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**Mohamed Boudiaf university of Msila**
**Faculty of Mathematics and computer sciences**
**Department of Mathematics**

# *Master memory*

**Field** : Mathematics and computer sciences
**Branch** : Mathematics
**Option** : Functional analysis

# Theme

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*Positive $p$-summing operators and disjoint $p$-summing operators*

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**Presented by :**
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**In front of the jury composed of :**

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University year 2021/2022

# Dedications

*In the name of ALLAH the most gracious the most merciful*

I dedicate this work to :

All the student

during academic year 2021/2022.

# Acknowledgments

First, I am so grateful to " **Allah** " for giving us protection and ability to start and finish this work.

I would like to express my most sincere thanks to my supervisor doctor **Rachid YAHI** for his precious advice and guidance.

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I do not forget to thank my teachers of our department of mathematics at M'sila University for their unceasing encouragement, support and help.

# Contents

# Notaions

| | |
|---|---|
| $\mathbb{K}$ | The field of real or complex numbers. |
| $\mathbb{R}_+$ | The field of non negative real numbers. |
| $p'$ | The conjugate of the number $p$ $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p'} = 1$ |
| $E^*$ | The topological dual of $E$. |
| $B_E$ | The closed unit ball of $E$ |
| $L(E;F)$ | The set of all linear operators. |
| $\mathcal{L}(E;F)$ | The sets of all continuous linear operators. |
| $w$ | The weak topology. |
| $w^*$ | The weak $^*$ topology. |
| $\mathcal{I}$ | The ideal of all linear operator. |
| $\ell_p(X)$ | the Banach space of all weakly $p$-summable sequences $(x_i)_{i=1}^{\infty}$ in $X$. |
| $T^*$ | The adjoint operator of $T$. |
| $\ell_p^{\omega}(X)$ | The Banach space of all weakly $p$-summable sequences $(x_i)_{i=1}^{\infty}$ in $X$ |
| $\Pi_p(X,Y)$. | . The collection of all $p$-summing operators between $X$ and $Y$. |
| $\gamma_p(E,X)$ | The space of all positive $p$-majorizing operators from $E$ to $X$ |
| $\Lambda_p(X;Y)$ | *The class of all positive $p$-summing linear operators from $X$ into $Y$* |
| $\psi_{(p,q)}(X,Y)$ | The class of all positive $(p,q)$-dominated operators from $X$ to $Y$. |
| $\mathcal{D}_p^+(X,F)$ | The class of all positive strongly $p$—summing operators between $X$ and $F$. |

## Abstract

In this research, we have studied some classes of operators from a Banach lattice $X$ into a Banach space $Y$, such as the class of positive $p$-summing operators, positive $p$-majorizing operator, positive $(p,q)$-dominated operators and disjoint $p$-summing operators,



**Keywords** : positive $p$-summing operators, positive $p$-majorizing operators, disjoint $p$-summing operators.



## ملخص

لقد قمنا في هذه المذكرة بدراسة بعض الأصناف من المؤثرات الخطية، المعرفة من بناخ لتيس نحو فضاء بناخي حيث تناولنا فيها: المؤثرات الموجبة $p$ جمعية: المؤثرات المنفصلة $p$ جمعية: ...الخ



**الكلمات المفتاحية:** المؤثرات الموجبة $p$ جمعية، المؤثرات المنفصلة $p$ جمعية:

# Introduction

In 1986 Oscar Blasco introduced the concept of positive $p$-summing operator from a Banach Lattice $X$ into a Banach space $E$, which is closely related to $p$-absolutely summing operators defined by Pietsch [14]. We mention that the class of positive $p$-summing operators have already considered by Shaefer in the case $p = 1$ under the name cone absolutely summing operator. A linear operator $T$ between a Banach lattice $X$ and a Banach space $Y$ is called positive $p$-summing operator for $1 \leq p \leq \infty$, if there exists a constant $C > 0$ *such that for every* $x_1, ..., x_n$ positive elements in $X$, *we have*

$$(\sum_{i=1}^{n} \|T(x_i)\|^p)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} (\sum_{i=1}^{n} |x^*(x_i)|^p)^{\frac{1}{p}}. \tag{1}$$

*We denote by* $\Lambda_p(X;Y)$ *the class of all positive* $p$*-summing linear operators from* $X$ *into* $Y$ *and* $\pi_p^+(T) = \inf\{C$ *verifying the inequality (2.5)*$\}$. obviously the space of $p$-absolutely summing operators $\Pi_p(X,Y)$ is included in $\Lambda_p(X;Y)$, and the same techniques as for $p$-absolutely summing operators lead us to see that for $p \leq q$ $\Lambda_p(X;Y) \subset \Lambda_q(X;Y)$

In 1972, Schaefer [17] introduced the concept of cone absolutely summing operators and majorizing operators, respectively, and established the duality relationships between cone absolutely summing operators and majorizing operators

In 2021 D. Chen et al. [5] extended the notion of majorizing operators to more general case and introduced the concept of positive $p$-majorizing operators. Furthermore, they generalized the duality relationships between cone absolutely summing operators and majorizing operators, and proved the duality relations between positive $p$-summing operators and positive $p'$-majorizing operators

The aim of this memory is to present some class of operators from a Banach lattice $X$ into a Banach space $Y$, so our work based in the article of **D. Chen et al.** (see[5]), this memory organized as follow

In the first chapter, we recall by some basic definitions concerning Banach lattice, sequences Banach spaces, tensor products, linear operator ideals, $p$-summing operators.

In the second chapter we present the new characterization of majorizing operators, Based on this characterization, we present the class of positive $p$-majorizing operators. The duality relations between positive $p$-summing operators and positive $p'$-majorizing operators is shown. By the end of this Chapter, we describe positive $p$-summing operators and positive $p$-majorizing operators in terms of tensor products equipped with suitable reasonable cross norms.

In the third chapter we study the concept of positive $(p, q)$-dominated operators that are positive analogues of $(p, q)$-dominated operators. we present a positive version of Kwapie´n's factorization theorem for positive $(p, q)$-dominated operators

The last chapter is devoted to study the class of disjoint p-summing operators, witch a new larger class of operators than positive $p$-summing operators:

Chapter 1

# Preliminaries

In this chapter, we present some concepts which will use it in the sequel of this Memory, as sequences Banach spaces, Banach Lattice, linear operator ideals. On based on the Elhadj Dahia's thesis [7]

## 1.1 Some Sequences Banach Spaces

Let $X$ be a Banach space over $\mathbb{K}$, (either $\mathbb{R}$ or $\mathbb{C}$) and $1 \leq p \leq \infty$. We write $p^*$ for the extended real number that satisfies $1/p + 1/p^* = 1$.

### 1.1.1 Absolutely and weakly $p$-summable sequences.

We denote by $\ell_p^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in $X$ with the norm

$$\|(x_i)_{i=1}^n\|_p = (\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}},$$

and by $\ell_{p,\omega}^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in $X$ with the norm

$$\|(x_i)_{i=1}^n\|_{p,\omega} = \sup_{\|\xi\|_{X^*}\leq 1} (\sum_{i=1}^n |\langle x_i, \xi\rangle|^p)^{\frac{1}{p}}.$$

Let $\ell_p(X)$ be the Banach space of all absolutely $p$ -summable sequences $(x_i)_{i=1}^\infty$ in $X$ with the norm

$$\|(x_i)_{i=1}^\infty\|_p = (\sum_{i=1}^\infty \|x_i\|^p)^{\frac{1}{p}}.$$

We denote by $\ell_p^\omega(X)$ the Banach space of all weakly $p$-summable sequences $(x_i)_{i=1}^\infty$ in $X$ with the norm

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} = \sup_{\|\xi\|_{X^*}\leq 1} \|(\xi(x_i))_{i=1}^\infty\|_p = \sup_{\|\xi\|_{X^*}\leq 1} (\sum_{i=1}^\infty |\xi(x_i)|^p)^{\frac{1}{p}}.$$

Notice that $\ell_p(X)$ is a linear subspace of $\ell_p^\omega(X)$ and

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} \leq \|(x_i)_{i=1}^\infty\|_p \text{ for all } (x_i)_{i=1}^\infty \in \ell_p(X).$$

If $p=\infty$ we are restricted to the case of bounded sequences and in $\ell_\infty(X)$ we use the sup norm. Then the spaces $\ell_\infty(X)$ and $\ell_\infty^\omega(X)$ coincide and

$$\|(x_i)_{i=1}^\infty\|_\infty = \|(x_i)_{i=1}^\infty\|_{\infty,\omega} \text{ for } (x_i)_{i=1}^\infty \in \ell_\infty(X).$$

If $X$ is finite dimensional with $\dim X = n$, then $\ell_p(X) = \ell_p^\omega(X)$ and

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} \leq \|(x_i)_{i=1}^\infty\|_p \leq n^{\frac{1}{p}} \|(x_i)_{i=1}^\infty\|_{p,\omega} \quad \text{for all } (x_i)_{i=1}^\infty \in \ell_p(X). \tag{1.1}$$

If we take $n=1$ in (1.1), or $X=\mathbb{K}$, then the spaces $\ell_p(\mathbb{K})$ and $\ell_p^\omega(\mathbb{K})$ coincide and we denote $\ell_p(\mathbb{K})$ by $\ell_p$. In this case we have

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} = \|(x_i)_{i=1}^\infty\|_p \quad \text{for all } (x_i)_{i=1}^\infty \in \ell_p. \tag{1.2}$$

We know (see [6, Theorem 2.1]) that $\left(\ell_p^n(X)\right)^* = \ell_{p^*}^n(X^*)$ isometrically i.e.,

$$\left\|(x_i)_{i=1}^n\right\|_p = \sup\left\{\left|\sum_{i=1}^n \langle x_i, x_i^*\rangle\right| : (x_i^*)_{i=1}^n \subset X^*, \left\|(x_i^*)_{i=1}^n\right\|_{p^*} \leq 1\right\}. \tag{1.3}$$

For the particular case $p = 1$ and $X == \mathbb{K}$ we have

$$\left\|(x_i)_{i=1}^n\right\|_1 = \sup\left\{\left|\sum_{i=1}^n \lambda_i x_i\right| : (\lambda_i)_{i=1}^n \subset \mathbb{K}, \left\|(\lambda_i)_{i=1}^n\right\|_\infty \leq 1\right\}. \tag{1.4}$$

Let $(x_i^*)_{i=1}^n \subset X^*$. Then it is also known (see [3, Lemma 2.1]) that

$$\left\|(x_i^*)_{i=1}^n\right\|_{p,\omega} = \sup_{\beta\in B_{X^{**}}} (\sum_{i=1}^n |\beta(x_i^*)|^p)^{\frac{1}{p}} = \sup_{x\in B_X} \left\|(x_i^*(x))_{i=1}^n\right\|_p. \tag{1.5}$$

### 1.1.2 Cohen strongly $p$-summable sequences.

The space of Cohen strongly $p$-summable sequences was introduced By Cohen in [8] in order to give a characterization of the class of strongly $p$-summing linear operators.

A sequences $(x_i)_{i=1}^\infty$ in a Banach space $X$ is Cohen strongly $p$-summing if the series $\sum\limits_{i=1}^\infty |\langle x_i, \xi_i\rangle|$ converges for all $(\xi_i)_{i=1}^\infty \in \ell_{p^*}^\omega(X^*)$. We denote by $\ell_p\langle X\rangle$ the space of Cohen strongly $p$-summable sequences in $X$ which is a Banach space (see [9, Page 233]) with the norm

$$\|(x_i)_{i=1}^\infty\|_{c,p} = \sup_{\left\|(\xi_i)_{i=1}^\infty\right\|_{p^*,\omega}\leq 1} \sum_{i=1}^\infty |\langle x_i, \xi_i\rangle|. \tag{1.6}$$

Notice that

$$\ell_p\langle X\rangle \subset \ell_p(X) \subset \ell_p^\omega(X).$$

Moreover, for all $(x_i)_{i=1}^\infty \in \ell_p\langle X\rangle$ ;

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} \leq \|(x_i)_{i=1}^\infty\|_p \leq \|(x_i)_{i=1}^\infty\|_{c,p}. \tag{1.7}$$

If $p = 1$ we have $\ell_1\langle X\rangle = \ell_1(X)$ with $\|.\|_{c,1} = \|.\|_1$ and if $p = \infty$ then $\ell_{c,\infty}\langle X\rangle = \ell_\infty(X)$ with $\|.\|_{c,\infty} = \|.\|_\infty$.

## 1.2 Banach lattices

Let recall some notations and results from . A Riesz space is a partially ordered real vector space $E$ which in addition is a lattice, i.e., any two elements $x; y \in E$ have a least upper bound, denoted by $x \vee y = sup\{x, y\}$ and a greatest lower bound, denoted by $x \wedge y = inf\{x, y\}$. For every $x \in E$, let $x^+ = x \vee 0, x^- = x \wedge 0$ and $|x| = x^+ \vee x^-$ be the positive part, the negative part and the absolute value of $x$, respectively. We have the identities $x = x^+ + x^-$ and $x = x^+ + x$. The set $E = \{x; x > 0\}$, is called the positive cone of $E$ and its elements are called positive .

**Example 1.2.1.** *Let $(\Omega, \Sigma, \mu)$ be a measure space. Define $\boldsymbol{C}$ to be the set of all realvalued $\Sigma$-measurable functions on $\Omega$ and order $\boldsymbol{C}$ by $f \leq g$ if $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$ and $f, g \in \boldsymbol{C}$. Let $\boldsymbol{B}$ be the space of all equivalence classes of functions in $\boldsymbol{C}$ for the equivalence relation $f \approxeq g$ if $f - g = 0$ $\mu$-a.e. Order $\boldsymbol{B}$ by $[f] \leq [g]$ if $f \leq g$*

**Definition 1.2.2.** ***(Banach Lattice)*** *Let $E$ be a Riesz space, A norm $\|\cdot\|$ on $E$ is a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ $\forall x, y \in E$. A Riesz space equipped with a lattice norm is known as a normed Riesz space. If a normed Riesz space is also norm complete, then it is referred to as a Banach lattice.*

## 1.3 Tensor products

In this section we shall recall some background on the theory of tensor product of two Banach spaces that we shall require, many of the results are well known. The proof of the majority of this results can be found in [10], [12, ] and [16].

Let $X$ and $Y$ be vector spaces. Each element of the (algebraic) tensor product $X \otimes Y$ has the form $\sum_{i=1}^{n} x_i \otimes y_i$ for some $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subset X$ and $(y_i)_{i=1}^n \subset Y$; such a representation is not unique. Let $G$ be a third vector space. For each bilinear map $T : X \times Y \longrightarrow G$, there is a unique linear map $T_L : X \otimes Y \longrightarrow G$ such that

$$T_L(x \otimes y) = T(x, y), \quad \text{for all } x \in X, y \in Y.$$

**Definition 1.3.1.** *Let $X, Y$ be Banach spaces. A norm $\alpha$ on the algebraic tensor product $X \otimes Y$ will be called a reasonable cross-norm if $\alpha$ satisfies the following conditions:*

*1) $\alpha(x \otimes y) \leq \|x\| \|y\|$ for all $x \in X$ and $y \in Y$.*

*2) For every $x^* \in X^*$ and $y^* \in Y^*$, the linear functional $x^* \otimes y^*$ on $X \otimes Y$ is continuous and $\|x^* \otimes y^*\| \leq \|x^*\| \|y^*\|$.*

If $X$ and $Y$ are Banach space and $\alpha$ is a reasonable crossnorm on $X$, then we will denote by $X \widehat{\otimes}_\alpha Y$ the completion of $X \otimes Y$ equipped with the norm $\alpha$.

First, an elementary but important fact:

**Proposition 1.3.2.** *If $\alpha$ is reasonable cross-norm on $X \otimes Y$ then $\alpha(x \otimes y) = \|x\| \|y\|$ for every $x \in X$ and $y \in Y$. Furthermore, $\|x^* \otimes y^*\| = \|x^*\| \|y^*\|$ for every $x^* \in X^*$ and $y^* \in Y^*$.*

### 1.3.1 The projective tensor product

The *projective norm* $\pi$ on $X \otimes Y$ is defined by

$$\pi(z) = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

where the infimum is taken over all representations $z = \sum_{i=1}^{n} x_i \otimes y_i$ of $z \in X \otimes Y$.

**Proposition 1.3.3.** *Let $X$ and $Y$ be Banach spaces.*

*(1) $\pi(x \otimes y) = \|x\| \|y\|$ for every $x \in X, y \in Y$.*

*(2) If $x^* \in X^*, y^* \in Y^*$, then $x^* \otimes y^*$ is a continuous linear functional on $X \widehat{\otimes}_\pi Y$ and $\|x^* \otimes y^*\| = \|x^*\| \|y^*\|$.*

The projective tensor product, $X\widehat{\otimes}_\pi Y$, is described in the following way: an element $z \in X\widehat{\otimes}_\pi Y$ has the representation

$$z = \sum_{n=1}^{\infty} x_i \otimes y_i \quad \text{with} \quad \sum_{n=1}^{\infty} \|x_i\| \, \|y_i\| < \infty.$$

### 1.3.2 The injective tensor product

The tensor product $X \otimes Y$ of two Banach spaces is canonically embedded into the Banach space $\mathcal{L}(X^*, Y^*; \mathbb{K})$ of continuous bilinear forms on $X^* \times Y^*$. This space induces another natural norm on $X \otimes Y$, the injective norm $\epsilon$.

**Definition 1.3.4.** *The injective norm on $X \otimes Y$ is defined by*

$$\epsilon(z) = \left\{ \left| \sum_{i=1}^{n} \langle x_i, x^* \rangle \langle y_i, y^* \rangle \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\}$$

*where $\sum_{i=1}^{n} x_i \otimes y_i$ is any representation of $z \in X \otimes Y$.*
*We denote by $X \otimes_\epsilon Y$ the tensor product $X \otimes Y$ with the injective norm. The completion, denoted by $X\widehat{\otimes}_\epsilon Y$; is called the injective tensor product of $X, Y$.*

Actualy the injective tensor product $X\widehat{\otimes}_\epsilon Y$ is the closure of $X \otimes Y$ in the space $\mathcal{L}(X^*, Y^*; \mathbb{K})$.

**Proposition 1.3.5.** *Let $X$ and $Y$ be Banach spaces.*

*(1) $\epsilon(z) \leq \pi(z)$ for all $z \in X \otimes Y$*

*(2) $\epsilon(x \otimes y) = \|x\| \, \|y\|$ for every $x \in X, y \in Y$.*

*(3) If $x^* \in X^*, y^* \in Y^*$, then $x^* \otimes y^*$ is a continuous linear functional on $X\widehat{\otimes}_\epsilon Y$ and $\|x^* \otimes y^*\| = \|x^*\| \, \|y^*\|$.*

Concerning the norm $\epsilon$ we take note of the frequently employed interpretation of members of the injective tensor product as operators. To be specific:

**Proposition 1.3.6.** *If $X$ and $Y$ are Banach spaces, then $X^*\widehat{\otimes}_\epsilon Y$ is a closed linear subspace of the space $\mathcal{L}(X, Y)$ of all continuous linear operators from $X$ to $Y$.*

The next result shows that the injective and the projective norms are the least and greatest reasonable crossnorm respectively.

**Proposition 1.3.7.** *Let $X$ and $Y$ be Banach spaces. A norm $\alpha$ on $X \otimes Y$ is a reasonable crossnorm if and only if*

$$\epsilon(u) \leq \alpha(z) \leq \pi(u)$$

*for every $z \in X \otimes Y$.*

## 1.4 Linear operator ideals

Recall that a linear operator $T \in \mathcal{L}(X, Y)$ is said to have finite rank if $T(X)$ is a finite dimensional subspace of $Y$. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(X, Y)$. An operator has rank one if and only if has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle \, y$$

i.e. if $u \in \mathcal{L}_f(X, Y)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset X^*$ and $(y_i)_{i=1}^n \subset Y$ (see [14, Page 25]).

**Definition 1.4.1.** *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces $X$ and $Y$ its components $\mathcal{I}(X, Y) := \mathcal{L}(X, Y) \cap \mathcal{I}$ satisfy:*

*(i) $\mathcal{I}(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$ which contains the finite rank operators.*

*(ii) The ideal property: if $v \in \mathcal{L}(G, X)$, $u \in \mathcal{I}(X, Y)$ and $w \in \mathcal{L}(Y, H)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(G, H)$.*

*If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies: (i') $(\mathcal{I}(X, Y), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces $E$ and $F$.*

*(ii')* $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1.$

*(iii') If* $v \in \mathcal{L}(G, X)$, $u \in \mathcal{I}(E, F)$ *and* $w \in \mathcal{L}(Y, H)$.

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\| \, ,$$

*then* $(\mathcal{I}, \|.\|_{\mathcal{I}})$ *is called a normed (Banach) operator ideal.*

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$ for the sup norm.

**Definition 1.4.2.** *(injective operator ideal)*
*A normed operator ideal* $(\mathcal{I}, \|.\|_{\mathcal{I}})$ *is said to be injective if for every metric injection* $i : Y \hookrightarrow G$ *and every* $u \in \mathcal{L}(X, Y)$ *it follows from* $i \circ u \in \mathcal{I}(X, G)$ *that* $u \in \mathcal{I}(X, Y)$. *Moreover*

$$\|i \circ u\|_{\mathcal{I}} = \|u\|_{\mathcal{I}} \, ,$$

The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one [**?**, Theorem 1.2.2].

## 1.5 Ideal of $p$-summing linear operators.

The theory of $p$-summing operators is based on a crucial criterion due to Pietsch [14]. We mention that the linear $p$-summing operators are the starting point in the study of Lipschitz $p$-summing mappings.

Let $1 \leq p < \infty$. A linear operator $T : X \longrightarrow Y$ is said to be $p$-summing if there exists a constant $C \geq 0$ such that for all finite sequence $(x_i)_{1 \leq i \leq n}$ in $X$

$$\left( \sum_{i=1}^{n} \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*} \leq 1} \left( \sum_{i=1}^{n} |\xi(x_i)|^p \right)^{\frac{1}{p}} . \tag{1.8}$$

The infimum of all such constants $C \geq 0$ is denoted by $\pi_p(T)$. The collection of all $p$-summing operators between $X$ and $Y$ is denoted by $\Pi_p(X, Y)$.

**Theorem 1.5.1.** *[4, Page 39] If $1 \leq p \leq q < \infty$, then $\Pi_p(X, Y) \subset \Pi_q(X, Y)$. Moreover, $\pi_q(T) \leq \pi_p(T)$ for every $u \in \Pi_p(X, Y)$.*

The following basic result about $p$-summing operators is due to A. Pietsch, and it characterizes the $p$-summability by means of a domination theorem.

**Theorem 1.5.2.** *(Pietsch Domination Theorem) [4, page 44]*
*Let $1 \leq p < \infty$ and $T \in \mathcal{L}(X, Y)$. Then $T$ is p-summing if and only if there exist a constant $C$ and a regular Borel probability measure $\mu$ on $B_{E^*}$ (with the weak star topology) so that*

$$\|T(x)\| \leq C \int_{B_{X^*}} |\langle x, x^* \rangle|^p \, d\mu(x^*), \qquad x \in E. \tag{1.9}$$

*In this case, $\pi_p(T)$ is the least of all the constants $C$ such that (1.9) holds.*

In order to adapt the previous result into a factorization theorem, we present basic examples of $p$-summing linear operators.

**Example 1.5.3.** *see [4, Example 2.9 (b),(d)]*
*(1) Let $K$ be a compact Hausdorff space, let $\mu$ be a positive regular Borel measure on $K$, and let $1 \leq p < \infty$. The canonical inclusion*

$$J_p : C(K) \longrightarrow L_p(\mu),$$

*is p-summing with $\pi_p(J_p) = \|J_p\| = \mu(K)^{\frac{1}{p}}$.*
*(2) Let $(\Omega, \Sigma, \mu)$ be a finite measure space and let $1 \leq p < \infty$. The formal inclusion map*

$$I_{\infty,p} : L_\infty(\mu) \longrightarrow L_p(\mu),$$

*is p-summing, with $\pi_p(I_{\infty,p}) = \mu(\Omega)^{\frac{1}{p}}$.*

We denote by $i_X$ the isometric embedding $X \longrightarrow C(B_{X^*})$ given by $i_X(x) = \langle x, . \rangle$.

**Corollary 1.5.4.** *[4, page 45] (Pietsch Factorization Theorem)*

*Let $1 \leq p < \infty$ and $T \in \mathcal{L}(X, Y)$. The following are equivalent*

*(i) $T$ is p-summing.*

*(ii) There exist a regular Borel probability measure $\mu$ on $B_{X^*}$ (with the weak star topology), a closed subspace $X_p$ of $L_p(\mu)$ and a linear continuous operator $\widetilde{u} : X_p \longrightarrow Y$ such that $J_p \circ i_X(X) \subset X_p$ and $\widetilde{u} \circ J_p \circ i_X(x) = T(x)$ for all $x \in E$.*

*In other words, if $\overline{J_p}$ is the map $i_X(X) \longrightarrow X_p$ induced by $J_p$, then the following diagram commutes:*

$$
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
i_X \downarrow & & \uparrow \widetilde{T} \\
i_X(X) & \xrightarrow{\overline{J_p}} & X_p \\
\cap & & \cap \\
C\left(B_{X^*}\right) & \xrightarrow{J_p} & L_p(\mu).
\end{array}
$$

*In addition, we may choose $\mu$ and $\widetilde{T}$ so that $\left\|\widetilde{T}\right\| = \pi_p(T)$.*

Chapter 2

# Positive $p$-summing operators

## 2.1 Positive $p$-summing linear operators

### 2.1.1 Cone absolutely summing operators

**Definition 2.1.1.** *An operator $T : X \to E$ is called cone absolutely summing(c.a.s for short) if for every positive unconditionally summable sequences $(x_n)n$ in $X$, the sequence $(T(x_n))_n$ is absolutely summable in $E$.*

Schaefer [18, Lemma 3.2,Proposition 3.3] and [17, Proposition 1] characterized c.a.s operators by various equivalences, in particular, an operator $T : X \to E$ is c.a.s if and only if there exists a constant $C > 0$ such that

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{X^*}\leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)^{\frac{1}{p}}. \tag{2.1}$$

for all finite families $(x_i)_{i=1}^n$ in $X_+$.

### 2.1.2 Positive $p$-majorizing operators

An operator $T$ beloning to $L(X, E)$ is called majorizing if there exists a constant $C > 0$ such that for every $x_1, x_2, ..., x_n$ in $X$

$$\left\| \sup_{1\leq i\leq n} |T(x_i)| \right\|_E \leq C \sup_{1\leq i\leq n} \|(x_i)\|_X . \tag{2.2}$$

**Theorem 2.1.2.** *[18] Let $X$, $Y$ be Banach lattices and $E$, $F$ be Banach spaces. Then*

1. *An operator $T : X \to E$ is c.a.s if and only if $T^*$ is majorizing.*

2. *An operator $S : F \to Y$ is majorizing if and only if $S^*$ is c.a.s.*

**Theorem 2.1.3.** *[5] The following statements are equivalent for an operator $S : E \to X$:*

1. *$S$ is majorizing.*

2. *There exists a constant $C > 0$ such that*

$$\left( \sum_{i=1}^{n} |\langle x_i^*, S(u_i) \rangle|^{p^*} \right)^{p^*} \leq C \, \|(x_i^*)_{i=1}^{\infty}\|_{p^*,\omega} \tag{2.3}$$

*for all finite families $(u_i)_{i=1}^{n}$ in $B_E$ and $(x_i^*)_{i=1}^{n}$ in $(X^*)_+$*

Theorem (2.1.3) leads to the following natural generalization of majorizing operators.

**Definition 2.1.4.** *We say that an operator $S : E \to X$ is positive $p$-majorizing if there exists a constant $C > 0$ such that*

$$\sum_{i=1}^{n} |\langle x_i^*, S(u_i) \rangle| \leq C \sup_{x^{**} \in B_{X^{**}}} (\sum_{i=1}^{n} |x^{**}(x_i^*)|^p)^{\frac{1}{p}} \tag{2.4}$$

*for all finite families $(u_i)_{i=1}^{n}$ in $B_E$ and $(x_i^*)_{i=1}^{n}$ in $(X^*)_+$*

We denote by $\gamma_p(E, X)$ the space of all positive $p$-majorizing operators from $E$ to $X$. It is easy to see that $\gamma_p(E, X)$ becomes a Banach space with the norm $\|.\|_{\gamma_p(E,X)}$ given by the infimum of the constants satisfying (2.4).

### 2.1.3 Positive $p$-summing linear operators

**Definition 2.1.5.** *Let $X$ be a Banach lattice and $Y$ be a Banach space . An operator $T : X \longrightarrow Y$ is positive $p$-summing for $1 \leq p \leq \infty$, if there exists a constant $C > 0$ such that for every $x_1, ..., x_n$*

*positive elements in* $X$*, we have*

$$(\sum_{i=1}^{n} \|T(x_i)\|^p)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} (\sum_{i=1}^{n} |x^*(x_i)|^p)^{\frac{1}{p}}. \tag{2.5}$$

*We denote by* $\Lambda_p(X;Y)$ *the class of all positive p-summing linear operators from* $X$ *into* $Y$ *and* $\pi_p^+(T) = \inf\{C$ *verifying the inequality (2.5)*$\}$.

**Proposition 2.1.6.** *Let* $X_1, X_2$ *be Banach spaces and for all* $1 \leq p \leq \infty$.
*If* $X_1 \subseteq X_2$, $\overline{X_1} = X_2$, *then*

$$\Lambda_p(X_2;Y) \subseteq \Lambda_p(X_1;Y).$$

**Proposition 2.1.7.** *Let* $Y$ *be a Banach space, we have*

- *If* $1 \leq p \leq \infty$, *then.*

$$\Lambda_p(L_1(\mu), Y) = \mathcal{B}(L_1(\mu), Y)$$

- *If* $1 \leq p \leq q \leq \infty$, *then .*

$$\Lambda_p(X;Y) \subset \Lambda_q(X;Y)$$

**Theorem 2.1.8.** *For all* $1 \leq p \leq \infty$, *we have*

$$\Lambda_p(L_{p'}(\mu);Y) = \Lambda_1(L_{p'}(\mu);Y).$$

Chapter 3

# Positive $(p, q)$-dominated operators

**Definition 3.0.1.** *Let $1 \leq p, q \leq \infty$ and let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. We say that an operator $T$ from a Banach lattice $X$ to a Banach lattice $Y$ is positive $(p, q)$-dominated if there exists a constant $C > 0$ such that*

$$(\sum_{i=1}^{n} | < y_i^*, Tx_i > |^r)^{\frac{1}{r}} \leq c \, \|(x_i)_{i=1}^n\|_p^w \, \|(y_i^*)_{i=1}^n\|_q^w$$

*for all finite families of $x_1, x_2, \ldots, x_n \in X_+$ and $y_1^*, y_2^*, \ldots, y_n^* \in (Y^*)_+$.*

*We put*

$$\|T\|_{\psi \ (p,q)} := \inf \mathrm{C}$$

*The class of all positive $(p, q)$-dominated operators from $X$ to $Y$ is denoted by $\psi_{(p,q)}(X, Y)$ .*

**Theorem 3.0.2.** *An operator $T : X \rightarrow Y$ is positive $(p, q)$-dominated if and only if there exists a constant $K > 0$ such that*

$$(\sum_{i=1}^{n} | < y_i^*, Tx_i > |^r)^{\frac{1}{r}} \leq c \, \|(|x_i|)_{i=1}^n\|_p^w \, \|(|y_i^*|)_{i=1}^n\|_q^w$$

*for all $x_1, x_2, \ldots, x_n \in X, y_1^*, y_2^*, \ldots, y_n^* \in Y^*$ and all $n \in \mathbb{N}$*

*Proof.* The sufficiency is trivial.

For the necessity, suppose that$T : X \rightarrow Y$ is positive $(p, q)$-dominated. Given any $x_1, x_2, \ldots, x_n \in$

$X, y_1^*$ and $y_2^*, \dots, y_n^* \in Y^*$.then one has

$$\begin{aligned}
(\sum_{i=1}^{n} | < y_i^*, Tx_i > |^r)^{\frac{1}{r}} &\leq (\sum_{i=1}^{n} | < y_i^*, Tx_i^+ > |^r)^{\frac{1}{r}} + (\sum_{i=1}^{n} | < y_i^*, Tx_i^- > |^r)^{\frac{1}{r}} \\
&\leq (\sum_{i=1}^{n} | < (y_i^*)^+, Tx_i^+ > |^r)^{\frac{1}{r}} + (\sum_{i=1}^{n} | < (y_i^*)^-, Tx_i^+ > |^r)^{\frac{1}{r}} \\
&+ (\sum_{i=1}^{n} | < (y_i^*)^+, Tx_i^- > |^r)^{\frac{1}{r}} \\
&\leq 4 \|T\|_{\psi\ (p,q)} \left\|(|x_i|)_{i=1}^n\right\|_p^w \left\|(|y_i^*|)_{i=1}^n\right\|_q^w
\end{aligned}$$

This completes the proof. □

The following theorem is a positive analogue of [14][ Theorem 17.4.2].

**Theorem 3.0.3** (Domination theorm). *Let $1 \leq p, q < \infty$ . An operator $T : X \to Y$ is positive $(p,q)$- dominated with constant $C$ if and only if there exist a probability measure $\mu$ on $(B_{X^*})_+$ and a probability measure $\nu$ on $(B_{Y^{**}})_+$ such that*

$$| < y^*, Tx > | \leq c \left[ \int_{(B_X^*)_+} < x^*, x >^p \, d\mu(x^*) \right]^{\frac{1}{p}} \left[ \int_{(B_y^{**})_+} < y^{**}, y^* >^q \, d\nu(y^{**}) \right]^{\frac{1}{q}}$$

*for all $x \in x_+$ and $y^* \in (Y^*)_+$*

The following is an immediate consequence of the last Theorem .

**Corollary 3.0.4.** *Let $1 \leq p, q < \infty$. An operator $T : X \to Y$ is positive $(p,q)$- dominated if and only if there exist a constant $K > 0$ , and probability measure $\mu$ on $(B_{X^*})_+$ and a probability measure $\nu$ on $(B_{Y^{**}})_+$ such that*

$$| < y^*, Tx > | \leq c \left[ \int_{(B_X^*)_+} < x^*, |x| >^p \, d\mu(x^*) \right]^{\frac{1}{p}} \left[ \int_{(B_y^{**})_+} < y^{**}, |y^*| >^q \, d\nu(y^{**}) \right]^{\frac{1}{q}}$$

*for all $x \in X_+$ and $y^* \in (Y^*)_+$*

**Theorem 3.0.5.** *An operator $T : X \to E$ is positive p-summing with $\|T\|_{\Lambda_p} \leq C$ if and only if there exist a probability measure $\mu$ on $(B_{X^*})_+$ such that*

$$\|Tx\| \leq C \left[\int_{(B_X^*)_+} < x^*, x >^p \, d\mu(x^*)\right]^{\frac{1}{p}}$$

*for all $x \in X_+$*

**Theorem 3.0.6.** *An operator $T : X \to E$ is positive p-summing if and only if there exist a constant $K$ and a probability measure $\mu$ on $(B_{X^*})_+$ such that*

$$\|Tx\| \leq K \left[\int_{(B_X^*)_+} < x^*, |x| >^p \, d\mu(x^*)\right]^{\frac{1}{p}}$$

*for all $x \in X$*

**Theorem 3.0.7.** $\psi_{(p,q)} = \Upsilon_{q^*} \circ \Lambda_p$

**Corollary 3.0.8.** *Let $T : X \to Y$ be an operator. The following are equivalent:*

*(1) $T$ is positive $(p, q)$-dominated;*

*(2) $T^*$ is positive $(q, p)$-dominated;*

*(3) $T^{**}$ is positive $(p, q)$-dominated.*

## 3.1 Positive strongly $p$-summing operators

**Definition 3.1.1.** *Let $1 \leq p < \infty$ . An operator $u : X \longrightarrow F$ is positive strongly $p-$summing if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ ,$(x_i)_{i=1}^n \subset X, (y_i^*)_{i=1}^n \subset F^*$ ,we have*

$$\sum_{i=1}^{n} |< u(x_i), y_i^* >| \leq C \left\|(x_i)_{i=1}^n\right\|_p \left\|(y_i^*)_{i=1}^n\right\|_{\ell_{p,|weak|}^*}^n (F^*) \qquad (3.1)$$

*The class of all positive strongly $p-$summing operators between $X$ and $F$ is denoted by $\mathcal{D}_p^+(X, F)$ , the infimum of all the constant $C$ in the inequality ( 3.1) defines a norm $d_p^+(.)$ on $\mathcal{D}_p^+(X, F)$ .*

**Theorem 3.1.2.** ***(Bu's theorem in the positive situation)*** *Let $1 < p, q < +\infty$ ; and let $H$ be a Hilbert space. Then*

$$\Pi_p^+(H, F) = \mathcal{D}_q^+(H, F)$$

.

Chapter 4

# Disjoint p-summing operators

**Definition 4.0.1.** *We say that an operator* $T : X \to E$ *is disjoint p-summing if there exists a constant* $C > 0$ *such that for any choice of finitely many pairwise disjoint elements* $x_1, ..., x_n$ *in* $X$*, we have*

$$(\sum_{i=1}^{n} \|Tx_i\|^p)^{\frac{1}{p}} \leq C \,\|(x_i)_{i=1}^n\|_p^w \,. \tag{4.1}$$

*We denote by* $\Gamma_p(X, E)$ *the space of all disjoint p-summing operators from* $X$ *to* $E$*. a standard argument shows that* $\Gamma_p(X, E)$ *becomes a Banach space with the norm* $\|.\|_{\Gamma_p}$ *given by the infimum of the constants satisfying (4.1) . For* $p = \infty$ *,* $\Gamma_p(X, E) = \mathcal{L}(X, E)$ *and* $\|T\|_{\Gamma_p} = \|T\|$ *.*

The following characterization of disjoint p-summing operators is straightforward.

**Proposition 4.0.2.** *An operator* $T : X \to E$ *is disjoint p-summing if and only if there exists a constant* $K > 0$ *such that for all pairwise disjoint positive elements* $x_1, ..., x_n$ *in* $X$ *, we have*

$$(\sum_{i=1}^{n} \|Tx_i\|^p)^{\frac{1}{p}} \leq k \,\|(x_i)_{i=1}^n\|_p^w \,.$$

*Proof.* It suffices to prove the sufficient part. Given any pairwise disjoint elements $x_1, ..., x_n$ in $X$. Then $x_1^+, ..., x_n^+$ are pairwise disjoint and hence, we get

$$(\sum_{i=1}^{n} \left\|Tx_i^+\right\|^p)^{\frac{1}{p}} \leq k \left\|(x_i^+)_{i=1}^n\right\|_p^w .$$

Similarly, we get

$$(\sum_{i=1}^{n} \left\|Tx_i^-\right\|^p)^{\frac{1}{p}} \leq k \left\|(x_i^-)_{i=1}^n\right\|_p^w .$$

Thus

$$
\begin{aligned}
(\sum_{i=1}^{n} \|Tx_i\|^p)^{\frac{1}{p}} &\leq (\sum_{i=1}^{n} \|Tx_i^+\|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} \|Tx_i^-\|^p)^{\frac{1}{p}} \\
&\leq k \left\|(x_i^+)_{i=1}^n\right\|_p^w + \left\|(x_i^-)_{i=1}^n\right\|_p^w \\
&= k \left( \sup_{a\in(B_{l_{p^*}^n})_+} \left\| \sum_{i=1}^{n} a_i x_i^+ \right\| + \sup_{a\in(B_{l_{p^*}^n})_+} \left\| \sum_{i=1}^{n} a_i x_i^- \right\| \right) \qquad (4.2)
\end{aligned}
$$

Since $x_1, ..., x_n$ are pairwise disjoint, we have

$$\left|\sum_{i=1}^{n} a_i x_i\right| = \sum_{i=1}^{n} a_i |x_i| = \sum_{i=1}^{n} a_i x_i^+ + \sum_{i=1}^{n} a_i x_i^-, \forall a = (a_i)_{i=1}^n \in (B_{l_{p^*}^n})_+.$$

This implies

$$\sup_{a\in(B_{l_{p^*}^n})_+} \left\| \sum_{i=1}^{n} a_i x_i^+ \right\| \leq \sup_{a\in(B_{l_{p^*}^n})_+} \left\| \sum_{i=1}^{n} a_i x_i \right\| \qquad (4.3)$$

and

$$\sup_{a\in(B_{l_{p^*}^n})_-} \left\| \sum_{i=1}^{n} a_i x_i^+ \right\| \leq \sup_{a\in(B_{l_{p^*}^n})_+} \left\| \sum_{i=1}^{n} a_i x_i \right\| \qquad (4.4)$$

Combining inequalities (4.2), (4.3) and (4.4), we get

$$(\sum_{i=1}^{n} \|Tx_i\|^p)^{\frac{1}{p}} \leq 2k \sup_{a\in B_{l_{p^*}^n}} \left\| \sum_{i=1}^{n} a_i x_i \right\| = 2k \left\|(x_i)_{i=1}^n\right\|_p^w .$$

□

An easy consequence of Proposition 4.0.2 is the following relationship between postive $p$-summing operators and disjoint $p$-summing operators.

**Corollary 4.0.3.** $\Lambda_p \subseteq \Gamma_p$

We end this section by discovering the relationship between disjoint $p$ -summing operator $T$ and its second dual $T^{**}$.

**Theorem 4.0.4.** *An operator $T : X \to E$ is disjoint p-summing if and only if so is $T^{**}$. In this case,* $\|T\|_{\Gamma_p} = \|T^{**}\|_{\Gamma_p}$ .

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