People's Democratic Republic Of Algeria
Ministry of Higher Education and Scientific Research

University Mohamed Boudiaf of M'sila
Faculty of Mathematics And Informatics
Department Of Mathematics

MASTER'S THESIS

Domain: Mathematics and Informatics
Faculty : Mathematics
Option : Mathematical and numerical analysis

A Thesis

Numerical treatment of the nonlinear Volterra integral equation by Newton-Kantorovich method

---

By:

GENDOUZ Aicha

Examination Committee :

M^s *KHIRANI* Amina	Prof,	University of M'sila	**President**.
M^s *GUECHI* Somia	Prof,	University of M'sila	**Supervisor**.
M^r *DJAIDJA* Noui	Prof,	University of M'sila	**Examiner**.

Academic Year

2021/2022

Contents

List of Tables

List of Figures

Acknowledgements

First and formost, I must acknowledge my limitless thanks to Allah,
the ever-thankful, for his help and bless.

Also, I would like to thank the virtuous, Pr. **GUECHI SOMIA** for supervising this memory and for the guidance she provided to complete this work.

I thank also to the jury members who will be kind enough to read and judge this modest worh. My special thank to, Pr. **KHIRANI AMINA** who kindly has agreed to chairthis jury. I wish to express my sincere thank to, Pr. **DJAIDJA NOUI** who kindly accepted to jury members of my memory. In addition I thank my parents for their patience until this moment.

Dedication

Dedicate this humble work to:

* To my dear parents, may God protect them.

* To my dear brothers, Samir, Abd alrahman, Mehammad Islam.

* To my relatives and famale friends, and especially my dear girlfriend Djamila.

* To everyone who didn't save up an effort to help me.

* And to everyone who contributed to teaching me, even letter, in my academic life.

List of symbols

The table below is a short list of symbols and notation used in this memory

$\mathbb{R}^n$: Set of n-tuples $x = (x_1, x_2, ..., x_n)$the integral.

X, Y, E, F : Metric spaces, Banach or Hilbert spaces.

A : Integral operator.

φ : Unknown function in the integral equation.

$k\left(x, t, \varphi\left(t\right)\right)$: Kernel of the integral equation.

$C\left(\left[a, b\right], \mathbb{R}^n\right)$: Set continuously differentiable functions. $\varphi : [a, b] \rightarrow \mathbb{R}^n$, ($[a, b] \subset \mathbb{R}^n$ open).

$L^p\left(\Omega; \mathbb{R}^n\right)$: Space of all measurable functions $\varphi : \Omega \rightarrow \mathbb{R}^n$, with $\int\limits_{\Omega} \left|\varphi\left(x\right)\right|^p dx < \infty$, ($\Omega \subset \mathbb{R}^n$ open, $1 \leq p < \infty$).

NVIE : Nonlinear Volterra integral equation.

NKMS : Newton-Kantorovich-modified Simpson method.

Introduction

Integral equations are very useful mathematical tools in both pure and applied mathematics, appear in various fields of science and numerous appllications such that elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queunig theory, electrical engineering, economics, medicine, ets.

A general form of an integral equation in $\varphi(x)$ is of the form

$$\varphi(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x, t, \varphi(t))\, dt,$$

where $k(x,t)$ is called the kernel of the integral equation, $\alpha(x)$ and $\beta(x)$ are the limits of integration. it is to be noted here that both the kernel $k(x,t)$ and the function $f(x)$ in the integral equation are given functions, and λ is a constant parameter.

If the lower limit of integration is constant and the upper one is variable we are in the case of Volterra integral equations which is our subject. This kind of equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equation in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908.

The aim of this work is numerical treatment of the nonlinear Volterra integral equation by a combination of the Newton-Kantorovich method and modified Simpson method, our work is divided as follows:

The two first chapters will present few basic concepts from general theorecal framework, such as compactness and we present examples of nonlinear integral operators and theory of fixed point. It also contains some classification of nonlinear integral equations, and existence and uniqueness theorems of the nonlinear integral equations.

In the third chapter we will talk about Newton-Kantorovich method with the propreties of its convergence and combine it with modified Simpson method to get all the approximations of the nonlinear integral equation of Volterra and give some examples solving by this method.

For more details, see ([14], [4], [1], [2], [8], [9], [16]) and others.

Chapter 1

Basic concepts

In this introductory chapter we recall some notions of compactness, and state some definitions and results that will be required later, for more details, see, ([1], [2], [3], [4], [5], [6], [7]). Let (X, d) be a metric space. Recall that a subset S of X is called compact if every sequence in S contains a convergent subsequence with a limit in S.

1.1 Compactness

Definition 1.1. *(Compact operators) Let $A : X \to Y$ be an operator between two normed space. A is called a compact operator (or completely continuous) if for all bounded sets $G \subseteq X$, $A(G)$is relatively compact in Y (i.e., the set $\overline{A(G)}$ is compact). We define*

$$K(X,Y) = \{A : X \to Y : A \text{ is compact}\}, K(X) = K(X,X).$$

In other words, A is compact operator if and only if for every bounded sequence $(\varphi_n) \subseteq X$, $(A\varphi_n)$ has a subsequence convergent in Y.

Remark 1.2. *Every compact operator is bounded (and hance continuous).*

Theorem 1.3. [3] *(Arzela-Ascoli) A subset G of $C(\Omega)$ is relatevly compact if and only if the following conditions are satisfied:*

(i) G is bounded, i.e., $\exists\ M > 0$ such that

$$|\varphi(x)| < M.$$

for all $x \in \Omega$ and $\varphi \in G$.

(ii) G is equicontinuous, i.e., for every $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for every x, $y \in \Omega$ and every $\varphi \in G$ we have

$$|x - y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \varepsilon.$$

Theorem 1.4. [4] *Let A be a bounded operator, and let B be a bounded operator. then AB and BA are compact.*

Proof. [4]let $Left(\varphi_n$ be a bounded sequence, since B is bounded, the sequence $(B\varphi_n)$ is bounded. Next, since subsequence, Which means that the operator AB is compact.

Similarly, since A is compact, tha sequence $(A\varphi_n)$ contains a convergent subsequence $\left(A\varphi_{n_k}\right)$. Now, since B is bounded (and thus continuous), the sequence $(BA\varphi_n)$ converges. Therefore, the operator BA is compact. □

Definition 1.5. [4] *(finit dimensional operator) An operator is called finit dimensional if its range if of finit dimensional.*

Theorem 1.6. [4] *Finite dimentional bounded operators are compact.*

Proof. [4] Let G be a bounded set, then $A(G)$ is also bounded and so $\overline{A(G)}$ is also bounded $(\dim A(X) < \infty)$ thence is compact. ☐

Theorem 1.7. [4] *A compact operator is a bounded operatour, the converse is false.*

Definition 1.8. *Let* $T = I - A$ *be a bounded operator with* I *is the identity and* A *a compact operator. The equation* $T\varphi = f$ *is called "equation of the second kind with compact operator" . Where the function* f *is given and* φ *is unknown.*

Remark 1.9. *If* $f = 0$ *the equation is a homogeneous equation. Otherwise this equation is called non–homogeneous equation.*

Theorem 1.10. [3] *The null space* $N(T)$ *is closed and finit dimensional.*

$$N(T) = \ker(T) = \{\varphi \in X; T\varphi = (I - A)\varphi = 0\}.$$

Proof. [3] It's clear that the kernel is a vectoriel subspace, let $\varphi_n \in \ker(T)$ a convergent subsequence then

$$T\varphi_n = 0 \Rightarrow T\varphi = 0.$$

this proves that $\ker(T)$ is closed.

Let $\varphi \in \ker(T)$ so $T(\varphi) = 0 \Rightarrow (I - A)\varphi = 0 \Rightarrow A\varphi = \varphi$. Then A coinsids with the identity in the subspace $N(T)$, A is compact from $N(T)$ to $N(T)$ as a result the kernel is finit demensional. ☐

Theorem 1.11. [3] *The range space* $R(T)$ *is closed.*

Theorem 1.12. [3] *Let* A *a compact operator on a normed space* X*, injectivity and surjectivity of* T *are equivalent.*

1.2 Integral operator

In this section we present four examples of nonlinear integral operators which are completely continuous on some spaces of continuous functions: the Fredholm integral operator, the Volterra integral operator AND the Hammerstein integral operator.

Theorem 1.13. [5] *Let* $k : [a,b]^2 \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous. Then the Fredholm operator associated to* k*,* $A : C([a,b], \mathbb{R}^n) \to C([a,b], \mathbb{R}^n)$ *given by*

$$A(\varphi)(x) = \int\limits_{\Omega} k(x,t,\varphi(t))\,dt, \;\; x,t \in \Omega;$$

is completely continuous.

Theorem 1.14. [6] *Let* $k : [a,b]^2 \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous. Then the Volterra operator associated to* k*,* $A : C([a,b], \mathbb{R}^n) \to C([a,b], \mathbb{R}^n)$ *given by*

$$A(\varphi(x)) = \int\limits_{a}^{x} k(x,t,\varphi(t))\,dt, \;\; x \in [a,b];$$

is completely continuous.

Proof. [6] We first prove that A is continuous. Let $\varphi_0 \in C([a,b], \mathbb{R}^n)$ and choose any number $R > \|\varphi_0\|_\infty$. Let $\varepsilon > 0$. Since K is uniformly continuous on the compact set $[a,b]^2 \times \overline{B}_R(0, \mathbb{R}^n)$, ther exists a constant $\delta_\varepsilon > 0$ such that for every $\varphi \in C([a,b], \mathbb{R}^n)$ satisfying$\|\varphi - \varphi_0\|_\infty < \delta_\varepsilon$ one has $\varphi(t) \in \overline{B}_R(0, \mathbb{R}^n)$ and

$$|k(x,t,\varphi(t)) - k(x,t,\varphi_0(t))| \leq \varepsilon$$

for all x, $t \in [a,b]$. Then

$$\begin{aligned} |A(\varphi(x)) - A(\varphi_0(x))| &\leq \int\limits_{a}^{x} |k(x,t,\varphi(t)) - k(x,t,\varphi_0(t))|\,dt \\ &\leq \varepsilon\,|b-a| \end{aligned}$$

for every $x \in [a,b]$. Hence

$$\|A(\varphi) - A(\varphi_0)\|_\infty \leq \varepsilon\,|b-a|$$

Whenever$\|\varphi - \varphi_0\|_\infty < \delta_\varepsilon$. Therefore A is continuous at φ_0.

Next, given a bounded subset Y of $C([a,b],\mathbb{R}^n)$, we shall prove that $A(Y)$ is relatively compact in $C([a,b],\mathbb{R}^n)$. According to the Ascoli-Arzela theorem, we have to show that $A(Y)$ is bounded and equicontinuous.

Indeed, since Y is bounded there exists a constant $c>0$ such that

$$|\varphi|_\infty \leq c \text{ for all } \varphi \in Y$$

It follows that for any $\varphi \in Y$ we have

$$|A(\varphi)|_\infty < M \ |b-a|$$

where

$$M = \max_{[a,b]^2 \times \overline{B}_c(0,\mathbb{R}^n)} |k(x,t,z)|$$

Hence the set $A(Y)$ is bounded in $C([a,b],\mathbb{R}^n)$.

On the other hand, using the uniform continuity of k on the compact $[a,b]^2 \times \overline{B}_c(0,\mathbb{R}^n)$, for each $\varepsilon>0$ there exists a $\delta_\varepsilon>0$ such that

$$|k(x,t,\varphi(t)) - k(x',t,\varphi(t))| \leq \varepsilon$$

for all $x,x',t \in [a,b]$ with $|x-x'| \leq \delta_\varepsilon$ and $\varphi \in Y$. This immediately yields

$$|A(\varphi(x)) - A(\varphi(x'))| \leq \varepsilon |b-a|$$

for all $x,x' \in [a,b]$ satisfying $|x-x'| \leq \delta_\varepsilon$ and $\varphi \in Y$. Thus $A(Y)$ is equicontinuous. □

Definition 1.15. *((p,q)–Carathédory) Let* $p \in [1,+\infty[$. *A function* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ *is said to be (p,q)–Carathédory if the following condition is satisfied:*

$$\begin{cases} \text{(i) if } 1 \leq p \leq \infty \text{ then } |f(x,z)| \leq g(x) + c|z|^{\frac{p}{q}} \\ \text{for a.e } x \in \Omega \text{ ,all } z \in \mathbb{R}^m \text{ and some } g \in L^q(\Omega;\mathbb{R}^+), c \in \mathbb{R}^+; \\ \text{(ii) if } p = \infty \text{ then for every } R>0 \text{ ther is a } g_R \in L^q(\Omega) \text{ with} \\ |f(x,z)| \leq g_R(x) \text{ for a.e } x \in \Omega \text{ and all } z \in \mathbb{R}^m \text{ with } |z| \leq R. \end{cases}$$

Definition 1.16. *Let* $\Omega \subset \mathbb{R}^n$*be an open set and* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$*be a given function. the application* $N_f : \varphi \longmapsto f(.,\varphi(.))$*, which to each function* $\varphi : \Omega \to \mathbb{R}^m$*associates the* $N_f(\varphi) : \Omega \to \mathbb{R}^n$*, defined by*

$$N_f(\varphi)(x) = f(x,\varphi(x)).$$

is called nemytskii operator associated with f.

Theorem 1.17. [6] *Let $p \in [1, +\infty]$ and $q \in [1, +\infty[$ Assume that the functon $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ is (p,q)–carathédory. Then the nemytskii operator $N_f : L^p(\Omega; \mathbb{R}^m) \to L^q(\Omega; \mathbb{R}^n)$ associated to f is well defined, continous and satisfies :*

$$\begin{cases} \text{(i) if } 1 \leq p \leq \infty : \|N_f(\varphi)\|_{L^q(\Omega;\mathbb{R}^n)} \leq \|g\|_{L^q(\varphi)} + c\,\|\varphi\|^{\frac{p}{q}}_{L^q(\Omega;\mathbb{R}^m)} \\ \text{for all } \varphi \in L^p(\Omega; \mathbb{R}^m)\,; \\ \text{(ii) for } p = \infty : \|N_f(\varphi)\|_{L^q(\Omega;\mathbb{R}^n)} \leq \|g\|_{L^q(\varphi)} + c\,\|\varphi\|^{\frac{p}{q}}_{L^q(\Omega;\mathbb{R}^m)} \\ \text{for all } \varphi \in L^\infty(\Omega; \mathbb{R}^m) \text{ with } \|\varphi\|_\infty \leq R \text{ and every } R > 0. \end{cases}$$

Definition 1.18. *Consider the full Hammerstein operator given by*

$$T(\varphi)(x) = \int_\Omega k(x,t)\, f(t, \varphi(t))\, dt, x \in \Omega. \tag{1.1}$$

this operator seems to be a coping of Fredholm's integral operator A kernel k and the Nemytskii N_f operator associated with f.

$$T = AN_f.$$

Theorem 1.19. [7] *Let Ω be an open set of $\mathbb{R}^n$, $k : \Omega \times \Omega \to \mathbb{R}^n$ et $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$. Let $p \in [1, +\infty]$, $q\,[1, +\infty[$ and Let $r \in]1, +\infty]$ the conjugate of q.*

suppose that the integral operator of Fredholm $A : L^p(\Omega; \mathbb{R}^n) \to L^q(\Omega; \mathbb{R}^m)$ of kernel k is well defined and completely continuous.In addition suppose f is function (p,q)–Carathédory.

then the integral operator of Hammerstein $T : L^p(\Omega; \mathbb{R}^n) \to L^q(\Omega; \mathbb{R}^m)$ given by (1.1) is well defined and completely continuous.

1.3 Fixed point theorems

This part contains basic theory of fixed point we will recall some important and different theorems such that Banach, Brouwer, Schauder, and the Schaefer fixed point theorems, these theorems are very useful in the next chapter.

The theory of fixed point is concerned with the conditions which guarantee that a map $A : X \to X$ of a topological space X into itself admits one or more fixed points, that is , points φ of X for which $\varphi = A(\varphi)$.

Theorem 1.20. [6] *Let A be a continuous mapping in Banach space X. Then the following statements hold true*

1- If there exist $\varphi; \psi \in X$ such that

$$\lim_{n\to\infty} A^n(\varphi) = \psi.$$

Then ψ is a fixed point for A, i.e., $A(\psi) = \psi$.

2- If $A(X)$ is a compact set in X and for each $\varepsilon > 0$ there exists a $\varphi_\varepsilon \in X$ such that

$$\|A(\varphi_\varepsilon) - \varphi_\varepsilon\| < \varepsilon.$$

Then A has a fixed point.

Proof. [6] Let $\psi_n = T^n(\varphi)$,$n = 1$,2,... If A is a continuous mapping then

$$A(\psi) = A\left(\lim_{n\to\infty} \psi_n\right) = \lim_{n\to\infty} A(\varphi_n) = \lim_{n\to\infty} \psi_{n+1} = \psi.$$

Which proves the first statement.

Assume that the assumptions of (2) are satisfied. Then for $n = 1$,2,... there are $\varphi_n \in X$ such that

$$\|A(\varphi_\varepsilon) - \varphi_\varepsilon\| < \frac{1}{n}.$$

$A(X)$ is a compact set implies that there exits a convergent subsequence $\left(A\left(\varphi_{n_k}\right)\right)_{k=1}^{\infty}$ of $(A(\varphi_n))_{n=1}^{\infty}$. Call the limit point φ. Then φ is a fixed point for A since also the sequence $\left(\varphi_{n_k}\right)_{k=1}^{\infty}$ converges to x according to (1) and A is continuous. □

Definition 1.21. *Let $(X,\ d)$ be a metric space. A mapping $A : X \to X$ is a contraction mapping or contraction, if there exists a constant c with $0 < c < 1$; such that*

$$d(A(\varphi), A(\psi)) < cd(\varphi, \psi)$$

for all $\varphi, \psi \in X$.

Theorem 1.22. [6] *(Banach's fixed point theorem) Let A be a contraction on a Banach space X. Then A has a unique fixed point.*

Theorem 1.23. [6] *Let A be a mapping on a Banach space X such that A^N is a contraction on X for some positive integer N. Then A has a unique fixed point (it is not necessary to assume that A is continuous).*

Proof. [6] Banach's fixed point theorem implies that there exists a unique fixed point for A^N. Call this element φ_0. Now just note that

$$\|A(\varphi_0) - \varphi_0\| = \left\|A^N(A(\varphi_0)) - A^N(\varphi_0)\right\| \leq c\,\|A(\varphi_0) - \varphi_0\|$$

implies that $A(\varphi_0) = \varphi_0$ sinse $0 < c < 1$. the uniqueness is clear since a fixed point for A is also a fixed point for A_N. $\square$

Theorem 1.24. [6] *(Brouwer's fixed point theorem) Assume that Ω is a compact convex subset of $\mathbb{R}^n$ and that $A : \Omega \to \Omega$ is a continuous mapping. Then A has a fixed point in Ω.*

Theorem 1.25. [6] *(Schauder's fixed point theorem) Assume that Ω is a convex compact set in a Banach space X and that $A : \Omega \to \Omega$ is a continuous mapping. Then A has a fixed point.*

Theorem 1.26. [6] *(Schaefer's fixed point theorem). Assume that X is a Banach space and that $A : \Omega \to \Omega$ is a continuous compact mapping. Moreover assume that the set*

$$\bigcup_{0\leq\lambda\leq1} \{x \in X : x = \lambda A(x)\}.$$

Is bounded then A has a fixed point.

Theorem 1.27. [6] *(Krasnoselskii's fixed point theorem). Assume that F is a closed bounded convex subset of a Banach space X. Furthermore assume that A_1 and A_2 are mappings from F into X such that*

1. *$A_1(x) + A_2(y) \in F$ for all $x, y \in F$,*
2. *A_1 is a contraction,*
3. *A_2 is continuous and compact.*

Then $A_1 + A_2$ has a fixed point in F.

Chapter 2

Linear and nonlinear Integral equations

In this chapter we present integral equations and we illustrate different criterions of classification of these equations, also we discuss existence of solutions of some kind of integral equations, see, [[5], [6], [8]]. Integral equations appear in many types. The types depend mainly on the characteristics which we had been spoke about it.

2.1 Classification

Integral equation is the equation in which the unknown function $\varphi(x)$ appears inside an integral sign. The most standard type of integral equation in $\varphi(x)$ is of the form

$$\varphi(x) = f(x) + \lambda \int\limits_{g(x)}^{h(x)} k(x, t, \varphi(t))\, dt,$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $k(x, t, \varphi(t))$ is a known function, of two variables x and t, called the kernel or the nucleus of the integral equation. The unknown function $\varphi(x)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $\varphi(x)$ appears inside and outside the integral sign. The functions $f(x)$ and $k(x, t, \varphi(t))$ are given in advance. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed. Integral equations appear in many forms.

The classification of integral equations it depends on many characteristics

-The first one in the linearity of the kernel $K(x, t, \varphi(t))$ with respect to the third variable

1.If $K(x, t, \varphi(t))$ is linear with respect to the third variable i.e

$$K(x, t, \varphi(t)) = K(x, t)\varphi(t),$$

the integral equation is called linear equation.

2.If $K(x, t, \varphi(t))$ is nonlinear with respect to the third variable i.e if the equation contains nonlinear functions of $\varphi(x)$ the integral equation is called nonlinear equation. In this case we find two form given by

$$\varphi(x) = f(x) + \lambda \int\limits_{\Omega} k(x, t, \varphi(t))\, dt,$$

$$\varphi(x) = f(x) + \lambda \int\limits_{\Omega} k(x, t) G(t, \varphi(t))\, dt,$$

namely Urysohn form and Hammerstein form respectively.

-Two other distinct ways that depend on the limits of integration are used to characterize integral equations, namely:

1. If the limits of integration are fixed, the integral equation is called a Fredholm integral equation given in the form:

$$\varphi(x) = f(x) + \lambda \int\limits_{a}^{b} k(x, t)\varphi(t)\, dt,$$

where a and b are constants.

2. If at least one limit is a variable, the equation is called a Volterra integral equation given in the form:

$$\varphi(x) = f(x) + \lambda \int_a^x k(x,t)\varphi(t)\,dt.$$

-Moreover, two other distinct kinds, that depend on the appearance of the unknown function $\varphi(x)$, are defined as follows:

1. If the unknown function $\varphi(x)$ appears only under the integral sign of Fredholm or Volterra equation, the integral equation is called a first kind Fredholm or Volterra integral equation respectively.

2. If the unknown function $\varphi(x)$ appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called a second kind Fredholm or Volterra equation integral equation respectively.

In all Fredholm or Volterra integral equations presented above, if $f(x)$ is identically zero, the resulting equation:

$$\varphi(x) = \int_a^b k(x,t,\varphi(t))dt.$$

$$\varphi(x) = \int_a^x k(x,t,\varphi(t))dt.$$

is called homogeneous Fredholm or homogeneous Volterra integral equation respectively.

Definition 2.1. *(Volterra-Fredholm Integral Equations) The Volterra-Fredholm integral equations arise from a combination of Volterra integral equations and Fredholm integral equations. The Volterra-Fredholm integral equations have two forms, namely*

$$\varphi(x) = f(x) + \lambda_1 \int_a^x k_1(x,t,\varphi(t))dt + \lambda_2 \int_a^b k_2(x,t,\varphi(t))dt,$$

and the mixed form

$$\varphi(x,t) = f(x,t) + \lambda \int_0^x \int_\Omega k(x,t,\upsilon,\omega,\varphi(\upsilon,\omega))d\upsilon d\omega, \quad (x,t) \in \Omega \times [0,X].$$

Where $k_1(x,t), k_2(x,t)$ *are two nonlinear functions of* $f(x), \varphi(x), f(x,t)$ *and* $k(x,t,\upsilon,\omega,\varphi(\upsilon,\omega))$ *are analytic funcions and* λ_1, λ_2 *and* λ *are arbitrary constants.*

Definition 2.2. *(Singular integral equation) An integral equation may be called singular if either:*

- *One of the limits of integration or both are infinite, or*
- *the kernel becomes unbounded at one or more points in the interval of integration,*

and it is said to be weakly-singular if the kernel becomes infinite at $t = x$.

Definition 2.3. *(Integro-differential equation) The nonlinear integro-differential equation appears in the form:*

$$\varphi^{(n)}(x) = f(x) + \lambda \int\limits_{\Omega} k(x, t, \varphi(t))\, dt, x, t \in \Omega,$$

and the standard form of the nonlinear integro differental equation of the first kind is given by

$$\int\limits_{\Omega} k_1(x, t, \varphi(t))\, dt + \int\limits_{\Omega} k_2\left(x, t, \varphi^{(n)}(t)\right) dt = f(x), k_2\left(x, t, \varphi^{(n)}(t)\right) \neq 0,$$

where $\varphi^{(n)}$ *indicates the nth derivative of* $\varphi(t)$, *the kernels* k, k_1, k_2 *and the function* f *are given real valued functions. The Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators.*

In the next section, we will present existence theorem for the solutions of nonlinear Volterra integral equations.

2.2 Existence and uniqness theorem for nonlinear Volterra integral equation

Fixed point theorems can also be used to prove the existence and uniqueness of solutions to nonlinear Volterra integral equations.

Theorem 2.4. [12] *Consider the following nonlineare Volterra integral equation:*

$$\varphi(x) = f(x) + \lambda \int\limits_{0}^{x} k(x, t, \varphi(t))\, dt. \tag{2.1}$$

Assume that the following conditions are true:

(i) $f : [0, +\infty[\to \mathbb{R}$ *is contiunous.*

(ii) *$k : [0,+\infty[\times [0,+\infty[\to \mathbb{R}$, is a continous function satisfies the lipschitz condition next:*

$$|k(x,t,\varphi_1) - k(x,t,\varphi_2)| \leq L\,|\varphi_1 - \varphi_2| \ \textit{ such that } x,t \in [0,+\infty[\ \textit{ and } \varphi_1, \varphi_2 \in \mathbb{R}.$$

Then, the equation (2.1) admet une unique solution $\varphi \in C([0,+\infty[,\mathbb{R})$.

Proof. [12] We choose the following standard

$$|g| = \sup_x \{|g(x)| \exp(-Lx)\}.$$

We determine the operator A as follows:

$$A\varphi(x) = f(x) + \lambda \int_0^x k(x,t,\varphi(t))\,dt.$$

In order to prove that equation (2.1) admits a solution, we must show that the operator A admits a fixed point.

First, we show that A is contractive.

$$\begin{aligned}
|A\varphi(x) - A\psi(x)| &\leq \sup_x \left\{ \exp(-Lx) \int_0^x |k(x,t,\varphi(t)) - k(x,t,\psi(t))|\,dt \right\} \\
&\leq L \sup_x \left\{ \exp(-Lx) \int_0^x |\varphi(t) - \psi(t)|\,dt \right\} \\
&\leq L \sup_x \left\{ \exp(-Lx) \int_0^x \exp(-Lt)\exp(Lt)\,|\varphi(t) - \psi(t)|\,dt \right\} \\
&\leq L\,|\varphi - \psi| \sup_x \left\{ \exp(-Lx) \int_0^x \exp(Lt)\,dt \right\} \\
&\leq L\,|\varphi - \psi| \sup_x \left\{ \exp(-Lx) \frac{\exp(Lx) - 1}{L} \right\} \\
&\leq (1 - \exp(-Lx))\,|\varphi - \psi|.
\end{aligned}$$

Because

$$(1 - \exp(-Lx)) \leq 1,$$

then , A is contractive, according Banach's principle the operator A admits a fixed point unique $\varphi \in C\,[0,+\infty[$,which is a unique solution of the integral equation (2.1). □

Example 2.5. [8] *Prove that the integral equation*

$$\varphi(x) = 1 + \int_0^x \frac{\sin(x - t)}{1 + \varphi^2(t)} dt, \quad 0 < x < 1 \tag{2.2}$$

admits a unique continuous solution on $[0, 1]$.

First note that the function $f(x) \equiv 1$ *is continue sur* $[0, 1]$*.To prove the existence and uniqueness of the solution to this equation, it is sufficient to prove that the nucleus* $k(x, t, \varphi)$ *the Lipschitz condition is fulfilled with respect to the third variable* φ*.We have*

$$\begin{aligned} |k(x, t, \varphi_1(t)) - k(x, t, \varphi_2(t))| &= \left| \frac{\sin(x - t)}{1 + \varphi_1^2(t)} - \frac{\sin(x - t)}{1 + \varphi_2^2(t)} \right| \\ &\leq \left| \frac{\varphi_1^2(t) - \varphi_2^2(t)}{(1 + \varphi_1^2(t))(1 + \varphi_2^2(t))} \right| \\ &\leq \left| \frac{\varphi_1(t) - \varphi_2(t)}{(1 + \varphi_1^2(t))(1 + \varphi_2^2(t))} \right| \|\varphi_1 - \varphi_2\| \\ &\leq \|\varphi_1 - \varphi_2\|. \end{aligned}$$

Because

$$\left| \frac{a + b}{(1 + a^2)(1 + b^2)} \right| \leq \frac{1}{2} \left| \frac{2a}{1 + a^2} \right| + \frac{1}{2} \left| \frac{2b}{1 + b^2} \right| \text{ for all } a, b \in \mathbb{R}.$$

So, equation (2.2) *admits a unique continuous solution.*

Theorem 2.6. *[13] Consider the following integral equation*

$$\varphi(x) = f(x) + \int_a^x k(x, t, \varphi(t)) dt. \quad (a \leq x \leq b) \tag{2.3}$$

Such that $k : [a, b] \times [a, b] \to \mathbb{R}$ *a continuous function satisfies the following conditions:*

1 $k(x, t, 0) = 0$ *for all:* $x, t \in [a, b]$

2 $\frac{\partial k(x,\, t,\, z)}{\partial z} < \left| \frac{1 - \|f\|}{b - a} \right|$

so for all $f \in C([a, b])$ *such that* $\|f\| < 1$ *the equation* (2.3) *admits a solution* $\varphi \in C([a, b])$.

Proof. [13] We Will show that $T(B(0, 1)) \subset B(0, 1)$ i.e. for if $\|\varphi\| \leq 1$, then $\|T\varphi\| \leq 1$.

Indeed:

$$\begin{aligned}
\|T\varphi\| &= \left\| f(x) + \int_a^x k(x,t,\varphi(t))\,dt \right\| \\
&\leq \|f(x)\| + \left\| \int_a^x k(x,t,\varphi(t))\,dt \right\| \\
&\leq \|f(x)\| + \int_a^x |k(x,t,\varphi(t))|\,dt \\
&\leq \|f(x)\| + \int_a^x |k(x,t,\varphi(t)) - k(x,t,0)|\,dt \\
&\leq \|f(x)\| + \|\varphi\| \frac{1-\|f(x)\|}{b-a}(b-a) < 1.
\end{aligned}$$

According to the Schauder theorem T admits a fixed point, hence the equation admits a solution. □

Theorem 2.7. *[5] Consider the nonlinear Volterra integral equation*

$$\varphi(x) = f(x) + \lambda \int_a^x k(x,t,\varphi(t))\,dt. \quad -\infty < a \leq x \leq b < +\infty \tag{2.4}$$

where f is continuous over $[a,b]$. Assume that the function $g(x,s,t)$ satisfies the following condition:

$$|g(x,s,t)| \leq V_1(x)\,V_2(x)\,\Phi(|x|), \qquad \left|\frac{\partial g}{\partial x}(x,s,t)\right| \leq V_1(x)\,V_2(x)\,\Psi(|x|),$$

where $V_1(.)$ is a positive and continuous function over $[a,b]$; $V_2(.)$ is a positive and integrable function over $[a,b]$, $\Psi(.)$is a positive and continuous function over $[0,+\infty[$. Finally , assume that the function $\Phi(.)$ is positive, continuous and satisfies the condition

$$\lim_{\varphi\to+\infty} \frac{\Phi(.)}{\varphi} = L < +\infty.$$

Under the above conditions, the equation(2.4)has a solution in $C[a,b]$.

Chapter 3

Numerical methods and examples

In this chapter we study solving nonlinear Volterra integral equation using a mix betwieen Newton-Kantorovich method and modified Simpson method. Moreover, many examples are presented to illustrate the accuracy, efficiency of the method and to confirm the ordre of convergence. For a more detailed, we refer to ([9], [6], [10], [11], [14], [16]).

3.1 Modified Simpson method

Modified Simpson have the form

$$\int_{x_{2j}}^{x_{2j+2}} f(x)\,dx \simeq \frac{x_{2j+1}-x_{2j}}{3}\left[f(x_{2j})+4f(x_{2j+1})+f(x_{2j+1})\right]. \tag{3.1}$$

Consider $\Omega=[a,b]$, let $x_0=a<x_1<\cdots<x_{2j-1}<x_{2j}<\cdots<x_{2n}=b$ be an equidistant subdivision of a step $h=x_{2j+1}-x_{2j}$ for $j=0,\ 1,\ 2,\ \cdots,\ n$. Our objective then, it's to approximate the solution of the integral equation to the nodes of even indices (at the point x_{2j}).

For further studies on modified method, we refer to [14].

3.2 Newton-Kantorovich method

Definition 3.1. *[9] A nonlinear operator P on the product space is said to be Fréchet differentiable at $\varphi\in\Omega$ if there exists a bounded linear operator F such that, for any $\varphi+h\in\Omega$,*

$$P(\varphi+h)=P(\varphi)+\langle F(\varphi),h\rangle+\omega(\varphi,h)$$

and

$$\lim_{h\to 0}\frac{\omega(\varphi,h)}{\|h\|}=0.$$

If such an operator F exists, it is unique. This operator, which is denoted by $P'(\varphi)$, is called the Fréchet derivative of P at φ.

Theorem 3.2. *[6] (Kantorovich) Let P be an operator defined on a Banach space E in to a Banach space F and Fréchet differentiable for $\varphi\in\Omega$ an open convex set in E, satisfies the following conditions*

(A1) $\|P'(\varphi)-P'(\Psi)\|\le L\|\varphi-\Psi\|$

(A2) $\left\|[P'(\varphi_0)]^{-1}\right\|\le M,\quad \varphi_0\in\Omega$

(A3) $\left\|[P'(\varphi_0)]^{-1}P(\varphi_0)\right\|\le N,\quad \varphi_0\in\Omega,$

with the constants L, M and N satisfying $LM<1$, $LMN<\frac{1}{2}$ then there exists a domain $\Omega_1=\left\{\varphi;\ \|\varphi-\varphi_0\|\le h=\frac{(\sqrt{2}-1)}{LM\sqrt{2}}\right\}\subset\Omega$ such that, the successive approximations

$$\varphi_{n+1}=\varphi_n-[P'(\varphi_n)]^{-1}P(\varphi_n),$$

are defined for all n*,* $\varphi_n \in \Omega_1$*,* $n = 1,2,\ldots$*and converge to the exact solution* $\varphi \in \Omega_1$ *which satisfies* $P(\varphi) = 0$*. Further*

$$\|\varphi_n - \varphi\| \leq \frac{C}{2^n(LMN)}, C \in \mathbb{R}_+^*, n = 1,2,3,\ldots$$

Proof. [6] Indeed, it is easy to see that

$$\begin{aligned} P(\varphi) - P(\psi) &= \int_0^1 P'(\varphi + y(\varphi - \psi))(\varphi - \psi)\,dy \\ &= \int_0^1 P'(\varphi + y(\varphi - \psi) - P'(\varphi))(\varphi - \psi)\,dy \\ &\quad + \int_0^1 P'(\varphi)(\varphi - \psi)\,dy \end{aligned}$$

$$\begin{aligned} \|P(\varphi) - P(\psi) - P'(\varphi)(\varphi - \psi)\| &= \left\| \int_0^1 (P'(\varphi + y(\varphi - \psi)) - P'(\varphi))(\varphi - \psi)\,dy \right\| \\ \|P(\varphi) - P(\psi) - P'(\varphi)(\varphi - \psi)\| &\leq \int_0^1 \|(P'(\varphi + y(\varphi - \psi)) - P'(\varphi))\| \, \|(\varphi - \psi)\|\,dy, \end{aligned}$$

using condition (A1), on Ω we obtian

$$\|P(\varphi) - P(\psi) - P'(\varphi)(\varphi - \psi)\| \leq \frac{L}{2}\|(\varphi - \psi)\|^2$$

.

Also, for $\varphi \in \Omega_1$ we get

$$\|P'(\varphi) - P'(\varphi_0)\| \leq Lh < \frac{1}{M}, \tag{3.2}$$

then, the relation (3.2) shows that $P'(\varphi)$ is inversible for all $\varphi \in \Omega_1$ and it comes

$$[P'(\varphi)]^{-1} = \left(I - [P'(\varphi_0)]^{-1}[P'(\varphi) - P'(\varphi_0)]^{-1}\right)[P'(\varphi_0)]^{-1},$$

or still

$$\left\|[P'(\varphi)]^{-1}\right\| \leq \frac{M}{(1 - LM)\|\varphi - \varphi_0\|}$$

Given the Newton function as

$$N(\varphi) = \varphi - [P'(\varphi)]^{-1} P(\varphi),$$

with φ and $N(\varphi)$ in Ω_1 and $\varphi_{n+1} = N(\varphi_n)$, we get

$$\|N(N(\varphi)) - N(\varphi)\| = [P'(N(\varphi))]^{-1} P(N(\varphi)).$$

Hence

$$P(N(\varphi)) \le \frac{L}{2} \|N(\varphi) - \varphi\|^2. \tag{3.3}$$

On the othed hand

$$[P'(N(\varphi))]^{-1} \le \frac{M}{(1 - LM)\|N(\varphi) - \varphi_0\|} \tag{3.4}$$

From the relations (3.3) and (3.4) we obtain

$$\|N(N(\varphi)) - N(\varphi)\| \le \frac{LM\|N(\varphi) - \varphi\|^2}{2(1 - LM)\|N(\varphi) - \varphi_0\|},$$

or still

$$\begin{aligned} \left\|\varphi_{n+1} - \varphi_n\right\| &\le \frac{LM\left\|\varphi_n - \varphi_{n-1}\right\|^2}{2(1 - LM)\left\|\varphi_n - \varphi_0\right\|} \\ &\le \frac{(LM)^n\left\|\varphi_1 - \varphi_0\right\|^{2n}}{2^n(1 - LM)^n\left\|\varphi_1 - \varphi_0\right\|^n}. \end{aligned}$$

From the relation $\left\|\varphi_q - \varphi_p\right\| \le \left\|\varphi_q - \varphi_{q-1}\right\| + \left\|\varphi_{q-1} - \varphi_{q-2}\right\| + \ldots + \left\|\varphi_{p+1} - \varphi_p\right\|$ it comes the sequence φ_n is Cauchy sequence in Banach space. Thus this sequence φ_n represents the Newton iterations are defined and converges to the solution φ in Ω_1. □

Let us apply the Newoton-Kantorovich method to solve a nonlinear Volterra integral equation of the second kind in the form

$$\varphi(x) = f(x) + \lambda \int_a^x k(x, t, \varphi(t))\, dt,\ x \in \Omega. \tag{3.5}$$

We obtain the following iteration process

$$\begin{cases} \varphi_n(x) = \varphi_{n-1}(x) + y_{n-1}(x), \quad n = 1,2,\ldots, \\ y_{n-1}(x) = \varepsilon_{n-1}(x) + \int_a^x \frac{\partial k}{\partial \varphi}\left(x, t, \varphi_{n-1}(t)\right) y_{n-1}(t)\, dt, \\ \varepsilon_{n-1}(x) = f(x) - \varphi_{n-1}(x) + \int_a^x k\left(x, t, \varphi_{n-1}(t)\right) dt. \end{cases} \tag{3.6}$$

We can to write (3.6) as

$$\begin{cases} \varphi_n(x) = \varphi_{n-1}(x) + y_{n-1}(x), \quad n = 1,2,\ldots, \\ y_{n-1}(x) = f(x) - \varphi_{n-1}(x) + \int_a^x k\left(x, t, \varphi_{n-1}(t)\right) dt \\ \qquad + \int_a^x \frac{\partial k}{\partial \varphi}\left(x, t, \varphi_{n-1}(t)\right) y_{n-1}(t)\, dt. \end{cases} \tag{3.7}$$

The lest algorithmis (3.7) based on the solution of the linear integral equation for the correction $y_{n-1}(x)$ with the karnel and right-hand side that vary from step to step. This process has a high rate of convergence, but it is rather complicated because we must solve a new equation at each step of iteration. to simplity the problem, we can replace the second equation of the algorithmis (3.7) by the equation

$$y_{n-1}(x) = f(x) - \varphi_{n-1}(x) + \int_a^x k\left(x, t, \varphi_{n-1}(t)\right) dt + \int_a^x \frac{\partial k}{\partial \varphi}(x, t, \varphi_0(t))\, y_{n-1}(t)\, dt. \tag{3.8}$$

Or by the equation

$$y_{n-1}(x) = f(x) - \varphi_{n-1}(x) + \int_a^x k\left(x, t, \varphi_{n-1}(t)\right) dt + \int_a^x \frac{\partial k}{\partial \varphi}(x, t, \varphi_m(t))\, y_{n-1}(t)\, dt. \tag{3.9}$$

Whose kernels do not vary. In the equation (3.8), φ_0 is the initial solution, and in the equation (3.9) m is fixed and satisfies the condition $m < n - 1$.

Application of NKMS

For Volterra integral equations we take and approximate the two integrals on the right-hand side of the second equation of the algorithms (3.7)with $j = 0$, 1,...,n

$$\begin{aligned} y(x_{2j}) &= f(x_{2j}) - \varphi(x_{2j}) + \int_a^{x_{2j}} k(x_{2j}, t, \varphi(t))\, dt + \int_a^{x_{2j}} \frac{\partial k}{\partial \varphi}(x_{2j}, t, \varphi(t))\, y(t)\, dt \\ &= f(x_{2j}) - \varphi(x_{2j}) + \sum_{i=0}^{j-1} \int_{x_{2i}}^{x_{2i+2}} \left(k(x_{2j}, t, \varphi(t)) + \frac{\partial k}{\partial \varphi}(x_{2j}, t, \varphi(t))\, y(t) \right) dt, \\ a &\le x_j \le b. \end{aligned}$$

By the numerical integration formulas of modified Simpson method(3.1), so we get

$$\begin{aligned} y(x_{2j}) &= f(x_{2j}) - \varphi(x_{2j}) + \sum_{i=0}^{j-1} \frac{h}{3}[k(x_{2j}, t_{2i}, \varphi(t_{2i})) \\ &\quad +4k(x_{2j}, t_{2i+1}, \varphi(t_{2i+1})) + k(x_{2j}, t_{2i+2}, \varphi(t_{2i+2}))] \\ &\quad + \sum_{i=0}^{j-1} \frac{h}{3}[\frac{\partial k}{\partial \varphi}(x_{2j}, t_{2i}, \varphi(t_{2i}))\, y(t_{2i}) \\ &\quad +4\frac{\partial k}{\partial \varphi}(x_{2j}, t_{2i+1}, \varphi(t_{2i+1}))\, y(t_{2i+1}) + \frac{\partial k}{\partial \varphi}(x_{2j}, t_{2i+2}, \varphi(t_{2i+2}))\, y(t_{2i+2})], \end{aligned} \tag{3.10}$$

we take $f(x_{2j}) = f_{2j}$, $\varphi(x_{2j}) = \varphi_{2j}$, $y(x_{2j}) = y_{2j}$, $k(x_{2j}, t_{2i}, \varphi(t_{2i})) = k_{2j,2i,2i}$,
and $\frac{\partial k}{\partial \varphi}(x_{2j}, t_{2i}, \varphi(t_{2i}))\, y(t_{2i}) = k'_{2j,2i,2i} y_{2i}$,

then the equation (3.10) becomes

$$\begin{aligned} y_{2j} &= f_{2j} - \varphi_{2j} + \sum_{i=0}^{j-1} \frac{h}{3}[k_{2j,2i,2i} + 4k_{2j,2i+1,2i+1} + k_{2j,2i+2,2i+2}] \\ &+ \sum_{i=0}^{j-1} \frac{h}{3}[k'_{2j,2i,2i}y_{2i} + 4k'_{2j,2i+1,2i+1}y_{2i+1} + k'_{2j,2i+2,2i+2}y_{2i+2}]. \end{aligned} \tag{3.11}$$

Since h sufficiently small, we approximate y_{2i+1} by $\frac{y_{2i}+y_{2i+2}}{2}$, the equation (3.11) becomes

$$\begin{aligned} y_{2j} &= g_{2j} - \varphi_{2j} + \sum_{i=0}^{j-1} \frac{h}{3}[k'_{2j,2i,2i}y_{2i} + 4k'_{2j,2i+1,2i+1}y_{2i+1} + k'_{2j,2i+2,2i+2}y_{2i+2}] \\ &= g_{2j} - \varphi_{2j} + \sum_{i=0}^{j-1} \frac{h}{3}[k'_{2j,2i,2i}y_{2i} + 4k'_{2j,2i+1,2i+1}\left(\frac{y_{2i}+y_{2i+2}}{2}\right) + k'_{2j,2i+2,2i+2}y_{2i+2}] \\ &= g_{2j} - \varphi_{2j} + \sum_{i=0}^{j-1} \frac{h}{3}[[k'_{2j,2i,2i} + 2k'_{2j,2i+1,2i+1}]y_{2i} + [2k'_{2j,2i+1,2i+1} + k'_{2j,2i+2,2i+2}]y_{2i+2}] \\ &= g_{2j} - \varphi_{2j} + \frac{h}{3}[\sum_{i=0}^{j-1}[k'_{2j,2i,2i} + 2k'_{2j,2i+1,2i+1}]y_{2i} + \sum_{i=0}^{j-1}[2k'_{2j,2i+1,2i+1} + k'_{2j,2i+2,2i+2}]y_{2i+2}] \\ &= g_{2j} - \varphi_{2j} + \frac{h}{3}[\sum_{i=0}^{j-1}[k'_{2j,2i,2i} + 2k'_{2j,2i+1,2i+1}]y_{2i} + \sum_{i=1}^{j}[2k'_{2j,2i-1,2i-1} + k'_{2j,2i,2i}]y_{2i}] \\ &= g_{2j} - \varphi_{2j} + \frac{h}{3}[2k'_{2j,2j-1,2j-1} + k'_{2j,2j,2j}]y_{2j} + \frac{h}{3}[2k'_{2j,0,0} + 2k'_{2j,1,1}]y_0 \\ &+ \frac{2h}{3}\sum_{i=1}^{j-1}[k'_{2j,2i-1,2i-1} + k'_{2j,2i,2i} + k'_{2j,2i+1,2i+1}]y_{2i}. \end{aligned}$$

Finally

$$\begin{aligned} [1 - \frac{h}{3}[2k'_{2j,2j-1,2j-1} + k'_{2j,2j,2j}]y_{2j} &= g_{2j} - \varphi_{2j} + \frac{h}{3}[2k'_{2j,0,0} + 2k'_{2j,1,1}]y_0 \\ &+ \frac{2h}{3}\sum_{i=1}^{j-1}[k'_{2j,2i-1,2i-1} + k'_{2j,2i,2i} + k'_{2j,2i+1,2i+1}]y_{2i} \end{aligned} \tag{3.12}$$

with

$$g_{2j} = f_{2j} + \sum_{i=1}^{j-1} \frac{h}{3}(k_{2j,2i,2i} + 4k_{2j,2i+1,2i+1} + k_{2j,2i+2,2i+2})$$

The evaluation of (3.12) on the x_{2j} gives a system of algebraic equations of the form

$$\left(I - \frac{h}{3}A\right)Y = B,$$

where the vectors Y, A and B respectively defined the components of y_{2j}, $2k'_{2j,2j-1,2j-1} + k'_{2j,2j,2j}$,

and $g_{2j} - \varphi_j + +\frac{2h}{3}\sum_{i=1}^{j-1}[k'_{2j,2i-1,2i-1} + k'_{2j,2i,2i} + k'_{2j,2i+1,2i+1}]y_{2i}+$ $\frac{h}{3}[2k'_{2j,0,0} + 2k'_{2j,1,1}]y_0$, for $j = 0,\ 1,...,n$.
And the initial aproximations $Y^{(0)} = \Phi^{(0)} = F$, and F defined the components of f_{2j}

$$\begin{cases} \Phi^{(K+1)} = \Phi^{(K)} + Y^{(K)} \\ \left(I - \frac{h}{3}A\right) Y^{(K+1)} = B, \end{cases} \quad K = 0,1,2,...$$

and by recurence, we can to calculate the vector of solutions Φ in all points x_{2j} for $j = 0,\ 1,...,n$.

3.3 Numerical results

In this section, implementation of the methods mentioned in this memory will be done. This implementation consists of computer programmes written in MATLAB, which approximates solutions to some example nonlinear Volterra integral equation of the second kind.

The format of the values outputted will in the floating-point format, $3,23e-03 = 3,23 \times 10^{-3}$.

Example 3.3. *[10]Consider the nonlinear Volterra integral equation of the form*

$$\varphi\left(x\right) = \sqrt{x} + (x+1)\exp(-2x) - \exp(-x) + \int_0^x \exp(-x-t)\varphi^2\left(t\right)dt, \quad 0 \leq x,t \leq 1, \quad (3.13)$$

with the exact solution $\varphi\left(x\right) = \sqrt{x}$.

Example 3.4. *[6]Consider the nonlinear Volterra integral equation of the form*

$$\varphi\left(x\right) = x + \cos\left(x\right) - 1 + \int_0^x \sin\left(\varphi\left(t\right)\right)dt, \quad (3.14)$$

with the exact solution $\varphi\left(x\right) = x$.

Example 3.5. *[16]Consider the nonlinear Volterra integral equation of the form*

$$\varphi\left(x\right) = x^2 - \frac{x^6}{6}\exp\left(-x\right) - 1 + \int_0^x t\exp(-x)\varphi^2\left(t\right)dt, \quad (3.15)$$

with the exact solution $\varphi\left(x\right) = x^2$.

Example 3.6. *[16]Consider the nonlinear Volterra integral equation of the form*

$$\varphi(x) = x + \int_0^x \varphi^2(t)\,dt, \tag{3.16}$$

with the exact solution $\varphi(x) = \tan(x)$.

Example 3.7. *[14]Consider the nonlinear Volterra integral equation of the form*

$$\varphi(x) = \sin(x) + \frac{1}{4}\sin^2(x) - \frac{1}{4}x^2 + \int_0^x (x-t)\varphi^2(t)\,dt, \tag{3.17}$$

with the exact solution $\varphi(x) = \sin(x)$.

Example 3.8. *[14]Consider the nonlinear Volterra integral equation of the form*

$$\varphi(x) = \frac{1}{4} + \frac{1}{2}x + \frac{1}{2}x^2 + \exp(x) - \frac{1}{4}\exp(2x) + \int_0^x (x-t)^2\varphi^2(t)\,dt, \tag{3.18}$$

with the exact solution $\varphi(x) = \exp(x)$.

Nodes	Exact solution	approximate solution	Error.NKMS
0	$0.0000e+00$	$0.0000e+00$	$0.0000e+00$
0.1	$3.1623e-01$	$3.1199e-01$	$4.2336e-03$
0.2	$4.4721e-01$	$4.3540e-01$	$1.1813e-02$
0.3	$5.4772e-01$	$5.3199e-01$	$1.5736e-02$
0.4	$6.3246e-01$	$6.1488e-01$	$1.7579e-02$
0.5	$7.0711e-01$	$6.8938e-01$	$1.7730e-02$
0.6	$7.7460e-01$	$7.5800e-01$	$1.6594e-02$
0.7	$8.3666e-01$	$8.2209e-01$	$1.4565e-02$
0.8	$8.9443e-01$	$8.8242e-01$	$1.2003e-02$
0.9	$9.4868e-01$	$9.3948e-01$	$9.2072e-03$
1	$1.0000e+00$	$9.9359e-01$	$6.4126e-03$

Table 3.1: Comparison of resultants, absolute error, by NKMS for NVIE (3.13) with h=0.1.

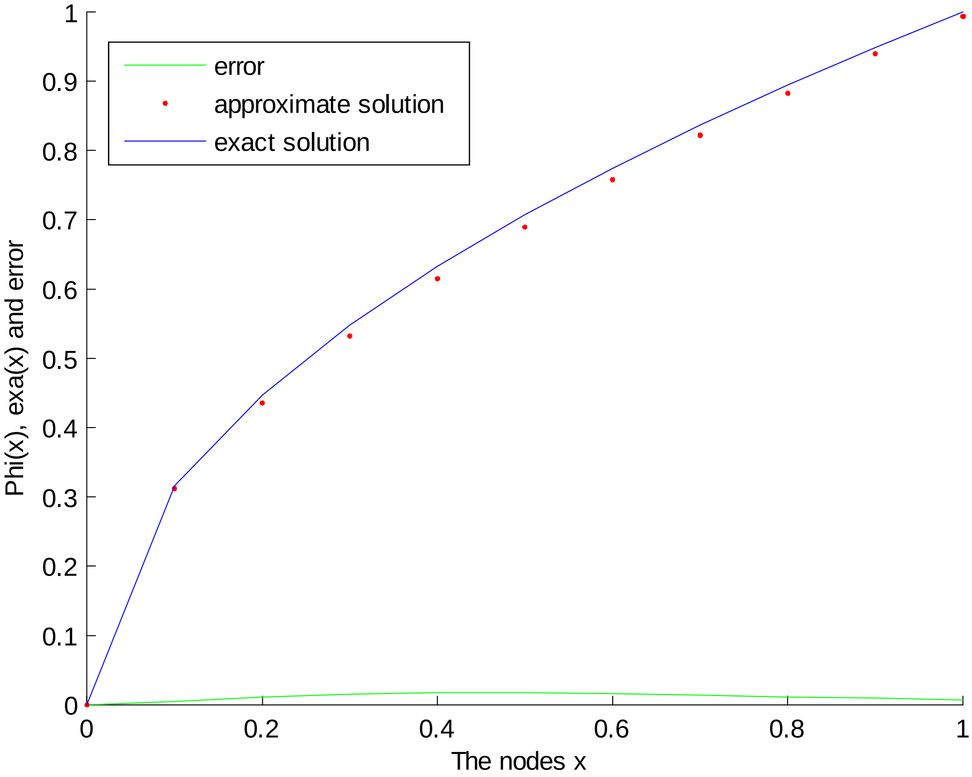

Figure 3.1: Comparison of resultants, absolute error, by NKMS for NVIE (3.13) with h=0.1.

Nodes	Exact solution	approximate solution	Error.NKMS
0	$0.0000e+00$	$0.0000e+00$	$0.0000e+00$
0.1	$1.0000e-01$	$9.5004e-02$	$4.9958e-03$
0.2	$2.0000e-01$	$1.8506e-01$	$1.4940e-02$
0.3	$3.0000e-01$	$2.7551e-01$	$2.4492e-02$
0.4	$4.0000e-01$	$3.6721e-01$	$3.2787e-02$
0.5	$5.0000e-01$	$4.6133e-01$	$3.8669e-02$
0.6	$6.0000e-01$	$5.5903e-01$	$4.0969e-02$
0.7	$7.0000e-01$	$6.6140e-01$	$3.8596e-02$
0.8	$8.0000e-01$	$7.6930e-01$	$3.0703e-02$
0.9	$9.0000e-01$	$8.8311e-01$	$1.6885e-02$
1	$1.0000e+00$	$1.0026e+00$	$2.5681e-03$

Table 3.2: Comparison of resultants, absolute error, by NKMS for NVIE (3.14) with h=0.1.

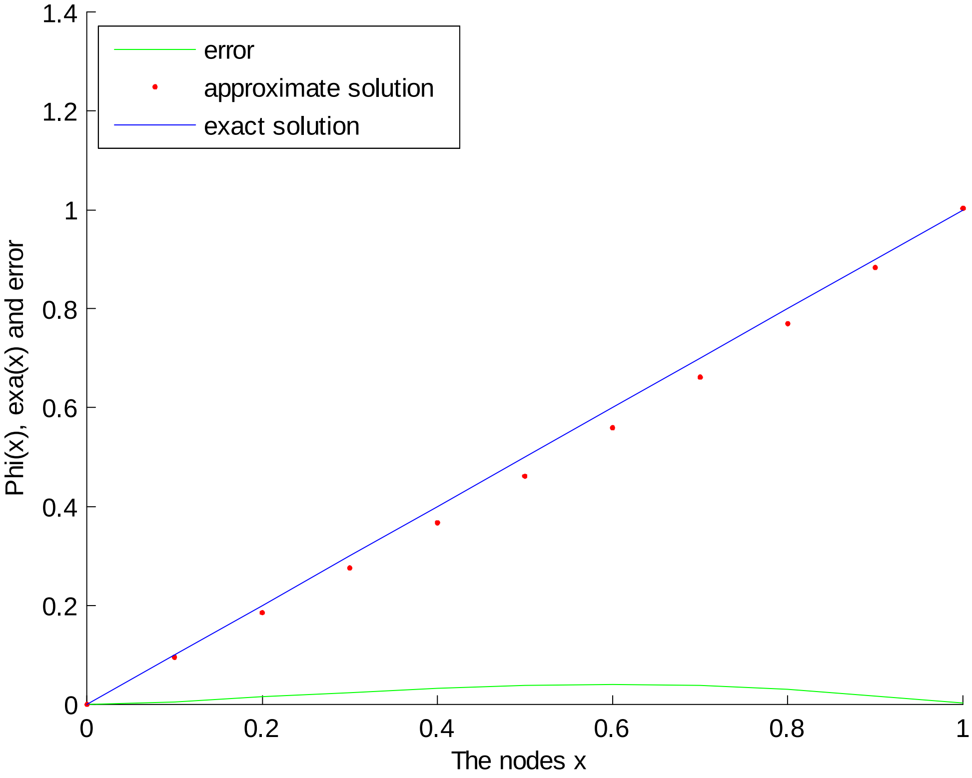

Figure 3.2: Comparison of resultants, absolute error, by NKMS for NVIE (3.14) with h=0.1.

Nodes	Exact solution	approximate solution	Error.NKMS
0	$1.0000e+00$	$1.0000e+00$	$0.0000e+00$
0.1	$1.0000e-02$	$9.9998e-03$	$1.5081e-07$
0.2	$4.0000e-02$	$3.9991e-02$	$8.5284e-06$
0.3	$9.0000e-02$	$8.9919e-02$	$8.1110e-05$
0.4	$1.6000e-01$	$1.5963e-01$	$3.7143e-04$
0.5	$2.5000e-01$	$2.4885e-01$	$1.1505e-03$
0.6	$3.6000e-01$	$3.5720e-01$	$2.7979e-03$
0.7	$4.9000e-01$	$4.8423e-01$	$5.7692e-03$
0.8	$6.4000e-01$	$6.2946e-01$	$1.0542e-02$
0.9	$8.1000e-01$	$7.9246e-01$	$1.7539e-02$
1	$1.0000e+00$	$9.7298e-01$	$2.7024e-02$

Table 3.3: Comparison of resultants, absolute error, by NKMS for NVIE (3.15) with h=0.1.

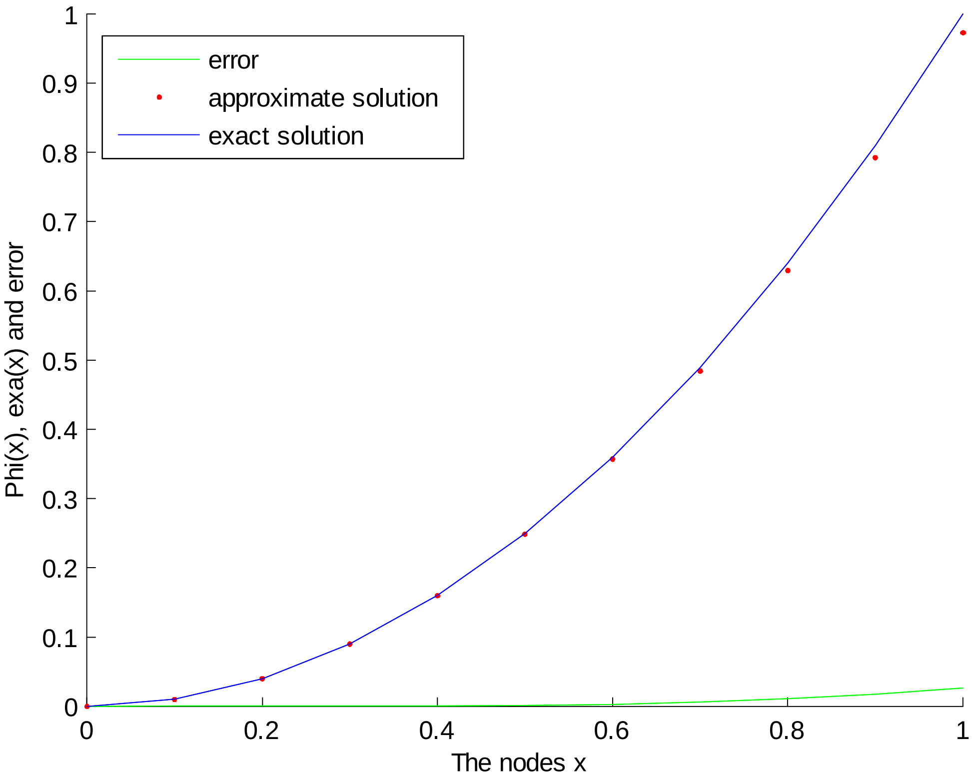

Figure 3.3: Comparison of resultants, absolute error, by NKMS for NVIE (3.15) with h=0.1.

Nodes	Exact solution	approximate solution	Error.NKMS
0	$0.0000e+00$	$0.0000e+00$	$0.0000e+00$
0.1	$1.0033e-01$	$1.0000e-01$	$3.3467e-04$
0.2	$2.0271e-01$	$2.0034e-01$	$2.3711e-03$
0.3	$3.0934e-01$	$3.0276e-01$	$6.5767e-03$
0.4	$4.2279e-01$	$4.0962e-01$	$1.3172e-02$
0.5	$5.4630e-01$	$5.2396e-01$	$2.2343e-02$
0.6	$6.8414e-01$	$6.5010e-01$	$3.4038e-02$
0.7	$8.4229e-01$	$7.9471e-01$	$4.7577e-02$
0.8	$1.0296e+00$	$9.6885e-01$	$6.0788e-02$
0.9	$1.2602e+00$	$1.1922e+00$	$6.7924e-02$
1	$1.5574e+00$	$1.5033e+00$	$5.4142e-02$

Table 3.4: Comparison of resultants, absolute error, by NKMS for NVIE (3.16) with h=0.1.

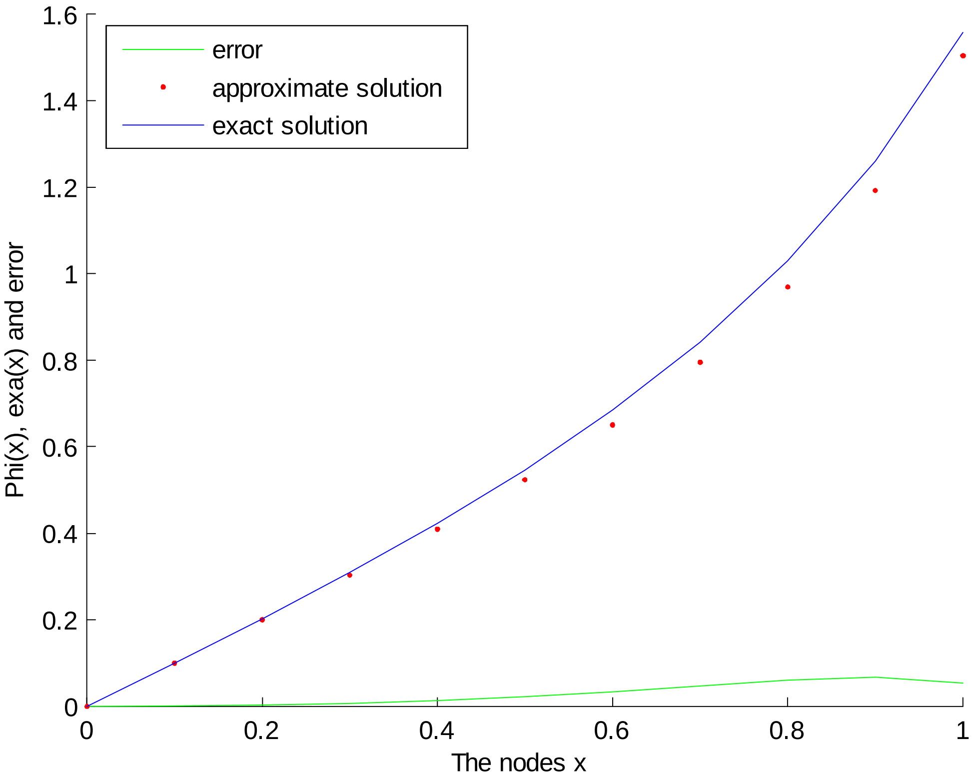

Figure 3.4: Comparison of resultants, absolute error, by NKMS for NVIE (3.16) with h=0.1.

Nodes	Exact solution	approximate solution	Error.NKMS
0	$0.0000e+00$	$0.0000e+00$	$0.0000e+00$
0.1	$9.9833e-02$	$9.9825e-02$	$8.3222e-06$
0.2	$1.9867e-01$	$1.9858e-01$	$9.1083e-05$
0.3	$2.9552e-01$	$2.9525e-01$	$2.7019e-04$
0.4	$3.8942e-01$	$3.8888e-01$	$5.3741e-04$
0.5	$4.7943e-01$	$4.7855e-01$	$8.7445e-04$
0.6	$5.6464e-01$	$5.6340e-01$	$1.2430e-03$
0.7	$6.4422e-01$	$6.4265e-01$	$1.5691e-03$
0.8	$7.1736e-01$	$7.1564e-01$	$1.7206e-03$
0.9	$7.8333e-01$	$7.8185e-01$	$1.4783e-03$
1	$8.4147e-01$	$8.4097e-01$	$5.0062e-04$

Table 3.5: Comparison of resultants, absolute error, by NKMS for NVIE (3.17) with h=0.1.

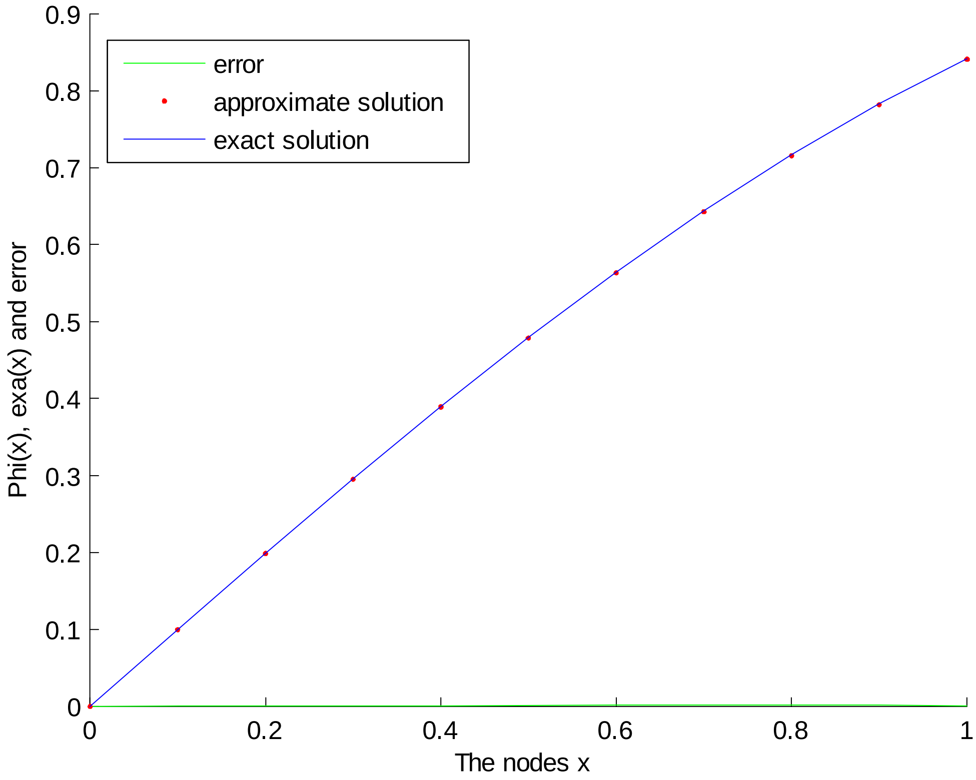

Figure 3.5: Comparison of resultants, absolute error, by NKMS for NVIE (3.17) with h=0.1.

Nodes	Exact solution	approximate solution	Error.NKMS
0	$1.0000e+00$	$1.0000e+00$	$0.0000e+00$
0.1	$1.1052e+00$	$1.1053e+00$	$1.5813e-04$
0.2	$1.2214e+00$	$1.2239e+00$	$2.4909e-03$
0.3	$1.3499e+00$	$1.3568e+00$	$6.9101e-03$
0.4	$1.4918e+00$	$1.5053e+00$	$1.3507e-02$
0.5	$1.6487e+00$	$1.6712e+00$	$2.2517e-02$
0.6	$1.8221e+00$	$1.8566e+00$	$3.4445e-02$
0.7	$2.0138e+00$	$2.0640e+00$	$5.0265e-02$
0.8	$2.2255e+00$	$2.2973e+00$	$7.1739e-02$
0.9	$2.4596e+00$	$2.5615e+00$	$1.0189e-01$
1	$2.7183e+00$	$2.8641e+00$	$1.4578e-01$

Table 3.6: Comparison of resultants, absolute error, by NKMS for NVIE (3.18) with h=0.1.

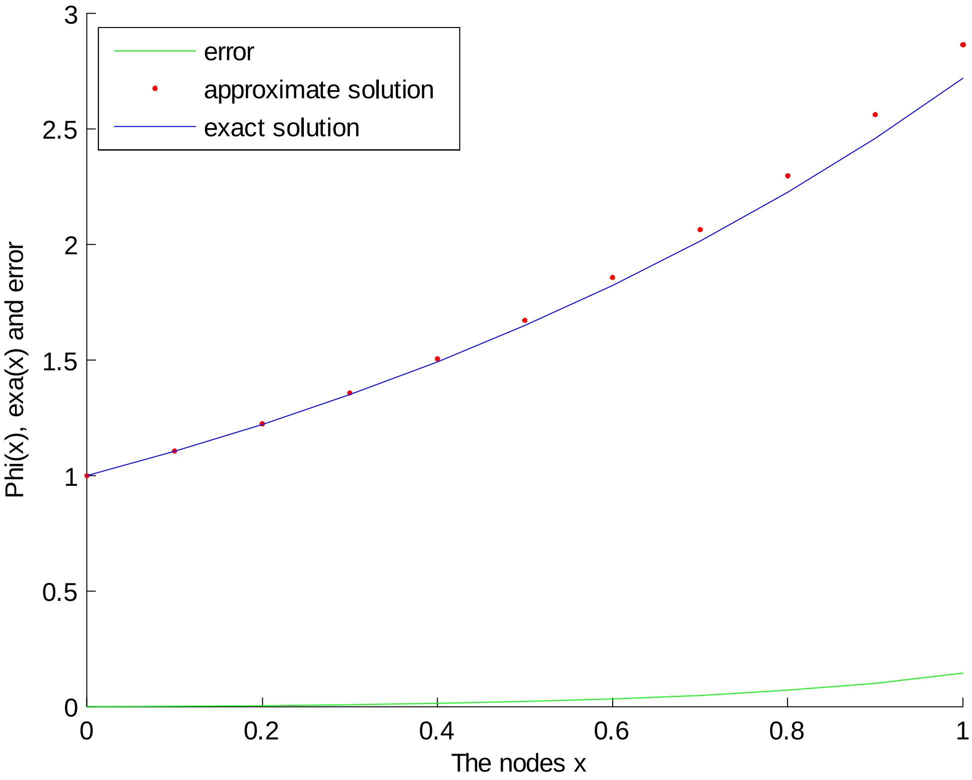

Figure 3.6: Comparison of resultants, absolute error, by NKMS for NVIE (3.18) with h=0.1.

Conclusion

There are various numerical method to solve nonlinear integral equations. Most of them transform the integral equation into a system of nonlinear equation is a problem.

In this work, the nonlinear system is solved using Newton-Kantorovich method in combination with modified Simpson method and, We have introduced the numerical method as an algorithms for the Volterra integral equation using MATLAB. The purpose of this work is to see the performance of this method.

According to the numerical results obtained from the illustrative exambles, we conclude that for h sufficiently small we obtain a good precision.

Bibliography

[1] D.charalambos, B.Owen, Positive operators, May 2006.

[2] R.Mesie, D.Vogt, Introduction to functional analysis, Jun 1996.

[3] M.Nadir, Cours sur les opératours et équations intégrales, University of Msila 2021.

[4] L.Qebnath, P.Mikusinski, Introduction to Hilbert space with applications, University of Central Florida, 1990.

[5] S.Guechi, Méthodes computationnelles pour la résolution des équations intégrales non linéaires, thesis of Doctorate, University of Msila, 2017.

[6] A.khirani, Etude des équations intégrales non linéaires de Volterra dans les espace fonctionnels, thesis of Doctorate, University of Msila, 2016.

[7] M.Chemcham, Etude des equations integrales non linéaire de type Hammerstein dans les espaces d'Orlicz, thesis of Doctorate, University of Biskra. Mohamed Khider, 2015.

[8] A.Rahmoune, Équations intégrales linéaires et non linéaires, 2016.

[9] M. A. Golberg, Numerical solution of integral équations, New York, 1990.

[10] S.Guechi, M.Nadir, Solutions of integral equations in the Urysohn form via some numerical methods, University of Msila, 2016.

[11] M.Nadir, S.Guech, The combination of modified Simpson and Newoton-Kantorovich methods for solving nonlinear integral equations, University of Msila, 2016.

[12] M.Guesba, Sur quelques équations intégral non linéaires, memory of magister, University of Ouargla, 2012.

[13] F.Bachiri, théorèmes du point fixe et Applications aux Equations integrales, memory of magister, University of Msila, 2017.

[14] M.Nadir, A.Rahmooune. Modified method for solving linear Volterra integral equations of second kind. IJMN (2007).

[15] A.Wazwaz, Linear and nonlinear integral equations, methods and applications. Higher Education Press Springer-Verlag Berlin Heidelberg, London, New York, 2011.

[16] B.Sehili, I.Teharbit, Résolution des équations integrales non linéaire, memory of magister, University of Bordj Bouarreridj, 2013.

Abstract

Many problems which arise in mathematical physics, engineering, biology, economics,...etc., lead to mathematical models described by nonlinear integral equation. In this work, the aim is solve numerically the nonlinear Volterra integral equation by the combination of Newton-Kantorovich method with modified Simpson method.This method transforms the nonlinear integral equation to a linear integral equation of the second kind which be solve by the modified Simpson method.

Keywords: Nonlinear Volterra integral equations, Fixed point theorem, Newton-Kantorovich method, modified Simpson method.

Résumé

De nombreux problèmes qui se posent en physique mathématique, ingénierie, biologie, économie,...ets., conduisent à des modèles mathématiques décrits par équation intégrale non linéaire. Dans ce travail, le but est de résoudre numériquement l'équation intégrale non linéaire de Volterra en combinant la méthode de Newton-Kantorovich et méthode de Simpson modifiée.Cette méthode transforme l'équation intégrale non linéaire en une équation intégrale linéaire de seconde espèce qui sera résolue par la méthode de Simpson modifiée.

Mots clés: Équations intégrales non linéaire de Volterra, Théorème du point fixe, Méthode de Newton-Kantorovich, Méthode de Simpson modifiée

الملخص

العديد من المشاكل التي تظهر في الفيزياء الرياضية و الهندسة و علم الاحياء و الاقتصاد ... الخ، تؤدي الى نماذج رياضية موصوفة بمعادلة تكاملية غير خطية. الهدف من هذه المذكرة هو حل معادلة فولتيرا التكاملية غير الخطية عدديا بالجمع بين طريقة نيوتن-كانتوروفيش و طريقة سمبسون المعدلة. تحول هذه الطريقة المعادلة التكاملية الغير خطية الى معادلة تكاملية خطية من الدرجة الثانية و التي تحل بطريقة سمبسون المعدلة.

الكلمات المفتاحية: معادلات فولتيرا التكاملية الغير خطية، نظرية النقطة الصامدة، طريقة نيوتن كانتوروفيش، طريقة سمبسون المعدلة .