example is a washing machine operating based on predefined washing cycles. The machine does not measure the cleanliness of the laundry; it simply follows a fixed sequence of operations (washing, rinsing, spinning) regardless of the actual result.

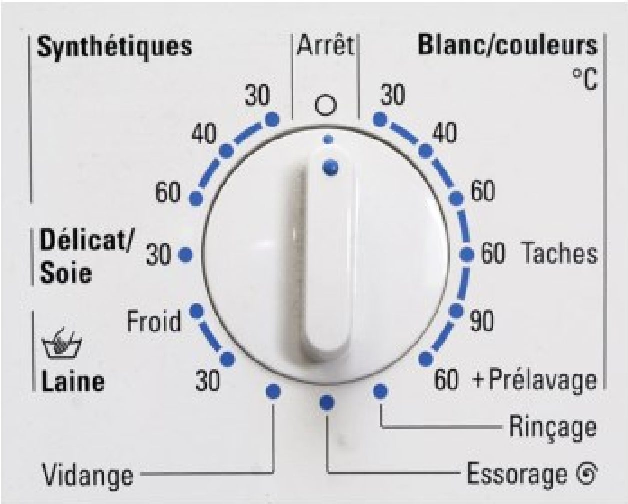


Figure 1-3 – External view of a washing cycle selector knob

Other examples include:

- A light switch controlling the flow of electric current
- Heating a room with an electric heater controlled by a rheostat (no temperature sensor)
- Position control of a missile launcher without measurement feedback
- Water filling based on a timer rather than a level sensor

CHARACTERISTICS AND LIMITATIONS OF OPEN-LOOP CONTROL

Using open-loop control requires prior knowledge of the system's behavior and how it responds to applied commands. This includes understanding the effect of disturbances, which is rarely feasible in real-world environments.

Because the output is not measured:

- The open-loop system cannot correct deviations caused by disturbances.
- The control action cannot be adjusted if conditions change.
- The desired output cannot be guaranteed with high precision.

Therefore, open-loop control is suitable only when:

- The system is perfectly known and well-modeled**,**
- Disturbances are negligible or predictable,
- Measuring the output is either impossible or too costly.

9.2 Closed-Loop Control

To overcome the limitations of open-loop control and reduce sensitivity to disturbances and parameter variations, it becomes essential to introduce a feedback mechanism**.**

In a closed-loop control system:

- The actual output of the system is continuously measured**.**
- This measurement is compared to the desired output (setpoint**)**.
- The difference between them, called the error, is used to generate the appropriate control signal.

This allows the system to self-correct and ensures that the output closely follows the setpoint despite disturbances or uncertainties.

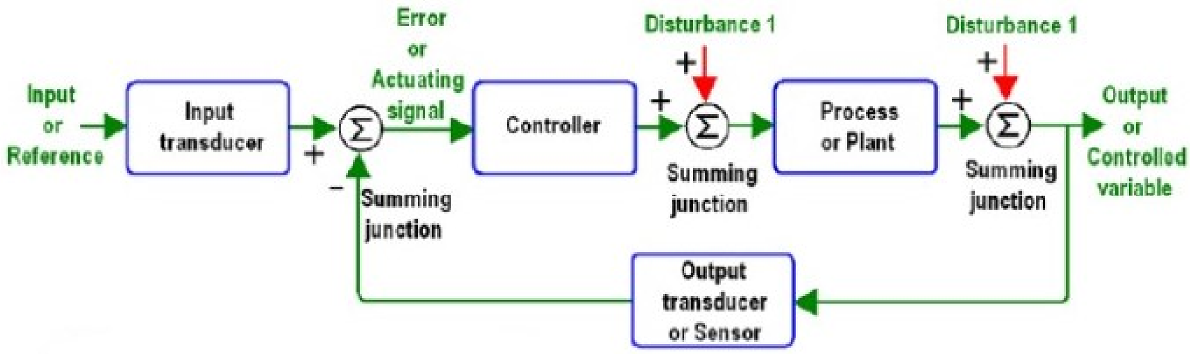


Figure 1-4 – Functional diagram of closed-loop control

ADVANTAGES OF CLOSED-LOOP CONTROL

Closed-loop control is especially advantageous when:

- The system is not perfectly known or too complex to model accurately,
- Unpredictable or unmeasurable disturbances affect the system,

- The system parameters vary over time (e.g., wear, friction, temperature changes),
- High precision and robustness are required.

The feedback mechanism continuously adapts the control action, resulting in:

- Greater accuracy
- Improved stability
- Better disturbance rejection
- Enhanced reliability in dynamic environments

10. COMPONENTS OF A FUNCTIONAL BLOCK DIAGRAM

- **Setpoint**

The setpoint is the target value—the quantified objective imposed on the control system by the operator or an external system. It represents the intention or desired outcome and serves as the absolute reference against which the system's performance is continuously compared. It is not a measurement but a static instruction (which only changes when explicitly commanded) that defines the desired steady-state of the process output.

- **Comparator**

The comparator is a purely functional summation point, often implemented electronically or in software, whose unique and crucial role is to quantify the difference between the objective and the actual value. It continuously performs the arithmetic operation Setpoint − Measurement to generate the error signal. It makes no decisions; it simply provides the essential information that will trigger corrective action.

- **Error – ε**

The error is the dynamic signal produced by the comparator. It represents not only the magnitude of the difference between setpoint and measurement but also its direction (indicated by its positive or negative sign). It is the informational "fuel" of the controller: zero error means the system is balanced and requires no action, while a non-zero error calls for corrective action.

- **Régulateur (Controller)**

The controller is the strategic brain of the control loop. Its function is to receive the error signal and, based on its internal algorithm (such as a PID: Proportional–Integral–Derivative), compute and generate an intelligent command signal. It does not "see" the setpoint or the output directly; it only reacts to the error. Its goal is to manipulate the process to eliminate the error as efficiently as possible (quickly, without overshoot, and with stability).

- **Action Path**

The action path is the set of components that physically execute the controller's command to influence the process. It is the "muscle" of the system, converting an information signal into a concrete physical action. It includes the actuator—an energy converter (e.g., the motorized throttle body)—and the process itself (the car's engine and drivetrain), which reacts to the actuator's action.

- **Feedback Path**

The feedback path is what closes the loop, giving the system a form of "self-awareness." Its function is to measure the current output and communicate it to the comparator in a usable form. It includes the sensor (e.g., wheel speed sensor converting physical rotation into an electrical signal) and the measurement itself. The quality, accuracy, and speed of this path are essential, because the system cannot properly control what it cannot properly measure.

- **Disturbances**

Disturbances are all uncontrolled and unpredictable external forces that affect the process and tend to push the output away from the setpoint. They are the fundamental reason closed-loop control exists. A good control system not only reaches its setpoint but effectively rejects disturbances. For cruise control, typical disturbances include climbing a hill (which slows the car), headwind, or changes in road rolling resistance.

CHAPTER 02 : THE LAPLACE TRANSFORM

1. INTRODUCTION :

Controlled systems are generally represented in the form of a functional diagram, where the relationship between input and output time signals is governed by a differential equation, whose solution can be more or less complex. The mathematical function of the system is often difficult to express.

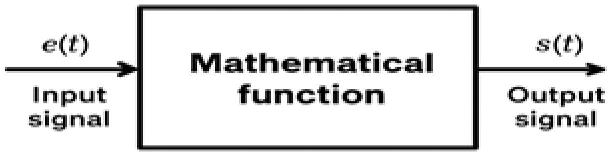


Figure 2-1. Input–Output Model of a Mathematical System

If an instrumented industrial process includes several elementary systems (for example: actuator, process, sensor…), each of them can be described by a differential equation. Connecting these elementary systems in series leads to a new differential equation that describes the complete system. However, a problem arises because differential equations cannot simply be multiplied together. A new mathematical approach becomes necessary, which consists of moving from the time-domain behavior of systems (characterized by functions defined over time t) to a symbolic domain where the variable is no longer time ttt but a complex symbolic variable called the Laplace operator, denoted s (s for the Anglo-Saxons). This process is called the Laplace transform.

The method for determining the output signal s(t) as a function of the input e(t) is then as shown in Figure 2-2.

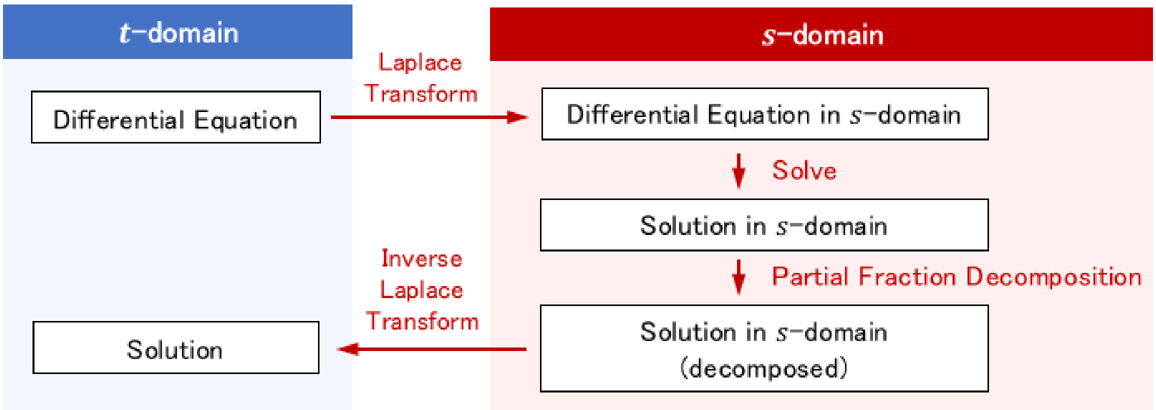


Figure 2-2. Solution Method Using the Laplace Transform

2. DEFINITION

The Laplace transform is an integral operation that converts a function of a real variable into a function of a complex variable. Through this transformation, a linear differential equation can be represented as an algebraic equation. It also allows particular functions (such as the Heaviside step function, the Dirac delta distribution, etc.) to be expressed in an elegant and convenient form. These capabilities make the Laplace transform highly useful and popular among engineers.

This transformation has given rise to the technique of *operational calculus* or *symbolic calculus*, which greatly simplifies the solution of linear differential equations that represent the systems we will study.

Let us consider a real function of a real variable $f(t)$ such that $f(t) = 0$ for t<0. The Laplace transform $L(f(t))$ is defined as the function ***F*** of the complex variable **s**:

$$F(s) = L(f(t))$$

The Laplace transform of a function $f(t)$ is defined as:

$$F(s) = L\big(f(t)\big) = \int_0^{\infty} f(t)\, e^{-st} dt$$

where:

- t: a real time variable
- s: à complex variable of the form $s = \sigma + jw$
- $f(t)$: the original function in the time domain
- $F(s)$: the transformed function in the Laplace domain

3- CONDITIONS FOR THE EXISTENCE OF THE LAPLACE TRANSFORM

For the Laplace transform of a function $f(t)$ to exist and be properly defined, the function must satisfy three fundamental conditions:

3-1. Piecewise Continuity

The function $f(t)$ must be piecewise continuous on every finite interval [0 ∞]. This means it may have a finite number of discontinuities, provided they are finite jumps or removable discontinuities, not infinite ones.

3-2. Exponential Order Condition

The function $f(t)$ must be of exponential order as t→+∞

This means there exist two positive real constants K and a such that:

$|f(t)| \leq Ke^{at}$ t→+∞

Below are some key advantages and benefits of using the Laplace transform in electrical and Mechanical engineering:

- **Simplified Mathematical Approach** Laplace transforms convert differential equations into algebraic equations, making it easier to analyze and solve complex systems.
- **Frequency domain representation** The transformed equation provides insight into the system's behavior in the frequency domain, enabling engineers to evaluate frequency response characteristics and stability.
- **Transfer function** determination Laplace transforms facilitate the calculation of transfer functions, which describe the relationship between input and output signals in a system. This information is crucial for designing filters and control systems.
- **Time-domain response** analysis by applying inverse Laplace transforms, electrical engineers can obtain the time-domain response of a system from its frequency-domain representation. This allows for understanding the system's transient behavior and response to input signals.
- **System stability analysis** Laplace transforms aid in determining the stability of systems by examining the poles and zeros of the transfer function. System stability is critical in electrical engineering, particularly in control systems design.

4. STANDARD INPUTS

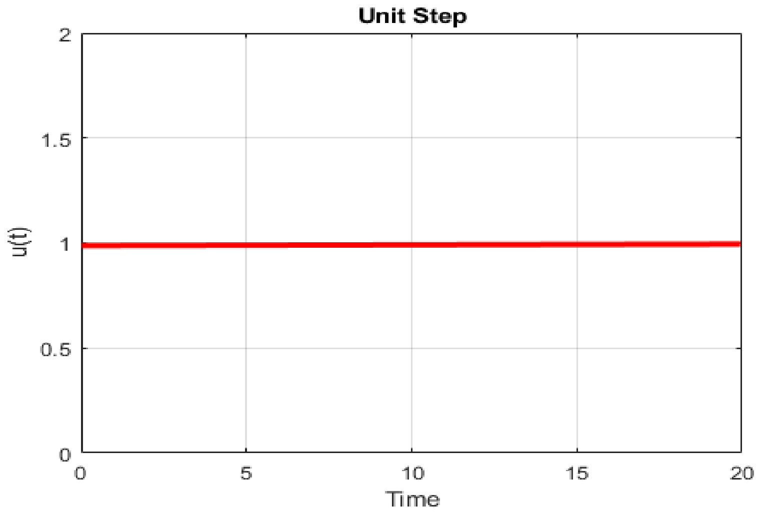
Certain standard input signals are used. They make it possible to define all the performance criteria of our control system. They are presented below.

4.1 Unit Step

It is a signal defined by: $\mathrm{u(t)} = \begin{cases} 1, & \mathrm{t} \geq 0 \\ 0, & \mathrm{x} < 0 \end{cases}$

The MATLAB program used to plot it is given below, along with its graph

```
t = [0:0.01:6*pi];
u = 1;
plot(t, u, 'b')
title('Unit Step')
xlabel('Time')
ylabel('u(t)')
grid on
```



We now calculate U(s) the Laplace transform of u(t).

$$U(s) = L\big(u(t)\big) = \int_0^{\infty} u(t)\, e^{-st} dt) = \int_0^{\infty} 1\, e^{-st} dt$$

$$U(s) = -\frac{1}{s}\left[e^{-st}{}_0^{\infty}\right]$$

$$U(s) = \frac{1}{s}$$

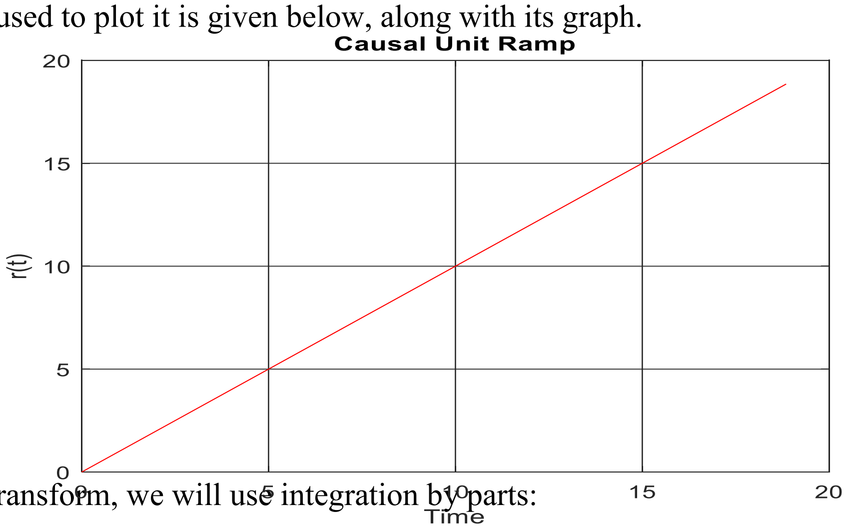
4.2 The causal ramp:

The MATLAB program used to plot it is given below, along with its graph. It is defined by:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & x < 0 \end{cases}$$

The MATLAB program used to plot it is given below, along with its graph.

```
t = [0:0.001:6*pi];
f = t;
plot(t, f, 'r')
title('Causal Unit Ramp')
xlabel('Time')
ylabel('r(t)')
grid on
```



To compute its Laplace transform, we will use integration by parts:

$$R(s) = L\big(r(t)\big) = \int_0^{\infty} r(t)\, e^{-st} dt) = \int_0^{\infty} t\, e^{-st} dt$$

$$R(s) = \left[-t\frac{1}{s}e^{-st}\right]_0^{\infty} - \int_0^{\infty} -\frac{1}{s}e^{-st}dt$$

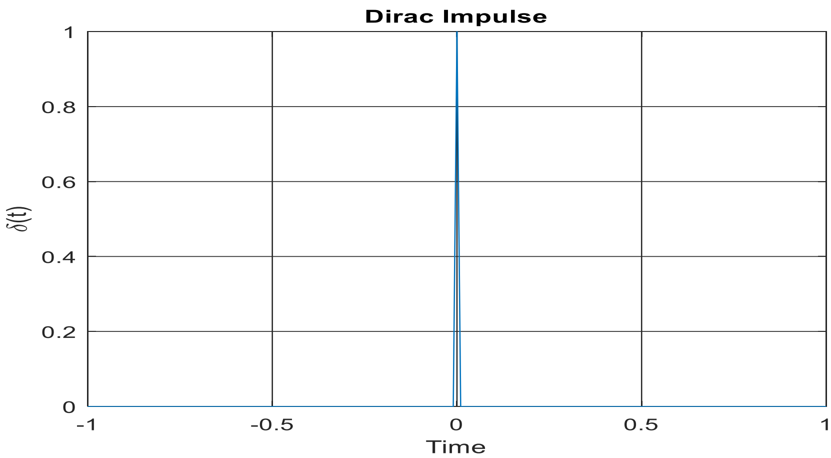
$$R(s) = \frac{1}{s^2}$$

4.3 Dirac Impulse

It is defined as a function that is zero for all t except at t=0, where it has an infinite value. L [δ(t)] = 1

The MATLAB program used to plot it is given below, along with its graph.

```
t = (-1:0.01:1)';
impulse = t = 0;
plot(t, impulse)
title('Dirac Impulse')
xlabel('Time')
ylabel('\delta(t)')
grid on
```



5- PROPERTIES OF THE LAPLACE TRANSFORM

The Laplace transform has many properties that are very useful for solving mathematical problems.

The properties we are going to present are essential because they allow us to easily compute the Laplace transforms of certain signals without directly using the definition of the Laplace transform.

5.1 Uniqueness of the Laplace Transform

The Laplace transform is a one-to-one mathematical operation under certain conditions. This means that if two functions have the **same** Laplace transform, then the two functions are identical almost everywhere on their domain.

More precisely:

If

$$L(f(t)) = L(g(t))$$

for all **s** in a common region of convergence, then

$f(t) = g(t)$ Almost everywhere for $t \geq 0$.

This property ensures that the Laplace transform can be inverted uniquely, which is fundamental for solving differential equations in control systems and signal processing.

5.2 Linearity of the Laplace Transform

The Laplace transform possesses an important property called linearity. This means that the transform preserves linear operations such as addition and multiplication by a constant. In mathematical terms:

If

$f(t)$ and $g(t)$ are functions, and a and b are constants, then:

$$L\{af(t) + bg(t)\} = a\ L\{f(t)\} + b\ L\{g(t)\}$$

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

Or simply:

$$L\{af(t) + bg(t)\} = aF(s) + bG(s)$$

5.3 Laplace Transform of the Derivative

The derivative of a function has a fundamental property under the Laplace transform. This property allows differential equations to be transformed into simple algebraic equations.

If $f(t)$ is differentiable, then:

$$L(f'(t)) = sF(s) - f(0)$$

where:

- $f'(t)$ is the derivative of the function
- $F(s) = L(f(t))$
- $f(0)$ is the initial value of the function at t=0

Explanation

This formula means:

✔ The Laplace transform of the first derivative equals s times the transform of the function minus its initial value.

✔ Thus, differentiation in the time domain becomes a simple algebraic operation in the s-domain.

This is why Laplace transforms are widely used to solve linear differential equations.

5.4 Laplace Transform of the Integral

Integration also has a very useful property under the Laplace transform. It converts an integral in the time domain into a simple division in the s-domain.

If $f\ (\ t\ )$ is integrable, then:

$$L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$$

where:

- $\int_0^t f(t)dt$ is the integral from 0 to t
- $F(s) = L(f(t))$

Explanation

This formula means:

✔ The Laplace transform of an integral equals the transform of the function divided by $\boldsymbol{s}$.

✔ Integration in the time domain becomes a simple algebraic division in the $\boldsymbol{s}$-domain.

✔ This is particularly useful in solving integral and differential-equation systems.

5.5 Time-Shift Theorem (Delay Theorem)

The delay theorem, or time-shift property of the Laplace transform, is essential for representing time-delayed signals in the s-domain.

If $f(t)$ has a Laplace transform $F(s)$ then the delayed function by $a > 0$:

$$f(t-a)u(t-a)$$

has the following Laplace transform:.

$$L(f(t-a)u(t-a) = e^{-as}F(s)$$

where:

- $u(t-a)$ is the heaviside step function
- a is the time delay
- e^{-as} is called the delay factor

Explanation

✔ A delay of a in the time domain

Corresponds to multiplying the transform by e^{-as} in the **s**-domain.

✔ This property is crucial for analyzing delayed systems, such as:

- control systems with response delay
- transport or dead-time models
- circuits with propagation delays

5.6 Scaling Theorem (Time-Scaling Property)

This theorem describes how changing the time scale of a function affects its Laplace transform.

It is also known as the time-scaling or time compression/dilation property.

If $f(t)$ has a Laplace transform $F(s)$ then:

$$L\big(f(at)\big) = \frac{1}{|a|}F(\frac{s}{a})$$

For causal systems, we usually consider a>0, giving:

$$L\big(f(at)\big) = \frac{1}{a}F(\frac{s}{a})$$

Explanation

✔ Time compression (when $a > 1$) stretches the transform in the $\boldsymbol{s}$-domain.

✔ Time expansion (when $0 < a < 10$) compresses the transform in the $\boldsymbol{s}$ domain.

In simple terms:

- Speeding up a signal raises its spectral content
- Slowing down a signal lowers its spectral content

This property is fundamental in control theory and signal processing.

5.7 Limit Theorems

The Laplace transform provides two essential theorems related to limits: the Final Value Theorem and the Initial Value Theorem**.**

They allow us to find initial and final values of a function directly from its Laplace transform without performing the inverse transform.

- **Final Value Theorem**

If all poles of $F(s)$ lie in the left half-plane (i.e., the system is stable), then:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

✔ Used to determine the steady-state value of a signal.

- **Initial Value Theorem**

If both $f(t)$ and $f'(t)$ have Laplace transforms, then:

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

✔ Provides the value of the function at $t = 0^+$

Important Notes

- The Final Value Theorem cannot be applied if $F(s)$ has a pole in the right half-plane or on the imaginary axis.
- These theorems are widely used in control engineering to check stability and steady-state behavior.

5.8 Plancherel's Theorem

Plancherel's theorem is a fundamental result in signal analysis that relates the total energy of a signal in the time domain to its energy in the Laplace (or Fourier) domain.

It shows that energy is preserved when transforming the signal, meaning that analysis in the frequency domain does not alter the total signal energy.

For a square-integrable function $f(t)$ (i.e., a signal with finite energy):

$$\int_0^\infty |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\sigma + jw)|^2 dw$$

where:

- $f(t)$: time-domain signal
- $F(s)$: Laplace transform of $f(t)$
- $s = \sigma + j\omega s$

Explanation

✔ The theorem states that the total signal energy remains the same in both domains.
✔ This guarantees that the Laplace domain preserves energetic properties of signals.
✔ It is widely used in signal processing, control theory, and stability analysis.

6- ELEMENTARY TABLE OF LAPLACE TRANSFORMS

The following table summarizes some commonly used Laplace transforms for standard functions. These are essential for solving differential equations and analyzing control systems.

Time Domain $f(t)$	Laplace Transform $F(s)$	Conditions
1	$\frac{1}{s}$	$\operatorname{Re}(s) > 0$
t^n (where n is a nonnegative integer)	$\frac{n!}{s^{n+1}}$	$\operatorname{Re}(s) > 0$
e^{at}	$\frac{1}{s-a}$	$\operatorname{Re}(s) > a$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\operatorname{Re}(s) > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$\operatorname{Re}(s) > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$$\operatorname{Re}(s) >
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$$\operatorname{Re}(s) >
$u(t)$ (unit step function)	$\frac{1}{s}$	$\operatorname{Re}(s) > 0$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$\operatorname{Re}(s) > a$

LAPLACE TRANSFORM EXERCISES

Exercise 1: Constant Function

Compute the Laplace transform of:

$$f(t) = 10$$

Solution:

$$L(10) = \int_0^{\infty} 10e^{-st}dt = 10\int_0^{\infty} e^{-st}dt$$

$$\int_0^{\infty} e^{-st}dt = \frac{1}{s}, s > 0$$

$$L(10) = \frac{10}{s}$$

Exercise 2: Exponential Function

Compute the Laplace transform of:

$$f(t) = e^{4t}$$

Solution

$$L(e^{4t}) = \int_0^{\infty} e^{4t}e^{-st}dt = \int_0^{\infty} e^{-(s-4)t}dt$$

$$\int_0^{\infty} e^{-(s-4)t}dt = \frac{1}{s-4} \quad , s > 4$$

$$L(e^{4t}) = \frac{1}{s-4}$$

Exercise 3: Polynomial Function

Compute the Laplace transform of: $f(t) = t^3$

Solution Using the formula

$$L(t^n) = \frac{n!}{s^{n+1}} \quad , n = 3$$

$$L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$$

Exercise 4: Derivative

Compute the Laplace transform of:

Function: $f(t) = t^2; \quad f'(t) = 2t$

Formula: $L\big(f'(t)\big) = sL(f(t) - f(0))$

Solution:

$$\begin{cases} f(0) = 0\,; L(t^2) = \dfrac{2}{s^3} \\ L\big(f'(t)\big) = s.\dfrac{2}{s^3} - 0 = \dfrac{2}{s^2} \end{cases}$$

CHAPTER 03 : TRANSFER FUNCTIONS

1. INTRODUCTION :

The transfer function is a fundamental and pervasive mathematical concept in the field of engineering and applied physics. It constitutes the standard modeling tool used to describe the dynamics of linear time-invariant systems (LTI), offering a powerful algebraic simplification compared to the complex differential equations that typically govern these systems in the time domain.

In simple terms, it acts as a mathematical "recipe" that predicts how a system (whether an electrical circuit, a mechanical mechanism, or a chemical process) will react to any given input, by characterizing the intrinsic relationship between its input signal and its output signal. By utilizing the Laplace transform, the transfer function allows engineers to analyze a system's stability, speed of response, and frequency characteristics, thereby facilitating the design of efficient control systems and high-performance filters.

2. KEY CONCEPTS

❖ **Input-Output Relationship**: The core purpose of a transfer function is to characterize how a system modifies an input signal to produce an output signal. This allows engineers to analyze complex dynamic systems using simple algebraic equations instead of difficult differential equations.

❖ **Linear Time-Invariant (LTI) Systems**: The concept of a transfer function is primarily applicable to LTI systems, where the output is proportional to the input (linear) and the system's characteristics do not change over time (time-invariant).

❖ **Laplace Transform**: The Laplace transform is the mathematical tool used to convert the time-domain differential equations that describe a system into the frequency domain (s-domain).

- **Poles and Zeros**:
  - **Zeros** are the values of the complex variable $'s'$ that make the numerator of the transfer function equal to zero, resulting in zero gain or no output.
  - **Poles** are the values of $'s'$ that make the denominator of the transfer function zero, causing the function to become infinite. The locations of poles and zeros on a complex plane provide insight into the system's stability and response characteristics (e.g., whether it will oscillate or decay).

❖ **Impulse Response**: The transfer function of a system is the Laplace transform of its unit impulse response, which is the system's natural reaction to a sudden, short-lived disturbance (like flicking a pendulum).

➢ **Applications**

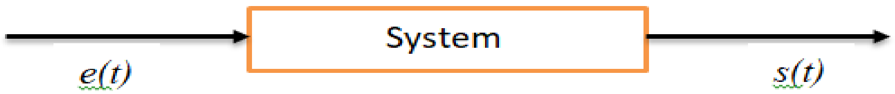
Transfer functions are widely used across various fields of engineering and science, including:

- **Control Systems**: For analyzing system dynamics, stability, and designing controllers.
- **Signal Processing**: In the design and analysis of filters
- **Electrical Engineering**: To describe the gain or impedance of circuits.
- **Mechanical Systems**: To model physical systems like a mass-spring-damper system.

3. NOTION OF TRANSFER FUNCTION:

3.1 Definition

Let us consider an arbitrary linear system with an input $e(t)$ and an output $s(t)$.



On suppose qu'il est régi par une équation différentielle de degré $\boldsymbol{n}$

We assume that it is governed by an n[th]-order differential equation.

$$a_n \frac{d^n s(t)}{dt^n} + \cdots + a_1 \frac{ds(t)}{dt} + a_0 s(t) = b_m \frac{d^m e(t)}{dt^m} + \cdots + b_1 \frac{de(t)}{dt} + b_0 e(t)$$

With $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}, \boldsymbol{m} \leq \boldsymbol{n}$

The coefficients $a_n, a_1, a_0, b_m, b_1, b_0$ are constants (linear system).

By applying the Laplace transform to both sides of the differential equation, and assuming all initial conditions are zero

$$\begin{cases} e(0) = e(0) = \dots\dots\dots\dots\dots\dots\dots\dots = e^m(0) = 0 \\ s(0) = s(0) = \dots\dots\dots\dots\dots\dots\dots = s^n(0) = 0 \end{cases}$$

we then obtain:

$$a_0 S(s) + a_1 s S(s) + \cdots \dots\dots + a_n s^n S(s) = b_0 E(s) + b_1 s E(s) + \cdots \dots\dots + b_m s^m E(s)$$

With

$$\begin{cases} L(e(t)) = E(s), L(e'(t)) = sE(s), L(e^m(t)) = s^m E(s) \\ L(s(t)) = S(s), L(s'(t)) = sS(s), L(s^m(t)) = s^n S(s) \end{cases}$$

By setting: $F(s) = \frac{S(s)}{E(s)}$

Therefore; $$F(s) = \frac{S(s)}{E(s)} = \frac{b_0 + b_1 s + \cdots \ldots\ldots + b_m s^m}{a_0 + a_1 s + \cdots \ldots\ldots + a_n s^n}$$

$F(s)$ is called the transfer function or transmittance of the system. This relationship is very useful for computing time responses of systems using the Laplace transform. One simply computes the system's transmittance, takes the Laplace transform of the input signal, and multiplies the two quantities. An inverse transform then gives the desired time response.

3.2 SHAPES OF THE TRANSFER FUNCTION

3.2.1 Open-Loop Transfer Function (OLTF)

Consider a continuous system represented in the adjacent diagram (Figure 1):

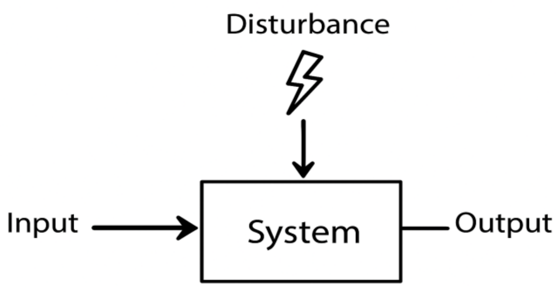


Figure 3-1. Continuous system".

For a system to respond correctly to the user's needs, it is important that the output remain unchanged regardless of external phenomena that might disturb it.
A non-feedback system (called an OPEN-LOOP system) has no control over the command signal applied at its input. The command signal (input) is independent of the output signal. An external action (a disturbance) can therefore modify the desired output of the system.

EXAMPLE:

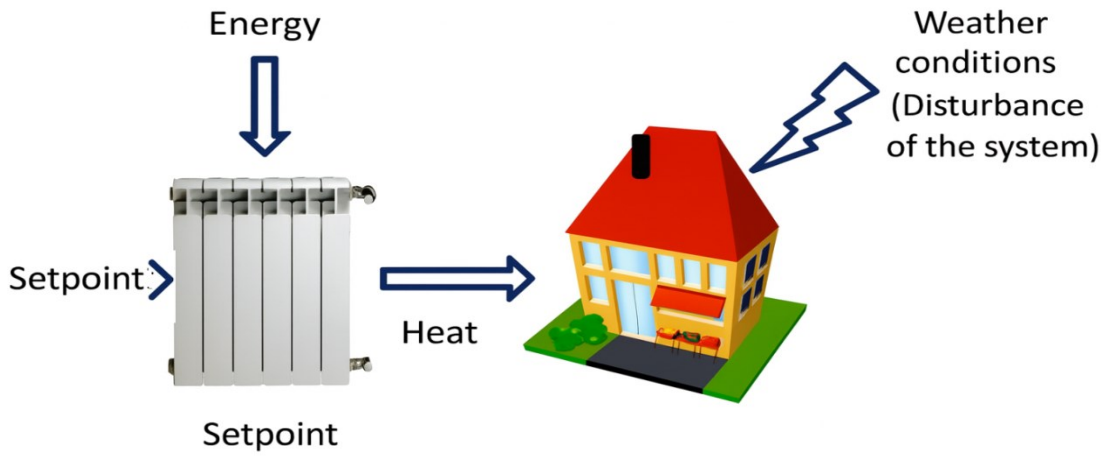


Figure 3-2. Temperature control without feedback.

Consider the control of the temperature $T(t)$ of a living space heated by central-heating radiators (Figure 3-2). To act on $T(t)$ the water flow in the radiators is adjusted using a valve, the setpoint determines the amount of heat produced. However, weather conditions disturb the system: if the outdoor temperature changes, the amount of heat required to maintain a constant indoor temperature is no longer the same. For such a system—an open-loop system—it is therefore impossible to determine in advance the correct value to assign to the setpoint. This is a control scheme without any feedback from the output variable.

3.2.2 Closed-Loop Transfer Function (CLTF)

In some exceptional cases, a control system can operate in open loop using only the setpoint signal. However, the closed loop (feedback) is capable of:

- Stabilizing a system that is unstable in open loop;
- Compensating for external disturbances;
- Compensating for internal uncertainties within the process itself.

Let us again consider the example of controlling the temperature $T(t)$ of a living space heated by central-heating radiators (Figure 3-3).

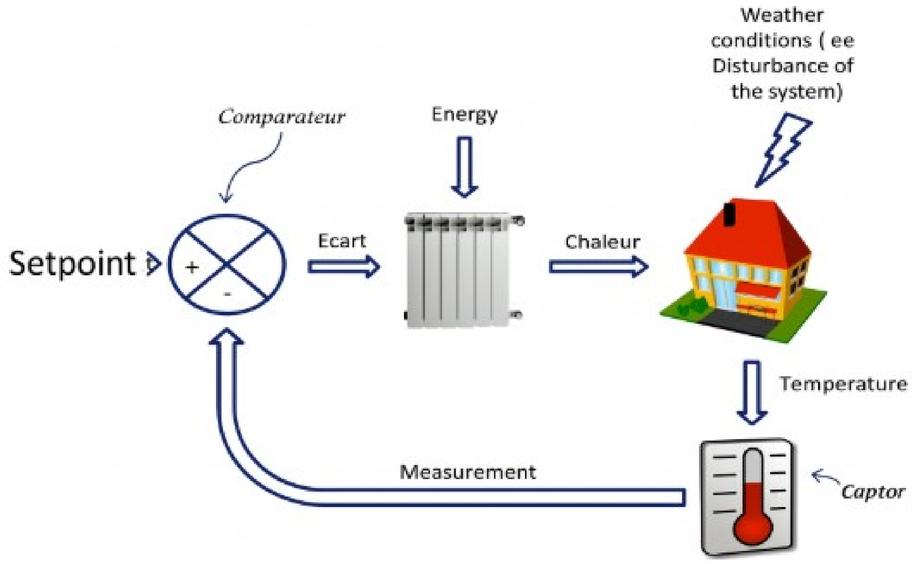


Figure 3-3. Temperature control with feedback

To maintain a constant temperature even when external conditions change, the indoor temperature must be controlled with respect to the imposed setpoint. For this, the amount of heat produced is no longer determined directly by the setpoint, but by the difference between the setpoint (the desired temperature) and the actual temperature inside the house (the measured temperature).

This example illustrates a closed-loop controlled system, whose purpose is to ensure that the system output follows the input setpoint. The indoor temperature (the physical quantity to be controlled) is measured by a sensor (here, a thermometer). The difference between the setpoint and the measurement is calculated by a comparator. The resulting error signal is then used to control heat production

3.2.3 FUNCTIONAL ORGANIZATION OF A CLOSED-LOOP CONTROL SYSTEM

The purpose of a feedback control loop is to ensure that the system output follows the input setpoint (Figure 4).

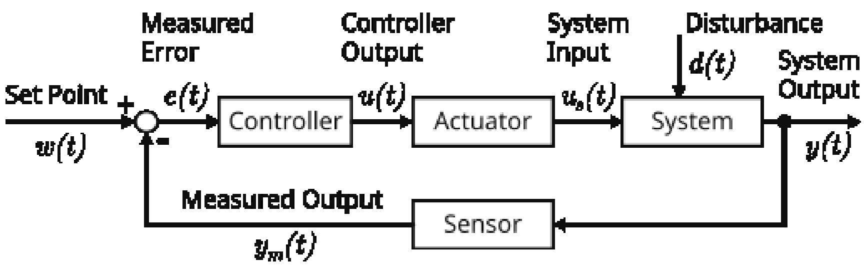


Figure 3-4. General Structure of a Closed-Loop System

To achieve this, the output is fed back to the input through a sensor and compared in a comparator. The difference between the input and the output (called the error or deviation) is calculated and becomes the control signal for the forward path (controller + system). In general, this system includes

- **System**
  It corresponds to the initial system that we want to control, pilot, and regulate in order to satisfy a need stated in the specifications.
- **Setpoint**
  The setpoint is fixed or desired by the operator; it is the adjustable quantity of the system. Its nature may differ from that of the system (S).
- **Sensor**
  The sensor extracts the regulated quantity (physical information) from the system and transforms it into a signal understandable by the regulator.
- **Measurement**
  It is provided by the sensor; generally, the measurement must be of the same physical nature as the setpoint.
- **Comparator or Error Detector**
  Compares the setpoint to the measurement and delivers an error signal proportional to the difference between these two quantities. This error signal, after amplification, will act on the power elements in such a way that the error tends to cancel out.
- **Deviation or Error**
  The deviation (error) represents the difference between the setpoint and the output measurement. This measurement can only be made on comparable quantities. Sometimes, there is an adapter before the comparator that allows adjusting the setpoint input to the measurement made by the sensor, in order to compare two quantities of the same nature.
- **Controller**
  Develops from the error signal the control order or adjustable quantity of the system. It includes an amplifier and mathematical functions for signal processing.
- **Regulator**
  The regulator is an assembly that includes the comparator and the controller.
- **Actuator**
  It is the action organ that supplies energy to the system to produce the desired effect. It is

sometimes associated with a pre-actuator that allows adapting the command (low power) and the energy.

- **Disturbance**

  Any physical phenomenon that intervenes on the system and modifies the state of the output is called a disturbance. A controlled system must be able to maintain the output at its level regardless of disturbances.

- **Output**

  The regulated output represents the physical phenomenon that the system must control, it is the reason for the system's existence.

- **Direct Action Chain**

  Contains all the power elements (requiring an external energy input) that perform the work. It generally includes many elements, notably amplifiers.

- **Feedback Chain**

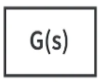
  Analyzes and measures the work performed and transmits to the comparator a physical quantity proportional to this work. It generally includes a sensor that provides a measurement of the output quantity.

4- SIMPLIFICATION OF FUNCTIONAL BLOCK DIAGRAMS.

It is often useful to modify the topology of a block diagram to make it easier to read or to highlight a particular structure; such diagrams are not always simple in form. Certain manipulations make it possible to reduce their complexity and thus to determine the overall transfer function. The elementary transformations are described below, knowing that more complex transformations can be derived from them.

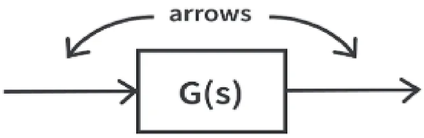
- **Blocks**

A block represents a system or a sub system. The block usually has the transfer function of that particular system of subsystem which it represents.



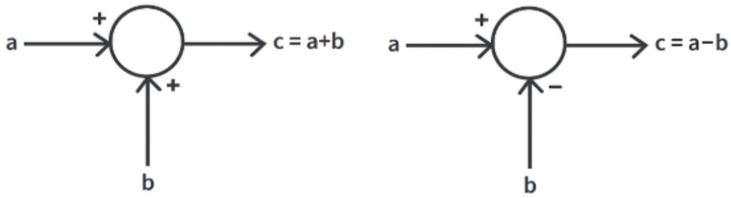
- **Arrows**

Arrows represent the direction of the flow of signal or information. This will tell us how the individual systems/subsystems are connected.

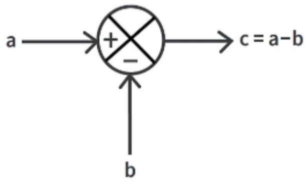


❖ **Summing Point**

The summing points add or subtract signals. It is a small circle with a small " + " *or* " − " near the entry of each signal telling us if the signals are being added or subtracted.

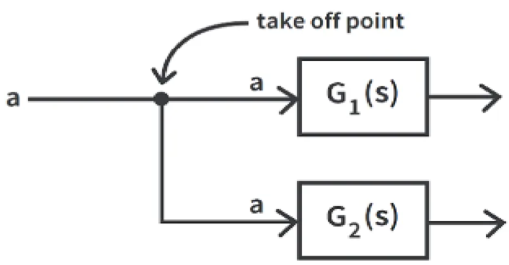


We can also include a small X inside the circle and write the " + −" signs inside. It would look like,



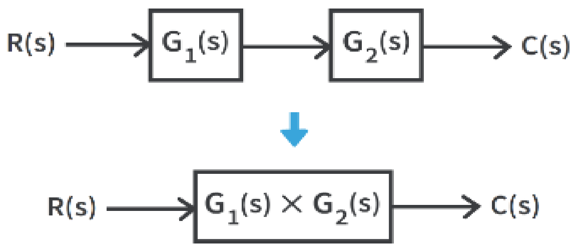
❖ **Take Off Point**

When we need to use the same signal to feed into multiple systems, we make use of take off points.



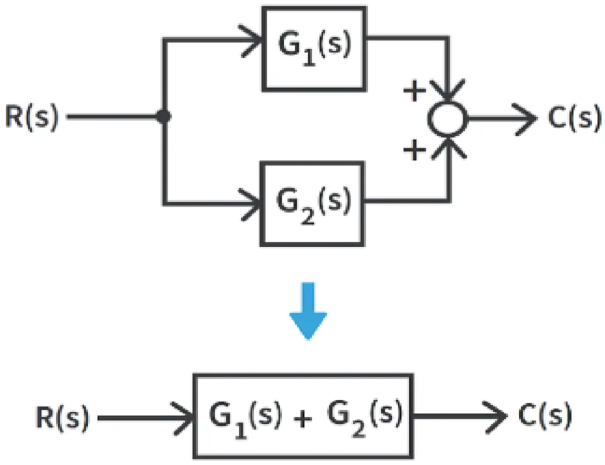
4- 1 Blocks in cascade.

When there are two or more blocks in a cascade (one next to the other), the resultant block would just be the product of the transfer functions of individual blocks.



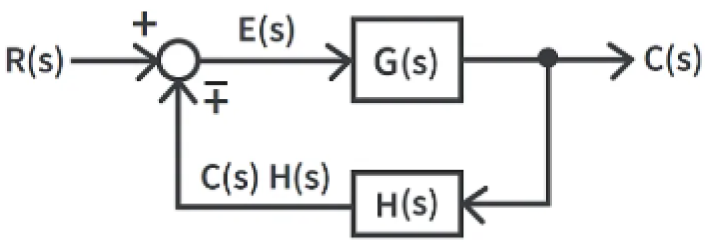
4- 2. Blocks in parallel.

When there are two or more blocks in parallel, the resultant block would just be the sum of the transfer functions of individual blocks.



4- 3. Eliminating a feedback loop.

Consider a simple feedback loop with a system block $G(s)$ and feedback block $H(s)$.



If we just look at the block $G(s)$ with $E(s)$ as input and $C(s)$ as output,

$$C(s) = G(s)E(s)$$

Where $E(s)$ is the difference or sum of the input and the feedback depending upon the type of feedback. For a feedback that is negative, $E(s)$ is the difference of the input and the feedback and for a feedback that is positive, $E(s)$ is the sum of the input and the feedback.

$$E(s) = R(s) \pm C(s)H(s)$$

Now,

$$C(s) = G(s)(R(s) \pm C(s)H(s))$$

$$C(s) = G(s)R(s) \pm G(s)C(s)H(s)$$

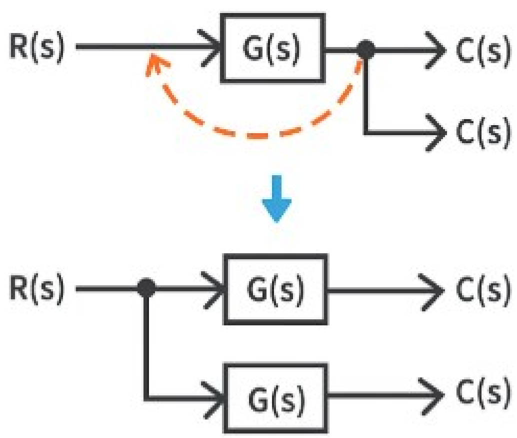
$$C(s) \pm G(s)C(s)H(s) = G(s)R(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

Hence, the above loop can be replaced by,

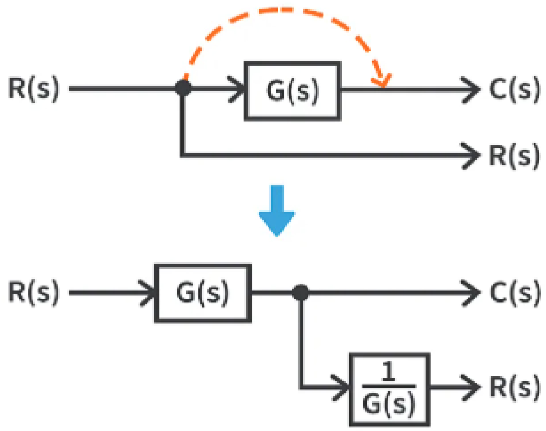
4- 4. Moving a take off point to the left of the block.

When we need to move a take off point to the left of a block, we introduce a block with the same transfer function in that branch of a take off point. The diagram below will make it clear.



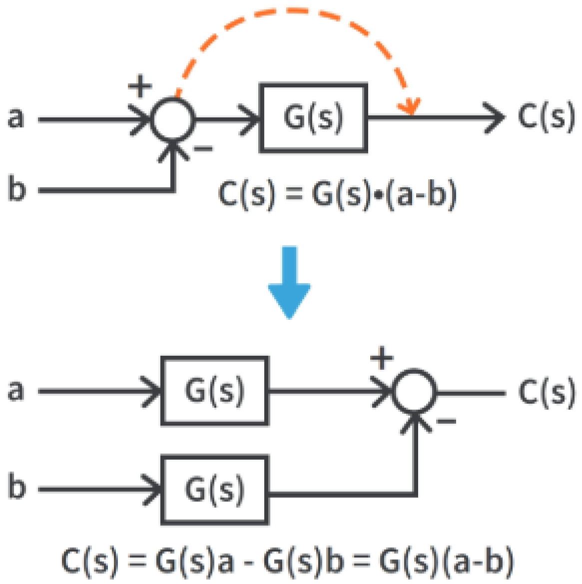
4- 5. Moving a take off point to the right of the block.

Similar to the previous one, when we have to move a take off point to the right of a block, we introduce a block with the reciprocal of the transfer function in that branch of the take off point.



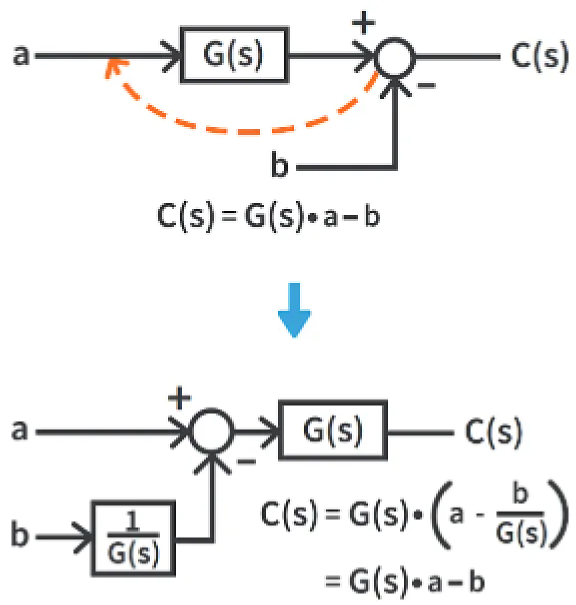
4- 6. Moving a summing point to the right of block.

When a summing point has to be moved to the right of the summing block, the following modifications are to be made



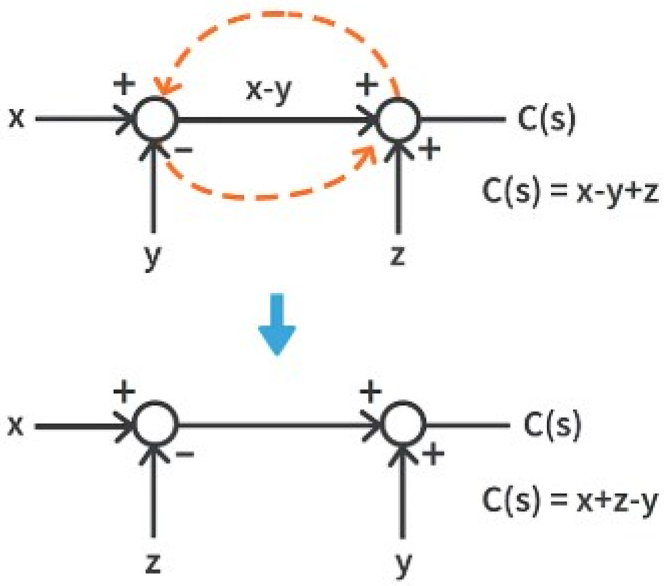
4- 7. Moving a summing point to the left of a block.

When a summing point has to be moved to the left of the summing block, the following modifications are to be made.



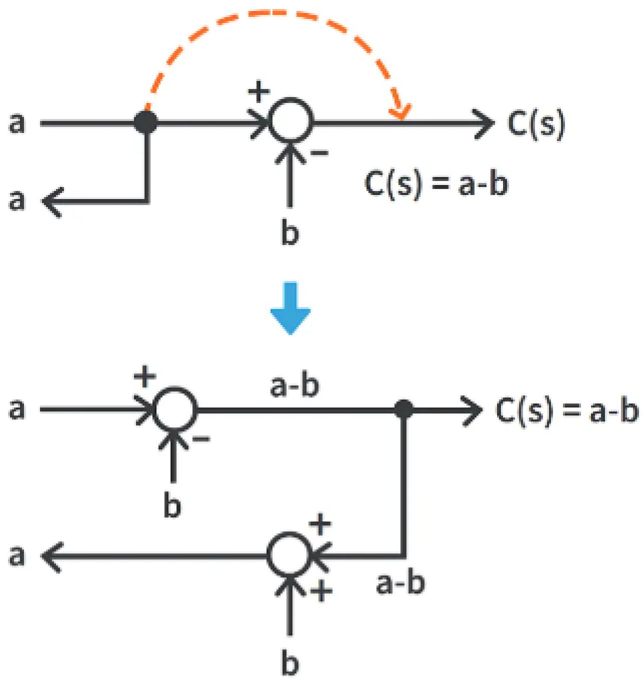
4- 8. Interchanging summing points.

The summing points can be interchanged without any modifications.



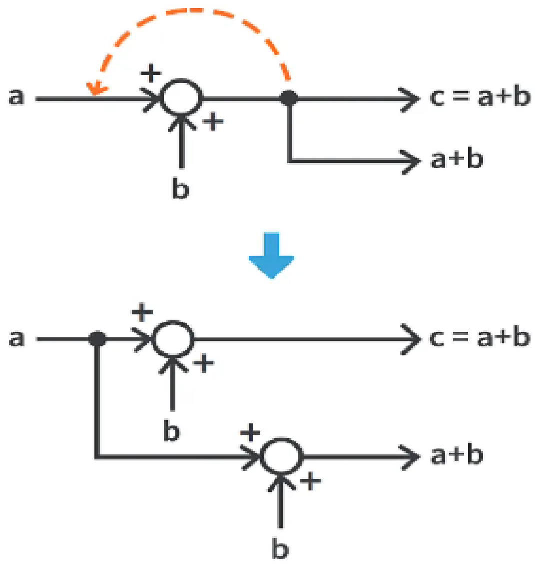
4- 9. Moving a take off point to the right of the summing point.

When we move the take off point to the right of the summing point, we need to compensate for the arithmetic changes so the value of the branch of the take off point as well as the output doesn't change.



4- 10. Moving a take off point to the left of the summing point.

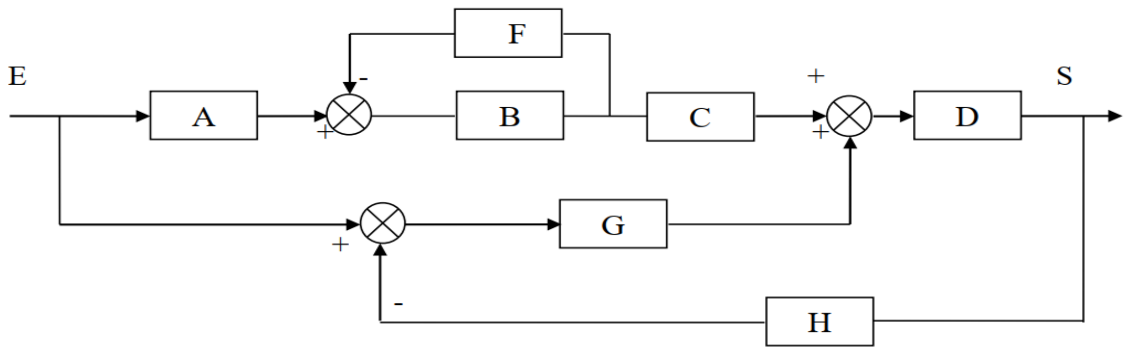
Similar to the previous rule, when we move the take off point to the left of the summing point, we need to compensate for the arithmetic changes as shown.



The idea behind these rules are just to keep the resulting values the same by slightly modifying the block diagram which shall compensate for the changes.

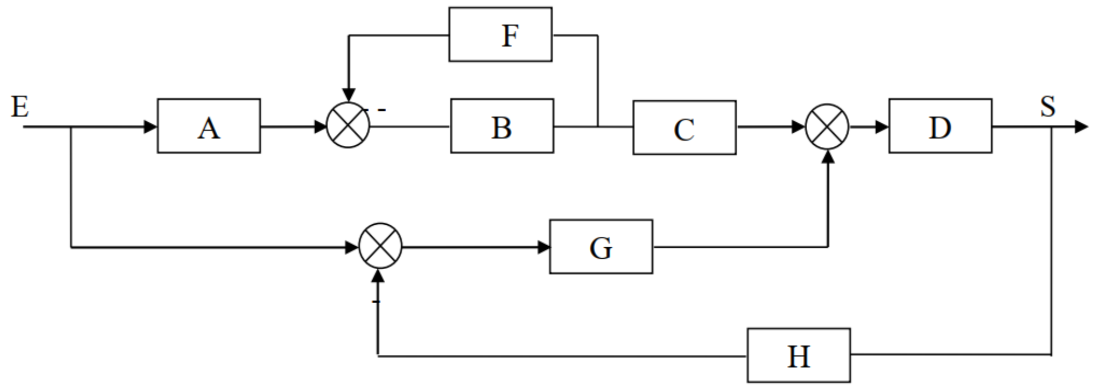
Example 01:

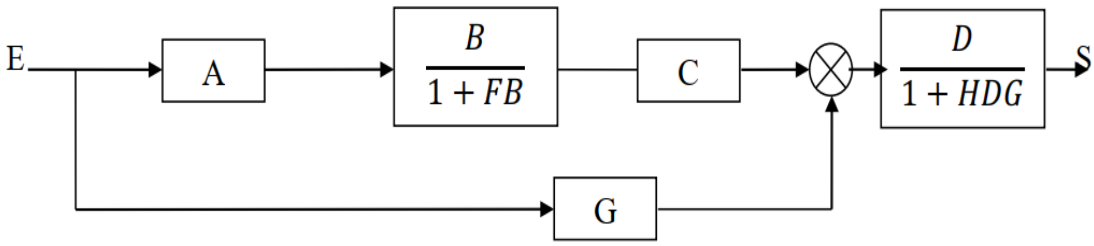
Consider the following block diagram:

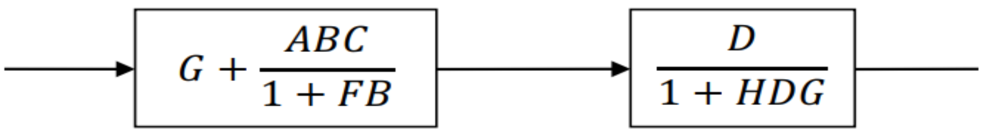


Calculate its transfer function S/E?

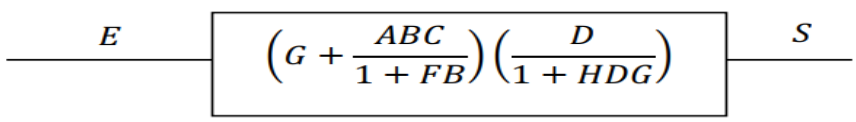
The solution







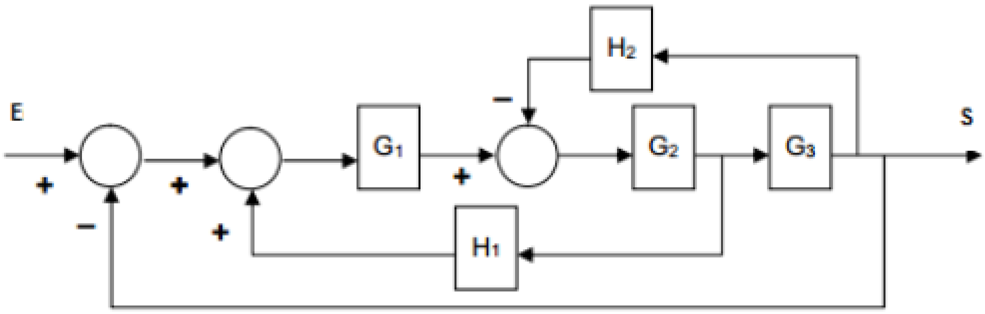
Finally: the cascade arrangement



Thus, the transfer function is

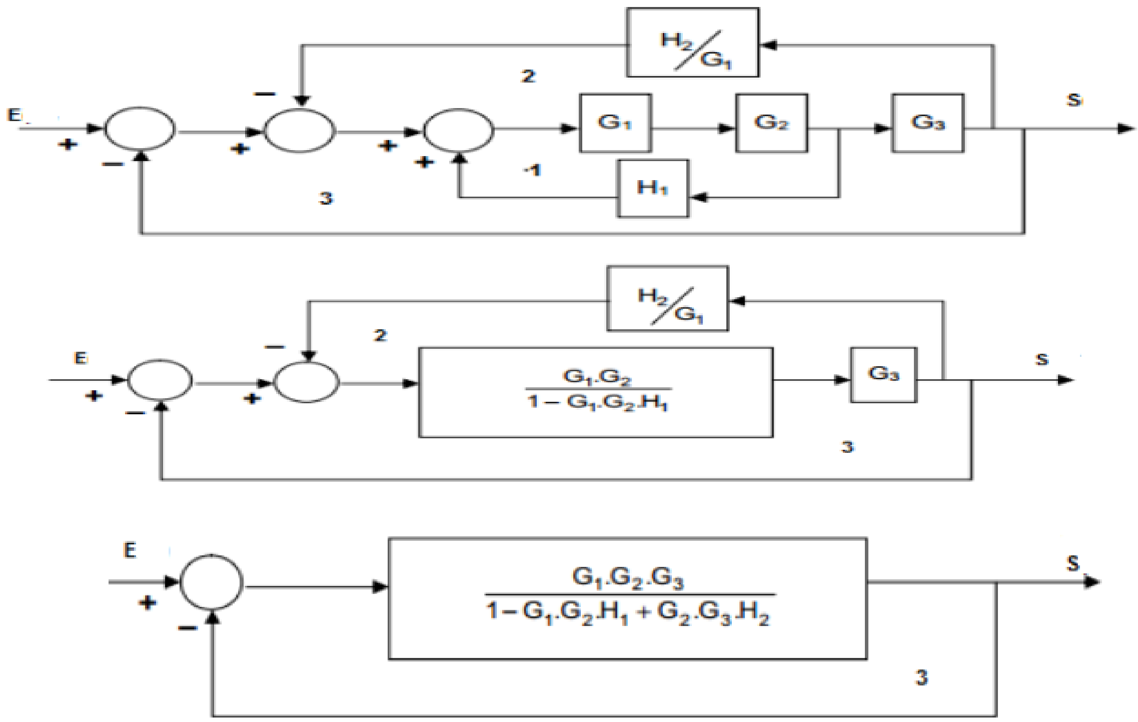
$$\frac{S}{E} = \frac{DG + FBDG + ABCD}{(1 + FB)(1 + HDG)}$$

Example 02: Calculate its transfer function S/E?



Solution:

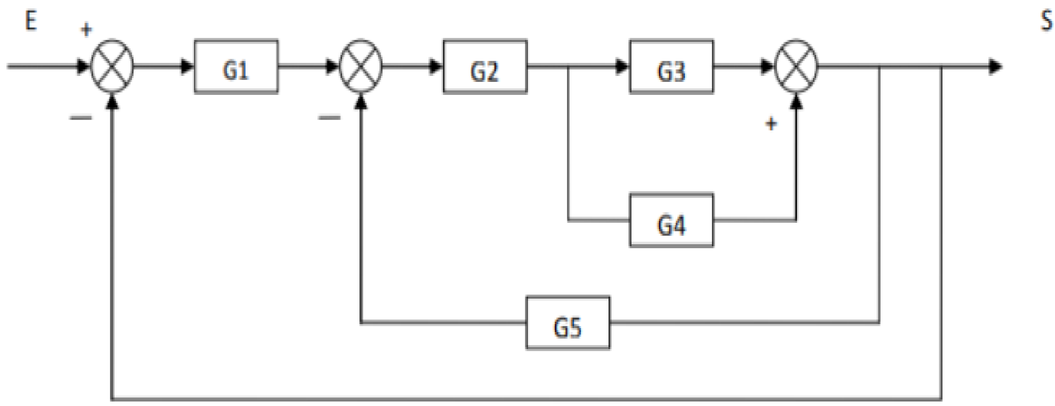
Once the comparator of loop 2 is moved, we simplify loop 1 and then loop 2 using the rule of elements in cascade, and finally we move on to loop 3



$$F(s) = \frac{G_1 G_2 G_3}{1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}$$

Example 03:

Simplify the following diagrams and provide their transfer functions $F(s) = \frac{S(s)}{E(s)}$?



Solution:

G1 and G2 are connected in series (cascade), giving G1·G2 . Then, there is a closed-loop formed by G3 and G4 :

$$\frac{G_3}{1 - G_3 G_4}$$

Then,

$$\frac{G_1 G_2 G_3}{1 - G_3 G_4}$$

with G5, we will obtain the final result

$$F(s) = \frac{S(s)}{E(s)} = \frac{\frac{G_1 G_2 G_3}{1 - G_3 G_4}}{1 - G_5 \frac{G_1 G_2 G_3}{1 - G_3 G_4}}$$

CHAPTER 04 : STUDY OF A FIRST-ORDER CONTROL SYSTEM

1. INTRODUCTION :

First-order systems constitute a fundamental category of dynamic systems whose behavior is governed by a first-order ordinary differential equation. Such an equation involves a first derivative of the output with respect to time and no derivative of higher order. Consequently, the order of the governing differential equation corresponds directly to the highest-order derivative present.

A defining feature of first-order systems is the presence of a single independent energy storage element. In general, the order of a system is equal to the number of its independent energy storage elements. Elements are considered independent only when they cannot be reduced to an equivalent storage element. For instance, two capacitors connected in parallel can be represented by a single equivalent capacitor, and therefore constitute one independent storage element.

First-order systems play a significant role in engineering analysis due to their broad applicability and their capacity to approximate more complex systems. Many physical processes naturally exhibit first-order behavior, such as the mass–damper system in mechanics or the thermal system in heat transfer. Moreover, higher-order systems can often be effectively approximated by first-order models when a dominant first-order mode governs their dynamics, thereby simplifying analysis while maintaining acceptable accuracy.

Typical first-order dynamics arise in several practical scenarios. Examples include tank-filling systems and mass–dashpot configurations, as illustrated in Figure. 4-1.

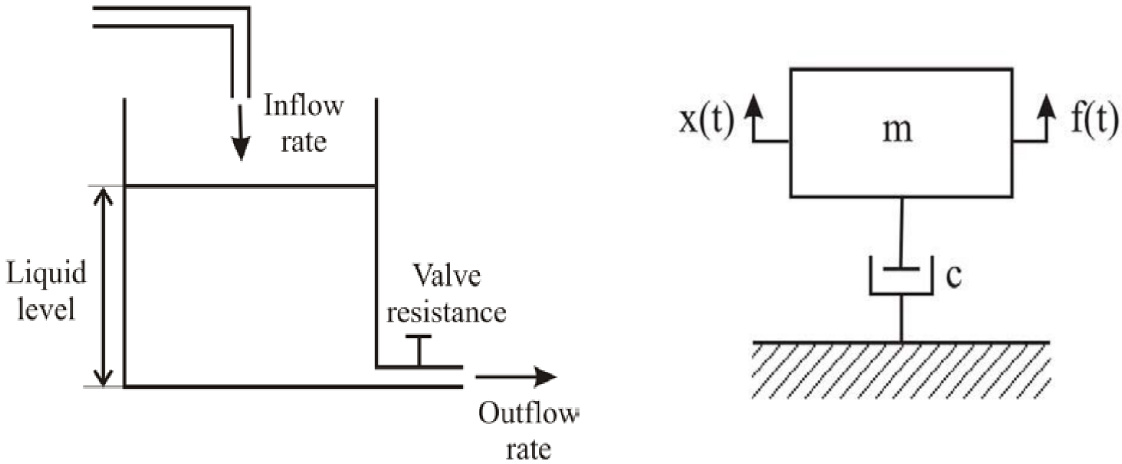


Figure 4-1 First-order tank-filling system and first-order mass–dashpot system.

Similarly, simple electrical networks such as the RC (resistor–capacitor) circuit, shown in Figure. 2, are described by first-order differential equations. The RC circuit, in particular, is frequently employed as a basic low-pass filter, making it a standard pedagogical and analytical model in control theory.

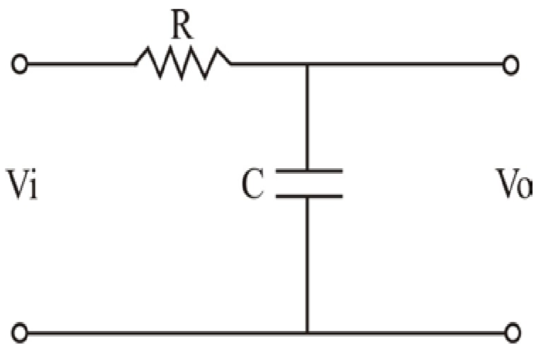


Figure. 4-2. First-order RC circuit.

2- FIRST ORDER SYSTEM MODEL

A first-order system is defined as a system governed by a linear first-order differential equation of the form:

$$T\frac{ds(t)}{dt} + s(t) = Ke(t)$$

By applying the Laplace transform to this equation, the transfer function of the first-order system can be obtained.

$$TsS(s) + S(s) = KE(s)$$

$$\frac{S(s)}{E(s)} = \frac{K}{Ts + 1}$$

The first order system has only one pole as shown

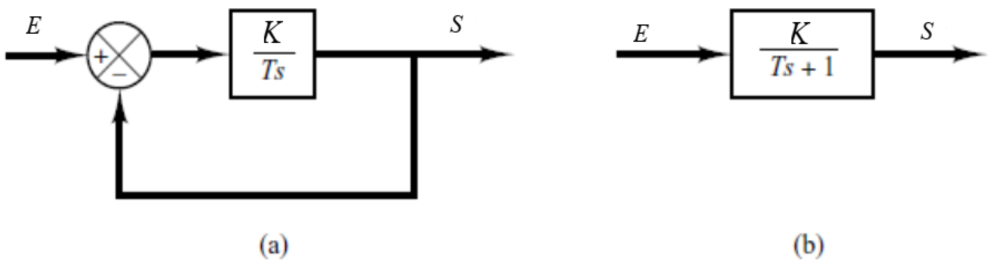


Figure 4-3: (a) Block Diagran of a first-order system; (b) Simplifed block Diagram

- Where *K* represents the static gain of the system, and *T* denotes its time constant.
- The time constant *T* characterizes the speed of response of a first-order system when subjected to a unit-step input. It indicates how quickly the system approaches its steady-state value.

- The static gain K corresponds to the ratio between the steady-state output and the applied input signal.

Example 1

For the first-order system shown below:

$$F(s) = \frac{6}{2s + 1}$$

- The static gain K is equal to **6**.
- The time constant T is equal to **2**.

Example 2

For the first-order system shown below:

$$F(s) = \frac{6}{s + 5}$$

$$F(s) = \frac{6/5}{1/5s + 1}$$

- The static gain K is equal to $\frac{6}{5}$ (or replace with the correct value if needed).
- The time constant T is equal to $\frac{1}{5}$

3- SYSTEM RESPONSES TO VARIOUS INPUT SIGNALS

3-1- Impulse Response of a First-Order System

Let us consider a first-order linear system. The system is subjected to a Dirac impulse $\boldsymbol{e(t) = \delta(t)}$, whose Laplace transform is given by $E(s) = 1$

$$\frac{S(s)}{E(s)} = \frac{K}{Ts + 1}$$

$$S(s) = \frac{K}{Ts + 1}$$

In order to express the system response in the time domain, it is necessary to compute the inverse Laplace transform of the above expression. The impulse response represents a fundamental characteristic of linear time-invariant systems and offers direct insight into their dynamic behavior. For a first-order system described by the transfer function

$$S(t) = \frac{K}{T} e^{\frac{-t}{T}}$$

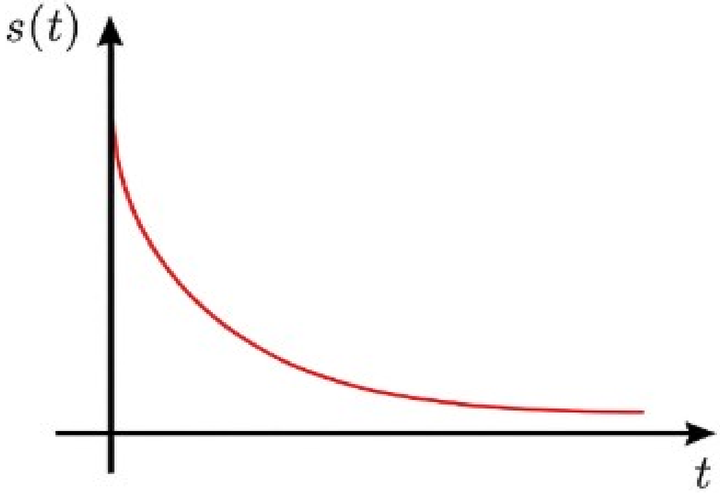


Figure 4-4: Impulse Response of a First-Order System

3-2- Unit Step Response for a First-Order System

Consider the first order system in figure 3

$$E(s) = \frac{1}{s}$$

$$S(s) = \frac{K}{Ts+1} \cdot \frac{1}{s}$$

In order to represent the response of the system in time domain we need to compute the inverse Laplace transform of the above equation, we have

Where $e(t) = 1$

$$s(t) = K - e^{-\frac{t}{T}}$$

Where $t = T$

$$(t) = K - e^{-1} = 0.632K$$

- **For example, assume $K = 10$, $T = 1.5s$**

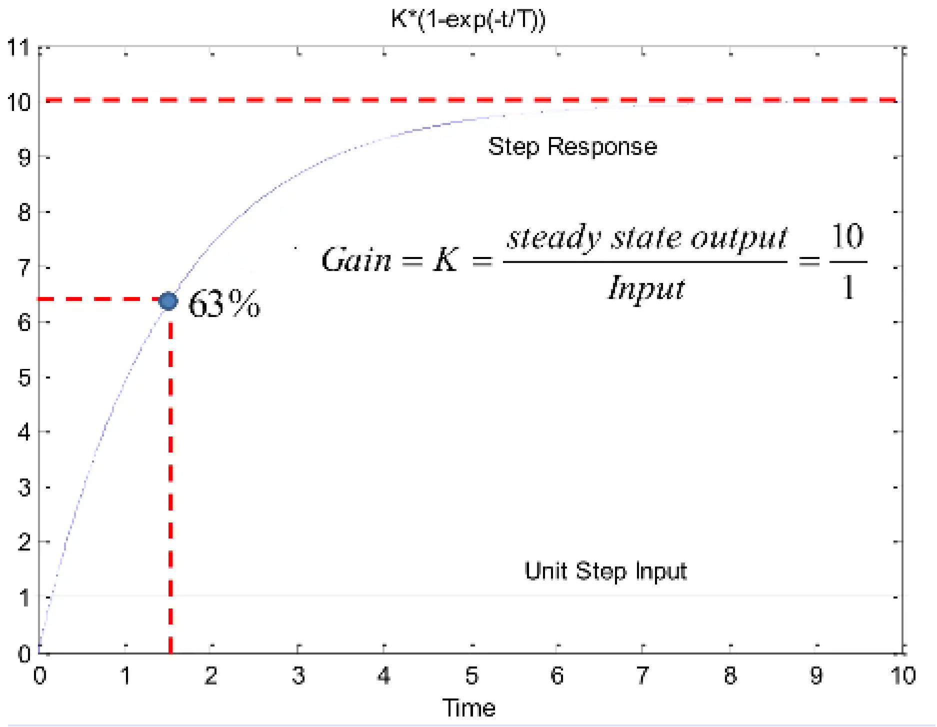


Figure 4-5: The step response specification of first order system

➢ **For example, assume $K = 10$, $T = 1 , 3 , 5 , 7s$**

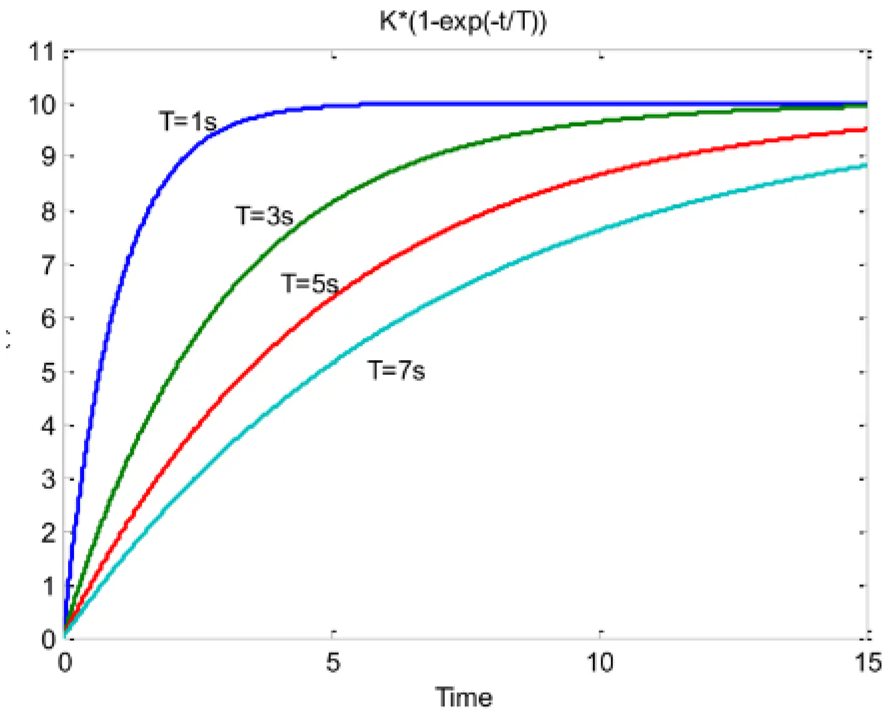


Figure 4-6: The step response at different value of *T*

➢ **For example, assume $K = 1 , 3 , 5 , 10$, $T = 1s$**

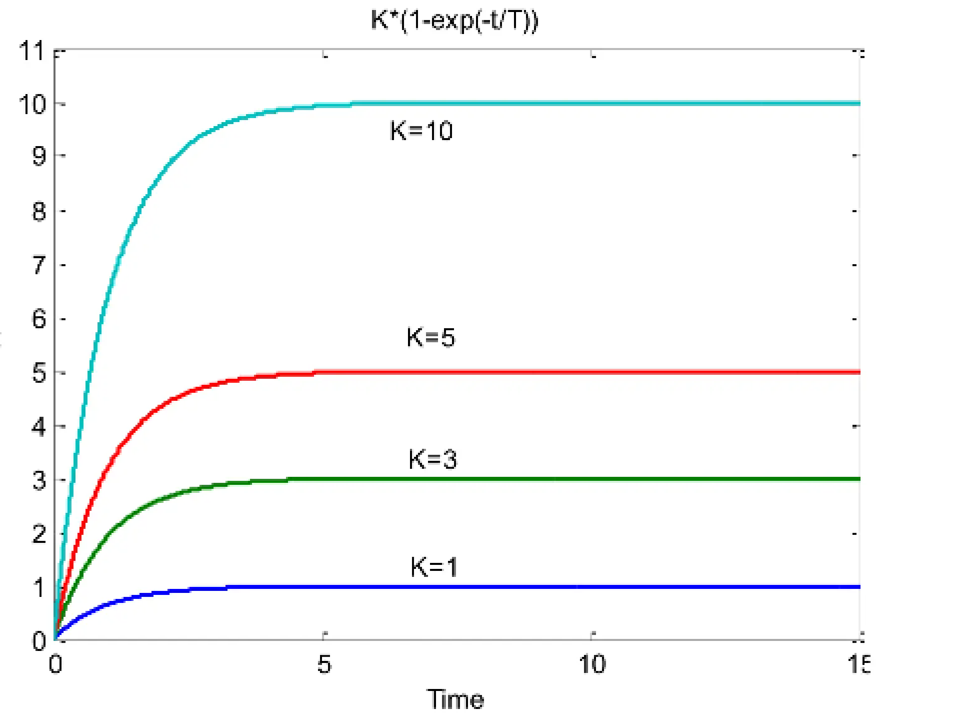


Figure 4-7: The step response at different value of *K*.

3.3- Response to a Ramp for a First-Order System

Let us consider a first-order linear system. The system input is a unit ramp:

$\boldsymbol{e(t)} = \boldsymbol{t(t)}$, whose Laplace transform is given by $E(s) = \frac{1}{s^2}$

$$S(s) = E(s)\frac{K}{Ts+1}$$

$$S(s) = \frac{1}{s^2}.\frac{K}{Ts+1}$$

In order to represent the response of the system in time domain we need to compute the inverse Laplace transform of the above equation, we have

$$s(t) = Kt - T + Te^{-\frac{t}{T}}.$$

- **The ramp response of system, if *K*=1 , *T*=1 is shown in figure 8**
- **The ramp response of system, if *K*=1 , *T*=3 is shown in figure 9**

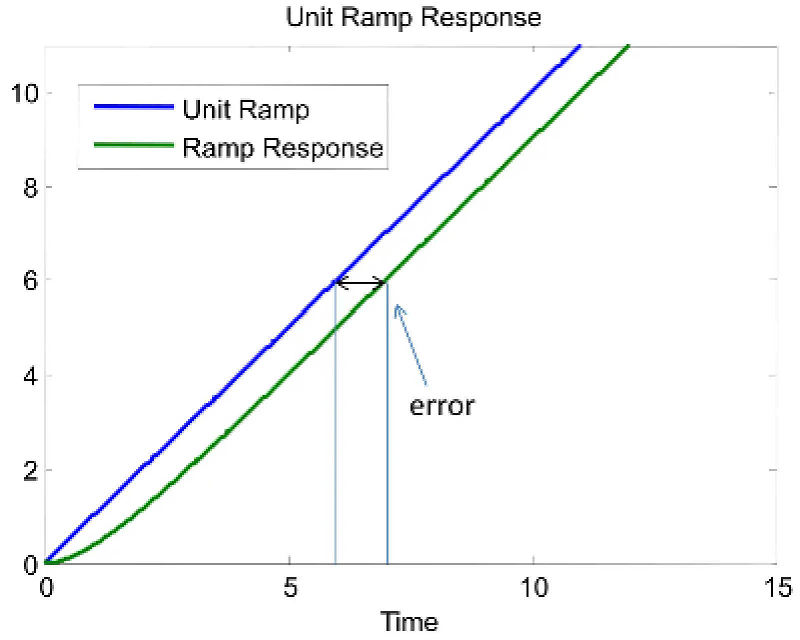


Figure 8: The ramp response for *K*=1, *T*=1

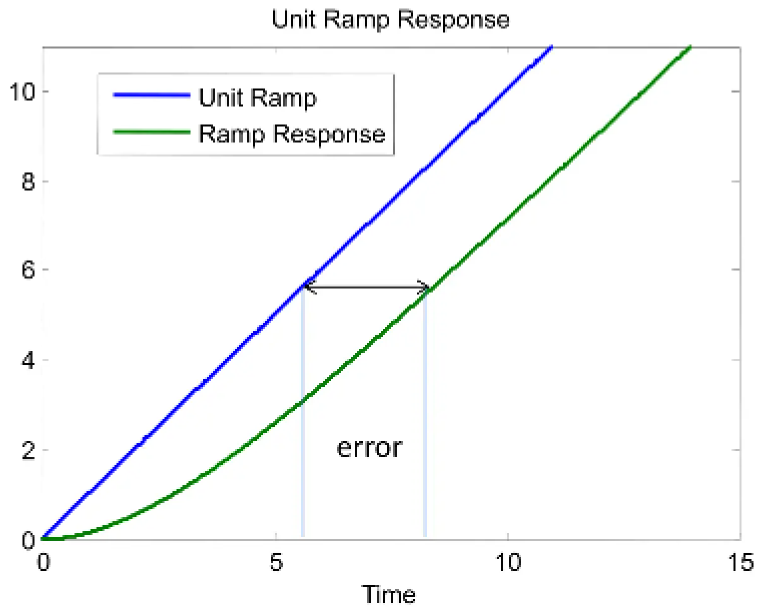


Figure 8: The ramp response for *K*=1, *T*=3

3.4- Harmonic Response for a First-Order System

This is the response of a system to a periodic input, such as sinusoidal functions. It allows the study of the system in steady-state conditions.

This topic will be further examined in Chapter 6, along with Bode and Nyquist diagrams.

EXAMPLE 1

Given system transfer function:

$$G(s) = \frac{0.5(1-s)}{(1+s)(1+0.5s)}$$

Questions:

1. Express the system as two first-order systems.
2. Determine and represent the poles and zeros in the complex plane.
3. Apply a unit step input $e(t) * 1$,Express s(t) and evaluate s(0) and s(t) as t →∞.

Solutions:

1- **Express as two first-order systems:**

$$G(s) = \frac{0.5(1-s)}{(1+s)(1+0.5s)} = \frac{A}{1+s} + \frac{B}{1+0.5s}$$

Using partial fraction decomposition:

$$0.5(1-s) = A(1+0.5s) + B(1+s)$$

Group terms

$$A + B + (0.5A + B)s = 0.5 - 0.5s$$

Equating coefficients**:**

$$\begin{cases} A + B = 0.5 \\ 0.5A + B = -0.5 \end{cases}$$

Solve the system: $A = 2$, $B = -1.5$

So: $G(s) = \frac{0.5(1-s)}{(1+s)(1+0.5s)} = \frac{2}{1+s} - \frac{1.5}{1+0.5s}$

2- Determine and plot poles and zeros in the complex plane

Poles: roots of denominator: $s = -1$, $s = -2$

Zero: root of numerator:$s = 1$

3- Response to a unit step input $e(t)$

Unit step input Laplace:$E(s) = \frac{1}{s}$

$$S(s) = G(s)E(s) = (\frac{2}{1+s} - \frac{1.5}{1+0.5s})\frac{1}{s}$$

Perform inverse Laplace:

$$\mathcal{L}^{-1}\left\{\frac{2}{s(1+s)}\right\}=2(1-e^{-t})$$

$$\mathcal{L}^{-1}\left\{\frac{-1.5}{s(1+0.5s)}\right\}=-3(1-e^{-0.5t})$$

$$s(t)=2(1-e^{-t})-3(1-e^{-0.5t})=-1+2e^{-t}+3e^{-0.5t}$$

Initial value: $s(0)=4$

Final value: $s(\infty)=-1$

EXAMPLE 2

Consider the unity-feedback system:

Forward transfer function: $G(s)=\frac{4}{1+2s}$

1. Determine the system type and the meaning of 4 and 2
2. Determine open-loop and closed-loop transfer functions

Solutions:

System type: First-order system

4 → system gain K

2 → time constant T=2

Determine open-loop and closed-loop transfer functions

Open-loop $G(s)=\frac{4}{1+2s}$

Closed-loop

$$\boldsymbol{F}(\boldsymbol{s})=\frac{\boldsymbol{G}(\boldsymbol{s})}{\mathbf{1}+\boldsymbol{G}(\boldsymbol{s})}=\frac{\frac{4}{1+2s}}{\mathbf{1}+\frac{4}{1+2s}}$$

$$\boldsymbol{F}(\boldsymbol{s})=\frac{2}{2.5+s}$$

CHAPTER 05 : STUDY OF A SECOND-ORDER CONTROL SYSTEM

1. INTRODUCTION :

Second-order systems play a central role in control engineering because they capture the essential dynamics of many real-world processes. Described by second-order differential equations, these systems incorporate inertia, energy storage, and damping effects, which together determine how the system responds to different inputs.

Their behavior is largely defined by two key parameters: the natural frequency and the damping ratio. These parameters dictate whether the system exhibits oscillations, overshoot, or fast stabilization, making second-order systems crucial for understanding transient and steady-state responses.

Studying these systems is essential not only because they appear frequently in mechanical, electrical, and electromechanical applications, but also because they form the foundation for analyzing more complex, higher-order systems. Many control strategies—such as PID and optimal control—depend on a clear understanding of second-order dynamics to ensure stability and desired performance.

This chapter provides an overview of second-order system modeling, their characteristic responses, and the influence of system parameters on their behavior, forming a basis for effective control system design.

2- PRACTICAL APPLICATIONS OF SECOND ORDER CONTROL SYSTEMS

Second order control systems are ubiquitous across various engineering disciplines.

- **Mechanical Systems**
- Suspension systems in vehicles
- Robotic arm joints
- Vibration control in structures
- **Electrical Systems**
- RLC circuits (Resistor-Inductor-Capacitor)
- Motor control systems
- Filters in signal processing
- **Thermal Systems**
- Temperature control in furnaces
- Heat exchanger regulation

3- ADVANTAGES AND LIMITATIONS

Advantages

- Well-understood behavior
- Capable of modeling many physical systems

- Facilitates controller design to meet specifi
- Translent and steady-state criteria

Limitations

- Assumes linearity and constant parameters
- Real systems may exhibit higher-order dynamics or nonlinearities
- Damping ratio and natural frequency may vary with operating conditions

4- DEFINITION AND TRANSFER FUNCTION

A second-order system is defined as a system governed by a linear second-order differential equation of the form:

$$T^2 \frac{d^2 s(t)}{d^2 t} + 2\xi T \frac{ds(t)}{dt} + s(t) = Ke(t)$$

where K : is the static gain, T : is the time constant, and ξ : is the damping ratio.

By applying the Laplace transform to this equation (assuming zero initial conditions), it can be expressed in the following form:

$$T^2 s^2 S(s) + 2\xi T s S(s) + S(s) = KE(s)$$

Thus, the transfer function of the second-order system is given by:

$$\frac{S(s)}{E(s)} = \frac{K}{T^2 s^2 + 2\xi T s + 1}$$

By setting: $T = \frac{1}{W_n}$ we obtain:

$$F(s) = \frac{S(s)}{E(s)} = \frac{K w_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

5- CHARACTERISTICS OF SECOND ORDER SYSTEMS

- **Underdamped, Critically Damped, and Overdamped Response:** The nature of the response relies upon at the places of the poles. If the poles are actual , the gadget is over-damped. If they're complicated conjugates, the gadget is under damped. For poles at the real axis with multiplicity 2, the machine is seriously damped.
- **Natural Frequency (w_n):** The natural frequency is a essential function of second-order system. This shows the frequency at which the system would oscillate if there were no damping. It is denoted by means of w_n and is related to the gap among the poles.

- **Peak Time, Rise Time, and Settling Time**: These parameters are important in evaluating the overall performance of the gadget. The top time is the time taken to attain the height of the reaction, the upw thrust time is the time taken to reach from 10% to 90% of the final value, and the settling time is the time required for the reaction to stay inside a positive percent (commonly 5%) of the very last cost.

- **Damping Ratio (ξ):** The damping ratio is a degree of the level of damping within the system. It is denoted via ζ and impacts the kind of response. A better damping ratio effects in a slower response but with less oscillation.

6- RELATIONSHIP BETWEEN ξ AND THE POLES OF THE SECOND-ORDER SYSTEM

1. ξ=0: Undamped system

$$s_{1,2} = \pm i w_n$$

- Poles are purely imaginary.
- Permanent oscillations with no damping.
- No asymptotic stability.

2. $0 < \xi < 1$: Underdamped system

$$s_{1,2} = -\xi w_n \pm i w_n \sqrt{1-\xi^2}$$

- Poles are complex conjugates.
- The real part is negative, indicating the system is stable.
- The response is oscillatory but decays over time.
- The smaller ξ is the larger the oscillations and overshoot.

3. $\xi = 1$: Critically damped system

$$s_1 = s_2 = -w_n$$

- Poles are real and coincident.
- The system returns to equilibrium as quickly as possible without oscillations.
- The optimal case for fast but non-oscillatory response.

4. $\xi > 1$: Overdamped system

$$s_{1,2} = -w_n(\xi \pm \sqrt{\xi^2 - 1}$$

- Poles are distinct real values.
- The response is non-oscillatory and slower than the critically damped system.
- While the system is stable, the dynamics are sluggish.

7. SECOND-ORDER SYSTEM RESPONSES TO DIFFERENT INPUT SIGNALS

7.1- Impulse Response of a Second-Order System

The impulse response of a second-order system describes the system's time-domain behavior when subjected to a Dirac delta (impulse) input $e(t) = \delta(t)$. It provides fundamental insight into the dynamic characteristics of the system, such as stability, damping, and natural frequency.

A standard second-order linear time-invariant system is represented by the transfer function:

$$E(s) = 1 \; and \; Kw_n^2 \neq 1 \;\Rightarrow S(s) = \frac{Kw_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

- $0 < \xi < 1: s(t) = K\left(\frac{w_n}{\sqrt{1-\xi^2}} e^{-\xi w_n t} sin\left(w_n t\sqrt{1-\xi^2}\right)\right) \qquad t \geq 0$
- $\xi = 1: s(t) = K(w_n^2 t e^{-w_n t}) \qquad t \geq 1$
- $\xi > 1: s(t) = K\frac{w_n}{2\sqrt{\xi^2-1}}\left(e^{-(\xi-\sqrt{\xi^2-1})w_n t} - e^{-(\xi+\sqrt{\xi^2-1})w_n t}\right) \qquad t \geq 1$.

with $K = 1$

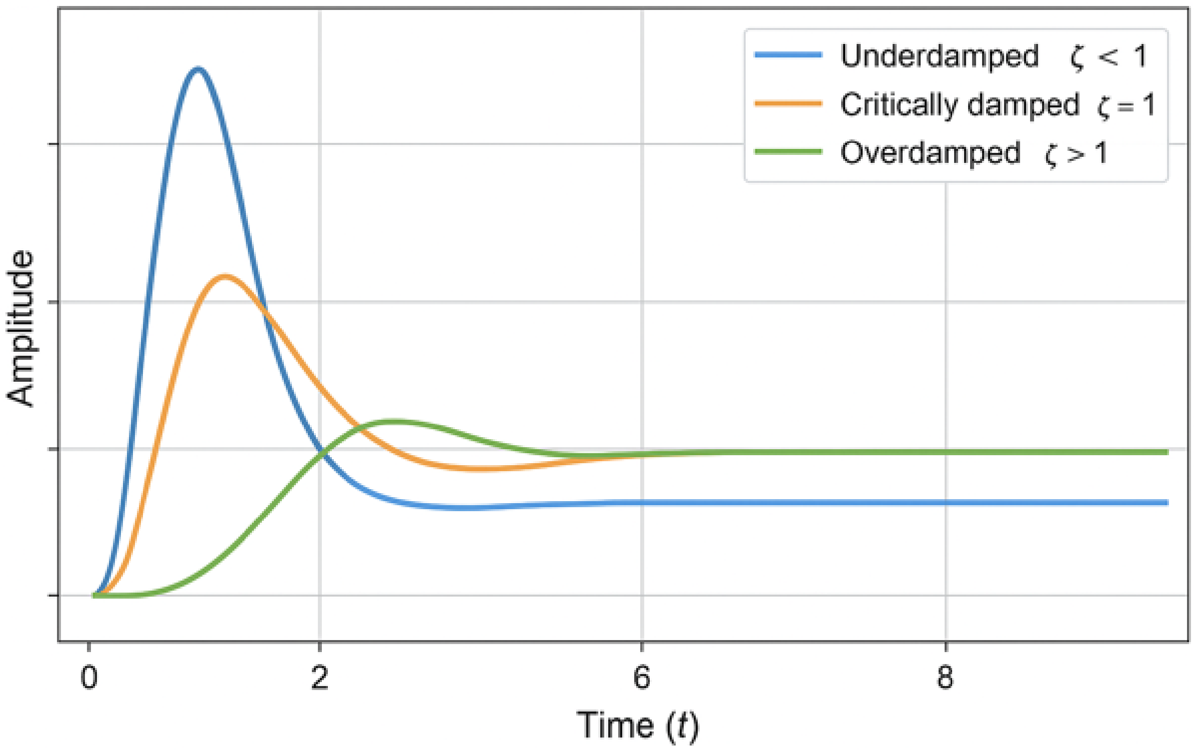


Figure 5-1- Impulse Response of a Second-Order System

7.2- Step Response of a Second-Order System

The step response of a second-order system describes the time-domain behavior of the system when the input is a unit step signal. It is one of the most important responses in control engineering, as it reveals key dynamic characteristics such as stability, speed of response, overshoot, and damping.

A standard second-order linear time-invariant system is defined by the transfer function**:**

$$\frac{S(s)}{E(s)} = \frac{K}{T^2s^2 + 2\xi Ts + 1}$$

For a unit step input $e(t) = u(t)$, its Laplace transform is $E(s) = \frac{1}{s}$

$$S(s) = E(s) * \frac{K}{T^2s^2 + 2\xi Ts + 1}$$

$$S(s) = \frac{1}{s} * \frac{K}{T^2s^2 + 2\xi Ts + 1}$$

- $\xi > 1$: $s(t) = 1 + \frac{1}{2}\frac{\xi}{\sqrt{\xi^2+1}}\left(e^{-(\xi-\sqrt{\xi^2+1})w_n t}\right)$
- $\xi = 1$: $s(t) = 1 - e^{-w_n t}(1 + w_n t)$

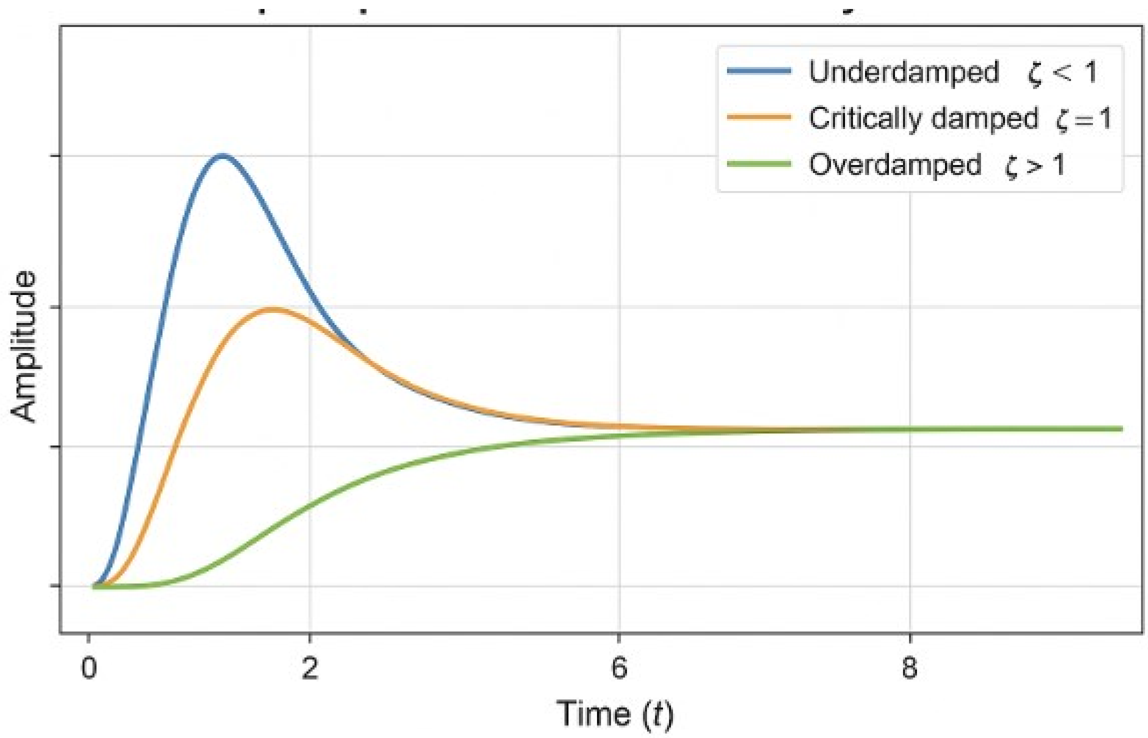


Figure 5-2- Step Response of a Second-Order System

7.2-1 Overshoot (Dépassement)

The maximum overshoot d is defined as the ratio between the maximum peak value and the steady-state value:

$$d = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

or in percentage form:

$$\text{Overshoot } \boldsymbol{d}\% = 100 * e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

7.2-2 Oscillation Period (Période)

The oscillation period T of the underdamped response is:

$$T = \frac{2\pi}{w_d} = \frac{2\pi}{w_n\sqrt{1-\boldsymbol{\xi}^2}}$$

- ✓ w_n is the natural frequency,
- ✓ $w_d = w_n\sqrt{1-\boldsymbol{\xi}^2}$ is the damped natural frequency.

7.2-3 Response Time / Settling Time (Temps de réponse)

The settling time $\boldsymbol{ts}$ is the time required for the response to remain within a specified tolerance band around the final value.

Common approximations:

- **5% criterion**:

$$\boldsymbol{t_s} \approx \frac{\mathbf{3}}{\xi w_n}$$

- **2% criterion (most commonly used):**

$$\boldsymbol{t_s} \approx \frac{\mathbf{4}}{\xi w_n}$$

❖ **For a second-order underdamped system** $(\mathbf{0} < \boldsymbol{\xi} < \mathbf{1})$:

OVERSHOOT $\boldsymbol{d}$

- It is the maximum peak above the steady-state value.
- Measured as:

$$d = \frac{s_{max} - s_\infty}{s_\infty}$$

$s(t)$: is the system response to a unit step input.

s_∞ ; represents the final steady-state value (the final horizontal line)

s_{max} ; denotes the first peak value of the response.

PERIOD *T*

Time between two successive peaks of oscillation.

$$T = \frac{2\pi}{w_n\sqrt{1-\xi^2}}$$

RESPONSE TIME (Settling Time) t_s

$$t_s(5\%) \approx \frac{3}{\xi w_n} \qquad t_s(2\%) \approx \frac{4}{\xi w_n}$$

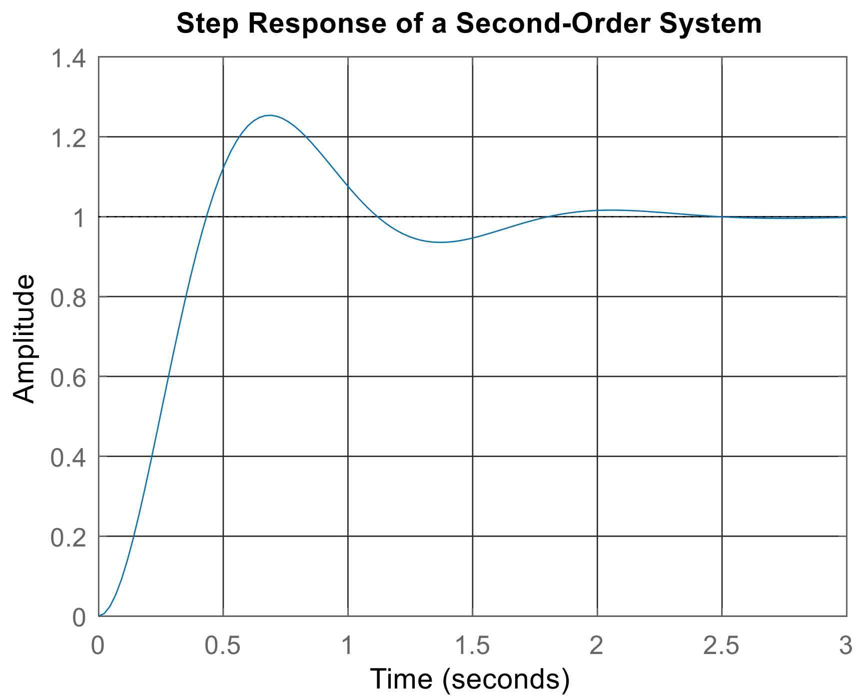


Figure 5-3- Step Response of a Second-Order System ($K = 1, w_n = 5, \xi = 0.4$)

7.3- Harmonic Response

The harmonic response is the response of a system to a periodic input, such as sinusoidal functions. It is used to analyze the system behavior in the steady-state regime.
This response will be studied in Chapter 6, using Bode and Nyquist diagrams.

EXAMPLE 1

Calculate the step response and the impulse response of the two systems defined by the following transfer functions:

$$G_1(s) = \frac{2(s+1)}{s(s+3)^2}$$

$$G_2(s) = \frac{s+4}{(s+1)(s^2-4s+4)}$$

SOLUTION

- **System $G_1(s)$**

a) Impulse Response

The impulse response $e(t) = \delta(t)$ is the inverse Laplace transform of $G_1(s)$

$$G_1(s) = \frac{S(s)}{E(s)} \Rightarrow S(s) = E(s) * G_1(s) \Rightarrow S(s) = G_1(s)$$

$$s(t) = L^{-1}(G_1(s))$$

We perform partial fraction decomposition:

$$G_1(s) = \frac{A}{s} + \frac{B}{3+s} + \frac{C}{(s+3)^2}$$

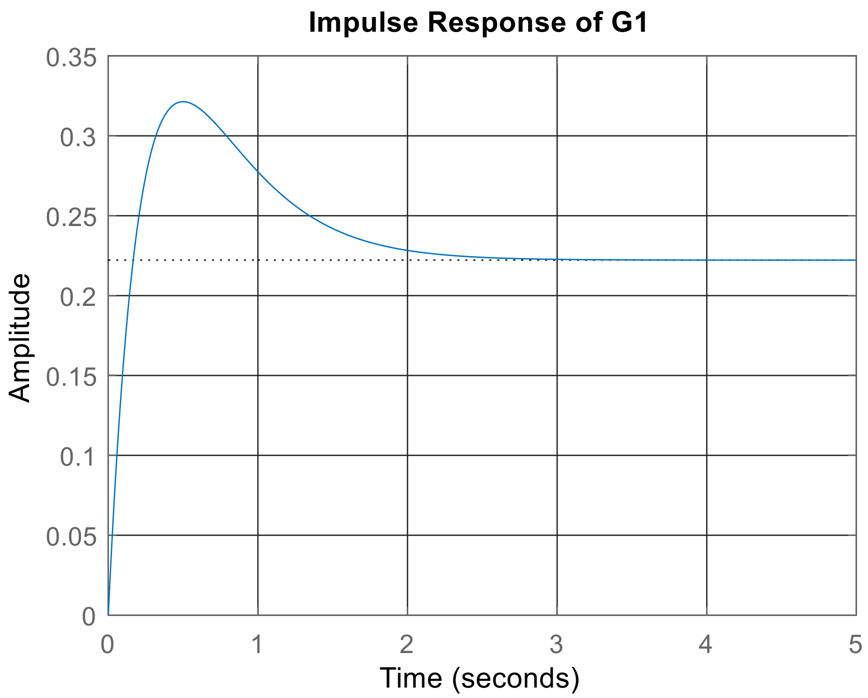
Solving gives: $A = \frac{2}{9}$, $B = -\frac{2}{9}$, $C = \frac{4}{3}$

Thus:

$$G_1(s) = \frac{2/9}{s} - \frac{2/9}{3+s} + \frac{4/3}{(s+3)^2}$$

Taking the inverse Laplace transform:

$$s_1(t) = \frac{2}{9} - \frac{2}{9}e^{-3t} + \frac{4}{3}te^{-3t} \qquad t \geq 0$$



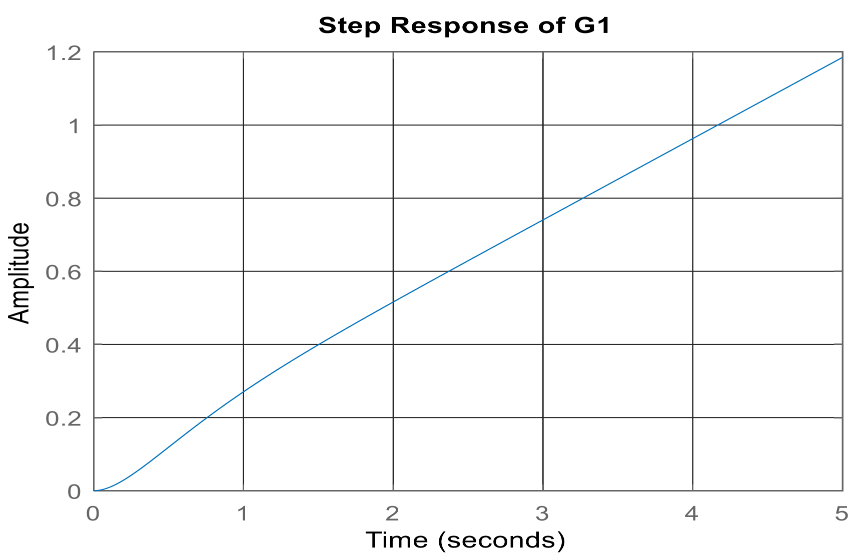
b) Step Response

The Laplace transform of a unit step $\boldsymbol{e(t)}$ is :$E(s) = \frac{1}{s}$:

$$G_1(s) = \frac{1}{s}\left(\frac{2/9}{s} - \frac{2/9}{3+s} + \frac{4/3}{(s+3)^2}\right)$$

After decomposition and inverse Laplace transform, we obtain:

$$s_1(t) = \frac{2}{9} - \frac{2}{9}e^{-3t} - \frac{2}{3}te^{-3t} \qquad t \geq 0$$



➢ **System $G_2(s)$**

$$G_2(s) = \frac{s+4}{(s+1)(s^2-4s+4)}$$

Note that:

$$s^2-4s+4=(s-2)^2$$

So

$$G_2(s) = \frac{s+4}{(s+1)(s-2)^2}$$

a) Impulse Response

We decompose:

$$G_2(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

Solving gives:

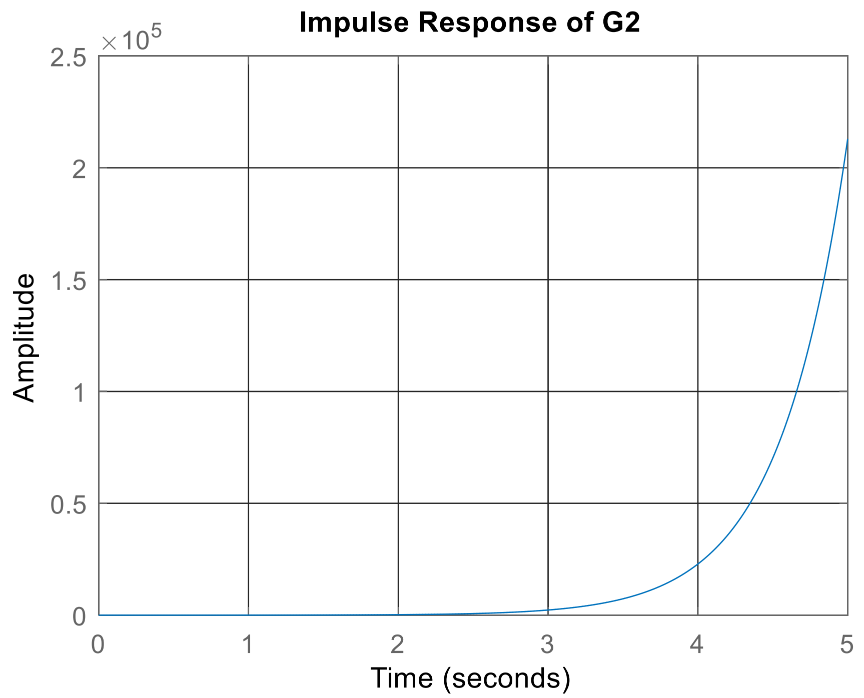
$$A=\frac{1}{9},\ \ B=-\frac{1}{9},\quad C=1$$

Thus:

$$G_2(s) = \frac{1/9}{s+1} - \frac{1/9}{s-2} + \frac{1}{(s-2)^2}$$

Inverse Laplace transform:

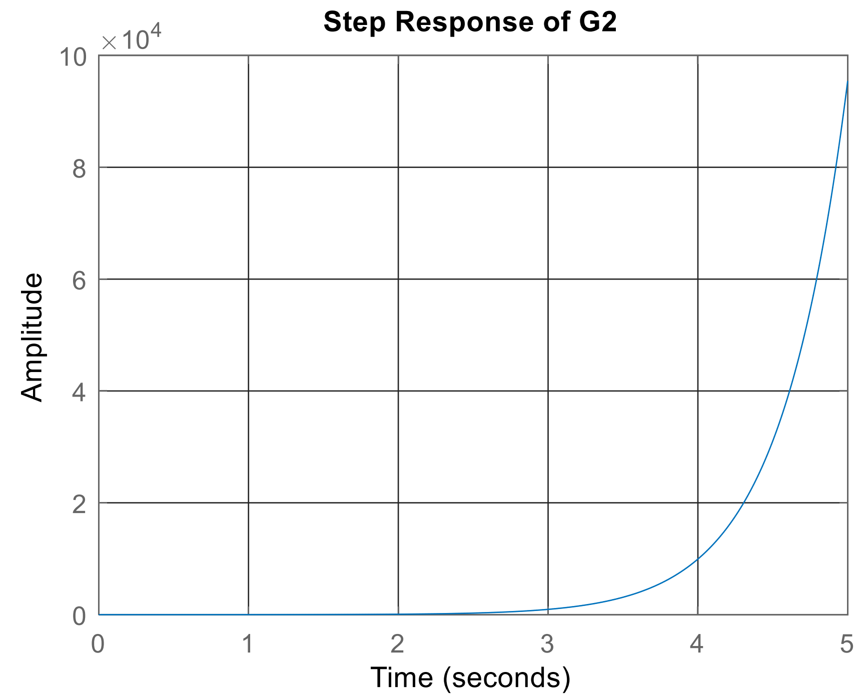
$$s_2(i) = \frac{1}{9}e^{-t} - \frac{1}{9}e^{2t} + te^{2t} \qquad t \geq 0$$



b) Step Response

The Laplace transform of a unit step $\boldsymbol{e(t)}$ is :$E(s) = \frac{1}{s}$:
After inverse Laplace transformation, the step response is:

$$s_2(i) = \frac{1}{9}(1 - e^{-t}) - \frac{1}{9}(e^{2t} - 1) + \frac{1}{2}t^2 e^{2t} \qquad t \geq 0$$



System	Impulse Response	Step Response	Stability
$G_1(s)$	Decaying exponential	Convergent	☐ Stable
$G_2(s)$	Growing exponential	Divergent	☐ Unstable

EXAMPLE 2

Consider a system whose transfer function is:

$$G(s) = \frac{S(s)}{E(s)} = \frac{-s+5}{s^2+5s+4}$$

Calculate the time response $s(t)$ of the system when the input $e(t)$ is:

1. A Dirac impulse.
2. A unit step.

SOLUTION

First, factor the denominator:

$$s^2 + 5s + 4 = (s+1)(s+4)$$

Thus,

$$G(s) = \frac{S(s)}{E(s)} = \frac{-s+5}{(s+1)(s+4)}$$

Using partial fraction decomposition:

$$G(s) = \frac{A}{s+1} + \frac{B}{s+4}$$

Solving: $-s + 5 = A(s+4) + B(s+1)$

Comparing coefficients:

$$\begin{cases} A + B = -1 \\ 4A + B = 5 \end{cases} \Rightarrow \left\{ \begin{cases} A = 2 \\ B = -3 \end{cases} \right.$$

So: $G(s) = \frac{2}{s+1} - \frac{3}{s+4}$

- **Response to a Dirac Impulse**

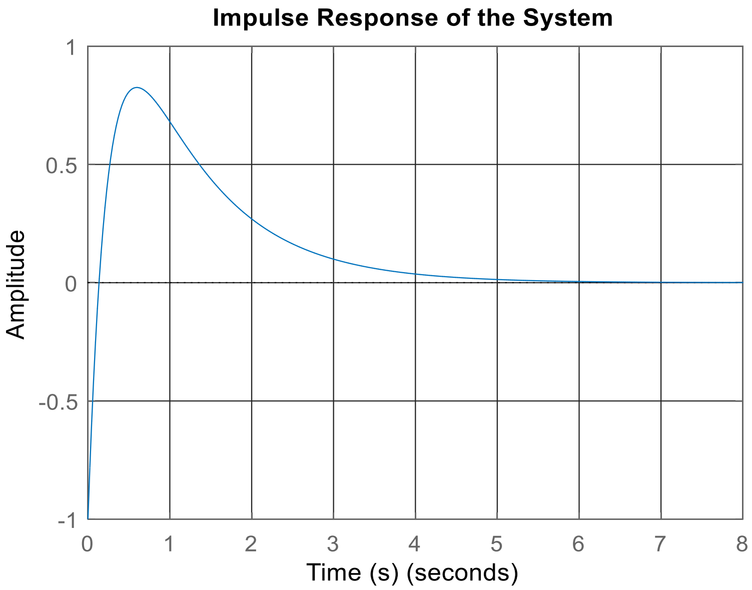
For a Dirac impulse input: $e(t) = \delta(t) \Rightarrow E(s) = 1$

So

$$G(s) = \frac{S(s)}{E(s)} \Rightarrow S(s) = E(s) * G_1(s) \Rightarrow S(s) = G(s)$$

$$s(t) = L^{-1}(G(s))$$

$$s(t) = 2e^{-t} - 3e^{-4t} \qquad t \geq 0$$



➢ **Response to a Unit Step**

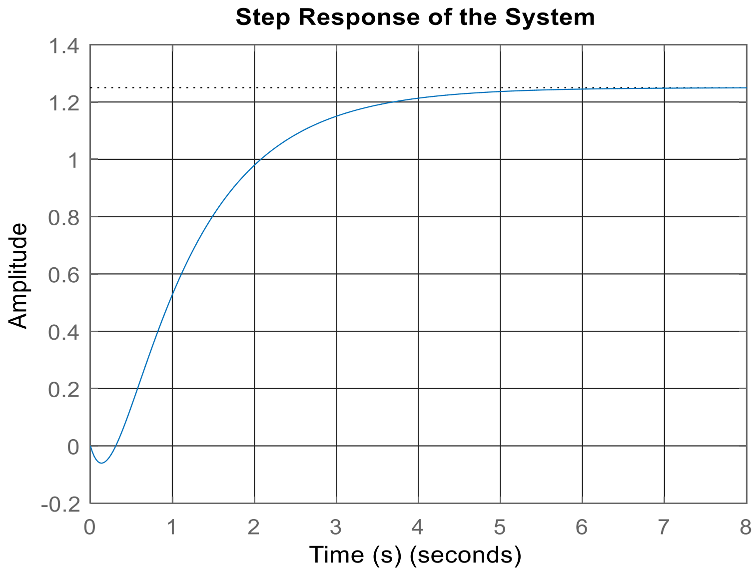
The Laplace transform of a unit step $\boldsymbol{e(t)}$ is :$E(s) = \frac{1}{s}$:
After inverse Laplace transformation, the step response is:

$$G(s) = \frac{S(s)}{E(s)} = \frac{1}{s}\left(\frac{-s+5}{(s+1)(s+4)}\right)$$

Partial Fraction Decomposition $\qquad G(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$

Solving gives: $\begin{cases} A = 5/4 \\ B = -7/3 \\ C = 13/12 \end{cases}$

Applying the inverse Laplace transform: $\quad s(t) = -\frac{5}{4} - \frac{7}{3}e^{-t} + \frac{13}{12}e^{-4t} \qquad t \geq 0$



CHAPTER 06 : BODE AND NYQUIST DIAGRAMS OF CONTROL SYSTEMS

1. INTRODUCTION :

The time-domain analysis of continuous linear time-invariant systems provides useful information about their general behavior and dynamic characteristics. However, this type of analysis is often insufficient to fully describe important properties such as stability margins, robustness, and frequency response.

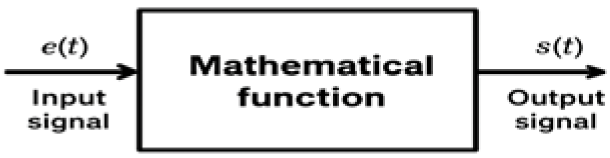
To overcome these limitations, frequency-domain analysis is used as a complementary approach in control engineering. This method studies the system response to sinusoidal inputs and allows a better understanding of how the system behaves over a range of frequencies.

In practice, input signals are rarely simple signals such as steps or ramps. According to Fourier theory, any signal can be represented as a sum of sinusoidal components with different amplitudes and frequencies. Therefore, analyzing the response of a system to sinusoidal inputs is fundamental.

The frequency response of a linear system describes its steady-state behavior when excited by a sinusoidal signal of a given frequency. Graphical tools such as the **Bode diagrams** and the **Nyquist diagram** are widely used to represent this response. They provide essential information for analyzing system stability, performance, and robustness, and are extensively used in the design and analysis of control systems…

2- HARMONIC RESPONSE OF LINEAR SYSTEMS

Consider a continuous-time linear time-invariant (LTI) system with input $e(t)$ and output $s(t)$, governed by a differential equation with constant coefficients:



$$a_n \frac{d^n s(t)}{dt^n} + \cdots + a_1 \frac{ds(t)}{dt} + a_0 s(t) = b_m \frac{d^m e(t)}{dt^m} + \cdots + b_1 \frac{de(t)}{dt} + b_0 e(t)$$

- **Harmonic Response**

If the input is sinusoidal:

$$e(t) = A\sin(wt)$$

then the output in steady state is also sinusoidal:

$$s(t) = B\sin(wt + \varphi)$$

where:

- ✓ B is the output amplitude,
- ✓ φ is the phase shift relative to the input.

The system's frequency response function is defined as:

$$F(jw) = \frac{S(jw)}{E(jw)}$$

Key Quantities

- **System Gain**:

$$G(w) = |F(jw)| = \frac{A}{B}$$

- **System Phase**:

$$\varphi(w) = \arg(F(jw))$$

❖ **Frequency Analysis**

The goal of frequency analysis is to study how the gain **and** phase change with the input frequency w.

Thus, one can represent:

- The magnitude $|F(jw)|$ versus frequency w
- The phase $\varphi(w)$ versus frequency w

These representations form the basis for plotting Bode and Nyquist diagrams, which are essential for:

- Analyzing system stability,
- Studying the system behavior in the frequency domain,
- Designing compensators or controllers effectively.

3- BODE DIAGRAMS

Bode diagrams are graphical representations of a system's frequency response. They illustrate how the magnitude (gain) and phase of a linear time-invariant (LTI) system vary with the input frequency. These diagrams are widely used in control systems and electronics to analyze system behavior and to design stable and robust controllers.

3-1- Purpose of Bode Diagrams

Bode diagrams allow us to visualize the frequency-dependent behavior of a system through two plots:

- **Magnitude Plot**

It represents the system gain:

$$G(w) = 20\log_{10}|F(jw)|$$

expressed in decibels (dB) versus the angular frequency ω\omegaω, usually on a logarithmic scale.

- **Phase Plot**

It represents the phase shift introduced by the system:

$$\varphi(w) = \arg(F(jw))$$

expressed in degrees versus the logarithmic frequency scale.

3-2- Importance of Bode Diagrams

Bode diagrams are essential tools because they allow us to:

- Quickly assess the frequency behavior of a system.
- Identify important frequencies such as cutoff frequencies, resonance peaks, and bandwidth.
- Evaluate the stability and robustness of feedback control systems.
- Facilitate the design of controllers or compensators (PID, lead, lag, etc.).

Thus, Bode diagrams play a crucial role in both analysis and design of linear control systems.

3-3- General Case Example

Consider a general linear time-invariant system described by the transfer function:

$$F(s) = K\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

where:

- K is the system gain,
- z_i are the zeros,
- p_j are the poles.

To obtain the Bode diagrams, we replace s by $j\omega$:

$$F(jw) = K\frac{\prod_{i=1}^{m}(jw+z_i)}{\prod_{j=1}^{n}(jw+p_j)}$$

- **Magnitude**

$$|F(jw)| = |K|\frac{\prod_{i=1}^{m}\sqrt{w^2+{z_i}^2}}{\prod_{j=1}^{n}\sqrt{w^2+{p_j}^2}}$$

In dB:

$$20\log_{10}|F(jw)| = 20\log_{10}|K| + \sum 20\log_{10}\sqrt{w^2+{z_i}^2} - \sum 20\log_{10}\sqrt{w^2+{p_j}^2}$$

❖ **Phase**

$$\varphi(w) = \sum tan^{-1}\left(\frac{w}{z_i}\right) - \sum tan^{-1}\left(\frac{w}{p_j}\right)$$

How to Draw Bode Magnitude and Phase Plots – General Case

4- HOW TO DRAW BODE MAGNITUDE AND PHASE PLOTS – GENERAL CASE

Consider the frequency response:

$$F(jw) = K\frac{\prod_{i=1}^{m}(jw + z_i)}{\prod_{j=1}^{n}(jw + p_j)}$$

where:

- $K > 0$ is the system gain
- $z_i > 0$ are the zeros
- $p_j > 0$ are the poles

4- 1- Rewrite in Bode standard form

Factor each term:

$$jw + z_i = z_i\left(1 + j\frac{w}{z_i}\right), jw + p_j = p_j\left(1 + j\frac{w}{p_j}\right)$$

Thus:

$$F(jw) = K'\frac{\prod_{i=1}^{m}\left(1 + j\frac{w}{z_i}\right)}{\prod_{j=1}^{n}\left(1 + j\frac{w}{p_j}\right)}$$

with:

$$K' = K\frac{\prod z_i}{\prod p_j}$$

The break (corner) frequencies are:

$$w_{z_i} = z_i \;,\; w_{p_j} = p_j$$

4- 2- Drawing the magnitude plot

➢ **Step 1: Initial level**

At low frequency:

$$G(w) = 20\log_{10}\left(K'\right)$$

Draw a horizontal line.

➢ **Step 2: Mark break frequencies**

Draw vertical reference lines at:

$$w = z_1, \dots \dots . z_m \;,\; w = p_1 \dots \dots \dots \dots \dots \dots , p_n$$

➢ **Step 3: Change the slope**

- Each **zero** → slope increases by **+20 dB/decade**
- Each **pole** → slope decreases by **−20 dB/decade**

➢ **Step 4: Final magnitude curve**

The result is a piecewise-linear curve on a logarithmic frequency axis.

4- 3- Drawing the phase plot

➢ **Step 1: Initial phase**

At low frequency:

$$\varphi(w) \approx 0^0$$

➢ **Step 2: Phase contribution of each term**

For each factor $\left(1 + j\frac{w}{a}\right)$

Frequency range	**Phase contribution**
$\boldsymbol{w < 0.1a}$	0^0
(0.1a to 10a)	linear transition
(\omega > 10a)	$+90^0$ (zero) / -90^0 (pole)

➢ **Step 3: Total phase**

$$\varphi(w) = \sum_{i=1}^{m} tan^{-1}\left(\frac{w}{z_i}\right) - \sum_{j=1}^{n} tan^{-1}\left(\frac{w}{p_j}\right)$$

Plot the sum of all phase contributions.

Summary table

Element	Magnitude effect	Phase effect
Gain (K')	Vertical shift	0°
Zero	+20 dB/dec	+90°
Pole	−20 dB/dec	−90°

4- 4- Key rules to remember

- Magnitude slopes change at break frequencies
- Each pole or zero affects two decades of phase
- The total Bode plot is obtained by adding all contributions

EXAMPLE 1 First-Order System

Transfer Function $F(s) = \frac{10}{1+0.5s}$

Identified Parameters

- **Gain:**

$$\mathrm{K} = 10 \;\Rightarrow 20\log_{10}(10) = 20dB$$

- **Time constant:** $T = 0.5$
- **Cutoff frequency:** $w_c = \frac{1}{T} = 2\; rad/s$
- **System type:** first-order low-pass.
- **Frequency Response :** $F(jw) = \frac{10}{1+j0.5w}$

❖ **Magnitude**

$$|F(jw)| = \frac{10}{\sqrt{1+(0.5w)^2}}$$

In dB:

$$\mathrm{G(w)} = 20\log_{10}|F(jw)| = 20\log_{10}(10) - 10\log_{10}(1+(0.5w)^2)$$

❖ **Asymptotic magnitude**

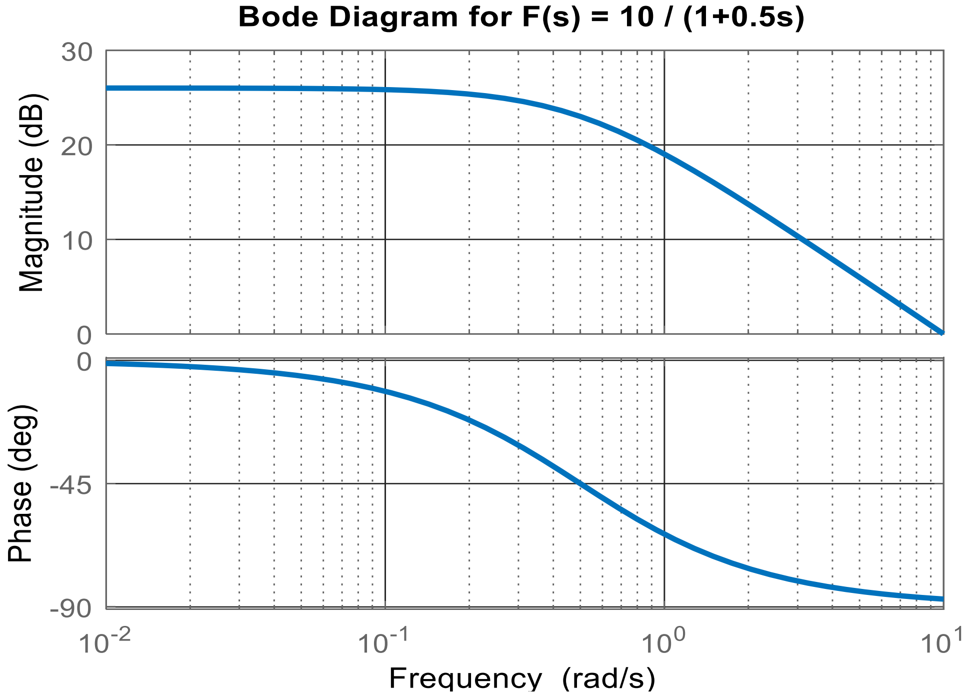
- $\omega < 2$: constant at 20 dB
- $\omega > 2$: slope −20 dB/decade

❖ **Phase**

$$\varphi(w) = -tan^{-1}(0.5w)$$

- **Asymptotic phase**
- $w < 0.2$: 0^0
- $0.2 \le w \le 20$: : transition

- $w > 20: \ -90^0$



EXAMPLE 2 Second-Order System

Transfer Function $F(s) = \frac{10}{(1+s)(1+0.1s)}$

Identified Parameters

- **Gain:**

$$K = 10 \ \Rightarrow 20\log_{10}(10) = 20dB$$

- **Poles::** $\boldsymbol{w_{p1}} = 1$ rad/s , $\boldsymbol{w_{p2}} = 10$ rad/s
- **System type:** second-order low-pass.
- **Frequency Response :** $F(jw) = \frac{10}{(1+jw)(1+0.1jw)}$

❖ **Magnitude**

$$|F(jw)| = \frac{10}{\sqrt{(1+w^2)(1+(0.1w)^2)}}$$

In dB:

$$G(w) = 20\log_{10}|F(jw)| = 20\log_{10}(10) - 10\log_{10}\big((1+w^2)(1+(0.1w)^2)\big)$$

❖ **Asymptotic magnitude**

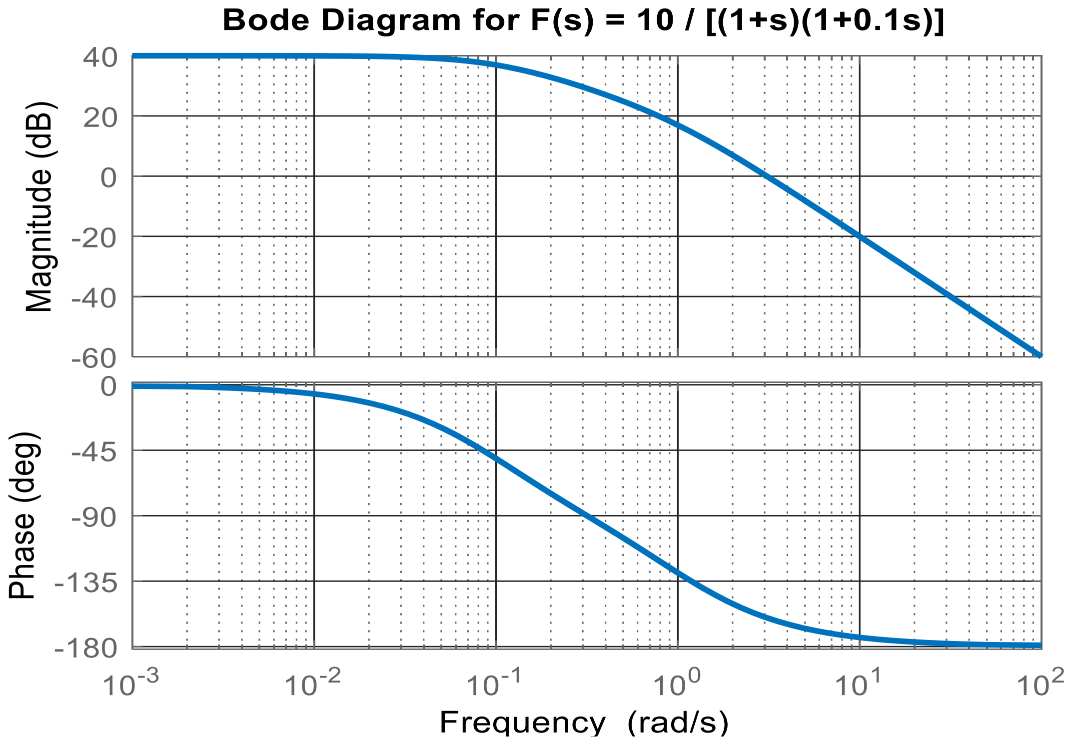
- $\omega < 1$: constant at 20 dB
- $1 < \omega < 10$: slope -20 dB/decade
- $\omega > 10$: slope -40 dB/decade

❖ **Phase**

$$\varphi(w) = -tan^{-1}(w) - tan^{-1}(0.1w)$$

- **Asymptotic phase**
- Initial phase: 0^0
- Each pole contributes : -90^0
- Final phase: -180^0

- **Phase transitions occur around:**
  - 0.1→10 rad/s (first pole)
  - 1→100 rad/s (second pole)



Comparison

System order	Magnitude slope	Final phase
1st order	−20 dB/dec	−90°
2nd order	−40 dB/dec	−180°

5- NYQUIST DIAGRAMS

The Nyquist diagram is a powerful graphical tool used in control systems engineering to analyze the stability and frequency response of linear time-invariant (LTI) systems. Unlike Bode diagrams, which represent the magnitude and phase separately as functions of frequency, the Nyquist diagram presents both characteristics simultaneously in the complex plane.

The Nyquist plot is obtained by mapping the open-loop transfer function as the frequency varies from zero to infinity

One of the main advantages of the Nyquist diagram is its ability to determine the closed-loop stability of a feedback system without explicitly calculating the closed-loop poles. This is achieved through the Nyquist stability criterion, which relates the number of encirclements of the critical point $(-1,0)$ in the complex plane to the number of poles of the open-loop transfer function located in the right half-plane.

Nyquist diagrams are particularly useful for:

- Assessing absolute and relative stability,
- Evaluating gain margin and phase margin,
- Analyzing systems with time delays or high-order dynamics,
- Complementing Bode and root-locus methods in frequency-domain analysis.

6- STEPS TO PLOT A NYQUIST DIAGRAM

6- 1. Determine the Transfer Function

- **Identify the System**: Start by defining the system you want to analyze. This could be an electronic circuit, a controller, or any other type of dynamic system.
- **Write the Transfer Function $F(s)$**: The transfer function represents the relationship between the input and output in terms of the complex variable s:

$$F(s) = \frac{S(s)}{E(s)}$$

6- 2. Calculate the Frequency Response

- **Substitute s**: Replace s with jw to obtain the frequency response

$$F(jw) = \frac{S(jw)}{E(jw)}$$

- **Calculate Values**: Evaluate $F(jw)$ for a range of frequencies ω from 0 to infinity. For example, compute values for $\omega = 0, 0.1, 0.5, 1, 2, 5, 10 \ldots ....$
- **Real and Imaginary Parts**: Express $F(jw)$ in terms of its real and imaginary components for each frequency.

6- 3. Plot the Diagram

- **Axes**: On the graph:

    - The horizontal axis represents the real part Re.
    - The vertical axis represents the imaginary part Im.
- **Plot Points**: For each value of ω, plot the point $(Re(F(jw)), Im(F(jw)))$.
- **Path Following**: As ω, the point traces a path that can be continuous or discontinuous, depending on the system.

6- 4. Close the Trajectory (if necessary)

- For systems with poles in the right half-plane, the curve may not close. If the response becomes periodic, close the path by returning to your initial point.

7- STEPS TO INTERPRET THE NYQUIST DIAGRAM

7- 1. Check System Stability

- **Encirclement of the Point** $(-1, 0)$:
    - Count the number of clockwise encirclements around the point $(-1, 0)$.
    - According to the Nyquist criterion, if a system has P poles in the right half-plane, the number of encirclements Z around the point $(-1, 0)$ must equal P for the system to be stable.

7-2. Analyze Gain and Phase

- **Gain at Key Frequencies**:
    - Identify points where the trace crosses the axes. These points will provide crucial information about the gain (magnitude):
        - If the value at a certain frequency is greater than 1, it indicates amplification.
        - If it is less than 1, it indicates attenuation.
- **Phase**:
    - The slope of the plot provides an indication of the phase. You will need to measure the angle formed by $F(jw)$ to determine the phase at each frequency.

7-3. Examine the Margins

- **Gain Margin**:
    - Determine the gain when the phase is -180° (phase crossover point).

- The gain margin is the difference between this gain and 0 dB.

- **Phase Margin**:
  - Find the phase when the gain is 1 (0 dB).
  - The phase margin is the difference between this phase and -180°.

7-4. Interpret the Results

- **Stability**: A system is stable if the Nyquist criterion is satisfied.
- **Performance**: The gain and phase margins provide insights into robustness, transient response, and frequency performance.

EXAMPLE 1 First-Order System

Transfer Function $F(s) = \frac{10}{1+0.5s}$

- **Step 1:** Frequency-domain representation

Replace ***s*** by ***jω***: $F(jw) = \frac{10}{1+j0.5w}$

- **Step 2:** Separate real and imaginary parts

Multiply numerator and denominator by the complex conjugate:

$$F(jw) = \frac{10(1-0.5jw)}{1+(0.5w)^2}$$

Hence:

$$Re\big(F(jw)\big) = \frac{10}{1+0.25w^2}.$$

$$Im\big(F(jw)\big) = \frac{-5w}{1+0.25w^2}$$

- **Step 3:** Key frequency points

- At $\omega = 0$: $F(j0) = 10$

starting point: $(10, 0)$

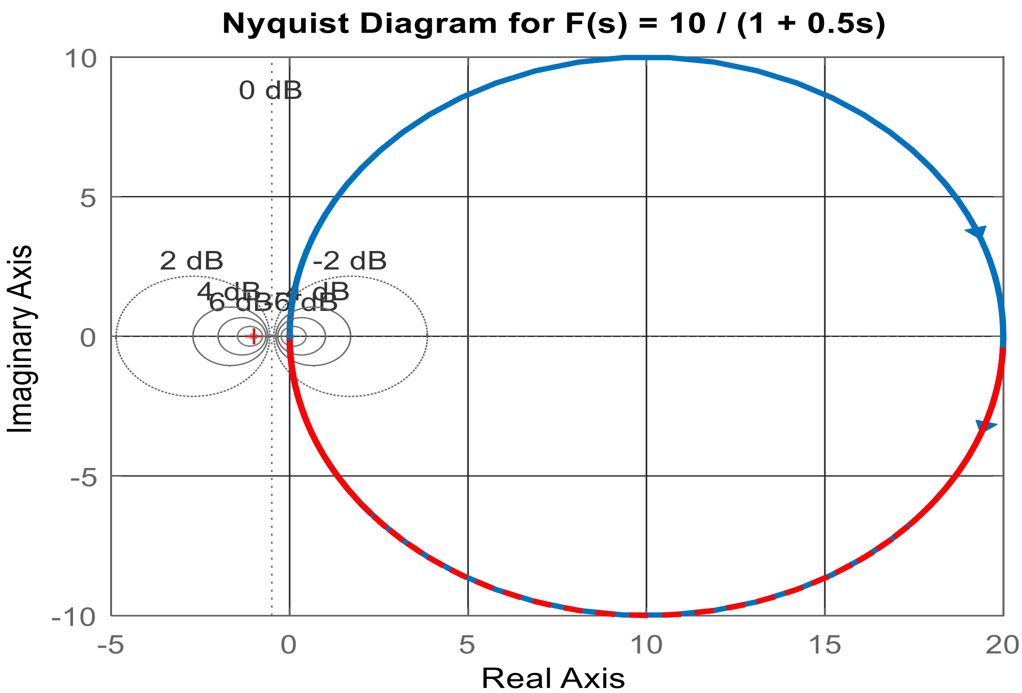
- As $\omega \to \infty$: $F(j\omega) \to 0$

the curve approaches the origin

- **Step 4:** Nyquist plot description
  - The Nyquist curve forms a semi-circle in the complex plane
  - The lower half corresponds to $\omega > 0$
  - The upper half (symmetric) corresponds to $w < 0$

- The critical point $(-1,0)$ is not encircled

Stability conclusion : The closed-loop system is stable.



EXAMPLE 2 Second-Order System

Transfer Function $F(s) = \frac{10}{(1+s)(1+0.1s)}$

- **Step 1:** Frequency-domain representation

Replace ***s*** by ***jω***: $F(jw) = \frac{10}{(1+jw)(1+0.1jw)}$

- **Step 2:** Denominator expansion

$$(1+jw)(1+0.1jw) = (1-0.1w^2) + j(1.1w)$$

Thus:

$$F(jw) = \frac{10}{(1-0.1w^2) + j(1.1w)}$$

- **Step 3:**Frequency behavior

- At $\omega = 0$: $F(j0) = 10$

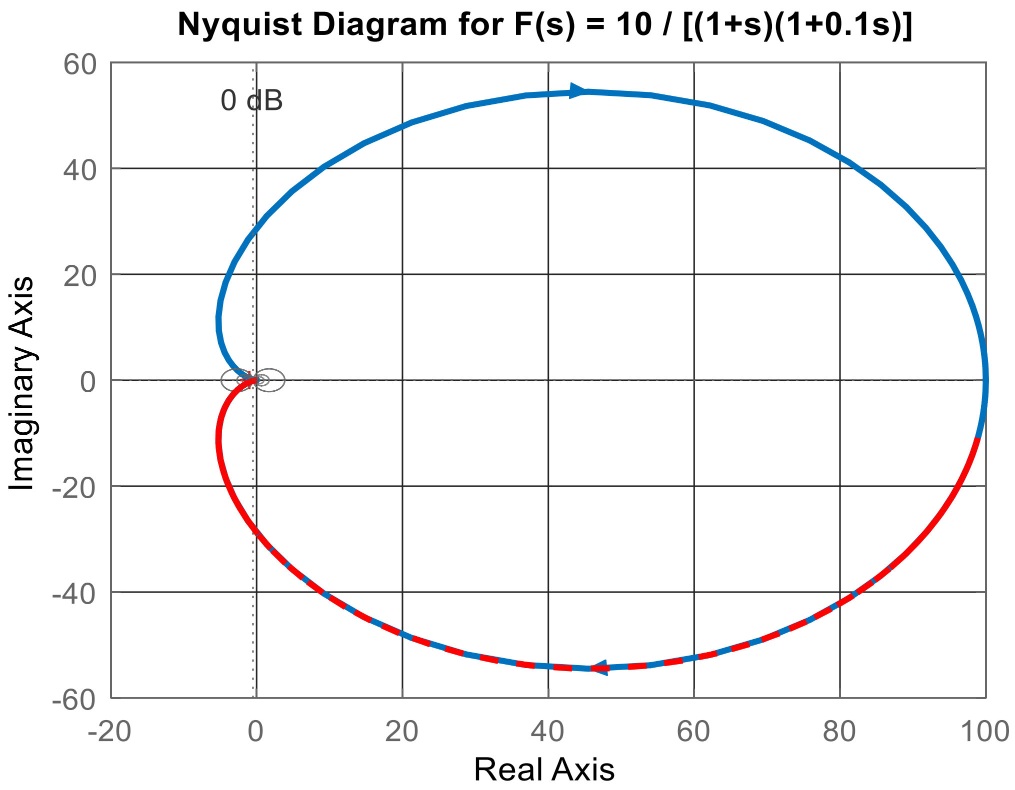
starting point: $(10, 0)$

- As $\omega \to \infty$: $F(j\omega) \to 0$

➢ **Step 4:** Nyquist plot description

- The curve is more distorted due to two poles
- The plot passes closer to the critical point
- There is no encirclement of $(-1, 0)$

Stability conclusion : The closed-loop system is stable, but with smaller stability margins than the first-order system.



CHAPTER 07 : STABILITY STUDY OF CONTROL SYSTEMS

1. INTRODUCTION :

Stability is one of the most important concepts in linear control systems. A control system cannot operate properly if it is unstable. Therefore, the study of stability aims to determine the conditions under which a system remains stable, when it becomes unstable, and how stability can be achieved through proper system design.

In general, stability is defined as the ability of a system to return to its equilibrium state after being subjected to a disturbance. A system is said to be stable if its output remains bounded for any bounded input.

The concept of stability can be easily understood using simple physical examples. If a ball is placed at the bottom of a curved surface, a small displacement will cause it to move and then return to its original position. This situation represents a stable system. If the ball is placed at the top of a curved surface, even a very small displacement will cause it to move away from its equilibrium point. This represents an unstable system. A third case occurs when the ball is placed on a flat surface; after being displaced, it remains in its new position. This situation is known as marginal (neutral) stability.

From a time-domain point of view, the response of a stable system converges to a constant value as time increases. In contrast, the response of an unstable system grows without bound. For a marginally stable system, the response remains constant or oscillates with constant amplitude.

This chapter focuses on the study of stability criteria for control systems. In particular, it introduces the Routh–Hurwitz stability criterion, which is an analytical method based on the characteristic equation, and the Nyquist stability criterion, which is a graphical method used in the frequency domain. These criteria are essential tools for analyzing and designing stable feedback control systems.

2. STABILITY ANALYSIS IN THE COMPLEX PLANE

The stability of a linear closed-loop control system can be determined by examining the location of its closed-loop poles in the complex s-plane. The position of these poles directly influences the time response and the overall behavior of the system.

If any closed-loop pole lies in the right-half of the sss-plane (RHS), whether the pole is real or complex (as illustrated in Figure 1), the system becomes unstable. In such cases, the corresponding modes dominate the system response as time increases. Consequently, the transient response either grows monotonically or oscillates with an increasing amplitude.

For an unstable system, once the power is applied, the output tends to increase with time. In practical situations, if no saturation, damping mechanism, or mechanical limit is present, the system response may grow without bound. Since real physical systems cannot sustain indefinitely increasing responses, such behavior may lead to system damage or complete failure.

Therefore, analyzing the pole locations in the complex plane provides a fundamental and effective method for assessing the stability of linear control systems.

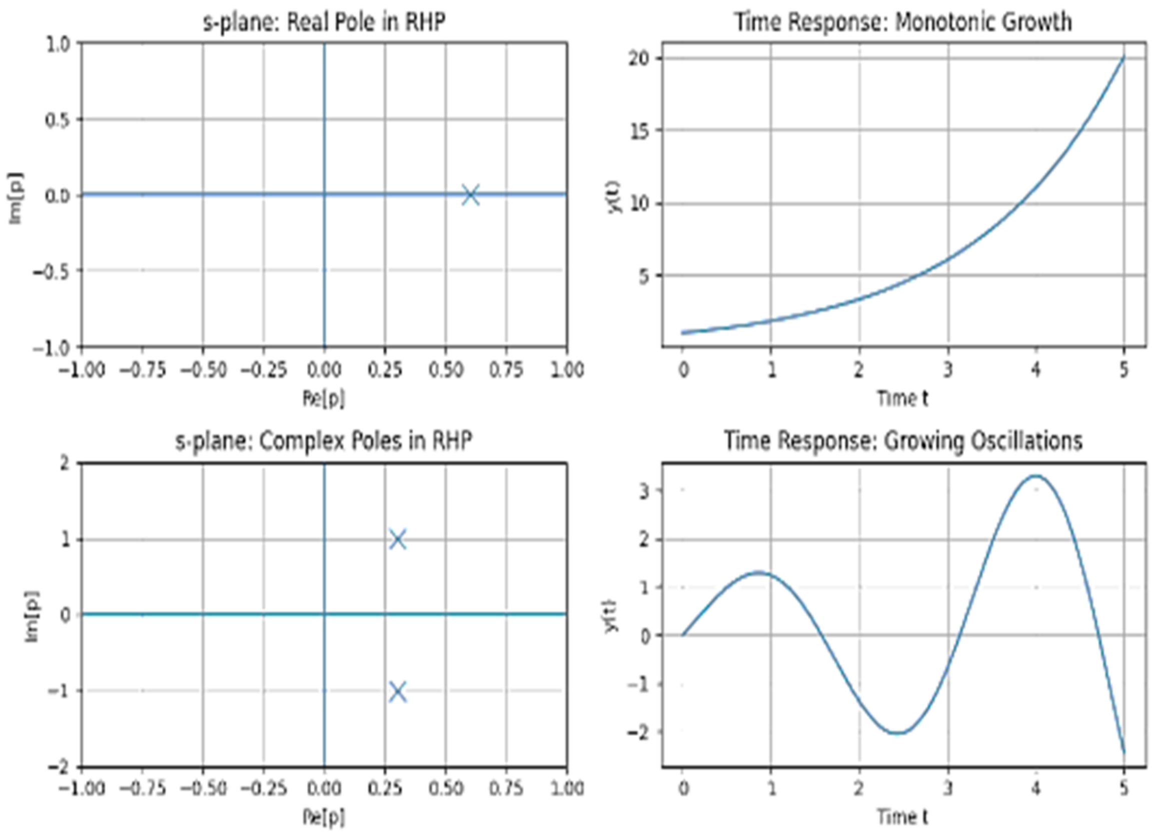


Figure. 7-1. Poles located in RHS gives unstable response

Consider a simple feedback system shown in Figure 7-2

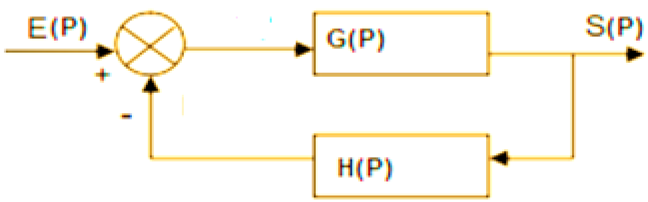


Figure 7-2 closed-loop control system

The Closed-Loop Transfer Function

$$F(s) = \frac{S(s)}{E(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

To study the stability of a control system, the denominator of the closed-loop transfer function must be set to zero in order to determine the poles (roots) of the system. This leads to the characteristic equation: $1 + T(s)$

Where $T(s) = G(s)H(s)$ is the open-loop transfer function.

The above equation is called the characteristic equation of the system.

The roots of the characteristic equation are known as the closed-loop poles. The location of these poles in the complex sss-plane determines the stability of the system, as illustrated in Figure 7-3.

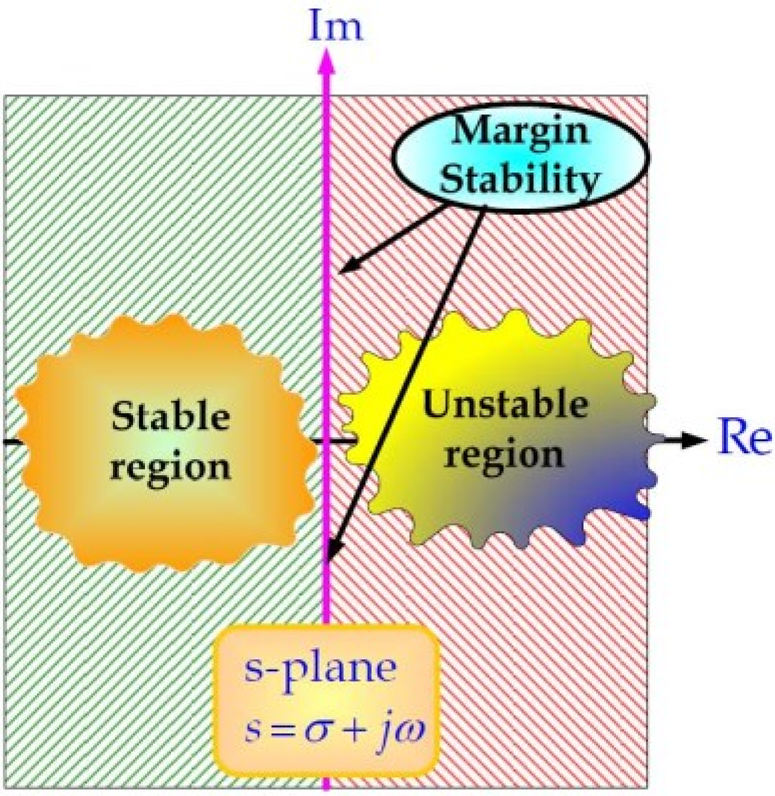


Figure 7-3 Stability condition based on the location of the closed loop poles

3. CLASSIFICATION OF SYSTEMS ACCORDING TO POLE LOCATIONS IN THE COMPLEX PLANE

Figure 7-4 illustrates how the location of system poles in the complex s-plane affects the dynamic behavior and stability of a control system. For a stable system, all poles lie in the left half of the complex plane. This results in a response that decays with time, where the trajectories (shown in green) converge toward the origin. This behavior occurs for both real negative poles and complex conjugate poles.

In the case of a marginally stable (oscillatory) system, the poles are located on the imaginary axis. The system response exhibits sustained oscillations with constant amplitude, represented by circular or elliptical trajectories (shown in blue). In this situation, the response neither converges to nor diverges from the origin.

An unstable system is characterized by the presence of at least one pole in the right half of the complex plane. This leads to a response that grows with time, where the trajectories (shown in red) diverge away from the origin. This behavior may result from real positive poles or complex conjugate poles with positive real parts.

This figure clearly demonstrates that the stability of a control system is directly determined by the location of its poles in the complex plane, which is a fundamental concept in stability analysis.

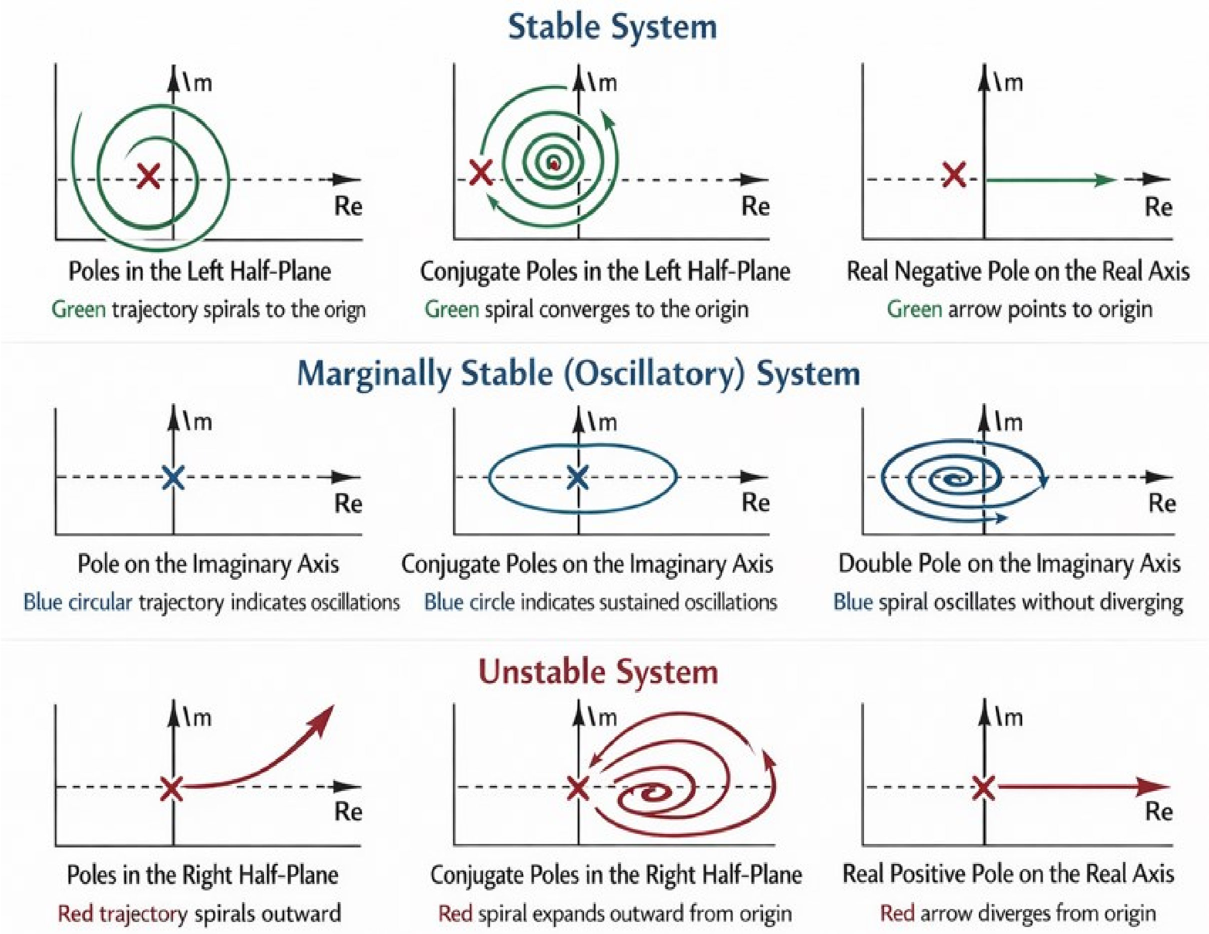


Figure 7-4 Stability Classification Based on Pole Locations in the Complex Plane

4. HURWITZ ALGEBRAIC STABILITY CRITERIA

The Hurwitz algebraic stability criterion provides a systematic and purely mathematical method for determining the stability of linear time-invariant control systems. Unlike graphical techniques, this criterion allows the stability of a system to be assessed directly from the coefficients of its characteristic polynomial, without explicitly computing the system poles. As a result, the Hurwitz criterion is particularly useful for analyzing high-order systems where root calculation may be difficult.

4-1-Characteristic Polynomial

The stability analysis of a linear control system is based on its characteristic polynomial, which is obtained from the denominator of the closed-loop transfer function. It is generally written as:

$$D(s) = 1 + T(s)$$

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

A necessary condition for stability is:

$$a_i > 0 \quad for\ all\ \ i = 0,1,2, \dots \dots \dots \dots n$$

If any coefficient is zero or negative**,** the system is unstable**,** and the Hurwitz criterion cannot be applied.

4-1-Hurwitz Matrix

The Hurwitz matrix is constructed using the coefficients of the polynomial:

$$A = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & 0 \\ 0 & a_n & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

Missing coefficients are replaced by zeros.

Let $\Delta_1 > 0$, $\Delta_2 > 0, \dots \dots \dots \dots \Delta_n > 0$ be the leading principal minors (determinants) of the Hurwitz matrix.

- If all Hurwitz determinants are strictly positive, all roots of the polynomial lie in the left half of the complex plane
- If at least one determinant is zero or negative, the system is unstable

EXAMPLE 1: Stable System

Polynomial:

$$D(s) = s^3 + 6s^2 + 11s$$

- **Step 0: Check coefficients**

  All coefficients: $1,6,11 > 0$ → Necessary condition satisfied.

- **Step 1:** Construct Hurwitz matrix

$$H = \begin{bmatrix} 6 & 6 & 0 \\ 1 & 11 & 0 \\ 0 & 6 & 6 \end{bmatrix}$$

- **Step 2:** Leading principal minors

$$\Delta_1 = 6 > 0, \qquad \Delta_2 = 60 > 0, \Delta_3 = 360 > 0$$

Conclusion: System stable.

EXAMPLE 2: Marginally Stable / Oscillatory System

Polynomial:

$$D(s) = s^2 + 4$$

- **Step 0: Check coefficients**

  All coefficients: 1,0,4 $\rightarrow a_1 = 0 \rightarrow$ Necessary condition fails, system cannot be stable.

- **Step 1:** Construct Hurwitz matrix

$$H = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

  - **Step 2:** Leading principal minors

$$\Delta_1 = 0, \qquad \Delta_2 = -4$$

Conclusion: System unstable, consistent with coefficient check.

EXAMPLE 3: Unstable System

Polynomial:

$$D(s) = s^3 + 2s^2 + s - 2$$

- **Step 0: Check coefficients**

  All coefficients: $1,2,1,-2$ $\rightarrow$ last coefficient negative $\rightarrow$ Necessary condition fails, system cannot be stable.

- **Step 1:** Construct Hurwitz matrix

$$H = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix}$$

  - **Step 2:** Leading principal minors

$$\Delta_1 = 2 > 0, \qquad \Delta_2 = 4 > 0, \Delta_3 = -8 < 0$$

Conclusion: System unstable, consistent with coefficient check.

5. ROUTH STABILITY CRITERION

The Routh Stability Criterion is a mathematical test used in control systems and signal processing to determine whether a linear time-invariant (LTI) system is stable, unstable, or marginally stable — without explicitly solving for the roots of its characteristic equation.

This makes it especially powerful when dealing with high-order polynomials, where finding roots is tedious or impractical.

5. 1. The Characteristic Equation

The starting point for the Routh criterion is the system's characteristic equation, which is the denominator of the closed-loop transfer function set to zero. A general n[th] order characteristic equation is represented as:

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0$$

A Necessary (but not sufficient) Condition for Stability: Before even starting the Routh array, a quick check can sometimes reveal instability. For a polynomial to have all its roots in the LHP, it is necessary that:

- All coefficients (a_n, a_{n-1}..., a_0) have the same sign (they must all be positive, assuming a_n is positive).
- There are no missing terms (i.e., all coefficients from s^n down to $s^0$are non-zero).

If either of these conditions is not met, the system is guaranteed to be unstable or, at best, marginally stable, and you don't need to proceed with the full test. However, if they are met, the system might still be unstable, and you must proceed with the Routh array.

5. 2. Constructing the Routh Array

The Routh criterion involves constructing a special array called the Routh array. The procedure is as follows:

❖ **Form the First Two Rows:**

- The first row consists of the coefficients of the even powers of s, starting with s^n.
- The second row consists of the coefficients of the odd powers of s, starting with s^{n-1}

For the polynomial $D(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0$:

$\boldsymbol{s^n}$	$\boldsymbol{a_n}$	$\boldsymbol{a_{n-2}}$	$\boldsymbol{a_{n-4}}$	**……..**
$\boldsymbol{s^{n-1}}$	a_{n-1}	a_{n-3}	a_{n-5}	…….

❖ **Calculate Subsequent Rows:**

The remaining rows are calculated using a specific pattern based on the two rows immediately preceding them. Let's say we want to calculate the elements of the s^{n-2} row, denoted as $b_1, b_2, b_3 \dots\dots. etc$

s^n	a_n	a_{n-2}	a_{n-4}	……..
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	…….
s^{n-2}	b_1	b_2	b_3	

The formulas are:

$$b_1 = \frac{(a_{n-1}*a_{n-2}-a_n*a_{n-3})}{a_{n-1}}$$

$$b_2 = \frac{(a_{n-1}*a_{n-4}-a_n*a_{n-5})}{a_{n-1}}$$ ...and so on.

This can be visualized as a negative determinant of 2x2 matrices, divided by the first element of the row above:

$$b_1 = -\left(\frac{1}{a_{n-1}}\right) * det([[a_n, a_{n-2}], [a_{n-1}, a_{n-3}]])$$

You continue this process to calculate the s^{n-3} row (using the s^{n-1} and s^{n-2} rows), and so on, until you reach the s^0row. The array will have $n+1$ rows.

5. 3. The Stability Criterion

Once the Routh array is complete, the stability of the system is determined by examining the first column of the array. The Routh Stability Criterion: The number of sign changes in the first column of the Routh array is equal to the number of roots of the characteristic equation that are in the right-half of the s-plane (RHP).

- ❖ If there are no sign changes in the first column, all roots are in the LHP (or on the jω-axis), and the system is stable (or marginally stable).
- ❖ If there are one or more sign changes, the system is unstable, and the number of sign changes tells you exactly how many unstable poles the system has.

EXAMPLE 1: Third-Order Stable System

Given the characteristic polynomial:

$$D(s) = s^3 + 5s^2 + 8s + 4$$

Tasks

1. Verify the necessary condition on the coefficients.
2. Construct the Routh array.
3. Determine the stability of the system.

Solution

❖ **Step 1: Check coefficients**

All coefficients are positive

❖ **Step 2: Routh array**

s^3	1	8
2	5	4
s^1	$\frac{5*8-1*4}{5}=\frac{36}{5}$	0
s^0	4	

❖ **Step 3: First column**

$1,\ 5,\ 36/5,\ 4\ (>0) \Rightarrow$ System is stable

EXAMPLE 1: Third-Order Unstable System

Given the characteristic polynomial:

$$D(s) = s^3 + 2s^2 + s - 3$$

Tasks

1. Check the necessary condition.
2. Construct the Routh array.
3. Determine the number of unstable poles.

Solution

❖ **Step 1: Check coefficients**

Coefficient $-3 < 0\ \rightarrow$ Necessary condition fails.

❖ **Step 2: Routh array**

s^3	1	1
2	2	-3
s^1	$\frac{2*1-1*(-3)}{2}=\frac{5}{2}$	0
s^0	-3	

❖ **Step 3: First column**

$1,\ 2,\ \frac{5}{2},\ -3\ \rightarrow$ One sign change $\Rightarrow\ 1$ unstable pole

EXAMPLE 3: Fourth-Order System with Special Case

Given the characteristic polynomial:

$$D(s) = s^4 + 2s^3 + 3s^2 + 6s + 2$$

Tasks

1. Check the coefficients
2. Construct the Routh array.
3. Comment on the stability.

Solution

❖ **Step 1: Check coefficients**

All coefficients are positive

❖ **Step 2: Routh array**

s^3	1	1	2
s^3	2	6-3	0
s^2	$\frac{2*3-1*6}{2}=0$	$\frac{2*2-1*0}{2}=2$	0
s^1	ζ	0	
s^0	2		

(A small positive ζ is used to replace zero.)

❖ **Step 3: First column**

$1,\ 2, \zeta, \zeta, \mathbf{2}$ Presence of zero $\Rightarrow$ system is not asymptotically stable

The system has roots on the imaginary axis.

6. GEOMETRIC STABILITY CRITERIA OF NYQUIST

The Nyquist stability criterion is a geometric method used to analyze the stability of linear time-invariant (LTI) control systems. Unlike algebraic criteria such as Routh–Hurwitz, the Nyquist criterion studies stability in the frequency domain and is particularly useful for systems with time delays or open-loop unstable poles.

The method is based on the Nyquist plot, which represents the mapping of the open-loop transfer function $T(jw) = G(jw)H(jw)$ in the complex plane as the frequency ω varies from 0 $to\ +\infty$ (and by symmetry from $-\infty\ to$ 0).

6-1. Principle of the Nyquist Criterion

Consider a closed-loop control system with transfer function:

$$F(s) = \frac{G(s)}{1 + G(s)H(s)}$$

The stability of the closed-loop system depends on the roots of the characteristic equation:

$$1 + G(s)H(s) = 0$$

The Nyquist criterion relates:

- the poles of the open-loop transfer function $G(s)H(s)$ in the right-half plane,
- to the encirclements of the critical point $(-1,0)$ in the complex plane.

6-2. Nyquist Criterion Statement

Let:

- P: number of poles of $G(s)H(s)$ in the right-half plane,
- N: number of encirclements of the point $(-1,0)$ by the Nyquist plot (clockwise encirclements are counted as negative),
- Z: number of zeros of $1 + G(s)H(s)$ in the right-half plane (i.e., unstable poles of the closed-loop system).

The fundamental Nyquist relation is:

$$Z = P + N$$

6-3. Stability Condition

The closed-loop system is stable if and only if:

$$Z = 0$$

Therefore:

$$N = -P$$

Important cases:

- If $P = 0$ (open-loop stable system), the Nyquist plot must not encircle the point $(-1,0)$
- If $P > 0$, the Nyquist plot must encircle the point $(-1,0)$ exactly P times in the clockwise direction.

6-4. Geometric Interpretation

- ❖ The critical point (−1,0) corresponds to a loop gain magnitude of unity and a phase shift of −180°.
- ❖ Incorrect encirclement of this point indicates instability.
- ❖ The distance of the Nyquist plot from the point (−1,0) provides insight into the system's relative stability.

EXAMPLE :

Given System

Consider the open-loop transfer function: $G(s) = \frac{K}{s(s+2)}$

We analyze the closed-loop stability using the Nyquist criterion.

- Poles: $s = 0, s = -2$
- Zeros: none

All poles are in the left-half plane or on the imaginary axis
Number of right-half-plane poles:

$P = 0$

ubstitute $s = j\omega$:

$$G(jw) = \frac{K}{jw(jw+2)}$$

Compute the denominator: $j\omega(2 + j\omega) = -w^2 + j2\omega$
Thus:

$$G(jw) = \frac{K}{-w^2 + j2\omega}$$

Multiply numerator and denominator by the complex conjugate:

$$G(jw) = \frac{K(-w^2 - j2\omega)}{w^4 + 4w^2}$$

Separate Real and Imaginary Parts

$$Re\big(G(jw)\big) = \frac{K}{w^2+4}.$$

$$Im\big(G(jw)\big) = \frac{2K}{w(w^2+4)}$$

At low frequency ($\omega \to 0+$)

$$\Re e \to -4K, \qquad Im \to -\infty$$

Nyquist plot starts at negative imaginary infinity

At high frequency ($\boldsymbol{\omega} \to \infty$)

$$\Re e \to 0-, Im \to 0-$$

Nyquist plot approaches the origin from the third quadrant

Real-axis crossing ($\boldsymbol{\Re e = 0}$)

$$-\frac{2K}{w(w^2+4)} = 0$$

This occurs only at: $\omega \to \infty$

No finite real-axis crossing

sketch the Nyquist Plot (Positive Frequencies)

For $\omega > 0$:

- Real part: negative
- Imaginary part: negative

Curve lies entirely in the third quadrant

Using symmetry, the negative-frequency part is the mirror image across the real axis.

The critical point is: (−1,0)

- The Nyquist curve does not encircle the point (−1,0)
- Therefore: $N = 0$

Apply Nyquist Criterion $Z = P + N \Rightarrow Z = 0 + 0 = 0$

$Z = 0 \Rightarrow$Closed-loop system is stable

This result holds for all positive values of K.

7. GEOMETRIC STABILITY CRITERIA OF BODE

The Bode stability criteria are geometric frequency-domain methods used to assess the stability of feedback control systems using the Bode diagrams (magnitude and phase plots). Unlike algebraic methods such as Routh–Hurwitz, Bode criteria provide a visual and quantitative evaluation of relative stability through stability margins.

These criteria are closely related to the Nyquist criterion, but they are often more intuitive and easier to apply in practical control design.

7.1. Principle of the Bode Method

Consider a closed-loop system:

$$F(s) = \frac{G(s)}{1 + G(s)H(s)}$$

The stability analysis is performed on the open-loop transfer function**:**

$$T(s) = G(s)H(s)$$

From the Bode plots of $T(j\omega)$, we examine:

- the magnitude $|\ T(j\omega)\ |$,
- the phase $\angle T(j\omega)$.

7.2. Key Frequencies

- Gain Crossover Frequency $\boldsymbol{w_c}$

$$|\boldsymbol{T(jw_c)}| = \mathbf{1(0\ dB)}$$

- Phase Crossover Frequency $\boldsymbol{w_\varphi}$

$$\angle \boldsymbol{T(j\omega\phi)} = \mathbf{-180^0}$$

7.3. Phase Margin (PM)

The phase margin is defined as:

$$PM = 180^0 + \angle T(jw_c)$$

- $PM > 0^0$: closed-loop system is stable
- $\boldsymbol{PM} = 0^0$: marginally stable
- $\boldsymbol{PM} < 0^0$: unstable

Recommended values: $30^0 \leq PM \leq \mathbf{60^0}$

7.4. Gain Margin (GM)

The gain margin is defined as: $GM = \frac{1}{|T(jw)|}$

In decibels: $GM_{dB} = -20\log_{10}|T(jw)|$

- $GM > 1$ ($or > 0\ dB$): Stable
- $GM = 1$ ($0\ dB$): Marginally stable
- $GM < 1$: unstable

Recommended value: $GM \geq 6\ dB$

7.5. Bode Geometric Stability Criterion

A closed-loop system is stable if and only if:

$$\begin{cases} PM > 0^0 \\ GM > 1 \end{cases}$$

These conditions ensure that the corresponding Nyquist plot does not encircle the critical point $(-1,0)$.

- ❖ The phase margin represents the angular distance to the instability condition.
- ❖ The gain margin represents the radial distance to the critical point.
- ❖ Larger margins indicate better robustness and higher relative stability.

7.6. Advantages of the Bode Stability Criteria

- ❖ Easy graphical interpretation
- ❖ Direct evaluation of stability margins
- ❖ Useful for controller tuning
- ❖ Widely used in industrial control

EXAMPLE :

Consider the open-loop transfer function: $G(s) = \frac{10}{s(1+0.5s)}$

From the Bode plot:

- $w_c \approx 3.2$ rad/s
- Phase at $w_c \approx -135^0$

$$PM = 180^0 - 135^0 = 45^0$$

The closed-loop system is stable with a satisfactory phase margin.

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