PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf University of M'sila
Faculty of Mathematics and Computer Science
Department of Mathematics

Master memory

Field : Mathematics and computer sciences
Branch : Mathematics
Option : Mathematical and Numerical Analysis

Theme

Operational Matrix via Boole Polynomials for Solving Linear Volterra-Fredholm Integro-Differential Equations

Presented by:
OUADAH Selma

Publicly supported on :June 12, 2026.

In front of the jury composed of:

KHAIRANI Amina	MCA,	Mohamed Boudiaf University of M′sila	**President**.
DJAIDJA Noui	MCA,	Mohamed Boudiaf University of M′sila	**Supervisor.**
SEGHIRI Fakheredine	MCB,	Mohamed Boudiaf University of M′sila	**Examiner.**

University year 2025/2026

الملخص:

تقدم هذه الأطروحة نهجاً عددياً لحل المعادلات التكاملية التفاضلية الخطيّة من نوع فولتيرا-فريدهولم ذات الرتب العليا بالاعتماد على متعددات حدود "بول "ومن خلال بناء مصفوفة مؤثرة للمشتقات والتكامل، يتم تحويل المعادلات الحاكمة إلى منظومة من المعادلات الجبرية. كما تم تقديم العديد من الأمثلة العددية لإثبات دقة وكفاءة وموثوقية الطريقة الحسابية المقترحة.

الكلمات المفتاحية :المعادلات التكاملية التفاضلية من نوع فولتيرا-فريدهولم؛ متعددات حدود بول؛ المصفوفة المؤثرة؛ التحليل العددي؛ معادلات الرتب العليا.

Abstract:

This thesis presents a numerical approach for solving high-order linear Volterra-Fredholm Integro-Differential Equations based on Boole polynomials. By constructing an operational matrix of derivatives and integration, the governing equations are transformed into a system of algebraic equations. Several numerical examples are provided to demonstrate the accuracy, efficiency, and reliability of the proposed computational method.

Keywords: Volterra-Fredholm Integro-Differential Equations; Boole Polynomials; Operational Matrix; Numerical Analysis; High-Order Equations.

Résumé :

Cette thèse présente une approche numérique pour la résolution d'équations intégro-différentielles linéaires de Volterra-Fredholm d'ordre élevé, basée sur les polynômes de Boole. En construisant une matrice opérationnelle de dérivation et d'intégration, les équations régissantes sont transformées en un système d'équations algébriques. Plusieurs exemples numériques sont présentés pour démontrer l'exactitude, l'efficacité et la fiabilité de la méthode de calcul proposée.

Mots-clés : Équations intégro-différentielles de Volterra-Fredholm ; Polynômes de Boole ; Matrice opérationnelle ; Analyse numérique ; Équations d'ordre élevé.

Acknowledgements

First of all, I thank Allah for making it possible for me to successfully complete this work.
I would like to express my profound gratitude and appreciation to the esteemed Professor DJAIDJA Noui, who has inspired me with his immense generosity, dedication, and exceptional mentorship. His patience, precision, and unwavering support have made him a true role model of devotion to the noble mission of education.
I also express my sincere gratitude to the members of the discussion committee professors KHAIRANI Amina and SEGHIRI Fakheredine, for kindly accepting to discuss this work and providing constructive feedback.
Lastly, I would like to express my sincere gratitude to the faculty members of the Faculty of Mathematics and Computer Science for their dedicated efforts, and to everyone who supported me throughout my academic journey.

Dedication

The journey was not short, nor was it meant to be. The dream was not close, nor was the path paved with ease but I arrived. Praise be to God in love, gratitude, and thankfulness.

I dedicate my graduation to the one whose name I bear with pride, to the one who supported me boundlessly and gave without expecting anything in return. To the one who cleared the thorns from my path to pave my way toward knowledge to my first teacher, my dear father.OUADAH Seddik.

To the heart that embraced me in prayers before embracing me in arms. To the steady pillar of strength through all my stumbles. To the very first person who believed in my abilities. To the tender and compassionate heart, my dear mother. GOUTTEL Fatima.

To those who stood by me in prosperity and adversity, who shared my moments of toil and joy, and who granted me strength: my dear brothers and sisters.

OUADAH Selma

Notations

$\mathbb{N}$	The set of natural numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$[a, b]$	Real interval.
$C([a, b])$	The space of continuous functions on $[a, b]$.
$L^2[a, b]$	The space of square-integrable functions on $[a, b]$
L^p	Lebesgue space, $1 \leq p$.
A	Linear operator.
$K_V(x, t)$	Kernel of the Volterra integral equation.
$K_F(x, t)$	Kernel of the Fredholm integral equation.
$u(x)$	The unknown function.
$u^{(k)}(x)$	The k-th order derivative of $u(x)$
$u_N(x)$	The approximate solution of order N
$B_i(x)$	The Boole polynomial of degree i
$\\| \cdot \\|$	A norm.
ε	Epsilon.
$\langle \cdot, \cdot \rangle$	Scalar product.
$\sum$	Summation sign.
$\int$	Integral sign.

Introduction

Integro-Differential Equations (IDEs) are essential for modeling complex phenomena in disciplines such as biology, physics, and engineering.

It is well known that solving these equations analytically is difficult, if not impossible; therefore, researchers have focused on developing numerical methods to obtain approximate solutions. One of the most frequently used numerical method is collocation method. In recent years, collocation methods such as Bessel, Chebyshev, Taylor polynomials and B-spline functions have been given for approximating the solutions of linear Volterra-Fredholm integro-differential equations.

In this thesis, we utilize a novel numerical technique based on collocation method and Boole polynomials to transform linear Volterra-Fredholm integro-differential equations into a system of matrix equations that can be easily solved.

This thesis is organized into three chapters, structured as follows:

Chapter 1: Introduces the fundamental concepts and essential defenitions.

Chapter 2: Prsents different types of Integral Equations and Integro-differential Equations. It also introduces Boole Polynomials and its properties.

Chapter 3: This chapter details the numerical solution technique for linear Volterra-Fredholm integro-differential equations with illustrative examples.

Chapter 1

Preliminaries

1.1 Functional Spaces:

1.1.1 Normed linear spaces

Definition 1.1.1. *(norms)[13]: let E be a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).A norm on the space E is called any function denoted by $\|.\|$ on E with values in $\mathbb{R}$, if it satisfies the following axioms:*

1. $\|u\| = 0$ *if and only if* $u = 0$.
2. $\|\lambda u\| = |\lambda|\|u\|$, *for every* $u \in E$ *and all scalars* $\lambda \in \mathbb{K}$. *(the homogeneity axiom)*
3. $\|u + v\| \leq \|u\| + \|v\|$ *for every* $u, v \in E$.*(the triangle inequality)*

Example 1.1.1. *define the function:* $\|u\| = |u|$

- *positivity:* $|u| \geq 0$ *and* $|u| = 0 \Leftrightarrow u = 0$
- *homogeneity:* $\|\lambda u\| = |\lambda u| = |\lambda||u| = |\lambda|\|u\|$
- *triangle inequality:* $\|u + v\| = |u + v| \leq |u| + |v| = \|u\| + \|v\|$

Definition 1.1.2. *(normed linear space)[13]: a vector space E endowed with a norm*

$$\begin{array}{rcl} \|.\| : E & \to & \mathbb{R}_+ \\ u & \mapsto & \|u\| \end{array}$$

Is called a normed linear space; the number $\|u\|$ is called the norm of $u \in E$.

1.1.2 Banach spaces

Definition 1.1.3. *(Cauchy sequences)[13]: a sequence (U_n) in a normed space $(E, \|.\|)$ is said to be a Cauchy sequence if, for every positive real number ϵ, there exists a natural number N such that for all indices $p, q \geq N$, the inequality $\|U_p - U_q\| < \epsilon$*

Remark 1.1.1. *every convergent sequence in a normed space is a Cauchy sequence. (the converse is not true)*

Definition 1.1.4. *(complete spaces)[13]:let* $(E, \|.\|)$ *be a normed vector space.*
The space E *is said to be complete if, every Cauchy sequence* (U_n) *in* E *converges in* E*.*

Definition 1.1.5. *(Banach spaces)[13]:let* $(E, \|.\|)$ *be a normed linear space, this space is called a Banach space if it is complete.*

Example 1.1.2. $L^p(a, b)$

1.1.3 Hilbert spaces

Definition 1.1.6. *(Iner product)[13]: Let* E *be a vector space. A real-valued function defined on* $E \times E$*, which assigns to each ordered pair* $(u, v) \in E \times E$ *a real number denoted by* $\langle u, v\rangle$*, is called an inner product on* E *if it satisfies the following axioms:*

1. $\langle u, u\rangle > 0$ *for every* $u \in E, u \neq 0$.
2. $\langle u, v\rangle = \langle v, u\rangle$ *for every* $u, v \in E$.
3. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ *for every* $u, v, w \in E$.
4. $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ *for every* $u, v \in E$ *and all scalars* λ.

A vector space E endowed with an inner product is called an inner product space.
Let E be an inner product space with inner product $\langle\cdot, \cdot\rangle$. The norm induced by the inner product is defined by

$$\|u\| = \sqrt{\langle u, u\rangle}, \quad \forall u \in E.$$

Definition 1.1.7. *(Hilbert space)[13]:a Hilbert space is a complete inner product space.*

Remark 1.1.2. *Every Hilbert space is a Banach space, (the converse is not true).*

Example 1.1.3. *The space* $L^2(\Omega)$*.*

$$L^2(\Omega) = \left\{\phi : \Omega \to \mathbb{R} \mid \int_\Omega |\phi(u)|^2 du < \infty\right\}$$

Inner product:

$$\langle \phi, \psi\rangle = \int_\Omega \phi(u)\psi(u)\, du$$

Norm induced by the inner product:

$$\|\phi\| = \sqrt{\langle \phi, \phi\rangle} = \left(\int_\Omega |\phi(u)|^2 du\right)^{1/2}$$

Example 1.1.4. *The space $L^p(\Omega)$.*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid \int_\Omega |f(x)|^p \, dx < \infty \right\}, \quad 1 \le p < \infty$$

Norm:

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}$$

$L^p(\Omega)$ is complete with respect to this norm, hence it is a Banach space.
For $p \neq 2$, the norm does not come from an inner product.
Therefore, $L^p(\Omega)$ with $p \neq 2$ is not a Hilbert space.

Definition 1.1.8. *(Orthogonality)[13]:Let H be a Hilbert space. Two vectors $u, v \in H$ are said to be orthogonal if*

$$\langle u, v \rangle = 0.$$

Property 1.1.1. *[13] A subset $G \subset H$ is said to be dense in H if every element of H can be approximated arbitrarily well by elements of G, that is:*

$$\forall v \in H, \ \forall \varepsilon > 0, \ \exists u \in G \ such \quad that \ \|v - u\| < \varepsilon.$$

Equivalent formulation:

The set G is dense in H if every element $v \in H$ is the limit of a sequence (u_n) of elements from G, i.e.:

$$\|u_n - v\| \to 0.$$

Definition 1.1.9. *(Hilbert Bases)[13]:Let H be a Hilbert space over $\mathbb{K}$($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and let $F = (e_i)_{i \in I}$ be a family of vectors. We say that F forms a Hilbert basis of H if the following two conditions are satisfied:*

1. *The family F is orthonormal, meaning that its elements are pairwise orthogonal and each vector has unit norm:*

$$\forall i \neq j, \ \langle e_i, e_j \rangle = 0, \quad and \quad \forall i, \ \|e_i\| = 1.$$

2. *The family F is complete, meaning that every element of H can be approximated arbitrarily well by finite linear combinations of elements of F; in other words, these combinations form a dense subset of H*

1.2 Linear operators

1.2.1 Bounded linear operators

Definition 1.2.1. *(Linear operators)[13]:Let E and F be normed spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).*
$A : E \to F$ is called a linear operator if it satisfies the following conditions:
$\forall u, v \in E, \quad \forall \lambda \in \mathbb{K}$

1. $A(u + v) = A(u) + A(v)$

2. $A(\lambda u) = \lambda A(u)$

Continuity of Linear Operators:

Definition 1.2.2. *[13] Let E and F be normed spaces, and let $A : G \subset E \to F$ be a linear operator. The operator A is said to be continuous at a point $u_0 \in G$ if for every sequence $(u_n) \subset G$ such that $u_n \to u_0$, we have $A(u_n) \to A(u_0)$, i.e.*

$$\lim_{n\to\infty} A(u_n) = A(u_0).$$

Remark 1.2.1. *An operator A is said to be continuous on G if it is continuous at every point of G, i.e., if it is continuous at each $u \in G$.*

Theorem 1.2.1. *[13] Let E and F be normed spaces, and let $A : G \subset E \to F$ be a linear operator. We say that A is continuous on G if it is continuous at the point $u_0 \in G$.*

Bounded Linear Operators

Definition 1.2.3. *[13] A linear operator $A : E \to F$ is said to be bounded if there exists a positive constant $C > 0$ such that:*

$$\|A(u)\|_F \leq C\|u\|_E, \quad \forall u \in E.$$

Theorem 1.2.2. *[13] Let $A : E \to F$ be a linear operator in normed spaces. A is continuous if and only if it is bounded.*

Property 1.2.1. *[13] Let $A : E \to F$ be a continuous linear operator between two normed spaces $(E, \|.\|_E)$ and $(F, \|.\|_F)$. Then*

$$\|A\| := \sup_{\|u\|\leq 1} \|A(u)\|_F < +\infty.$$

1.2.2 Compact operators

Definition 1.2.4. *(Compact operator)[13]:Let $A : E \to F$ a linear operator.. We say that A is a compact operator if it maps every bounded subset $G \subset E$ into a relatively compact subset $A(G)$ in F. That is to say, the closure of $A(G)$ is compact.*
Relatively compact sets: *A subset $G \subset E$ is said to be relatively compact if for every sequence (u_n) in G, there exists a subsequence (u_{nk}) that converges in F.*

Property 1.2.2. *[13] Let $A : E \to F$ be a bounded linear operator between two normed spaces. If E is finite-dimensional, then A is a compact operator. In other words, every bounded linear operator defined on a finite-dimensional normed space is compact.*

Proof. Since E is finite-dimensional, its image under A, denoted by $A(E)$, is also finite-dimensional. In fact, we have

$$\dim A(E) \leq \dim E.$$

Thus, the range of A is finite-dimensional. It follows that any bounded sequence in E is mapped by A into a sequence in a finite-dimensional space, which necessarily admits a convergent subsequence. Therefore, A is a compact operator. □

Remark 1.2.2.

- *The identity operator* Id *on a normed space E is compact if and only if E is finite-dimensional.*
- *Every compact operator is a bounded operator, but the converse is false. For example, let E be an infinite-dimensional normed space and consider the identity operator $A = I : E \to E$. Then I is bounded, since*

$$\|Ix\| = \|x\| \quad \forall x \in E,$$

But it is not compact.

Theorem 1.2.3. *[13]*
A linear operator $A : E \to F$ is compact if and only if for every bounded sequence $(\varphi_n) \subset E$, the sequence $(A\varphi_n) \subset F$ admits a convergent subsequence.

Theorem 1.2.4. *[13] Let A_1 and A_2 be compact operators. Then, for any scalars α and β, the operator*

$$A = \alpha A_1 + \beta A_2$$

Is also compact.

Proof. Let (φ_n) be a bounded sequence in E. Consider the sequence $(A\varphi_n)$ in F, defined by

$$A\varphi_n = \alpha A_1\varphi_n + \beta A_2\varphi_n, \quad \varphi_n \in E, \ n \in \mathbb{N}.$$

Since A_1 and A_2 are compact operators, the sequences $(A_1\varphi_n)$ and $(A_2\varphi_n)$ each possess convergent subsequences. Furthermore, a linear combination of convergent subsequences remains convergent. Consequently, $(A\varphi_n)$ admits a convergent subsequence, and therefore A is compact. □

Theorem 1.2.5. *[13] The composition AB of two bounded operators A and B is compact if at least one of the operators A or B is compact.*

Proof. Let (φ_n) be a bounded sequence in E. If B is a bounded operator, then the sequence $(B\varphi_n)$ is also bounded. As A is compact, the sequence $(A(B\varphi_n))$ admits a convergent subsequence. So, the operator AB is compact.

On the other hand, if B is compact, then from the sequence $(B\varphi_n)$ one can extract a convergent subsequence $(B\varphi_{n_k})$. Given that A is continuous (as it is bounded), it follows that the sequence $(A(B\varphi_{n_k}))$ is convergent. Therefore, AB is compact □

Theorem 1.2.6. *[13] Let E be a normed space and F a Banach space. Let (A_n) be a sequence of compact operators from E into F such that A_n converges to $A : E \to F$ in the operator norm, i.e.,*

$$\lim_{n\to\infty} \|A_n - A\| = 0.$$

Then A is a compact operator.

Corollary 1.2.1. *In any infinite-dimensional normed space, the unit ball $B(0, 1)$ is not compact.*

1.2.3 Integral linear operators

Integral operators play a fundamental role in functional analysis, as they are particularly effective in simplifying complex functions and facilitating their analysis.

Definition 1.2.5. *(Integral operators):[13] An integral operator is defined as a linear operator $A : E \to F$, where E and F are normed spaces, and is given by*

$$A\varphi(u) = \int_{G_2} k(u, v)\, \varphi(v)\, dv, \quad u \in G_1$$

Where:

- *$k(u, v)$ is a measurable function on $G_1 \times G_2$,*
- *$\varphi(v)$ is a measurable function on G_2.*

The measurable function $k(u, v)$ is called the kernel of the integral operator A

Definition 1.2.6. *(Norms of integral operators)[13] Let A be an integral operator defined on $L^p(G_1)$. Then, for conjugate exponents p and q (i.e., $\frac{1}{p} + \frac{1}{q} = 1$ with $1 \leq p, q \leq \infty$), the norm of the operator A is given by:*

$$\|A\|_p \begin{cases} \left(\int \left(\int |k(u,v)|^q dv\right)^{p/q} du\right)^{1/p}, & 1 < p < \infty \\ \int \sup |k(u,v)| du, & p = 1 \\ \sup \int |k(u,v)| dy, & p = \infty \end{cases}$$

Theorem 1.2.7. *[13] Let A be an integral operator with finite norm $\|A\|_p < 1$.*
Then the integral operator A is a continuous linear operator from $L^p(G_2)$ into $L^p(G_1)$.
Moreover, we have:

$$\|A\varphi\|_p \leq \|A\|_p \, \|\varphi\|_p, \qquad \|A\varphi\|_\infty \leq \|A\|_\infty \, \|\varphi\|_\infty.$$

Remark 1.2.3. *The norm of the integral operator A for $p = q = 2$ is:*

$$\|A\|_2 = \left(\int_{G_1} \int_{G_2} |k(u, v)|^2 \, du \, dv\right)^{\frac{1}{2}} < 1.$$

Proposition 1.2.1. *[13] The condition $\|A\|_p < 1$ imposed on the norm of the integral operator A guarantees its continuity; however, it is only a sufficient condition and not a necessary one.*

Chapter 2

Integral and integro-differential equations

2.1 Integral and integro-differential equations

2.1.1 Integral equations

- **Volterra Integral equations:** [9]
  The general form of a Volterra linear integral equation is:

$$\varphi(x)u(x) = f(x) + \lambda \int_a^x k(x,t)\, u(t)\, dt$$

  If $\varphi(x) = 0$

$$f(x) = \int_0^x K(x,t)\, u(t)\, dt.$$

  It is called Volterra integral equations of the first kind.

  If $\varphi(x) = 1$.

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt.$$

  It is called Volterra integral equations of the second kind

- **Fredholm integral equations:**[9]
  For Fredholm integral equations, the limits of integration are fixed.
  The general Fredholm linear integral equation is written as:

$$\varphi(x)u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt$$

  If $\varphi(x) = 0$

$$f(x) = \int_a^b K(x,t)\, u(t)\, dt.$$

  This is called the Fredholm integral equation of the first kind.

If $\varphi(x) = 1$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt.$$

This is called the Fredholm integral equation of the second kind.

Remark 2.1.1. *If the unknown function $u(x)$ appears in a nonlinear way inside the integral, for example through terms such as $u^2(t)$, $\sin(u(t))$, or $\ln(u(t))$, then the resulting equation is classified as a nonlinear integral equation.*

Remark 2.1.2. *if f(x) = 0, the equation is called homogeneous.*

- **Volterra-Fredholm integral equations:**[9]
  A Volterra-Fredholm integral equation is formed by combining separate Volterra and Fredholm integrals, as the form:

$$u(x) = f(x) + \lambda_1 \int_a^x k_V(x,t)\, u(t)\, dt + \lambda_2 \int_a^b k_F(x,t)\, u(t)\, dt, \quad x \in [a,b],$$

Where $k_V(x,t)$, $k_F(x,t)$, and $f(x)$ are known functions, $u(x)$ is the unknown function, and λ_1 and λ_2 are non-zero constants.

Example 2.1.1.

$$u(x) = 1 - x + \int_0^x e^{x-t} u(t)\, dt + \int_0^1 (x-t)^2 u(t)\, dt$$

A mixed integral equation is defined as an equation of the form:

$$\varphi(x) = f(x) + \lambda \int_a^x \int_a^b k(s,t)\, \varphi(t)\, dt\, ds$$

Where the functions k_V, k_F, and f are known, while $\varphi(x)$ is the unknown function to be determined.

Example 2.1.2.

$$u(x,t) = x + t^3 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)\, d\xi\, d\tau$$

2.1.2 Integro-Differential Equations

- **Volterra Integro-Differential Equations:**[9]
  A Volterra integro-differential equation is generally formulated as:

$$\sum_{k=0}^{m} P_k(x)\, u^{(k)}(x) = f(x) + \lambda \int_a^x k_V(x,t)\, u(t)\, dt$$

together with the initial conditions:

$$u^{(k)}(a) = \alpha_k, \quad 0 \le k \le m-1$$

Example 2.1.3.

$$u'(x) + 3u(x) = \sin(x) + \int_0^x e^{x-t} u(t)\,dt, \quad u(0) = 2$$

(This is a first-order Volterra integro-differential equation)

- **Fredholm integro-differential equation:**[9]
  A Fredholm integro-differential equation is generally formulated as:

$$\sum_{k=0}^{m} P_k(x)\,u^{(k)}(x) = f(x) + \lambda \int_a^b K_F(x,t)\,u(t)\,dt$$

  Whith the initial conditions:

$$u^{(k)}(a) = \alpha_k, \quad 0 \leq k \leq m-1$$

- **Volterra–Fredholm integro-differential equations:**
  The Volterra–Fredholm integro-differential equation is a mathematical formulation that integrates differential operators with both Volterra and Fredholm integral operators. These elements can be combined into a unified representation given by

$$\sum_{k=0}^{m} P_k(x)\,u^{(k)}(x) = f(x) + \lambda_1 \int_a^x k_V(x,t)\,u(t)\,dt + \lambda_2 \int_a^b k_F(x,t)\,u(t)\,dt,$$

  subject to the initial conditions

$$u^{(k)}(a) = \alpha_k, \quad 0 \leq k \leq m-1.$$

Example 2.1.4.

$$\begin{cases} u''(x) = e^x + \int_0^x (x-t)\,u(t)\,dt + \int_0^1 \cos(x+t)\,u(t)\,dt \\ u(0) = 1, \quad u'(0) = 0. \end{cases}$$

$$\begin{cases} u''(x) + u(x) = x + \int_0^x t\,u(t)\,dt + \int_0^1 u(t)\,dt \\ u(0) = 0, \quad u'(0) = 1. \end{cases}$$

2.2 Boole polynomials

Boole polynomials constitute a significant class of polynomial families in numerical analysis, named after the mathematician George Boole.

Definition 2.2.1. *(Boole polynomials)[3]:The generated function of Boole polynomials is defined as:*

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{2(1+t)^x}{2+t} \tag{2.1}$$

Thus, the general form of the Boole polynomials can be written as follows:

$$B_n(x) = \sum_{m=0}^{n} \frac{(-1)^m}{2^m} \binom{x}{n-m}$$

Initial Values of Boole Polynomials:
Based on the generating function, the first values of the Boole polynomials can be derived as follows:

$$B_0(x) = 1$$
$$B_1(x) = x - \frac{1}{2}$$
$$B_2(x) = x^2 - 2x + \frac{1}{2}$$
$$B_3(x) = x^3 - \frac{9}{2}x^2 + 5x - \frac{3}{4}$$
$$B_4(x) = x^4 - 8x^3 + 20x^2 - 16x + \frac{3}{2}$$
$$B_5(x) = x^5 - \frac{25}{2}x^4 + 55x^3 - 100x^2 + 64x - \frac{15}{4}$$

2.2.1 Fundamental properties of Boole polynomials

1. **Orthogonality Property:**
   Boole polynomials are orthogonal with respect to a specific weight function $w(x)$ over a defined interval $[a, b]$. This property ensures that the system matrix (in the collocation method) is either a diagonal or near-diagonal matrix (Sparse Matrix).

2. **Recurrence Relation:**
   Boole polynomials are characterized by a recurrence relation between their terms, which enhances computational efficiency. This facilitates building programming algorithms very rapidly and with minimal computational effort, allowing the calculation of any term $B_n(x)$ in terms of its preceding components. The relation is expressed as:

   $$B_n(x+1) + B_n(x) = 2x^n$$

3. **Symmetry and Regularity:**
   These polynomials possess a regular and uniform distribution of zeros (roots) within their domain of definition.

4. **Interrelation with Euler Polynomials:**
   Boole polynomials are closely related to Euler polynomials $E_n(x)$. In many cases, they can be represented under specific conditions as follows:

   $$B_n(x) = E_n(x)$$

   This allows researchers to leverage the established characteristics and boundaries of Euler polynomials.

Chapter 3

Numerical Methods

3.1 Numerical solution techniques for Volterra-Fredholm integro-differential equations

We consider the linear Volterra-Fredholm integro-differential equation :

$$\sum_{k=0}^{m} P_k(x)\, u^{(k)}(x) = f(x) + \lambda_1 \int_a^x K_V(x,t)\, u(t)\, dt + \lambda_2 \int_a^b K_F(x,t)\, u(t)\, dt \quad a \leq t, x \leq b. \tag{3.1}$$

With the initial conditions:

$$u^{(k)}(a) = \alpha_k \quad k = 0, 1, 2, \ldots, m-1. \tag{3.2}$$

The approximate Boole solution of eq (3.1) is expressed in the form of a finite Boole series as

$$u(x) \approx u_N(x) = \sum_{i=0}^{N} a_i B_i(x) \tag{3.3}$$

Where $B_i(x)$ denotes the Boole polynomials of degree i, and a_i are the unknown Boole coefficients to be determined.

3.1.1 Matrix relations of the linear Volterra-Fredholm integro-differential equation

$$D(x) = f(x) + \lambda_1\, I_V + \lambda_2\, I_F$$

Derivatives matrix:
We transform eq (3.1) into a system of matrix equations.
Firstly, the matrix form of the Boole polynomial $B(x)$ is written as

$$B(x) = Z(x)H^T. \tag{3.4}$$

Where

$$B(x) = [B_0(x)\ B_1(x)\ B_2(x)\ \cdots\ B_N(x)]$$
$$Z(x) = [1\ x\ x^2\ \cdots\ x^N]$$

$$H = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ -\frac{1}{2} & 1 & 0 & \cdots \\ \frac{1}{2} & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The matrix relation of the Boole series form eq (3.3) is written as

$$u_N(x) = B(x)A \tag{3.5}$$

Where A is the vector of unknown Boole coefficients:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_N \end{bmatrix}^T$$

By substituting eq (3.4) into eq (3.5), we obtain:

$$u_N(x) = Z(x)H^T A \tag{3.6}$$

and its kth derivative is given by:

$$u_N^{(k)}(x) = B^{(k)}(x)A. \tag{3.7}$$

So,

$$u_N^{(k)}(x) = Z^{(k)}(x)H^T A. \tag{3.8}$$

where

$$Z'(x) = Z(x)E, \qquad Z''(x) = Z(x)E^2, \qquad Z^{(3)}(x) = Z(x)E^3..$$

Therefore, the k-th derivative of the vector $Z(x)$ is given by the general relation

$$Z^{(k)}(x) = Z(x)E^k. \tag{3.9}$$

By using the matrix form (3.8) and the matrix relation (3.9),We obtain the matrix relation:

$$u_N^{(k)}(x) = Z(x)E^k H^T A \tag{3.10}$$

Where the matrix E is the derivative transition matrix of $Z(x)$.

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Example 3.1.1. *if we set $N = 2$, we get:*

$$Z(x) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}$$

$$[1,\ x,\ x^2] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = [0,\ 1,\ 2x]$$

Thus, this is the differentiation matrix E when $N = 2$.

Therefore, the operational differentiation matrix D can be written as follows:

$$D(x) = \sum_{k=0}^{m} P_k(x)\, Z(x) E^k H^T A$$

Integration matrix:

In (3.1) the kernel functions $K_V(x,t)$ and $K_F(x,t)$ corresponding to the Volterra and Fredholm parts, respectively, are defined for the Taylor and Boole polynomials as follows:

For Taylor polynomials:

$$K_V(x,t) = Z(x)\, K_V^t\, Z^T(t), \quad K_F(x,t) = Z(x)\, K_F^t\, Z^T(t). \tag{3.11}$$

For Boole polynomials:

$$K_V(x,t) = B(x)\, K_V^B\, B^T(t), \quad K_F(x,t) = B(x)\, K_F^B\, B^T(t). \tag{3.12}$$

where

$$K_V^t(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} (k_{mn}^t)^V\, x^m t^n, \qquad K_F^t(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} (k_{mn}^t)^F\, x^m t^n, \tag{3.13}$$

$$K_V^B(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} (k_{mn}^B)^V\, B_m(x) B_n(t), \qquad K_F^B(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} (k_{mn}^B)^F\, B_m(x) B_n(t), \tag{3.14}$$

and

$$(k_{mn}^t)^V = \frac{1}{m!\,n!}\frac{\partial^{m+n} K_V(0,0)}{\partial x^m \partial t^n}, \qquad (k_{mn}^t)^F = \frac{1}{m!\,n!}\frac{\partial^{m+n} K_F(0,0)}{\partial x^m \partial t^n}, \quad m,n = 0,1,2,\ldots,N. \tag{3.15}$$

By equating the kernel functions $K_V(x,t)$ and $K_F(x,t)$ expressed in terms of Taylor polynomials and Boole polynomials, we obtain:

$$K_V^B = (H^T)^{-1}\, K_V^t\, H^{-1}, \quad K_F^B = (H^T)^{-1}\, K_F^t\, H^{-1}. \tag{3.16}$$

Using eqs (3.4) and (3.16), eq (3.12) can be evaluated in the following form:

$$\begin{aligned} K_V(x,t) &= B(x)\, K_V^B\, B^T(t) \\ &= Z(x) H^T (H^T)^{-1}\, K_V^t\, H^{-1} (Z(t) H^T)^T \\ &= Z(x) K_V^t Z^T(t) \\ K_F(x,t) &= B(x)\, K_F^B\, B^T(t) \\ &= Z(x) H^T (H^T)^{-1}\, K_F^t\, H^{-1} (Z(t) H^T)^T \\ &= Z(x) K_F^t Z^T(t) \end{aligned}$$

The Volterra and Fredholm integrals can be expressed in the following form:

$$\begin{aligned} I_V &= \int_a^x K_V(x,t)\, u(t)\, dt \\ &= Z(x) K_V^t \left(\int_a^x Z^T(t)\, Z(t)\, dt\right) H^T A \\ &= Z(x) K_V^t Q_1(x) H^T A \end{aligned}$$

where

$$Q_1(x) = \int_a^x Z^T(t)\, Z(t)\, dt$$

$$\begin{aligned} I_F &= \textstyle\int_a^b K_F(x,t)\, u(t)\, dt \\ &= Z(x) K_F^t \left(\textstyle\int_a^b Z^T(t)\, Z(t)\, dt\right) H^T A \\ &= Z(x) K_F^t Q_2 H^T A \end{aligned}$$

where

$$Q_2 = \int_a^b Z^T(t)\, Z(t)\, dt$$

By substituting the matrix relation (3.10), the Volterra and Fredholm integrals corresponding to the Taylor polynomial into eq (3.1), the resulting matrix equation is obtained:

$$\sum_{k=0}^{m} P_k(x)\, Z(x)\, E^k\, H^T\, A = f(x) + \lambda_1\, Z(x)\, K_V^t\, Q_1(x)\, H^T\, A + \lambda_2\, Z(x)\, K_F^t\, Q_2\, H^T\, A \tag{3.17}$$

3.1.2 Collocation method

By using the collocation points x_i, defined as:

$$x_i = a + \frac{b-a}{N}\, i, \quad i = 0, 1, \ldots, N. \tag{3.18}$$

The system of the matrix equations is gained as

$$\sum_{k=0}^{m} P_k(x_i)\, Z(x_i)\, E^k\, H^T\, A = f(x_i) + \lambda_1\, Z(x_i)\, K_V^t\, Q_1(x_i)\, H^T\, A + \lambda_2\, Z(x_i)\, K_F^t\, Q_2\, H^T\, A \tag{3.19}$$

Or, in brief, the fundamental matrix equation can be expressed as follows:

$$\left\{ \sum_{k=0}^{m} P_k\, Z\, E^k\, H^T - \lambda_1\, Z\, K_V^t\, Q_1\, H^T - \lambda_2\, Z\, K_F^t\, Q_2\, H^T \right\} A = F \tag{3.20}$$

Where

$$P_k = \begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}$$

$$F = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix},$$

$$Z = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}$$

$$Q_1 = \begin{bmatrix} Q_1(x_0) & 0 & \cdots & 0 \\ 0 & Q_1(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_1(x_N) \end{bmatrix}$$

The fundamental matrix relation (3.19) can be expressed as follows:

$$GA = F \quad or \left[G; F\right] \tag{3.21}$$

Where

$$G = \sum_{k=0}^{m} P_k\, Z\, E^k\, H^T - \lambda_1\, Z\, K_V^t\, Q_1\, H^T - \lambda_2\, Z\, K_F^t\, Q_2\, H^T$$

$$[G; F] = \left[\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,n} & f(x_0) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,n} & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{n,0} & g_{n,1} & \cdots & g_{n,n} & f(x_n) \end{array}\right]$$

According to the relation (3.7), the corresponding matrix form of the conditions (3.2) is written as:

$$Z(a)E^k H^T A = \alpha_k, \quad k = 0, 1, \ldots, m-1. \tag{3.22}$$

We write the matrix corresponding to conditions (3.2), we obtain

$$\psi_k A = \alpha_k, \quad \text{or} \quad [\psi_k, \alpha_k] = [\psi_{k,0}, \psi_{k,1}, \psi_{k,2}, \ldots, \psi_{k,n}; \alpha_k], \qquad k = 0, 1, 2, \ldots, m-1. \tag{3.23}$$

where

$$\psi_k = Z(a)\, E^k\, H^T$$

To reach the solution of the eq (3.1) under conditions (3.2), the last rows matrices (3.21) are replaced with m rows of the matrix (3.23).
So, the new augmented matrix form is obtained as:

$$[G^*; F^*] = \left[\begin{array}{ccc|c} g_{0,0} & \cdots & g_{0,N} & f(x_0) \\ g_{1,0} & \cdots & g_{1,N} & f(x_1) \\ \vdots & \ddots & \vdots & \vdots \\ g_{(N-m),0} & \cdots & g_{(N-m),N} & f(x_{N-m}) \\ \psi_{0,0} & \cdots & \psi_{0,N} & \alpha_0 \\ \vdots & \ddots & \vdots & \vdots \\ \psi_{(m-1),0} & \cdots & \psi_{(m-1),N} & \alpha_{m-1} \end{array}\right]$$

If,

$$\text{rank}(G^*) = \text{rank}[G^*; F^*] = N + 1$$

then the augmented matrix solution is written as:

$$A = (G^*)^{-1}F. \tag{3.24}$$

Finally, the unknown Boole coefficients from the matrix (3.24) solution are found and placed in the series (3.3).

3.2 Numerical examples

Example 1: Consider the linear Volterra-Fredholm integro-differential equation:

$$\begin{cases} u''(x) + xu'(x) - xu(x) - \int_0^1 \sin(x)e^{-t}\, u(t)\, dt + \frac{1}{2}\int_0^x \cos(x)e^{-t}\, u(t)\, dt = f(x) \\ u(0) = 1, \qquad u'(0) = 1 \end{cases}$$

Where

$$f(x) = e^x - \sin(x) + \frac{1}{2}x\cos(x)$$

and the exact solution given by:

$$u_{\text{ex}}(x) = e^x$$

We present the approximate solution (Solapp) obtained by the Boole collocation method, the error is calculated for N =10

Table 3.1: Comparison of the exact and approximate solutions

x	Solex	Solapp	err
0	1.0000e+00	1.0000e+00	2.2204e-16
1.0000e-01	1.1052e+00	1.1052e+00	6.7502e-14
2.0000e-01	1.2214e+00	1.2214e+00	1.6831e-13
3.0000e-01	1.3499e+00	1.3499e+00	2.6290e-13
4.0000e-01	1.4918e+00	1.4918e+00	3.5949e-13
5.0000e-01	1.6487e+00	1.6487e+00	4.5564e-13
6.0000e-01	1.8221e+00	1.8221e+00	5.4889e-13
7.0000e-01	2.0138e+00	2.0138e+00	6.5015e-13
8.0000e-01	2.2255e+00	2.2255e+00	7.1188e-13
9.0000e-01	2.4596e+00	2.4596e+00	1.7870e-12
1.0000e+00	2.7183e+00	2.7183e+00	2.8217e-11

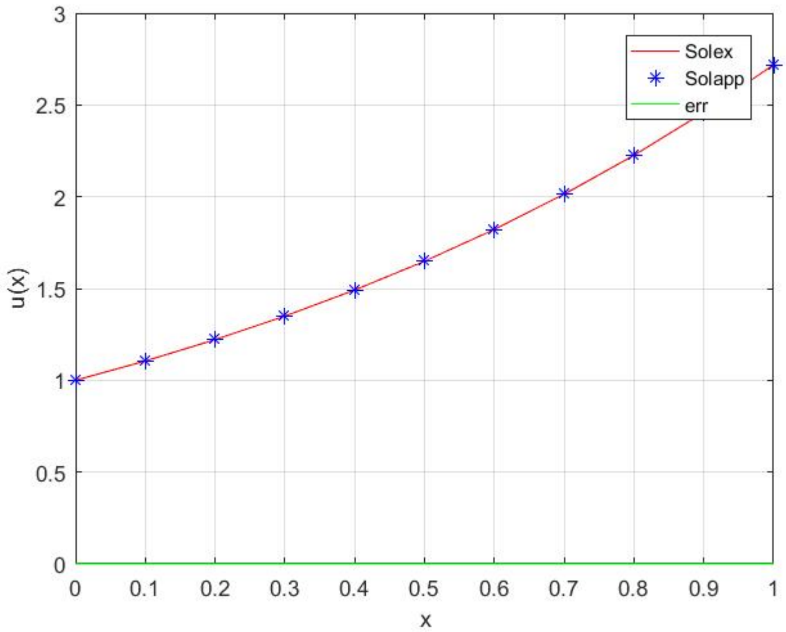

Figure 3.1: A graph illustrating the exact solution, the approximate solution, and the error

Example 2: Consider the linear Volterra-Fredholm integro-differential equation:

$$\begin{cases} u^{(5)}(x) - xu''(x) + xu(x) - \frac{1}{2}\int_0^1 e^{2x+t}u(t)\,dt - \int_0^x xe^t u(t)\,dt = f(x) \\ u(0) = 1, \qquad u'(0) = -1, \qquad u''(0) = 1, \qquad u'''(0) = -1, \qquad u(4)(0) = 1 \end{cases}$$

Where

$$f(x) = -e^{-x} - \frac{1}{2}e^{2x} - x^2$$

the exact solution is:

$$u_{\text{ex}}(x) = e^{-x}$$

The error is calculated for N =10

Table 3.2: Comparison of the exact and approximate solutions

x	Solex	Solapp	err
0	1.0000e+00	1.0000e+00	5.5511e-15
1.0000e-01	9.0484e-01	9.0484e-01	4.7740e-15
2.0000e-01	8.1873e-01	8.1873e-01	2.6756e-14
3.0000e-01	7.4082e-01	7.4082e-01	2.0417e-13
4.0000e-01	6.7032e-01	6.7032e-01	7.3364e-13
5.0000e-01	6.0653e-01	6.0653e-01	1.9120e-12
6.0000e-01	5.4881e-01	5.4881e-01	4.1291e-12
7.0000e-01	4.9659e-01	4.9659e-01	7.5588e-12
8.0000e-01	4.4933e-01	4.4933e-01	6.6178e-12
9.0000e-01	4.0657e-01	4.0657e-01	4.5368e-11
1.0000e+00	3.6788e-01	3.6788e-01	3.7769e-10

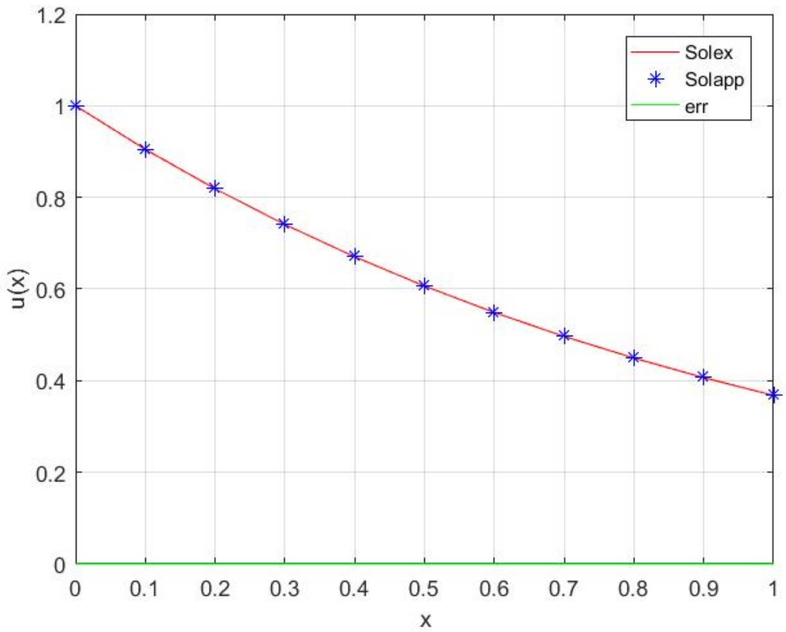

Figure 3.2: A graph illustrating the exact solution, the approximate solution, and the error

Example 3: Consider the linear Volterra-Fredholm integro-differential equation:

$$\begin{cases} u'''(x) + u'(x) - \int_0^x tu(t)dt - \int_0^1 u(t)dt = f(x) \\ u(0) = 0, \qquad u'(x) = 0, \qquad u''(0) = -2 \end{cases}$$

Where

$$f(x) = -\frac{1}{5}x^5 + \frac{1}{4}x^4 + 3x^2 - 2x + \frac{73}{12}$$

and the solution exact is:

$$u_{\text{ex}}(x) = x^3 - x^2$$

The error is calculated for N =4

Table 3.3: Comparison of the exact and approximate solutions

x	Solex	Solapp	err
0	0	1.7463e-16	1.7463e-16
2.5000e-01	-4.6875e-02	-4.6875e-02	3.3307e-16
5.0000e-01	-1.2500e-01	-1.2500e-01	4.7184e-16
7.5000e-01	-1.4063e-01	-1.4062e-01	4.9960e-16
1.0000e+00	0	7.1355e-16	7.1355e-16

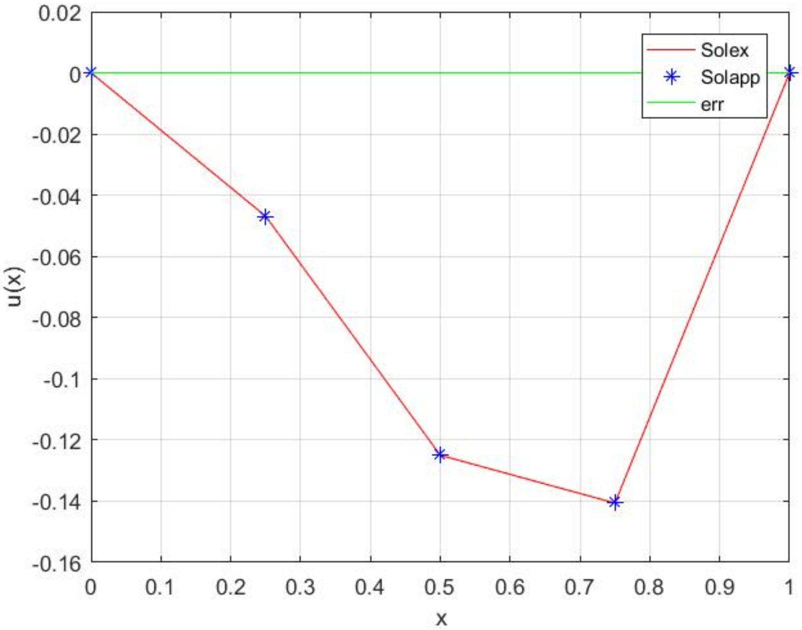

Figure 3.3: A graph illustrating the exact solution, the approximate solution, and the error

Conclusion

In this thesis, we propose a numerical scheme for the resolution of linear Volterra-Fredholm integro-differential equations. The approach employs a collocation method based on Boole polynomials to approximate the solutions. By projecting the integro-differential equation onto a finite-dimensional subspace and constructing operational matrices for differentiation and integration, the original problem is reformulated as a system of linear matrix equations, enabling an efficient and straightforward solution procedure. The effectiveness of the method is validated through several illustrative examples, which confirm its accuracy, computational efficiency, and broad applicability.

Bibliography

[1] Da, H. G., & Bier, K. E. A Numerical Technique Based on Boole Polynomials for Solving Fredholm Integro-Differential Equations. In 3rd International Students Science Congress (p. 299).

[2] Yzba, ., ahn, N., & Yildirim, A. (2012). A collocation approach for solving high-order linear FredholmVolterra integro-differential equations. Mathematical and Computer Modelling, 55(3-4), 547-563.

[3] Bier, K. E., & Da, H. G. (2021). Boole approximation method with residual error function to solve linear Volterra integro-differential equations. Celal Bayar University Journal of Science, 17(1), 59-66.

[4] ztrk, Y., & Glsu, M. (2021). An operational matrix method to solve linear Fredholm-Volterra integro-differential equations with piecewise intervals. Mathematical Sciences, 15(2), 189-197.

[5] Rahmani, L., Rahimi, B., & Mordad, M. (2011). Numerical solution of Volterra-Fredholm integro-differential equation by block pulse functions and operational matrices. Math. Notes, 4(2), 37-48.

[6] BER, K. E., & Da, H. G. (2023). Numerical solutions of differential equations having cubic nonlinearity using Boole collocation method. Turkish Journal of Mathematics, 47(2), 732-745.

[7] Shahoodh, M. K. (2021). Using Touchard polynomials method for solving Volterra-Fredholm integro-differential equations,". Tikrit Journal for Pure Science, 26(5), 92-96.

[8] Mirzaee, F., & Hadadiyan, E. (2016). Numerical solution of VolterraFredholm integral equations via modification of hat functions. Applied Mathematics and Computation, 280, 110-123.

[9] Wazwaz, A. M. (2011). Volterra-Fredholm Integral Equations. In Linear and Nonlinear Integral Equations: Methods and Applications (pp. 261-283). Berlin, Heidelberg: Springer Berlin Heidelberg.

[10] Elmaci, D., & Baykus Savasaneril, N. (2022). The Lucas Polynomial solution of linear Volterra-Fredholm integral equations. Matrix Sci. Math, 6(1), 21-25.

[11] ler Acar, N., & Dacolu, A. (2019). A projection method for linear FredholmVolterra integro-differential equations. Journal of Taibah University for Science, 13(1), 644-650.

[12] Deepmala, M. V., Marasi, H. R., Shabanian, H., & Nosraty, M. (2017). Solution of volttra-Fredholm integro-differential equations using chebyshev collocation method. Global Journal of Technology and Optimization, 8(210), 1-4.

[13] M. NADIR .Cours danalyse fonctionnelle ,universit de Msila Algrie 2004