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TITLE

VIETA-LUCAS MATRIX METHOD FOR SOLVING LINEAR SECOND-ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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ملخص

تهدف هذه الأطروحة إلى إيجاد حل عددي لمعادلات فريدهولم التفاضلية التكاملية الخطية من الدرجة الثانية باستخدام كثيرات حدود فيتا-لوكاس مع تقديم امثلة عددية مختلفة لضمان دقة وكفاءة الطريقة الحسابية المقترحة.

كلمات مفتاحية :

معادلات فريدهولم التفاضلية التكاملية الخطية ، كثيرات حدود فيتا-لوكاس.

Abstract :

This thesis aims to find a numerical solution for second-order linear Fredholm integro-differential equations using Vieta-Lucas polynomials with various numerical examples to ensure the accuracy and efficiency of the proposed computational method.

Keywords :

Linear Fredholm integro-differential equations, Vieta-Lucas polynomials.

Résumé :

Cette thèse vise à trouver la solution numérique des équations intégro-différentielles de Fredholm linéaires du second ordre en utilisant les polynômes de Vieta-Lucas avec divers exemples numériques pour assurer la précision et l'efficacité de la méthode de calcul proposée.

Mot-clés :

Equations intégro-différentielles linéaires de Fredholm, polynômes de Vieta-Lucas.

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Dédicace

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To the companions from the first step to the last step, to those who were during the lean years as clouds, to the friends of my life.

I dedicate to you all the joy of my graduation

List of sympols

Notation	Name
$\mathbb{N}$	The set of natural numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$[a, b]$	Real interval.
$\|\cdot\|$	A norm.
λ	Numerical parameter.
ε	epsilon.
$C([a, b])$	The space of continuous functions on [a,b].
$\langle , \rangle$	Scalar product.
$\sum$	Summation sign.
$\int$	Integral sign.
A	Linear operator.
$K(x, t)$	Kernel of the integral equation.
$FIDEs$	Fredholm integro-differential equations

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Introduction

In recent years, integro-differential equations have emerged as essential mathematical tools in modeling diverse phenomena across disciplines such as physics, biology, and engineering.

The Fredholm integro-differential equation arises from converting boundary value problems in differential equations to integro-differential equations,These equations are characterized by the existence of one or more of the derivatives outside the integral sign. It's often difficult or impossible to find analytical solutions for these equations, researchers have focused on developing numerical methods to obtain approximate solutions.Several authors have contributed to this area. For example [4] employed the Chebyshev Third Kind Polynomial, [5] used the Lucas Collocation method,[8] applied the Euler polynomials with least squares method.[9] employed Legendre polynomial .

The Fredholm integro-differential equation of order $m \geq 1$ can be written as:

$$\sum_{k=0}^{m} P_k(x)u^{(k)}(x) = f(x) + \lambda \int_a^b k_f(x,t)u(t)\,dt \tag{0.0.1}$$

This equation is accompanied by initial conditions:

$$u^{(k)}(a) = \alpha_k, \quad \text{for } 0 \leq k \leq m-1, \tag{0.0.2}$$

where the functions $k_f(x,t)$, $f(x)$ and $P_k(x)$ are known. $u(x)$ is the unknown function to be determined.a, b and λ are constants.

In this thesis we introduce and explore a novel numerical technique based on **vieta-Lucas polynomials**, a class of orthogonal polynomials known for their recurrence properties and advantageous convergence behavior. By employing a Vieta-Lucas matrix collocation method, the work presents a robust and efficient framework for approximating the solutions to linear second-order Fredholm integro-differential equations.

This work consists of three chapters that cover different aspects of the numerical solution of Fredholm integro-differential equations, It is organized as follows:

The first chapter: Presenting fundamental concepts as well as essential definitions and theorems that will be applied in the subsequent chapters.

The second chapter: Certain types of integral equations as well as Fredholm integro-differential equations .

In third chapter: Vieta-Lucas polynomial and its properties,numerical resolution methods for Fredholm integro-differential equations with illustrative examples.

Chapter 1

Preliminary

This chapter is mainly devoted to presenting fundamental concepts as well as essential definitions and theorems that will be applied in the subsequent chapters.

1.1 Functional spaces

1.1.1 Normed vector space

(Normes) Let F a vector space over field $\mathbb{K}= \mathbb{R}$ or $\mathbb{C}$,we call a norm on the space Fany function denoted $\| \cdot \|$ defined on F with values in $\mathbb{R}_+$, such that:
For all $x, y \in F$ and $\lambda \in \mathbb{K}$

(i) $\|x\| \geq 0$,

(ii) $\|x\| = 0$ if and only if $x = 0$,

(iii) $\lambda\| \, x\| = |\lambda|\|x\|$,

(iv) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

A vector space F with a norm $(F, \|.\| \ )$ is called a normed space

Examples:

1. The absolute value is a norm on $\mathbb{R}$.

2. The modulus is a norm on $\mathbb{C}$.

3. The following applications are norms on $\mathbb{R}^n$:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}, \quad \|x\|_\infty = \sup_{1 \leq i \leq n} |x_i|.$$

4. Let $F = (C([a,b]), \mathbb{R})$ be the space of real-valued continuous functions on the interval $([a,b])$ for any $g \in F$, the following functions define a norm on F :

$$\|g\|_1 = \int_a^b |g(x)|\,dx, \quad \|g\|_2 = \left(\int_a^b |g(x)|^2\,dx \right)^{\frac{1}{2}}, \quad \|g\|_\infty = \max_{a \leq x < b} |g(x)|.$$

1.1.2 Banach space

(Cauchy sequence). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of normed space $(F, \|\cdot\|)$, we say that sequence x_n is Cauchy if:

$$\forall \varepsilon > 0, \exists N_\varepsilon, \forall p, q \geq N_\varepsilon, \quad \|x_p - x_q\| < \varepsilon \tag{1.1.1}$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a normed space $(F, \|\cdot\|)$, then x_n is said to be cnvergent to $x_0 \in F$ if:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Longrightarrow \|x_n - x_0\| \leq \varepsilon \tag{1.1.2}$$

A normed space $(F, \|\cdot\|)$ is complete if every Cauchy sequence of elements in F converges to an element in F, meaning there is a point $x \in F$, such that the sequence (x_n) converges to x in the norm $\|\cdot\|$.

The advantage of complete spaces is that, in such spaces, it is enough to verify that a sequence is Cauchy in order to conclude that it converges. You do not need to explicitly know the limit of the sequence; the completeness of the space guarantees that the sequence will converge to some limit within the space.

(banach space).A normed vector space $(F, \|\cdot\|)$ is called a Banach space if it is complete, meaning that every Cauchy sequence (x_n) in F converges to some limit $x \in F$.

Every finite-dimensional normed vector space is automatically complete, meaning it is a Banach space.

1.1.3 Hilbert space

An inner product on a real or complex vector space H is a function

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$$

that assigns to each pair $(x, y) \in H \times H$ a complex number $\langle x, y \rangle$, and satisfies certain properties (typically linearity in one argument, conjugate symmetry, and positive-definiteness). Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space H. Then for all $x, y, z \in H$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$ for real vector spaces), the inner product satisfies:

1. **Linearity in the first argument**:

$$\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle \tag{1.1.3}$$

2. **Conjugate symmetry:**

$$\langle x, y \rangle = \overline{\langle y, x \rangle} \tag{1.1.4}$$

3. **Positive-definiteness:**

$$\langle x, x \rangle \geq 0, \quad \text{and} \quad \langle x, x \rangle = 0 \iff x = 0 \tag{1.1.5}$$

Example 1.1.1 *Here is some examples of inner product space which demonstrate that expression $\|x\| = \sqrt{\langle x, x \rangle}$ defines a norm.*

1. *The inner product for $\mathbb{C}^n$ is given by*

$$\langle x, y \rangle = \sum_{1}^{n} x_j \overline{y_j}$$

where $\overline{y_i}$ denotes the complex conjugate of y_i.

2. *Let extension for infinite vectors, let l_2 be*

$$l_2 = \left\{ sequences \ \{x_j\}_1^\infty \ : |\sum_1^n |x_j|^2 < \infty \right\}.$$

3. *Let $C([a,b])$ be a space of continous function on the interval $[a,b] \subset \mathbb{R}$:*

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx \ \ and \ \ \|f\|_2 = \left(\int_a^b |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

A Hilbert space is a vector space equipped with an inner product that is complete with respect to the norm induced by that inner product.

1.2 Notions on operators

1.2.1 Linear operators

Consider two normed vector spaces E and F, and let $A : E \to F$ be a mapping. The operator A is said to be **linear** if it satisfies the following property:

For any vectors $\varphi_1, \varphi_2 \in E$ and any scalars $\lambda_1, \lambda_2 \in \mathbb{K}$ (where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$), we have:

$$A(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 A(\varphi_1) + \lambda_2 A(\varphi_2). \tag{1.2.1}$$

In other words, A preserves vector addition and scalar multiplication.

(Continuous linear operators). Let E and F be normed spaces, and let A be a linear operator defined on a subset $G \subset E$, mapping into F. The operator A is said to be continuous at a point $x_0 \in G$ if the following condition holds:

- For every sequence (x_n) in G that converges to x_0, the sequence $A(x_n)$ converges to $A(x_0)$.

In other words:

$$\lim_{n\to\infty} A(x_n) = A\left(\lim_{n\to\infty} x_n\right) = A(x_0). \tag{1.2.2}$$

We can say about the operator A that it is continuous in the set G if it's continuous at every point of G.

1.2.2 Bounded linear operators

Let $A : E \to F$ be a linear operator between normed spaces E and F. The operator A is said to be **bounded** if there exists a constant $C > 0$ such that:

$$\|A(x)\|_F \leq C\|x\|_E, \quad \forall x \in E. \tag{1.2.3}$$

In other words, the norm of $A(x)$ in F is controlled by the norm of x in E, uniformly over all $x \in E$.

The ***norm*** *of a linear operator A, defined as the smallest constant C that satisfies inequality (1.10), is expressed as:*

$$\|A\| = \sup_{x\neq 0} \frac{\|A(x)\|_F}{\|x\|_E} = \sup_{\|x\|=1} \|A(x)\|_F = \sup_{\substack{x\in E \\ x\neq 0, \|x\|\leq 1}} \|A(x)\|_F. \tag{1.2.4}$$

A is continuous linear operator equiv A is bounded.

1.2.3 Compact linear operators

(Compact linear operator). Let A be a linear operator from a normed space E into a normed space F. The operator A is said to be a **compact linear operator** (or **completely continuous operator**) if, for every bounded subset $\Omega \subset E$, the image $A(\Omega) \subset F$ has compact closure in F; that is, $\overline{A(\Omega)}$ is compact in F. In simpler terms, A maps bounded sets in E to relatively compact sets in F.

(finite dimensional domain). Suppose A is a bounded linear operator acting from a normed space E into another normed space F. If the domain space E has finite dimension (that is, $\dim E < \infty$), then A is a compact operator. In other words, every bounded linear operator defined on a finite-dimensional normed space is automatically compact.

proof. Since the space E is finite-dimensional, i.e., $\dim E < \infty$, it follows that the image $A(E)$ is also finite-dimensional. Specifically,

$$\dim A(E) \leq \dim E. \tag{1.2.5}$$

Therefore, the operator A has finite-dimensional range, and hence A is compact. ■

Let A_1 and A_2 be compact operators. Then for any scalars α and β, the operator

$$A = \alpha A_1 + \beta A_2 \tag{1.2.6}$$

is also compact.

proof. Let $\{\varphi_n\}$ be a bounded sequence in E, and consider the sequence $A\varphi_n$ in F, where

$$A\varphi_n = \alpha A_1\varphi_n + \beta A_2\varphi_n, \quad \varphi_n \in E,\ n \in \mathbb{N}. \tag{1.2.7}$$

Since both A_1 and A_2 are compact operators, the sequences $\{A_1\varphi_n\}$ and $\{A_2\varphi_n\}$ each have convergent subsequences. The linear combination of these convergent subsequences also converges. Therefore, $\{A\varphi_n\}$ has a convergent subsequence. Hence, A is compact. ∎

The identity operator I_d on a normed space E is compact **if and only if** E is finite-dimensional.

A bounded operator A on a normed space E is not necessarily compact. For example, the identity operator $A = I$ on an infinite-dimensional normed space is bounded but not compact.

1.2.4 Integral linear operators

Integral operators are key tools in functional analysis, especially useful for transforming complex functions into simpler forms for easier analysis.

An **integral operator** is any linear operator A defined on a normed space E, taking values in another normed space F, and expressed as:

$$A\varphi(x) = \int_{G_2} k(x,y)\varphi(y)\,dy, \quad x \in G_1, \tag{1.2.8}$$

where:

- $k(x,y)$ is a measurable function on the measurable set $G_1 \times G_2$,
- $\varphi(y)$ is a measurable function on G_1.

The function $k(x,y)$ is called the **kernel** of the integral operator A.

The **norm** of an integral operator A defined on the space $L^p(G_1)$, where p and q are conjugate exponents (i.e., $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \leq p, q \leq \infty$), is given by:

$$\|A\|_p = \begin{cases} \left(\int_{G_1}\left(\int_{G_2}|k(x,y)|^q\,dy\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}, & \text{for } 1 < p < \infty, \\ \int_{G_1}\operatorname{ess\,sup}_y |k(x,y)|\,dx, & \text{for } p = 1, \\ \operatorname{ess\,sup}_x \int_{G_2}|k(x,y)|\,dy, & \text{for } p = \infty. \end{cases} \tag{1.2.9}$$

Here, $k(x, y)$ is the **kernel** of the integral operator A.

Let A be an integral operator such that its norm $\|A\|_p$ is finite:

$$\|A\|_p < \infty. \tag{1.2.10}$$

then the operator A is a **continuous linear operator** from $L^p(G_1)$ to $L^p(G_2)$. Moreover, for every function $\varphi \in L^p(G_1)$, we have:

$$\|A\varphi\|_p \leq \|A\|_p \cdot \|\varphi\|_p. \tag{1.2.11}$$

Chapter 2

Integral Equations

In this chapter, we will discuss certain types of integral equations as well as Fredholm integro-differential equations.

2.1 Classification of integral equations

Integral equations can be linear or nonlinear and are closely related to differential equations. The two main types are Volterra and Fredholm equations, which may also be homogeneous or inhomogeneous. Additionally, we encounter integro-differential and singular integral equations in various applications.

We will describe each using essential definitions and properties.

2.1.1 Volterra integral equations

The general form of a Volterra linear integral equation is:

$$\phi(x)u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)\,dt \tag{2.1.1}$$

If $\phi(x) = 1$, the equation becomes:

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)\,dt \tag{2.1.2}$$

This is the Volterra equation of the second kind.

If $\phi(x) = 0$, we get:

$$f(x) + \lambda \int_a^x k(x,t)u(t)\,dt = 0 \tag{2.1.3}$$

This is known as the Volterra equation of the first kind.

2.1.2 Fredholm integral equations

The general Fredholm linear integral equation is written as:

$$\phi(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \tag{2.1.4}$$

Here, a and b are constants, and $u(x)$ appears linearly under the integral. If $\phi(x) = 1$, the equation becomes:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \tag{2.1.5}$$

This is the **Fredholm equation of the second kind**.

If $\phi(x) = 0$, we obtain:

$$f(x) + \lambda \int_a^b k(x,t)u(t)\,dt = 0 \tag{2.1.6}$$

This is the **Fredholm equation of the first kind**.

Integral equations play a key role in many scientific fields such as engineering, physics, and biology. Many boundary value problems from differential equations can be reformulated as Volterra or Fredholm integral equations.

If the unknown $u(x)$ appears nonlinearly under the integral, such as in expressions like $u^2(t)$, $\sin(u(t))$, or $\ln(u(t))$, then the equation is classified as **nonlinear**. Examples:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u^2(t)\,dt$$

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\sin(u(t))\,dt$$

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\ln(u(t))\,dt$$

Moreover, if $f(x) = 0$, the equation is called **homogeneous**; otherwise, it is **nonhomogeneous**.

2.1.3 Volterra-Fredholm integral equations

The Volterra-Fredholm integral equation is defined as an equation of the form:

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)\,dt + \lambda_2 \int_a^b k_2(x,t)U(t)\,dt, \quad x \in [a,b],$$

where the functions $k_1(x,t)$, $k_2(x,t)$, and $f(x)$ are given, $u(x)$ is the unknown function, and λ_1 and λ_2 are non-zero constants.

2.2 Classification of integro-differential equations

2.2.1 Volterra integro-differential equations

A Volterra integro-differential equation is generally expressed as:

$$\sum_{k=0}^{m} P_k(x)u^{(k)}(x) = f(x) + \lambda \int_a^x k_v(x,t)u(t)\,dt, \tag{2.2.1}$$

with initial conditions:

$$u^{(k)}(a) = \alpha_k, \quad \text{for } 0 \leq k \leq m-1,$$

where $u(x)$ is the unknown function, $P_k(x)$, $k_v(x,t)$, and $f(x)$ are given functions, λ is a parameter, $u^{(k)}$ is the k^{th} derivative of $u(x)$.

Examples

1.
$$\begin{cases} u'(x) + 2u(x) = \cos(x) + \int_0^x e^{x-t}u(t)\,dt, \\ u(0) = 1 \end{cases}$$

This is a first-order Volterra integro-differential equation.

2.
$$\begin{cases} u''(x) - xu'(x) = \sin(x^2) - \int_0^x tu(t)\,dt, \\ u(0) = 0, \quad u'(0) = 1 \end{cases}$$

This is a second-order Volterra integro-differential equation.

2.2.2 Fredholm integro-differential equation

The Fredholm integro-differential equation of order $m \geq 1$ can also be written as:

$$\sum_{k=0}^{m} P_k(x)u^{(k)}(x) = f(x) + \lambda \int_a^b k_f(x,t)u(t)\,dt \tag{2.2.2}$$

with initial conditions:

$$u^{(k)}(a) = \alpha_i, \quad 0 \leq k \leq m-1$$

Examples

1.
$$\begin{cases} u'(x) + 3u(x) = e^x + \displaystyle\int_0^1 \cos(x+t)u(t)\,dt \\ u(0) = 2 \end{cases}$$

This is a first-order Fredholm integro-differential equation.

2.
$$\begin{cases} u''(x) - 2u'(x) + u(x) = \ln(1+x) + \displaystyle\int_0^1 (x-t)u(t)\,dt \\ u(0) = 0, \quad u'(0) = 1 \end{cases}$$

This is a second-order Fredholm integro-differential equation.

2.2.3 Volterra-Fredholm integro-differential equations

The Volterra-Fredholm integro-differential equation is a mathematical expression that combines both the Volterra and Fredholm integral operators along with differential operators. These components can be unified in a single integral form, expressed as:

$$\begin{cases} +\displaystyle\sum_{k=0}^{m} P_k(x)u^{(k)}(x) = f(x) + \lambda_1 \int_a^x k_v(x,t)\,u(t)\,dt + \lambda_2 \int_a^b k_f(x,t)\,u(t)\,dt, \\ u^{(k)}(a) = \alpha_k, \quad \text{for } 0 \leq k \leq m-1. \end{cases} \tag{2.2.3}$$

where $u^{(k)}(x)$ denotes the k^{th} derivative of $u(x)$ and $k_v(x,y)$, $k_f(x,y)$, $P_k(x)$ and $f(x)$ are known functions, λ_1 and λ_2 are non-zero parameters, and $u(x)$ is the unknown function to be determined.

Example:

$$\begin{cases} u''(x) = \sin(x) + \displaystyle\int_0^x (x-t)^2 u(t)\,dt + \int_0^1 e^{x+t} u(t)\,dt, \\ u(0) = 0, \quad u'(0) = 1 \end{cases}$$

It is a Volterra-Fredholm integro-differential equation of second order.

2.3 Analytical solution of Fredholm integro-differential equations

There are several analytical methods for solving Fredholm integro-differential equations of the second kind. We will focus on the following methods:
The Adomian decomposition method and the series solution method.

2.3.1 The Adomian decomposition method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian. ADM provides the solution in the form of an infinite series of components. The idea behind the Adomian decomposition method is to transform the Fredholm integro-differential equation into an integral equation.

Consider the second-order Fredholm integro-differential equation:

$$u''(x) = f(x) + \int_a^b k(x,t)\,u(t)\,dt \tag{2.3.1}$$

With initial conditions

$$u(0) = \alpha_0, \quad u'(0) = \alpha_1$$

Integrating both sides of 2.3.1 from 0 to x, we obtain:

$$u(x) = a_0 + a_1 x + \mathcal{L}^{-1}\left[f(x)\right] + \mathcal{L}^{-1}\left(\int_a^b k(x,t)\, u(t)\, dt\right) \tag{2.3.2}$$

Where the initial conditions are used, and $\mathcal{L}^{-1}$ is a double integral operator, $\varphi(x)$ is expressed as :

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{2.3.3}$$

substituting 2.3.3 into 2.3.2 we find:

$$\sum_{n=0}^{\infty} u_n(x) = a_0 + a_1 x + \mathcal{L}^{-1}\left[f(x)\right] + \mathcal{L}^{-1}\left(\int_a^b k(x,t) \sum_{n=0}^{\infty} u_n(t)\, dt\right) \tag{2.3.4}$$

hence

$$u_0(x) + u_1(x) + u_2(x) + \cdots = a_0 + a_1(x) + \mathcal{L}^{-1}\left[f(x)\right] + \mathcal{L}^{-1}\left(\int_a^b k(x,t) u_0(t)\, dt\right) +$$

$$\mathcal{L}^{-1}\left(\int_a^b k(x,t) u_1(t)\, dt\right) + \mathcal{L}^{-1}\left(\int_a^b k(x,t) u_2(t)\, dt\right) + \ldots$$

Note that, $u_0(x)$ is defined by all terms not included under the integral sign, that is:

$$u_0(x) = a_0 + a_1(x) + \mathcal{L}^{-1}[f(x)]$$

$$u_{j+1}(x) = \mathcal{L}^{-1}\left(\int_a^x k(x,t)\, u_j(t)\, dt\right), \quad j = 0, 1, \ldots$$

Using 2.3.3, the obtained series converges to the exact solution.

Solve the Fredholm integro-differential problem using the Adomian decomposition method.

$$u'''(x) - x + \int_0^1 xtu(t)dt, \quad u(0) = u'(0) = u''(0) = 1. \tag{2.3.5}$$

We integrate both sides of Eq 2.3.5 three times from 0 to x, and applying the initial conditions, we get

$$u(x) = e^x - \frac{1}{3!}x^3 + \frac{1}{3!}x^3\left(\int_0^1 tu(t)dt\right). \tag{2.3.6}$$

As before, we establish the recurrence relation

$$u_0(x) = e^x - \frac{1}{3!}x^3 . u_{k+1}(x) = \frac{1}{3!}x^3 \int_0^1 tu_k(t)dt. \quad k \geqslant 0 \tag{2.3.7}$$

Consequently, this provides

$$u_0(x) = e^{-x} - \frac{1}{3!}x^3,$$

$$u_1(x) = \frac{1}{3!}x^3 \int_0^1 tu_0(t)\,dt = \frac{29}{180}x^3,$$

$$u_2(x) = \frac{1}{3!}x^3 \int_0^1 tu_1(t)\,dt = \frac{29}{5400}x^3,$$

$$u_3(x) = \frac{1}{3!}x^3 \int_0^1 tu_2(t)\,dt = \frac{29}{162000}x^3,$$

The series form of the solution is provided by

$$u(x) = e^x - \frac{1}{3!}x^3 + \frac{29}{180}x^3 \left(1 + \frac{1}{30} + \frac{1}{900} + \dots\right). \tag{2.3.8}$$

The infinite geometric series has $a_1 = 1$,$r = \frac{1}{3}$. The infinite geometric series' sum may be found using

$$S = \frac{1}{1 - \frac{1}{30}} = \frac{30}{29}. \tag{2.3.9}$$

This result givesthe precise answer

$$u(x) = e^x.$$

2.3.2 The series method of solutions

The series method is a useful method that derives mainly from the Taylor series for analytic functions for the Fredholm integro-differential equation.

A real function $u(x)$ has a Taylor series representation:

$$u(x) = \sum_{k=0}^{n} \frac{u^{(k)}(b)}{k!}(x - b)^k \tag{2.3.10}$$

An analytic function is one that can be differentiated infinitely many times, and its Taylor series centered at any point b converges to the function $u(x)$ near b.

To simplify, the Taylor series at $b = 0$ is:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2.3.11}$$

Assume the Fredholm integro-differential equation:

$$u^{(m)}(x) = F(x) + \lambda \int_a^b k(x,t)u(t)\,dt, \quad u^{(j)}(0) = a_j, \quad 0 \leq j \leq m-1 \tag{2.3.12}$$

If $u(x)$ is analytic, it has a Taylor expansion as in 2.3.12, and the coefficients a_n can be found recursively.

Substituting the Taylor series into the equation gives:

$$(\sum_{n=0}^{\infty} a_n x^n)^{(m)} = T[f(x)] + \lambda \int_a^b k(x,t)(\sum_{n=0}^{\infty} a_n t^n)dt \tag{2.3.13}$$

This is equivalent to:

$$(a_0 + a_2 x^2 + a_3 x^3 + \dots)^{(k)} = T[f(x)] + \lambda \int_a^b k(x,t)(a_0 + a_2 t^2 + \dots)\,dt \tag{2.3.14}$$

where $T[f(x)]$ is the Taylor series for $f(x)$.

By comparing coefficients of powers of x, the a_n can be determined.

Use the series solution method to solve the Fredholm integro-differential problem.

$$u'(x) = 4 + 4x + \int_{-1}^{1} (1 - xt)u(t)dt, \quad u(0) = 1. \tag{2.3.15}$$

Changing $u(x)$ to the series

$$u'(x) = (\sum_{n=0}^{\infty} a_n x^n)' \tag{2.3.16}$$

into each side of Equation 2.3.15 results in

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = 4 + 4x + \int_{-1}^{1} ((1 - xt) \sum_{n=0}^{\infty} a_n t^n) dt. \tag{2.3.17}$$

Observe that, given the initial conditions, $a_0 = 1$. When the integral on the right side is evaluated, it provides

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 6 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + \frac{2}{7}a_6 + \frac{2}{9}a_8 + \left(4 - \frac{2}{3}a_1 - \frac{2}{5}a_3 - \frac{2}{7}a_5 - \frac{2}{9}a_7\right) x. \tag{2.3.18}$$

The coefficients of like powers of x in both sides of 2.3.18 can be equated to yield

$$a_1 = 6, \quad a_n = 0, \quad n \geqslant 2. \tag{2.3.19}$$

The exact solution is given by

$$u(x) = 1 + 6x, \tag{2.3.20}$$

where we used $a_0 = 1$ from the initial condition.

Chapter 3

Numerical solution of LFIDE

Vieta-Lucas polynomial and its properties,numerical resolution methods for Fredholm integro-differential equations with illustrative examples. This is what we will cover in this chapter.

3.1 Vieta-Lucas polynomials

Orthogonal Polynomials

Definition 3.1.1 *Two functions $f(x)$ and $g(x)$ are orthogonal if :*

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx = 0$$

where $w(x) > 0$ acts as the weight function.

3.1.1 Vieta-Lucas polynomials

Let $|x| \leq 2$. The Vieta–Lucas polynomials $VL_n(x)$, $(n \in \mathbb{N})$, are defined as:

$$VL_n(x) = 2\cos(n\theta), \quad \theta = \cos^{-1}\left(\frac{x}{2}\right), \quad \theta \in [0, \pi] \tag{3.1.1}$$

We can build these polynomials using the recurrence relation:

$$VL_n(x) = xVL_{n-1}(x) - VL_{n-2}(x), \quad \text{for } n = 2, 3, \ldots \tag{3.1.2}$$

with initial values:

$$VL_0(x) = 2, \quad VL_1(x) = x$$

Another way to define $VL_n(x)$ is through the power series:

$$VL_n(x) = \sum_{i=0}^{[\frac{n}{2}]} (-1)^i \binom{n-i}{i} \frac{n}{n-i} x^{n-2i}, \quad n = 2, 3, \ldots \tag{3.1.3}$$

where $[\frac{n}{2}]$ is the ceiling function.

Using the previous recursion, the next five VL polynomials can be determined as follows:

$$\begin{aligned}
VL_2(x) &= -2 + x^2, \\
VL_3(x) &= -3x + x^3, \\
VL_4(x) &= 2 - 4x^2 + x^4, \\
VL_5(x) &= 5x - 5x^3 + x^5, \\
VL_6(x) &= -2 + 9x^2 - 6x^4 + x^6.
\end{aligned}$$

The Vieta–Lucas polynomials are orthogonal with respect to the inner product:

$$\langle VL_m(x), VL_n(x) \rangle = \int_{-2}^{2} \frac{1}{\sqrt{4-x^2}} VL_m(x)\, VL_n(x)\, dx \tag{3.1.4}$$

This integral evaluates to:

$$\int_{-2}^{2} \frac{1}{\sqrt{4-x^2}} VL_m(x)\, VL_n(x)\, dx = \begin{cases} 0, & \text{if } m \neq n,\ m, n \neq 0 \\ 4\pi, & \text{if } m = n = 0 \\ 2\pi, & \text{if } m = n \neq 0 \end{cases} \tag{3.1.5}$$

The VL polynomial $u(x) = VL_n(x)$ satisfies the following differential equation:

$$(4 - x^2)u''(x) - xu'(x) + n^2 u(x) = 0, \quad n \in \mathbb{N}. \tag{3.1.6}$$

proof. Differentiating $u(x)$ with respect to x, we have:

$$u' = \frac{du}{dx} = \frac{du}{d\varphi}\frac{d\varphi}{dx} = n\frac{\sin(n\varphi)}{\sin\varphi}.$$

Similarly, for the second derivative:

$$u'' = \frac{d}{dx}(u') = \frac{d}{dx}\left(\frac{du}{d\varphi}\frac{d\varphi}{dx}\right) = -n^2\frac{\cos(n\varphi)}{2\sin^2\varphi} + n\frac{\cos\varphi\sin(n\varphi)}{2\sin^3\varphi}.$$

Substituting u', u'', $u = 2\cos(n\varphi)$, and $x = 2\cos\varphi$ into the given differential equation, and simplifying, all terms cancel out. Thus, the proof is completed. ■

3.1.2 Shifted Vieta-Lucas polynomials

To solve the linear Fredholm integro-differential equation of second order with the Vieta-Lucas polynomials family, we are required to transform them on the interval $[0, 1]$. In this regard, we make use of the change on the differential equation, variable $x = 4t - 2$ that means that:

$$(x - x^2)\frac{d^2}{dt^2}VL_n^*(x) - (x - \frac{1}{2})\frac{d}{dx}VL_n^*(x) = 0, \quad n \in \mathbb{N}. \tag{3.1.7}$$

In this case $VL_n^*(x) = VL_n(4t - 2)$, The shifted VL polynomials defined on $[0, 1]$. Given 3.1.3, the unique solution of 3.1.6 may be used to represent the explicit form of shifted VL polynomials. Following a few adjustments, we get

$$VL_0^*(x) = 2, \quad VL_n^*(x) = 2n\sum_{r=0}^{n}(-1)^{n-r}\frac{4^r(n+r-1)!}{(n-r)!(2r)!}x^r, \quad n \in \mathbb{N}. \tag{3.1.8}$$

Consequently, the following is the construction of a few shifted VL polynomials on $[0, 1]$

$$\begin{aligned}
VL_1^*(x) &= -2 + 4x,\\
VL_2^*(x) &= +2 - 16x + 16x^2,\\
VL_3^*(x) &= -2 - 36x - 96x^2 + 64x^3,\\
VL_4^*(x) &= +2 - 64x + 320x^2 - 512x^3 + 256x^4,\\
VL_5^*(x) &= -2 + 100x - 800x^2 + 2240x^3 - 2560x^4 + 1024x^5.
\end{aligned}$$

The weight function $w(x)$ for the shifted VL polynomials on $[0, 1]$ is therefore expressed as $w^*(x) = \dfrac{1}{\sqrt{x - x^2}}$. As a result, the orthogonal condition 3.1.5 for the shifted VL polynomials is written as

$$\int_0^1 VL_n^*(x)VL_m^*(x)w^*(x)dx = \begin{cases} 0, & \text{if } m \neq n,\ m, n \neq 0 \\ 4\pi, & \text{if } m = n = 0 \\ 2\pi, & \text{if } m = n \neq 0 \end{cases} \tag{3.1.9}$$

3.2 Numerical solution of LFIDE

We consider the linear Fredholm integro-differential equations of second order :

$$\sum_{k=0}^{k=2} P_k(x)u^{(k)}(x) = f(x) + \int_0^1 k_f(x,t)u(t)dt \tag{3.2.1}$$

with initial conditions

$$u^{(k)}(0) = \alpha_k, k = 0, 1 \tag{3.2.2}$$

First, we consider the approximate solution of 3.2.1 in the form of shifted Vieta-Lucas series given below as:

$$u_n(x) = \sum_{i=0}^{n} a_i VL_i^*(x) = VL^* A \quad x \in [0,1]\,. \tag{3.2.3}$$

where

$$VL^*(x) = \begin{bmatrix} VL_0^*(x) & VL_1^*(x) & \cdots & VL_n^*(x) \end{bmatrix}$$

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_n \end{bmatrix}^t$$

The vector of VL^* basis functions is constructed as follows in order to create the VL^* matrix scheme:

$$VL^*(x) = \begin{bmatrix} VL_0^*(x) & VL_1^*(x) & \cdots & VL_n^*(x) \end{bmatrix} = T(x)H. \tag{3.2.4}$$

where

$$T(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^n \end{bmatrix}.$$

and H is a matrix $(n+1)\times(n+1)$:

$$H = \begin{pmatrix} 2 & -2 & 2 & \cdots & 2(-1)^{n-1} & 2(-1)^n \\ 0 & 4 & -16 & \cdots & 2(n-1)\dfrac{(-1)^{n-2}(n-1)!4}{(n-2)!2!} & 2n\dfrac{(-1)^{n-1}n!4}{(n-1)!2!} \\ 0 & 0 & 16 & \cdots & 2(n-1)\dfrac{(-1)^{n-3}(n)!4^2}{(n-3)!4!} & 2n\dfrac{(-1)^{n-2}(n+1)!4^2}{(n-2)!4!} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2(n-1)\dfrac{(-1)^0(2n-3)!4^{n-1}}{0!(2n-2)!} & 2n\dfrac{(-1)^1(2n-2)!4^{n-1}}{1!(2n-2)!} \\ 0 & 0 & 0 & \cdots & 0 & 2n\dfrac{(-1)^0(2n-1)!4^n}{0!(2n)!} \end{pmatrix}$$

For example, if we set $n = 2$, we get:

$$H = \begin{pmatrix} 2 & -2 & 2 \\ 0 & 4 & -16 \\ 0 & 0 & 16 \end{pmatrix}$$

From the matrix relation Eq. 3.2.4, it follows that

$$u_n(x) = \sum_{i=0}^{n} a_i VL_i^*(x) = T(x)HA \tag{3.2.5}$$

Our objective is to use the shifted VL^*-collocation matrix technique to find the $(n+1)$ unknown coefficients $a_i, i = 0, 1, \ldots, n$.

3.2.1 Derivatives matrix

The matrix expressions of $u_n'(x)$ and $u_n''(x)$ evaluated at the collocation points are given by

$$u_n'(x) = T(x)MHA, \quad u_n''(x) = T(x)M^2HA. \tag{3.2.6}$$

proof. After making a single differentiation from 3.2.11, we get:

$$\frac{d}{dx}u_n(x) = [\frac{d}{dx}T(x)]HA. \tag{3.2.7}$$

The derivatives of $T(x)$ may be expressed as a product of themselves and the powers of a sparse matrix $M_n = (m_{i,j})_{i,j=0}^{n}$ according to a simple computation

$$\frac{d^k}{dx^k}T(x) = T(x)M_n^k, \quad k \geqslant 1, \quad m_{i,j} = \begin{cases} j, & \text{if } j = i+1 \\ 0, & otherwise \end{cases} \tag{3.2.8}$$

In order to represent the first derivative of the unknown solution, we so combine 3.2.7 and 3.2.8 as follows:

$$\frac{d}{dx}u_n(x) = T(x)MHA. \tag{3.2.9}$$

Analogously, the second-order derivative of $u_n(t)$ may be expressed as follows:

$$\frac{d^2}{dx^2}u_n(x) = T(x)M^2HA. \tag{3.2.10}$$

Thus, in relations 3.2.9 and 3.2.10 the Chebyshev nodes **??** are replaced. Lastly, two vectors, $u_n^{(1)}$ and $u_n^{(2)}$, are defined as uuu

$$u_n^{'} = \begin{pmatrix} u_n'(x_0) \\ u_n'(x_1) \\ \vdots \\ u_n'(x_n) \end{pmatrix}, \quad u_n'' = \begin{pmatrix} u_n''(x_0) \\ u_n''(x_1) \\ \vdots \\ u_n''(x_n) \end{pmatrix}.$$

■

Vieta-Lucas matrix collocation technique

In this section, we will apply the shifted VL^* operational matrix of derivatives together with collocation method to solve numerically the $LFIDE$. We achieve this by letting the

solution of $LFIDE$ as in equation 3.2.1 can be approximated by the first n terms shifted VL^* polynomials. In this case, we have

$$u(x) \approx u_n(x) = T(x)HA. \tag{3.2.11}$$

We now link our approximation method to a collection points $x_i = \dfrac{i}{n+1}, i = 0, 1, ..., n.$ The system of equations that results is:

$$u_n(x_i) = T(x_i)HA, \quad i = 0, 1, 2, \ldots, n.$$

The former equations can be represented as

$$u_n(x_i) = \begin{pmatrix} u_n(x_0) \\ u_n(x_1) \\ \vdots \\ u_n(x_n) \end{pmatrix}, \quad T(x_i) = \begin{pmatrix} T(x_0) \\ T(x_1) \\ \vdots \\ T(x_n) \end{pmatrix}. \tag{3.2.12}$$

By using Eq.**??** and 3.2.11, we have the matrix relation

$$u_n^{(m)}(x) = T(x)M^m HA, \quad m = 0, 1, 2, \ldots \tag{3.2.13}$$

Furthermore, Eq. 3.2.1's kernel function $k(x, t)$ is built in matrix form as follows

$$k(x, t) = T(x)\mathbf{K}T(t) \tag{3.2.14}$$

Where $\mathbf{K} = [k_{ij}], \quad i, j = 0, 1, \ldots, n$

$$k_{ij} = \frac{1}{i!j!} \cdot \frac{\partial^{i+j} k(0,0)}{\partial x^i \partial t^j}$$

$$\int_0^1 k(x,t)u(t)dt = T(x)\mathbf{K}QHA \tag{3.2.15}$$

Where

$$Q = [q_{ij}] = \int_0^1 T^t(x)T(x)dx,$$

$$q_{ij} = \frac{b^{i+j+1} - a^{i+j+1}}{i+j+1} = \frac{1}{i+j+1} \quad i, j = 0, 1, \ldots, n$$

The matrix equation system is obtained by taking the following steps:

$$u''(x_i) + P_1(x_i)u'(x_i) + P_2(x_i)u(x_i) = f(x_i) + \int_0^1 k(x_i,t)u(t_i)dt \tag{3.2.16}$$

By replacing the matrix relations Eqs. 3.2.13 and3.2.15 into Eq. 3.2.16, the basic matrix equation corresponding to the FIDEs is constructed:

$$T(x)M^2HA + P_1T(x)MHA + P_2T(x)HA = f(x) + \int_0^1 k(x,t)T(t)HAdt \tag{3.2.17}$$

or briefly,

$$\left(TM^2H + P_1TMH + P_2TH - QH\right) A = F \tag{3.2.18}$$

where

$$Q = \int_0^1 k(x,t)T(t)dt.$$

$$P_1 = \begin{pmatrix} p_1(x_0) & 0 & \cdots & 0 \\ 0 & p_1(x_1) & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & p_1(x_n) \end{pmatrix}, P_2 = \begin{pmatrix} p_2(x_0) & 0 & \cdots & 0 \\ 0 & p_2(x_1) & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & p_2(x_n) \end{pmatrix},$$

$$A = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}, T = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & ... & x_1^2 \\ . & . & . & . & . \\ 1 & x_n & x_n^2 & ... & x_n^n \end{pmatrix}$$

Additionally, the basic matrix equation 3.2.18 may be written as

$$\mathbf{GA} = \mathbf{F} \quad or \quad [\mathbf{G}; \mathbf{F}] \tag{3.2.19}$$

where

$$\mathbf{G} = [\varphi_{i,j}] = TM^2H + P_1TMH + P_2TH - QH. \quad i, j = 0, 1, ..., n$$

$$[\mathbf{G}; \mathbf{F}] = \begin{pmatrix} \varphi_{0,0} & \varphi_{0,1} & \cdots & \varphi_{0,n} & ; & f(x_0) \\ \varphi_{1,0} & \varphi_{i,1} & \cdots & \varphi_{i,n} & ; & f(x_1) \\ . & . & . & . & ; & \vdots \\ \varphi_{n,0} & \varphi_{n,1} & \cdots & \varphi_{n,n} & ; & f(x_n) \end{pmatrix}$$

We write the matrix corresponding to conditions 3.2.2, we obtain

$$\psi_i A = \alpha_i, \quad or \quad [\psi_i; \alpha_i] = [\psi_{i,0}, \psi_{i,1}, \psi_{i,2}, ..., \psi_{i,n}; \alpha_i], i = 0, 1$$

where

$$\psi_i = T(0)M^iH, \quad i = 0, 1$$

For two initial conditions, the augmented matrix becomes:

$$[\mathbf{G}; \mathbf{F}] = \begin{pmatrix} \varphi_{0,0} & \varphi_{0,1} & \cdots & \varphi_{0,n} & ; & f(x_0) \\ \varphi_{1,0} & \varphi_{i,1} & \cdots & \varphi_{i,n} & ; & f(x_1) \\ . & . & . & . & ; & \vdots \\ \varphi_{n-2,0} & \varphi_{n-2,1} & \cdots & \varphi_{n-2,n} & ; & f(x_{n-2}) \\ \psi_{0,0} & \psi_{0,1} & \cdots & \psi_{0,n} & ; & \alpha_0 \\ \psi_{1,0} & \psi_{1,1} & \cdots & \psi_{1,n} & ; & \alpha_1 \end{pmatrix} \tag{3.2.20}$$

If rank$([G; F]) = n + 1$, then equation 3.2.1 admits a unique solution, which can be effectively obtained using the shifted Vieta Lucas series expansion as presented in equation 3.2.11

3.3 Numerical examples

Example 01.Consider the linear Fredholm integro-differential equation :

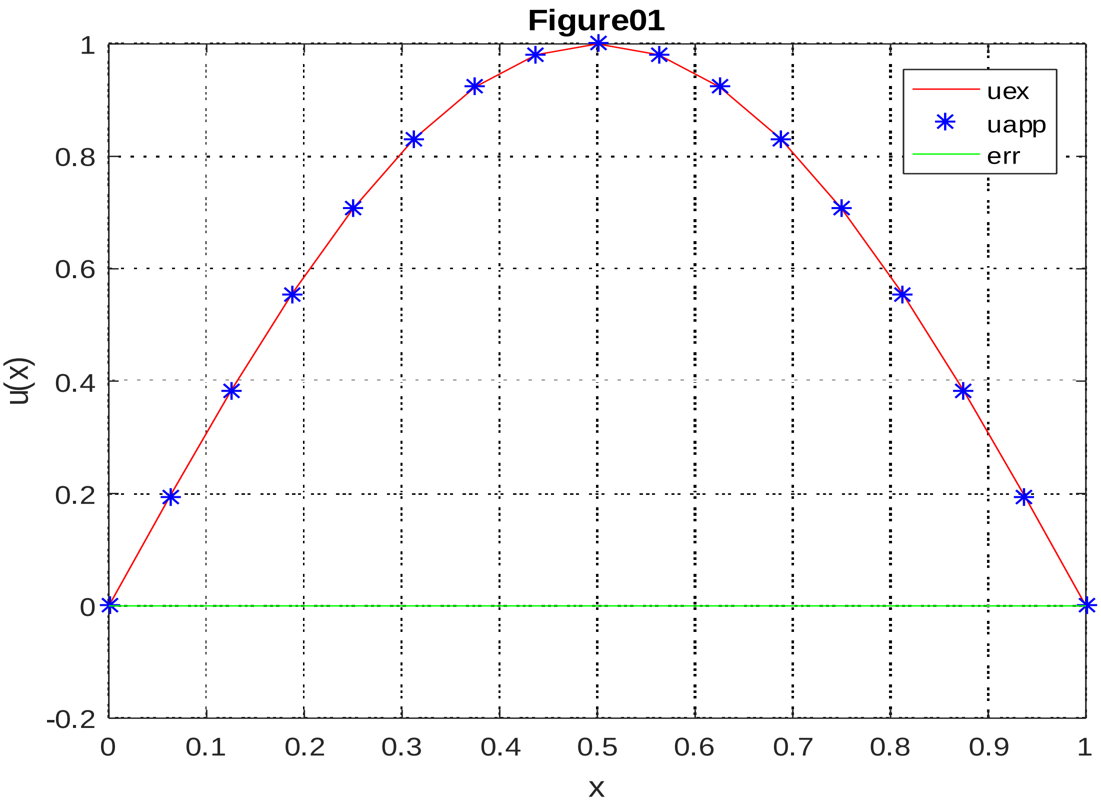
$$\begin{cases} u''(x) + xu'(x) + \pi^2 u(x) = \pi x \cos(\pi x) - \dfrac{(2x-1)}{\pi} + \int_0^1 (x+t)u(t)\,dt \\ u(0) = 0, \quad u'(0) = \pi \end{cases}$$

the exact solution given by:

$$u_{ex}(x) = \sin(\pi x)$$

Table 01. *We present the approximate solution u_{app} obtained by the Veita-Lucas collocation method, the error is calculated for* $N = 8$ and $N = 16$

Val of x	$u_{ex}(x)$	u_{app}	$Err.forN = 8$	$Err.forN = 16$
0	0	-1.4019e-16	1.4019e-16	9.3191e-15
1.2500e-01	3.8268e-01	3.8268e-01	7.2361e-07	1.4988e-15
2.5000e-01	7.0711e-01	7.0711e-01	1.7274e-06	3.1086e-15
3.7500e-01	9.2388e-01	9.2388e-01	2.4684e-06	1.1102e-15
5.0000e-01	1.0000e+00	1.0000e+00	2.9004e-06	3.4417e-15
6.2500e-01	9.2388e-01	9.2388e-01	3.0037e-06	1.8874e-15
7.5000e-01	7.0711e-01	7.0711e-01	2.5953e-06	3.8858e-15
8.7500e-01	3.8268e-01	3.8268e-01	5.1089e-06	2.3315e-15
1.0000e+00	1.2246e-16	-6.3729e-05	6.3729e-05	4.3596e-13



Example 02.Consider the integro-differential equation of Fredholm:

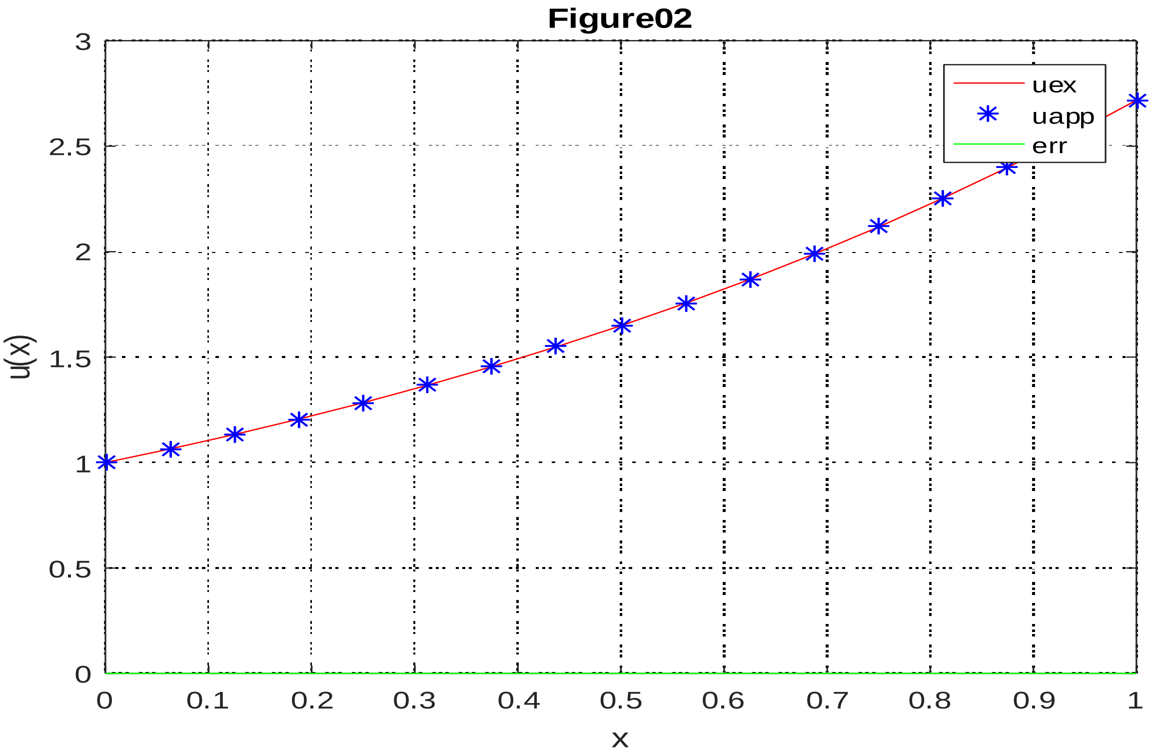
$$\begin{cases} u'(x) - u(x) = \dfrac{1-e^{x+1}}{x+1} + \int_0^1 e^{xt} u(t)\, dt \\ u(0) = 1, \end{cases}$$

the exact solution given by:

$$\varphi_{ex}(x) = e^x$$

Table 02. *We present the approximate solution u_{app} obtained by the Veita-Lucas collocation method, the error is calculated for* $N = 8$ and $N = 16$

Val of x	$u_{ex}(x)$	u_{app}	$Err.forN = 8$	$Err.forN = 16$
0	1.0000e+00	1.0000e+00	1.2212e-15	2.0317e-13
1.2500e-01	1.1331e+00	1.1331e+00	1.1645e-08	1.9851e-13
2.5000e-01	1.2840e+00	1.2840e+00	2.6103e-08	2.3870e-13
3.7500e-01	1.4550e+00	1.4550e+00	4.3710e-08	2.0006e-13
5.0000e-01	1.6487e+00	1.6487e+00	6.5129e-08	1.6942e-13
6.2500e-01	1.8682e+00	1.8682e+00	9.1463e-08	2.5424e-13
7.5000e-01	2.1170e+00	2.1170e+00	1.2604e-07	3.4195e-13
8.7500e-01	2.3989e+00	2.3989e+00	1.7948e-07	3.5083e-13
1.0000e+00	2.7183e+00	2.7183e+00	2.8741e-07	8.4910e-13



Example 03.Consider thelinear integro-differential equation of Fredholm:

$$\begin{cases} u'(x) - 2xu(x) = -2x^3 - 2x^2 + \dfrac{23}{6}x + 1 + \int_0^1 2xtu(t)\,dt \\ u(0) = -1, \end{cases}$$

the exact solution given by:

$$u_{ex}(x) = x^2 + x - 1$$

Table 03. *We present the approximate solution u_{app} obtained by the Veita-Lucas collocation method, the error is calculated for $N = 8$*

Val of x	$u_{ex}(x)$	u_{app}	$Err.forN = 8$
0	-1.0000e+00	-1.0000e+00	5.5511e-16
1.2500e-01	-8.5938e-01	-8.5938e-01	1.1102e-16
2.5000e-01	-6.8750e-01	-6.8750e-01	1.1102e-16
3.7500e-01	-4.8438e-01	-4.8438e-01	5.5511e-17
5.0000e-01	-2.5000e-01	-2.5000e-01	5.5511e-17
6.2500e-01	1.5625e-02	1.5625e-02	2.4286e-17
7.5000e-01	3.1250e-01	3.1250e-01	5.5511e-17
8.7500e-01	6.4063e-01	6.4063e-01	2.2204e-16
1.0000e+00	1.0000e+00	1.0000e+00	9.9920e-16

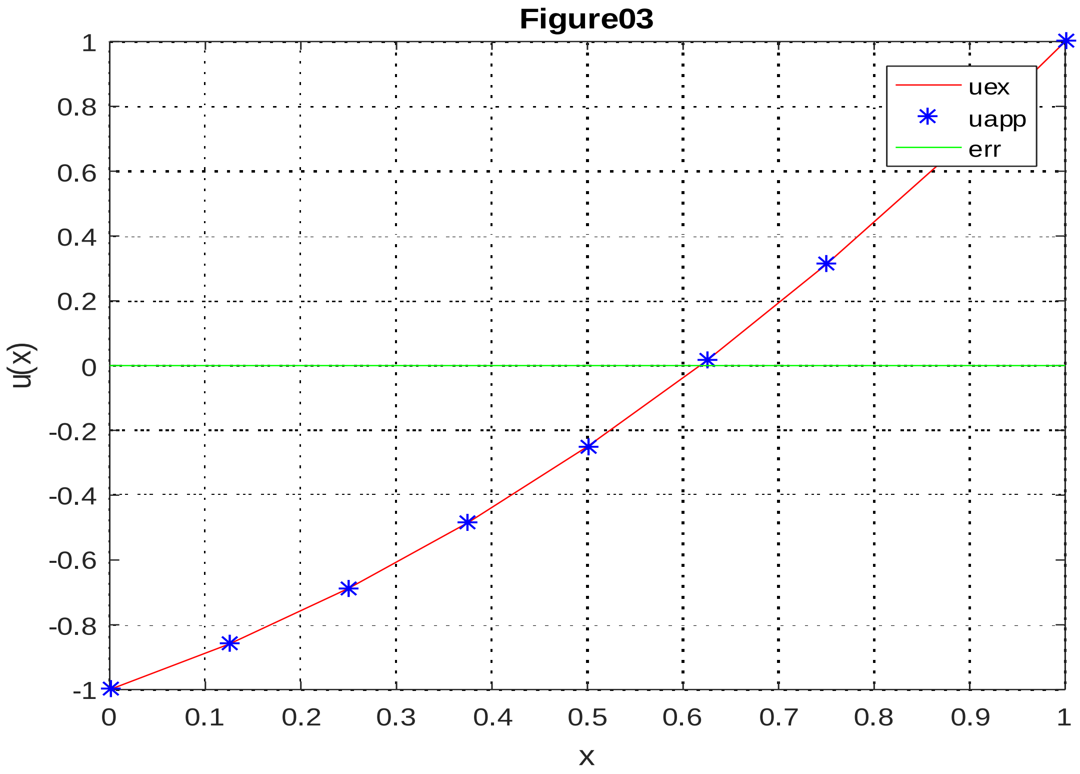


Figure03
uex
uapp
err
u(x)
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Conclusion

In this thesis, we have explored both the theoretical and numerical approaches for solving linear second-order Fredholm integro-differential equations (LFIDEs). We began with a foundational overview of functional spaces and operator theory, setting the stage for the classification and analysis of integral and integro-differential equations. Analytical methods, including the Adomian Decomposition Method and the Series Solution Method, were applied to selected examples, illustrating their applicability and limitations in solving LFIDEs.

The core contribution of this work lies in the development and implementation of a numerical method based on Vieta-Lucas polynomials. These orthogonal polynomials, with their advantageous properties and recurrence relations, were employed to construct an efficient collocation scheme. By transforming the problem domain and leveraging the orthogonality of the shifted Vieta-Lucas basis, we derived a systematic matrix formulation that allows for accurate numerical approximation of the solution.

The results demonstrate that the Vieta-Lucas matrix method is not only effective in handling second-order LFIDEs but also offers good convergence behavior and computational efficiency. This method provides a viable alternative to traditional numerical techniques and can be extended to a broader class of integro-differential problems.

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