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Two-dimensional surface waves

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Contents

Introduction

Wave interaction with infinite periodic arrays of spheres has been studied extensively in a variety of physical settings. It is important to draw a distinction between three-dimensional arrays which fill the whole of space and oneand two-dimensional arrays which allow for wave propagation towards or away from the spheres. It is these latter cases that we are concerned with here.

Our brief is organized as follows :
The first chapter will be devoted to the Some basics about partial differential equations,we cite for exemple some mathematical physics equations,Derivat Partialives,Types of Partial Differential Equations,Partial Differential Equations (PDEs),Dimension and order a PDE,Linearity and homogeneity,Construction of a partial differential equation,Partial differential equation of the second order,TYPES OF SECOND-ORDER EQUATIONS.
The second chapter of this thesis is dedicted to the Fluid mechanics,for exemple,of definition the fluid mechanics,the types of fluid mechanics,as well as the Fluid properties and the equation of fluid mechanics.
In the third chapter ,we study the Surface waves,Free-surface flows,Two-dimensional flows,Linear waves,Nonlinear Water Waves.

Chapter 1

Some basics about partial differential equations

1.1 Introduction

A partial differential equation (PDE) is a relation relating an Unknown function u of several variables to its partial derivatives. We also find different physical, engineering, biological phenomena and economic applications that modeled with EDPs. Indeed, in these domains, the phenomena are often modeled by mathematical systems involving PDEs.

Definition 1 *: A first-order partial differential equation of unknown* u *of* n *independent variables* $x_1 ..., x_n$ *is an equation of the form.*

$$F(x_1 ..., x_n; u_{x_1} ..., u_{x_n}) = 0 \tag{1.1}$$

This is the most general PDE in several independent variables of first order. The order of an equation is the highest derivative that appears. The most general second-order PDE in several independent variables is

$$F(x_1 ..., x_n; u_{x_1} ..., u_{x_n}; u_{x_1 x_1} ..., u_{x_1 x_n}, ..., u_{x_n x_1}, ..., u_{x_n x_n}) = 0 \tag{1.2}$$

A solution of a PDE is a function $u(x_1, ..., x_n)$ that satisfies the equation identically, at least in some region of the $x_1, ..., x_n$ variables. When solving an ordinary differential equation (ODE), one sometimes reverses the roles of the independent and the dependent variables for instance, for the separable ODE $\frac{du}{dx} = u3$. For PDEs, the distinction between the independent variables and the dependent variable (the unknown) is always maintains [1]

1.2 some mathematical physics equations

1.2.1 transport equation

$$\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0 & où \\ \qquad u(x,t) & \end{cases} \tag{1.3}$$

1.2.2 Burgers equation

$$\begin{cases} \frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = 0 & où \\ \qquad u(x,t) & \end{cases} \tag{1.4}$$

1.2.3 wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0 & où \\ \qquad u(x,y,z,t) & \end{cases} \tag{1.5}$$

1.2.4 Heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0 & où \\ \qquad u(x,y,z,t) & \end{cases} \tag{1.6}$$

1.2.5 Laplace's equation or the p otential

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 & où \\ \qquad u(x,y,z,t) & \end{cases} \tag{1.7}$$

1.2.6 Euler _Bernoulli equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + c^4\frac{\partial^4 u}{\partial x^4} = 0 & où \\ \qquad u(x,t) & \end{cases} \tag{1.8}$$

1.3 Partial Derivatives

We will gradually introduce some notions on the functions of several real Variables .we limit ourselves for the statements to the case of functions of Two variables ,but the notion which follow can be easily generalized to functions of n real variables , where n is any integer (greater than 2). For the moment , we only examine the properties of the partial maps associated with such a function f.[2]

1.4 Types of Partial Differential Equations

1.4.1 Ordinary différentielles Equations (ODE)

Ordinary differential equations are found in all fields of physique (electricity, mechanics, thermal ,etc.). It is a relation between a function unknown and its Derivatives .The unknown function depends on only one variable,[3]
An ODE is a relation of the type

$$F(y; u(y); u'(y); u''(y), ..., u^n(y)) = 0 \tag{1.9}$$

Between the variable $y \epsilon R$(sometimes $y \epsilon I \subset R$) and the derivatives of the unknown function at the point ysuch that

$$\begin{cases} F : (y, z) \rightarrow F(y, z) \\ \quad R^{n+2} \rightarrow R \end{cases} \tag{1.10}$$

With $z = (z_0; z_1, ..., z_n) \epsilon R^{n+1}$, n is the number of variables. ,[2]

1.4.2 Establishment of courant electric current in a coil:

Consider coil inductance$L = 12$ mHand resistance $R = 0,6\Omega$. At the instant, the Current in the coil is zero and a step is applied voltage $U = 3V$.

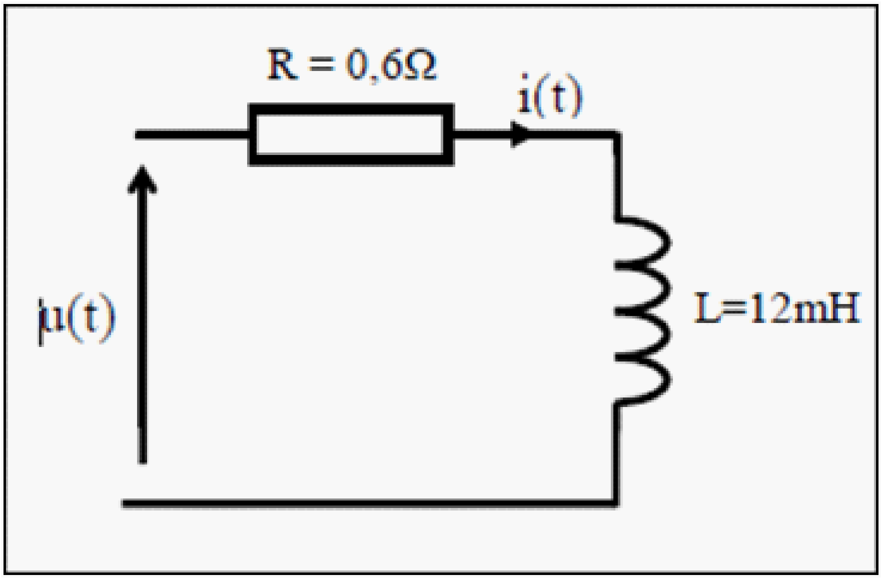
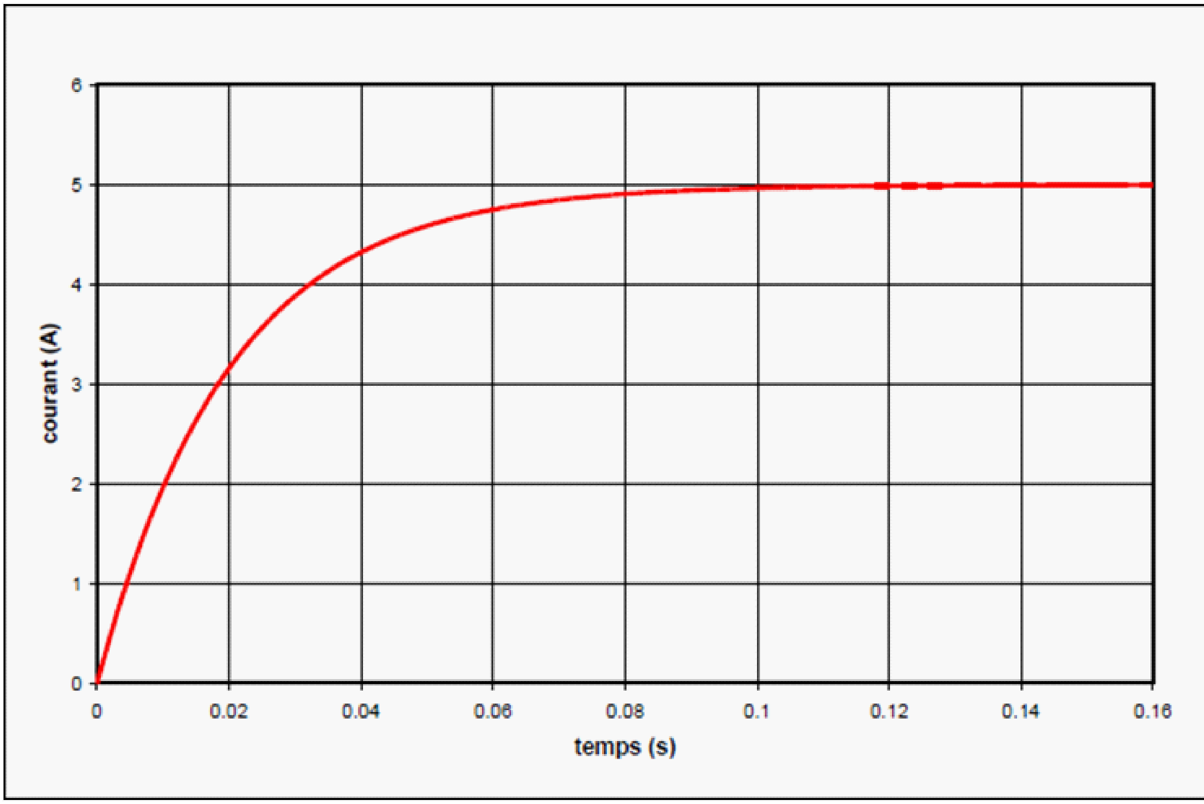


Figure 1 : Establishment of electric current in a coil
The evolution of the electric current is given by a differential equation of the 1^{st}order (see License electricity course) :

$$\begin{cases} L\frac{di(t)}{dt} + Ri(t) = u(t) & avec \\ \qquad\qquad i(t = 0) = 0 \end{cases} \tag{1.11}$$



Solving this differential equation gives :

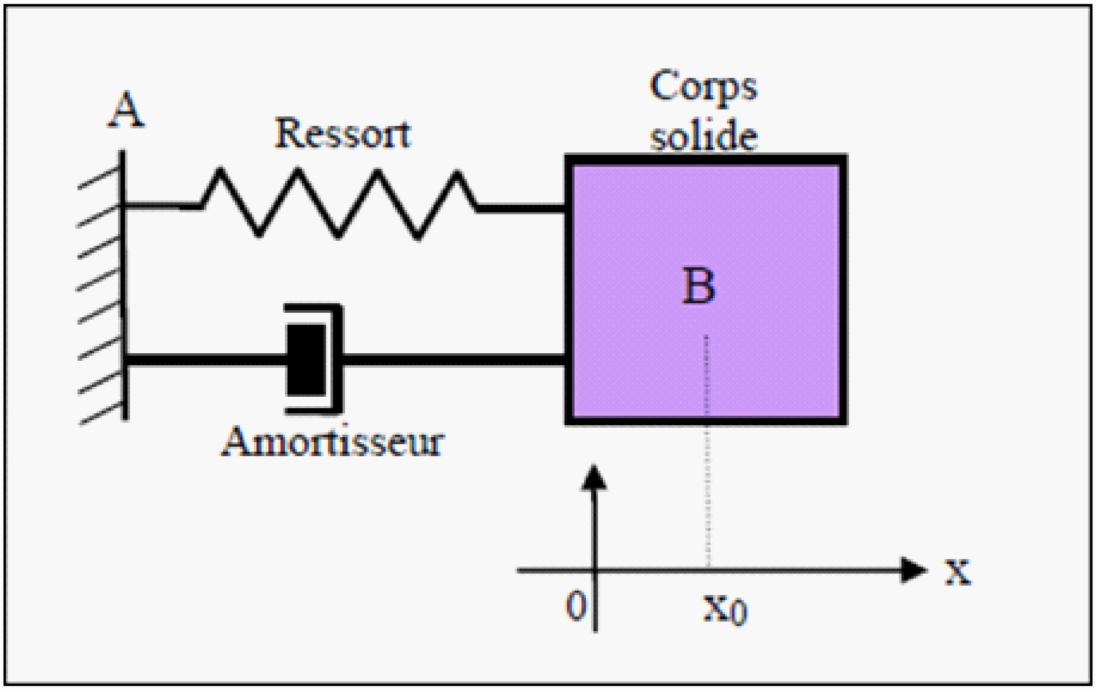
$$\begin{cases} i(t) = \frac{U}{R}\left(1 - e^{-\frac{t}{\tau}}\right) & avec \\ \tau = \frac{L}{R} & \end{cases} \tag{1.12}$$

La figure 2 represents the evolution temporal of the current. Figure 2 :Time evolution of the current in a coil

1.4.3 Oscillator in point mechanics :

We consider in figure 3 a solid body of mass $m = 100g$ and center of inertia B. the body is connected to point A by a spring with turns of stiffness $k = 125N.m^{-1}$ The spring is paralleled on a friction damper viscous $f = 1kg.s^{-1}$(force proportionalto the speed and which opposes the movement). We assume that the body can move along x without friction on the contact surface. Figure 3 : Oscillator in point mechanics

We move the body away from its equilibrium position $(x = 0)$ to the right



($x = x_0 = 10cm$) and let go without initial speed. This is to determine the evolution of the body position as a function of time .
The projection of the fundamental relation of the dynamics on the axis of x us leads to the following 2^{nd} order differential equation (see course of license mechanics):

$$\begin{cases} \frac{d^2x}{dt^2} + \frac{f}{m}\frac{dx}{dt} + \frac{k}{m}x = x_0, & with \\ x(t=0) = x_0, & and \\ \qquad\qquad \frac{dx}{dt}\Big|_{t=0} & \end{cases} \tag{1.13}$$

The resolution of the second order differential equation (pseudo-periodic) gives :

$$\begin{cases} x(t) = x_0 e^{-\lambda t}(\cos\Omega t + \frac{\lambda}{\Omega}\sin\Omega t), \qquad with \\ \qquad\qquad and \qquad\qquad 2\lambda = \frac{f}{m}, \\ \qquad\qquad \Omega = \sqrt{(\frac{f}{m}) - \lambda^2} \end{cases} \tag{1.14}$$

[3]

1.5 Partial Differential Equations (PDEs)

The generalization of the previous description with the involvement of multi-class functions able makes it possible to build the concept of PDE. Begin by

defining the definition of a 1^{st} order PDE.
A partial differential equation of the1^{st} order of unknown u of n independent variables dents $y_1, \ldots, y_n$ is an equation of the form

$$f\left(y_1, ..., y_n, u, \frac{\partial u}{\partial y_1}, ..., \frac{\partial u}{\partial y_n}\right) = 0 \tag{1.15}$$

Where $(y_1, \ldots, y_n)\epsilon$ open Ω of R^n

For a function of two variables, the definition is given by
The general form of a PDE of order 2 is

$$f\left(y, z, u, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 u}{\partial z \partial y}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}\right) = 0 \tag{1.16}$$

For $(y, z)\epsilon$ open Ω of R^2 ,[2]

In general, the modeling of physical phenomena is based on the resolution of partial differential equations.
These equations correspond to the mathematical translation to the laws of physics :
- fluid mechanics :Navies- Stocks equations
- electromagnetism :Maxwell
- thermal :heat equations
- quantum mechanics: Schrödinger equations

In engineering sciences, the modeling of a physical device (engine electricity , for example) makes it possible to predict its behavior and to study the influence performance parameters. The validity of the mode must be compared, when possible, to experimental measurements. In a large majority of cases, the partial differential equations are non-linear (because of the properties of the materials) and the computer must be used to solve them (numerical calculation software).
When possible (linear equations), it is interesting to solve the model equations analytically ("by hand").In this case ,the solution obtained makes it possible to see the influence of the various parameters. It can be used for a first optimization study of the studied device .
A partial differential equations relates an unknown function to its derived. the unknown function depends on several variables (variables of space and time).Examples of linear partial differential equations:

$$1\cdot \frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} \qquad \text{1D heat (diffusion) equation} \tag{1.17}$$

$$2\cdot \frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2} \qquad \text{wave equation (propagation) 1D} \tag{1.18}$$

$$3\cdot \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \qquad \text{Laplace's equation in 2D (Cartesian coordinates)} \tag{1.19}$$

$$4\cdot\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad \text{Poisson's equation in 2D (cylindrical)} \tag{1.20}$$

$$\frac{\partial^2 u}{\partial t^2}=c\left(\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad \text{3D wave equation} \tag{1.21}$$

[3]

1.6 Dimension and order a PDE

·The dimension of a partial differential equation is the number of independent variables dents on which the unknown function depends.
· The order of a partial differential equation is the highest degree of differentiation present In the equation.

1.7 Linearity and homogeneity

The notion of linearity for PDEs involves different operators. Of which is a factor variation is a content factor of the partial derivatives of different functions. A PDE of an unknown u is said to be linear if it can be put in the form

$$Lu = f \tag{1.22}$$

Or
L is a differential linear operator, hi f is a function of independent variables no de need over a domain of R^n. If $f \equiv 0$ the equation is said to be linear homogeneous. Otherwise it is non-homogenous. ,[2]

1.8 Construction of a partial differential equation

1.8.1 First-order linear equation

The simplest possible PDE is $\frac{\partial u}{\partial x}=0$ [where $u=u(x,y)$]. Its general Solution is$u=f(y)$, where f is any function of one variable. For instance,$u=y^2-y$ and $u=e^y$ cosy are two solutions. Because the solutions don't depend on x, they are constant on the lines $y=constant$ in the xy plane.

The constant coefficient equation

Let us solve

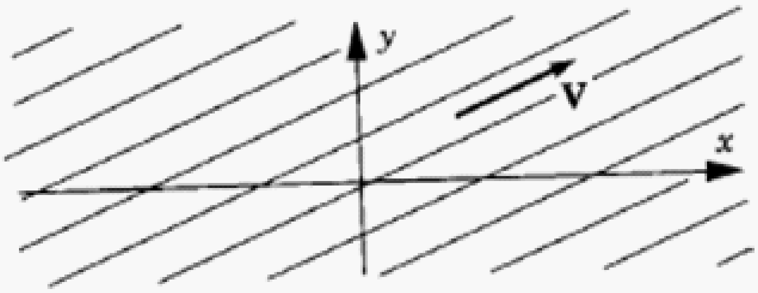


Figure 1

$$au_x + bu_y = 0 \tag{1.23}$$

Where a and b are constants not both zero.

The quantity $au_x + bu_y$ is the directional derivative ofu in the direction of the vector $V(a, b) = ai + bj$.It must always be zero.
This means that $u(x, y)$ must be constant in the direction of V. The vector $(b, -a)$ isorthogonal to V. The lines parallel to V(see Figure 1) have the equations $bx - ay = constant$.(They are called the characteristic lines.)The solution is constant on each such line. Therefore,$u(x, y)$ depends on $bx - ay$ only. Thus the solution is

$$u(x, y) = f(bx - ay) \tag{1.24}$$

Where f is any function of one variable. Let's explain this conclusion more explicitly. On the line $bx - ay = c$, the solution u has a constant value. Call

this value f (c). Then $u(x, y) = f(c) = f(bx - ay)$. Since c is arbitrary, we have formula (2) for all values of x and y. In xyu space the solution defines surface that is made up of parallel horizontal straight lines like a sheet of corrugated iron.Coordinate Method Change variables (or "make a change of coordinates";

Figure 2) to

$$x' = ax + byy' = bx - ay \tag{3}$$

Replace all x and y derivatives by x' and y' derivatives. By the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \tag{1.25}$$

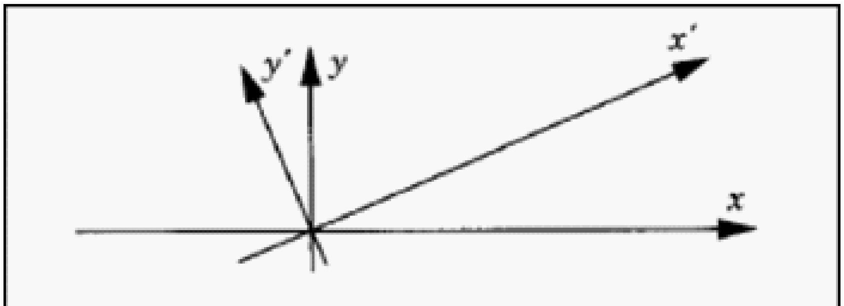


Figure 2

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} = au_{x'} - bu_{y'} \tag{1.26}$$

Hence $au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2 + b^2)u_{x'}$. So,since $a^2 + b^2 \neq 0$, the equation takes the form $u_{x'} = 0$ in the new (primed)variables. Thus the solution is $u = f(y') = f(bx - ay)$, with f an arbitrary function of one variable. This is exactly the same answer as before! ,[3]

1.8.2 Nonlinear first order PDEs

In this section, we are interested in solving PDEs having the following form

$$F\left(y, z, u, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0 \tag{1.27}$$

In equation (4) is often written using standard notation $p = \frac{\partial u}{\partial y}$ and $q = \frac{\partial u}{\partial z}$ .For simplicity, partial derivatives will be denoted by $u_y, u_z, \ldots$..The most appropriate example Is that of the icon equation, which is used in geometric.

1.9 Partial differential equation of the second order

Many physical problems related to fluid mechanics, heat transfer, dynamics and electromagnetism are described by second-order PDFs. As a result ,the study of their methods of resolution is very important to solve the problems of the world real.

1.9.1 Classification of equations

We call semi-linear partial differential equation of the second order, of unknown u
On an open set Ω of R^2,an equation of the for

$$a(y,z)\frac{\partial^2 u}{\partial y^2}+b(y,z)\frac{\partial^2 u}{\partial y\partial z}+c(y,z)\frac{\partial^2 u}{\partial z^2}=F\left(y,z,u,\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right) \tag{1.28}$$

wherea,b,care given function, and F a function, de need in an open set of R^5.
The classification of second-order PDEs comes from the classification of the quadratic equation conic sections in analytical geometry. The equation

$$ay^2+byz+cz^2+dy+ez+f=0 \tag{1.29}$$

represents the hyperbola, or the ellipse according to the sign of b^2-4ac (positive, nulls or negative).
Then, The classification of equation (ε) depends on the coefficients $a(y,z), b(y,z)$and $c(y,z)$in one given point (y,z).Consequently,we give the following definition.
Let $\Delta(y,z)=b^2(y,z)-4a(y,z)c(y,z)$ we have the following cases

$$1\cdot \text{If } \Delta(y,z)>0 \qquad \text{the equation is said to be hyperbolic.} \tag{1.30}$$

$$2\cdot \text{If } \Delta(y,z)=0 \qquad \text{the equation iscalled parabolic.} \tag{1.31}$$

$$3\cdot \text{If } \Delta(y,z)<0 \qquad \text{the equation is said elliptical.} \tag{1.32}$$

[2]

1.10 TYPES OF SECOND-ORDER EQUATIONS

In this section we show how the Laplace, wave, and diffusion equations are in some sense typical among all second-order PDEs. However, these three equations are quite different from each other. It is natural that the Laplace equation $u_{xx}+u_{yy}=0$ and the wave equation$u_{xx}-u_{yy}=0$ should have very different properties. After all, the algebraic equation $x^2+Y^2=1$ represents a circle, whereas the equation $x^2+Y^2=1$ represents a hyperbola. The parabola is somehow in between. In general, let's consider the PDE

$$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+a_1u_x+a_2u_y+a_0u=0 \tag{1.33}$$

This is a linear equation of order two in two variables with six real constant coefficients. (The factor 2 is introduced for convenience.)[1]

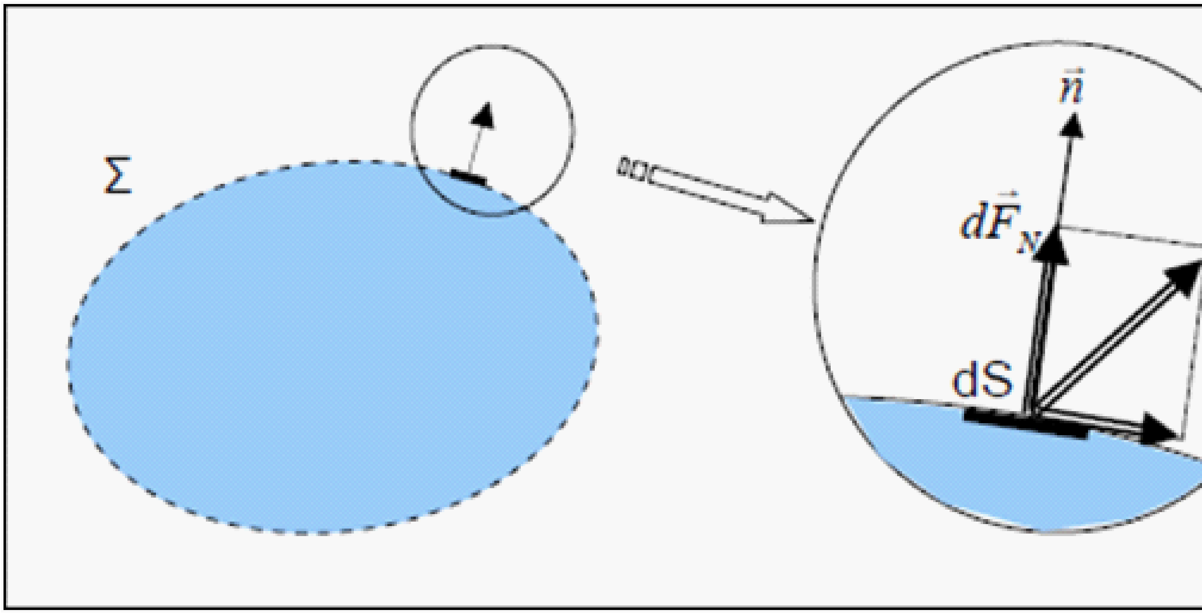
Chapter 2

Fluid mechanics

2.1 INTRODUCTION

Fluid mechanics is the science of the laws of fluid flow. She is the basis for the dimensioning of fluid lines and control mechanisms.fluid transfer.It is a branch of phisics that studies the flow of fluid, i.e. liqiuds and gases when they are subjected to forces or constraints. It inclludes two large sub-branches:
-the statics of fluid, or hydrostatics which studies fluids at rest. This is historically the beginning of fluid mechanics, with the thrust of Archimedes and the study of pressure.
-fluid dynamics, which studies fluids in motion. Like others branches of fluid mechanics.
There are also other branches related to fluid mechanics:
hydraulics, hydrodynamics, aerodynamics, etc. A new approach has seen the day since a few decades: computational fluid mechanics (CDF or computational fluid Dynamics), which simulates the flow of fluids by solving the equations that govern them using very powerful
computers:the supercomputers.
Fluid mechanics has many applications in various fields such as naval engineering, aeronautics, but also meteorology,climatology or oceanography.

Definition 2 *A fluid can be considered as a substance made up of a large number of meterial particles, very small and free to move one by one reation to others . It is therefore a continuous material medium, deformable, without rigidity.and which can flow. The cohesive forces between elementary particles are very weak so that fluid is a formless body of its own which takes on the shape of the container that contains it, for exemple:molten metals are fluids which allwo by molding to obtain raw parts of shapes complex.*
We insist on the fact that a fluid is supposed to be a continuous medium:even if we chooses a very small volume element, it will always be much large than the size of the molecules that constitute it. For exemple, a droplet of fog, no matter how small on our scale, is still huge on the scale molecular. It will al-



ways be considered as a continuous medium. From fluids, a distinction is often made between liquids and gases.Fluid can also be classified into two families relatively by their viscosity.Viscosity is one of their physico-chemical characteristics which will be defined later in the course and which defines the internal friction of fluids.The fluids can be classified into two large families:The family of fluids "newtonian" (like water,air and most gases) and that of "non-newtonian" fluids. "Newtonians" fluids have a viscosity constant or which can var only according to the temperature. The second family is made up of "non-Newtonian" fluid which have the particularity of having their viscosity which varies according to the speed and the stresses they undergo when these flow. This course is limited to fluids only Newtonian which will be classified as follows.

2.2 types of fluid mechanics

2.2.1 perfect fluid

Consider a fluid system, i.e. a volume delimited by a closed surface Σ fictitious or not.

Consider $d\overrightarrow{F}$ the interaction force at the elementary dS de Normale $\overrightarrow{n}$ between the fluid and the external medium .
We can always decompose $d\overrightarrow{F}$ into two components:
- a $d\overrightarrow{F_T}$ compose tangential to dS.

-a component $d\overrightarrow{F_N}$ normal to dS

In fluid mechanics, a fluid is said to be perfect if it is possible to describe its movement without taking into account the effects of friction. That is to say when the component $d\overrightarrow{F_T}$ is zero. That is, the force $d\overrightarrow{F}$ is normal to the element of surface dS.

2.2.2 Real fluid

Unlike a perfect fluid, which is only a model to simplify calculations, peactically non-existent in nature, in a real fluid the tangential forces of internal friction which oppose the relative sliding of the fluid layers are taken into account. This phenomenon of viscous friction apppears during the fluid mouvement.It is only at rest, that we will admit that real fluid behaves as a perfect fluid, and it is assumed that the contact forces element on which they are exerted. The statics of real fluids confuses with the statics of perfect fluids .

2.2.3 Incompressible fluid

A fluid is said to be incompressible when the volume occupied by a given mass does not vary with external pressure. Liquids can be considered as incompressible fluids (water, oil, etc.)

2.2.4 compressible fluid

A fluid is said to be compressible when the volume occupied by a given mass varies according to the external pressure . Gases are compressible fluids. For exemple, state, air, hydrogen, methane in gaseous state , are considered as compressible fluids .[1]

2.3 Fluid properties

Definition 3 *A fluid is a body which is easily deformable. The liquids and gases are fluids, as well as more complex bodies such as polymers or edible fluids.They deform and flow easily. A fluid encompasses mainly two physical states: the gaseous state and the liquid state .*

2.3.1 Unit system

The units of measurement used in this documement are those of the international system (SI).
The main units of this system are listed in the following table:
Table : Main units in the international system (SI)

Length	Mass	Time	pressure	Force	Energy	Power
Meter	Kilogram	Second	Pascal	Newton	Joule	Watt
(m)	(Kg)	(s)	(Pa)	(N)	(J)	(W)
L	M	T	$ML^{-1}T^{-2}$	MLT^{-2}	ML^2T^{-2}	ML^2T^{-3}

2.3.2 Properties of fluids

All fluids have characteristics that describe their conditions physics in a given stste .Among these characteristics, which are called properties of fluids, we have:

Compressibility

Compressibility is the character of fluid volume variation which a variation of pressure (dp). the fluid volume undergoes a decrease in volume (dv).The increase in pressure leads to a decrease in volume.
The compressibility cofficient is :

$$\beta = -\frac{dv/v}{dp} = -\frac{dv}{dpv}(pa\text{-}^1), (m^2/N) \tag{2.1}$$

$\beta =$ compressibility coefficient (m^2/N)
$v =$ flluid volume (m^3)
$dv =$ change in volume (m^3)
$dp =$ pressurevariation (N/m^2)

volume characterized by their density

$$\rho = \frac{masse}{volume} = \frac{m}{v} \tag{2.2}$$

$\beta =$ compressibility coefficient (m^2/N)
$v =$ flluid volume (m^3)
$dv =$ change in volume (m^3)
$M =$ fluid mass (kg)
$\rho =$ volumice mass (kg/m^3)

Fluids	Mercury	Se a water	Pure water	oil	Petrol	butan	aire
$\rho = (kg/m^3)$	13600	1000	1000	900	700	2	2.193

Density density:it measures the ratio of the density of the fluid to a body of reference. It is a unitless quantity defined by:

$$d = \frac{\rho}{\rho_{réf}} \tag{2.3}$$

The reference body depends
Water: for solids and liquids

Air: for gases
Exemples :

$$d_{eau} = \frac{1000}{1000} = 1 \tag{2.4}$$

$$d_{essence} = \frac{700}{1000} = 0.7 \tag{2.5}$$

Liquids are characterized by a relatively high density;

$$\rho_{liquide} >> \rho_{gaz} \tag{2.6}$$

For gases, density depends on temperature and pressure.

2.3.3 Volume weight

(specific weight)

$$\varpi = N/m^3 \tag{2.7}$$

It represents the force of attraction exerted by the eath on the unit of volume, i.e. the weight of the unit of volume.

$$\left\{ \begin{array}{c} \varpi = \frac{G}{V} = \frac{Mg}{V} = \frac{\rho V g}{V} \\ \varpi = \rho g \\ N/m^3 \end{array} \right. \tag{2.8}$$

Mass volume

It is the volume occupied by the unit mass of a substance, it is the inverse of the mass voluminous

$$\left\{ \begin{array}{c} V = \frac{V}{M} = \frac{V}{\rho V} = \frac{1}{\rho} \\ (m^3/kg) \end{array} \right. \tag{2.9}$$

2.3.4 Viscosity

The viscosity of a fluid is the property of resisting tangential forces which tend to cause the layers of fluid to move relative to each other.when the fluid moves in parallel layers ; the proportionality factor is the coefficient of viscosity dynamic, (μ) and we then write:

$$\tau = \mu \frac{du}{dy} \tag{2.10}$$

The kinematic viscosity,v,is defined as the ratio between the viscosity dynamics and density.

$$v = \frac{\mu}{\rho} \tag{2.11}$$

In the SI system,the unit of dynamic viscosity is $(Pa.s)$ or (kg/ms) or Pl

$Pa.s$: Pascal second

Pl: Poiseuille wich $1Pa.s = 1$, $Pl = 1kg/ms$

In the CGS system the unit is the Poise (Po) with $1Po = 10^{-1}Pl$

In the SI systeme, the unit of kinematic viscosity, v, is (m^2/s) ; in the CGS system the unit is the stokes where $1 stokes = 1\ cm^2/s = 10^{-4}m^2/s$

2.4 The equations of fluid mechanics

We start with a brief introduction to the equations of fluid mechanics. For further details see for example Batchelor [8] or Acheson [1].
All the fluids considered in this book are assumed to be inviscid and tohave constant density ρ (i.e. to be incompressible).Conservatio of momentum yields the Euler equations

$$\frac{Du}{Dt} = -\frac{1}{\rho}\nabla p + X \tag{2.12}$$

where u is the vector velocity, p is the pressure and X is the body force. Here

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u.\nabla \tag{2.13}$$

is the material derivative. We assume that the body force X derives from a potential Ω, i.e. that

$$X = -\nabla\Omega \tag{2.14}$$

In most applications considered in this book, the flow is assumed to be irrotational. Therefore

$$\nabla \times u = 0 \tag{2.15}$$

Relation (2.15) implies that we can introduce a potential function φ such that

$$u = \nabla\phi \tag{2.16}$$

Conservation of mass gives

$$\nabla.u = 0 \tag{2.17}$$

Then (2.16) and (2.17) imply that φ satisfies Laplace's equation

$$\nabla^2\phi = 0 \tag{2.18}$$

Flows that satisfy (2.15)–(2.18) are referred to as potential flows. Using the identity

$$u.\nabla u = \frac{1}{2}\nabla(u.u) + (\nabla \times u) \times u \tag{2.19}$$

(2.15) and (2.13) yield

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{1}{2}\nabla(u.u) \tag{2.20}$$

Substituting (.2.20) into (2.12) and using (2.14) and (2.16) we obtain

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{u.u}{2} + \frac{P}{\rho} + \Omega\right) = 0 \tag{2.21}$$

After integration, (2.21) gives the well-known Bernoulli equation

$$\frac{\partial \phi}{\partial t} + \frac{u.u}{2} + \frac{P}{\rho} + \Omega = F(t) \tag{2.22}$$

Here $F(t)$ is an arbitrary function of t. It can be absorbed in the definition of φ, and then (2.22) can be rewritten as

$$\frac{\partial \phi}{\partial t} + \frac{u.u}{2} + \frac{P}{\rho} + \Omega = B \tag{2.23}$$

where B is a constant. For steady flows (2.23) reduces to

$$\frac{u.u}{2} + \frac{P}{\rho} + \Omega = B \tag{2.24}$$

Chapter 3

Surface waves

3.1 Introduction

Only longitudinal and transverse waves can propagate in a homogeneous, isotropic and unlimited medium. If the medium is bounded, another type of waves, surface waves, can be guided along the surface of the medium. These waves usually form the principal phase of seismograms. There are two types of surface elastic waves.

3.2 Free-surface flows

The concept of a free surface by contrasting both flows, the past flow a rigid sphere (see Figure 1) and the flow passing through the bubble (see Figure2) . Both flows are assumed to be constant and to approach a uniform stream at a constant velocity U as $x^2 + y^2 + z^2 \to \infty$; the effects of gravity are neglected. They can be interpreted as flows due to a rigid sphere or as bubble rising at a constant velocity U, when viewed in a frame of reference moving with the sphere or the bubble. The pressure p_b in the bubble is constant. We know by S the surface of a sphere or bubble and by n the outdoor unit normal. The flow through a sphere can be formulated as :

$$\begin{cases} \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \\ \quad\quad outside\ S \end{cases} \tag{3.1}$$

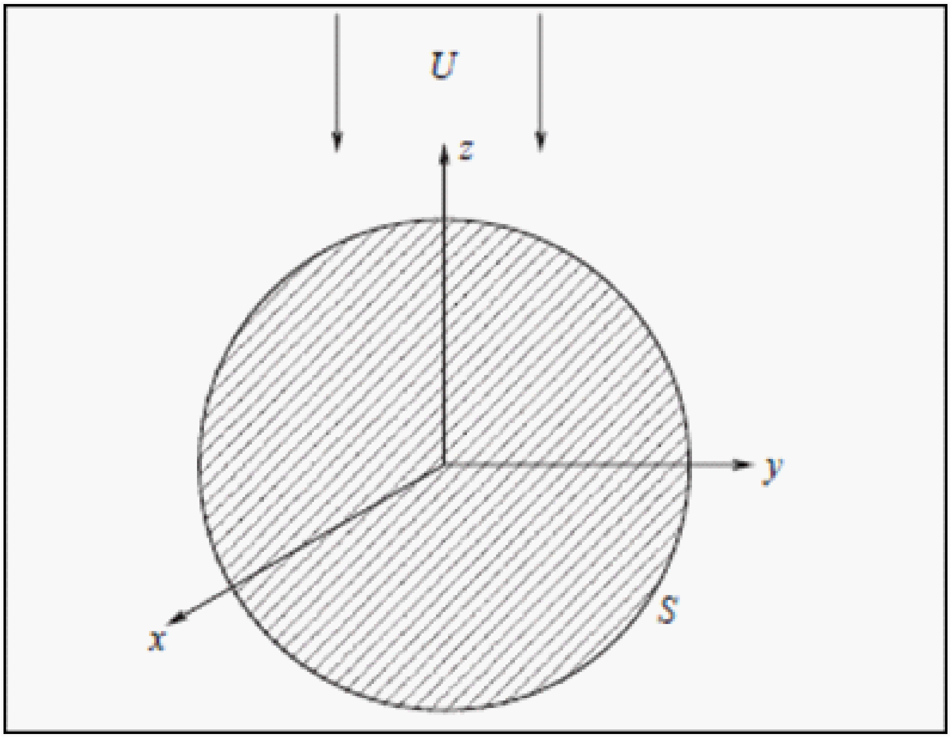
Free-surface flOW

Figure 1: flow through the rigid sphere.

The surface $S$of the sphere is described by $x^2 + y^2 + z^2 = R^2$, where R is the radius of the sphere.

Figure 2: The flow past a bubble.

The surface S of the bubble was previously unknown and it should be looked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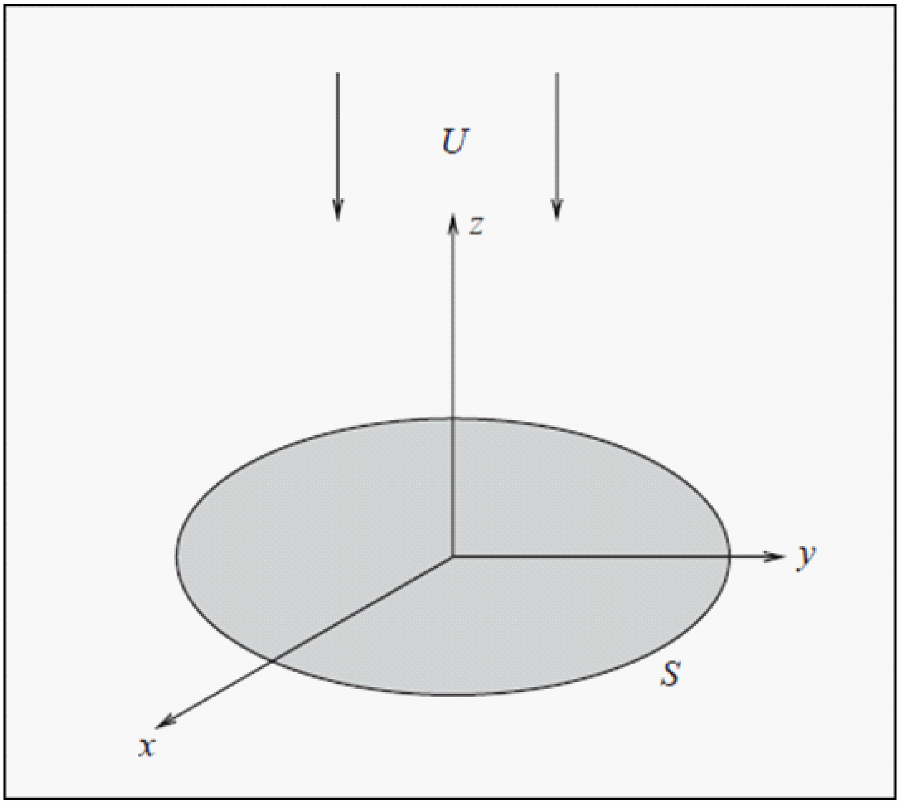
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for as part of the solution$_\infty$

$$\begin{cases} \frac{\partial\phi}{\partial n} = 0 \\ on \qquad s \end{cases} \tag{3.2}$$

$$\begin{cases} (\phi_x, \phi_y, \phi_z) \rightarrow (0, 0, -u) \qquad\qquad as \\ \qquad\qquad x^2 + y^2 + z^2 \rightarrow \infty \end{cases} \tag{3.3}$$

Equation (3.1) is Laplace's equation $\left(\nabla^2\varphi = 0\right)$ expressed in Cartesian coordinates.The boundary condition (3.2) is known as the kinematic boundary condition. It states that the normal component of the velocity vanishes on S. Equations (3.1) –(3.3) form a linear boundary value problem whose solution is

$$\phi = -U\left[z + \frac{R^3 z}{(2x^2 + y^2 + z^2)^{3/2}}\right] \tag{3.4}$$

The radius of the sphere is R.

Oserving, we conclude solving equation (3.4) without using the Bernoulli theorem and equation $(\frac{u.u}{2} + \frac{p}{\rho} + \Omega = B)$can be written for the current problem as

$$\frac{1}{2}({\phi^2}_x + {\phi^2}_y + {\phi^2}_z) + \frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p_\infty}{\rho} \tag{3.5}$$

Here P_∞ denotes the pressure as $x^2 + y^2 + z^2 \rightarrow \infty$. Equation (3.5) holds everywhere outside the sphere. In deriving (3.5) we have set

$\Omega = 0$ in $(\frac{u.u}{2} + \frac{p}{\rho} + \Omega = B)$ and evaluated B by taking the limit $x^2 + y^2 + z^2 \rightarrow \infty$ in $(\frac{u.u}{2} + \frac{p}{\rho} + \Omega = B)$.Then, using (3.3) gives

$B = U^2/2 + p_\infty/\rho$.

Equation ((3.5)) is nonlinear but it is only used if we want to calculate the pressure p inside the fluid. In other words the main problem is to find φ by solving the linear set of relations (3.1)–(3.3). We may then substitute the values (3.4) of φ into the nonlinear equation (3.5) if we wish to compute the pressure.

We now show that we need to use the nonlinear boundary condition (3.5) to solve for the potential φ for a flow past the bubble of Figure 2. This implies that, because of it nonlinearity, the flow past a bubble is a much harder problem to solve than the flow past a sphere. The potential function φ still satisfies (3.1)–(3.3). However, the main difference is that the shape of the surface S of the bubble is not known and has to be found as part of the solution. In other words the equation of the surface S is no longer given as it was for the flow past a sphere. Therefore we need an extra equation to find S. This equation uses (3.5) and can be derived as follows. First we relate the pressure p on the fluid side of S to the pressure p_b inside the bubble by using the concept of surface tension. If we draw a line on a fluid surface (such as S), the fluid on the right of the line is found to exert a tension T, per unit length of the line, on the fluid to the left. We call T the surface tension coefficient. It depends on the fluid and also on the temperature. It can be shown that

$$p - p_b = TK = T(\frac{1}{R_1} + \frac{1}{R_2}) \tag{3.6}$$

Here R_1 and R_2 are the principal radii of curvature of the fluid surface: they are counted positive when the centers of curvature lie inside the fluid. The quantity

$$K = (\frac{1}{R_1} + \frac{1}{R_2}) \tag{3.7}$$

is referred to as the mean curvature of the fluid surface. In most applications presented in this book the surface tension T is assumed to be constant.

We now apply the Bernoulli equation (3.5) to the fluid side of the surface S and use (3.6). This gives

$$\begin{cases} \frac{1}{2}({\phi^2}_x + {\phi^2}_y + {\phi^2}_z) + \frac{T}{\rho}K = \frac{1}{2}U^2 + \frac{p_\infty - p_b}{\rho} \\ \qquad\qquad on \qquad\qquad\qquad\quad S \end{cases} \tag{3.8}$$

Equation (3.8) is known as the dynamic boundary condition. This is the extra equation needed to find S. To solve the bubble problem we seek the function φ and the equation of the surface S such that (3.1)–(3.3) and (3.8) are satisfied. It is a nonlinear problem that requires the solution of a partial differential equation (here the Laplace equation (3.1)) in a domain whose boundary (here S) has to be found as part of the solution. This is a typical free-surface flow problem. In this book we will describe various analytical and numerical methods for investigating such nonlinear problems.

We note that the problem of Figure 2 is an idealized one, in which the viscosity and gravity and the wake behind the bubble are neglected. Bubbles with wakes and the effect of including gravity will be considered in Section 3.4.3. Readers interested in the effects of viscosity are referred to,

The dynamic boundary condition (3.8) is valid for steady flows with $\Omega = 0$. Combining ($\frac{\partial\phi}{\partial t}\frac{u.u}{2} + \frac{p}{\rho} + \Omega = B$)and (3.6) we find that the general form of the dynamic boundary condition (for unsteady flows) with $\Omega \neq 0$ is

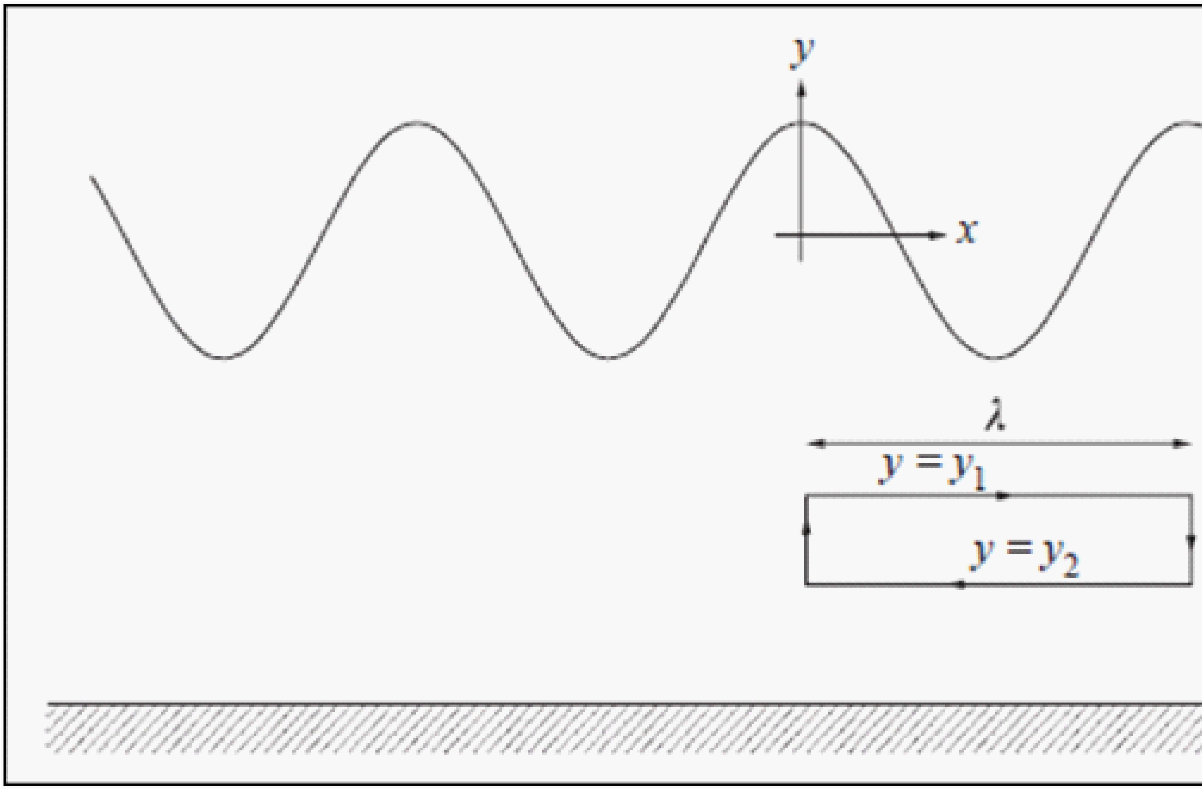
$$\frac{\partial\phi}{\partial t}\frac{u.u}{2} + \Omega + \frac{T}{\rho}K = B \tag{3.9}$$

Here B is the Bernoulli constant. For steady flows, (3.9) reduces

$$\frac{u.u}{2} + \Omega + \frac{T}{\rho}K = B \tag{3.10}$$

3.3 Two-dimensional flows

cross section of the cylinder is the semicircle shown in Figure 3. As we shall see, many interesting free-surface flows can be modeed as two-dimensional flows. We then introduce Cartesian coordinates x and y with the y-axis directed vertically upwards (at present we reserve the letter z to denote the complex quantity $x + iy$). In most applications considered in this book, the potential Ω (see



$(X = -\nabla\Omega))$ is due to gravity. Assuming that the acceleration of gravity g is acting in the negative y-direction, we write

$$\Omega = gy \tag{3.11}$$

An example is the two-dimensional free-surface flow past a semicircular obstacle at the bottom of a channel (see Figure 2). This two-dimensional configuration provides a good approximation to the three-dimensional free-surface flow past a long half-cylinder perpendicular to the plane of the figure (except near the ends of the cylinder).

Figure3: Two-dimensional free-surface flow past a submerged semicircle.

For two-dimensional potential flows, $(\nabla \times u = 0)$ and $(\nabla \cdot u = 0)$ become

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{3.12}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.13}$$

Here uand vare the x and y components of the velocity vector u. We can introduce a stream function ψ by noting that (3.13) is satisfied

$$u = \frac{\partial \psi}{\partial y} \tag{3.14}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{3.15}$$

It then follows from (3.12) that

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{3.16}$$

For two-dimensional flows, equations ($u = \nabla \phi.$) and ($\nabla^2 \phi = 0.$) give

$$u = \frac{\partial \phi}{\partial x} \tag{3.17}$$

$$v = \frac{\partial \phi}{\partial y} \tag{3.18}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{3.19}$$

Combining (3.14), (3.15), (3.17) and (3.18) we obtain

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{3.20}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{3.21}$$

Equations (3.20) and (3.21) can be recognized as the classical Cauchy–Riemann equations. They imply that the complex potential

$$f = \phi + i\psi \tag{3.22}$$

is an analytic function of $z = x + iy$ in the flow domain. This result is particularly important since it implies that two-dimensional potential flows can be investigated by using the theory of analytic functions. This applies in particular to all two-dimensional potential free-surface flows with or without gravity and/or surface tension included in the dynamic boundary condition. It does not apply, however, to ax symmetric and three-dimensional free surface flows. Since the derivative of an analytic function is also an analytic function, it follows that the complex velocity

$$u - iv = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x} = \frac{df}{dz} \tag{3.23}$$

is also an analytic function of $z = x + iy$.
The theory of analytic functions will be used intensively in the following chapters to study two-dimensional free-surface flows. In particular the following important tools will be useful.
The first tool is conformal mappings. These are changes of variable defined by analytic functions. For example, if $h(t)$ is an analyticfunction of t, the change of variables $z = h(t)$ enables us to seek the complex velocity $u - iv$ as an analytic function of t (since an analytic function of an analytic function is also an analytic function). Such conformal mappings are used to redefine a problem in a new complex t-plane in which the geometry is simpler than in the original z-plane.
The second tool is Cauchy's theorem: If $h(z)$ is analytic throughout a simply connected domain D then, for every closed contour C within

$$\int_c h(z)dz = 0 \tag{3.24}$$

The third tool is the Cauchy integral formula: Let $h(z)$ be analytic everywhere within and on a closed contour C, taken in the positive sense (counterclockwise).Then the integral

$$\frac{1}{2i\pi}\int_c \frac{h(z)}{z - z_0}dz = 0 \tag{3.25}$$

takes the following values:

$$\begin{cases} 0 \quad \text{if} \quad z_0 \\ is\ outside\ \ C \end{cases} \tag{3.26}$$

$$\begin{cases} h(z) \quad \text{if} \quad z_0 \\ \quad is\ inside\ \ C \end{cases} \tag{3.27}$$

$$\begin{cases} 2h(z) \quad \text{if} \quad z_0 \\ \quad is\ on\ \ C \end{cases} \tag{3.28}$$

When z_0 is on C the integral (3.25) is a Cauchy principal value.
We now show that for steady flows the stream function ψ is constant along streamlines. A streamline is a line to which the velocity vectors are tangent. Let us describe a streamline in parametric form by $x = X(s), y = Y(s)$, where s is the arc length. Then we have

$$-vX'(s) + uY'(s) = 0 \tag{3.29}$$

where the primes denote derivatives with respect to s. Using (3.14) and (3.15) we have

$$\frac{\partial\psi}{\partial x}X'(s) + \frac{\partial\psi}{\partial y}Y'(s) = \frac{\partial\psi}{\partial s} = 0 \tag{3.30}$$

which implies that ψ is constant along a streamline. For steady flows the kinematic boundary condition implies that a free surface is a streamline.The stream function is therefore constant along a free surface.For two-dimensional flows the dynamic boundary condition (1.9) becomes

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\phi_x^2+\phi_y^2)+gy+\frac{T}{\rho}K=B \tag{3.31}$$

If we denote by θ the angle between the tangent to the free surface and the horizontal then the curvature K can be defined by

$$K=-\frac{d\theta}{ds} \tag{3.32}$$

where again s denotes the arc length. In particular if the (unknown) equation of the free surface is $y=\eta(x,t)$ then of the free surface is $y=\eta(x,t)$ then

$$\begin{cases} \tan\theta=\eta_x & \textit{and} \\ \frac{dx}{ds}=\frac{1}{(1+\eta_x^2)^{3/2}} & \end{cases} \tag{3.33}$$

Using (3.32), (3.33) and the chain rule gives the formula

$$K=\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \tag{3.34}$$

3.4 Linear waves

3.4.1 The water-wave equations

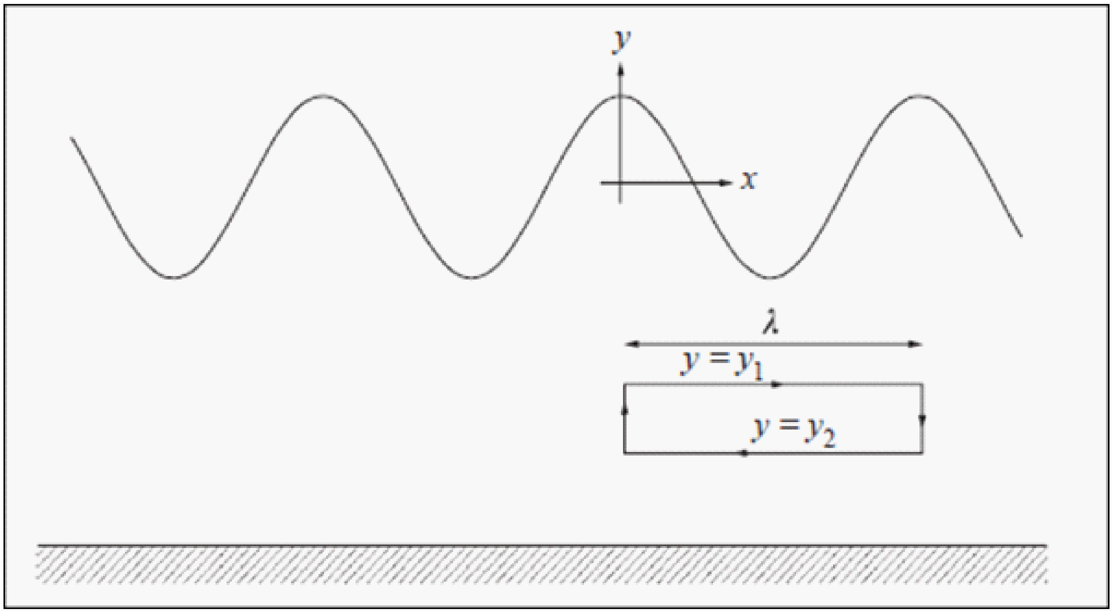
Many free-surface flows involve waves on their free surfaces. When dissipation is neglected and the flow is assumed to be two-dimensional, these waves often approach uniform wave trains in the far field (see for example Figure 3). Therefore a fundamental problem in the theory of free-surface flows is the study of a uniform train of two-dimensional waves of wavelength λ extending from $x=-\infty$ to $x=\infty$ and travelling at a constant velocity c. The flow configuration is illustrated in Figure 4

Fig4: A two-dimensional train of waves viewed in a frame of reference moving

with the wave. The free-surface profile has wavelength λ. The fluid is bounded below by a horizontal bottom with equation $y=-h$. Also Shown is the rectangular contour used in (3.43).For convenience we have chosen a frame of reference moving with the wave,so that the flow is steady. Using the notation of Section 3, we formulate the problem as

$$\begin{cases} \phi_{xx}+\phi_{yy}=0 & \textit{on} \\ \qquad -h<y>\eta(x) & \end{cases} \tag{3.35}$$

$$\begin{cases} \phi_y=\phi_{x\eta x} & \textit{on} \\ \qquad y=\eta(x) & \end{cases} \tag{3.36}$$



$$\begin{cases} \phi_y = 0 \qquad \qquad \qquad \textit{on} \\ \qquad y = -h \end{cases} \tag{3.37}$$

$$\begin{cases} \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy - \frac{T}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = B \qquad \qquad \textit{on} \\ \qquad y = \eta(x) \end{cases} \tag{3.38}$$

$$\nabla\phi(x+\lambda) = \eta(x) \tag{3.39}$$

$$\eta(x+\lambda) = \eta(x) \tag{3.40}$$

$$\int_0^\lambda \eta(x)dx = 0 \tag{3.41}$$

$$\begin{cases} \frac{1}{\lambda}\int_0^\lambda \phi(x)dx = c \qquad \qquad \textit{on} \\ \qquad y = constant \end{cases} \tag{3.42}$$

Here g is the acceleration of gravity (assumed to act in the negative y direction),T is the surface tension, ρ is the density, $y = -h$ is the equation of the bottom and $y = \eta(x)$ is the equation of the (unknown) free surface. Equations (3.36) and (3.37) are the kinematic boundary conditions on the free surface and on the bottom respectively. Equation (3.38) is the dynamic boundary condition on the free surface. We have used (3.31) and the formula (3.34) for the curvature of a curve $y = \eta(x)$. Relations (3.39) and (3.40) are periodicity conditions, which require the solution to be periodic with wavelength λ. Equation (3.41) fixes the origin of the y-coordinates as the mean water level. Finally, (3.42) defines the velocity c as the average value of $u = \phi_x$ at a level $y = constant$in the fluid. The value of c is independent of the constant chosen; this can be seen by applying Stokes' theorem to the vector velocity (u, v) using a contour C consisting of two horizontal lines $y = y1$,$y = y2$ and two vertical lines separated by a wavelength (see Figure 4). Since the flow is irrotational,

$$\int_0 udx + vdy = 0 \tag{3.43}$$

The contributions from the two vertical lines cancel by periodicity and (3.43) Gives

$$\int_0^\lambda [u]_{y=y1}\, dx = \int_0^\lambda [u]_{y=y2}\, dx \tag{3.44}$$

Since $y1$ and $y2$are arbitrary, the integral on the left-hand side of (3.42) is independent of the level $y = constant$ chosen in the fluid.
The relations (3.35)–(3.42) are referred to as the water-wave equations because they model waves travelling at the interface between water and air (although they apply also to other fluids).

3.4.2 Linear solutions for water waves

A trivial solution of the system (3.35)–(3.42) is

$$\begin{cases} \phi = cx, \quad \eta(x) = 0 & and \\ \qquad\qquad B = \frac{c^2}{2}. \end{cases} \tag{3.45}$$

This solution describes a uniform stream with constant velocity c, bounded below by a horizontal bottom and above by a flat free surface.Linear waves are obtained by seeking a solution as a small perturbation of the exact solution (3.45). Therefore we write

$$\phi(x, y) = cx + \varphi(x, y) \tag{3.46}$$

and assume that both $|\varphi(x, y)|$ and $|\eta(x)|$ are small. Substituting (3.46) into (3.35)–(3.42) and dropping nonlinear terms in ϕ and η, we obtain the linear System

$$\begin{cases} \phi_{xx} + \phi_{yy} = 0, \\ \qquad -h < y > 0 \end{cases} \tag{3.47}$$

$$\begin{cases} \phi_y = c\eta_x, \\ \qquad y = 0 \end{cases} \tag{3.48}$$

$$\begin{cases} \varphi_y = 0, \\ \qquad y = -h \end{cases} \tag{3.49}$$

$$\begin{cases} -\frac{T}{\rho}\eta_{xx} + c\varphi_x + g\eta = 0, \\ y = 0 \end{cases} \tag{3.50}$$

$$\nabla\varphi(x+\lambda, y) = \nabla\varphi(x, y) \tag{3.51}$$

$$\eta(x+\lambda) = \eta(x) \tag{3.52}$$

$$\int_0^\lambda \eta(x)dx = 0 \tag{3.53}$$

$$\begin{cases} \frac{1}{\lambda}\int_0^\lambda \varphi(x)dx = c & on \\ y = constant. \end{cases} \tag{3.54}$$

We choose the origin of x at a crest and assume that the wave is symmetric about $x = 0$. Thus we impose the conditions

$$\varphi(-x, y) = -\varphi(x, y) \tag{3.55}$$

$$\eta(-x) = \eta(x) \tag{3.56}$$

Using the method of separation of variables, we seek a solution of (3.47) in the formed

$$\varphi(x, y) = X(x)Y(y). \tag{3.57}$$

Substituting (3.57) into (3.47), (3.55) and (3.49) yields

$$X(-x) = -X(x) \tag{3.58}$$

the ordinary differential equations

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = constant = -\alpha^2 \tag{3.59}$$

and the boundary condition

$$Y'(-h) = 0 \tag{3.60}$$

Here we have chosen a negative separation constant in (3.59), so that the solution is periodic in x. Solutions of the two ordinary differential equations (3.59) satisfying (3.58) and (3.60) are written as

$$X(x) = \sin\blacksquare\alpha x \tag{3.61}$$

$$Y(y) = \cos h\alpha y + h \tag{3.62}$$

The periodicity condition (3.51) implies that

$$\alpha = nk \tag{3.63}$$

where n is a positive integer and

$$k=\frac{2\Pi}{\lambda} \tag{3.64}$$

is the wave number. Multiplying (3.61) and (3.62) and taking a linear combination of the solutions corresponding to the values (3.63) of α, we obtain

$$\varphi(x,y)=\sum_{n=1}^{\infty} B_n\ \cos\blacksquare hnk(y+h)\ \sin nkx \tag{3.65}$$

Here the B_n are constants.

Using the periodicity and the symmetry conditions (3.56) and (3.52), we express $\eta(x)$ as the Fourier series

$$\eta(x)=A_0+\sum_{n=1}^{\infty} A_n \cos\blacksquare nkx \tag{3.66}$$

where the A_n are constants. The condition (3.48) implies that $A_0=0$.

Substituting (3.65) and (3.66) into (3.53) and equating the coefficients of $\sin nkx$ yields

$$cA_n=-B_n\ \sin h\blacksquare nkh, \qquad\qquad n=1,2,\ldots \tag{3.67}$$

Similarly substituting (3.65) and (3.66) into (3.50) gives

$$\frac{T}{\rho}A_n n^2k^2+gA_n+cB_n nk\ \cos\blacksquare hnkh=0, \qquad\qquad n=1,2,\ldots \tag{3.68}$$

Eliminating B_n between (3.67) and (3.68) yields

$$(g+\frac{T}{\rho}n^2k^2-\frac{c^2nk}{\sin h\blacksquare nkh}\cos\blacksquare hnkhA_n=0, \qquad\qquad n=1,2,\ldots \tag{3.69}$$

Since we seek a nontrivial periodic solution $\eta(x)\neq 0$, we can assume without loss of generality that $A_1\neq 0$; then (3.68) with $n=1$ implies that

$$c^2=(\frac{g}{k}+\frac{T}{\rho})\tan\blacksquare hkh \tag{3.70}$$

Relation (3.69) for $n>1$ gives

$$A_n=0, \qquad\qquad n=2,3,\ldots \tag{3.71}$$

provided that

$$g+\frac{T}{\rho}n^2k^2-\frac{c^2nk}{\sin h\blacksquare nkh}\cos\blacksquare hnkh\neq 0, \qquad n=2,3,\ldots \tag{3.72}$$

If (3.72) is satisfied, the solution of the linear problem is

$$\varphi = -\frac{cA_1}{\sin h\blacksquare nkh}\cos\blacksquare hnkh(y+h)\sin\blacksquare kx \tag{3.73}$$

$$\eta = A_1\cos\blacksquare kx \tag{3.74}$$

If the condition (3.72) is not satisfied for some integer value m of n, the solution of the linear problem is

$$\varphi = -\frac{cA_1}{\sin h\blacksquare kh}\cos\blacksquare hk(y+h)\sin\blacksquare kx - \frac{cA_m}{\sin h\blacksquare mkh}\cos\blacksquare hmk(y+h)\sin m\blacksquare kx \tag{3.75}$$

$$\eta_1 = A_1\cos\blacksquare kx + A_m\cos m\blacksquare kx \tag{3.76}$$

where Am is an arbitrary constant. In the theory of linear waves, it is usually assumed that $Am = 0$. However, when we are developing nonlinear theories for water waves, i.e. improving the linear approximations (3.75) and (3.76) by adding nonlinear corrections or solving the fully nonlinear problem (3.47)–(3.54) numerically, we shall see that $Am \neq 0$. Two consequences are the existence of many different families of nonlinear periodic gravity–capillary waves and the existence of solitary waves with oscillatory tails.
The velocity c is called the phase velocity and equation (3.70) is the (linear) dispersion relation. Relation (**??**) implies that waves of different wave numbers and therefore of different wavelengths travel at different phase velocities c.It is convenient to rewrite (3.70) in the dimensionless form

$$F^2 = (\frac{1}{kh} + \tau kh)\tan\blacksquare hkh \tag{3.77}$$

Where

$$F = \frac{c}{(gh)^{1/2}} \tag{3.78}$$

is the Froude number and

$$\tau = \frac{T}{\rho gh^2} \tag{3.79}$$

[1]

3.5 Nonlinear Water Waves

3.5.1 Long Wave Reflection

We start this section with a simple exercise for a linear hyperbolic wave reflecting at an interface. It is instructive to consider the linear acoustic equations

$$\begin{cases} \rho(x)u_1 + p_x = 0 \\ \frac{1}{k(x)}p_t + u_x = 0 \end{cases} \tag{3.80}$$

It is an easy exercise to adapt the calculations here presented for the acoustic equations to the shallow water model presented in Eq.($u_t(h(x)\eta_x = 0, \eta_t + u_x = 0$). Here we consider a one-dimensional heterogeneous acoustic medium along the x-axis. The material density is given by $\rho(x)$, where $\rho(x) = \rho 1$ when $x < 0$ and $\rho(x) = \rho 2$ when $x > 0$. The material compressibility is given through the variable coefficient $1/k(x)$, which is also discontinuous at $x = 0$, due to two different constant values of the bulk modulus,$k1 \neq k2$. The velocity at a point x in the variable medium is denoted by $u(x,t)$, while $p(x,t)$ is the pressure. We will choose initial conditions so that a pulse-shaped disturbance in pressure and velocity will travel from left to the right, and interact with the medium's interface at $x = 0$.

3.5.2 Reduced Modeling

The mathematical exercise of using system ($u_t(h(x)\eta_x = 0, \eta_t + u_x = 0$) with a non-smooth topography follows from that of the acoustic model. It is a good warm up problem in order to consider a layered topography. Nevertheless in the presence of a rapidly varying (non-smooth) topography, the shallow water system is not asymptotically valid as a reduced model arising from the potential theory equations [13]. We will outline the reasons why this is so. Consider the potential theory formulation arising from Euler's equations [26].Let variables with physical dimensions be denoted with a tilde. We introduce the length scales σ (a typical pulse width or wavelength), h_0 (a typical depth), a (a typical wave amplitude), lb (the horizontal length scale for bottom irregularities) and L (the total length of the rough region). The acceleration due to gravity is denoted by g and the reference shallow water speed is $c_0 = gh_0$. Dimensional variables are then defined in a standard fashion [24, 26] by having

$$\tilde{x} = \sigma x, \qquad \widetilde{y} = h_0 y, \qquad \widetilde{t} = \left(\frac{\sigma}{c_0}\right) t$$

$$\widetilde{\eta} = a\eta, \widetilde{\phi} = \left(\frac{g\sigma a}{c_0}\right) \phi, \qquad \widetilde{h} = h_0 H \frac{\widetilde{x}}{l_b}$$

Conclusion 4 *This short paper should be concluded by reminding the reader that the conjectures made here are based on observations of numerical 'evidence' and may not be true, in general. However, for symmetric waves with one crest per wavelength, these conjectures should be true at least up to a finite (not small) amplitude, i.e. not only for infinitesimal waves.*