PEOPLE's DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf university of Msila
Faculty of Mathematics and computer sciences
Department of Mathematics

Master memory

Field : Mathematics and computer sciences
Branch : Mathematics
Option : Functional Analysis

Theme

Study of some operators on Herz-type Hardy spaces with variable exponents

---

Presented by :
M^S *Amina* TOURI

Publicly supported on : ../09/2020.

In front of the jury composed of :

Aissa DJERIOU	MCA,	university of Msila	**President**.
Rabah HERAIZ	MCA,	university of Msila	**Supervisor.**
Abed Elhamid TALLAB	MCB,	university of Msila	**Examiner.**

University year 2019/2020

Acknowledgements

I cannot begin and finish my work without thanking the greatest and the most powerful "Allah".

The Prophet Muhammed,may Allah bless him and grant him peace said,"*Allah does not thank the person who does not thank people.*"

Foremost, I would like to express my sincere gratitude and appreciation to my supervisor the professor *Rabah HERAIZ* for the continuous support of my study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better supervisor and mentor for my study.

Besides my advisor, I would like to thank the rest of my thesis committee: Prof *Aissa DJERIOU* to accept this task and to give interest to my workand prof *Abed Elhamid TALLAB* to accept being the examiner of this thesis.

Last but not the least, I would like to thank my family: my parents **salah elddine** and **soumia**, for giving birth to me at the first place and supporting me spiritually throughout my life and my only sister **chaima** for encouraged me .

Thanks

Contents

List of abbreviations and symbols

• By $\mathbb{R}^n$ we denote the n-dimensional real Euclidean space.

• By $\mathbb{N}$ we denote the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

• By $\mathbb{Z}$ we denote the set of all integer numbers.

• The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by

$$x \cdot y = x_1 y_1 + ... + x_n y_n.$$

• For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

• The expression $f \lesssim g$ means that $f \leq c\, g$ for some independent constant c (and non-negative functions f and g).

• $f \approx g$ means $f \lesssim g \lesssim f$.

• For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we write $|x| = \sqrt{x_1^2 + ... + x_n^2}$.

• $R(u) := \{x \in \mathbb{R}^n : u/2 \leq |x| < u\}$ and $R_k := R(2^k)$.

• The notation $X \hookrightarrow Y$ stands for continuous embeddings from X to Y, where X and Y are quasi-normed spaces.

• As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.

• For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r.

• "i.e." stands simply for "in other words"

• We use c for various positive constant, i.e. a constant whose value may change from appearance to appearance.

• By supp$f$ we denote the support of the function f , i.e., the closure of its non-zero set.

• If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E .

χ_E denotes its characteristic function.

• By $\mathcal{D}(\mathbb{R}^n)$ we denote the space of functions with continuous derivatives of all orders and compact support.

• By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$. The topology in the complete locally convex space $\mathcal{S}(\mathbb{R}^n)$ is generated by

$$p_N(\varphi) := \sup_{x\in\mathbb{R}^n} (1+|x|)^N \sum_{|\alpha|\leq N} |\partial^\alpha \varphi(x)|, \quad N = 1, 2, 3, ...$$

• By $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$.

• For $0 < p \leq \infty$. The classical *Lebesgue space* $L^p(\Omega)$ is the class of all measurable functions f on Ω normed by (quasi-normed for $p < 1$)

$$\|f \mid L^p(\Omega)\| = \Big(\int_\Omega |f(x)|^p \, dx\Big)^{1/p} < \infty, \quad 0 < p < \infty$$

and

$$\|f \mid L^\infty(\Omega)\| = \operatorname*{ess-sup}_{x\in\Omega} |f(x)| < \infty.$$

• Given $p(\cdot) : \Omega \subset \mathbb{R}^n \to]c, \infty[$, we define the conjugate exponent function $p'(\cdot)$ by the formula

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \ x \in \Omega$$

with the convention that $1/\infty = 0$. Since $p(\cdot)$ is a function, the notation $p'(\cdot)$ can be mistaken for the derivative of $p(\cdot)$, but we will never use the symbol "$'$" in this sense.

• The notation p' will also be used to denote the conjugate of p a constant exponent.

• We define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

• By ℓ^q, $0 < q \leq \infty$, we denote the space of all (complex) sequences $\{a_k\}_{k\in\mathbb{Z}}$ equipped with the quasi-norm

$$\left\|\{a_k\}_{k\in\mathbb{Z}} \mid \ell^q\right\| = \Big(\sum_{k=-\infty}^{\infty} |a_k|^q\Big)^{1/q}$$

(with the usual modification if $q = \infty$).

• Let $L^1_{\text{loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$.

• Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy, \ \forall x \in \mathbb{R}^n.$$

• By $BMO(\mathbb{R}^n)$ be the collection of all locally integrable functions f such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_{Q} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx < \infty,$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Introduction

In this memory, we study Herz-type Hardy spaces with variables exposants (all the three parameters are variables) which are a generalization of classical Herz-type Hardy spaces. Since the early 1990s, these spaces obtained a great development in the past few years and played important roles in harmonic analysis. Recall that function spaces with variable exponents have been intensively studied in the recent years by a significant number of authors.

These spaces based on the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$, with a variable exponent function $p(\cdot) : \Omega \to (0, \infty)$, consists of all measurable functions f such that

$$\int_{\Omega} |f(x)|^{p(x)} < \infty,$$

as a generalization of classical Lebesgue spaces, variable exponent Lebesgue spaces were introduced by Orlicz in 1931.

After this, we introduce Herz spaces with variable exponents, recall that the classical Herz spaces goes back to the authors Beurling and Herz in the sixty's of the last century (Beurling first introduced in 1964 and Herz in 1968 defined this spaces). Based on variable Herz spaces, we present variable Herz-type Hardy.

Also, in this memory, we present the atomic decompositions of variable Herz and variable Herz-type Hardy. Using these decompositions, we present some results of H. Wang, L. Zongguang and F. Zunwei [20] and R. Heraiz [12] concrning integral operators.

First, for $0 < \sigma < n$, for an appropriate function f, the commutator with m-order of fractional integral operators $I^m_{\sigma,b}$, $(m = 1, 2,...)$ is defined by

$$I^m_{\sigma,b}(f)(x) := \int_{\mathbb{R}^n} \frac{(b(x) - b(y))^m}{|x - y|^{n-\sigma}} f(y)dy$$

We denote $I^{1}_{\sigma,b}$ by $[b, I_\sigma]$ and $I^{0}_{\sigma,b}$ by the fractional integral operator I_σ, respectively.

Second, we define Marcinkiewicz integral operator μ by

$$\mu(f)(x) := \left(\int_0^\infty |F_\Omega f(x)|^2 \frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}}$$

where

$$F_\Omega f(x) := \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \ \mathrm{d}y.$$

This class of operators was first defined by Stein in 1958 and under the conditions above, Stein proved that μ is of type (p,p) for $1 < p \leq 2$ and of weak type $(1,1)$. Benedek et al. showed that μ of type (p,p) with $1 < p < \infty$.

In the past two decades, some results concerning integral operators generalized to variable function spaces (in particular, variable Herz-type Hardy spaces).

Our memory consists of three chapters. In the first chapter, we give some basic properties of variable Lebesgue spaces and the mixed variable Lebesgue's-sequence spaces, after this we define variable Herz-type and variable Herz-type Hardy spaces.

In the second chapter, we recall notations and definition of central atomic decomposition and we present the characterizations atomic decomposition of variable Herz-type Hardy spaces and their proof.

In the last chapter, we present some results of H. Wang, L. Zongguang and F. Zunwei [20] and R. Heraiz [12] concerning fractional integral operators I_σ and Marcinkiewicz integral operators μ.

Chapiter 1

Herz-type Hardy spaces with variable exponents

In this chapter, we present Herz-type Hardy with variable exponents which covers classical Herz-type Hardy spaces. We start our first section by recalling some necessary preliminaries on variable Lebesgue spaces and mixed variable Lebesgue's-sequence spaces. After this section, we give some basic properties of variable Herz spaces. In the last section of this chapter, we define Herz-type Hardy spaces with variable exponent, where all the three parameters are variables.

1.1 Variable Lebesgue spaces

In this section we give a definition of the variable Lebesgue spaces which are a generalization of the classical Lebesgue spaces, also we establish their structural properties as Banach function spaces, many properties in classical Lebesgue spaces have been generalized to this spaces.

1.1.1 Variable exponents

In this subsection, we begin with the basic properties and notation of variable exponent. Given an open set $\Omega \subset \mathbb{R}^n$. We put

$$\mathcal{P}_0(\Omega) := \{p \text{ mesurable: } p(\cdot) : \Omega \to [c, \infty[\text{ for some } c > 0\}.$$

The elements of $\mathcal{P}_0(\Omega)$ are called exponent functions or simply exponents.

Notation 1.1.1 *We denote by*

$$\mathcal{P}(\Omega) := \{p \textit{ mesurable: } p(\cdot) : \Omega \subset \mathbb{R}^n \to [1, \infty[\} .$$

Given $p \in \mathcal{P}_0(\Omega)$ *and a set* $E \subseteq \Omega$ *, let*

$$p^-(E) = \operatorname*{ess\,inf}_{x \in E} p(x), \quad p^+(E) = \operatorname*{ess\,sup}_{x \in E} p(x).$$

If the domain $E = \Omega = \mathbb{R}^n$ *we will simply write*

$$p^- = p^-(\mathbb{R}^n), \quad p^+ = p^+(\mathbb{R}^n).$$

Definition 1.1.1 *Given* Ω *and* $p \in \mathcal{P}_0(\Omega)$. *The variable Lebesgue space* $L^{p(\cdot)}(\Omega)$ *to be the set of all measurable functions* f *such that* $\rho_{p(\cdot)}(f/\lambda) < \infty$ *for some* $\lambda > 0$.

$$L^{p(\cdot)}(\Omega) := \left\{ f \textit{ measurable} : \exists \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) = \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} \leq 1 \right\},$$

equipped with the following quasi-norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1 \right\}.$$

If the set on the right-hand side is empty we define $\|f\|_{L^{p(\cdot)}(\Omega)} = \infty$. *If* $\Omega = \mathbb{R}^n$, *we will often write* $\|f\|_{p(\cdot)}$ *instead of* $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Theorem 1.1.1 *Given* Ω *and* $p \in \mathcal{P}_0(\Omega)$. *The function* $\|f\|_{L^{p(\cdot)}(\Omega)}$ *defines a quasi-norm on* $L^{p(\cdot)}(\Omega)$ *and* $L^{p(\cdot)}(\Omega)$ *is quasi Banach spaces.*

Proof. First, we prove that $L^{p(\cdot)}(\Omega)$ is vector spaces.
Since $\rho_{p(\cdot)}(0) = 0$, then we have $0 \in L^{p(\cdot)}(\Omega)$.
Let $f \in L^{p(\cdot)}(\Omega)$ and $\alpha \in \mathbb{R}^*$. There exists $\lambda > 0$ such that

$$\rho_{p(\cdot)}(\lambda f) < \infty.$$

We put $\lambda_0 = \frac{\lambda}{|\alpha|}$.

$$\begin{aligned} \rho_{p(\cdot)}(\lambda_0 \alpha f) &= \rho_{p(\cdot)}(\lambda_0 |\alpha| f) \\ &= \rho_{p(\cdot)}(\lambda f) < \infty. \end{aligned}$$

Which shows that $\alpha f \in L^{p(\cdot)}(\Omega)$. It suffices to show that if $f, g \in L^{p(\cdot)}(\Omega)$, then $f+g \in L^{p(\cdot)}(\Omega)$. By the convexity of $\rho_{p(\cdot)}$,

$$\begin{aligned}\rho_{p(\cdot)}(\lambda(f+g)) &= \rho_{p(\cdot)}\left((\frac{1}{2}2\lambda f + (1-\frac{1}{2})2\lambda g)\right)\\ &\leq \frac{1}{2}\rho_{p(\cdot)}(2\lambda f) + \frac{1}{2}\rho_{p(\cdot)}(2\lambda g) \to 0 \text{ if } \lambda \to 0.\end{aligned}$$

Now, we show that $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ is a norm. Let $f \in L^{p(\cdot)}(\Omega)$. There exists $\mu > 0$ such that $\rho_{p(\cdot)}(\mu f) < 1$. This shows that $\|f\|_{L^{p(\cdot)}(\Omega)} < \infty$.

We also have $\|0\|_{L^{p(\cdot)}(\Omega)} = 0$.

For $\alpha \in \mathbb{R}$, we have:

$$\begin{aligned}\|\alpha f\|_{L^{p(\cdot)}(\Omega)} &= \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{\alpha f}{\lambda}\right) \leq 1\right\}\\ &= \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{|\alpha| f}{\lambda}\right) \leq 1\right\}\\ &= \inf\left\{\tau |\alpha| > 0 : \rho_{p(\cdot)}\left(\frac{f}{\tau}\right) \leq 1\right\}\\ &= |\alpha| \inf\left\{\tau > 0 : \rho_{p(\cdot)}\left(\frac{f}{\tau}\right) \leq 1\right\}\\ &= |\alpha| \|f\|_{L^{p(\cdot)}(\Omega)}.\end{aligned}$$

We now show the triangular inequality. Let $f, g \in L^{p(\cdot)}(\Omega)$ and $\|f\|_{L^{p(\cdot)}(\Omega)} < \gamma$ and $\|g\|_{L^{p(\cdot)}(\Omega)} < \delta$. Then

$$\left\|\frac{f}{\gamma}\right\|_{L^{p(\cdot)}(\Omega)} \leq 1 \text{ and } \left\|\frac{g}{\delta}\right\|_{L^{p(\cdot)}(\Omega)} \leq 1$$

By the convexity of $\rho_{p(\cdot)}$, we have:

$$\begin{aligned}\rho_{p(\cdot)}\left(\frac{f+g}{\gamma+\delta}\right) &= \rho_{p(\cdot)}\left(\frac{\gamma}{\gamma+\delta}\frac{f}{\gamma} + \frac{\delta}{\gamma+\delta}\frac{g}{\delta}\right)\\ &\leq \frac{\gamma}{\gamma+\delta}\rho_{p(\cdot)}\left(\frac{f}{\gamma}\right) + \frac{\delta}{\gamma+\delta}\rho_{p(\cdot)}\left(\frac{g}{\delta}\right)\\ &\leq \frac{\gamma}{\gamma+\delta} + \frac{\delta}{\gamma+\delta} = 1.\end{aligned}$$

Therefore,

$$\|f+g\|_{L^{p(\cdot)}(\Omega)} \leq \gamma + \delta.$$

which implies

$$\|f+g\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{L^{p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)}.$$

If $\|f\|_{L^{p(\cdot)}(\Omega)} = 0$, then $\rho_{p(\cdot)}(\alpha f) \leq 1$ for all $\alpha > 0$. By the convexity of $\rho_{p(\cdot)}$,

$$\begin{aligned} \rho_{p(\cdot)}(\lambda f) &= \rho_{p(\cdot)}\left(\tau \frac{\lambda f}{\tau}\right) \\ &= \rho_{p(\cdot)}\left(\tau \frac{\lambda f}{\tau} + (1-\tau)0\right) \\ &\leq \tau \rho_{p(\cdot)}\left(\frac{\lambda f}{\tau}\right) \\ &\leq 1, \end{aligned}$$

for all $\lambda > 0$ and for all $\tau \in (0;1]$. Then $x = 0$. ■

Definition 1.1.2 *Soient $\Omega \subset \mathbb{R}^n$ et $p \in \mathcal{P}_0(\Omega)$,on définit $L^{p(\cdot)}_{loc}(\Omega)$ par:*

$$L^{p(\cdot)}_{loc}(\Omega) := \left\{ f \text{ muserabe : } f \in L^{p(\cdot)}(K) \text{ pour tout } K \subset \Omega,\ K \text{ compact} \right\}.$$

1.1.2 Logarithmic Hölder continuity

In this subsection, we introduce the most important condition on the exponent in the study of variable exponent spaces, the log-Hölder continuity condition, this condition has emerged as the right one to guarantee regularity.

Definition 1.1.3 *We say that a function $g : \Omega \to \mathbb{R}$ is* locally log-Hölder continuous *on Ω, abbreviated $g \in C^{\log}_{loc}(\Omega)$, if there exists $c_{\log}(g) > 0$ such that*

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\ln(e + 1/|x-y|)} \tag{1.1.1}$$

for all $x, y \in \Omega$. If $0 \in \Omega$ and

$$|g(x) - g(0)| \leq \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \Omega$, then we say that g is log-Hölder continuous at the origin *(or has a* log decay at the origin*). If, for some $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$, there holds*

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \Omega$, then we say that g is log-Hölder continuous at infinity *(or has a* log *decay at infinity), abbreviated $g \in C^{\log}(\Omega)$.*

Remark 1.1.1 *We note that all functions g are* log*-Hölder at infinity always belong to L^{∞}.*

Notation 1.1.2 *The notation $\mathcal{P}^{\log}(\Omega)$ is used for all those exponents $p \in \mathcal{P}(\Omega)$ which are locally* log*-Hölder continuous and have a* log *decay at infinity. The class $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ is defined analogously. If Ω is unbounded, then we define p_∞ by*

$$p_\infty := \lim_{|x| \to \infty} p(x).$$

1.1.3 Fundamental inequalities in variable Lebesgue spaces

The following theorem is the generalization of the classical Hölder's inequality in variable Lebesgue spaces. The classical Hölder's inequality is that for all p, $1 \leq p \leq \infty$, given $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$

$$\int_\Omega |f(x)g(x)|\, dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)} .$$

This inequality is true for variable exponents with a constant on the right-hand side, see for example [4, Theorem 2.33].

Theorem 1.1.2 *Given Ω and $p \in \mathcal{P}(\Omega)$. Then there exists a constant K such that for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leq K \|f\|_{L^{p(\cdot)}(\Omega)} \|f\|_{L^{p'(\cdot)}(\Omega)} .$$

where

$$K = (1/p^- + 1 - 1/p^+).$$

The generalized Hölder's inequality in the classical Lebesgue holds in variable Lebesgue spaces. For the proof of the following corollary, see [3, Corollary 2.28].

Lemma 1.1.1 *Let $\Omega \subset \mathbb{R}^n, p \in \mathcal{P}(\Omega)$ and $s > 0$ such that $sp^- > 1$. Then*

$$\||f|^s\|_{L^{p(\cdot)}(\Omega)} = \|f\|^s_{L^{sp(\cdot)}(\Omega)} .$$

Proof. We have

$$\begin{aligned}\|f\|_{L^{sp(\cdot)}(\Omega)} &= \inf\left\{\lambda>0:\rho_{sp(\cdot)}\left(\frac{f}{\lambda}\right)\leq 1\right\}\\ &= \inf\left\{\lambda>0:\rho_{p(\cdot)}\left(\frac{f^s}{\lambda^s}\right)\leq 1\right\}\\ &= \inf\left\{t^{\frac{1}{s}}>0:\rho_{p(\cdot)}\left(\frac{f^s}{t}\right)\leq 1\right\}\\ &= \||f|^s\|^{\frac{1}{s}}_{L^{p(\cdot)}(\Omega)}.\end{aligned}$$

This finishes the proof. ■

1.2 Variable Herz spaces

In this section, we give the definition of Herz spaces with variable exponent.

1.2.1 The mixed variable Lebesgue-sequence space

In this sebsection, we present a functional spaces create by Alexander Almeida and Peter Hästö, which allows us to define variable Herz spaces. This new spaces as a generalization of the iterated function space $\ell^q(L^{p(\cdot)})$ for the case of variable q.

Definition 1.2.1 *Let $p,q\in\mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular*

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left((f_v)_v\right):=\sum_v \inf\left\{\lambda_v>0:\rho_{p(\cdot)}\left(f_v/\lambda_v^{\frac{1}{q(\cdot)}}\right)\leq 1\right\}.$$

Here we use the convention $\lambda^{\frac{1}{\infty}}=1$. The quasi-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}:=\sum_v \inf\left\{\mu>0:\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right)\leq 1\right\}. \tag{1.2.1}$$

To motivate this definition, we mention that

$$\|(f_v)_v\|_{\ell^q(L^{p(\cdot)})}=\left\|\|(f_v)_v\|_{p(\cdot)}\right\|_{\ell^q}$$

if $q\in(0,\infty]$ is constant.

Remark 1.2.1 *If $q^+ < \infty$, then*

$$\inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(f/\lambda^{\frac{1}{q(\cdot)}}\right) \leq 1\right\} = \left\| |f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Since the right-hand side expression is much simpler, we use this notation to stand for the left-hand side even when $q^+ = \infty$. For instance, we often use the notation

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left((f_v)_v\right) := \sum_v \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}$$

for the modular

The following theorem presents the $\ell^{q(\cdot)}(L^{p(\cdot)})$ quasi-norm, see H. Kempka and J. Vybíral [14, Theorem 1].

Theorem 1.2.1 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a quasi-norm on the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$.*
Let $p, q \in \mathcal{P}(\mathbb{R}^n)$. If $p(x) \geq 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $q \geq 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$ a.e. on $\mathbb{R}^n$, or if $1 \leq q(x) \leq p(x) < \infty$ a.e. on $\mathbb{R}^n$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a norm on the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$.

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$.

The following lemma is a Hardy-type inequality, see [8, Lemma 2].

Lemma 1.2.1 *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ and $\left\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ belong to ℓ^q, and

$$\left\|\{\delta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} + \left\|\{\eta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} \leq c\, I,$$

with $c > 0$ only depending on a and q.

1.2.2 Definition of variable Herz spaces

For convenience, we set

$$B_k := B(0,2^k)\,, \quad R_k := B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}\,, \quad k \in \mathbb{Z}.$$

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various operators in harmonic analysis. In classical L^p spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for $p \in \mathcal{P}^{\log}$ we have

$$\|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \approx |B|. \tag{1.2.2}$$

Also,

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}}, \quad x \in B \tag{1.2.3}$$

for small balls $B \subset \mathbb{R}^n$ ($|B| \leq 2^n$), and

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p_\infty}} \tag{1.2.4}$$

for large balls ($|B| \geq 1$), with constants only depending on the log-Hölder constant of p (see, for example, [5, Section 4.5]).

For characteristic functions defined on (dyadic) annuli we have similar norm estimates, without requiring the log-Hölder continuity at every point.

The following lemma plays an important role in the proof of the main results of this paper, where is a generalization of (1.2.2), (1.2.3) and (1.2.4) to the case of dyadic annuli, see A. Almeida and D. Drihem [1, Lemma 2.2].

Lemma 1.2.2 *Let $p \in \mathcal{P}(\mathbb{R}^n)$ be log-Hölder continuous at infinity, and $R = B(0,r) \backslash B(0,\frac{r}{2})$. If $|R| > 2^{-n}$, then*

$$\| \chi_R \|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}}$$

with the implicit constants independent of r and $x \in R$.
The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, p is log-Hölder continuous, both at the origin and at infinity.

Definition 1.2.2 *Let* $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ *and* $\alpha : \mathbb{R}^n \to \mathbb{R}$. *The* homogeneous Herz space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all* $f \in L_{loc}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\})$ *such that*

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} := \left\| \left(2^{k\alpha(\cdot)} f\,\chi_k\right)_{k\in\mathbb{Z}} \right\|_{\ell^{p(\cdot)}(L^{q(\cdot)})} < \infty. \tag{1.2.5}$$

Obviously, If α and p, q are constant, then $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ are the classical Herz spaces and if $\alpha(\cdot) = 0, p(\cdot) = q(\cdot)$ then $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n)$ coincide with $L^{q(\cdot)}(\mathbb{R}^n)$.

The next two propositions give some basic embeddings between Herz spaces. See [6] and [1, Proposition 3.5].

Proposition 1.2.1 *Let* $p, q, \theta \in \mathcal{P}_0(\mathbb{R}^n)$ *and* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *be log-Hölder continuous, both at the origin and at infinity. If* $(p-\theta)^- \geq 0$, *then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),\theta(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}. \tag{1.2.6}$$

Proof. By a simple consequence of the embedding $\ell^{\theta(\cdot)}(L^{q(\cdot)}) \hookrightarrow \ell^{p(\cdot)}(L^{q(\cdot)})$, we obtain the proof of this proposition. ■

Proposition 1.2.2 *Let* $\alpha \in L^\infty(\mathbb{R}^n)$, $p_0, p_1 \in \mathcal{P}(\mathbb{R}^n)$ *and* $q \in (0, \infty]$. *If* $p_0(\cdot) \leq p_1(\cdot)$ *and* $1/p_0 - 1/p_1$ *be log-Hölder continuous, both at the origin and at infinity, then*

$$\dot{K}_{p_1(\cdot)}^{\alpha(\cdot)+n/p_0(\cdot)-n/p_1(\cdot),q} \hookrightarrow \dot{K}_{p_0(\cdot)}^{\alpha(\cdot),q}.$$

Proof. By Hölder inequality in $L^{p_0(\cdot)}(\mathbb{R}^n)$, we have the following estimate

$$\begin{aligned} \left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p_0(\cdot)} &\leq \left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p_0(\cdot)} \|\chi_k\|_{t(\cdot)} \text{ avec } \frac{1}{p_0(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{t(\cdot)} \\ &\leq \left\|2^{k\alpha(\cdot)} f\chi_k \|\chi_k\|_{t(\cdot)}\right\|_{p_0(\cdot)}, \end{aligned}$$

for any $k \in \mathbb{Z}$. By Lemma 1.2.2, we get the equivalence

$$2^{k\alpha(\cdot)} f\chi_k \|\chi_k\|_{t(\cdot)} \approx 2^{k\alpha(x)} f\chi_k |R_k|^{1/t(x)}$$

where the constants are independent of k and $x \in R_k$.

Since $1/p_0 - 1/p_1$ be log-Hölder continuous, both at the origin and at infinity, then

$$|R_k|^{1/t(x)} \approx 2^{\frac{kn}{t(x)}},$$

we obtain the estimate

$$\left\|2^{k\alpha(\cdot)} f\chi_k \left\|\chi_k\right\|_{t(\cdot)}\right\|_{p_0(\cdot)} \leq c\left\|2^{k(\alpha+n/p_0(\cdot)-n/p_1(\cdot))} f\chi_k\right\|_{p_1(\cdot)}$$

For $f \in \dot{K}_{p_1(\cdot)}^{\alpha(\cdot)+n/p_0(\cdot)-n/p_1(\cdot),q}$, we have

$$\begin{aligned}
\|f\|_{\dot{K}_{p_0(\cdot)}^{\alpha(\cdot),q}} &= \left(\sum_{k=-\infty}^{+\infty}\left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p_0(\cdot)}^q\right)^{\frac{1}{q}} \\
&\leq c\left(\sum_{k=-\infty}^{+\infty}\left\|2^{k(\alpha+n/p_0(\cdot)-n/p_1(\cdot))} f\chi_k\right\|_{p_1(\cdot)}^q\right)^{\frac{1}{q}} \\
&\leq c\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot)+n/p_0(\cdot)-n/p_1(\cdot),q}}.
\end{aligned}$$

■

The next proposition gives a new equivalent norm in $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}$- spaces, see D. Drihem and F. Seghiri in [8, Proposition 1].

Proposition 1.2.3 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are a log-Hölder continuous, both at the origin and at infinity, then*

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} \approx \Big(\sum_{k=-\infty}^{-1}\left\|2^{k\alpha(0)} f\,\chi_k\right\|_{p(\cdot)}^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty}\left\|2^{k\alpha_\infty} f\,\chi_k\right\|_{p(\cdot)}^{q_\infty}\Big)^{1/q_\infty}.$$

Next, we give the following lemma, which is a generalization Hölder inequality in variable Herz spaces [7].

Lemma 1.2.3 *Let $\alpha_i \in L^\infty(\mathbb{R}^n)$ and $p_i, q_i \in \mathcal{P}(\mathbb{R}^n)$, $i = 1, 2$, $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, $1/q(\cdot) = 1/q_1(\cdot) + 1/q_2(\cdot)$. If α_i and q_i are* log-*Hölder continuous, both at the origin and at infininity, Then there exists a constant C such that for all $f \in \dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$ and $g \in \dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)$, $fg \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ and*

$$\|fg\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\|g\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2(\cdot)}(\mathbb{R}^n)}.$$

Proof. The proof follows immediately by applying Proposition 1.2.3, and Hölder's inequality in $\ell^{q(0)}(\mathbb{R}^n)$, $\ell^{q_\infty}(\mathbb{R}^n)$ and in $L^{p(\cdot)}(\mathbb{R}^n)$. ■

Remark 1.2.2 *If the exposants are constants, this result is from [9, Lemma (Hölder's inequality)].*

1.3 Variable Herz-type Hardy spaces

The Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent is the set of all tempered distributions on $\mathbb{R}^n$ for which $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} := \|\mathcal{M}(f)\|_{p(\cdot)}$ is finite. Based on this spaces, we introduce Herz-type Hardy spaces which obtained a great development in the past few years and played important roles in harmonic analysis.

1.3.1 Definition and basic properties of variable Herz-type Hardy spaces

In this subsection, we will give the definition of Herz-type Hardy spaces with variable exponent $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}$.

Definition 1.3.1 *Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with supp $\varphi \subseteq B_0$, such that*

$$\int_{\mathbb{R}^n} \varphi(x)dx \neq 0 \text{ and } \varphi_t(\cdot) = t^{-n}\varphi\left(\frac{\cdot}{t}\right) \text{ for any } t > 0.$$

Let $\mathcal{M}_\varphi(f)$ be the grand maximal function of f defined by

$$\mathcal{M}_\varphi(f)(x) := \sup_{t>0} |\varphi_t * f(x)|.$$

The variable Herz-type Hardy spaces $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}$ are defined in the following way.

Definition 1.3.2 *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The* homogeneous Herz-type Hardy space $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{M}_\varphi(f) \in \dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ and we put*

$$\|f\|_{H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}} := \|\mathcal{M}_\varphi(f)\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}}.$$

The spaces $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}$, where the three parameters are variables, have been first studied in [8]. Many of the results from the fixed situation have variable counterparts. If $q \in \mathcal{P}_0(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}$ with $1 < p^- \leq p^+ < \infty$, and let α and q are log-Hölder continuous, both at the origin and at infinity, such that $\alpha \in L^\infty(\mathbb{R}^n)$ and

$$-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n\big(1 - \frac{1}{p^-}\big),$$

then

$$H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}\left(\mathbb{R}^n\right)\cap L_{loc}^{p(\cdot)}\left(\mathbb{R}^n\setminus\{0\}\right)=\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}\left(\mathbb{R}^n\right).$$

One recognizes immediately that if α, p and q are constants, then the spaces $H\dot{K}_p^{\alpha,q}$ are just the usual Herz-type Hardy spaces were recently studied in [10] and [16]. See [11] for further results.

Chapiter 2

Atomic decomposition of variable Herz and variable Herz-type Hardy spaces

Atomic decomposition plays a fundamental role in the harmonic analysis, it is a powerful tool for dealing with duality theorems, interpolation theorems and some fundamental inequalities in harmonic analysis.

2.1 Atomic decomposition of variable Herz spaces

In this section, we establish characterizations of the $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$-spaces in terms of central atomic decompositions, which make it convenient to study the boundedness of singular integral operators and Marcinkiewcz integras operators in the next chapter.

Definition 2.1.1 *Let* $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$ *and* $s \in \mathbb{N}_0$. *A function* a *is said to be a central* $(\alpha(\cdot), p(\cdot))$*-atom, if*

(i) $\operatorname{supp} a \subset \overline{B}(0,r) = \{x \in \mathbb{R}^n : |x| \leq r\}, r > 0.$

(ii) $\|a\|_{p(\cdot)} \leq |\overline{B}(0,r)|^{-\alpha(0)/n}, \qquad 0 < r < 1.$

(iii) $\|a\|_{p(\cdot)} \leq |\overline{B}(0,r)|^{-\alpha_\infty/n}, \qquad r \geq 1.$

(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \quad |\beta| \leq s.$

Remark 2.1.1 *A function a on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies the conditions (iii), (vi) above and* $\operatorname{supp} a \subset \overline{B}(0,r), r \geq 1$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 2.1.1, then the corresponding central $(\alpha(\cdot), p(\cdot))$-atom is called a dyadic central $(\alpha(\cdot), p(\cdot))$-atom.

The following theorem is proved by D. Drihem and R. Heraiz in [7] which is a generalization of atomic decomposition of variable Herz spaces if α, p and q are not all variables.

Theorem 2.1.1 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q are be* log*-Hölder continuous, both at the origin and at infinity with $p^+, q^+ < \infty$, the following two statements are equivalentes*
(i)- $f \in \dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$
(ii)- f can be represented by

$$f(x) = \sum_{k=-\infty}^{+\infty} \lambda_k a_k(x), \tag{2.1.1}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in B_k and

$$\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{+\infty} |\lambda_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}} \leq c \, \|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}.$$

Moreover, the norms $\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}}$ and $\inf\Big(\big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\big)^{\frac{1}{q(0)}} + \big(\sum_{k=0}^{+\infty} |\lambda_k|^{q_\infty}\big)^{\frac{1}{q_\infty}}\Big)$ are equivalent, where the infimum is taken all over all decompositions of f as in (2.1.1).

Proof. We first prove (i) implies (ii). For every $f \in \dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, write

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{+\infty} f(x)\chi_k(x) \\ &= \sum_{k=-\infty}^{+\infty} \left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)} \frac{f(x)\chi_k(x)}{\|2^{k\alpha(\cdot)} f\chi_k\|_{p(\cdot)}} = \sum_{k=-\infty}^{+\infty} \lambda_k a_k(x), \end{aligned}$$

where $\lambda_k = \left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}$ and $a_k(x) = \frac{f(x)\chi_k(x)}{\left\|2^{k\alpha(\cdot)} f\chi_k\right\|_{p(\cdot)}}$. It is obvious that $\operatorname{supp} a_k \subset B_k$ and

$$\|a_k\|_{p(\cdot)} \approx \begin{cases} 2^{-k\alpha(0)}, & \text{if } k \leq -1 \\ 2^{-k\alpha_\infty}, & \text{if } k \geq 0 \end{cases}.$$

Thus, each a_k is a central $(\alpha(\cdot), p(\cdot))$- atoms with the support B_k and

$$\begin{aligned}
&\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{+\infty} |\lambda_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
= &\Big(\sum_{k=-\infty}^{-1} \big\|2^{k\alpha(\cdot)} f\chi_k\big\|_{p(\cdot)}^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{+\infty} \big\|2^{k\alpha(\cdot)} f\chi_k\big\|_{p(\cdot)}^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
\approx &\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \|f\chi_k\|_{p(\cdot)}^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{+\infty} 2^{k\alpha_\infty q_\infty} \|f\chi_k\|_{p(\cdot)}^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
\approx &\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}.
\end{aligned}$$

Now we prove (ii) implies (i). Let $f(x) = \sum_{k=-\infty}^{+\infty} \lambda_k a_k(x)$ be a decomposition of f which satisfies the hypothesis (ii) of Theorem 2.1.1. For each $j \in \mathbb{Z}$, by the Minkowski inequality

$$\big\|f\chi_j\big\|_{p(\cdot)} \leq \sum_{k=j}^{\infty} |\lambda_k| \, \|a_k\|_{p(\cdot)}. \tag{2.1.2}$$

By Proposition 1.2.3 and from (2.1.2), it follows that $\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is bounded by

$$\begin{aligned}
&\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)}\Big(\sum_{j=k}^{+\infty} |\lambda_j| \, \|a_j\|_{p(\cdot)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{+\infty} 2^{k\alpha_\infty q_\infty}\Big(\sum_{j=k}^{+\infty} |\lambda_j| \, \|a_j\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\
= \; &I_1 + I_2.
\end{aligned}$$

For I_1, we divide the sum $\sum_{j=k}^{+\infty} \cdots$ into two parts,

$$\sum_{j=k}^{-1} \cdots + \sum_{j=0}^{+\infty} \cdots.$$

I_1 is bounded by $I_1^a + I_1^b$, where

$$I_1^a = \Big(\sum_{k=-\infty}^{-1} \Big(2^{k\alpha(0)} \sum_{j=k}^{-1} |\lambda_j| \, \|a_j\|_{p(\cdot)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}}$$

and

$$I_1^b = \Big(\sum_{k=-\infty}^{-1} \Big(2^{k\alpha(0)} \sum_{j=0}^{+\infty} |\lambda_j| \, \|a_j\|_{p(\cdot)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}}.$$

Since $0 < \alpha(0) < \infty$, then by Lemma 1.2.1 (with $0 < a = 2^{-\alpha(0)} < 1$), we have

$$I_1^a \leq c\Big(\sum_{k=-\infty}^{-1} \Big(\sum_{j=k}^{-1} |\lambda_j| 2^{-(j-k)\alpha(0)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \leq c\Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\Big)^{\frac{1}{q(0)}}.$$

Since $0 < \alpha_\infty < \infty$, then by the embedding $\ell^{q_\infty} \hookrightarrow \ell^\infty$

$$\begin{aligned} I_1^b &\leq c\Big(\sum_{k=-\infty}^{-1}\Big(2^{k\alpha(0)}\sum_{j=0}^{+\infty}|\lambda_j|2^{-j\alpha_\infty}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \\ &\leq c\sup_{j\geq 0}|\lambda_j|\Big(\sum_{k=-\infty}^{-1}\Big(2^{k\alpha(0)}\sum_{j=0}^{+\infty}2^{-j\alpha_\infty}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}} \leq c\Big(\sum_{k=0}^{+\infty}|\lambda_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}}. \end{aligned}$$

Thus, we have the desired estimate for I_1.

For I_2, we have

$$I_2 = \Big(\sum_{k=0}^{+\infty}\Big(2^{k\alpha_\infty}\sum_{j=k}^{+\infty}|\lambda_j|\,\|a_j\|_{p(\cdot)}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}} \leq \Big(\sum_{k=0}^{+\infty}\Big(\sum_{j=k}^{+\infty}|\lambda_j|2^{-(j-k)\alpha_\infty}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}}.$$

Since $0 < \alpha_\infty < \infty$, then by Lemma 1.2.1 (with $0 < a = 2^{-\alpha_\infty} < 1$), we have

$$I_2 \leq \Big(\sum_{k=0}^{+\infty}|\lambda_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}} \leq \Big(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k=0}^{+\infty}|\lambda_k|^{q_\infty}\Big)^{\frac{1}{q_\infty}}.$$

This finishes the proof. ■

2.2 Atomic decomposition of variable Herz-type Hardy spaces

In this section, we present the characterization of Herz-type Hardy spaces in term of central atomic decompositions, which use to study the boundedness of Marcinkiewcz integrals operator on these spaces, the following theorem is [8, Theorem 4].

Theorem 2.2.1 *Let α and q are be* log*-Hölder continuous, both at the origin and at infinity and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty}\lambda_k a_k,$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with supp$a \subset B_k$ *and*

$$\Bigg(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Bigg)^{1/q(0)} + \Bigg(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Bigg)^{1/q_\infty} \leq c\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha(\cdot) \geq n(1-\frac{1}{p^-})$ *and* $s \geq [\alpha^+ + n(\frac{1}{p^-}-1)]$*, and if holds, then* $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$*, and*

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}} \approx \inf\left\{\Big(\sum_{k=-\infty}^{-1}|\lambda_k|^{q(0)}\Big)^{1/q(0)} + \Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f *as above.*

Chapiter 3

Boundedness of some operators on variable exponent Herz-type Hardy spaces

In this chapter, we present some results of H. Wang, L. Zongguang and F. Zunwei [20] and R. Heraiz [12] concerning fractional integral operators I_σ and Marcinkiewicz integral operators μ which are the boundedness of this class of operators from variable Herz spaces into itself when the parameters α, p and q satisfies some conditions. Also, we present the boundedness of this operators on variable Herz-type Hardy spaces..

3.1 Boundedness of fractional integral and their commutators on Variable Herz-type Hardy spaces

In this section, we present the boundedness of fractional integral operators and their commutators on variable Herz-type Hardy spaces. We only study the case where pne parameter p is variable.

3.1.1 Some lemmas

Here we present three lemmas used to prove the results of H. Wang, L. Zongguang and F. Zunwei [20]

The three following lemmas are from [20, Lemma 0.5].

Lemma 3.1.1 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, k be a positive integer and B be a ball in $\mathbb{R}^n$. Then we that for all $b \in BMO(\mathbb{R}^n)$and all $i, j \in \mathbb{Z}$ with $j > i$,*

$$\frac{1}{c}\|b\|_{BMO}^k \leq \sup_B \frac{1}{\|\chi_B\|_{p(\cdot)}}\|(b-b_B)^k\chi_B\|_{p(\cdot)} \leq c\|b\|_{BMO}^k,$$

$$\|(b-b_{B_i})^k\chi_{B_j}\|_{p(\cdot)} \leq c(j-i)^k\|b\|_{BMO}^k\|\chi_{B_j}\|_{p(\cdot)}$$

Lemma 3.1.2 *Suppose that $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_1^+ < \frac{n}{\sigma}$ and $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\sigma}{n}$. Then for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\|I_\sigma(f)\|_{p_2(\cdot)} \leq c\|f\|_{p_1(\cdot)}.$$

Lemma 3.1.3 *Suppose that $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_1^+ < \frac{n}{\sigma}$, $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\sigma}{n}$ and $b \in BMO(\mathbb{R}^n)$. Then for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\|[b, I_\sigma](f)\|_{p_2(\cdot)} \leq C\|b\|_{BMO}\|f\|_{p_1(\cdot)}.$$

3.1.2 Variable Herz estimate of fractional integral operators

First, we treat the boundedness of I_σ on variable Herz-type Hardy spaces.

Theorem 3.1.1 *Suppose that $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_1^+ < \frac{n}{\sigma}$ and $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\sigma}{n}$, $\alpha \in L^\infty(\mathbb{R}^n)$. Let $0 < q_1 \leq q_2 < \infty$ and $n(1 - \frac{1}{p_1^-}) \leq \alpha < \infty$. Then I_σ is bounded from $H\dot{K}_{p_1(\cdot)}^{\alpha,q_1}(\mathbb{R}^n)$ to $\dot{K}_{p_2(\cdot)}^{\alpha,q_2}(\mathbb{R}^n)$.*

Proof of Theorem 3.1.1. We must show that

$$\|I_\sigma(f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha,q_2}(\mathbb{R}^n)} \leq c\|f\|_{H\dot{K}_{p_1(\cdot)}^{\alpha,q_1}(\mathbb{R}^n)}$$

for all $f \in H\dot{K}_{p_1(\cdot)}^{\alpha,q_1}(\mathbb{R}^n)$. Using Theorem 2.2.1, we may assume that

$$f = \sum_{i=-\infty}^{+\infty} \lambda_i a_i$$

where $\lambda_i \geq 0$ and a_i's are $(\alpha(\cdot), p_1(\cdot))$- atom with supp$a_i \subseteq B_i$. Using the definition of $\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)$-norm, we have

$$\begin{aligned}
\|I_\sigma(f)\|_{\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{+\infty} 2^{k\alpha q_2} \|I_\sigma(f)\chi_k\|^{q_2}_{p_2(\cdot)}\right\}^{1/q_2} \\
&\leq \left\{\sum_{k=-\infty}^{+\infty} 2^{k\alpha q_2} \left(\sum_{i=-\infty}^{k-2} |\lambda_i| \|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2}\right\}^{1/q_2} \\
&\quad + \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha q_2} \left(\sum_{i=k-1}^{+\infty} |\lambda_i| \|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2}\right\}^{1/q_2} \\
&= : E_1 + E_2.
\end{aligned}$$

Now we estimate E_1. ,We can subtract the Taylor expansion of $|x-y|^{-n+\sigma}$ at x, we obtain

$$\begin{aligned}
|I_\sigma(a_i)(x)| &\leq \int_{B_i} \frac{|a_i(y)|\,|y|^{s+1}}{|x|^{n-\sigma+s+1}} dy \\
&\leq c2^{-k(n-\sigma+s+1)+i(s+1)} \int_{B_i} |a_i(y)|\,dy,
\end{aligned}$$

Applying Hölder inequality, we get

$$|I_\sigma(a_i)(x)| \leq c2^{-k(n-\sigma+s+1)+i(s+1)} \|a_i\|_{p_1(\cdot)} \left\|\chi_{B_i}\right\|_{p_1'(\cdot)}. \tag{3.1.1}$$

On the other hand

$$I_\sigma(\chi_{B_k})(x) \geq \int_{B_k} \frac{dy}{|x-y|^{n-\sigma}} \chi_{B_k}(x) \geq c2^{k\sigma}\chi_{B_k}(x). \tag{3.1.2}$$

By (3.1.1), (3.1.2) and Lemma 3.1.2, gives

$$\begin{aligned}
\|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)} &\leq c2^{-k(n-\sigma+s+1)+i(s+1)} \|a_i\|_{p_1(\cdot)} \left\|\chi_{B_i}\right\|_{p_1'(\cdot)} \|\chi_k\|_{p_2(\cdot)} \\
&\leq c2^{-k(n+s+1)+i(s+1)} \|a_i\|_{p_1(\cdot)} \left\|\chi_{B_i}\right\|_{p_1'(\cdot)} \left\|I_\sigma(\chi_{B_k})\right\|_{p_2(\cdot)} \\
&\leq c2^{-k(n+s+1)+i(s+1)} \|a_i\|_{p_1(\cdot)} \left\|\chi_{B_i}\right\|_{p_1'(\cdot)} \left\|\chi_{B_k}\right\|_{p_1(\cdot)}
\end{aligned}$$

Thus we obtain

$$E_1 \leq c\left\{\sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-2} |\lambda_i|\, 2^{(i-k)(n+s+1-\alpha-n/p_{1(0)})}\right)^{q_2}\right\}^{1/q_2},$$

• if $1 < q_1 < \infty$, since $s+1-\alpha^+ + n(1-\frac{1}{p_1^-}) > 0$ and $q_1 \leq q_2$, then by the Hölder inequality we have

$$\begin{aligned}
E_1 &\leq c\left\{\sum_{k=-\infty}^{+\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|^{q_1}\, 2^{(i-k)(n+s+1-\alpha-n/p_{1(0)})\frac{q_1}{2}}\right)^{\frac{q_2}{q_1}}\left(\sum_{i=-\infty}^{k-2}2^{(i-k)(n+s+1-\alpha-n/p_{1(0)})\frac{q_1'}{2}}\right)^{\frac{q_2}{q_1'}}\right\}^{1/q_2}\\
&\leq c\left\{\sum_{k=-\infty}^{+\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|^{q_1}\, 2^{(i-k)(n+s+1-\alpha-n/p_{1(0)})\frac{q_1}{2}}\right)\left(\sum_{i=-\infty}^{k-2}2^{(i-k)(n+s+1-\alpha-n/p_{1(0)})\frac{q_1'}{2}}\right)^{\frac{q_1}{q_1'}}\right\}^{1/q_1}\\
&\leq c\left\{\sum_{k=-\infty}^{+\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|^{q_1}\, 2^{(i-k)(n+s+1-\alpha-n/p_{1(0)})\frac{q_1}{2}}\right)\right\}^{1/q_1}\\
&= c\left\{\sum_{j=-\infty}^{+\infty}|\lambda_j|^{q_1}\left(\sum_{k=j+2}^{+\infty}2^{(j-k)(n+s+1-\alpha-n/p_{1(0)})\frac{q_1}{2}}\right)\right\}^{1/q_1}\\
&\leq c\left\{\sum_{j=-\infty}^{+\infty}|\lambda_j|^{q_1}\right\}^{1/q_1}.
\end{aligned}$$

By the atomic decomposition of Herz-type Hardy spaces, we have

$$E_1 \leq c\left(\sum_{k=-\infty}^{+\infty}|\lambda_k|^{q_1}\right)^{1/q_1} \leq c\|f\|_{H\dot{K}^{\alpha,q_1}_{p_1(\cdot)}}.$$

• if $0 < q_1 \leq 1$, by the injection $\ell_{q_1} \hookrightarrow \ell_1$, we have

$$\begin{aligned}
E_1 &\leq c\left\{\sum_{k=-\infty}^{+\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\, 2^{(i-k)(n+s+1-\alpha-n/p_{1(0)})}\right)^{q_2}\right\}^{1/q_2}\\
&\leq c\left\{\sum_{k=-\infty}^{+\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|^{q_1}\, 2^{(i-k)(n+s+1-\alpha-n/p_{1(0)})q_1}\right)^{\frac{q_2}{q_1}}\right\}^{1/q_2}\\
&\leq c\left\{\sum_{k=-\infty}^{+\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|^{q_1}\, 2^{(i-k)(n+s+1-\alpha-n/p_{1(0)})q_1}\right)\right\}^{1/q_1}\\
&= c\left\{\sum_{j=-\infty}^{+\infty}|\lambda_j|^{q_1}\left(\sum_{k=j+2}^{+\infty}2^{(j-k)(n+s+1-\alpha-n/p_{1(0)})q_1}\right)\right\}^{1/q_1}\\
&\leq c\left(\sum_{k=-\infty}^{+\infty}|\lambda_k|^{q_1}\right)^{1/q_1} \leq c\|f\|_{H\dot{K}^{\alpha,q_1}_{p_1(\cdot)}}.
\end{aligned}$$

Let us estimate E_2. By Lemma 3.1.2, we have

$$\begin{aligned} E_2 &= \left\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha q_2} \left(\sum_{i=k-1}^{+\infty} |\lambda_i| \, \|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)} \right)^{q_2} \right\}^{1/q_2} \\ &\leq c \left\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha q_2} \left(\sum_{i=k-1}^{+\infty} |\lambda_i| \, \|a_i\|_{p_2(\cdot)} \right)^{q_2} \right\}^{1/q_2} \\ &\leq c \left\{ \sum_{k=-\infty}^{+\infty} \left(\sum_{i=k-1}^{+\infty} |\lambda_i| \, 2^{(k-i)\alpha} \right)^{q_2} \right\}^{1/q_2} \\ &\leq c \left\{ \sum_{k=-\infty}^{+\infty} \left(\sum_{i=k-1}^{+\infty} |\lambda_i| \, 2^{(k-i)\alpha} \right)^{q_1} \right\}^{1/q_1} \\ &\leq c \left\{ \sum_{i=-\infty}^{+\infty} |\lambda_i|^{q_1} \right\}^{1/q_1} \leq c\|f\|_{H\dot{K}^{\alpha,q_1}_{p_1}}. \end{aligned}$$

A combination of estimations of E_1 and E_2 finish the proof ot Theorem 3.1.1.

3.1.3 Variable-type Hardy estimate of commutator of fractional integral operators

We present the boundedness of $[b, I_\sigma]$ on variable Herz-type Hardy spaces.

Theorem 3.1.2 *Suppose that $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_1^+ < \frac{n}{\sigma}$ and $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\sigma}{n}$. Let $n(1 - \frac{1}{p_1^-}) \leq \alpha < \infty$ and $0 < q_1 \leq q_2 < \infty$ and $b \in BMO(\mathbb{R}^n)$. Then $[b, I_\sigma]$ is bounded from $H\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)$.*

Proof of Theorem 3.1.2. We must show that

$$\|[b, I_\sigma]f\|_{\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} \leq c\,\|f\|_{H\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(\mathbb{R}^n)}$$

for all $f \in H\dot{K}^{\alpha_1,q_1}_{p_1(\cdot)}(\mathbb{R}^n)$. Using Theorem 2.2.1, we may assume that

$$f = \sum_{i=-\infty}^{+\infty} \lambda_i a_i$$

where $\lambda_i \geq 0$ and a_i's are $(\alpha, p_1(\cdot))$- atom with supp$a_i \subseteq B_i$. Using definition of $\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)$ −norm , we have

$$\begin{aligned}
\|[b,I_\sigma]f\|_{\dot{K}^{\alpha,q_2}_{p_2(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{+\infty} 2^{k\alpha q_2}\|[b,I_\sigma]f\chi_k\|^{q_2}_{p_2(\cdot)}\right\}^{1/q_2} \\
&\leq \left\{\sum_{k=-\infty}^{+\infty} 2^{k\alpha q_2}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\,\|[b,I_\sigma]f\chi_k\|_{p_2(\cdot)}\right)^{q_2}\right\}^{1/q_2} \\
&\quad +\left\{\sum_{k=-\infty}^{+\infty} 2^{k\alpha q_2}\left(\sum_{i=k-1}^{+\infty}|\lambda_i|\,\|[b,I_\sigma]f\chi_k\|_{p_2(\cdot)}\right)^{q_2}\right\}^{1/q_2} \\
&= : F_1+F_2.
\end{aligned}$$

Let us estimate F_1. As in E_1, we use the Taylor expansion of $|x-y|^{-n+\sigma}$ at x and the s-order vanishing moments of a_i with $s \geq \left[\alpha^+ - n(1-\frac{1}{p_1^-})\right]$, we have

$$\begin{aligned}
&|[b,I_\sigma](a_i)| \\
\leq\ & \int_{B_i}|b(x)-b(y)|\frac{|a_i(y)|\,|y|^{s+1}}{|x|^{n-\sigma+s+1}}dy \\
\leq\ & c2^{-k(n-\sigma+s+1)+i(s+1)}\int_{B_i}|a_i(y)|\,|b(x)-b(y)|\,dy \\
\leq\ & c2^{-k(n-\sigma+s+1)+i(s+1)}\left(|b(x)-b_{B_i}|\int_{B_i}|a_i(y)|\,dy+\int_{B_i}|a_i(y)|\,|b_{B_i}-b(y)|\,dy\right),
\end{aligned}$$

we use the Hölder inequality, the last expression is bounded by

$$c2^{-k(n-\sigma+s+1)+i(s+1)}\|a_i\|_{p_1(\cdot)}\left(|b(x)-b_{B_i}|\,\|\chi_{B_i}\|_{p_1'(\cdot)}+\left\||b_{B_i}-b(y)|\,\chi_{B_i}\right\|_{p_1'(\cdot)}\right), \qquad (3.1.3)$$

by (3.1.2) and Lemma 3.1.1, we obtain

$$\begin{aligned}
&\|[b,I_\sigma](a_i)\chi_k\|_{p_2(\cdot)} \\
\leq\ & c2^{-k(n-\sigma+s+1)+i(s+1)}\|a_i\|_{p_1(\cdot)}\Big(\||b(x)-b_{B_i}|\,\chi_k\|_{p_2(\cdot)}\|\chi_{B_i}\|_{p_1'(\cdot)} \\
&+\left\||b_{B_i}-b(y)|\,\chi_{B_i}\right\|_{p_1'(\cdot)}\|\chi_k\|_{p_2(\cdot)}\Big) \\
\leq\ & c2^{-k(n-\sigma+s+1)+i(s+1)}\|a_i\|_{p_1(\cdot)}\Big((k-i)\|b\|_{BMO}\|\chi_{B_k}\|_{p_2(\cdot)}\|\chi_{B_i}\|_{p_1'(\cdot)} \\
&+\|b\|_{BMO}\|\chi_{B_i}\|_{p_1'(\cdot)}\|\chi_k\|_{p_2(\cdot)}\Big) \\
\leq\ & c(k-i)2^{-k(n-\sigma+s+1)+i(s+1)}\|a_i\|_{p_1(\cdot)}\|b\|_{BMO}\|\chi_{B_i}\|_{p_1'(\cdot)}\|\chi_{B_k}\|_{p_2(\cdot)} \\
\leq\ & c(k-i)2^{-k(n+s+1)+i(s+1)}\|a_i\|_{p_1(\cdot)}\|b\|_{BMO}\|\chi_{B_i}\|_{p_1'(\cdot)}\|I_\sigma(a_i)\chi_k\|_{p_2(\cdot)} \\
\leq\ & c(k-i)2^{-k(n+s+1)+i(s+1)}\|a_i\|_{p_1(\cdot)}\|b\|_{BMO}\|\chi_{B_i}\|_{p_1'(\cdot)}\|\chi_k\|_{p_1(\cdot)}.
\end{aligned}$$

By (1.2.3) and Lemma 1.2.2, we have

$$\begin{aligned} F_1 &= \left\{ \sum_{k=-\infty}^{+\infty} 2^{k\alpha q_2} \left(\sum_{i=-\infty}^{k-2} |\lambda_i| \left\| [b, I_\sigma] (a_i) \chi_k \right\|_{p_2(\cdot)} \right)^{q_2} \right\}^{1/q_2} \\ &\leq c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-2} |\lambda_i| (k-i) 2^{(i-k)(s+1+n-\alpha-n/p_1(0))} \right)^{q_2} \right\}^{1/q_2} \\ &\leq c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-2} |\lambda_i| (k-i) 2^{(i-k)(s+1+n-\alpha-n/p_1(0))} \right)^{q_2} \right\}^{1/q_2} \end{aligned} \tag{3.1.4}$$

• if $1 < q_1 < \infty$, since $s+1-\alpha^+ + n(1-\frac{1}{p_1^-}) > 0$ and $q_1 \leq q_2$, then by the Hölder inequality, we have (3.1.4) is bounded by

$$\begin{aligned} & c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-2} |\lambda_i|^{q_1} (k-i)^{q_1} 2^{(i-k)(s+1+n-\alpha-n/p_1(0)) \frac{q_1}{2}} \right)^{\frac{q_2}{q_1}} \right\}^{1/q_2} \\ & \times \left\{ \sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-2} (k-i)^{q_1'} 2^{(i-k)(s+1+n-\alpha-n/p_1(0)) \frac{q_1'}{2}} \right)^{\frac{q_2}{q_1'}} \right\}^{1/q_2} \\ \leq\ & c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{+\infty} |\lambda_i|^{q_2} \left(\sum_{i=-\infty}^{k-2} (k-i)^{q_1} 2^{(i-k)(s+1+n-\alpha-n/p_1(0)) \frac{q_1}{2}} \right)^{\frac{q_2}{q_1}} \right\}^{1/q_2} \\ \leq\ & c \|b\|_{BMO} \left\{ \sum_{k=-\infty}^{+\infty} |\lambda_i|^{q_1} \left(\sum_{i=-\infty}^{k-2} (k-i)^{q_1} 2^{(i-k)(s+1+n-\alpha-n/p_1(0)) \frac{q_1}{2}} \right) \right\}^{1/q_1} \\ =\ & c \|b\|_{BMO} \left\{ \sum_{i=-\infty}^{+\infty} |\lambda_j|^{q_1} \left(\sum_{i=-k+2}^{+\infty} (i-k)^{q_1} 2^{(i-k)(s+1+n-\alpha-n/p_1(0)) \frac{q_1}{2}} \right) \right\}^{1/q_1}, \end{aligned}$$

by the atomic decomposition theorem, we obtaint that F_1 is bounded by

$$F_1 \leq c \|b\|_{BMO} \left(\sum_{k=-\infty}^{+\infty} |\lambda_k|^{q_1} \right)^{1/q_1} \leq c \|f\|_{H\dot{K}^{\alpha, q_1}_{p_1(\cdot)}}.$$

• if $0 < q_1 \leq 1$, by the injection $\ell_{q_1} \hookrightarrow \ell_{q_2} \hookrightarrow \ell_1$, we have (3.1.4) is bounded by

$$\begin{aligned}
& c\,\|b\|_{BMO}\left\{\sum_{k=-\infty}^{+\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|\,(k-i)2^{(i-k)(s+1+n-\alpha-n/p_1(0))}\right)^{q_2}\right\}^{1/q_2} \\
\leq\;& c\,\|b\|_{BMO}\left\{\sum_{k=-\infty}^{+\infty}\left(\sum_{i=-\infty}^{k-2}|\lambda_i|^{q_1}\,(k-i)^{q_1}2^{(i-k)(s+1+n-\alpha-n/p_1(0))q_1}\right)\right\}^{1/q_1} \\
\leq\;& c\,\|b\|_{BMO}\left\{\sum_{i=-\infty}^{+\infty}|\lambda_i|^{q_1}\left(\sum_{k=i+2}^{+\infty}(k-i)^{q_1}2^{(i-k)(s+1+n-\alpha-n/p_1(0))q_1}\right)\right\}^{1/q_1} \\
\leq\;& c\,\|b\|_{BMO}\left\{\sum_{i=-\infty}^{+\infty}|\lambda_i|^{q_1}\right\}^{1/q_1} \leq c\|f\|_{H\dot{K}^{\alpha,q_1}_{p_1(\cdot)}}.
\end{aligned}$$

Let us estimate F_2. By Lemma 3.1.1, we have

$$\begin{aligned}
F_2 \;=\;& \left\{\sum_{k=-\infty}^{+\infty}2^{k\alpha q_2}\left(\sum_{i=k-1}^{+\infty}|\lambda_i|\,\|[b,I_\sigma]\,(a_i)\,\chi_k\|_{p_2(\cdot)}\right)^{q_2}\right\}^{1/q_2} \\
\leq\;& c\,\|b\|_{BMO}\left\{\sum_{k=-\infty}^{+\infty}2^{k\alpha q_2}\left(\sum_{i=k-1}^{+\infty}|\lambda_i|\,\|a_i\|_{p_1(\cdot)}\right)^{q_2}\right\}^{1/q_2} \\
\leq\;& c\,\|b\|_{BMO}\left\{\sum_{k=-\infty}^{+\infty}\left(\sum_{i=k-1}^{+\infty}|\lambda_i|\,2^{(k-i)\alpha}\right)^{q_2}\right\}^{1/q_2} \\
\leq\;& c\left(\sum_{k=-\infty}^{+\infty}|\lambda_k|^{q_1}\right)^{1/q_1} \leq c\|f\|_{H\dot{K}^{\alpha,q_1}_{p_1(\cdot)}}.
\end{aligned}$$

A combination of estimations of F_1 and F_2 finish the proof ot Theorem 3.1.2.

3.2 Boundedness of Marcinkiewicz integrals on variable exponent Herz-type Hardy spaces

In this section, we present some results of R. Heraiz [12] concerning Marcinkiewicz integral operators μ which are the boundedness of this class of operators from $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ when the parameters α, p and q satisfies some conditions. Also, we present the boundedness of μ on variable Herz-type Hardy spaces $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$..

3.2.1 Preliminaries

Definition 3.2.1 *For $0 < \beta \leq 1$, the Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is defined as*

$$Lip_\beta(\mathbb{R}^n) := \left\{ f : \|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n; x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty \right\}$$

Given $\Omega \in Lip_\beta(\mathbb{R}^n)$ be a homogeneous function of degree zero and

$$\int_{\mathrm{s}^{n-1}} \Omega\left(z\right) \ \mathrm{d}\sigma(z) = 0$$

where $z = x/|x|$ for any $x \neq 0$ and S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure.

Definition 3.2.2 *The Marcinkiewicz integral μ is defined by*

$$\mu(f)\left(x\right) := \left(\int_0^\infty \left|F_\Omega f\left(x\right)\right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

where

$$F_\Omega f\left(x\right) := \int_{|x-y|\leq t} \frac{\Omega(x-y)}{\left|x-y\right|^{n-1}} f\left(y\right) \ dy.$$

It is well known that the operator μ was first defined by Stein [17]. First he proved that if $\Omega \in Lip_\beta(S^{n-1})$ $(0 < \beta \leq 1)$, then μ is of type (p,p) for $1 < p \leq 2$ and of weak type $(1,1)$. Benedek et al. [2] showed that μ of type (p,p) with $1 < p < \infty$. Recently, the boundedness of Marcinkiewicz integral operators μ on variable function spaces have attracted great attention (see [18], [19], ...). In [12], R. Heraiz generalize some results concerning Marcinkiewicz integral operators μ on variable Herz spaces $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(R^n)$ and variable Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Here we present three lemmas used to prove the results of R. Heraiz.

The following Lemma presents the $L^{p(\cdot)}$-boundedness of μ.

Lemma 3.2.1 *([15]) Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then there exists a constant C such that for any $f \in L^{p(\cdot)}\left(\mathbb{R}^n\right)$*

$$\|\mu(f)\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

The next Lemma treats the boundedness of fractional integral on variable Lebesgue space.

Lemma 3.2.2 *([20]) Suppose that $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_1^+ < \frac{n}{\sigma}$ and $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\sigma}{n}$. Then for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, we have*

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|\cdot - y|^{n-\sigma}} dy \right\|_{p_2(\cdot)} \leq c \, \|f\|_{p_1(\cdot)} .$$

The last Lemma presents the boundedness of homogeneous function of degree zero.

Lemma 3.2.3 *([18]) If $a > 0, 1 \leq s \leq \infty, 0 < d \leq s$ and $-n + (n-1)d/s < \tau < \infty$, then*

$$\left(\int_{|y| \leq a|x|} |y|^{\tau} |\Omega(x-y)|^{d} dy \right)^{\frac{1}{d}} \leq c \, |x|^{(\tau + n/d)} \, \|\Omega\|_{L^s(\mathrm{S}^{n-1})} .$$

3.2.2 Variable Herz estimate of Marcinkiewicz integral operators

In this subsection, we present two results concerning the Marcinkiewicz integral operator μ. In the first, we show that μ is bounded from $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ for α, p and q satisfies some conditions. Next, we present the boundedness of μ on variable Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Here, we present the first results of R. Heraiz [12] which is the boundedness for Marcinkiewicz integral operators μ to the case of variable Herz spaces (all exponents are variables). One of our main results can be stated as follows.

Theorem 3.2.1 *([12]) Suppose that $0 < \tau \leq 1, p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty, \Omega \in L^s(S^{n-1}), s > (p')^-$ and $\alpha \in L^{\infty}(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q have a* log *decay at the origin such that*

$$-\frac{n}{p(0)} - \frac{n}{s} - \tau < \alpha(0) < n - \frac{n}{p(0)} - \frac{n}{s} - \tau \text{ and } -\frac{n}{p_{\infty}} - \frac{n}{s} - \tau < \alpha_{\infty} < n - \frac{n}{p_{\infty}} - \frac{n}{s} - \tau$$

then μ is bounded from $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ (or $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$) to $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ (or $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$).

Proof. *Let $f \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, we can write*

$$f = \sum_{j \in \mathbb{Z}} f \chi_j = \sum_{j \in \mathbb{Z}} f_j .$$

My goal is to show that

$$\|\mu(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq c \, \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}$$

for all $f \in \dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. *We have* $\mu(f)$ *in* $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$*-norm is equivalent to*

$$\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left\|\mu(f)\chi_k\right\|_{p(\cdot)}^{q(0)}\right\}^{1/q(0)} + \left\{\sum_{k=0}^{+\infty} 2^{k\alpha_\infty (q)_\infty} \left\|\mu(f)\chi_k\right\|_{p(\cdot)}^{(q)_\infty}\right\}^{1/(q)_\infty},$$

this expression can be estimated by

$$\begin{aligned}
\|\mu(f)\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \lesssim\ & \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=-\infty}^{k-2} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)} \\
&+ \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=k-2}^{k+1} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)} \\
&+ \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=k+2}^{\infty} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q(0)}\right\}^{1/q(0)} \\
&+ \left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=-\infty}^{k-2} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty} \\
&+ \left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=k-2}^{k+1} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty} \\
&+ \left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=k+2}^{\infty} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty} \\
=\ & : U_1 + U_2 + U_3 + U_4 + U_5 + U_6.
\end{aligned}$$

First, we estimate U_1 *and* U_4*. Using the triangular property*

$$\begin{aligned}
|\mu f_j(x)| \ \le\ & \left(\int_0^{|x|} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)\ dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
&+ \left(\int_{|x|}^{\infty} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_j(y)\ dy\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \\
: \ & = I_1 + I_2.
\end{aligned}$$

We notice that when $x \in R_k$, $y \in R_j$ and $j \le k-2$. So we know that $|x-y| \approx |x| \approx 2^k$, and by mean value theorem, we have

$$\begin{aligned}
\left|\frac{1}{|x-y|^2} - \frac{1}{|x|^2}\right| &= \left|\frac{|x|^2 - |x-y|^2}{|x|^2 |x-y|^2}\right| \\
&\le \frac{|(|x| - |x-y|)| \, (|x| + |x-y|)}{|x|^2 |x-y|^2} \\
&\le \frac{c\,|y|}{|x-y|^3}. \qquad (3.2.1)
\end{aligned}$$

By (3.2.1), the Minkowski inequality and the generalized Hölder inequality, we have

$$\begin{aligned}
I_1 &\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_j(y)| \left| \int_{|x-y|}^{|x|} \frac{dt}{t^3} \right|^{\frac{1}{2}} dy \\
&\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_j(y)| \frac{|y|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} \, dy \\
&\lesssim \frac{2^{j/2}}{|x|^{n+1/2}} \int_{\mathbb{R}^n} |\Omega(x-y)| \, |f_j(y)| \, dy \\
&\lesssim 2^{-nk} \|f_j\|_{p(\cdot)} \left\|\Omega(x-\cdot)\chi_j\right\|_{p'(\cdot)}. \qquad (3.2.2)
\end{aligned}$$

Noting that $|x-y| \approx |x|$, then the estimation of I_2 is the same as before and we obtain

$$I_2 \lesssim 2^{-nk} \|f_j\|_{p(\cdot)} \left\|\Omega(x-\cdot)\chi_j\right\|_{p'(\cdot)}.$$

Concluding that that each term (I_1 and I_2) is no more than

$$c2^{-nk} \|f_j\|_{p(\cdot)} \left\|\Omega(x-\cdot)\chi_j\right\|_{p'(\cdot)},$$

Also, by Hölder inequality and Lemma 3.2.3, we obtain

$$\begin{aligned}
\left\|\Omega(x-\cdot)\chi_j\right\|_{p'(\cdot)} &\le \left\|\Omega(x-\cdot)\chi_j\right\|_s \left\|\chi_j\right\|_{\theta(\cdot)} \\
&\le 2^{-j\tau} \left(\int_{R_j} |y|^{s\tau} |\Omega(x-y)|^s \, dy \right)^{\frac{1}{s}} \left\|\chi_j\right\|_{\theta(\cdot)} \\
&\lesssim 2^{-j\tau} 2^{k(\tau+\frac{n}{s})} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \left\|\chi_j\right\|_{\theta(\cdot)}
\end{aligned}$$

where $\frac{1}{p'(\cdot)} = \frac{1}{s} + \frac{1}{\theta(\cdot)}$. Since $\theta \in P_\infty^{\log}(R^n)$, we have for any j

$$\left\|\chi_j\right\|_{\theta(\cdot)} \approx \left\|\chi_j\right\|_{p'(\cdot)} |R_j|^{-\frac{1}{s}},$$

which gives

$$\|\mu(f_j)\chi_k\|_{p(\cdot)} \lesssim 2^{-nk}2^{-(\frac{n}{s}+\tau)(j-k)} \|f_j\|_{p(\cdot)} \|\chi_j\|_{p'(\cdot)} \|\chi_k\|_{p(\cdot)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})}. \tag{3.2.3}$$

Estimation of U_1*. Note that in this case* k *and* j *are negative integers, by Lemma 1.2.2 and since* $\Omega \in L^s(S^{n-1})$*, so we have*

$$\begin{aligned}\|\mu(f_j)\chi_k\|_{p(\cdot)} &\lesssim 2^{(n-\frac{n}{p(0)}-\tau-\frac{n}{s})(j-k)} \|f_j\|_{p(\cdot)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\ &\lesssim 2^{(n-\frac{n}{p(0)}-\tau-\frac{n}{s})(j-k)} \|f_j\|_{p(\cdot)},\end{aligned}$$

which gives

$$\begin{aligned}U_1 &\lesssim \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n-\frac{n}{p(0)}-\tau-\frac{n}{s})q(0)} \|f_j\|_{p(\cdot)}^{q(0)}\right)\right\}^{1/q(0)} \\ &= c\left\{\sum_{k=-\infty}^{-1} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n-\alpha(0)-\frac{n}{p(0)}-\tau-\frac{n}{s})q(0)} \left(2^{j\alpha(0)q(0)} \|f_j\|_{p(\cdot)}^{q(0)}\right)\right)\right\}^{1/q(0)},\end{aligned}$$

since $\alpha(0) - n + \frac{n}{p(0)} - \frac{n}{s} - \tau < 0$*, then by Hardy-type inequality, we have*

$$U_1 \leq c\left\{\sum_{j=-\infty}^{-1} 2^{j\alpha(0)q(0)} \|f_j\|_{p(\cdot)}^{q(0)}\right\}^{1/q(0)} \leq c\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Estimation of U_4*. We can split*

$$\sum_{j=-\infty}^{k-2} \|\mu(f_j)\chi_k\|_{p(\cdot)} = \sum_{j=-\infty}^{-1} \cdots + \sum_{j=0}^{k-2} \cdots$$

then U_4 *can be estimated by*

$$U_4^1 + U_4^2,$$

where

$$U_4^1 := \left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=-\infty}^{-1} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty}$$

and

$$U_4^2 := \left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q_\infty} \left(\sum_{j=0}^{k-2} \|\mu(f_j)\chi_k\|_{p(\cdot)}\right)^{q_\infty}\right\}^{1/q_\infty}$$

Let estimate U_4^1*. We have in this case* $j < 0 \leq k$*. By Lemma 1.2.2, we obtain*

$$\|\mu(f_j)\chi_k\|_{p(\cdot)} \leq c2^{k(\frac{n}{p_\infty}-n+\frac{n}{s}+\tau)} 2^{j(n-\frac{n}{s}-\tau-\frac{n}{p(0)})} \|f_j\|_{p(\cdot)}$$

therefore, U_4^1 *is bounded by*

$$c\sup_{j\leq 0}2^{\alpha(0)}\left\|f_j\right\|_{p(\cdot)}\left\{\sum_{k=0}^{\infty}2^{k(\alpha_\infty+\frac{n}{p_\infty}-n+\frac{n}{s}+\tau)q_\infty}\left(\sum_{j=-\infty}^{-1}2^{j(n-\frac{n}{s}-\tau-\frac{n}{p(0)})}\right)^{q_\infty}\right\}^{1/q_\infty},$$

From the embedding $\ell^{q(0)}\hookrightarrow\ell^\infty$ *and since* $\alpha_\infty+\frac{n}{p_\infty}-n+\frac{n}{s}+\tau<0<n-\frac{n}{s}-\tau-\frac{n}{p(0)}$, *it follows*

$$U_4^1\leq c\left(\sum_{j=-\infty}^{-1}2^{\alpha(0)q(0)j}\left\|f_j\right\|_{p(\cdot)}^{q(0)}\right)^{1/q(0)}\leq c\left\|f\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Thus, by the method for estimating U_1, *we can estimate* U_4^2 *when we replace* $\alpha(0)$, $p(0)$ *and* $q(0)$ *by* α_∞, p_∞ *and* q_∞ *respectively, then* U_4^2 *is bounded by*

$$\begin{aligned}&\left\{\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k-2}2^{(j-k)(n-\frac{n}{p_\infty}-\tau-\frac{n}{s}-\alpha_\infty)}\left(2^{j\alpha_\infty}\left\|f_j\right\|_{p(\cdot)}\right)\right)^{q_\infty}\right\}^{1/q_\infty}\\ \leq\ & c\left(\sum_{k=0}^{\infty}2^{j\alpha_\infty q_\infty}\left\|f_j\right\|_{p(\cdot)}^{q_\infty}\right)^{1/q_\infty}\leq c\left\|f\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

Let us estimate U_2+U_5. *By the* $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$*-boundedness of* μ, *we have*

$$\begin{aligned}H_2+H_5&\lesssim\left(\sum_{k=-\infty}^{-1}\left\|2^{k\alpha(0)}f\chi_k\right\|_{p(\cdot)}^{q(0)}\right)^{1/q(0)}+\left(\sum_{k=0}^{\infty}\left\|2^{k\alpha_\infty}f\chi_k\right\|_{p(\cdot)}^{q_\infty}\right)^{1/q_\infty}\\&\lesssim\left\|f\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Let us estimate U_3. *It is possible to prove the following estimation (similar to the estimate for* I_1 *and* I_2*)*

$$|\mu(f_j)|\leq c2^{-nj}\left\|f_j\right\|_{p(\cdot)}\left\|\Omega(x-\cdot)\chi_j\right\|_{p'(\cdot)},\tag{3.2.4}$$

for the detailed proof of this estimation, see [18, p.259-260]. By Hölder inequality and Lemma 3.2.3, the right-hand of (3.2.4) is bounded by

$$c2^{-j\tau}\left(\int_{R_j}|y|^{s\tau}\left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}}\left\|\chi_j\right\|_{\theta(\cdot)}\lesssim 2^{-nj}2^{k(\tau+\frac{n}{s})}\left\|\chi_j\right\|_{p'(\cdot)}\left\|\Omega\right\|_{L^s(\mathrm{S}^{n-1})},$$

which gives

$$\left\|\mu(f_j)\chi_k\right\|_{p(\cdot)}\lesssim 2^{-nj}2^{(k-j)(\tau+\frac{n}{s})}\left\|f_j\right\|_{p(\cdot)}\left\|\chi_j\right\|_{p'(\cdot)}\left\|\chi_k\right\|_{p(\cdot)}.$$

We split

$$\sum_{j=k+2}^{\infty}\left\|\mu(f_j)\chi_k\right\|_{p(\cdot)}=\sum_{j=k+2}^{-1}\cdots+\sum_{j=0}^{\infty}\cdots$$

Then U_3 is bounded by

$$\begin{aligned} & \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=k+2}^{-1} \|\mu(f_j)\chi_k\|_{p(\cdot)} \right)^{q(0)} \right\}^{1/q(0)} \\ & + \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \left(\sum_{j=0}^{\infty} \|\mu(f_j)\chi_k\|_{p(\cdot)} \right)^{q(0)} \right\}^{1/q(0)} \\ = & : U_3^1 + U_3^2 \end{aligned}$$

For U_3^1, since j and k are negative integers, we have

$$U_3^1 \lesssim \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{j=k+2}^{-1} 2^{(\alpha(0)+\frac{n}{p(0)}+\tau+\frac{n}{s})(k-j)} 2^{j\alpha(0)} \|f_j\|_{p(\cdot)} \right)^{q(0)} \right\}^{1/q(0)}$$

Since $\alpha(0) + \frac{n}{p(0)} + \tau + \frac{n}{s} > 0$, by Hardy lemma, we obtain

$$\begin{aligned} U_3^1 & \lesssim \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(0)} \|f\chi_k\|_{p(\cdot)}^{q(0)} \right)^{1/q(0)} \\ & \lesssim \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

For U_3^2, since $k < 0 \leq j$, then we have

$$\begin{aligned} \|\mu(f_j)\chi_k\|_{p(\cdot)} & \leq c 2^{(k-j)(\tau+\frac{n}{s})} \|f_j\|_{p(\cdot)} 2^{-\frac{nj}{p_\infty}} 2^{\frac{nk}{p(0)}} \\ & \leq c 2^{k(\tau+\frac{n}{s}+\frac{n}{p(0)})} 2^{j\alpha_\infty} \|f_j\|_{p(\cdot)} 2^{-j(\frac{n}{p_\infty}+\alpha_\infty+\tau+\frac{n}{s})}, \end{aligned} \tag{3.2.5}$$

by Hölder inequality in ℓ^1 and since $\gamma = \frac{n}{p_\infty} + \alpha_\infty + \tau + \frac{n}{s} > 0$, we have

$$\begin{aligned} U_3^2 & \lesssim \left\{ \sum_{k=-\infty}^{-1} 2^{k(\tau+\frac{n}{s}+\frac{n}{p(0)})} \left(\sum_{j=0}^{\infty} 2^{j\alpha_\infty} \|f_j\|_{p(\cdot)} 2^{-j\gamma} \right)^{q(0)} \right\}^{1/q(0)} \\ & \lesssim \left(\sum_{k=-\infty}^{-1} 2^{k\eta q(0)} \right)^{1/q(0)} \left(\sum_{j=0}^{\infty} 2^{-j\gamma q'_\infty} \right)^{1/q'_\infty} \left(\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty} \|f_j\|_{p(\cdot)}^{q_\infty} \right)^{1/q_\infty} \\ & \lesssim \left(\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty} \|f_j\|_{p(\cdot)}^{q_\infty} \right)^{1/q_\infty} \\ & \lesssim \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}, \end{aligned}$$

where $\eta = \tau + \frac{n}{s} + \frac{n}{p(0)} > 0$.

Let us estimate U_6*. In this case, since* k *and* j *are non-negative integers, by (3.2.5) and Lemma 1.2.2, we have*

$$\left\|\mu(f_j)\chi_k\right\|_{p(\cdot)} \leq c2^{(-n+\frac{n}{p_\infty}+\tau+\frac{n}{s})(k-j)} \left\|f_j\right\|_{p(\cdot)},$$

which gives

$$U_6 \lesssim \left\{\sum_{k=0}^{\infty}\left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n-\gamma)q_\infty}\left(2^{j\alpha_\infty q_\infty}\left\|f_j\right\|_{p(\cdot)}^{q_\infty}\right)\right)\right\}^{1/q_\infty},$$

since $n-\gamma>0$*, by Hardy type Lemma, we obtain*

$$U_6 \leq \left\{\sum_{j=0}^{\infty} 2^{j\alpha_\infty q_\infty}\left\|f_j\right\|_{p(\cdot)}^{q_\infty}\right\}^{1/q_\infty} \leq c\left\|f\right\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

∎

3.2.3 Boundedness of Marcinkiewicz integral operators on variable Herz-type Hardy spaces.

In this subsection, we present the second result of R. Heraiz [12] which is the boundedness of Marcinkiewicz integral operators with homogeneous kernel on variable Herz-type Hardy spaces.

Theorem 3.2.2 *([12])Suppose that* $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $p_1^+ < 2n$ *and* $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{1}{2n}$*,* $\alpha \in L^\infty(\mathbb{R}^n)$*,* $q_1, q_2 \in \mathcal{P}_0(R^n), \Omega \in L^s(S^{n-1})$ *with* $s > (p_1')^-$*. If* α, q_1 *and* q_2 *are* log*-Hölder continuous, both at the origin and at infinity such that*

$$\alpha(\cdot) \geq n(1-\frac{1}{p_1^-}), q_1(0) \leq q_2(0) \text{ and } (q_1)_\infty \leq (q_2)_\infty.$$

Then μ *is bounded from* $H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)$ *to* $\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(\mathbb{R}^n)$*.*

Proof. Let $f \in H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)$, we can write

$$f = \sum_{i=-\infty}^{+\infty} \lambda_i a_i$$

where $\lambda_i \geq 0$ and a_i's are $(\alpha(\cdot), p_1(\cdot))$- atom with $suppa_i \subseteq B_i$.
My goal is to show that

$$\|\mu(f)\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)} \leq c\,\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$$

for all $f \in H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$. Using the definition of $\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)$-norm, we have

$$\begin{aligned}
&\|\mu(f)\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)} \\
\approx\ & \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\,\|\mu(f)\chi_k\|_{p_2(\cdot)}^{q_2(0)}\right\}^{1/q_2(0)} \\
&+\left\{\sum_{k=0}^{+\infty} 2^{k\alpha_\infty(q_2)_\infty}\,\|\mu(f)\chi_k\|_{p_2(\cdot)}^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \\
\leq\ & \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left(\sum_{i=-\infty}^{k-3}|\lambda_i|\,\|\mu(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)} \\
&+\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-2}^{\infty}|\lambda_i|\,\|\mu(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)} \\
&+\left\{\sum_{k=0}^{+\infty} 2^{k\alpha_\infty(q_2)_\infty}\left(\sum_{i=-\infty}^{k-3}|\lambda_i|\,\|\mu(a_i)\chi_k\|_{p_2(\cdot)}\right)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \\
&+\left\{\sum_{k=0}^{+\infty} 2^{k\alpha_\infty(q_2)_\infty}\left(\sum_{i=k-2}^{+\infty}|\lambda_i|\,\|\mu(a_i)\chi_k\|_{p_2(\cdot)}\right)^{(q_2)_\infty}\right\}^{1/(q_2)_\infty} \\
=\ & :V_1+V_2+V_3+V_4.
\end{aligned}$$

Let us estimate V_1. Using the triangular property, we have

$$\begin{aligned}
|\mu(a_i)(x)| \ &\leq\ \left(\int_0^{|x|}\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}a_i(y)\ \mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}} \\
&\quad+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}a_i(y)\ \mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}} \\
:&=W_1+W_2.
\end{aligned}$$

First we we use that in this case $x \in R_k$, $y \in B_i$ and $i \leq k-3$, we have

$$\begin{aligned} c|x| &\leq |x-y| \leq C|x| \\ &\text{and} \\ c2^k &\leq |x| \leq C2^k \end{aligned}$$

i.e

$$|x-y| \approx |x| \approx 2^k.$$

By (3.2.2), we have

$$W_1 \lesssim \int_{B_i} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} \frac{|y|^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} |a_i(y)| \ \mathrm{d}y,$$

using the m-order vanishing moment condition of a_i (see condition (iv) in the Definition of Atom) with $m \geq \left[\alpha^+ - n(1-\frac{1}{p_1^-})\right]$, we can subtract the Taylor expansion of $|x-y|^{\frac{1}{2}-n}$ at x, we obtain

$$\begin{aligned} W_1 &\leq \int_{B_i} \frac{|y|^{m+1}}{|x|^{n+m+\frac{1}{2}}} |a_i(y)| \, |\Omega(x-y)| \, dy \\ &\leq c2^{-k(n+m+\frac{1}{2})+i(m+1)} \int_{B_i} |a_i(y)| \, |\Omega(x-y)| \, dy, \end{aligned}$$

The estimation of W_2 is the same as before since we never use $|x-y| \approx |x|$, we have

$$|\mu(a_i)(x)| \leq c2^{-k(n+m+\frac{1}{2})+i(m+1)} \int_{B_i} |a_i(y)| \, |\Omega(x-y)| \, dy,$$

as the same reason in the proof of (3.2.3), we get

$$\begin{aligned} \|\mu(a_i)\chi_k\|_{p_2(\cdot)} &\lesssim 2^{-(n-\frac{1}{2})k} 2^{\beta(i-k)} \|a_i\|_{p_1(\cdot)} \|\chi_i\|_{p_1'(\cdot)} \|\chi_k\|_{p_2(\cdot)} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \\ &\lesssim 2^{-(n-\frac{1}{2})k} 2^{\beta(i-k)} \|a_i\|_{p_1(\cdot)} \|\chi_i\|_{p_1'(\cdot)} \|\chi_k\|_{p_2(\cdot)}, \end{aligned} \tag{3.2.6}$$

where $\beta = (1+m-\frac{n}{s}-\tau)$.

Also, to estimate (3.2.6), we need the following inequality (see [13, p.350] for $\sigma = \frac{1}{2}$)

$$\int_{B_k} \frac{\chi_{B_k}(y)}{|x-y|^{n-\frac{1}{2}}} dy \geq \int_{B_k} \frac{dy}{|x-y|^{n-\frac{1}{2}}} \chi_{B_k}(x) \geq c2^{\frac{k}{2}} \chi_{B_k}(x). \tag{3.2.7}$$

(3.2.6), (3.2.7) and Lemma 3.2.2, gives

$$\begin{aligned}
\|\mu(a_i)\chi_k\|_{p_2(\cdot)} &\leq 2^{-(n-\frac{1}{2})k}2^{\beta(i-k)}\|a_i\|_{p_1(\cdot)}\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}\|\chi_k\|_{p_2(\cdot)} \\
&\leq 2^{-nk}2^{\beta(i-k)}\|a_i\|_{p_1(\cdot)}\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}\left\|\int_{B_k}\frac{\chi_{B_k}(y)\,dy}{|\cdot-y|^{n-\frac{1}{2}}}\right\|_{p_2(\cdot)} \\
&\leq 2^{-nk}2^{\beta(i-k)}\|a_i\|_{p_1(\cdot)}\left\|\chi_{B_i}\right\|_{p_1'(\cdot)}\left\|\chi_{B_k}\right\|_{p_1(\cdot)},
\end{aligned}$$

Now it follows from Lemma 1.2.2

$$\begin{aligned}
V_1 &= \left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left(\sum_{i=-\infty}^{k-3}|\lambda_i|\,\|\mu(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)} \\
&\leq c\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=-\infty}^{k-3}|\lambda_i|\,2^{(i-k)(\beta-(\alpha+n/p_1)(0))}\right)^{q_2(0)}\right\}^{1/q_2(0)},
\end{aligned}$$

Now we can choose m large enough such that $\beta-\alpha^{+}+n(1-\frac{1}{p_1^{-}})>0$, by Hardy type Lemma, we obtain

$$V_1 \leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_2(0)}\right)^{1/q_2(0)} \leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_1(0)}\right)^{1/q_1(0)} \leq c\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}}.$$

Let us estimate V_2. By Lemma 3.2.1 and applying the size condition of a_i (conditions (ii) and (iii) in Definition of atom), we have

$$\begin{aligned}
V_2 &= \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-2}^{+\infty}|\lambda_i|\,\|\mu(a_i)\chi_k\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\
&\leq c\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-2}^{+\infty}|\lambda_i|\,\|a_i\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\
&\leq c\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left(\sum_{i=k-2}^{-1}|\lambda_i|\,\|a_i\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\
&\quad +c\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left(\sum_{i=0}^{+\infty}|\lambda_i|\,\|a_i\|_{p_2(\cdot)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\
&\leq c\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=k-2}^{-1}|\lambda_i|\,2^{(k-i)\alpha(0)}\right)^{q_2(0)}\right\}^{1/q_2(0)}\\
&\quad +c\left\{\sum_{k=-\infty}^{-1}\left(\sum_{i=0}^{+\infty}|\lambda_i|\,2^{(k-i)\alpha^-+k(\alpha(0)-\alpha^-)+i(\alpha^--\alpha_\infty)}\right)^{q_2(0)}\right\}^{1/q_2(0)}
\end{aligned}$$

for $k<0\leq i$ and since $\alpha^-\leq\min(\alpha(0),\alpha_\infty)$, we have

$$k(\alpha(0)-\alpha^-)+i(\alpha^--\alpha_\infty)\leq 0.$$

By Lemma 1.2.2, we obtain

$$V_2\leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_2(0)}\right)^{1/q_2(0)}\leq c\left(\sum_{k=-\infty}^{-1}|\lambda_k|^{q_1(0)}\right)^{1/q_1(0)}\leq c\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}}.$$

If we replace $\alpha(0)$, $p_2(0)$ and $q_2(0)$ by α_∞ respectively, we then obtain the estimatation of V_3 and V_4 by the same arguments used in the estimation of V_1 and V_2. A combination of estimations of V_1, V_2, V_3 and V_4 completes the proof of Theorem. ■

Bibliographie

[1] A. Almeida and D. Drihem, Maximal, potential and singular type operators on Herz spaces with variable exponents, J. Math. Anal. Appl. **394** (2012), 781-795.

[2] A. Benedek. A. -P. Calderon. and R. Panzone, Convolution operators on Banach space valued functions, Proc. Nat. Acad. Sci. U.S.A., 48(1962), 356–365.

[3] D. Cruz-Uribe and A. Fiorenza, Variable Lebesgue spaces. Birkhäuser, Berlin, 2013.

[4] D. Cruz-Uribe, A. Fiorenza, M. Ruzhansky and J. Wirth, Variable Lebesgue space and hyperbolic systems, edited by Sergey Tikhonov. Advanced Courses in Mathematics CRM Barcelona, vol. **20**, Birkhäuser, Basel, 2013. ISBN 978-3-0348-0839-2.

[5] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, **2017**, Springer Verlag, Berlin, 2011.

[6] D. Drihem and R. Heraiz, Herz type Besov spaces of variable smoothness and integrability, Kodai Math. J. **40** (2017) 31–57.

[7] D. Drihem and R. Heraiz, Boundedness of some bilinear operators on variable Herz-type Hardy spaces, J. of Pseudo-Diff. Op. and App. 10 (2019) 601-648.

[8] D. Drihem and F. Seghiri, Notes on the Herz-type Hardy spaces of variable smoothness and integrability, Math. Ineq and Appl. **19** (2016), 145-165.

[9] L. Grafakos, X. Li and D. Yang, Bilinear operators on Herz-type Hardy spaces, Trans. Amer. Math. Soc. **350** (1998), 1249-1275.

[10] E. Hernandez and D. Yang, Interpolation of Herz-type Hardy spaces, Illinois J. Math. **42** (1998), 564-581.

[11] R. Heraiz, Some properties of variable Herz type-Besov spaces and application, Phd thesis, univ. of M'sila, algeria, 2017.

[12] R. Heraiz, On the Boundedness of Marcinkiewcz integrals on on variable Herz-type Hardy spaces, Kyungpook Math. J. 59 (2019) 259-275.

[13] M. Izuki, Fractional integrals on Herz–Morrey spaces with variable exponent, Hiroshima Math. J. 40(2010), 343–355.

[14] H. Kempka and J. Vybíral, A note on the spaces of variable integrability and summability of Almeida and Hästö, Proc. Amer. Math. Soc. **141** (2013), 3207-3212.

[15] Z. Liu, H.Wang, Boundedness of Marcinkiewicz integrals on Herz spaces with variable exponent. Jordan J. Math. Stat. 5(2012), 223–239.

[16] S. Lu and D. Yang, Some characterizations of weighted Herz-type Hardy spaces and their applications, Acta Math. Sinica (New Ser.) **13** (1997), 45-58.

[17] E. M. Stein, On the functions of Littlewood–Paley, Lusin, and Marcinkiewicz, Trans. Amer. Math. Soc. 88(1958), 430–466.

[18] H. Wang, Commutators of Marcinkiewicz integrals on Herz spaces with variable exponent, Czech. Math. J., 66(2016), 251–269.

[19] H. Wang, The continuity of commutators on Herz-type Hardy spaces with variable exponent, Kyoto J. Math., 56(2016), 559–573.

[20] H. Wang, L. Zongguang and F. Zunwei, Boundedness of Fractional Integrals on Herz-type Hardy Spaces with variable exponent, Advances in Mathematics(China) 46(2017), 252–260.

ملخص المذكرة:

في هذه المذكرة قمنا بدراسة فضاءات "هارز-هاردي" ذات الأدلة المتغيرة، حيث أعطينا بعض الخصائص الأساسية لهذه الفضاءات، وفي الأخير درسنا محدودية بعض المؤثرات التكاملية في فضاءات "هارز-هاردي باستعمال التفكيك الذري لهذه الفضاءات.

الكلمات المفتاحية: فضاءات هارز، فضاءات هارز-هاردي، مؤثرات تكاملي الشاذ، مؤثرات مارسينكويكز، أدلة متغيرة، التفكيك الذري.

Résumé du mémoire:

Dans cette mémoire, nous avons étudié les espaces Herz type Hardy avec des exposants variables où nous avons donné certaines propriétés de base de ces espaces et finalement, nous avons étudié la continuité de certains opérateurs intégraux dans les espaces de Herz-Hardy en utilisant la décomposition atomique de ces espaces.

Mots clés : Espaces de Herz, Espaces de Herz-Hardy, Opérateurs intégraux singuliers, Opérateurs de Marcinkiewicz, Exposants variables, Décomposition atomique.

Abstract of memory:

In this memory, we studied the Herz type Hardy spaces with variable exponents where we gave some basic properties of these spaces and finally we studied the boundedness of some integral operators in Herz-Hardy by using atomic decomposition of these spaces.

Key words: Herz spaces, Herz-type Hardy spaces, singular Integral operators, Marcinkiewicz operators, Variable exponent, Atomic decomposition