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Particular fuzzy subsets on topologies generated by fuzzy relations

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Contents

Introduction

The concept of fuzzy sets was first introduced by Lotfi Zadeh[30] in 1965 as a way to deal with uncertainty and imprecision in data . Fuzzy sets allowed for the representation of degrees of membership of the elements in the interval [0, 1] whereas crisp sets only allowed for binary membership.

Following the development of fuzzy sets, researchers began to explore that idea of using fuzzy relations to represent and reason about relationships between elements in a more flexible and nuanced way. In 1971, L. Zadeh[31] introduced the concept of fuzzy relation, which is a generalization of a crisp relation that allows for degrees of membership or truth values between elements.

However, fuzzy relations are a key tool in fuzzy topology which was introduced for the first time by Chang[5] in 1968. He defined it as a collection of subsets of a given set that satisfies certain condition based on the fuzzy relation. In 2009, Knoblauch[11] had introduced a topology generated by a binary relations. Later, Mishra and Srivastava[23] extend this concept and generate fuzzy topology by fuzzy relation in 2016. They studied the notion of fuzzy topology generated by fuzzy relation and proved several properties which became a basic for many researchers in their studies.

Recently, K. Saadaoui, S. Milles and L. Zedam [21] have given more importance to the study of lattice structure of fuzzy open sets on this topology by providing various chracteristics and properties. They concentrated on the notion of fuzzy ideal (resp. fuzzy filter) on this topology generated by fuzzy relation. Moreover, they provided a chracterization of this notion of fuzzy ideal (resp. fuzzy filter) based on the meet and join operations of the introduced lattice.

In this memoir, we study special particular fuzzy subsets which are the ideals and filters on the lattice of open sets in topology generated by fuzzy relation. Also, we investigate some properties of those notions and some characterizations.

This memoir is organized as follows :

- **In the first chapter**, we recall binary relations, ordered sets and lattices and ideals and filters on a crisp lattice, and we will give definition of a crisp set, fuzzy set, operations of fuzzy set, characteristics of fuzzy set, triangular norms, triangular conorms, projection and Cartesian product, and we will give definition of fuzzy relations their operations and types of relations.

- **In the second chapter**, the purpose of this chapter is to provide to fuzzy topology generated by fuzzy relation. Also, we study the lattice structure of fuzzy open sets on this topology. To that end, we investigate the notion of fuzzy ideal (resp. fuzzy filter) on topology generated by fuzzy relation.

Chapter 1

Generalities on fuzzy sets and fuzzy relations

In this chapter, we recall binary relations, ordered sets and lattices and ideals and filters on a crisp lattice, and we will give definition of a crisp set, fuzzy set, operations of fuzzy set, characteristics of fuzzy set, triangular norms, triangular conorms, projection and Cartesian product, and we will give definition of fuzzy relations their operations and types of relations.

1.1 Binary relations, ordered sets and lattices

This section contains the basic definitions and properties of binary relations, posets, lattices.

A binary relation on a set X is a subset of X^2, i.e., it is a set of couples $(x, y) \in X^2$. For a relation $R \subseteq X^2$, we often write xRy instead of $(x, y) \in R$. Two elements x and y of a set X equipped with a relation R are called comparable elements, denoted by $x \nparallel y$, if it holds that xRy or yRx. Otherwise, they are called incomparable elements, denoted by $x \parallel_R y$, or simply $x \parallel y$ when no confusion can occur. We denote by R^c the complement of the relation R on X, i.e., for any $x, y \in X$, xR^cy denotes the fact that $(x, y) \notin R$. We denote by R^t the transpose of the relation R on X, i.e., for any $x, y \in X$, xR^ty denotes the fact that yRx. We denote by R^d the dual of the relation R on X, i.e., for any $x, y \in X$, xR^dy denotes the fact that yR^cx. A relation R on a set X is said to be included in a relation S on the same set X, denoted by $R \subseteq S$, if, for any $x, y \in X$, xRy implies that xSy. The union of two relations R and S on a set X is the relation $R \cup S$ on X defined as $R \cup S = \{(x, y) \in X^2 \mid xRy \vee xSy\}$. Similarly, the intersection of two relations R and S on a set X is the relation $R \cap S$ on X defined as $R \cap S = \{(x, y) \in X^2 \mid xRy \wedge xSy\}$. If $R \cap S = \emptyset$, then R and S are called disjoint relations. The composition of two relations R and S on a set X is the relation $R \circ S$ on X defined as $R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}$. For any $n \in \mathbb{N}^*$, the n-th power relation R^n of R is recursively defined as follows:

$$(R^1 = R) \wedge (\forall n \geqslant 1)(R^{n+1} = R^n \circ R)\,.$$

A binary relation R on a set X is called:

(i) Reflexive, if, for any $x \in X$, it holds that xRx;

(ii) Irreflexive, if, for any $x \in X$, it holds hat $xR^c x$;

(iii) Symmetric, if, for any $x, y \in X$, it holds that xRy implies that yRx;

(iv) Antisymmetric, if, for any $x, y \in X$, it holds that xRy and yRx imply that $x = y$;

(v) Asymmetric, if, for any $x, y \in X$, it holds that xRy implies that $yR^c x$;

(vi) Transitive, if, for any $x, y, z \in X$, it holds that xRy and yRz imply that xRz;

(vii) Complete, if, for any $x, y \in X$, either xRy or yRx holds.

A binary relation R on a set X is called:

(i) A pseudo-order relation, if it is reflexive and antisymmetric;

(ii) A strict order, if it is irreflexive and transitive;

(iii) An order relation, if it is reflexive, antisymmetric and transitive;

(iv) A total order relation, if it is reflexive, antisymmetric, transitive and complete;

For more details on binary relations, we refer to [6, 20, 22].

A partial order (order, for short) is a binary relation $\leqslant$ over a set X which is reflexive ($a \leqslant a$, for any $a \in X$), antisymmetric ($a \leqslant b$ and $b \leqslant a$ implies $a = b$, for any $a, b \in X$) and transitive ($a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$, for any $a, b, c \in X$). A set with an order relation is called an ordered set (also called a poset). Further, $\{x, y\}^u$ denotes the set of all upper bounds of x and y, while $\{x, y\}^l$ denotes the set of all lower bounds of x and y, i.e., $\{x, y\}^u = \{z \in X \mid x \leqslant z \;\wedge\; y \leqslant z\}$ and $\{x, y\}^l = \{z \in X \mid z \leqslant x \;\wedge\; z \leqslant y\}$.

A strict order is a binary relation $<$ on a set X that is irreflexive ($a < a$ does not hold for any $a \in X$), asymmetric (if $a < b$, thus $b < a$ does not hold for any $a, b \in X$) and transitive. A given binary relation $\sim$ on a set X is said to be an equivalence relation if it is reflexive, symmetric ($a \sim b$ implies $b \sim a$, for any $a, b \in X$) and transitive. If $\leqslant$ is an order, then the corresponding strict order $<$ is the irreflexive kernel given by:

$$a < b \text{ if } a \leqslant b \text{ and } a \neq b.$$

Conversely, if $<$ is a strict order, then the corresponding order $\leqslant$ is the reflexive closure given by

$$a \leqslant b \text{ if } a < b \text{ or } a = b.$$

Two elements x and y of X are called comparable if $x \leqslant y$ or $y \leqslant x$; otherwise they are called incomparable, and we write $x \parallel y$. Using the strict order $<$, the relation (x is covered by y) denoted as $x \ll y$, if $x < y$ and there exists no $z \in X$ such that $x < z < y$. A poset can be conveniently represented by a Hasse diagram, displaying the covering relation $<$. Note that $x < y$ if there is a sequence of connected lines upwards from x to y.

For more details about order, strict order and equivalence relations we refer to [22].

Definition 1.1. [6] *Let $(X, \leqslant_1)$ and $(Y, \leqslant_2)$ two posets, then the mapping $f : X \to Y$ is monotone (or order-preserving) if $x \leqslant_1 y \Rightarrow f(x) \leqslant_2 f(y)$, for any $(x, y \in X)$.*

Definition 1.2. [6] *Let P be an ordered set. Then P is a chain if for any $x, y \in P$, either $x \leq y$ or $y \leq x$ (that is, if any two elements of P are comparable). Alternative names for a chain are linearly ordered set and totally ordered set.*

Zorn's lemma is a result in set theory that appears in proofs of some non-constructive existence theorems throughout mathematics.

Theorem 1.1. *(Zorn's lemma) Let P be a non-empty ordered set in which every nonempty chain has an upper bound. Then P has a maximal element.*

Many important properties of an order set $(L, \leq)$ are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of X. Particularly, we are interested in two of the most important classes of ordered sets defined in this way. They are a lattice and a complete lattice. We often write $x \vee y$ instead of $\sup\{x, y\}$ when it exists and $x \wedge y$ instead of $\inf\{x, y\}$ when it exists. Similarly, we write $\bigvee S$ (the join of S) and $\bigwedge S$ (the meet of S) instead of $\sup S$ and $\inf S$ when they exist.

Definition 1.3. [6] *Let $(X, \leq)$ be an ordered set.*

(i) If $x \vee y$ exists for any $x, y \in X$, then $(X, \leq)$ is called a $\vee$-semi-lattice;

(ii) If $x \wedge y$ exists for any $x, y \in X$, then $(X, \leq)$ is called a $\wedge$-semi-lattice;

(iii) $(X, \leq)$ is called a lattice if it is both a $\wedge$-semi-lattice and a $\vee$-semi-lattice;

(iv) If $\bigvee S$, $\bigwedge S$ exist for any $S \subseteq X$, then $(X, \leq)$ is called a complete lattice.

A bounded lattice is a lattice that additionally has a greatest element 1 and a smallest element 0, which satisfy $0 \leq x \leq 1$, for any x in X.

A lattice $(L, \leq, \wedge, \vee)$ is distributive if the following additional condition holds

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\,, \text{ for any } x, y, z \in L\,.$$

This means that the meet operation preserves non-empty finite joins. It is known that the above condition is equivalent to its dual

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\,, \text{ for any } x, y, z \in L\,.$$

Definition 1.4. [6] *Let L and L' be two lattices. A mapping $f : L \to L'$ is called an homomorphism if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$, for any $x, y \in L$. If f is a bijection, then f is called an isomorphism.*

1.2 Ideals and filters on a crisp lattice

In this section, we present the notion of crisp ideals (resp. filters) and their types on a crisp lattice as proper, prime and maximal.

Definition 1.5. [6] *A nonempty subset I on a lattice L is called an ideal of L if, for any $x, y \in L$, the following conditions are satisfied:*

1. if $y \in I$ and $x \leq y$, then $x \in I$,

2. if $x, y \in I$ implies $x \vee y \in I$.

The definition can be more compactly stated by declaring an ideal to be a non-empty down-set closed under join.

A dual ideal is called a filter. Specifically, a non-empty subset of L determined by the following definition.

Definition 1.6. [6] *A nonempty subset F on a lattice L is called a filter if, for any $x, y \in L$, the following conditions are satisfied:*

1. *if $y \in F$ and $y \leq x$, then $x \in F$,*
2. *if $x, y \in F$ implies $x \wedge y \in F$.*

The set of all ideals (resp. filters) of L is denoted by $\mathcal{I}(L)$ (resp. $\mathcal{F}(L)$), and carries the usual inclusion order.

More precisely, an ideal or filter is called proper if it does not coincide with L. More precisely, an ideal I of a lattice with 1 is proper if and only if $1 \notin I$, and dually, a filter F of a lattice with 0 is proper if and only if $0 \notin F$. For any $a \in L$, the set $\downarrow a$ is an ideal (also known as the principal ideal generated by a). Dually, $\uparrow a$ is a principal filter.

Definition 1.7. [6] *An ideal I on a lattice L is called a prime ideal if, $x \wedge y \in L$, then $x \in I$ or $y \in I$, for any $x, y \in L$.*

Definition 1.8. [6] *A filter F on a lattice L is called a prime filter if, $x \vee y \in F$, then $x \in F$ or $y \in F$, for any $x, y \in L$.*

Definition 1.9. [6] *Let L be a lattice. A proper ideal (resp. filter) A is said to be a maximal ideal (resp. maximal filter or more usually known as an ultrafilter) if the only ideal (resp. filter) properly containing A is L.*

Example 1.1. *(i) The following are ideals in $\mathcal{P}(X)$;*

(a) All subsets not containing a fixed element of X;

(b) All finite subsets (this ideal is non-principal if X is infinite);

(ii) Let $(X, \mathfrak{T})$ be a topological space and let $x \in X$. Then the set $\{V \subseteq X \mid (\exists U \in \mathfrak{T}) x \in U \subseteq V\}$ is a filter in $(X, \mathfrak{T})$.

1.3 T-norms and T-conorms

In this section, we give the definitions for triangular norm and triangular conorm, we also support them with some examples.

Tringular norms

Definition 1.10. [28] *Triangular norm is a binary operation* T *on the unit interval* $[0,1]$*, i.e., it is a function* $T : [0,1]^2 \longrightarrow [0,1]$ *: the following four axioms are satisfied :*

(T1) Commutativity i.e., $T(x,y) = T(y,x)$ *;*

(T2) Associativity i.e., $T(x,T(y,z)) = T(T(x,y),z)$ *;*

(T3) Monotonicity i.e., $T(x,y) \leq T(x,z)$ *whenever* $y \leq z$ *;*

(T4) Boundary condition i.e., $T(x,1) = x$*.*

Conditions (T4) and (T3) imply that for any t-norm T it holds that $T(x,y) \leq x$, $T(x,y) \leq y$, $T(x,y) \leq Min(x,y)$ and $T(x,0) = 0$.

Example 1.2. *The following four operations are the most common t-norms:*

(T5) *Minimum t-norm* $T_M = min(x,y)$;

(T6) *Lukasiewiez t-norm* $T_L = max(x+y-1,0)$;

(T7) *Product t-norm* $T_P = xy$;

(T8) *Einstein t-norm* $T_E = \dfrac{xy}{(2-x-y+xy)}$;

(T9) *Drastic product:*

$$T_D(x,y) = \begin{cases} x, & \text{if } \ y = 1; \\ y, & \text{if } \ x = 1; \\ 0, & \text{if } \ x,y < 1. \end{cases}$$

Triangular conorms

Definition 1.11. [28] *A triangular conorm is a binary operation* S *on the unit interval* $[0,1]$*, i.e., it is a function* $S : [0,1]^2 \longrightarrow [0,1]$ *: the following four axioms are satisfied :*

(S1) Commutativity : $S(x,y) = S(y,x)$ *;*

(S2) Associativity : $S(x,S(y,z)) = S(S(x,y),z)$ *;*

(S3) Monotonicity : $S(x,y) \leq S(x,z)$ *whenever* $y \leq z$ *;*

(S4) Boundary condition : $S(x, 0) = x$.

Example 1.3. *(1) Maximum t-conorm* $S_M = max(x, y)$;

(2) Lukasiewiez t-conorm $S_L = min(x + y, 1)$;

(3) Probabilistic sum $S_P = x + y - xy$;

(4) Einstein t-conorm ; $S_E = \frac{x+y}{1+xy}$

(T8) *Drastic sum:*

$$S_D(x, y) = \begin{cases} 1, & if \ \ (x, y) \in [0, 1)^2 \\ \max\{x, y\}, & otherwise. \end{cases}$$

1.4 Fuzzy sets

Fuzzy sets were introduced by Zadeh [30] as a generalization of crisp set. In this section, we recall the definition fuzzy sets, its operation and characteristic fuzzy set.

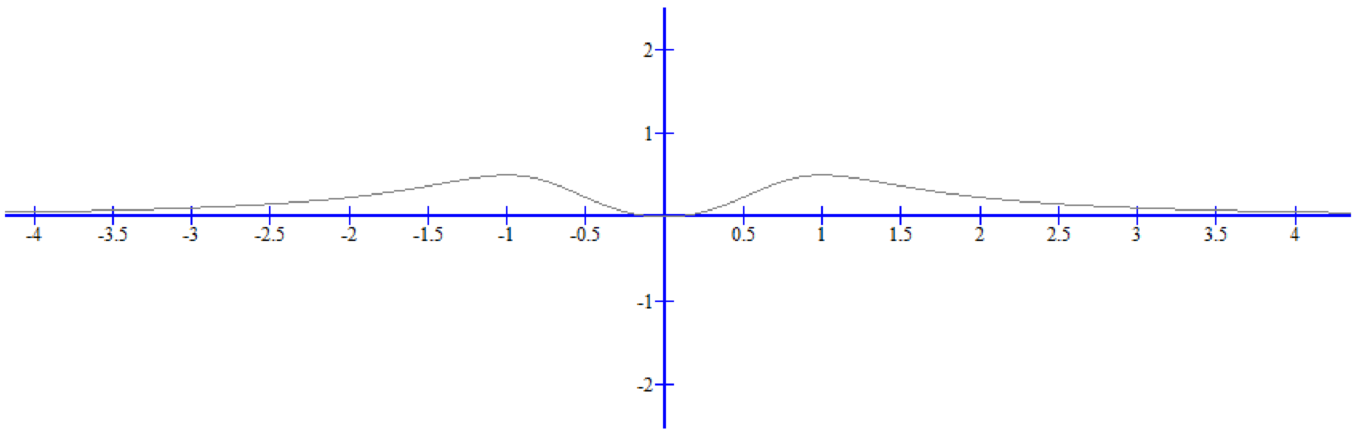
1.4.1 Definitions

Definition 1.12. [30] *Let* X *be a non empty set. A fuzzy set* $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ *is characterized by a membership function* $\mu_A : X \to [0, 1]$, *where* $\mu_A(x)$ *is interpreted as the degree of membership of the element* x *in the fuzzy subset* A *for each* $x \in X$.

Example 1.4. *(1) Let* $X = \{a, b, c\}$ *be a universal set.* $A = \{(a, 0.2), (b, 0.8), (c, 1)\}$ *a fuzzy subset of* X *;*

(2) Let $X = \mathbb{R}$, *and* A *is a fuzzy subset of* X, *defined by :*

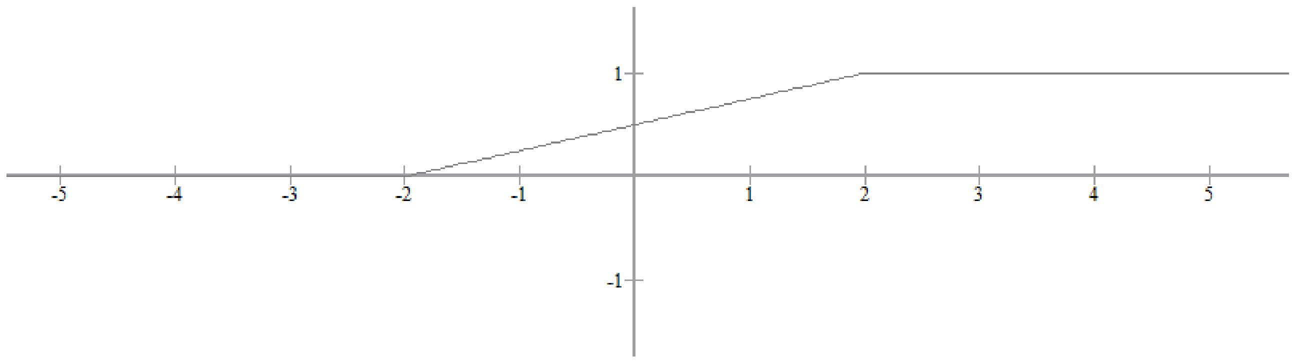
$$\mu_A(x) = \frac{x^2}{1 + x^4};$$



graph of μ_A

(3) Let $X = \mathbb{R}$, and A is a fuzzy subset of X, defined by :

$$\mu_A(x) = \begin{cases} 0, & if \;\; x \leq -2; \\ \dfrac{x+2}{4}, & if \;\; -2 \leq x \leq 2; \\ 1, & if \;\; 2 \leq x. \end{cases}$$



graph of μ_A

1.4.2 Operations of fuzzy sets

In this subsection, we give the definition of operations on fuzzy sets as equality, inclusion, intersection, union, complement, sum and product.

Definition 1.13 (Equality)**.** [30] Let X be a non empty set and let A and B two fuzzy subsets, we say that $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

Definition 1.14 (Inclusion). [30] Let X be a non empty set and let A and B two fuzzy subsets, we say that $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ for all x in X.

Definition 1.15 (Intersection). [30] Let X be a non empty set and let A and B two fuzzy subsets, the intersection defined by for all $x \in X$

$$\mu_{A\cap B}(x) = min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x).$$

Definition 1.16 (Union). [30] Let X be a non empty set and let A and B two fuzzy subsets, the union defined by for all $x \in X$

$$\mu_{A\cup B}(x) = max\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x).$$

Definition 1.17 (Complement). [30] The complement of a fuzzy set A is de noted by $C(A)$ and is defined by : for all $x \in X$

$$\mu_{C(A)}(x) = 1 - \mu_A(x).$$

Definition 1.18 (Sum). [30] Let X be a non empty set and let A and B two fuzzy subsets, the sum defined by for all $x \in X$

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \times \mu_B(x).$$

Definition 1.19 (Product). [30] Let X be a non empty set and let A and B two fuzzy subsets, the product defined by for all $x \in X$

$$\mu_{A\times B}(x) = \mu_A(x)\mu_B(x).$$

Example 1.5. *Let* $X = \{a, b, c\}$*, and let* $A = \{(a, 0.2), (b, 0.6), (c, 0.5)\}$*, and* $B = \{(a, 0.7), (b, 0.1), (c, 1)\}$ *we have :*

1. $A \cap B = \{(a, 0.2), (b, 0.1), (c, 0.5)\}$;
2. $A \cup B = \{(a, 0.7), (b, 0.6), (c, 1)\}$;
3. $A \times B = \{(a, 0.14), (b, 0.06), (c, 0.5)\}$;
4. $A + B = \{(a, 0.76), (b, 0.74), (c, 1)\}$;
5. $C(A) = \{(a, 0.8), (b, 0.4), (c, 0.5)\}$.

1.4.3 Characteristics of fuzzy sets

In this subsection, we give the definition of the characteristics of fuzzy sets as α-cuts, support, kernel, height and cardinality.

Definition 1.20 (α-cuts)**.** [30, 31] Let A be a fuzzy set in X and let $\alpha \in]0,1]$, The α-cut of A, denoted A_α. We mean all elements of X that belong to A to a degree of at least α. That is A_α is a classical set defined by

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

Definition 1.21 (Support)**.** [30, 31] The support of a fuzzy set A, denoted by $Supp(A)$, we mean all elements of X that belong to a nonzero degree. That is $Supp(A)$is a calssical set defined by

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Definition 1.22 (Kernel)**.** [30, 31] The ker of a fuzzy set A, denoted by $Ker(A)$, we mean all elements of X that belong to a equal one. That is $Ker(A)$is a calssical set defined by

$$Ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

Definition 1.23 (Height)**.** [30, 31] The height of a fuzzy set A is the largest membership grade of any element in A.

$$H(A) = Max\mu_A(x).$$

Definition 1.24 (Cardinality)**.** [30, 31] Cardinality of a finite fuzzy set A, denoted $\mid A \mid$ is defined as

$$\mid A \mid = \sum_{x \in X} \mu_A(x).$$

Example 1.6. *1) Let* $X = \{a, b, c, d\}$*, and* $A = \{(a, 0.6), (b, 1), (c, 0.3), (d, 0)\}$;

- $A_{0.5} = \{a, b\}$;
- $Supp(A) = \{a, b, c\}$;
- $ker(A) = \{b\}$;
- $H(A) = 1$;
- $|A| = 1.9$.

2) Let $X = [0, 1]$ *with* $\alpha, \beta \in \mathbb{R}$ *and let* $a, b \in \mathbb{R}$*. We define the fuzzy set* A *on* X *by*

$$\mu_A(x) = \begin{cases} 0\ , & if\ x < a-\alpha\ or\ b+\beta < x; \\ 1\ , & if\ \ a < x < b; \\ 1+(\frac{x-a}{\alpha})\ , & if\ \ a-\alpha < x < a\,; \\ 1-(\frac{b-x}{\beta})\ , & if\ \ b < x < b+\beta. \end{cases}$$

Then $Ker(A) = [0,1]$, $Supp(A) = [a-\alpha, b+\beta]$ *and* $H(A) = 1$.

1.4.4 Cartesian product and projection on fuzzy sets

In this subsection contains definition of certesian product and projection on fuzzy sets.

Definition 1.25 (Cartesian product)**.** [2] The cartesian product applied to n fuzzy sets can be defined as follows : Let $\mu_{A_1}, \mu_{A_2}, ...\mu_{A_n}$, be membership functions of $A_1, A_2, ..., A_n$. Then, the membership degree of $(x_1, x_2, ...x_n) \in X_1 \times X_2 \times ... \times X_n$ on the fuzzy set $A_1 \times A_2 \times ... \times A_n$ is,

$$\mu_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n) = min\left\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\right\}.$$

Example 1.7. *Lets* $X_1 = \{a, b, c, \}$, $X_2 = \{\alpha, \beta\}$ *and lets* A_1, A_2 *two fuzzy subset respectively defined on* X_1 *and* X_2 *given by:*
$A_1 = \{(a, 0.1), (b, 0.4), (c, 0.8)\}$;
$A_2 = \{(\alpha, 0.2), (\beta, 0.6)\}$.
So, we get:

$$A_1 \times A_2 = \{((a,\alpha), 0.1), ((a,\beta), 0.1), ((b,\alpha), 0.2), ((b,\beta), 0.4), ((c,\alpha), 0.2), ((c,\beta), 0.6)\}.$$

Definition 1.26 (Projection)**.** [2] The projection on X_1 of the fuzzy set A of $X_1 \times X_2 \times ... \times X_n$ is the fuzzy set $Proj_{X_1}(A)$ of X_1, whose membership function is defined by: for any $x_1 \in X_1$,

$$\mu_{ProjX_{1(A)}}(x_1) = sup_{x_2\in X_2, x_3\in X_3, ..., x_n\in X_n}(\mu_A(x_1, x_2, ..., x_n)).$$

Example 1.8. *Let* $X = X_1 \times X_2$ *the set of reference such that* X_1 *and* X_2 *two sets, we consider* $A_1 \times A_2 = A$ *given by:*
$A = \{((a,\alpha), 0.1), ((a,\beta), 0.1), ((b,\alpha), 0.2), ((b,\beta), 0.4), ((c,\alpha), 0.2), ((c,\beta), 0.6)\}$.
So, we get:

$$\begin{aligned} Proj_{X_1}(A) &= \{(a, max(0.1, 0.1)), (b, max(0.2, 0.4)), (c, max(0.2, 0.6))\}; \\ &= \{(a, 0.1), (b, 0.4), (c, 0.6)\}. \end{aligned}$$

1.5 Fuzzy relations

In this section, we give the definitions and the operation of fuzzy relations, and we give examples and properties.

Definition 1.27. [31] Let X and Y be two nonempty sets. A fuzzy binary relation (a fuzzy relation, for short) from X to Y is a fuzzy subset of $X \times Y$, i.e, is an expression R given by

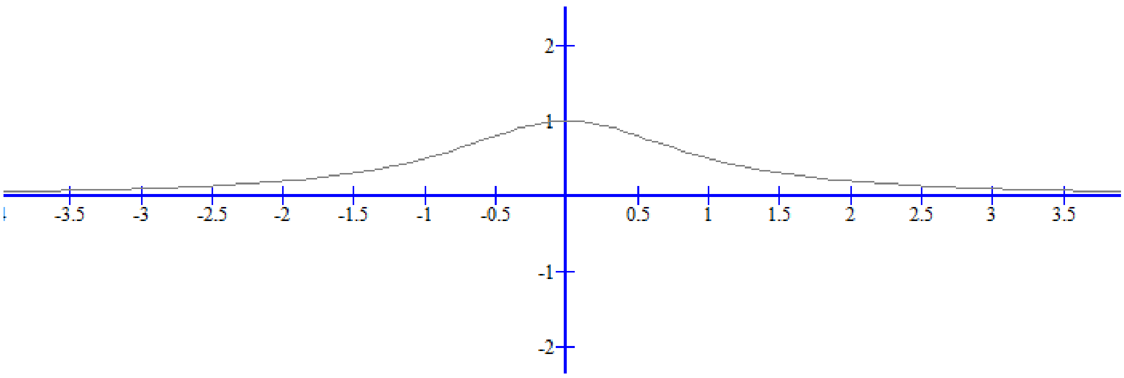
$$R = \{< (x, y), \mu_R(x, y) > |(x, y) \in X \times Y\}$$

where

$$\mu_R : X \times Y \to [0, 1]$$

for any $(x, y) \in X \times Y$. The value $\mu_R(x, y)$ is called the degree of membership of (x, y) in R.

Example 1.9. *Let $X = X_1 \times X_2$ the crisp set, we consider the fuzzy relation given by* $R(x, y) = \dfrac{1}{1 + +x^2 + y^2}$.



graph of $R(x, 0)$

Operations on fuzzy relations

In this section, we give definition of operations on fuzzy relations as inclusion, transpose, intersection and union.

Definition 1.28 (Inclusion)**.** [31] Let R, P two fuzzy relation. R is said to be contained in P(or we say that P contains R), denoted by $R \subseteq P$, if for any $(x, y) \in X \times Y$ it hold that $\mu_R(x, y) \leq \mu_P(x, y)$.

Definition 1.29 (Transpose)**.** [31] The inverse R^t of R is the fuzzy relation from Y to X defined by

$$R^t = \{< (x,y), \mu_{R^t}(x,y) > |(x,y) \in X \times Y\}.$$

Definition 1.30 (The intersection)**.** [31] The intersection of two fuzzy relation R and P is defined as

$$R \cap P = \{< (x,y), min(\mu_R(x,y), \mu_P(x,y) > |(x,y) \in X \times Y\}.$$

Definition 1.31 (Union)**.** [31] The union of two fuzzy relation R and P is defined as

$$R \cup P = \{< (x,y), max(\mu_R(x,y), \mu_P(x,y) > |(x,y) \in X \times Y\}.$$

Example 1.10. *Let R and P two fuzzy relations in $X \times X \to [0,1]$, such that $X = \{x,y,z\}$.*

R	x	y	z
x	1	0.7	0.4
y	0.5	1	0.2
z	0.1	0.6	1

P	x	y	z
x	1	0.3	0
y	0.7	0.2	1
z	0.3	0	0.4

Then

$R \cup P$	x	y	z
x	1	0.7	0.4
y	0.7	1	1
z	0.3	0.6	1

$R \cap P$	x	y	z
x	1	0.3	0
y	0.5	0.2	0.2
z	0.1	0	0.4

Proposition 1.1. [30, 31] Let R, P and Q be three fuzzy relation from a universe X to a universe Y

(1) If $R \subset P$, then $R^t \subset P^t$;

(2) $(R \cup P)^t = R^t \cup P^t$;

(3) $(R \cap P)^t = R^t \cap P^t$;

(4) $(R^t)^t = R$;

(5) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ and $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

(6) $R \cup P \supseteq R, R \cup P \supseteq P, R \cup P \subseteq R, R \cup P \subseteq P$;

(7) If $R \supseteq P$ and $R \supseteq Q$, then $R \supseteq P \cup Q$;

(8) If $R \subseteq P$ and $R \subseteq Q$, then$R \subseteq P \cap Q$.

Proof.

1 - If $R \subset P$, then $\mu_R(x,y) \leq \mu_P(x,y)$ for all $(x,y) \in X^2$, $\mu_R(y,x) \leq \mu_P(y,x)$. Hence $R^t \subset P^t$;

2-

$$\begin{aligned}(R \cup P)^t &= ((R \cup P)(x,y))^t = (R \cup P(y,x)\\ &= (R(y,x) \vee P)(y,x)) = (R(x,y))^t \vee (P(x,y))^t\\ &= R^t \cup P^t;\end{aligned}$$

3- $(R^t)^t = (R(x,y)^t)^t = R(y,x)^t = R(x,y) = R.$ □

Definition 1.32. [30, 31] Let R be a fuzzy relation. Then

(i) Reflexivity : $R(x,x) = 1$, for any $x \in X$;

(ii) Symmetry : for any $x, y \in X$ $R(x,y) = R(y,x)$;

(iii) Antisymemety: for any $x, y \in X$, if $x \neq y$ then $R(x,y) = 0$ *or* $R(y,x) = 0$;

(iv) Transitivity : if $R(x,y) \geq \max\{\min R(x,z), R(z,y)\}$ for all $x, y, z \in X$.

Definition 1.33. *Let X be a non empty crisp set and R be an fuzzy order or partial fuzzy order if it is reflexive, antisymmetric and transitive.*
A non empty set X with a fuzzy order R defined on it is called a fuzzy ordered set and is denoted by (X, μ_R). It easily follows that each partially ordered set $(X, \leq)$ and each fuzzy ordered set (X, R) can be viewed as fuzzy ordered sets.

Example 1.11. *Let $X = \{1,2,3,4,5\}$, the fuzzy relation R defined on $X \times X$ by*

$$R = \{< (x,y), \mu_R(x,y) > |(x,y) \in X\}.$$

R(x,y)	*1*	*2*	*3*	*4*	*5*
1	*1*	*0*	*0*	*0.55*	*0.4*
2	*0*	*1*	*0*	*0.35*	*0.45*
3	*0*	*0*	*1*	*0*	*0.7*
4	*0*	*0*	*0*	*1*	*0*
5	*0*	*0*	*0*	*0*	*1*

Then, R is fuzzy order relation.

Example 1.12. *Let $m, n \in \mathbb{N}$. Then the following fuzzy relation R on $\mathbb{N}$ is a fuzzy order, where*

$$R(m,n) = \begin{cases} 1, \; if \; m = n; \\ 1 - \frac{m}{n} \;, \quad if \; m < n; \\ 0 \;, \quad if m > n. \end{cases}$$

Definition 1.34. *A fuzzy order R on a universe X is called complete (or total) if for any $x, y \in X$ it holds that*

$$R(x,y) > 0 \quad or \quad R(y,x) > 0.$$

Definition 1.35. *A fuzzy ordered set (X, μ_R) in which R is linear is called a linearly fuzzy ordred set or a fuzzy chain.*

Chapter 2

Fuzzy ideals and filters on topology generated by fuzzy relations

The purpose of this chapter is to provide to fuzzy topology generated by fuzzy relation. Also, we study the lattice structure of fuzzy open sets on this topology. To that end, we investigate the notion of fuzzy ideal (resp. fuzzy filter) on topology generated by fuzzy relation.

2.1 Fuzzy topology generated by fuzzy relation

In this section, we recall the concept of fuzzy topology given by Chang [5]. Moreover, we recall the concept of fuzzy topology generated by fuzzy relation given by Mishra and Srivastava [23] and some basic properties.

Definition 2.1 (Fuzzy topology)**.** [5] *A fuzzy topology (FT, for short) on a nonempty set X is a family τ of fuzzy sets on X which satisfies the following axioms:*

(i) $\emptyset, X \in \tau$ *;*

(ii) $G_1 \cap G_2 \in \tau$ *for any* $G_1, G_2 \in \tau$*;*

(iii) $\bigcup G_i \in \tau$ *for any* $\{G_i : i \in J\} \subseteq \tau$.

In this case, the pair (X, τ) is called a fuzzy topological space (FTS, for short) and any FS in τ is known as a fuzzy open set (FOS, for short) in X. The complement of a fuzzy open set is called a fuzzy closed set (FCS, for short) in X.

Example 2.1. *Let $X = \{x, y, z\}$ and $A, B, C \in FS(X)$ such that*

$A = \{\langle x, 0.6\rangle, \langle y, 0.9\rangle, \langle z, 0.4\rangle\}$;

$B = \{\langle x, 0.5\rangle, \langle y, 0.8\rangle, \langle z, 0.4\rangle\}$;

$C = \{\langle x, 0.4\rangle, \langle y, 0.7\rangle, \langle z, 0.4\rangle\}$;

Then, $\tau = \{\emptyset, X, A, B, C\}$ is a fuzzy topology on X.

The notion of fuzzy topology generated by fuzzy relation was previously proposed by Mishra and Srivastava [23].

Definition 2.2. [19]Let X be a nonempty crisp set and $R = \{\langle (x, y), \mu_R(x, y)\rangle \mid x, y \in X\}$, be an fuzzy relation on X. Then for any $x \in X$, the fuzzy sets $\mathcal{L}_x$ and $\mathcal{R}_x$ are defined by
$\mu_{\mathcal{L}_x}(y) = \mu_R(y, x)$, for any $y \in X$;
$\mu_{\mathcal{R}_x}(y) = \mu_R(x, y)$, for any $y \in X$.
They are called the lower and the upper contour, respectively, of x.

We denote by τ_1, the fuzzy topology generated by the set of all lower contours and τ_2, the fuzzy topology generated by the set of all upper contours. Consequently, we denote by τ_R, the fuzzy topology generated by S the set of all lower and upper contours and it's called the fuzzy topology generated by R.

Definition 2.3. [19] Let R be a fuzzy relation on the set X and τ_R is the fuzzy topologies generated by R and let U_1, U_2 are two fuzzy open sets on τ_R. The U_1 is said to be contained in U_2 (in symbols, $U_1 \sqsubseteq U_2$) if $\mu_{U_1}(x_i) \leq \mu_{U_2}(x_i)$ for any $x_i \in X$.
In this case, we also say that U_1 is smaller than U_2.

Example 2.2. *Let $X = \{x, y\}$ and R be a fuzzy relation on X given by*

Table 2.1: fuzzy relation for Example 2.2

R	x	y
x	0.3	0.9
y	0.2	0.7

Then, $\mathcal{L}_x$, $\mathcal{L}_y$, $\mathcal{R}_x$ *and* $\mathcal{R}_y$ *are the fuzzy sets on* X *given by :*

$$\begin{aligned}\mathcal{L}_x &= \{\langle x, 0.3\rangle; \langle y, 0.2\rangle\};\\ \mathcal{L}_y &= \{\langle x, 0.9\rangle; \langle y, 0.7\rangle\};\\ \mathcal{R}_x &= \{\langle x, 0.3\rangle; \langle y, 0.9\rangle\};\\ \mathcal{R}_y &= \{\langle x, 0.2\rangle; \langle y, 0.7\rangle\}.\end{aligned}$$

We note that, $\mathcal{R}_y \subset \mathcal{R}_x$ *and* $\mathcal{R}_y \subset \mathcal{L}_y$. *Then the fuzzy topology* $\tau_{\mathcal{R}}$ *is generated by* $S = \{\mathcal{L}_x, \mathcal{L}_y\} \cup \{\mathcal{R}_x, \mathcal{R}_y\}$. *Thus,* $\tau_{\mathcal{R}} = \{\emptyset, X, \mathcal{L}_x, \mathcal{L}_y, \mathcal{R}_x, \mathcal{R}_y, \mathcal{L}_x \cap \mathcal{R}_y, \mathcal{L}_y \cap \mathcal{R}_x, \mathcal{L}_x \cup \mathcal{R}_y, \mathcal{L}_y \cup R_x\}$, *where*
$\mathcal{L}_x \cap \mathcal{R}_y = \{\langle x, 0.2\rangle; \langle y, 0.2\rangle\}$, $\mathcal{L}_y \cap \mathcal{R}_x = \{\langle x, 0.3\rangle; \langle y, 0.7\rangle\}$;
$\mathcal{L}_x \cup \mathcal{R}_y = \{\langle x, 0.3\rangle; \langle y, 0.7\rangle\}$, *and* $\mathcal{L}_y \cup \mathcal{R}_x = \{\langle x, 0.9\rangle; \langle y, 0.9\rangle\}$.

Example 2.3. *Let* $X = \{x, y, z\}$ *and* R *be a fuzzy relation on* X *given by*

R	x	y	z
x	1	0.3	0
y	0	1	0.9
z	0.6	0	1

Then, L_x, L_y, L_z, R_x, R_y *and* R_z *are the fuzzy sets on* X *given by :*

$$\begin{aligned}L_x &= \{\langle x, 1\rangle; \langle y, 0\rangle; \langle z, 0.6\rangle\}\ ;\\ L_y &= \{\langle x, 0.3\rangle; \langle y, 1\rangle; \langle z, 0\rangle\}\ ;\\ L_z &= \{\langle x, 0\rangle; \langle y, 0.9\rangle; \langle z, 1\rangle\}\ ;\\ R_x &= \{\langle x, 1\rangle; \langle y, 0.3\rangle; \langle z, 0\rangle\}\ ;\\ R_y &= \{\langle x, 0\rangle; \langle y, 1\rangle; \langle z, 0.9\rangle\}\ ;\\ R_z &= \{\langle x, 0.6\rangle; \langle y, 0\rangle; \langle z, 1\rangle\}.\end{aligned}$$

The fuzzy topology $\tau_{\mathcal{R}}$ is generated by $S = \{L_x, L_y, L_z\} \cup \{R_x, R_y, R_z\}$. Thus,

$$
\begin{aligned}
\tau_{\mathcal{R}} \;=\; & \Big\{\emptyset, X, L_x, L_y, L_z, R_x, R_y, R_z, \{\langle x,0.3\rangle;\langle y,0\rangle;\langle z,0\rangle\}, \{\langle x,0\rangle;\langle y,0\rangle;\langle z,0.6\rangle\} \\
& \{\langle x,1\rangle;\langle y,0\rangle;\langle z,0\rangle\}, \{\langle x,0.6\rangle;\langle y,0\rangle;\langle z,0.6\rangle\}, \\
& \{\langle x,0\rangle;\langle y,0.9\rangle;\langle z,0\rangle\}, \{\langle x,0.3\rangle;\langle y,0.3\rangle;\langle z,0\rangle\}, \{\langle x,0\rangle;\langle y,1\rangle;\langle z,0\rangle\}, \\
& \{\langle x,0\rangle;\langle y,0.9\rangle;\langle z,0.9\rangle\}, \\
& \{\langle x,0\rangle;\langle y,0\rangle;\langle z,1\rangle\}, \{\langle x,0\rangle;\langle y,0.3\rangle;\langle z,0\rangle\}, \{\langle x,0\rangle;\langle y,0\rangle;\langle z,0.9\rangle\}, \\
& \{\langle x,0.6\rangle;\langle y,0\rangle;\langle z,0\rangle\}, \{\langle x,1\rangle;\langle y,1\rangle;\langle z,0.6\rangle\} \\
& \{\langle x,1\rangle;\langle y,0.9\rangle;\langle z,1\rangle\}, \{\langle x,1\rangle;\langle y,0.3\rangle;\langle z,0.6\rangle\}, \{\langle x,1\rangle;\langle y,1\rangle;\langle z,0.9\rangle\}, \\
& \{\langle x,1\rangle;\langle y,0\rangle;\langle z,1\rangle\}, \{\langle x,0.3\rangle;\langle y,1\rangle;\langle z,1\rangle\}, \{\langle x,1\rangle;\langle y,1\rangle;\langle z,0\rangle\}, \\
& \{\langle x,0.3\rangle;\langle y,1\rangle;\langle z,0.9\rangle\}, \{\langle x,0.6\rangle;\langle y,1\rangle;\langle z,1\rangle\}, \\
& \{\langle x,0\rangle;\langle y,1\rangle;\langle z,1\rangle\}, \{\langle x,0.6\rangle;\langle y,0.9\rangle;\langle z,1\rangle\}, \{\langle x,1\rangle;\langle y,1\rangle;\langle z,0.9\rangle\}, \\
& \{\langle x,1\rangle;\langle y,0.3\rangle;\langle z,1\rangle\}\Big\}.
\end{aligned}
$$

In next proposition, we studied fuzzy topology generated by symmetric fuzzy relation.

Proposition 2.1. [23]If R is a symmetric fuzzy relation, then $\tau_1 = \tau_2$.

Proof. Since R is a symmetric fuzzy relation, so $R(x,y) = R(y,x)$, for each $x, y \in X$. This implies that $R_x(y) = L_x(y)$, for each $x, y \in X$ and hence $R_x = L_x$, for each $x \in X$. Thus the topologies τ_1 and τ_2, which are generated by $L_x : x \in X$ and $R_x : x \in X$, respectively, are same. □

After that, we studied fuzzy topology generated by fuzzy preorder relation.

Proposition 2.2. [23] If R is a fuzzy preorder relation, then

(1) If $A \in \tau_1$, then $A \supseteq \bigcup\limits_{x:A(x)=1} R_x$;

(2) If $A \in \tau_2$, then $A \supseteq \bigcup\limits_{x:A(x)=1} L_x$.

Proof.

(1) To show that $A \supseteq \bigcup\limits_{x:A(x)=1} L_x$, let $y \in \bigcup\limits_{x:A(x)=1} L_x$. This implies that there exists some x such that $A(x) = 1$ and $y_r \in L_x$. So $r < R(y,x)$. Now, since A is open and $A(x) = 1$, so $x_r \in A$ and there exists a basic fuzzy open set $\bigcap_{i=1}^{n} L_{x_i}$ such that $x_r \in \bigcap_{i=1}^{n} L_{x_i} \subseteq A$
$\Rightarrow r < R(x, xi)$, for each $i = 1, 2, ..., n$.

$\Rightarrow r < minR(y,x), R(x,xi) \leq R(y,xi), foreachi = 1,2,...,n$

$\Rightarrow y_r \in L_{x_i} foreachi = 1,2,...,n$

$\Rightarrow y_r \in \bigcap_{i=1}^{n} L_{x_i} \subseteq A$

$\Rightarrow y_r \in A$

$\Rightarrow A \supseteq \bigcup_{x:A(x)=1} L_x.$

(2) The proof is similar to that of part 1. □

In this theorem, we provide $U_y(x) = V_x(y)$ in fuzzy topological space.

Theorem 2.1. [23] Let (X, τ) be a fuzzy topological space. The fuzzy topology τ is generated by a fuzzy relation R if and only if it has a subbase $U_x, V_x : x \in X$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$.

Proof. First assume that τ is generated by some fuzzy relation R, then obviously it has a subbase $U_x, V_x : x \in X$ where $U_x = L_x$ and $V_x = R_x$, for each $x \in X$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$. Conversely, assume that τ has a subbase $U_x, V_x : x \in X$ such that $U_y(x) = V_x(y)$, for each $x, y \in X$. Now to show that τ is generated by some fuzzy relation R, define a fuzzy relation $R : X \times X \to I$ by $R(x,y) = U_y(x) = V_x(y)$, for each $(x,y) \in X \times X$. Then for $x \in X$, $L_x(y) = R(y,x) = U_x(y)$ and $R_x(y) = R(x,y) = V_x(y)$,for each $y \in X$ which implies that $L_x = U_x$ and $R_x = V_x$, for each $x \in X$. So from the hypothesis of the theorem, we have that the family $L_x, R_x : x \in X$ is a subbase for τ . Hence τ is generated by the fuzzy relation R. □

2.2 Lattice of fuzzy open sets on fuzzy topology generated by fuzzy relation

In this section, we study the lattice structure of fuzzy open sets on topology generated by fuzzy relation.

Definition 2.4. [19] Let τ_R the fuzzy topology on the set X generated by R and let U_1 ,U_2 are two fuzzy open sets on τ_R . The intersection of U_1 and U_2 (in symbols, $U_1 \sqcap U_2$) is a fuzzy open set V such that $\mu_V(x_i) = min(\mu_{U_1}(x_i), \mu_{U_2}(x_i))$ for any $x_i \in X$.
Furthermore, $\sqcap_{i \in I} U_i$ is the smallest fuzzy open set on X containing all U_i.

Next, we give the union of fuzzy open sets on fuzzy topology generated by fuzzy relation.

Definition 2.5. [19] Let τ_R the fuzzy topology on the set X generated by R and let U_1 ,U_2 are two fuzzy open sets on τ_R . The union of U_1 and U_2 (in symbols, $U_1 \sqcup U_2$) is an fuzzy open set V such that $\mu_V(x_i) = max(\mu_{U_1}(x_i), \mu_{U_2}(x_i))$ for any $x_i \in X$.
Furthermore, $\bigsqcup_{i\in I} U_i$ is greater fuzzy open set on X containing all U_i.

The following theorem provides the lattice structure of fuzzy open sets on a fuzzy topology generated by fuzzy relation.

Theorem 2.2. [19] Let X be a finite set and τ_R be a fuzzy topology generated by fuzzy relation R and let U_i be a fuzzy open sets on τ_R. Then the family

$$\mathfrak{L} = \{U_i \mid U_i \text{ is a fuzzy open sets on } \tau_R\},$$

is a lattice on X

Proof. Suppose that $\{U_i\}$ is a set of fuzzy open sets on τ_R. Definition of fuzzy topology guaranties that $\{U_i\}$ is a nonempty set. Now, let U_1 and U_2 be two fuzzy open sets. It is easy to cheek that $U_1 \sqsubseteq U_1$ i.e., the fuzzy reflexivity, and if we suppose that $U_1 \sqsubseteq U_2$ and $U_2 \sqsubseteq U_1$, it follows that $U_1 = U_2$ i.e., the fuzzy-antisymmety. In order to verify the fuzzy transitivity, we suppose that $U_1 \sqsubseteq U_2$ and $U_2 \sqsubseteq U_3$, it follows that $U_1 \sqsubseteq U_3$ i.e., the fuzzy-transitivity. Hence, $(\mathfrak{L}, \sqsubseteq)$ is a fuzzy poset on X. Moreover, the least upper bound (resp. the greatest lower bound) of U_1 and U_2 is coincides with the intersection of fuzzy open sets (resp. the union of fuzzy open sets) i.e., $U_1 \wedge U_2 = U_1 \sqcap U_2$ (resp. $U_1 \vee U_2 = U_1 \sqcup U_2$). Thus, we can conclude that $(\mathfrak{L}, \sqsubseteq)$ is a lattice on X. □

Proposition 2.3. [19] Let X be a finite set and $\mathfrak{L} = \{U_i\}$, is the lattice of all fuzzy open sets on fuzzy topology τ_R generated by fuzzy relation R. Then $\mathfrak{L}$ is complete.

Proof. Let $\mathfrak{L} = \{U_i\}$ be the lattice of fuzzy open sets on fuzzy topology τ_R generated by fuzzy relations R. Suppose that $A = \{U_j\}$ is a subset of $\mathfrak{L}$ under the fuzzy inclusion between fuzzy open sets defined above. The fact that $\mathfrak{L}$ is a finite lattice, this implies that $\sqcap U_j \in \mathfrak{L}$ which implies that A has an infimum. Hence, $\mathfrak{L}$ is complete. □

Corollaire 2.1. [19] Let $\mathfrak{L}$ be the complete lattice of all fuzzy open sets on fuzzy topology generated by fuzzy relation, then $\mathfrak{L}$ is bounded. Indeed, the least element of $\mathfrak{L}$ is $0_{\mathfrak{L}} = \emptyset = \sqcap U_i$ and the greatest element of $\mathfrak{L}$ is $1_{\mathfrak{L}} = X = \sqcup U_i$.

The following proposition discuss the distributivity property for the lattice of all fuzzy open sets on fuzzy topology generated by fuzzy relation.

Proposition 2.4. [19] Let $\mathfrak{L}$ be the lattice of fuzzy open sets on fuzzy topology τ_R generated by fuzzy relation R, then $\mathfrak{L}$ is distributive.

Proof. Suppose that U_1, U_2 and U_3 are a fuzzy open sets on fuzzy topology generated by fuzzy relation R and. We will show that $U_1 \sqcap (U_2 \sqcup U_3) = (U_1 \sqcap U_2) \sqcup (U_1 \sqcap U_3)$. We have

$$\begin{aligned} \mu_{U_1 \sqcap (U_2 \sqcup U_3)}(x) &= \mu_{U_1}(x) \wedge (\mu_{U_2}(x) \vee \mu_{U_3(x)}), \\ &= (\mu_{U_1}(x) \wedge \mu_{U_2}(x)) \vee (\mu_{U_1}(x) \wedge \mu_{U_3}(x)), \\ &= \mu_{(U_1 \sqcap U_2)}(x) \vee \mu_{(U_1 \sqcap U_3)}(x), \\ &= \mu_{(U_1 \sqcap U_2) \sqcup (U_1 \sqcap U_3)}(x). \end{aligned}$$

Hence, $U_1 \sqcap (U_2 \sqcup U_3) = (U_1 \sqcap U_2) \sqcup (U_1 \sqcap U_3)$. We conclude that $\mathfrak{L}$ is a distributive lattice. □

Corollaire 2.2. [19] Since $\mathfrak{L}$ is a distributive lattice, then it holds that $\mathfrak{L}$ is modular.

Proof. [19] Let U_1, U_2 and U_3 be a fuzzy open sets on τ_R. Since $\mathfrak{L}$ is a distributive lattice, then it follows that $U_1 \sqcup (U_2 \sqcap U_3) = (U_1 \sqcup U_2) \sqcap (U_1 \sqcup U_3)$. Since $U_1 \sqsubseteq U_3$, then it holds that $U_1 \sqcup U_3 = U_3$. This implies that $U_1 \sqcup (U_2 \sqcap U_3) = (U_1 \sqcup U_2) \sqcap U_3$. Hence, $\mathfrak{L}$ is modular. □

In 1958, Hartmanis proved that the lattice of all topologies on a finite set is complemented. In the following proposition, we prove that the lattice of fuzzy open sets on a topology generated by fuzzy relation is also complemented.

Proposition 2.5. [19] Let $\mathfrak{L}$ be the lattice of open fuzzy sets on fuzzy topology τ_R generated by the fuzzy relations R, then $\mathfrak{L}$ is complemented.

Proof. Indeed, every element U_{i_0} has a complement U_{j_0} such that $U_{i_0} \sqcap U_{j_0} = 0_{\mathfrak{L}}$ and $U_{i_0} \sqcup U_{j_0} = 1_{\mathfrak{L}}$. Hence, $\mathfrak{L}$ is complemented. □

Corollaire 2.3. [19] Since $\mathfrak{L}$ is a distributive lattice and complemented with a least element $0_{\mathfrak{L}} = \emptyset$ and a greatest element $1_{\mathfrak{L}=X}$, then $\mathfrak{L}$ is a boolean algebra denoted by $(\mathfrak{L}, \sqcap, \sqcup, 0_{\mathfrak{L}}, 1_{\mathfrak{L}})$.

Proof. The proof is directly from Proposition 2.4 and Proposition 2.5. □

2.3 Fuzzy ideal and filter on topology generated by fuzzy relation

This section contains the basic definitions and properties of fuzzy ideals and filters on the lattice of fuzzy open sets.

2.3.1 Definitions

Definition 2.6. [19] Let $\mathfrak{L}$ be the lattice of fuzzy open sets on fuzzy topology τ_R generated by the fuzzy relation R. A fuzzy subset I on $\mathfrak{L}$ is called a fuzzy ideal if for all $A, B \in \mathfrak{L}$ the following conditions hold:

(i) $\mu_I(A \sqcup B) \geq \mu_I(A) \wedge \mu_I(B)$;

(ii) $\mu_I(A \sqcap B) \geq \mu_I(A) \vee \mu_I(B)$.

Definition 2.7. [19] Let $\mathfrak{L}$ be the lattice of fuzzy open sets on fuzzy topology τ_R generated by the fuzzy relation R. A fuzzy subset F on $\mathfrak{L}$ is called a fuzzy filter if for all $A, B \in \mathfrak{L}$ the following conditions hold:

(i) $\mu_F(A \sqcup B) \geq \mu_F(A) \vee \mu_F(B)$;

(ii) $\mu_F(A \sqcap B) \geq \mu_F(A) \wedge \mu_F(B)$.

The following proposition expresses the relationship between a fuzzy ideal and a fuzzy filter on a lattice of fuzzy open sets. Its proof is straightforward.

Proposition 2.6. [19] Let $\mathfrak{L}$ be the lattice of fuzzy open sets, $\mathfrak{L}^d$ be its order-dual lattice and $A \in S(\mathfrak{L})$. Then it holds that A is a fuzzy ideal on $\mathfrak{L}$ if and only if A is a fuzzy filter on $\mathfrak{L}^d$ and conversely.

We need also the following result.

Proposition 2.7. [19] Let $\mathfrak{L}$ be the lattice of fuzzy open sets, A and B are two fuzzy sets on $\mathfrak{L}$. Then it holds that

(1) If A and B are two fuzzy ideals on $\mathfrak{L}$, then $A \cap B$ is a fuzzy ideal on $\mathfrak{L}$;

(2) If A and B are two fuzzy filters on $\mathfrak{L}$, then $A \cap B$ is a fuzzy filter on $\mathfrak{L}$.

Proof. (1) I and J are two fuzzy ideals on $\mathfrak{L}$, then

(i)

$$\begin{aligned}\mu_{I\cap J}(A \sqcup B) &= \mu_I(A \sqcup B) \wedge \mu_J(A \sqcup B)\\ &\geq (\mu_I(A) \wedge \mu_I(B)) \wedge (\geq \mu_J(A) \wedge \mu_J(B))\\ &=\geq (\mu_I(A) \wedge \geq \mu_J(A)) \wedge (\mu_I(B) \wedge \mu_J(B))\\ &= \mu_{I\cap J}(A) \wedge \mu_{I\cap J}(B);\end{aligned}$$

(ii)

$$\begin{aligned}\mu_{I\cap J}(A \sqcap B) &= \mu_I(A \sqcap B) \wedge \mu_J(A \sqcap B)\\ &\geq (\mu_I(A) \vee \mu_I(B)) \wedge (\mu_J(A) \vee \mu_J(B))\\ &\geq (\mu_I(A) \wedge \mu_J(A)) \vee (\mu_I(B) \wedge \mu_J(B))\\ &= \mu_{I\cap J}(A) \vee \mu_{I\cap J}(B)\\ &= \mu_{I\cap J}(A) \vee \mu_{I\cap J}(B).\end{aligned}$$

Then, $A \cap B$ is a fuzzy ideal on $\mathfrak{L}$.

(2) Same way for intersection two fuzzy filters.

□

2.3.2 Basic characterization of fuzzy ideals and filters on a lattice of fuzzy open sets

In this subsection, we provide interesting characterization of fuzzy ideals (resp. filters) on the lattice of fuzzy open sets in terms of its meet and its join operations.

Theorem 2.3. [19] Let $\mathfrak{L}$ be the lattice of fuzzy open sets. Then it holds that I is a fuzzy ideal on $\mathfrak{L}$ if and only if the following condition is satisfied:

$$\mu_I(A \sqcup B) = \mu_I(A) \wedge \mu_I(B), \text{ for any } A, B \in \mathfrak{L}\,.$$

Proof. Suppose that I is a fuzzy ideal on $\mathfrak{L}$, then for any $A, B \in \mathfrak{L}$ it holds that $\mu_I(A \sqcup B) \geq \mu_I(A) \wedge \mu_I(B)$. Since $A \sqsubseteq A \sqcup B$ and $B \sqsubseteq A \sqcup B$, it follows from Definition 2.6 (ii) that

$$\mu_I(A) = \mu_I(A \sqcap (A \sqcup B)) \geq \mu_I(A) \vee \mu_I(A \sqcup B) \geq \mu_I(A \sqcup B)\,.$$

In the same manner, $\mu_I(B) \geq \mu_I(A \sqcup B)$. Hence, $\mu_I(A) \wedge \mu_I(B) \geq \mu_I(A \sqcup B)$. Thus, $\mu_I(A \sqcup B) = \mu_I(A) \wedge \mu_I(B)$.

Conversely, suppose that $\mu_I(A \sqcup B) = \mu_I(A) \wedge \mu_I(B)$ for any $A, B \in \mathfrak{L}$. Then it is easy to see that $\mu_I(A \sqcup B) \geq \mu_I(A) \wedge \mu_I(B)$ for any $A, B \in \mathfrak{L}$. Next, we will show that $\mu_I(A \sqcap B) \geq \mu_I(A) \vee \mu_I(B)$ for any $A, B \in \mathfrak{L}$. Let $A, B \in \mathfrak{L}$, since $A \sqcup (A \sqcap B) = A$ and $B \sqcup (A \sqcap B) = B$ then it holds that $\mu_I(A \sqcup (A \sqcap B)) = \mu_I(A)$ and $\mu_I(B \sqcup (A \sqcap B)) = \mu_I(B)$. From hypothesis it follows that $\mu_I(A) \wedge \mu_I(A \sqcap B) = \mu_I(A)$ and $\mu_I(B) \wedge \mu_I(A \sqcap B) = \mu_I(B)$. Hence, $\mu_I(A \sqcap B) \geq \mu_I(A)$ and $\mu_I(A \sqcap B) \geq \mu_I(B)$. Thus, $\mu_I(A \sqcap B) \geq \mu_I(A) \vee \mu_I(B)$, for any $A, B \in \mathfrak{L}$. Therefore, I is a fuzzy ideal on $\mathfrak{L}$. □

In the same manner, the following theorem provides a characterization of fuzzy filters on the lattice of fuzzy open sets in terms of its meet operation.

Theorem 2.4. [19] Let $\mathfrak{L}$ be the lattice of fuzzy open sets. Then it holds that F is a fuzzy filter on $\mathfrak{L}$ if and only if the following condition is satisfied:

$$\mu_F(A \sqcap B) = \mu_F(A) \wedge \mu_F(B), \text{ for any } A, B \in \mathfrak{L}.$$

Proof. The proof is a direct application of Proposition 2.6 and Theorem 2.3. □

As corollaries of the above theorems, we obtain the following interesting properties of fuzzy ideals and fuzzy filters on a lattice of fuzzy open sets.

Corollaire 2.4. [19] Let $\mathfrak{L}$ be the lattice of fuzzy open sets and I be a fuzzy ideal on $\mathfrak{L}$. Then for any $A, B \in \mathfrak{L}$ it holds that
If $A \sqsubseteq B$, then $\mu_I(A) \geq \mu_I(B)$, (i.e., the mapping μ_I is antitone).

Corollaire 2.5. [19] Let $\mathfrak{L}$ be the lattice of fuzzy open sets and F be a fuzzy filter on $\mathfrak{L}$. Then for any $A, B \in \mathfrak{L}$ it holds that
If $A \sqsubseteq B$, then $\mu_F(A) \leq \mu_F(B)$, (i.e., the mapping μ_F is monotone).

Conclusion

In this memoir, we have studied a particular subsets which are the fuzzy ideals and fuzzy filters on a lattice of open sets in topology generated by fuzzy relation. Also, we have studied some characterizations of these ideals and filters with some related results.

Due to the usefulness of these notions, we think it makes sense to study some kinds of fuzzy ideals (resp. fuzzy filters) on fuzzy topology generated by fuzzy relation.

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ملخص

في هذه المذكرة، قمنا بدراسة مجموعات جزئية وهي المثاليات والمرشحات داخل الشبكةالمكونة من المفتوحات لطوبولوجيا المولدة بعلاقة ضبابية.

كما قدمنا مميزات هذه المجموعات الجزئية مع بعض النتائج الخاصة منها.

الكلمات مفتاحية

العلاقات الضبابية، الطوبولوجيا الضبابية، الشبكة، المثالي الضبابي، المرشح الضبابي.

Abstract

In this memoire, we study a particular subsets which are the ideals and filters on a lattice of open sets in topology generated by fuzzy relation.

Moreover, we study characterizations of these particular subsets with some related results.

Key words

Fuzzy relation, Fuzzy topology, Lattice, Fuzzy ideal, Fuzzy filter.

Résumé

Dans ce mémoire, nous étudions des sous-ensembles particuliers qui sont les idéaux et les filtres sur un treillis d'ensembles ouverts en topologie générés par une relation floue.

De plus, nous étudions des caractérisations de ces sous-ensembles particuliers avec des résultats connexes.

Mot-clés

Relation floue, Topologie floue, Treillis, Ideal floue, Filter floue,