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Filters and ideals in implicative semigroups

Persented by :
M^s BELOUADAH Samia

M^s KHADRAOUI Naima Safa

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in front of the jury :

AMROUNE Abdelaziz	Prof.,	University of M'sila	**Chairperson**.
OUMHANI Ali	M.C.A,	ENS Bousaada	**supervisor**.
ZIANE Brahim	M.C.B,	ENS Bousaada	**Examinator**.

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BELOUADAH Samia and KHADRAOUI Naima Safa

Contents

List of Symbols

Notation	Name
$.$	A binary operation.
$*$	Implication.
$\circ$	The composition operation.
$\leq$	A binary ordered relation.
$\geq$	A binary dual ordered relation.
$\wedge$	The greatest lower bound.
$\vee$	The least upper bound.
$\cup$	The union operation.
$\cap$	The intersection operation.
$\subseteq$	The subset operation.
$\langle S\rangle$	The ordered filter generated by a the ordered filter generated by S .
$\langle\{a\}\rangle$	The ordered filter generated by $\{a\}$.
ϕ	The empty set.

Introduction

The idea of implicative semigroups introduced by Chan and Shum in [3] that is a generalization of implicative semilattices which studied by W.C. Nemitz in [12]. By a negatively ordered semigroup, we mean a partially ordered semigroup S such that $x.y \leq x$ and $x.y \leq y$ for all $x, y \in S$. A negatively ordered semigroup S endowed with an additional binary operation $"* : S \times S \to S''$ such that for any elements x, y, z in S: $z \leq x * y$ if and only if $z \cdot x \leq y$ is called a negatively ordered implicative semigroup and the operation $*$ is called an implication. For the general development of implicative semilattice theory the ordered filters play an important role which is shown by Nemitz [12]. Motivated by this, Chan and Shum [3] established some elementary properties. Jun, Meng, and Xin [9] discussed ordered filters of implicative semigroups. Also Jun [8] stated implicative ordered filters of implicative semigroups, and introduced the notion of positive implicative ordered filters in implicative semigroups. He show that every positive implicative ordered filter is both an ordered filter and an implicative ordered filter. Also, examples that an ordered filter (an implicative ordered filter) may not be a positive implicative ordered filter are given. Jun et al in [9], have studied ordered filters of commutative implicative semigroups. And study how to generate an ordered filter by a set. Following the idea of general lattice theory, He have introduced the notions of prime ordered filters. In [7] Jun and Kim have introduced the notion of ideals in implicative semigroups. By introducing special subsets of an implicative semigroups, and provided a condition for the special subset to be an ideal. Moreover, two characterizations of ideals have established. In [3] Chan and Shum have studied homomorphisms between implicative negatively partially ordered semigroups.

This work is structured as follows. In Chapter 1, we provide a basic introduction to negatively ordered implicative semigroup. And, we recall some definitions, basic concepts and their fundamental properties, such as negatively ordered implicative semigroup, semilattice, lattice, ordered filter, positive implicative ordered filter, and we give some examples. In Chapter 2, we discuss some characterizations on positive implicative ordered filter. At the end of this Chapter, we will give the notion of filters generated by a subset of an implicative semigroup and some properties are given. Chapter 3, focus to the idea of ideals in implicative semigroups, and then state the characterizations of the ideals. Finally, we study the notion of homomorphisms between implicative semigroups.

Chapter 1

Basic notions

In this short chapter, we will briefly review some basic notations, definitions and propositions. The presentations of these topics is intentionally brief for two reasons, the first is that the readers are likely familiar with these topics, and second, we include only the necessary materiel needed to start working on this subject.

1.1 Basic Definitions

Definition 1.1 *A semigroup is a pair* $(S, \diamond)$ *in which* S *is a non-empty set and* "$\diamond$" *is a binary associative operation on* S*. i.e., the equation* $(x \diamond y) \diamond z = x \diamond (y \diamond z)$ *holds for all* $x, y, z \in S$*.*

Example 1.2 $(\mathbb{N}, +)$ *is a semigroup*

1. *for all* $x, y, z \in \mathbb{N}, \ x + y \in \mathbb{N}$*,*
2. *for all* $x, y, z \in \mathbb{N}, \ (x + y) + z = x + (y + z)$*.*

Definition 1.3 *A group is a set* G *with a binary operation*

$$\diamond : G \times G \to G$$

$$(a, b) \mapsto a \diamond b$$

Satisfying the following conditions:

1. *(associativity) For all* $a, b, c \in G$*,* $(a \diamond b) \diamond c = a \diamond (b \diamond c)$

2. *(existence of a neutral element) There exists an elemnt* $e \in G$ *such that* $a \diamond e = a = e \diamond a$*, for all* $a \in G$*,*
3. *(existence of inverses) For each* $a \in G$*, there exists an* $a' \in G$ *such that* $a \diamond a' = e = a' \diamond a$

Example 1.4 $(\mathbb{Z}, +), (\mathbb{C}, +), (\mathbb{R}, +)$ *are groups.*

Definition 1.5 *A subsemigroup of* $(S, \diamond)$ *is a non-empty subset* T *of* S *which is closed under the multiplication of* S*, i.e., it satisfies* $T \diamond T \subseteq T$*. In other words,* T *is a semigroup under the multiplication of* S *restricted to* T*.*

Example 1.6 $(2\mathbb{N}, +)$ *is a subsemigroup of* $(\mathbb{N}, +)$*. And for every idempotent* a *of a semigroup* S*,* $\langle a \rangle$ *is a subsemigroup of* S*.*

Definition 1.7 *Let* E *be a non-empty set, and let* $\leq$ *be a relation on* E*.* $\leq$ *is a partial order if:*

1. *(Reflexive) For all* $x \in E, x \leq x$*.*
2. *(Antisymmetric) For all* $x, y \in E$*, if* $x \leq y$ *and* $y \leq x$*, then* $x = y$*.*
3. *(Transitive) For all* $x, y, z \in E$*, if* $x \leq y$ *and* $y \leq z$*, then* $x \leq z$*.*

Example 1.8 *Let* X *be a set and let* $P(X)$ *be the power set of* X*, i.e., the set of all subsets of* X*. The inclusion relation is a partial order on* $P(X)$*.*
Subsets A *and* B *of* X *are related under set inclusion if* $A \subset B$*.*
If $A \subset X$*, then* $A \subset A$*, the relation is reflexive.*
Suppose $A, B \subset X$*, if* $A \subset B$ *and* $B \subset A$*, then by definition of set equality,* $A = B$*. Hence, the relation is antisymmetric.*
Finally, suppose $A, B, C \subset X$*. If* $A \subset B$ *and* $B \subset C$*, then* $A \subset C$*. So, the relation is transitive.*

Definition 1.9 *[2] A meet semilattice (join semilattice) is a poset* $(L, \leq)$ *such that every two-element subset* x, y *has an infimum (a supremum); this element is usually denoted by* $x \wedge y$ $(x \vee y)$ *and called the meet (join) of* x *and* y*. A lattice is a poset which is both a meet semilattice and a join semilattice.*

The following axioms are valid in a meet-semilattice and in a join-semilattice respectively

1. Commutativity, i.e.,

$(l1)$ $x \wedge y = y \wedge x$

$(l1)'$ $x \vee y = y \vee x.$

2. Associativity, i.e.,

$(l2)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

$(l2)'$ $(x \vee y) \vee z = x \vee (y \vee z).$

3. Idempotency, i.e.,

$(l3) x \wedge x = x$

$(l3)' x \vee x = x.$

And in a lattice we have the absorption axioms

$(l4)$ $x \wedge (x \vee y) = x$

$(l4)'$ $x \vee (x \wedge y) = x.$

Conversely, suppose $\wedge$, $\vee$ are binary operations on the non-empty set L. If $\wedge$ fulfils $(l1)$-$(l3)$ then L can be made a meet semi-lattice such that:

$(l5)$ $x \leq y \Leftrightarrow x \wedge y = x$

and if $\vee$ satisfies $(l1)'$ - $(l3)'$ then L be a join semi-lattice via the definition:

$(l6)$ $x \leq y \Leftrightarrow x \vee y = y$

A meet semi-lattices and join semi-lattices can also be defined as algebraic systems fulfils axioms $(l1)$ -$(l3)$ and $(l1)'$ - $(l3)'$, respectively, while lattices can be defined as algebraic structure $(L, \wedge, \vee)$ fulfils $(l1)$ -$(l3)$ and $(l1)'$ - $(l3)'$, see [2].

1.2 Negatively partially ordered semigroups

We recall some definitions and results.

Definition 1.10 *[3] A negatively partially ordered semigroup (briefly, n.p.o. semigroup) is a non empty set S with a partial ordering $\leq$ and a binary operation $\diamond$ such that for all $x, y, z \in S$, we have:*

(1) $(x \diamond y) \diamond z = x \diamond (y \diamond z)$,

(2) $x \leq y$ *implies* $x \diamond z \leq y \diamond z$ *and* $z \diamond x \leq z \diamond y$,

(3) $x \diamond y \leq x$ *and* $x \diamond y \leq y$.

An n.p.o. semi-group $(S, \leq, \diamond)$ is called implicative if there is an additional binary operation $\circledcirc : S \times S \to S$ such that for any elements x, y, z of S,

(4) $z \leq x \circledcirc y$ *if and only if* $z \diamond x \leq y$.

The operation $\circledcirc$ is called implication. An implicative n. p. o. semigroup is simply called an implicative semigroup .

An implicative semigroup $(S, \leq, \diamond, \circledcirc)$ is called commutative if it fulfils

$x \diamond y = y \diamond x$ for all $x, y \in S$, that is $(S, \diamond)$ is a commutative semigroup.

In any implicative semigroup $(S, \leq, \diamond, \circledcirc), x \circledcirc x = y * y$ for every $x, y \in S$ and this element is the greatest element, denote 1 of $(S, \leq)$.

Definition 1.11 *[3] Let S be an implicative semigroup and F be a nonempty subset of S. Then F is called an ordered filter of S if*

$(F1)$ $x \diamond y \in F$ for every $x, y \in F$. that is F is a subsemigroup of S.

$(F2)$ if $x \in F$ and $x \leq y$, then $y \in F$.

Definition 1.12 *[12] An implicative semi-lattice is a structure $(E, \leq, \wedge, \circledcirc, , 1)$ in which E is a nonempty set, $\leq$ is a partial order on E, $\wedge$ is a greatest lower bound with respect to $\leq$, and $\circledcirc$ is a binary composition in E such that for all $x, y, z \in E$, we have*

$$z \leq x \circledcirc y \Leftrightarrow z \wedge x \leq y$$

Then $\circledcirc$ is called implication operation.

Example 1.13 *The system* $([0,1], \leq, \min, \circledcirc, 1)$ *is an implicative semilattice such that*

$x \circledcirc y = \sup\{u \in [0,1], \ \min(x,u) \leq y\}$.

The next proposition extends the fundamental properties of implicative semi-lattices obtained by H. B. Curry [4] and W. C. Nemitz [12] to the structure of n. p. o. implicative semigroups.

Proposition 1.14 *[3] Let* S *be an implicative semigroup. Then for every* $x, y, z \in S$*, the following hold:*

(6) $x \leq 1, \ x \circledcirc x = 1, \ x = 1 \circledcirc x$

(7) $x \leq y \circledcirc (x \diamond y)$

(8) $x \leq x \circledcirc x^2$,

(9) $x \leq y \circledcirc x$,

(10) if $x \leq y$ *then* $x \circledcirc z \geq y \circledcirc z$ *and* $z \circledcirc x \leq z \circledcirc y$,

(11) $x \leq y$ *if and only if* $x \circledcirc y = 1$,

(12) $x \circledcirc (y \circledcirc z) = (x \diamond y) \circledcirc z$,

(13) if S *is commutative then* $x \circledcirc y \leq (s \diamond x) \circledcirc (s \diamond y)$ *for all* s *in* S.

Remark 1.15 *[8] Let* S *be a commutative implicative semigroup, then for any* $x, y, z \in S$ *it hold that*

(14) $x \circledcirc (y \circledcirc z) = y \circledcirc (x \circledcirc z)$,

(14)' $((x \circledcirc y) \circledcirc y) \circledcirc y) = x \circledcirc y$,

(15) $y \circledcirc z \leq (x \circledcirc y) \circledcirc (x \circledcirc z)$,

(15)' $y \circledcirc z \leq (z \circledcirc x) \circledcirc (y \circledcirc x)$,

(16) $x \leq (x \circledcirc y) \circledcirc y$.

Example 1.16 *Let* $S = \{1, x, y\}$ *be a semigroup with Cayley table and Hasse diagram as follows:*

$\diamond$	1	x	y
1	1	y	y
x	y	y	y
y	y	y	y

Table 1.1:

$\circledcirc$	1	x	y
1	1	x	y
x	1	1	y
y	1	1	1

Table 1.2:

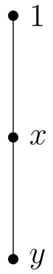

Figure 1.1:

Then it is clear that $a \circledcirc a = 1$ for all $a, b \in S$. Moreover, it is easy to see that S is an implicative n. p. o. semigroup. However, $1 \diamond x = y \neq x$, so the greatest element 1 is not the multiplicative identity of S.

The following example show that not every n.p.o. semigroup with multiplicative identity admits the implicative structure.

Example 1.17 *Let S be the set $\{0, 1, x, y\}$ with Cayley table and Hasse diagram as follows :*

.	1	x	y	0
1	1	x	y	0
x	x	0	0	0
y	b	0	y	0
0	0	0	0	0

Table 1.3:

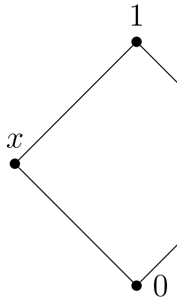

Figure 1.2:

Then it is easy to check that S *is an n. p. o. semigroup. Now, let* $a*b$ *for all* $a, b \in S$*. Clearly* $a \cdot a = 0 < b, b \cdot a = 0 < b$*. Therefore* $a \leq a*b$ *and* $b \leq a*b$*. This means that* $a*b = 1$*, the greatest element of* S*. However,* $1 \cdot a = a \leq b$*. This implies that* $a*b$ *does not exist in* S*, hence* S *is not implicative.*

1.3 On ordered filters of implicative semigroups

In this section, we first study how to generate an ordered filter by a set.

Definition 1.18 *[3] Let* S *be an implicative semigroup and* F *be a non-empty subset of* S*. Then* F *is called an orderd filter of* S *if*

$(F1)$ *for every* $x, y \in F$, $x \diamond y \in F$, *(*F *is a subsemigroup of* S*).*

$(F2)$ *if* $x \in F$ *and* $x \leq y$*, then* $y \in F$*.*

The following result gives an equivalent condition of an ordered filter.

Proposition 1.19 *[9] Let* S *be an implicative semigroup. Then a non-empty subset* F *of* S *is an ordered filter if and only if it hold:*

$(F3)$ $1 \in F$,

$(F4) x \circledcirc y \in F$ *and* $x \in F$*, imply* $y \in F$*.*

Proof. Let F be an ordered filtre of S. Since 1 is the greatest element of S, then from $(F2)$ it comes that $1 \in F$. Hence F satisfies $(F3)$. For every $a, b \in S$, since $b \circledcirc a \leq b \circledcirc a$, by (4). we have

(17) $(b \circledcirc a) \diamond b \leq a$.

Let $x \circledcirc y \in F$ and $x \in F$. By $(F1)$ we know $(x \circledcirc y) \diamond x \in F$. Using (17), we get $(x \circledcirc y) \diamond x \leq y$. It follows from $(F2)$ that $y \in F$. So, that F satisfies $(F4)$.

Conversely suppose F satisfies $(F3)$ and $(F4)$. If $x \in F$ and $x \leq y$, then by (11) we obtain $x \circledcirc y = 1 \in F$. From $(F4)$ it follows that $y \in F$. Therefore F satisfies $(F2)$. Now let $x \in F$ and $y \in F$. Then by (7), $x \leq y \circledcirc (x \diamond y)$. Hence by $(F2)$ we obtain $y \circledcirc (x \diamond y) \in F$. Combining $y \in F$ and using (ii), we conclude that $x \diamond y \in F$. So, F satisfies $(F1)$. Hence, the proof is complete.

□

Lemma 1.20 *[6] Suppose that S is an implicative semigroup and F is a non-empty subset of S. Then F is an ordered filter if and only if, for all $x, y \in F$ and $z \in S$:*

(F5) $x \leq y \circledcirc z$ implies $z \in F$.

Definition 1.21 *[8] Let S be an implicative semigroup. A nonempty subset F of S is called an implicative ordered filter of S if it satisfies (F3) and*

(I) $x \circledcirc (y \circledcirc z) \in F$ and $x \circledcirc y \in F$ imply $x \circledcirc z \in F$, for all $x, y, z \in S$.

Next, we define a positive implicative ordered filter in an implicative semigroup.

Definition 1.22 *[8] Let S be an implicative semigroup. A nonempty subset F of S is called a positive implicative ordered filter of S if it satisfies (F3) and*

(PI) $x \circledcirc ((y \circledcirc z) \circledcirc y) \in F$ and $x \in F$ imply $y \in F$ for all $x, y, z \in S$.

Example 1.23 *Let $S = \{1, x, y, z, t\}$ be an implicative semigroup with Cayley tables (1.4 and 1.5) and Hasse diagram 1.3 as follows:*

$\diamond$	1	x	y	z	t
1	1	x	y	z	t
x	x	x	t	z	t
y	y	t	y	t	t
z	z	z	t	z	t
t	t	t	t	t	t

Table 1.4:

$\circledcirc$	1	x	y	z	t
1	1	x	y	z	t
x	1	1	y	z	t
y	1	x	1	z	z
z	1	1	y	1	y
t	1	1	1	1	1

Table 1.5:

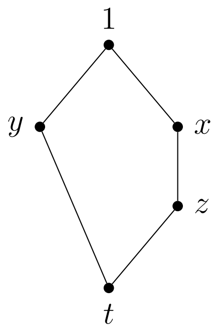

Figure 1.3:

We can easly check that $F := \{1, x, y\}$ *is a positive implicative ordered filter of* S.

Theorem 1.24 [8] *Let* S *be an implicative semigroup. Then any positive implicative ordered filter of* S *is an ordered filter.*

Proof. Let F be a positive implicative ordered filter of S and let $x \circledcirc y \in F$. For $x \in F$ and by using (6) we get $x \circledcirc ((y \circledcirc y) \circledcirc y) = x \circledcirc y \in F$ and $x \in F$. It follows from (PI) that $y \in F$, hence F is an ordered filter of S. □

Theorem 1.25 [8] *Let* S *be a commutative implicative semigroup. Then any positive implicative ordered filter of* S *is an implicative ordered filter.*

Proof. Suppose that F be a positive implicative ordered filter. Suppose that $x \circledcirc (y \circledcirc z) \in F$ and $x \circledcirc y \in F$. Then

$$\begin{aligned} x \circledcirc (y \circledcirc z) &= y \circledcirc (x \circledcirc z) && \textbf{by } (14) \\ &\leq (x \circledcirc y) \circledcirc (x \circledcirc (x \circledcirc z)) && \textbf{by } (15). \end{aligned} \tag{1.1}$$

Since F is an ordered filter (see Theorem 1.24), it follows from Lemma 1.20 that $x \circledcirc (x \circledcirc z) \in F$. On the other hand, note that

$$\begin{aligned} ((x \circledcirc \circledcirc z) \circledcirc z) \circledcirc (x \circledcirc z) &= x \circledcirc (((x \circledcirc z) \circledcirc z) \circledcirc z) && \textbf{by}(14) \\ &= x \circledcirc (x \circledcirc z) \in F && \textbf{by}(14)'. \end{aligned} \tag{1.2}$$

Hence $1 \circledcirc (((x \circledcirc z) \circledcirc z) \circledcirc (x \circledcirc z)) \in F$. Since F is a positive implicative ordered filter, we have $x \circledcirc z \in F$ by (PI). Thus the proof is complete. ☐

Remark 1.26 *[8] The converse of Theorems 1.24 and 1.25 may not be true as shown in the following example.*

Example 1.27 *Let S be a commutative implicative semigroup as in Example 1.23. We know that $G = \{1, b\}$ is an implicative ordered filter, and hence an ordered filter. But it is not a positive implicative ordered filter, since $b \circledcirc ((a \circledcirc d) \circledcirc a) \in G$ and $b \in G$,but $a \notin G$.*

Chapter 2

Some characterizations of ordered filters

In this chapter, some characterizations of positive implicative ordered filters in implicative semigroups established by Y. B. Jun and K. H. Kim in their papers "Positive implicative ordered filters of implicative semigroups". In this way we show that every positive implicative ordered filter is an ordered filter and an implicative ordered filter. Also an examples that an ordered filter (an implicative ordered filter) may not be a positive implicative ordered filter are given. Moreover we give equivalent conditions of positive implicative ordered filters.

2.1 Properties positive implicative ordered filter

Now we give equivalent conditions for every ordered filter (implicative ordered filter) to be a positive implicative ordered filter.

Theorem 2.1 *[8] Let S be an implicative semigroup and let F be an ordered filter of S. Then F is a positive implicative ordered filter if and only if, for all $x, y \in S$*

(F6) $(x \circledcirc y) \circledcirc y \in F$ implies $x \in F$.

Proof. Assume that F is a positive implicative ordered filter and let $(x \circledcirc y) \circledcirc x \in F$ for all $x, y \in S$. Then, by (6), we have $1 \circledcirc ((x \circledcirc y) \circledcirc x) \in F$. Since $1 \in F$,it follows from (PI) that $x \in F$, and (F6) holds.

Conversely, suppose that F satisfies (F6). Let $x \circledcirc ((y \circledcirc z) \circledcirc y) \in F$ and $x \in F$ for all $x, y, z \in S$. Then $(y \circledcirc z) \circledcirc y \in F$ by (F4), which implies $y \in F$ by (F6). Hence F

is a positive implicative ordered filter. Hence the proof is complete. □

Theorem 2.2 *[8] Let S be a commutative implicative semigroup. If F is a positive implicative ordered filter, then it satisfies*

(F7) $(x \circledcirc y) \circledcirc y \in F$ implies $(y \circledcirc x) \circledcirc x \in F$ for all $x, y \in S$.

Proof. Let F be a positive implicative ordered filter and let $(x \circledcirc y) \circledcirc y \in F$ for all $x, y \in S$. Since $x \leq (y \circledcirc x) \circledcirc x$ by Proposition 1.14 (9), it follows from Proposition 1.14 (10) that

$$((y \circledcirc x) \circledcirc x) \circledcirc y \leq x \circledcirc y. \tag{2.1}$$

Then

$$\begin{aligned}(x \circledcirc y) \circledcirc y &\leq (y \circledcirc x) \circledcirc ((x \circledcirc y) \circledcirc x) && \textbf{by } (15) \\ &= (x \circledcirc y) \circledcirc ((y \circledcirc x) \circledcirc x) && \textbf{by } (14) \\ &\leq (((y \circledcirc x) \circledcirc x) \circledcirc y) \circledcirc ((y \circledcirc x) \circledcirc x) && \textbf{by } (2.1) \text{ and } (10)\end{aligned} \tag{2.2}$$

By (F2) and (6) we have $1 \circledcirc ((((y \circledcirc x) \circledcirc x) \circledcirc y) \circledcirc ((y \circledcirc x) \circledcirc x)) \in F$, hence $(y \circledcirc x) \circledcirc x \in F$ by (PI). Thus the proof is complete. □

Lemma 2.3 *[8] Let S be an implicative semigroup. If F is an implicative ordered filter of S, then we have for all $x, y \in S$*

(F8) $x \circledcirc (x \circledcirc y) \in F$ implies $x \circledcirc y \in F$.

Theorem 2.4 *[8] Let S be a commutative implicative semigroup and let F be an implicative ordered filter of S satisfying (F7). Then F is a positive implicative ordered filter of S.*

Proof. Let F be an implicative ordered filter satisfying (F7) and let $(x \circledcirc y) \circledcirc x \in F$ for all $x, y \in S$ It is sufficient to show that $x \in F$ by Theorem 2.1. From (15)' we have $(x \circledcirc y) \circledcirc x \leq (x \circledcirc y) \circledcirc ((x \circledcirc y) \circledcirc y)$. Then, by (F2), we have $(x \circledcirc y) \circledcirc ((x \circledcirc y) \circledcirc y) \in F$, hence $(x \circledcirc y) \circledcirc y \in F$ by Lemma 2.3. The fact that F satisfies (F7), we have

$$(y \circledcirc x) \circledcirc x \in F. \tag{2.3}$$

On the other hand, by Proposition 1.14 (6), (9), and (10) we get

$$(x \circledcirc y) \circledcirc x \leq y \circledcirc x = 1 \circledcirc (y \circledcirc x) \tag{2.4}$$

and hence $1 \circledcirc (y \circledcirc x) \in F$ by (F2). Since $1 \in F$, it follows from (F4) that $y \circledcirc x \in F$. Hence $x \in F$ follows from 2.4 and (F4). Thus the proof is complete. □

Corollary 2.5 *[8] Let S be a commutative implicative semigroup and let F be an implicative ordered filter of S. Then F is a positive implicative ordered filter if and only if F satisfying the condition (F7).*

The following lemmas will be needed in the sequel.

Lemma 2.6 *[8] Let S be a commutative implicative semigroup and let F and G be an ordered filters of S such that $F \subseteq G$. If F is an implicative ordered filter, then G is also an implicative ordered filter.*

Lemma 2.7 *[8] Let S be a commutative implicative semigroup and let F be a nonempty subset of S. Then F is an implicative ordered filter if and only if F is an ordered filter and satisfying for all $x, y, z \in S$*

(F9) $x \circledcirc (y \circledcirc z) \in F$ implies $(x \circledcirc y) \circledcirc (x \circledcirc z) \in F$.

Finally, we give the following theorem.

Theorem 2.8 *[8] Let S be a commutative implicative semigroup. If F is a positive implicative ordered filter of S, then every ordered filter G which contain F is also a positive implicative ordered filter.*

Proof. Since any positive implicative ordered filter is an implicative ordered filter, by Lemma 2.6 and since G is an implicative ordered filter. So, it is sufficient to prove that G satisfying (F7). Suppose that $a = (x \circledcirc y) \circledcirc y \in G$ for all $x, y \in S$. Since

$a \circledcirc ((x \circledcirc y) \circledcirc y) = 1 \in F$ and F is an implicative ordered filter, from (F9) and (14) we have

$$(x \circledcirc (a \circledcirc y)) \circledcirc (a \circledcirc y) = (a \circledcirc (x \circledcirc y)) \circledcirc (a \circledcirc y) \in F. \tag{2.5}$$

Since F is a positive implicative ordered filter, by (F7) we get

$$((a \circledcirc y) \circledcirc x) \circledcirc x \in F \subseteq G. \tag{2.6}$$

On the other hand

$$\begin{aligned} (x \circledcirc y) \circledcirc y = a & \\ &\leq (a \circledcirc y) \circledcirc y && \textbf{by } (16) \\ &\leq (((a \circledcirc y) \circledcirc x) \circledcirc x) \circledcirc ((y \circledcirc x) \circledcirc x) && \textbf{by } (15)'. \end{aligned} \tag{2.7}$$

By (F2) we obtain $(((a \circledcirc y) \circledcirc x) \circledcirc x) \circledcirc ((y \circledcirc x) \circledcirc x) \in G$, and by (F4) we conclude that $(y \circledcirc x) \circledcirc x \in G$. Hence G satisfying (F7). Hence the proof is complete. ☐

2.2 Filters generated by a subset of an implicative semigroup

Remark 2.9 *[9] Let G is a non-empty family of ordered filters of an implicative semigroup S. Then $F = \cap G$ is also an ordered filter of S.*

Let δ be a subset of an implicative semigroup S. The least ordered filter containing δ is called the ordered filter generated by δ, written $\langle \delta \rangle$.

Noticing that S is clearly an ordered filter containing δ.

Remark 2.10 *[9] Let S be an implicative semigroup and let δ and σ be subsets of S. Then the following hold:*

(18) $\langle \{1\} \rangle = \{1\}, \langle \phi \rangle = \{1\}$,

(19) $\langle S \rangle = S$,

(20) $\delta \subseteq \sigma$ *implies* $\langle \delta \rangle \subseteq \langle \sigma \rangle$,

(21) $x \leq y$ *implies* $\langle \{y\} \rangle \subseteq \langle \{x\} \rangle$,

(22) if δ *is an ordered filter of* S, *then* $\langle \delta \rangle = \delta$.

In the following we give a description of elements of $\langle\delta\rangle$ *in case of commutative implicative semigroups.*

Proposition 2.11 *[9] If* σ *is a non-empty subset of a commutative implicative semigroup* S*, then*

$$\langle\delta\rangle = \{x \in S : a_n \circledcirc (... \circledcirc (a_1 \circledcirc x)...) = 1,\ \textit{for some } a_1, a_2, ..., a_n \in \sigma\}$$

Proof. Denote $\sigma = \{x \in S : a_n \circledcirc (... \circledcirc (a_1 \circledcirc x)...) = 1,\ \text{for some } a_1, a_2, ..., a_n \in \sigma\}$.

We first prove that σ is an ordered filter. Let $x \in \sigma$ and $x \leq y$. Then there are $a_1, \ldots, a_n \in \delta$ such that $a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc x) \ldots) = 1$. If follows from (10) that $1 = a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc x) \ldots) \leq a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc y) \ldots)$. Since 1 is the greatest element of S, therefore $a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc y) \ldots) = 1$. This shows $y \in \sigma$, and hence σ satisfies $(F2)$.

Now let $x \in \sigma$ and $y \in \sigma$. By (7), $x \leq y \circledcirc (x \diamond y)$. Using $(F2)$ we have $y \circledcirc (x \diamond y) \in \sigma$. hence there are $a_1, \ldots, a_n \in \delta$ such that $a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc (y \circledcirc (x \diamond y))) \ldots) = 1$. By Remark 1.15 (14) it follows that $y \circledcirc (a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc (x \diamond y)) \ldots)) = 1$. By (11) we get

(23) $y \leq a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc (x \diamond y)) \ldots)$.

As $y \in \sigma$, we know that $b_m \circledcirc (\cdots \circledcirc (b_1 \circledcirc y) \ldots) = 1$, for some $b_1, \ldots, b_m \in \delta$.

Leftly $\circledcirc-$ multiplying both sides of (23) by $b_1, \ldots, b_m$ and using (10), we obtain $1 = b_m \circledcirc (\cdots \circledcirc (b_1 \circledcirc y) \ldots) \leq b_m \circledcirc (\cdots \circledcirc (b_1 \circledcirc (a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc (x \cdot y)) \ldots))) \ldots)$.

So, $b_m \circledcirc (\cdots \circledcirc (b_1 \circledcirc (a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc (x \diamond y)) \ldots))) \ldots) = 1$. This means that $x \diamond y \in \sigma$, that is σ satisfies $(F1)$.

Finally we prove that σ is the least ordered filter containing δ. Let $a \in \delta$. Then $a \circledcirc a = 1$ by (6). Hence $a \in \sigma$, which shows that $\delta \subseteq \sigma$. Now assume F is every ordered filter containing δ and let $x \in \sigma$. Then there exist $a_1, \ldots, a_n \in \delta$ such that $a_n \circledcirc (\cdots \circledcirc (a_1 \circledcirc x) \ldots) = 1 \in F$. From $\delta \subseteq F$ and $a_n \in \delta$, we get $a_n \in F$. By $(F1)$, we obtain $a_n \diamond (a_n \circledcirc (a_{n-1} \circledcirc (\cdots \circledcirc (a_1 \circledcirc x) \ldots))) \in F$, and $a_n \diamond (a_n \circledcirc (a_{n-1} \circledcirc (\cdots \circledcirc (a_1 \circledcirc x) \ldots))) \leq a_{n-1} \circledcirc (\cdots \circledcirc (a_1 \circledcirc x) \ldots)$ by (17).

From $(F2)$ we have $a_{n-1} \circledcirc (\cdots \circledcirc (a_1 \circledcirc x) \ldots) \in F$. Repeating the above arguments, we conclude that $x \in F$. This shows $\sigma \subseteq F$. Therefore σ is the least ordered filter containing δ. Thus the proof is complete. $\square$

For any natural number n, we define $x \circledcirc^n y$ recursively as follows:

$$x \circledcirc^1 y = x \circledcirc y \text{ and } x \circledcirc^{n+1} y = x \circledcirc (x \circledcirc^n y).$$

We denote $\langle \{a_1, a_2, \dots, a_n\} \rangle = \langle a_1, a_2, \dots, a_n \rangle$ for short.

The following corollary is immediate from proposition 2.11.

Corollary 2.12 *[9] Let S be a commutative implicative semigroup and let $a \in S$. Then*

$$\langle a \rangle = \{x \in S : a \circledcirc^n x = 1, \ for\ some\ natural\ number\ n\}.$$

Proposition 2.11 Provides us with an equivalent condition for ordered filters in the commutative case .

Theorem 2.13 *[9][9] Let F be a non-empty subset of a commutative implicative semigroup S. Then F is an ordered filter if and only if*

(24) $b \circledcirc (a * x) = 1$ implies $x \in F$

for all $a, b \in F$ and all $x \in S$.

Proof. Let F be an ordered filter and let $a, b \in F$ and $x \in S$. If $b \circledcirc (a \circledcirc x) = 1$, then $x \in \langle F \rangle = F$ by (22) and proposition 2.11. Thus F satisfies (24).

Conversely assume that F satisfies (24) and let $a \in F$. By Proposition 1.14 (6) and (11) we have $a \circledcirc (a \circledcirc 1) = a \circledcirc 1 = 1$. Hence $1 \in F$ by (24). If $x \in F$ and $x \leq y$ then we obtain $1 \circledcirc (x \circledcirc y) = 1 \circledcirc 1 = 1$. By Proposition 1.14(6) and (11) by (24), we have $y \in F$. Hence, F satisfies $(F2)$. Let $x, y \in F$. Then $x \circledcirc (y \circledcirc (x \diamond y)) = 1$ by Proposition 1.14 (7). So $x \diamond y \in F$, which is $(F1)$. Therefore F is an ordered filter of S. $\square$

Proposition 2.14 *[9] Let S be a commutative implicative semigroup, F is an ordered filter and "a" is a fixed element of S. Then*

$\langle F \cup \{a\} \rangle = \{x \in S : a \circledcirc^n x \in F, \ for\ some\ natural\ number\ n\}$

Proof. Denote $B = \{x \in S : a \circledcirc^n x \in F, for\ some\ natural\ number\ n\}$.

In order to prove that B is an ordered filter of S, let $x \in B$ and $x \leq y$. Then $a \circledcirc^n x = u$ for some natural number n and $u \in F$. Hence $a \circledcirc^n (u \circledcirc x) = u \circledcirc (a \circledcirc^n x) = u \circledcirc u = 1$.

Since $x \leq y$ implies $a \circledcirc^n (u \circledcirc x) \leq a \circledcirc^n (u \circledcirc y)$, we have $1 \leq a \circledcirc^n (u \circledcirc y)$. It follows that $1 = a \circledcirc^n (u \circledcirc y) = u \circledcirc (a \circledcirc^n y)$.

So that $u \leq a \circledcirc^n y$. By $(F2)$ we have $a \circledcirc^n y \in F$, and so $y \in B$. This shows that B satisfies $(F2)$. Let $x, y \in B$. Then $x \leq y \circledcirc (x \diamond y)$.

Using $(F2)$ we get $y \circledcirc (x \diamond y) \in B$. Hence there exists a natural number n and $u \in F$ such that $u = a \circledcirc^n (y \circledcirc (x \diamond y)) = y \circledcirc (a \circledcirc^n (x \diamond y))$. It follows from Proposition 1.14 (6) and (14) that $y \circledcirc (a \circledcirc^n (u \circledcirc (x \diamond y))) = 1$ so that

(25) $y \leq a \circledcirc^n (u \circledcirc (x \diamond y))$.

Since $y \in B$, there exists a natural number m and $v \in F$ such that $a \circledcirc^m y = v$, which implies that $a \circledcirc^m (v \circledcirc y) = v \circledcirc (a \circledcirc^m y) = 1$. It follows from (25) that

$$\begin{aligned} 1 &= a \circledcirc^m (v \circledcirc y) \\ &\leq a \circledcirc^m (v \circledcirc (a \circledcirc^n (u \circledcirc (x \diamond y)))) \\ &= v \circledcirc (u \circledcirc (a \circledcirc^m (a \circledcirc^n (x \diamond y)))) \\ &= v \circledcirc (u \circledcirc (a \circledcirc^{m+n} (x \diamond y))) \end{aligned}$$

Since 1 is the greatest element of S, we have $v \circledcirc (u \circledcirc (a \circledcirc^{m+n} (x \diamond y))) = 1$.

Clearly, $u, v \in F$, we have $a \circledcirc^{m+n} (x \diamond y) \in F$ by Theorem 2.13. This means that $x \diamond y \in B$. Thus B satisfies $(F1)$. We have proved that B is an ordered filter. As $a * a = 1 \in F$, therefore $a \in B$. Let $x \in F$. Then $x \leq a \circledcirc x$ by (9). If we use $(F2)$, then $a \circledcirc x \in F$, and so $x \in B$. This shows that $F \cup \{a\} \subseteq B$.

Finally let A be each ordered filter containing F and a. If $x \in B$, then there exists a natural number n such that $a \circledcirc^n x \in F \subseteq A$. combining $a \in A$ we have $(a \circledcirc (a \circledcirc^{n-1} x)) \diamond a = (a \circledcirc^n x) \diamond a \in A$ by (F1).

On the other hand, we know that $(a \circledcirc (a \circledcirc^{n-1} x)) \diamond a \leq a \circledcirc^{n-1} x$ by (17). So $a \circledcirc^{n-1} \in A$. Repeating the procedure above, we conclude that $x \in A$. This proves that

$B \subseteq A$. Therefore B is the least ordered filter containing F and a. This completes the proof. $\square$

In order to introduce the notion of prime ordered filters in implicative commutative semigroups, we give the following definition.

If, in an implicative semigroup $(S; \leq, \diamond, \circledcirc)$, for every $x, y \in S$ there exists a least upper bound of x and y ($lub\,\{x, y\}$ for short) in S, we say, as usual, that S is an upper *semilattice.* we denote $lub\,\{x, y\}$ by $x \vee y$.

2.3 Prime ordered filters in implicative semigroup

Let $(S; \leq, \diamond, \circledcirc)$ be an implicative semigroup and upper semilattice and let F be an ordered filter of S. Then F is called a prime ordered filter if for any x, y in S, $x \vee y \in F$ implies $x \in F$ or $y \in F$.

Lemma 2.15 [9] *Let $(S; \leq, \diamond, \circledcirc)$ be a commutative implicative semigroup and upper semilattice. Suppose $a \circledcirc^m x = b \circledcirc^n x = 1$ where m and n are natural numbers. Then there is a natural number k such that $(a \vee b) \circledcirc^k x = 1$.*

Proof. Assume that $m \leq n$. Since $a \circledcirc^m x = 1$ implies $a \circledcirc^n x = 1$, it is sufficient to show that if $a \circledcirc^n x = b \circledcirc^n x = 1$ then there exists a natural number k such that $(a \vee b) \circledcirc^k x = 1$. We proceed by induction on n. If $n = 1$ then $a \circledcirc x = b \circledcirc x = 1$, and so $a \leq x$ and $b \leq x$. It follows that $a \vee b \leq x$, that is, $(a \vee b) \circledcirc x = 1$. Thus the lemma is true for $n = 1$. Suppose that the assertion holds for a natural number n, that is, $a \circledcirc^n x = b \circledcirc^n x = 1$ implies that there exists a natural number p such that $(a \vee b) \circledcirc^p x = 1$. If $a \circledcirc^{n+1} x = b \circledcirc^{n+1} x = 1$ then $1 = a \circledcirc^{n+1} x = a \circledcirc (b \circledcirc^n (a \circledcirc^n x))$ and $1 = b \circledcirc^{n+1} x = b \circledcirc (b \circledcirc^n (a \circledcirc^n x))$

In view of the first inductive step, we have by (14)

$$\begin{aligned} 1 &= (a \vee b) \circledcirc (b \circledcirc^n (a \circledcirc^n x)) \\ &= b \circledcirc^n (a \circledcirc^n ((a \vee b) \circledcirc x)) \\ &= b \circledcirc (b \circledcirc^{n-1} (a \circledcirc^n ((a \vee b) \circledcirc x))) \end{aligned}$$

From $b \circledcirc^{n+1} y = 1$ it easily follows that $a \circledcirc (b \circledcirc^{n-1} (a \circledcirc^n ((a \vee b) \circledcirc x))) = 1$

Using the first inductive step again, we get by (14)

$$1 = (a \vee b) \circledcirc (b \circledcirc^{n-1} (a \circledcirc^{n} ((a \vee b) \circledcirc x)))$$

$$= b \circledcirc^{n-1} (a \circledcirc^{n} ((a \vee b) \circledcirc^{2} x))$$

We repeat the above procedure n times we get

$1 = a \circledcirc^{n} ((a \vee b) \circledcirc^{n+1} x)$

By the similar way we obtain $1 = b \circledcirc^{n} ((a \vee b) \circledcirc^{n+1} x)$

By the inductive hypothesis there is a natural number p such that

$1 = (a \vee b) \circledcirc^{p} ((a \vee b) \circledcirc^{n+1} x) = (a \vee b) \circledcirc^{p+n+1} x.$

Taking $k = p + n + 1$, we obtain the required assertion.

□

In the situation considered in Proposition 2.14, we denote $\langle F \cup \{a\} \rangle$ by $F(a)$ for conveniece.

Theorem 2.16 *[9] Let S be a commutative implicative semigroup and upper semilattice, F an ordered filter of S and let $a, b \in S$. Then $F(a) \cap F(b) = F(a \vee b)$.*

Proof. If $x \in F(a) \cap F(b)$, then there are natural numbers m and n such that $a \circledcirc^{m} x \in F = \langle F \rangle$ and $b \circledcirc^{n} x \in F = \langle F \rangle$ respectively. It follows from proposition 2.11 that there are $u \in F$ and $v \in F$ such that $1 = u \circledcirc (a \circledcirc^{m} x) = a \circledcirc^{m} (u \circledcirc x)$ and $= v * (b \circledcirc^{n} x) = b \circledcirc^{n} (v \circledcirc x)$ respectively. By Proposition 1.14 (6), (11) and Remark 1.15 (14), we know that $1 = v \circledcirc (u \circledcirc (a \circledcirc^{m} x)) = a \circledcirc^{m} (v \circledcirc (u \circledcirc x))$ and $1 = u \circledcirc (v \circledcirc (b \circledcirc^{n} x)) = b \circledcirc^{n} (u \circledcirc (v \circledcirc x)) = b \circledcirc^{n} (v \circledcirc (u \circledcirc x))$.

By remark 1.15 (14) and 2.15, we conclude that there exists a natural number t such that $1 = (a \vee b) \circledcirc^{t} (v \circledcirc (u \circledcirc x)) = v \circledcirc (u \circledcirc ((a \vee b) \circledcirc^{t} x))$. Since $u, v \in F$, therefore $(a \vee b) \circledcirc^{t} x \in \langle F \rangle = F$. Which it means that $x \in F(a \vee b)$. Hence we have proved that $F(a) \cap F(b) \subseteq F(a \vee b)$

The reverse inclusion follows from (20) and (21). Thus the proof is complete. □

Corollary 2.17 *[9] Let S and F be as in theorem 2.16 and let $a, b \in S$. If $a \vee b \in F$ then*

$$F(a) \cap F(b) = F.$$

Lemma 2.18 *[9] Let S and P be as in theorem 2.16. If P is prime then, for all ordered filters H, G of S, $H \cap G \subseteq P$ implies $H \subseteq P$ or $G \subseteq P$.*

Proof. Let H, G two filters of the implicative semigroup S such that $H \cap G \subseteq P$.

Suppose that $H \nsubseteq P$ and $G \nsubseteq P$, so there exists $a \in H$ and $a \notin P$ and there exists $b \in G$ and $b \notin P$. Since H and G are filters, then $a \vee b \in H \cap G \subseteq P$, and since P is prime $a \in P$ or $b \in P$, which is contradiction.

□

Chapter 3

Ideals of implicative semigroups

In this chapter we study the notion of ideals in implicative semigroups by introducing special subsets of an implicative semigroups, we provide a condition for the special subset to be an ideal. we establish also two characterizations of ideals. The majority of the results are in the paper titled "On iIdeals of implicative semigroups" by Young Bae Jun and Kyung Ho Kim see [7].

In what follows let S denote an implicative semigroup unless otherwise specified.

3.1 Definition of ideal in implicative semigroups

We begin by defining the notion of ideals of S.

Definition 3.1 *[7] A subset I of S is called an ideal of S if*

$(I1)$ $x \in S$ and $a \in I$ imply $x \circledcirc a \in I$,

$(I2)$ $x \in S$ and $a, b \in I$ imply $(a \circledcirc (b \circledcirc x)) \circledcirc x \in I$.

Example 3.2 *Consider an implicative semigroup $S = \{1, x, y, z, t, 0\}$ with Cayley tables (Table 3.1 and Table 3.2) and Hasse diagram (3.1) as follows:*

$\diamond$	1	x	y	z	t	0
1	1	x	y	z	t	0
x	x	y	y	t	0	0
y	y	y	y	0	0	0
z	z	t	0	z	t	0
t	t	0	0	t	0	0
0	0	0	0	0	0	0

Table 3.1:

$\circledcirc$	1	x	y	z	t	0
1	1	x	y	z	t	0
x	1	1	x	z	z	t
y	1	1	1	z	z	z
z	1	x	y	1	x	y
t	1	1	x	1	1	x
0	1	1	1	1	1	1

Table 3.2:

We know that $\{1, x, y\}$ *is an ideal of* S*, but* $\{1, x\}$ *is not an ideal of* S*, since* $(x \circledcirc (x \circledcirc y)) \circledcirc y = y \notin \{1, x\}$

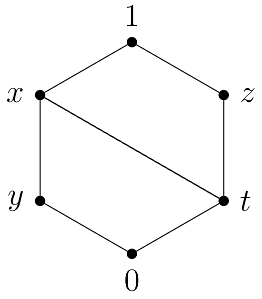

Figure 3.1:

3.2 Properties of ideal in implicative semigroups

Lemma 3.3 [7] *Any ideal of* S *contains* 1.

Proof. let I be an ideal of S, then, for all $a \in I$ we have $a \in S$ too and by using $(I1)$ we get $a \circledcirc a \in I$ implies $1 \in I$ ☐

Lemma 3.4 [7] *If* I *is an ideal of* S*, then* $(a \circledcirc x) \circledcirc x \in I$ *for all* $a \in I$ *and* $x \in S$.

Proof. We take $b = a$ and $a = 1$ in $(I2)$ we get $(1 \circledcirc (a \circledcirc x)) \circledcirc x \in I$ implies $(a \circledcirc x) \circledcirc x \in I$ for all $a \in I$ and $x \in S$. $\square$

Corollary 3.5 *[7] Let I be an ideal of S. If $a \in I$ and $a \leq x$, then $x \in I$.*

Proof. let $a \in I$ and $x \in S$ be such that $a \leq x$. we have $a \leq x$ if and only if $a \circledcirc x = 1$, then $x = 1 \circledcirc x = (a \circledcirc x) \circledcirc x$. From Lemma 3.4, we get: $(a \circledcirc x) \circledcirc x \in I$ that implies $x \in I$. $\square$

Lemma 3.6 *[7] Let I be a subset of S such that*

(I3) $1 \in I$

(I4) $x \circledcirc (y \circledcirc z) \in I$ *and* $y \in I$ *imply* $x \circledcirc z \in I$ *for all* $x, y, z \in S$.

If $a \in I$ *and* $a \leq x$ *then,* $x \in I$.

Proof. Let $a \in I$ and $x \in S$ be such that $a \leq x$. Then $x \circledcirc (a \circledcirc 1) = x \circledcirc 1 = 1 \in I$ by (6) and $(I3)$, and so $x = x \circledcirc 1 \in I$ by $(I4)$. Thus the proof is complete. $\square$

In the following we give some characterizations of ideals.

Theorem 3.7 *[7] Let S be a commutative implicative semigroup. A subset I of S is an ideal of S if and only if it satisfy conditions $(I3)$ and $(I4)$.*

Proof. Let I be an ideal of S. Then $1 \in I$ by Lemma 3.3 . Let $x, y, z \in S$ be such that $x \circledcirc (y \circledcirc z) \in I$ and $y \in I$. By Lemma 3.4, we get $(y \circledcirc z) \circledcirc z \in I$. From (6), (15) and $(I2)$ that :

$$x \circledcirc z = 1 \circledcirc (x \circledcirc z) = (((y \circledcirc z) \circledcirc z) \circledcirc ((x \circledcirc (y \circledcirc z)) \circledcirc (x \circledcirc z))) \circledcirc (x \circledcirc z) \in I$$

.

Conversely, suppose that I satisfies the conditions $(I3)$ and $(I4)$. Let $x \in S$ and $a \in I$. Since $x \circledcirc (a \circledcirc a) = x \circledcirc 1 = 1 \in I$ by $(I3)$, it follows from $(I4)$ that $x \circledcirc a \in I$, that is, $(I1)$ holds.

Since $(a \circledcirc x) \circledcirc (a \circledcirc x) = 1 \in I$, we have $(a \circledcirc x) \circledcirc x \in I$ by $(I4)$. Note from (15) that:

$$((a \circledcirc x) \circledcirc x) \circledcirc ((b \circledcirc (a \circledcirc x)) \circledcirc (b \circledcirc x)) = 1$$

that is,

$$(a \circledcirc x) \circledcirc x \leq (b \circledcirc (a \circledcirc x)) \circledcirc (b \circledcirc x)$$

for all $b \in I$. Thus, by Lemma 3.4, we have $(b \circledcirc (a \circledcirc x)) \circledcirc (b \circledcirc x) \in I$. Using $(I4)$, we conclude that $(b \circledcirc (a \circledcirc x)) \circledcirc x \in I$ which proves $(I2)$. Hence I is an ideal of S. $\square$

3.3 Some characterizations of ideals in implicative semigroups

For any $u, v \in S$, consider a set

$$E(u,v) = \{z \in S \mid u \circledcirc (v \circledcirc z) = 1\} \tag{3.1}$$

In Example 3.2, the set $S(1,a) = \{1,a\}$ is not an ideal of S. Hence we know that $E(u,v)$ may not be an ideal of S in general.

Theorem 3.8 *[7] Let S satisfy the left self-distributive law under $\circledcirc$, that is, $x \circledcirc (y \circledcirc z) = (x \circledcirc y) \circledcirc (x \circledcirc z)$ for all $x, y, z \in S$. For any $u, v \in S$, the set $E(u,v)$ is an ideal of S.*

Proof. Let $x \in S$ and $a, b \in E(u,v)$. Then

$$u \circledcirc (v \circledcirc (x \circledcirc a)) = (u \circledcirc (v \circledcirc x)) \circledcirc (u \circledcirc (v \circledcirc a)) = (u \circledcirc (v \circledcirc x)) \circledcirc 1 = 1,$$

$$\begin{aligned} u \circledcirc (v \circledcirc ((a \circledcirc (b \circledcirc x)) \circledcirc x)) &= (u \circledcirc (v \circledcirc (a \circledcirc (b \circledcirc x)))) \circledcirc (u \circledcirc (v \circledcirc x)) \\ &= ((u \circledcirc (v \circledcirc a)) \circledcirc (u \circledcirc (v \circledcirc (b \circledcirc x)))) \circledcirc (u \circledcirc (v \circledcirc x)) \\ &= (1 \circledcirc ((u \circledcirc (v \circledcirc b)) \circledcirc (u \circledcirc (v \circledcirc x)))) \circledcirc (u \circledcirc (v \circledcirc x)) \\ &= (u \circledcirc (v \circledcirc x)) \circledcirc (u \circledcirc (v \circledcirc x)) = 1 \end{aligned}$$

Hence $x \circledcirc a \in E(u,v)$ and $(a \circledcirc (b \circledcirc x)) \circledcirc x \in E(u,v)$, which shows that $E(u,v)$ is an ideal of S. $\square$

Lemma 3.9 *[7] Let S be an implicative semigroup. If $y \in S$ satisfies $y \circledcirc z = 1$ for all $z \in S$, then $E(x,y) = E(y,x)$ for all $x \in S$.*

Proof. The proof is straightforward. □

Example 3.10 *Let $S = \{1, x, y, z, t\}$ be an implicative semigroup with Cayley tables (3.3 and 3.4) and Hasse diagram 3.2 as follows:*

$\diamond$	1	x	y	z	t
1	1	x	y	z	t
x	x	x	t	z	t
y	y	t	y	t	t
z	z	z	t	z	t
t	t	t	t	t	t

Table 3.3:

$\circledcirc$	1	x	y	z	t
1	1	x	y	z	t
x	1	1	y	z	t
y	1	x	1	z	z
z	1	1	y	1	y
t	1	1	1	1	1

Table 3.4:

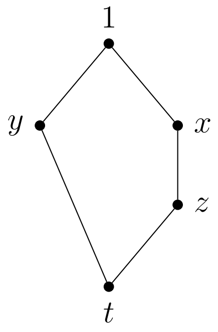

Figure 3.2:

It is easy to check that S *satisfies the left self-distributive law under* $\circledcirc$*, that is,* $a \circledcirc (b \circledcirc c) = (a \circledcirc b) \circledcirc (a \circledcirc c)$ *for all* $a, b, c \in S$*. By Lemma 3.9 we have* $E(a, t) = E(t, a) = S$ *for all* $x \in S$*. Furthermore we know that* $E(1, 1) = \{1\}, E(1, x) = E(x, 1) = E(x, x) = \{1, x\}, E(1, y) = E(y, 1) = E(y, y) = \{1, y\}, E(1, z) = E(x, z) = E(z, 1) = E(z, x) = E(z, z) = \{1, x, z\}, E(y, x) = \{1, x, y\}$*, and* $E(z, y) = S$ *are ideals of* S*. Using the set* $E(u, v)$*, we describe a characterization of ideals.*

Theorem 3.11 *[7] Let* S *be a commutative implicative semigroup and let* I *be a non-empty subset of* S*. Then* I *is an ideal of* S *if and only if* $E(u, v) \subseteq I$ *for all* $u, v \in I$*.*

Proof. Suppose that I is an ideal of S and let $u, v \in I$. If $z \in E(u, v)$, then $u \circledcirc (v \circledcirc z) = 1 \in I$ and so $z = 1 \circledcirc z = (u \circledcirc (v \circledcirc z)) \circledcirc z \in I$ by (I2). Hence $E(u, v) \subseteq I$.

Conversely, suppose that $E(u, v) \subseteq I$ for all $u, v \in I$. Note that $1 \in E(u, v) \subseteq I$. Let $x, y, z \in S$ be such that $x \circledcirc (y \circledcirc z) \in I$ and $y \in I$. Since

$$(x \circledcirc (y \circledcirc z)) \circledcirc (y \circledcirc (x \circledcirc z)) = (y \circledcirc (x \circledcirc z)) \circledcirc (y \circledcirc (x \circledcirc z)) = 1 \tag{3.2}$$

We have $x \circledcirc z \in E(x \circledcirc (y \circledcirc z), y) \subseteq I$. Applying Theorem 3.7, we conclude that I is an ideal of S. ☐

Theorem 3.12 *[7] Let* S *be a commutative implicative semigroup. If* I *is an ideal of* S*, then*

$$I = \cup_{u,v \in I} E(u, v)$$

Proof. Let I be an ideal of S and let $x \in I$. Clearly, $x \in E(x, 1)$ and so

$$I \subseteq \cup_{x \in I} E(x, 1) \subseteq \cup_{u,v \in I} E(u, v)$$

Now let $y \in \cup_{u,v \in I} E(u, v)$. Then there exist $a, b \in I$ such that $y \in E(a, b)$. It follows from Theorem 3.11 that $y \in I$. Hence $\cup_{u,v \in I} E(u, v) \subseteq I$. Thus the proof is complete. ☐

Corollary 3.13 *[7] If* I *is an ideal of a commutative implicative semigroup* S*, then*

$$I = \cup_{w \in I} E(w, 1)$$

Chapter 4

Homomorphism between implicative semigroups

In this chapter, the notion of homomorphism between implicative ordered semigroups is studied. So we generalize Nemitz's results on implicative semilattices to these semigroups, exploring their properties. The most of the results are in the paper of M. W. Chan and K. P. Shum, titled:"Homomorphisms of implicative semigroups" see[3].

Let $(S, \diamond, \leq, \circledcirc)$ and $(S', \circ, \leq, \circledcirc)$ be two implicative n.p.o. semigroups. and Let γ be a mapping from $(S, \diamond, \leq, \circledcirc)$ onto $(S', \circ, \leq, \circledcirc)$ such that $\gamma(x \circledcirc y) = \gamma a(x) \circledcirc \gamma(y)$ for all elements x and y of S. Then γ is called an implicative homomorphism of S onto S'.

Theorem 4.1 *[3] Let $(S, \diamond, \leq, \circledcirc)$ and $(S', \circ, \leq, \circledcirc)$ be two implicative semi-groups. Let γ be an implicative homomorphism from S onto S'. Then the following properties hold*

1. *$\gamma(1) = 1'$, where $1'$ is the multiplicative identity as well as the greatest element of S.*
2. *γ is* ***isotonic****, that is, if $x \leq y$ then $\gamma(x) \leq \gamma(y)$*
3. *γ is a semigroup homomorphism, that is, $\gamma(x \diamond y) = \gamma(x) \circ \gamma(y)$.*
4. *$\mathcal{F} = \gamma^{-1}(1')$ is an ordered filter of S.*
5. *γ is semigroup isomorphism if and only if $\mathcal{F} = \{1\}$.*

Proof.

1. By Proposition 1.14 (6), it can be verified that $\gamma(1) = \gamma(1 \circledcirc 1) = \gamma(1) \circledcirc \gamma(1) = 1'$.

2. Suppose that $x \leq y$. Then, because γ is an implicative homomorphism, we have $\gamma(x) \circledcirc \gamma(y) = \gamma(x \circledcirc y) = \gamma(1) = 1'$. By Proposition 1.14 (11), we thus have $\gamma(x) \leq \gamma(y)$.

3. We first show that $\gamma(x \diamond y) \leq \gamma(x) \circ \gamma(y)$. Since γ is an onto mapping between S and S', so given $\gamma(x) \circ \gamma(y) \in S'$, there exists z in S' such that $\gamma(z) = \gamma(x) \circ \gamma(y)$. Since γ is also an implicative homomorphism, we have
$$\begin{aligned} \gamma(x \diamond y) \circledcirc \gamma(z) &= \gamma(x \diamond y \circledcirc z) \\ &= \gamma[x \circledcirc (y \circledcirc z)] \\ &= \gamma(x) \circledcirc \gamma(y \circledcirc z) \\ &= \gamma(x) \circledcirc [\gamma(y) \circledcirc \gamma(z)] \\ &= [\gamma(x) \circ \gamma(y)] \circledcirc \gamma(z) \\ &= \gamma(z) \circledcirc \gamma(z) \\ &= 1'. \end{aligned}$$
Consequently, $\gamma(x \diamond y) \leq \gamma(z) = \gamma(x) \circ \gamma(y)$. Next, we show that $\gamma(x) \circ \gamma(y) \leq \gamma(x \diamond y)$. As $\gamma(y) \circledcirc \gamma(x \diamond y) = \gamma(y \circledcirc (x \diamond y))$, so by Proposition 1.14 (7), we have $x \leq y \circledcirc (x \diamond y)$. This implies that $\gamma(x) \leq \gamma(y \circledcirc (x \diamond y))$ that is $\gamma(x) \leq \gamma(y) \circledcirc \gamma(x \diamond y)$. Thus, $\gamma(x) \circ \gamma(y) \leq \gamma(x \diamond y)$. Hence we conclude that $\gamma(x \diamond y) = \gamma(x) \circ \gamma(y)$. This proves (3.).

4. Suppose that $x \in \mathcal{F}$ and $y \in \mathcal{F}$. Then $\gamma(x \diamond y) = \gamma(x) \circ \gamma(y) = 1' \circ 1' = 1'$ and so $x \diamond y \in \mathcal{F}$. If $x \diamond y \in \mathcal{F}$. Then since S is negatively ordered semigroup, $x \geq x \diamond y$. Thus, $\gamma(x) \geq \gamma(x \diamond y) = 1'$ and so $\gamma(x) = 1'$. Consequently, $x \in \mathcal{F}$. Similarly, we can show that $y \in \mathcal{F}$. Moreover, suppose that $x \in \mathcal{F}$ with $x \leq y$. Then, by the isotonic property of γ, we have $\gamma(y) \geq \gamma(x) = 1'$. This implies that $y \in \mathcal{F}$. Hence $\mathcal{F}$ is an order filter of S.

5. Suppose that $\mathcal{F} = 1$ and $\gamma(x) = \gamma(y)$. Then we have $\gamma(x \circledcirc y) = \gamma(x) \circledcirc \gamma(y) = \gamma(x) \circledcirc \gamma(x) = 1'$.

This means that $x \circledcirc y \in \mathcal{F}$, that is, $x \circledcirc y = 1$ and $x \leq y$. By using similar arguments, we can show that $y \circledcirc x = 1$ and $y \leq x$. Hence $x = y$. In other words, we have proved that γ is a semigroup isomorphism between S and S'. The proof is completed.

□

Corollary 4.2 [3] *Let S be an implicative semilattice. Then the following results are true for any elements x, y, z of S.*

1. $x \leq 1,\ x \circledcirc x = 1,\ x = 1 \circledcirc x.$
2. $x \leq y$ *if and only if* $x \circledcirc y = 1.$
3. $y \leq x \circledcirc y.$
4. *If* $x \leq y$, *then* $x \circledcirc z \geq y \circledcirc z$, *and* $z \circledcirc x \leq z \circledcirc y.$
5. $x \circledcirc (y \circledcirc z) = (x \wedge y) \circledcirc z.$
6. $x \circledcirc (y \wedge z) = (x \circledcirc y) \wedge (x \circledcirc z).$
7. $x \circledcirc (y \circledcirc z) = (x \circledcirc y) \circledcirc (x \circledcirc z).$
8. *If S is a distributive lattice with a least upper bound V, then* $(x \vee y) \circledcirc z = (x \circledcirc z) \wedge (y \circledcirc z).$

Proof. We only prove (8.) since the others are straightforward. Let $x, y, z, t \in S$. Then

$$\begin{aligned} t \in (x \vee y) \circledcirc z &\Leftrightarrow t \wedge (x \vee y) \leq z \\ &\Leftrightarrow (t \wedge x) \vee (t \wedge y) \leq z \text{ (for S is distributive)} \\ &\Leftrightarrow t \wedge x \leq z \text{ and } t \wedge y \leq z \\ &\Leftrightarrow t \leq (x \circledcirc z) \wedge (y \circledcirc z) \end{aligned}$$

So

$$(x \vee y) \circledcirc z = (x \circledcirc z) \wedge (y \circledcirc z).$$

□

Returning to the situation of implicative semilattices, we have the following Corollary.

Corollary 4.3 *[3] Let S and S' be two implicative $\wedge$-semilattices. And Let γ be a mapping from S into S', such that for all elements x and $y \in S, \gamma(x \circledcirc y) = \gamma(x) \circledcirc \gamma(y)$ Let $\mathcal{F} = \gamma^{-1}(1')$. Then the following properties hold*

1. $\gamma(1) = 1'$.
2. *γ is an isotonic homomorphism.*
3. *$\gamma(x \wedge y) = \gamma(x) \wedge \gamma(y)$ for any elements x and y of S.*
4. *$\mathcal{F}$ is a (semilattice) filter of S .*
5. *γ is an* ***injective homomorphism*** *if and only if the filter $\mathcal{F}$ of S is of the form $\mathcal{F} = \{1\}$.*
6. *If S and S' are both lattices, and if γ is surjective mapping, then, $\gamma(x \vee y) = \gamma(x) \vee \gamma(y)$, for any element x and y of S.*

Proof. The proofs of (1.) and (2.) follow verbatim of Theorem 4.1(1.) and (2.) and hence omitted (3.) Clearly $x \wedge y \leq x$ and $x \wedge y \leq y$ imply that $\gamma(x \wedge y) \leq \gamma(x) \wedge \gamma(y)$. On the other hand, we have $x \circledcirc (y \circledcirc (x \wedge y)) = (x \wedge y) \circledcirc (x \wedge y) = 1$. This implies that $\gamma(x) \circledcirc \gamma(y \circledcirc (x \wedge y)) = 1'$ and so $\gamma(x) \leq \gamma(y \circledcirc (x \wedge y)) = \gamma(y) \circledcirc \gamma(x \wedge y)$. This means that $\gamma(x) \wedge \gamma(y) \leq \gamma(x \wedge y)$. Consequently $\gamma(x \wedge y) = \gamma(x) \wedge \gamma(y)$

(4.) It is trivial to see that $x \wedge y \in \mathcal{F}$ for any $x \in \mathcal{F}$ and $y \in \mathcal{F}$. Now assume $x \in \mathcal{F}$ and $y \geq x$. Then by (2), $\gamma(y) \geq \gamma(x)$. As $x \in \mathcal{F} = \gamma^{-1}(1'), \gamma(x) = 1'$.

This implies that $\gamma(y) = 1'$, hence $y \in \mathcal{L}^{-1}(1') = \mathcal{F}$. Thus $\mathcal{F}$ is an order filter of S.

(5.) Let $\mathcal{F} = \mathcal{L}^{-1}(1')$. Then $\gamma(\mathcal{F}) = 1'$. Also, by (1.), $\gamma(1) = 1'$. Thus $\mathcal{F} = \{1\}$ since γ is a monomorphism of S into S'. For the converse part, see (5.) of Theorem 4.1.

(6.) By the isotonicity of γ, we have $\gamma(x) \vee \gamma(y) \leq \gamma(x \vee y)$. On the other hand, $\gamma(x) \wedge \gamma(x) = \gamma(x)$. So, by definition, $\gamma(x)$ is the greatest element in $\gamma(x) \circledcirc \gamma(x)$ with respect to“$\leq$" in S'. Also,

$$\gamma(x \vee y) \leq 1' = \gamma(1) = \gamma(x \circledcirc x) = \gamma(x) \circledcirc \gamma(x).$$

This implies that $\gamma(x \vee y) \leq \gamma(x)$. Similarly $\gamma(x \vee y) \leq \gamma(y)$. Thus $\gamma(x \vee y) = \gamma(x) \vee \gamma(y)$.

□

Conclusion

In the presented thesis, we considered the negatively partially ordered implicative semigroups (briefly n. p. o. implicative semigroups), which is the basis for our study. while the major thrust of this thesis has been to shed some light on what professors from southeast asian universities works on for years.

We tried to study this structure of semigroups and discuss the following research questions: what's a positive implicative ordered filter, how can we generate an ordered filter by a set, what's a prime ordered filter, how can we introduce the notion of ideals in implicative semigroup and what conditions can we provide for a subset to be an ideal.

Returning to the definition of n. p. o. implicative semigroup we studied also the homomorphisms between these semigroups.

The propositions and proofs mentioned in our work do correspond with other professor's works.

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ملخص

في هذه المذكرة، تم تقديم بعض خصائص المرشحات المرتبة الضمنية الإيجابية في بنية الزمرة شبه الضمنية. وقد ناقشنا بعض خصائص المرشحات التي تم إنشاؤها بواسطة مجموعة جزئية من شبه زمرة ضمنية. كما نعطي شرطا على مجموعة جزئية خاصة لتكون مثالي. وأخيرا قمنا بدراسة فكرة التماثل بين الزمر شبه الضمنية.

Abstract

In this work, some characterizations of positive implicative ordered filters in the structure of implicative semigroups are given. And we have discussed some properties of filters generated by a subset of an implicative semigroup. Also we give a condition for a special subset to be a ideal.

Finally we study the notion of homeomorphisms between implicative semigroups.

Keywords: Implicative semigroups, filters (ideals) of implicative semigroups, negatively ordered implicative semigroups, homomorphism of implicative semigroup.

Résumé

Dans ce travail, quelques caractérisations d'implicatif positif filtres ordonnés dans la structure d'implicatif semi-groupe sont données. Et nous avons étudier certaines propriétés d'un filtre engendré par un sous-ensemble d'un implicatif semi-groupe. Nous donnons également une condition pour qu'un sous-ensemble spécial soit un idéal. Enfin nous étudions la notion d'homomorphismes entre implicatifs semi-groupes.