PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

MOHAMED BOUDIAF UNIVERSITY -M'SILA
Faculty of Mathematics and Computer Science
Department of Mathematics

Master's degree in Mathematics

Domain : Mathematics and Informatics
Speciality : Mathematics
Option : Algebra and Discrete Mathematics

Titled

Algebraic Linear Differential Equations on Boolean Lattices

Presented by :
Amouria Guidah

Defended on : 10/06/2025.

Before the jury composed of :

Chair :	M^r *Abdelkrim* Mehenni	M.C.B, University of M'Sila
Supervisor :	M^r *Mourad* Yettou	M.C.A, University of M'Sila
Co-Supervisor :	M^r *Abdelaziz* Amroune	Prof, University of M'Sila
Examiner :	M^r *Zouareg* Yahia	M.C.B, ENSH of Blida

University year : 2024/2025

Contents

إهداء

وآخِرُ دَعْوَاهُمْ أَنِ الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِين

الحمد لله والشكر لله عز وجل، الذي أنار لي دربي، وفتح لي أبواب العلم، وأمدّني بالصبر والإرادة لإتمام هذه المذكرة، فله الحمد والشكر، حمدًا طيبًا مباركًا يليق بجلاله. وإلى من كانت رحمته بابًا لا يُغلق، وعطاؤه بلا حدود، إلى الكريم الذي إذا أعطى أدهش، وإذا وهب أذهل، سبحانه وتعالى. واستنادًا إلى قول النبي ﷺ: *" من لا يشكر الناس لا يشكر الله".*

إنّ الوفاء يقتضي أن يُردَّ الفضل إلى أهله، ولذلك أتقدّم بجزيل الشكر والامتنان إلى من كان له الفضل – بعد الله تعالى – في إخراج هذه المذكرة العلمية، **الدكتور مراد يطو**، الذي لم يبخل عليَّ بتوجيهاته السديدة، فكان نعم الداعم والموجّه في جميع مراحل إعداد هذه المذكرة. فجزاه الله عني خير الجزاء، وأمدّه بدوام الصحة والعافية، وجعل ما قدّمه لي في ميزان حسناته.

وإنه لمن دواعي فخري واعتزازي أن تُناقش هذه المذكرة أمام لجنة المناقشة المكونة من البرفيسور عبد العزيز عمرون و الدكتور زوارق يحيي و الدكتور مهني عبد الكريم الذين أتوجّه إليهم بأسمى آيات الشكر والتقدير على قبولهم دعوة مناقشة هذه المذكرة، وإسهامهم بإرشاداتهم القيّمة وتوجيهاتهم المتميزة، فجزاهم الله عني خير الجزاء.

إلى أستاذتي التي كانت نعمة وهبني الله إياها، والتي كانت سببًا وعونًا ونورًا، شكرًا لكِ على توجيهك وصبرك ودعمك المتواصل، وستظلين حاضرة في ذاكرتي دائمًا، فلكِ منّي كل الامتنان والتقدير **''حسيني إلهام''.**

إلى من كان له كل الفضل والتقدير على عطائه النبيل ودعمه المستمر طوال مسيرتي الجامعية، بعد توفيق الله تعالى، أتوجّه له بخالص الشكر والامتنان. لقد كنتَ مثالًا يُحتذى به في العلم والتواضع، فلك مني كل التقدير والعرفان. دمتَ نِبراسًا للعلم والوفاء **''محمد''.**

وفي هذه اللحظة التي تغمرني فيها مشاعر الفخر والامتنان، أُهدي هذا العمل إلى من ربّاني وكافح من أجلي، إلى المصباح الذي أنار دربي، إلى من أحمل اسمه بكل فخر واعتزاز... طاب بك العمر يا سيّد الرجال، وطِبتَ لي عمرًا. أسأل الله أن يمدّ في عمرك لترى ثمارًا قد آن أوان قطافها، فهي ثمرة تعبك وصبرك ودعائك الذي لم ينقطع...**أبي الغالي ''ساعد قدح''.**

إلى ملاكي في الحياة، ومعنى الحب، وقُرَّة عيني، وأعزّ ما أملك... إلى بسمة الحياة وسرّ الوجود، إلى من كانت دعواتها سرَّ نجاحي، وحنانها بلسمًا لجراحي، إلى غاليتي وجنّة قلبي التي رافقتني وأرشدتني في كل خطواتي ...**أمي الغالية ''فاطمة الزهراء يوسف''،** حفظكِ الله ورعاكِ، يا أمان عمري وركني الثابت الذي لا يميل.

وإلى من رزقني الله بهم سندًا وملاذًا أولا وآخرا، إلى من أزالوا عن طريقي أشواك الفشل، وكانوا العون في كل لحظة إلى إخوتي وأخواتي : **محمد، بوعلام، صلاح الدين، زينب، وفاء،** وسندي في الحياة. إلى **زوجة أخي عائشة،** وإلى عمّاتي العزيزات **منيرة وخديجة،** إلى كل أصدقائي دون استثناء، أنتم النور الذي زاد الطريق إشراقًا.

وإلى رفيقة دربي، التي كانت خير عون وسند، وشاركتني التعب والفرح والنجاح... لكِ كل الشكر، فوجودكِ كان نعمة في طريقي **''رجاء زقعار''.**

لا يسعني إلا أن أعبّر عن خالص امتناني وتقديري لكل من كان له دور، ولو بكلمة، في دعمي ومساندتي خلال هذه الرحلة العلمية. فلكم جزيل الشكر ووافر الدعاء، **و**الحمد لله على حسن الختام والتمام.

Introduction

Derivations and integrations have played a fundamental role in the development of mathematical theories throughout the ages. Physicists have relied on modelling natural phenomena with linear and non-linear differential equations and relied on solving these equations using the properties of derivations and integrations. Therefore, scientists worked on transferring those two concepts to several algebraic structures, for example, rings, fields, ordered sets, and lattices.

In 1957, the author E.C. Posner introduced the notion of derivations in prime rings [14]. Derivations have recently investigated on ring structures (see e.g. [4] and [6, 7]). In 1975, Szász has transferred the concept of derivation to lattice structures [16]. On a given lattice L, he has defined a derivation d on a lattice L as a function verifies these two conditions:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \text{ and } d(x \vee y) = d(x) \vee d(y), \text{ for every } x, y \in L.$$

Derivations have been further investigated in ([9], 2001) for several classes of lattices. Then only the first condition has been considered for the definition of a derivation, since the second condition is always satisfied for isotone derivations on distributive lattices, see the article of Xin et al. ([19], 2008).

The concept of derivations has recently applied in several fields such as partially ordered sets in [3, 2, 1, 25]; distributive lattices in [5, 21, 24]; bounded hyperlattices in [17]; residuated lattices in [11, 23]; integration on lattices in [22]; the subgroup lattice of finite groups in [12]; and pseudo L-algebras in [10].

Recently, M. Yettou et al. ([22], 2023) have introduced the notion of integration i_d with respect to a given derivation d on a lattice L as a function $i_d : L \to \mathcal{P}(L)$ defined by

$$i_d(x) = d^{-1}(x) = \{z \in L \mid dz = x\} \text{ for every } x \in L.$$

They have given the definitions of integrable elements and their integral sets. They have investigated several characterizations and properties of integrations on lattices. Also, they have shown a lattice structure to the family of integral sets with respect to a given integration. Further, they have provided a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets. As an application for the integration, they have used the integrable elements of a Boolean lattice to define algebraic linear differential equations on Boolean lattices.

Moreover, they have determined the necessary and sufficient conditions to solve those differential equations.

In this memory, we focus on studying those type of algebraic linear differential equations on Boolean lattices. To that end, we organize our memory into three chapters as follows:

- We devote the first chapter for recalling basic concepts and properties of lattices.
- In chapter 2, we focus our study to present the concepts of derivations and integrations on lattices and their properties.
- The third chapter is the main goal of this memory. Here, we present the newly notion of algebraic linear differential equations on Boolean lattices. Further, we establish the necessary and sufficient conditions to solve those differential equations. As an application, we use Python Programs for solving those linear differential equations.

Chapter 1

Lattice Structures

In this chapter, we recall basic concepts and properties of lattices. For more information on lattices, we refer the reader to these references [8, 13, 15].

1.1 Generalities on lattices

Definition 1.1. *A partially ordered set (poset, for short) is a set S with a binary relation $\leqslant$ such that each of the following axioms is satisfied:*

- *Reflexive: If $x \in S$, then $x \leqslant x$;*
- *Antisymmetric: If $x, y \in S$, such that $x \leqslant y$ and $y \leqslant x$, then $x = y$;*
- *Transitive: If $x, y, z \in S$, such that $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.*

Example 1.1. *Let $\mathbb{N}^*$ be the set of positive integers and $\mid$ be the divisibility relation. So this relation $\mid$ is:*

- *Reflexive, $\forall x \in \mathbb{N}^*,\ x \mid x$ (x divides x);*
- *Antisymmetric, $\forall x, y \in \mathbb{N}^*$, if $x \mid y$ and $y \mid x$, then $x = y$;*
- *Transitive, $\forall x, y, z \in \mathbb{N}^*$, if $x \mid y$ and $y \mid z$, then $x \mid z$.*

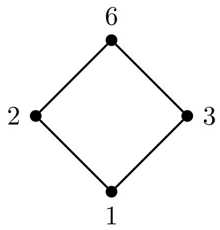
Thus, $\mid$ is a partially ordered on $\mathbb{N}^*$. So the structure $(\mathbb{N}^*, \mid)$ is a poset.

Definition 1.2 (bounded posets)**.** *A poset $(P, \leqslant)$ is called bounded, if it has a least and a greatest element, respectively denoted by 0 and 1, such that $0 \leqslant x \leqslant 1$, for any $x \in P$.*
Usually, the notation $(P, \leqslant, 0, 1)$ is used to describe a bounded poset.

Example 1.2. *Let $D(6)$ be the set of positive divisors of 6 and $\mid$ be the divisibility order. The poset $(D(6), \mid)$ has 1 as the least element and 6 as the greatest element. Indeed, 1 divides all the elements of $D(6)$ and any element of $D(6)$ divides 6. Thus, the structure $(D(6), \mid, 1, 6)$ is a bounded poset.*

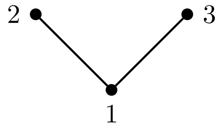
Definition 1.3 (Hasse diagram of posets)**.** *The Hasse diagram of a finite poset* $(P, \leqslant)$ *is a picture of the digraph whose vertices are the elements of* P *and which has line segments between some their vertices. If an element* $y \in P$ *covers an element* $x \in P$ *(i.e.,* $x < y$ *and* $\forall z \in P$ *if* $x \leqslant z \leqslant y$*, then* $z = x$ *or* $z = y$*) we get the vertex* y *is higher up than the vertex* x *and they are connected with a line segment.*

Example 1.3. *The Hasse diagram of the bounded poset* $(D(6), |, 1, 6)$ *is given in the bellow figure.*



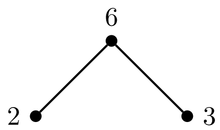
Definition 1.4 (meet semi-lattices)**.** *A meet semi-lattice is a poset* $(P, \leqslant)$ *such that for any two elements* $x, y \in S$ *have a greatest lower bound in* P *denoted* $x \wedge y$*.*

Example 1.4. *Let* $(P_1 = \{1, 2, 3\}, |)$ *be a poset ordered by the divisibility order and shown by the following Hasse diagram.*



This poset is a meet semi-lattice, because for every two elements x *and* y *in* $P_1 = \{1, 2, 3\}$ *they have the greatest lower bound* $x \wedge y$ *in* P_1*.*

Example 1.5. *Let* $(P_2 = \{2, 3, 6\}, |)$ *be a poset ordered by the divisibility order and shown by the following Hasse diagram.*



This poset is not a meet semi-lattice, because 2 *and* 3 *has not a greatest lower bound in* P_2*.*

Definition 1.5 (join semi-lattices)**.** *A join semi-lattice is a poset* $(P, \leqslant)$ *such that for every two elements* $x, y \in S$ *have a least upper bound in* P *denoted* $x \vee y$*.*

Example 1.6. *Let* $(P_2 = \{2, 3, 6\}, |)$ *be the poset given in Example 1.5 is a join semi-lattice. Because for every two elements* x *and* y *of* P_2 *they have the least upper bound* $x \vee y$ *in* P_2*.*

Example 1.7. *The poset* $(P_1 = \{1, 2, 3\}, |)$ *given in Example 1.4 is not a join semi-lattice, because* 2 *and* 3 *have not a least upper bound in* P_1*.*

Definition 1.6 (lattices)**.** *A lattice is both meet and join semi-lattices.*

Usually, the notation $(L, \wedge, \vee)$ *is used to describe a lattice.*

Example 1.8. 1. *The poset of real numbers $(\mathbb{R}, \leqslant)$ ordered by the usual order $\leqslant$ is a lattice, where min and max are its meet and join operations;*

2. *The bounded poset $(D(6), |, 1, 6)$ given in Example 1.3 is a lattice such that gcd and lcm are respectively its meet and join operations;*

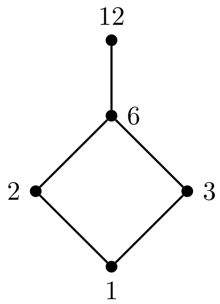
3. *Let S be a set, the power set of S denoted $\mathcal{P}(S)$ equipped with the inclusion relation $\subset$ has a structure of a lattice, where "$\cap$" is the meet operation and "$\cup$" is the join operation.*

1.1.1 Sub-lattices and lattice morphisms

Definition 1.7 (sub-lattices)**.** *Let $(L, \wedge, \vee)$ be a lattice and M be a non-empty sub-set of L. Then M is a sub-lattice of L if and only if:*

$$\text{for every } a, b \in M, \text{ then } a \wedge b \in M \text{ and } a \vee b \in M.$$

Example 1.9. *Let $L = \{1, 2, 3, 6, 12\}$ be the lattice given by the Hasse diagram in the bellow figure ordered by the divisibility order.*



The sub-set $M_1 = \{3, 6, 12\}$ is a sub-lattice of L, but $M_2 = \{2, 3\}$ is not a sub-lattice of L. Because, for the elements 2 and 3 of M_2, we have $2 \wedge 3 = 1 \notin M_2$ and $2 \vee 3 = 6 \notin M_2$.

Definition 1.8 (bounded lattices)**.** *A lattice $(L, \wedge, \vee)$ is called bounded if it has a least and a greatest element denoted by 0 and 1 respectively. Usually, the structure $(L, \wedge, \vee, 0, 1)$ is used to describe a bounded lattice.*

Example 1.10. *The poset $(D(6), |, 1, 6)$ given in Example 1.3 is a bounded lattice.*

Definition 1.9 (lattice morphism)**.** *A mapping $f : L_1 \to L_2$ between two lattices L_1 and L_2 is called a lattice morphism if:*

$$f(x \wedge y) = f(x) \wedge f(y) \text{ and } f(x \vee y) = f(x) \vee f(y) \text{ for each } x, y \in L.$$

Example 1.11. *The identity mapping $f(x) = x$ of a lattice L is a lattice morphism. Because, for any $x, y \in L$, we have:*

$$f(x \vee y) = x \vee y = f(x) \vee f(y) \text{ and } f(x \wedge y) = x \wedge y = f(x) \wedge f(y).$$

Example 1.12. *Let $L_1 = D(6) = \{1, 2, 3, 6\}$ and $L_2 = D(24) = \{1, 2, 3, 4, 6, 8, 12, 24\}$ be two lattices ordered by the divisibility order. Let $f : L_1 \to L_2$ be a mapping defined in the following table:*

x	1	2	3	6
$f(x)$	4	8	12	24

Then f is a lattice morphism. Indeed:

(1) *if $(x, y) = (1, 1)$, then*
$f(1 \vee 1) = f(1) \vee f(1) = 4 \vee 4 = 4 = f(1)$ *and* $f(1 \wedge 1) = f(1) \wedge f(1) = 4 \wedge 4 = 4 = f(1)$;

(2) *if $(x, y) = (1, 2)$, then*
$f(1 \vee 2) = f(1) \vee f(2) = 4 \vee 8 = 8 = f(2)$ *and* $f(1 \wedge 2) = f(1) \wedge f(2) = 4 \wedge 8 = 4 = f(1)$;

(3) *if $(x, y) = (1, 3)$, then*
$f(1 \vee 3) = f(1) \vee f(3) = 4 \vee 12 = 12 = f(3)$ *and* $f(1 \wedge 3) = f(1) \wedge f(3) = 4 \wedge 12 = 4 = f(1)$;

(4) *if $(x, y) = (1, 6)$, then*
$f(1 \vee 6) = f(1) \vee f(6) = 4 \vee 24 = 24 = f(6)$ *and* $f(1 \wedge 6) = f(1) \wedge f(6) = 4 \wedge 24 = 4 = f(1)$;

(5) *if $(x, y) = (2, 2)$, then*
$f(2 \vee 2) = f(2) \vee f(2) = 8 \vee 8 = 8 = f(2)$ *and* $f(2 \wedge 2) = f(2) \wedge f(2) = 8 \wedge 8 = 8 = f(2)$;

(6) *if $(x, y) = (2, 3)$, then*
$f(2 \vee 3) = f(2) \vee f(3) = 8 \vee 12 = 24 = f(6)$ *and* $f(2 \wedge 3) = f(2) \wedge f(3) = 8 \wedge 12 = 4 = f(1)$;

(7) *if $(x, y) = (2, 6)$, then*
$f(2 \vee 6) = f(2) \vee f(6) = 8 \vee 24 = 24 = f(6)$ *and* $f(2 \wedge 6) = f(2) \wedge f(6) = 8 \wedge 24 = 8 = f(2)$;

(8) *if $(x, y) = (3, 3)$, then*
$f(3 \vee 3) = f(3) \vee f(3) = 12 \vee 12 = 12 = f(3)$ *and* $f(3 \wedge 3) = f(3) \wedge f(3) = 12 \wedge 12 = 12 = f(3)$;

(9) *if $(x, y) = (3, 6)$, then*
$f(3 \vee 6) = f(3) \vee f(6) = 12 \vee 24 = 24 = f(6)$ *and* $f(3 \wedge 6) = f(3) \wedge f(6) = 12 \wedge 24 = 12 = f(3)$;

(10) *if $(x, y) = (6, 6)$, then*
$f(6 \vee 6) = f(6) \vee f(6) = 24 \vee 24 = 24 = f(6)$ *and* $f(6 \wedge 6) = f(6) \wedge f(6) = 24 \wedge 24 = 24 = f(6)$.

1.1.2 Ideals and filters of lattices

Definition 1.10 (Ideals)**.** *Let L be a lattice. A non-empty subset I of L is called an ideal if I satisfies the following two conditions:*

- *if $a, b \in I$ implies $a \vee b \in I$;*
- *if $a \in L$ and $b \in I$ such that $a \leqslant b$ imply $a \in I$.*

Example 1.13. *Let $L = D(30)$ be the lattice of positive divisors of 30 shown by the Hasse diagram in Figure 1.1. Let $S_1 = \{1, 2, 3, 6\}$ and $S_2 = \{2, 3, 6\}$ be two sub-sets of L. We have S_1 is an ideal of L, but S_2 is not because $1 \in L$ and $2 \in S_2$ such that $1 \leqslant 2$ but $1 \notin S_2$.*

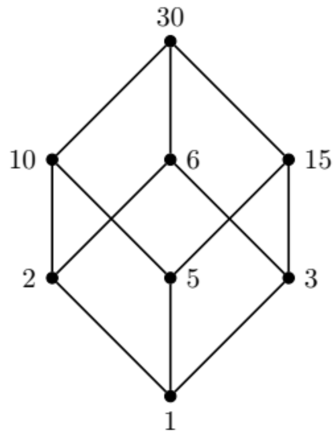


Figure 1.1: The Hasse diagram of the lattice $(D(30), |, gcd, lcm)$.

Definition 1.11 (Filters)**.** *Let L be a lattice. A non-empty subset F of L is called A filter if F satisfies the following two conditions:*

- *if $a, b \in I$ implies $a \wedge b \in I$;*
- *if $a \in L$ and $b \in F$ such that $b \leqslant a$ imply $a \in F$.*

Example 1.14. *Let $L = D(30)$ be the lattice given in Example 1.13. Let $M_1 = \{2, 6, 10, 30\}$ and $M_2 = \{6, 10, 30\}$ be two sub-sets of L. We have M_1 is a filter of L, but M_2 is not because $6, 10 \in L$ but $6 \wedge 10 = 2 \notin M_2$.*

1.2 Algebraic properties of some classes of lattices

Here we present algebraic properties of important classes of lattices, as distributive, modular and Boolean lattices.

Theorem 1.1. *The operations $\vee$ and $\wedge$ of a lattice L satisfy the following properties, for all $x, y, z \in L$:*

- *Commutative: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;*
- *Associative: $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;*
- *Idempotent: $a \wedge a = a$ and $a \vee a = a$;*
- *Absorption: $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.*

1.2.1 Distributive lattices

Definition 1.12. *A lattice $(L, \wedge, \vee)$ is called distributive, if one of the following two equivalent conditions holds:*

(D_1) : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, *for all* $a, b, c \in L$;

(D_2) : $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, *for all* $a, b, c \in L$.

Example 1.15. *1) The lattice of real numbers* $(\mathbb{R}, min, max)$ *is distributive;*

2) Let $L = D(30)$ *be the lattice given in Example 1.13. Then* L *is distributive;*

3) The lattice of the power set of a set $(\mathcal{P}(S), \cap, \cup)$ *is distributive.*

Example 1.16. *Let* M_3 *be the diamond lattice and* N_5 *be the pentagon lattice shown in the following figures:*

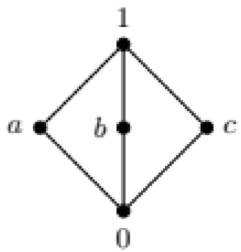


Figure 1.2: The Hasse diagram of the diamond lattice M_3

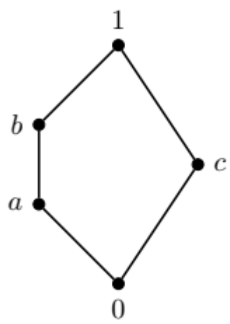


Figure 1.3: The Hasse diagram of the pentagon lattice N_5

Those lattices M_3 *and* N_5 *are not distributive. Indeed,*

- *In* M_3*:*

$$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$$
$$a \vee 0 \neq 1 \wedge 1$$
$$a \neq 1.$$

- *In N_5:*

$$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$$
$$a \vee 0 \neq b \wedge 1$$
$$a \neq b.$$

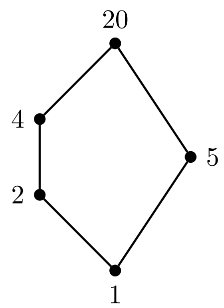
Theorem 1.2 (Characterization of distributive lattices)**.** *Let $(L, \vee, \wedge)$ be a lattice. Then L is distributive if and only if it has not a sub-lattice of the form of M_3 or N_5.*

1.2.2 Modular lattices

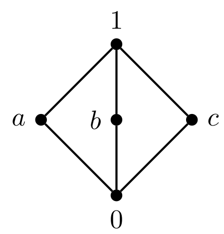
Definition 1.13. *A lattice $(L, \wedge, \vee)$ is said to be modular if for all $a, b, c \in L$:*

$$a \leqslant c \text{ implies } a \vee (b \wedge c) = (a \vee b) \wedge c.$$

Example 1.17. *(1) Let $N_5 = \{1, 2, 4, 5, 20\}$ be the pentagon lattice ordered by the divisibility order. The fact that $2 \leqslant 4$ but $2 \vee (5 \wedge 4) = 2 \vee 1 = 2$ and $(2 \vee 5) \wedge 4 = 20 \wedge 4 = 4$. Then $2 \vee (5 \wedge 4) \neq (2 \vee 5) \wedge 4$. Thus, N_5 is not modular;*



(2) Let $(M_3 = \{0, a, b, c, 1\}, \leqslant, \wedge, \vee)$ be the diamond lattice. Then M_3 is modular, but it is not distributive.



Theorem 1.3 (Characterization of modular lattices)**.** *Let $(L, \vee, \wedge)$ be a lattice. Then L is modular if and only if it has not a sub-lattice of the form of N_5.*

Theorem 1.4. *If L is a modular (resp. distributive) lattice, then every sub-lattice of L is modular (resp. distributive).*

Theorem 1.5. *Every distributive lattice is modular.*

Proof. Assume that L is distributive and $a, b, c \in L$ such that $a \leqslant c$, we have $a \vee c = c$. Then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c.$$

Hence L is modular. □

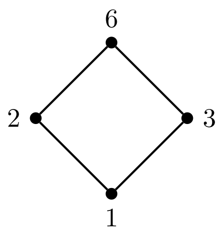
1.2.3 Boolean lattices

Definition 1.14 (Complemented lattice)**.** *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice. L is said to be complemented if for each $a \in L$, there is $b \in L$ such that*

$$a \wedge b = 0 \text{ and } a \vee b = 1.$$

Example 1.18. *The lattice $D(6)$ of positive divisors of 6 forms a complemented lattice. Indeed,*

- $1 \wedge 6 = 1 = 0_{D(6)}$ *and* $1 \vee 6 = 6 = 1_{D(6)}$*;*
- $2 \wedge 3 = 1 = 0_{D(6)}$ *and* $2 \vee 3 = 6 = 1_{D(6)}$*.*



Example 1.19. *The lattice of positive divisors of 12 is not complemented, because we can not get an element $b \in D(12)$ satisfies $6 \wedge b = 1$ and $6 \vee b = 12$.*

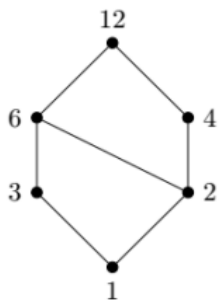


Figure 1.4: The Hasse diagram of $D(12)$

Theorem 1.6. *Let L be a complemented lattice. If L is distributive, then every element $a \in L$ has a unique complement denoted a'.*

Definition 1.15. *A bounded lattice $(B, \wedge, \vee, 0, 1)$ is called Boolean lattice if it is distributive and complemented. Usually, the notation $(B, \wedge, \vee, 0, 1, ')$ used to describe a Boolean lattice.*

Example 1.20. *1) The chain* $(\{0,1\}, min, max)$ *is a Boolean lattice;*

2) The lattice $(D(6), |, gcd, lcm, 1, 6,')$ *is a Boolean lattice;*

3) The lattice $(D(12), gcd, lcm)$ *is not a Boolean lattice, because it is not complemented as we have seen in Example 1.19.*

Theorem 1.7. *Let* $(B, \wedge, \vee, 0, 1,')$ *be a Boolean lattice and* $a, b \in B$*. Then*

1) $1^{'} = 0, 0^{'} = 1$ *and* $(a')' = a$.

2) ***De Morgan's laws:***
$(a \wedge b)^{'} = a^{'} \vee b^{'}$ *and* $(a \vee b)^{'} = a^{'} \wedge b^{'}$.

3) ***Order and complements:***
$a \leq b \Leftrightarrow a \wedge b^{'} = 0 \Leftrightarrow a^{'} \vee b = 1;$
$a \leqslant b \Leftrightarrow b^{'} \leqslant a'.$

4) ***Addition operation:***
$(a + b) = (a \wedge b') \vee (a' \wedge b) = (a \vee b) \wedge (a' \vee b').$
This operation is commutative and associative, also for any $a \in B$:

- $a + a = 0;$
- $a + 1 = a'.$

Theorem 1.8. *If* B *is a finite Boolean lattice, then* B *has* 2^n *elements with* n *is a positive integer.*

Theorem 1.9. *The lattice* $D(n)$ *ordered by the divisibility order* $|$ *is a Boolean lattice if and only if* n *is not divided by any square of a prime number. It means,* n *has the form* $p_1 \cdots p_s$ *such that* p_i *are distinct prime numbers.*
In this case, the operations of the Boolean lattice $D(n)$ *characterized for any* $a, b \in D(n)$ *as follows:*

- $a \wedge b = gcd(a, b);$
- $a \vee b = lcm(a, b);$
- $a^{'} = \frac{n}{a};$
- $a \cdot b = a \wedge b;$
- $a + b = (a \wedge b') \vee (a' \wedge b) = lcm(gcd(a, \frac{n}{b}), gcd(\frac{n}{a}, b)).$

Example 1.21. *1) The lattice* $D(6)$ *given in Example 1.20 is a Boolean lattice because* $6 = 2 \times 3$ *is not divided by any square of a prime number;*

2) The lattice $D(30)$ *is a Boolean lattice because* $30 = 2 \times 3 \times 5;$

3) The lattice $D(210)$ given in the following Hasse diagram is a Boolean lattice because

$$210 = 2 \times 3 \times 5 \times 7.$$

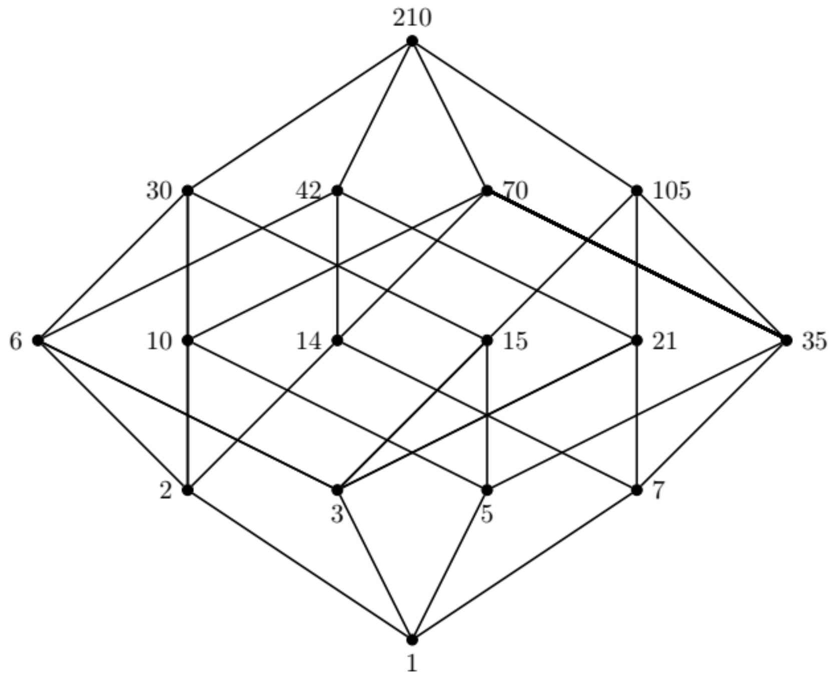


Figure 1.5: The Hasse diagram of the Boolean lattice $(D(210), gcd, lcm, 1, 210,')$.

Example 1.22. *1. The lattice $D(60)$ is not a Boolean lattice because $60 = 2^2 \times 3 \times 5$ is divided by a square of a prime number 2^2;*

2. The lattice $D(18)$ is not a Boolean lattice because $18 = 2 \times 3^2$.

Chapter 2

Derivations and Integrations on Lattices

This chapter is devoted to study the concepts of derivations and integrations on lattices. More details on those two concepts can be found in [19, 18, 20, 22].

2.1 Derivations on lattices

In this section, we present the concept of derivations on lattices and their important properties.

2.1.1 Definitions and examples

Definition 2.1. *[20, 19] Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a derivation on L if it satisfies the following condition:*

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)), \text{ for all } x, y \in L.$$

In the rest of this memory, we write dx instead of $d(x)$.

Example 2.1. *[20]*

1) The identity function f of L (i.e., $f(x) = x$, for any $x \in L$) is a derivation on L;

2) Let $(L, \leqslant, \wedge, \vee, 0)$ be a lattice with the least element $0 \in L$. The null function g of L (i.e., $g(x) = 0$, for any $x \in L$) is a derivation on L.

Example 2.2. *Consider $L = \{0, a, b, 1\}$ be the lattice given by the bellow Hasse diagram. Let $d = L \to L$ be a function defined in the following table:*

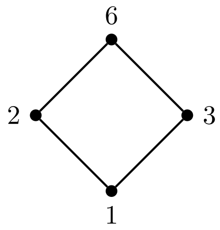
x	0	a	b	1
dx	0	b	b	0

The function d is a derivation on L. Indeed, for all $x, y \in L$: $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$. Because:

- *if* $(x, y) = (0, 0)$, *then*
  $d(0 \wedge 0) = d0 = 0$ *and* $(d0 \wedge 0) \vee (0 \wedge d0) = 0 \vee 0 = 0$;
- *if* $(x, y) = (0, a)$, *then*
  $d(0 \wedge a) = d(0) = 0$ *and* $(d0 \wedge a) \vee (0 \wedge da) = 0 \vee 0 = 0$;
- *if* $(x, y) = (0, b)$, *then*
  $d(0 \wedge b) = d(0) = 0$ *and* $(d0 \wedge b) \vee (0 \wedge db) = (0 \wedge b) \vee (0 \wedge b) = 0 \vee 0 = 0$;
- *if* $(x, y) = (0, 1)$, *then*
  $d(0 \wedge 1) = d(0) = 0$ *and* $(d0 \wedge 1) \vee (0 \wedge d1) = (0 \wedge 1) \vee (0 \wedge 0) = 0 \vee 0 = 0$;
- *if* $(x, y) = (a, a)$, *then*
  $d(a \wedge a) = d(a) = b$ *and* $(da \wedge a) \vee (a \wedge da) = (b \wedge a) \vee (a \wedge b) = b \vee b = b$;
- *if* $(x, y) = (a, b)$, *then*
  $d(a \wedge b) = d(b) = b$ *and* $(da \wedge b) \vee (a \wedge db) = (b \wedge a) \vee (a \wedge b) = b \vee b = b$;
- *if* $(x, y) = (a, 1)$, *then*
  $d(a \wedge 1) = d(a) = b$ *and* $(da \wedge 1) \vee (a \wedge d1) = (b \wedge 1) \vee (a \wedge 0) = b \vee 0 = b$;
- *if* $(x, y) = (b, b)$, *then*
  $d(b \wedge b) = d(b) = b$ *and* $(db \wedge b) \vee (b \wedge db) = (b \wedge b) \vee (b \wedge b) = b \vee b = b$;
- *if* $(x, y) = (b, 1)$, *then*
  $d(b \wedge 1) = d(b) = b$ *and* $(db \wedge 1) \vee (b \wedge d1) = (b \wedge 1) \vee (b \wedge 0) = b \vee 0 = b$;
- *if* $(x, y) = (1, 1)$, *then*
  $d(1 \wedge 1) = d(1) = 0$ *and* $(d1 \wedge 1) \vee (1 \wedge d1) = (0 \wedge 1) \vee (1 \wedge 0) = 0 \vee 0 = 0$.

Example 2.3. *Let $(L = D(6), |, gcd, lcm)$. We define a function $d : D(6) \to D(6)$ in the following table:*

x	1	2	3	6
dx	1	3	2	6



Then d is not a derivation on $D(6)$. Because if $x = 2$ and $y = 3$, then

$$d(2 \wedge 3) = 1 \neq (d2 \wedge 3) \vee (2 \wedge d3) = 6.$$

2.1.2 Properties of derivations on lattices

Proposition 2.1. *[20, 19] Let d be a derivation on L. Then the following holds:*

(i) $dx \leqslant x$, *for any* $x \in L$;

(ii) $d(dx) = dx$, *for any* $x \in L$;

(iii) If L has a least element $0 \in L$, then $d0 = 0$.

Proof. (i) For all $x \in L$, we have

$$dx = d(x \wedge x) = (dx \wedge x) \vee (x \wedge dx) = dx \wedge x.$$

So $dx \leqslant x$.

(ii) In one hand, the first property (i) guarantees that $d(dx) \leqslant dx$, so $d(dx) \leqslant x$. Then $x \wedge d(dx) = d(dx)$. On the other hand, we have $dx \leqslant x$, then $dx = x \wedge dx$. The fact that d is a derivation implies that

$$\begin{aligned} d(dx) &= d(x \wedge dx) \\ &= (dx \wedge dx) \vee (x \wedge d(dx)) \\ &= dx \vee (x \wedge d(dx)) \\ &= dx \vee d(dx) \\ &= dx. \end{aligned}$$

Therefore, $d(dx) = dx$.

(iii) From (i), we have $dx \leqslant x$, for all $x \in L$. So if $x = 0$, then 0 is the least element of L, then $d0 = 0$.

□

The second property in the above Proposition leads to the following corollary.

Corollary 2.1. *[18] Define $d^2(x) = d(dx)$ for all $x \in L$. Then we have $d^2 = d$.*

Proposition 2.2. *[19] Let d be a derivation on L. Then for any $x, y \in L$:*

$$dx = dx \vee (x \wedge d(x \vee y)).$$

Proof. Let $x, y \in L$. Since $dx \leqslant x$ and $x \leqslant x \vee y$, it holds that $dx \leqslant x \vee y$. So $dx \wedge (x \vee y) = dx$. Then

$$\begin{aligned} dx &= d((x \vee y) \wedge x) \\ &= (d(x \vee y) \wedge x) \vee ((x \vee y) \wedge dx) \\ &= (d(x \vee y) \wedge x) \vee dx \\ &= dx \vee (x \wedge d(x \vee y)). \end{aligned}$$

□

Proposition 2.3. *[19] If a lattice L has a greatest element 1 and d is a derivation on L, then*

$$dx = (x \wedge d1) \vee dx, \text{ for all } x \in L.$$

Proof. Let $x \in L$, then

$$\begin{aligned} dx &= d(x \wedge 1) \\ &= (dx \wedge 1) \vee (x \wedge d1) \\ &= dx \vee (x \wedge d1). \end{aligned}$$

□

Corollary 2.2. *[19] Let L be a lattice with a greatest element 1 and d is a derivation on L. Then $d1 = 1$ if and only if d is an identity derivation.*

Corollary 2.3. *[19] Let L be a lattice with a greatest element 1 and d be a derivation on L. Then we have*

1) if $x \geqslant d1$, then $dx \geqslant d1$;

2) if $x \leqslant d1$, then $dx = x$.

Proposition 2.4. *[18] Let d be a derivation on L. If $y \leqslant x$ and $dx = x$, then $dy = y$.*

2.2 Classes of derivations

Next, we provide two classes of derivations on lattices, which are isotone and principal derivations.

2.2.1 Isotone derivations

Definition 2.2. *[19] A derivation d on L is called isotone if it satisfies the following condition:*

$$dx \leqslant dy \text{ whenever } x \leqslant y, \text{ for any } x, y \in L.$$

Example 2.4. *The given derivation d of Example 2.2 is isotone.*

Theorem 2.1. *[18] Let d be a derivation. Then the following are equivalent:*

i) d *is an isotone;*

ii) $d(x \wedge y) = dx \wedge y$;

iii) $d(x \wedge y) = dx \wedge dy$.

Theorem 2.2. *[18] Let L be a distributive lattice and d be a derivation on L. Then the following are equivalent:*

1) d *is isotone;*

2) $d(x \wedge y) = dx \wedge dy$;

3) $d(x \vee y) = dx \vee dy$.

Now, we present a characterization of distributive lattices based on their isotone derivations.

Theorem 2.3. *[19] Let L be a lattice. Then the following are equivalent:*

(i) L *is distributive;*

(ii) Every isotone derivation d of L satisfies $d(x \vee y) = dx \vee dy$, for any $x, y \in L$.

2.2.2 Principal derivations

Theorem 2.4. *[19] Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $\alpha \in L$ and $d_\alpha : L \to L$ be a function such that*

$$d_\alpha(x) = \alpha \wedge x, \text{ for any } x \in L.$$

Then d_α is a derivation on L.

Proof. Let $x, y \in L$, then $d_\alpha(x \wedge y) = \alpha \wedge (x \wedge y) = \alpha \wedge x \wedge y$. Also

$$\begin{aligned}(d_\alpha(x) \wedge y) \vee (x \wedge d_\alpha(y)) &= ((\alpha \wedge x) \wedge y) \vee (x \wedge (\alpha \wedge y)) \\ &= (\alpha \wedge x \wedge y) \vee (\alpha \wedge x \wedge y) \\ &= (\alpha \wedge x \wedge y).\end{aligned}$$

Thus $d_\alpha(x \wedge y) = (d_\alpha(x) \wedge y) \vee (x \wedge d_\alpha(y))$. Hence d_α is a derivation on L. □

Definition 2.3. *[19] d_α is called principal derivation on L.*

Example 2.5. *The lattice $L = D(6)$ has four principal derivations are d_1, d_2, d_3 and d_6 defined in this table:*

x	1	2	3	6
$d_1(x)$	1	1	1	1
$d_2(x)$	1	2	1	2
$d_3(x)$	1	1	3	3
$d_6(x)$	1	2	3	6

Proposition 2.5. *[18] Every principal derivation d_α is isotone.*

Proof. Let d_α be a principal derivation on a lattice L and $x, y \in L$ with $x \leqslant y$. Then we have $d_\alpha(x) = x \wedge \alpha \leqslant y \wedge \alpha = d_\alpha(y)$, hence d_α is isotone. □

2.3 Integrations on lattices

In this section, we give the concept of integrations on lattices and their important properties. More information on this concept can be got in [22].

Definition 2.4. *[22] Let $(L, \wedge, \vee)$ be a lattice and d be a derivation on L. Let $i_d : L \to \mathcal{P}(L)$ be a function defined as $i_d(x) = d^{-1}(x) = \{z \in L \mid dz = x\}$ for any $x \in L$. The function i_d is called the integration with respect to d (the d-integration, for short) on L. The set $i_d(x)$ is called the integral set of x with respect to d (the d-integral set of x, for short).*

Definition 2.5. *[22] Let $(L, \wedge, \vee)$ be a lattice and i_d be a d-integration on L. An element x of L is called an integrable element with respect to i_d (a d-integrable element, for short) if $i_d(x) \neq \emptyset$.*

Example 2.6. *[22]*

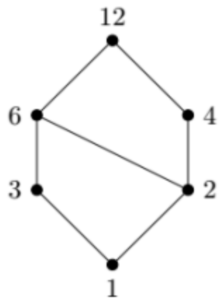


Figure 2.1: The Hasse diagram of the lattice $(D(12), gcd, lcm)$

Let $(D(12), gcd, lcm)$ be the lattice of positive divisors of 12 ordered by the divisibility order relation $\mid$ and given by the Hasse diagram in Figure 2.1. Let d be a principal derivation on $D(12)$ defined as $d(x) = d_6(x) = gcd(6, x)$, for any $x \in D(12)$. The d-integration function i_d on $D(12)$ is defined in the following table:

x	1	2	3	4	6	12
$d(x)$	1	2	3	2	6	6
$i_d(x)$	$\{1\}$	$\{2,4\}$	$\{3\}$	$\emptyset$	$\{6,12\}$	$\emptyset$

Corollary 2.4. *[22] From Example 2.6, we conclude that $1, 2, 3, 6$ are d-integrable elements of $D(12)$ but $4, 12$ are not.*

Remark 2.1. *[22] In any lattice $(L, \wedge, \vee, 0)$ with 0, the least element 0 is an integrable element with respect to any d-integration on L. Indeed, let i_d be a d-integration on L. Since $d0 = 0$ (see Proposition 2.1) we conclude that $i_d(0) \neq \emptyset$. Hence 0 is a d-integrable element of L.*

Lemma 2.1. *[22] Let $(L, \wedge, \vee)$ be a lattice and i_d be a d-integration on L. Then an element x of L is a d-integrable element if and only if x is a fixed point of d, i.e., $d(x) = x$.*

Proof. To prove the direct implication, suppose that x is a d-integrable element of L. Then there exists $z \in L$ such that $dz = x$. Applying the proposition 2.1 (ii), we obtain $dx = d(dz) = dz = x$. Thus, x is a fixed point of d. The converse implication is immediate. □

Remark 2.2. *[22] Let $(L, \wedge, \vee)$ be a lattice and d be a derivation on L. Consider $Fix_d(L) = \{x \in L \mid d(x) = x\}$ is the set of fixed points of d, then it is a down-set. Moreover, if d is isotone, then $Fix_d(L)$ is an ideal of L.*

Proposition 2.6. *[22] Let $(L, \wedge, \vee)$ be a lattice, i_d be a d-integration on L and x be a d-integrable element of L. Then it holds that:*

(i) x is the least element of $i_d(x)$;

(ii) if $y \leqslant x$, then y is also a d-integrable element of L for any $y \in L$.

Proof. Suppose that x is a d-integrable element of L.

(i) On the one hand, Lemma 2.1 guarantees that $d(x) = x$, so $x \in i_d(x)$. On the other hand, let $z \in i_d(x)$, then $dz = x$. Since d is a derivation on L, it holds from proposition 2.1 that $dz \leqslant z$. Hence, $x \leqslant z$. Thus, x is the least element of $i_d(x)$.

(ii) Using Lemma 2.1 guarantees that $x \in Fix_d(L)$. Let $y \in L$ such that $y \leqslant x$. The fact that $Fix_d(L)$ is a down-set (see remark 2.2) implies $y \in Fix_d(L)$, so $d(y) = y$. Therefore, y is also a d-integrable element of L.

□

Proposition 2.7. *[22] Let $(L, \wedge, \vee)$ be a lattice and i_d be a d-integration on L such that d is isotone. The following implications hold:*

(i) if $y_1 \in i_d(x_1)$ and $y_2 \in i_d(x_2)$, then $y_1 \wedge y_2 \in i_d(x_1 \wedge x_2)$ for any $x_1, x_2, y_1, y_2 \in L$;

(ii) if $y_1 \in i_d(x_1)$, $y_2 \in i_d(x_2)$ and L is distributive, then $y_1 \vee y_2 \in i_d(x_1 \vee x_2)$ for any $x_1, x_2, y_1, y_2 \in L$.

Proof. Let $x_1, x_2, y_1, y_2 \in L$ such that $y_1 \in i_d(x_1)$ and $y_2 \in i_d(x_2)$, then $d(y_1) = x_1$ and $d(y_2) = x_2$.

(i) The fact that d is an isotone derivation on L implies from theorem 2.2 that $d(y_1 \wedge y_2) = dy_1 \wedge dy_2 = x_1 \wedge x_2$. Thus, $y_1 \wedge y_2 \in i_d(x_1 \wedge x_2)$.

(ii) Since L is distributive and d is an isotone derivation on L, we conclude from theorem 2.2 that $d(y_1 \vee y_2) = dy_1 \vee dy_2 = x_1 \vee x_2$. Therefore, $y_1 \vee y_2 \in i_d(x_1 \vee x_2)$.

□

Proposition 2.8. *[22] Let $(L, \wedge, \vee)$ be a lattice and i_{d_1}, i_{d_2} be two integrations on L such that $d_1 \leqslant d_2$ (i.e., $d_1(x) \leqslant d_2(x)$ for any $x \in L$). If x is a d_1-integrable element of L, then x is also a d_2-integrable element.*

Proof. Let x be a d_1-integrable element of L. Then from Lemma 2.1, we have x is a fixed point of d_1, i.e., $d_1(x) = x$. On the one hand, the fact that $d_1 \leqslant d_2$ implies $x = d_1(x) \leqslant d_2(x)$. Hence, $x \leqslant d_2(x)$. On the other hand, since d_2 is a derivation on L, we obtain from Proposition 2.1 that $d_2(x) \leqslant x$. Thus, $d_2(x) = x$, i.e., x is also a fixed point of d_2. Therefore, Lemma 2.1 guarantees that x is also a d_2-integrable element. □

Proposition 2.9. *[22] Let $(L, \wedge, \vee, 0)$ be a lattice with a least element 0 and i_d be a d-integration on L. Let x be a d-integrable element of L. Then the following statements hold:*

(i) $m \wedge i_d(x) \subseteq i_d(m \wedge x) \subseteq i_d(m \wedge dx)$ for any $m \in i_d(0)$, where $m \wedge i_d(x) = \{m \wedge z | z \in i_d(x)\}$;

(ii) if L is distributive and d is isotone, then $m \vee i_d(x) \subseteq i_d(x)$ for any $m \in i_d(0)$, where $m \vee i_d(x) = \{m \vee z | z \in i_d(x)\}$.

Proof. Let x be a d-integrable element of L and $m \in i_d(0)$. We know from Proposition 2.6 *(ii)* that $m \wedge x$ is also a d-integrable elements of L, because $m \wedge x \leqslant x$. Hence, $i_d(m \wedge x) \neq \emptyset$.

(i) Let $y \in m \wedge i_d(x)$, then $y = m \wedge z$ with $dz = x$. Since d is a derivation on L and $dm = 0$, it follows that

$$dy = d(m \wedge z) = (dm \wedge z) \vee (m \wedge dz) = m \wedge dz = m \wedge x.$$

Hence, $y \in i_d(m \wedge x)$. Thus, $m \wedge i_d(x) \subseteq i_d(m \wedge x)$. Next, let $t \in i_d(m \wedge x)$, then $dt = m \wedge x$. Using the fact that d is a derivation on L and Proposition 2.1 we obtain

$$dt = d(dt) = d(m \wedge x) = (dm \wedge x) \vee (m \wedge dx) = m \wedge dx.$$

Hence, $t \in i_d(m \wedge dx)$. Therefore, $i_d(m \wedge x) \subseteq i_d(m \wedge dx)$. Consequently,

$$m \wedge i_d(x) \subseteq i_d(m \wedge x) \subseteq i_d(m \wedge dx).$$

(ii) Let $y \in m \vee i_d(x)$, then $y = m \vee z$ with $dz = x$. Since L is distributive and d is isotone derivation, it follows from Theorem 2.3 that $dy = d(m \vee z) = dm \vee dz = dz = x$. Hence, $y \in i_d(x)$. Therefore, $m \vee i_d(x) \subseteq i_d(x)$.

$\square$

Chapter 3

Algebraic Linear Differential Equations on Boolean Lattices

This chapter is the main goal of this memory. Here, we present the newly notion of algebraic linear differential equations on Boolean lattices. Further, we establish the necessary and sufficient conditions to solve those equations [22].

3.1 Solving algebraic linear differential equations

Definition 3.1. *[22] Let $(B, \wedge, \vee, 0, 1,')$ be a Boolean lattice and $\leqslant$ be its order relation. Let i_d be an integration on B and a, b be two elements of B. A linear differential equation with respect to d is any equation with the form*

$$a \cdot d(x) + b = 0. \tag{3.1}$$

where $a \cdot d(x) = a \wedge d(x)$ and $x + y = (x \wedge y^{'}) \vee (x^{'} \wedge y)$, for any $x, y \in B$.

Theorem 3.1. *[22] Let $(B, \wedge, \vee, 0, 1,')$ be a Boolean lattice and i_d be an integration on B. Then the equation (3.1) has a solution if and only if $b \leqslant a$ and b is a d-integrable element of B.*

Proof. Let x_0 be a solution of the equation (3.1), then $a \cdot d(x_0) + b = 0$, so $a \cdot d(x_0) + b + b = b$. Thus, $a \cdot d(x_0) + 0 = b$, hence $a \cdot d(x_0) = b$. Therefore

$$a \wedge d(x_0) = b.$$

Then, $b \leqslant a$. On an other hand, the fact that d is a derivation on B implies that

$$\begin{aligned} d(b) &= d(a \wedge d(x_0)) \\ &= (d(a) \wedge d(x_0)) \vee (a \wedge d(d(x_0))) \\ &= (d(a) \wedge d(x_0)) \vee (a \wedge d(x_0)) \\ &= (d(a) \wedge d(x_0)) \vee b. \end{aligned}$$

Thus, $b \leqslant d(b)$. Also, from Proposition 2.1 (i) we have $d(b) \leqslant b$. Then $d(b) = b$, i.e., b is a fixed point of d. Therefore, Lemma 2.1 guarantees that b is a d-integrable element of B. Conversely, suppose that $b \leqslant a$ and b is a d-integrable element of B. Then $a \cdot b = a \wedge b = b$ and $d(b) = b$. Hence, $a \cdot d(b) + b = a \cdot b + b = b + b = 0$. Thus, b is a solution of the equation (3.1). □

Corollary 3.1. *[22] In view of proof of Theorem 3.1, we conclude that if the equation (3.1) has solutions, then b is always a solution for this equation (3.1).*

Example 3.1. *[22] Let $(D(30), gcd, lcm, 1, 30, ')$ be the Boolean lattice of positive divisors of 30 ordered by the divisibility order relation $|$ and given by the Hasse diagram in Figure 3.1. Let d be a principal derivation on $D(30)$ defined as $d(x) = gcd(6, x)$ for any $x \in D(30)$. The d-integration function i_d on $D(30)$ is defined in the following table:*

x	1	2	3	5	6	10	15	30
$d(x)$	1	2	3	1	6	2	3	6
$i_d(x)$	$\{1,5\}$	$\{2,10\}$	$\{3,15\}$	$\emptyset$	$\{6,30\}$	$\emptyset$	$\emptyset$	$\emptyset$

Let $10 \cdot d(x) + 2 = 0$ and $10 \cdot d(x) + 5 = 0$ be two linear differential equations on $D(30)$. Since $2 \mid 10$ and 2 is a d-integrable element of $D(30)$, then Theorem 3.1 guarantees that $10 \cdot d(x) + 2 = 0$ has solutions. The solutions are given by this set:

$$S = \{2, 6, 10, 30\}.$$

Moreover, the fact that $5 \mid 10$ but 5 is not a d-integrable element of $D(30)$, then Theorem 3.1 guarantees that the $10 \cdot d(x) + 5 = 0$ has not solutions in $D(30)$.

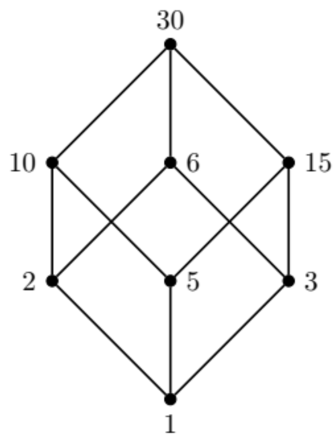


Figure 3.1: The Hasse diagram of the lattice $(D(30), gcd, lcm, 1, 30, ')$.

Theorem 3.2. *[22] Let $(B, \wedge, \vee, 0, 1, ')$ be a Boolean lattice and i_d be an integration on B. If x_0 is a solution of the equation (3.1), then $b \leqslant d(x_0) \leqslant a + b + 1$.*

Proof. [22] Let $x_0 \in B$ is a solution of the equation (3.1), i.e., $a \cdot d(x_0) + b = 0$. Then $a \cdot d(x_0) = b$, so on the one hand $b \leqslant d(x_0)$. On the other hand,

$$\begin{aligned}(a+b+1)\cdot d(x_0) &= a\cdot d(x_0) + b\cdot d(x_0) + 1\cdot d(x_0)\\ &= b + b + d(x_0)\\ &= d(x_0).\end{aligned}$$

Thus, $d(x_0) \leqslant (a+b+1)$. Therefore, $b \leqslant d(x_0) \leqslant a+b+1$. □

Example 3.2. *Let $(D(210), gcd, lcm, 1, 210,')$ be the Boolean lattice of positive divisors of 210 ordered by the divisibility order relation | and shown in Figure 3.2.*

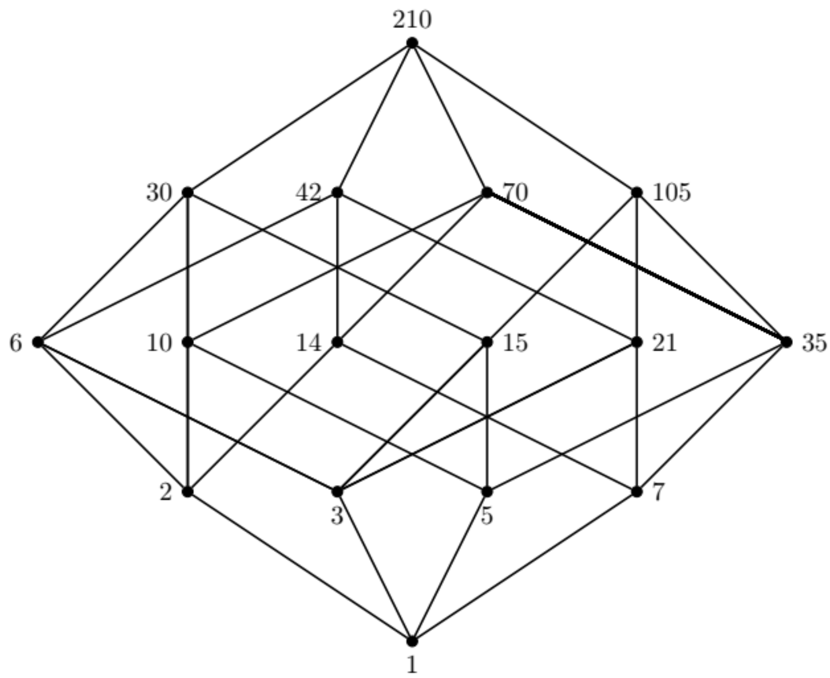


Figure 3.2: The Hasse diagram of the Boolean lattice $(D(210), gcd, lcm, 1, 210,')$.

Let d be a principal derivation on $D(210)$ defined as $d(x) = gcd(15, x)$ for any $x \in D(210)$. The d-integration function i_d on $D(210)$ is defined in the following table:

x	1	2	3	5	6	7	10	14
$d(x)$	1	1	3	5	3	1	5	1
$i_d(x)$	$\{1,2,7,14\}$	$\emptyset$	$\{3,6,21,42\}$	$\{5,10,35,70\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x	15	21	30	35	42	70	105	210
$d(x)$	15	3	15	5	3	5	15	15
$i_d(x)$	$\{15,30,105,210\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Let $30 \cdot d(x) + 5 = 0$ and $30 \cdot d(x) + 6 = 0$ be two linear differential equations on $D(210)$. Since $5 \mid 30$ and 5 is a d-integrable element of $D(210)$, then Theorem 3.1 guarantees that $30 \cdot d(x) + 5 = 0$

has solutions. The solutions are given by this set:

$$S = \{5, 10, 35, 70\}.$$

Moreover, the fact that $6 \mid 30$ *but* 6 *is not a d-integrable element of* $D(210)$*, then Theorem 3.1 guarantees that the* $30 \cdot d(x) + 6 = 0$ *has not solutions in* $D(210)$*.*

3.2 Python programs for solving linear differential equations of Boolean lattices

Here, we use Python Programs for solving those linear differential equations of Boolean lattices.

```
import math
from tabulate import tabulate

n = int(input("  ادخل قيمةn:"))
L = sorted([i for i in range(1, n+1) if n % i == 0])
print("الشبكة هي")
print(f"L = {{{', '.join(str(x) for x in L)}}}")

alpha = int(input("  ادخل قيمةalpha:"))
print("عبارة المشتقة الأساسية هي")
print(f"dx = {alpha}.x")

def d(x):
    return math.gcd(alpha, x)

id_table = {}
for x in L:
    dx = d(x)
    if dx not in id_table:
        id_table[dx] = []
    id_table[dx].append(x)

table_data = []
seen_dx = set()

for x in L:
    dx = d(x)
    if dx not in seen_dx:
        i_dx = "{" + ", ".join(str(i) for i in id_table[dx]) + "}"
        seen_dx.add(dx)
    else:
        i_dx = "Ø"
    table_data.append([x, dx, i_dx])

headers = ["x", "dx", "i_dx"]
print(tabulate(table_data, headers=headers, tablefmt="fancy_grid",
stralign="center"))

result = {x for x in L if d(x) == x}
print("العناصر القابلة للمكاملة هي:", result)

a = int(input("  ادخل قيمةa:"))
b = int(input("  ادخل قيمةb:"))

def meet(x):
    return math.gcd(alpha, x)

def solve_boolean_eq(a, b):
    if b > a or a % b != 0 or meet(b) != b:
```

```
        return []
    solutions = []
    for dx in id_table:
        if math.gcd(a, dx) == b:
            solutions.extend(id_table[dx])
    return sorted(solutions)

print(f"المعادلة التفاضلية من الشكل:")
print(f"{a}.dx + {b} = 0")

bool_eq_solution = solve_boolean_eq(a, b)

if bool_eq_solution:
    print(f"حلول هذه المعادلة التفاضلية هي: {{{', '.join(str(x) for x
in bool_eq_solution)}}}")

else:
    print("هذه المعادلة التفاضلية لا تملك حلول")
```

ادخل قيمة n:210
الشبكة هي
L = {1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210}
ادخل قيمة alpha:15
عبارة المشتقة الأساسية هي
dx = 15.x

x	dx	i_dx
1	1	{1, 2, 7, 14}
2	1	Ø
3	3	{3, 6, 21, 42}
5	5	{5, 10, 35, 70}
6	3	Ø
7	1	Ø
10	5	Ø
14	1	Ø
15	15	{15, 30, 105, 210}
21	3	Ø
30	15	Ø
35	5	Ø
42	3	Ø
70	5	Ø
105	15	Ø
210	15	Ø

العناصر القابلة للمكاملة هي : {1, 3, 5, 15}
ادخل قيمة a:30
ادخل قيمة b:5
المعادلة التفاضلية من الشكل:
30.dx + 5 = 0
حلول هذه المعادلة التفاضلية هي: {5, 10, 35, 70}

Bibliography

[1] A.Y. Abdelwanis and A. Boua, *On generalized derivations of partially ordered sets.* Communications in Mathematics **27** (2019), 69-78.

[2] A. Y. Abdelwanis and S. Ali, Skew derivations on partially ordered sets, Indian Journal of Pure and Applied Mathematics, 52(2021), 1256-1262.

[3] A. Y. Abdelwanis and S. Ali, Symmetric bi-derivations on posets. Indian Journal of Pure and Applied Mathematics, 54(2)(2023), 421-427.

[4] S. Ali, N. N. Rafiquee, and V. Varshney, Certain types of derivations in rings: a survey. , Journal of the Indonesian Mathematical Society, 30(2) (2024)

[5] A. Amroune, L. Zedam and M. Yettou, (F, G)*-derivations on a lattice.* Kragujevac Journal of Mathematics **46** (2022), 773-778.

[6] M. Ashraf, S. Ali and C. Haetinger, *On derivations in rings and their applications.* The Aligarh Bulletin of Mathematics **25** (2006), 79–107.

[7] I. Banič, *Integrations on rings.* Open Mathematics **15** (2017), 365–373.

[8] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order.* 2nd edition, Cambridge University Press, Cambridge, 2002.

[9] L. Ferrari, *On derivations of lattices.* Pure Math. Appl. **12** (2001), 365–382.

[10] Y.Q. Guo and X.L. Xin, *On Derivation of Pseudo L-Algebras.* Journal of Mahani Mathematical Research Center **14.1** (2025).

[11] P. He, X.L. Xin and J. Zhan, *On derivations and their fixed point sets in residuated lattices.* Fuzzy Sets and Systems **303** (2016), 97–113.

[12] A. Humam and P. Astuti. On the Structure of Characteristic Subgroup Lattices of Finite Abelian p-Groups. Jordan Journal of Mathematics and Statistics, **15(3)** (2023), 435–444.

[13] B. Kolman, R.C. Busby and S.C. Ross, *Discrete Mathematical Structures.* 4th edition, Prentice Hall PTR, 2000.

[14] E.C. Posner, *Derivations in prime rings.* Proc. Am. Math. Soc. **8** (1957), 1093-1100.

[15] S. Roman, *Lattices and Ordered Sets.* Springer Science+Business Media, New York, 2008.

[16] G. Szász , *Derivations of lattices*, Acta Sci. Math. **37** (1975), 149–154.

[17] J. Wang, Y. Jun, X.L. Xin, T.Y. Li and Y. Zou, *On derivations of bounded hyperlattices.* Journal of Mathematical Research with Applications **36** (2016), 151–161.

[18] X.L. Xin, *The fixed set of a derivation in lattices.* Fixed Point Theory and Application **218** (2012), 1–12.

[19] X.L. Xin, T.Y. Li and J.H. Lu, *On derivations of lattices.* Information Sciences **178** (2008), 307–316.

[20] M. Yettou, doctoral thesis: (F,G)-derivations on lattices, University of M'sila, Algeria, 2020.

[21] M. Yettou and A. Amroune, *Homoderivations on a lattice.* Jordan Journal of Mathematics and Statistics **15.2** (2022), 325–338.

[22] M. Yettou, A. Amroune and L. Zedam, *Integrations on lattices.* Miskolc Mathematical Notes **24.1** (2023); 515–528.

[23] M. Zaoui, D. Gretete and B. Fahid, (f,g)*-Derivations in Residuated Lattices.* Journal of Algebraic Systems **13.1** (2025), 21–36.

[24] L. Zedam, M. Yettou and A. Amroune, f*-fixed points of isotone* f*-derivations on a lattice.* Discussiones Mathematicae-General Algebra and Applications **39** (2019); 69–89.

[25] H. Zhang and Q. Li, *On derivations of partially ordered sets.* Mathematica Slovaca **67** (2017), 17-22.

ملخص:

في هذه المذكرة، ركزنا على دراسة مفهوم المعادلات التفاضلية الخطية الجبرية على الشبكات البوليانية. ولتحقيق ذلك، قسمنا مذكرتنا إلى ثلاثة وحدات. خصصنا الوحدة الأولى لاستعراض المفاهيم الأساسية وخصائص الشبكات. علاوة على ذلك، عرضنا بعض فئات الشبكات، كالشبكات التوزيعية،المعيارية، و البوليانية. في الوحدة الثانية، ركزنا دراستنا على مفاهيم الاشتقاقات والتكاملات على الشبكات وخصائصها. أما الوحدة الثالثة، فهي الهدف الرئيسي لهذه المذكرة، حيث قدمنا المفهوم الجديد للمعادلات التفاضلية الخطية الجبرية على الشبكات البوليانية وحددنا الشروط اللازمة والكافية لحل هذه المعادلات التفاضلية. كذلك قمنا ببرمجة برامج باستعمال البايثون لحل هذه المعادلات التفاضلية.

الكلمات المفتاحية: مجموعة مرتبة، شبكة، شبكة بوليانية، مشتقة، تكامل، معادلة تفاضلية خطية جبرية.

Abstract:

In this memory, we have focused on studying the concept of algebraic linear differential equations on Boolean lattices. To that end, we have organized our thesis into three chapters. We have devoted the first chapter for recalling basic concepts and properties of lattices. Moreover, we have presented some classes of lattices, distributive, modular and Boolean lattices. In chapter 2, we have focused our study on the concepts of derivations and integrations on lattices, and their properties. The third chapter is the main goal of this thesis, where we provided the newly notion of algebraic linear differential equations on Boolean lattices. Further, we have established the necessary and sufficient conditions to sole those differential equations. In addition, we have used Python Programs for solving those linear differential equations.

Key words: Poset, lattice, Boolean lattice, derivation, integration, algebraic linear differential equation.

Résumé:

Dans ce mémoire, nous nous sommes concentrés sur l'étude du concept d'équations différentielles linéaires algébriques sur treillis booléens. À cette fin, nous avons organisé notre mémoire en trois chapitres. Le premier chapitre a été consacré au rappel des concepts et propriétés fondamentaux des treillis. De plus, nous avons présenté quelques classes de treillis : distributives, modulaires et booléens. Le deuxième chapitre a porté sur les concepts de dérivations et d'intégrations sur les treillis, ainsi que sur leurs propriétés. Le troisième chapitre, principal objectif de cette mémoire, a fourni la nouvelle notion d'équations différentielles linéaires algébriques sur réseaux booléens. De plus, nous avons établi les conditions nécessaires et suffisantes pour résoudre ces équations différentielles. Nous avons aussi utilisé des programmes Python pour résoudre ces équations différentielles linéaires.

Mots clés : Poset, treillis, treillis booléen, derivation, integration, équation différentielle linéaire algébrique.