جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

PEOPL's DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf university of Msila
Faculty of Mathematics and computer sciences
Department of Mathematics

Master memory

Field : Mathematics and computer sciences
Branch : Mathematics
Option : Partial Differential Equations and Applications

Theme

Existence results for a boundary value problem of p-Kirchhoff type on the half-line

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Presented by :
GUETTOUCHE Imane

Publicly supported on :June 16, 2026.

In front of the jury composed of :

Bouafia Dahmane	Grade,	Mohamed Boudiaf University of M′sila	**President**.
Mokhtari Abdelhak	Grade,	Mohamed Boudiaf University of M′sila	**Supervisor.**
Sengouga Abdelmouhsen	Grade,	Mohamed Boudiaf University of M′sila	**Examiner.**

University year 2025/2026

Acknowledgments

I would like to express my sincere thanks and gratitude to my esteemed professors who accompanied and supported me throughout my academic journey, especially: **Dr. Mokhtari Abdelhak**, my supervisor, for his continuous support, valuable advice, and guidance, which were essential in completing this work. And **Dr. Abdelkebir Saad**, for his help, attention, and encouragement, which had a positive impact on the completion of this thesis. I would also like to express my deepest thanks to **Dr.Bouafia Dahmane** , the president of the examination committee, for accepting to evaluate this work, and to **Dr.Sengouga Abdelmouhsen**, the examiner, for their interest and participation in the defense of this thesis. Finally, I would like to thank everyone who supported me, directly or indirectly, with a word or advice.

الإهداء

بسم الله الرحمن الرحيم

{من قال أنا لها نالها و أنا لها و إن أبت رغما عنها أتيت بها}

الحمد لله وكفى والصلاة والسلام على الحبيب المصطفى وأهله ومن وفى الحمد لله على توفيقه وفضله الذي انار لنا درب العلم والمعرفة واعاننا على إتمام هدا العمل المتواضع بعد رحلة طويلة من التعب والاجتهاد بعد سنوات من السهر والمثابرة

"اما بعد "

اهدي هذا التخرج الى نفسي أولا الى تلك الروح التي خاضت الكثير من المعارك بصمت وتحملت التعب والضغط والخذلان لكنها لم تستسلم الى نفسي التي كانت تؤمن دائما ان بعد كل صعوبة فرج الى نفسي التي كانت السند حين لم يكن احد يعلم بما اشعر به

الى الذي زين اسمي باجمل الألقاب الى الذي دعمني واعطاني بلا مقابل الى الذي تحمل قساوة الحياة من اجلنا كنت دائما القدوة التي افتخر بها والى من انتظر هذه اللحظة ليفتخر بي

(ابي الغالي) حفظك الله ورعاك

الى تلك الانسانة العظيمة التي طالما تمنت ان تقر عينيها برؤيتي في يوم كهذا الى الغالية التي زينت حياتي بضياء البدر الى الداعم الأول لتحقيق هذا الطموح الى المحفز الأول لاكمال هذا المشوار الى من كانت دعواتها تحيطني الى القلب الحنون

(امي الغالية) اطال الله في عمرك

الى سندي في الحياة الى اجمل نعمة رزقني الله بها شكرا لوجودكم الدائم الى جانبي ولدعمكم ومحبتكم التي كانت تمنحني القوة والأمان اطال الله في عمركم دمتم لي سندا لا يميل

اخوتي (ايناس نعمة اية نور اسلام)

الى اللواتي شاركنني أيام الدراسة والإقامة الجامعية الى اللواتي كانو معي في كل لحظة في الفرح والحزن حفظكن الله ورزقكن فرحا دائما كل واحدة باسمها ومقامها

(صديقاتي)

الى كل من مد لي يد العون من بعيد او من قريب الى كل من ساندني بكلمة طيبة او بدعوة صادقة او موقف جميل اللهم بارك لهم في حياتهم وارزقهم يارب شكرا جزيلا

وفي الختام الحمد لله رب العالمين على هذه التجربة العلمية المليئة بالاعمال والتحديات

Contents

Introduction **1**

1 Preliminary **2**

1.1 Functional Spaces . . . 2

1.1.1 L^p Spaces . . . 2

1.1.2 $W^{1,p}(\Omega)$ Spaces . . . 2

1.1.3 Sobolev Embeddings . . . 4

1.1.4 $W_0^{1,p}(\Omega)$ Space . . . 4

1.2 Differentiable of Functional and Critical Points . . . 5

1.2.1 Gâteaux differentiability . . . 5

1.2.2 Fréchet differentiability . . . 5

1.2.3 Critical Points . . . 6

1.3 **O**perators on Banach Spaces . . . 7

2 Variational Methods for a class of p-Kirchhoff Problem on the half-line **9**

2.1 Introduction . . . 9

2.2 EMBEDDING THEOREM . . . 11

2.3 Main Result . . . 14

2.4 Example . . . 24

3 A Double phase problem on the half-line **26**

3.1 Musielak-Orlicz space . . . 26

3.2 Embedding . . . 28

3.3 Main Result . . . 29

3.4 Example . . . 39

Conclusion **40**

Notation

Throughout this work, we use the following notations.

- $\mathbb{N}$: set of natural numbers.
- $\mathbb{R}$: set of real numbers.
- $\mathbb{R}_+$: set of positive real numbers.
- $(0, +\infty)$: the half-line equipped with the Lebesgue measure.
- $u : (0, +\infty) \to \mathbb{R}$: measurable function.
- u': derivative of u.
- $C((0, +\infty))$: space of continuous functions on $(0, +\infty)$.
- $C^1((0, +\infty))$: space of continuously differentiable functions on $(0, +\infty)$.
- $L^p(0, +\infty)$: Lebesgue space endowed with the norm

$$\|u\|_{L^p} = \left(\int_0^{+\infty} |u(t)|^p \, dt \right)^{\frac{1}{p}}.$$

- $L^H(0, +\infty)$: Musielak–Orlicz space associated with the generalized N-function

$$H(t, s) = s^p + a(t) s^q.$$

- $W^{1,H}(0, +\infty)$: Musielak–Orlicz–Sobolev space defined by

$$W^{1,H}(0, +\infty) = \left\{ u \in L^H(0, +\infty) \; ; \; u' \in L^H(0, +\infty) \right\}.$$

- ρ_H: modular associated with H, defined by

$$\rho_H(u) = \int_0^{+\infty} H(t, |u(t)|) \, dt.$$

- $\|\cdot\|_H$: Luxemburg norm in $L^H(0,+\infty)$.
- $\|\cdot\|_{W^{1,H}}$: norm on $W^{1,H}(0,+\infty)$, given by

$$\|u\|_{W^{1,H}} = \|u\|_H + \|u'\|_H.$$

- $C_{l,r}(0,+\infty)$: weighted continuous function space.
- $E \hookrightarrow F$: continuous embedding of E into F.
- $E \hookrightarrow\hookrightarrow F$: compact embedding of E into F.
- $u_n \rightharpoonup u$: weak convergence.
- $u_n \to u$: strong convergence.
- $\langle\cdot,\cdot\rangle$: duality pairing.
- $J : W^{1,H}(0,+\infty) \to \mathbb{R}$: the associated energy functional.
- J': Gâteaux derivative of J.
- $a.e.$: almost everywhere.
- $C_0^\infty(0,+\infty)$: space of infinitely differentiable functions with compact support.
- $\|u\|_\infty$: supremum norm of u.

Introduction

The study of nonlinear differential equations is an important area in modern mathematical analysis due to its role in describing many physical, engineering and mechanical phenomena. Significant progress has been achieved through variational methods ([4]) and critical point theory, which provide powerful tools for proving existence and multiplicity results. Among the most studied problems are boundary value problems on unbounded domains, especially on the half-line $[0, +\infty)$, because of the lack of compactness compared with bounded domains.

The aim of this memory is to study the existence and multiplicity of solutions for a class of nonlinear Kirchhoff problems ([11]) defined on the half-line. The approach is based on variational methods, the Palais–Smale condition, and Krasnoselskii genus theory, which are key tools in critical point theory to obtain infinitely many nontrivial solutions.

This work also concerns double phase problems ([10] [16]) of the form

$$H(x,t) = t^p + a(x)t^q, \qquad 1 < p < q, \quad a(x) \geq 0,$$

which arise in elasticity, nonlinear mechanics, and composite materials. These models describe nonhomogeneous behaviors that cannot be treated by classical growth conditions. The half-line setting makes the analysis more delicate due to compactness issues.

This memoir is divided into three chapters.

The first chapter is devoted to the preliminary results, Sobolev spaces ([7]), and some basic concepts required for this work. These results provide the theoretical framework used in the study of the problems considered in the subsequent chapters.

The second chapter studies a Kirchhoff problem on the half-line in $W_0^{1,p}(0, +\infty)$, and proves, via variational methods ([4]) and genus theory, the existence of infinitely many pairs of nontrivial solutions.

The third chapter studies a double phase problem in the Musielak–Sobolev space $W^{1,H}(0, +\infty)$ ([19]), with a generalized N-function ([6]) of (p, q)-growth, and establishes the existence of infinitely many pairs of nontrivial solutions via variational methods.

Chapter 1

Preliminary

1.1 Functional Spaces

1.1.1 L^p Spaces

Definition 1.1. [7] Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 < p < \infty$. We define

$$L^p(\Omega) = \{g : \Omega \to \mathbb{R}^N \quad g\, is\, measurable\, and\; |f(x)| \leq c \qquad \text{a.e. in } \Omega\}$$

The norm in $L^p(\Omega)$ is given by:

$$\|g\|_{L^p} = \|g\|_p = \left(\int_\Omega |g(x)|^p \; dx\right)^{\frac{1}{p}}$$

Definition 1.2. [7] For $p = \infty$, we define the lebesgue space as follow :

$$L^\infty(\Omega) = \{g : \Omega \to \mathbb{R}^N \; ; \; g\, is\, measurable\, and\; |g(x)| \leq c \qquad \text{a.e. in } \Omega\}$$

where c is a positive constant. The norm in $L^\infty(\Omega)$ is defined by :

$$\|g\|_{L^\infty} = \|g\|_\infty = \inf\{c : |g(x)| \leq c \text{ a.e. in } \Omega\}$$

Theorem 1.1. *[7] For any $1 \leq p \leq \infty$, the space $L^p(\Omega)$ is a Banach space. Moreover,*

- *$L^p(\Omega)$ is reflexive for $1 < p < \infty$,*
- *$L^p(\Omega)$ is separable for $1 \leq p < \infty$.*

1.1.2 $W^{1,p}(\Omega)$ Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

We denote by $\mathcal{D}(\Omega)$ the set of functions of class $C^\infty(\Omega)$ with compact support contained in Ω.

Definition 1.3. The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \; ; \; \exists\, g_i \in L^p(\Omega) \text{ such that } \int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega g_i \varphi \, dx, \; \forall \varphi \in \mathcal{D}(\Omega), \; i = 1, \ldots, N \right\}.$$

For $u \in W^{1,p}(\Omega)$, we set

$$\frac{\partial u}{\partial x_i} = g_i, \quad i = 1, \ldots, N,$$

and we define the gradient of u by

$$\nabla u = \operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N} \right).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}$$

, or equivalently

$$\|u\|_{W^{1,p}} = \left(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p\right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty.$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

The space $H^1(\Omega)$ is endowed with the scalar product

$$(u, v)_{H^1(\Omega)} = \int_\Omega uv \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx.$$

The associated norm

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)^{\frac{1}{2}}$$

is equivalent to the $W^{1,2}(\Omega)$ norm.

Proposition 1.1. *The space $W^{1,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$, reflexive for $1 < p < \infty$, and separable for $1 \leq p < \infty$. In particular, $H^1(\Omega)$ is reflexive, separable, and a Hilbert space.*

1.1.3 Sobolev Embeddings

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces.

Notation 1.1. 1. We say that E is continuously embedded into F if the canonical injection $j : E \to F$ is continuous, that is, if there exists a constant $c > 0$ such that

$$\|x\|_F \leq c\|x\|_E \quad \text{for all } x \in E.$$

In this case, we write $E \hookrightarrow F$.

2. We say that E is compactly embedded into F if the canonical injection $j : E \to F$ is compact, that is, for every bounded sequence (u_n) in E, there exists a subsequence (u_{n_k}) converging in F. In this case, we write $E \hookrightarrow\hookrightarrow F$.

If $1 \leq p < N$, the Sobolev exponent associated with p is defined by

$$p^* = \frac{Np}{N-p} \qquad \text{or equivalently} \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

If $p \geq N : p^* = \infty$

Theorem 1.2. *[7] Let $1 \leq p \leq \infty$, and assume that Ω is an open set of class C^1 with bounded boundary, or $\Omega = \mathbb{R}^N_+$. Then:*

1. $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, *If* $p < N \quad \forall q \in [1, p^*)$.

2. $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $If p = N \quad \forall q \in [p, +\infty)$.

3. $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. $If p > N$

1.1.4 $W^{1,p}_0(\Omega)$ Space

Definition 1.4. Let $1 \leq p \leq \infty$. We denote by $W^{1,p}_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. We note that

$$W^{1,p}_0(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}} = \left\{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega\right\}.$$

Moreover,

$$H^1_0(\Omega) = W^{1,2}_0(\Omega).$$

The space $W^{1,p}_0(\Omega)$ provided with norm induced by $W^{1,p}(\Omega)$, $H^1_0(\Omega)$ is a Hilbert space for the scalar product of $H^1(\Omega)$. We put $\|u\|_{H^1_0} = \|u\|_{H^1}$.

Remark 1.1. When $\Omega = \mathbb{R}^N$, the space $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, and therefore

$$W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N).$$

Proposition 1.2 (Poincaré Inequality). *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then there exists a constant $c > 0$ such that*

$$\|u\|_{W^{1,p}(\Omega)} \leq c \|\nabla u\|_{L^p(\Omega)} \quad \textit{for all } u \in W_0^{1,p}(\Omega).$$

In other words, on $W_0^{1,p}(\Omega)$, the quantity $\|\nabla u\|_{L^p(\Omega)}$ defines a norm equivalent to the $W^{1,p}(\Omega)$ norm.

1.2 Differentiable of Functional and Critical Points

In what follows, we present the two principal definitions of differentiable and their main properties. We start with the directional derivative. for more details see [4] [15]

1.2.1 Gâteaux differentiability

Definition 1.5. Let E be a Banach space, $\Omega \subset E$ an open set, and let $I : \Omega \to \mathbb{R}$ be a functional. We say that I is *Gâteaux differentiable* (or *G-differentiable*) at $u \in \Omega$ if there exists $A \in E'$ (linear and continuous), denoted by $I'_G(u)$, such that for all $v \in E$,

$$\lim_{t \to 0} \frac{I(u + tv) - I(u)}{t} = \langle A, v \rangle. \tag{1.1}$$

If I is Gâteaux differentiable at u, then there exists a unique linear functional $A \in E'$ satisfying (1.1). It is called the *Gâteaux differential* of I at u and is denoted by $I'_G(u)$.

Definition 1.6. If the functional I is Gâteaux differentiable at every u of an open set $U \subset E$, we say that I is Gâteaux differentiable on U. The map

$$I'_G : E \to E', \quad u \mapsto I'_G(u),$$

is called the *Gâteaux derivative* of I.

1.2.2 Fréchet differentiability

Definition 1.7. Let E be a Banach space, $\Omega \subset E$ an open set, and let $I : \Omega \to \mathbb{R}$ be a functional. We say that I is *Fréchet differentiable* at $u \in \Omega$ if there exists $A_u \in E'$ such that

$$\lim_{\|v\| \to 0} \frac{I(u + v) - I(v) - A_u(v)}{\|v\|} = 0. \tag{1.2}$$

Thus, for a Fréchet differentiable functional I, we have :

$$I(u+v) - I(u) = A_u(v) + o(|v\|), \quad \text{where} \quad \lim_{\|v\|\to 0} \frac{o(\|v\|)}{\|v\|} = 0.$$

If a functional I is differentiable at u, then A_u is unique.

Definition 1.8. Let $I : \Omega \to \mathbb{R}$ be differentiable at $u \in \Omega$. The unique element A_u such that (1.2) holds is called the *Fréchet differential* of I at u, and is denoted by $I'(u)$. Then we can write

$$I(u+v) - I(u) = \langle I'(u), v\rangle + o(\|v\|), \text{as } \|v\| \to 0.$$

Definition 1.9. Let $\Omega \subset E$ be an open set.

1. If the functional I is differentiable at every $u \in \Omega$, we say that I is differentiable on Ω.
2. The map $I' : \Omega \to E'$ that sends $u \in \Omega$ to $I'(u) \in E'$ is called the Fréchet derivative of I. Note that I' is in general a nonlinear map.
3. If the derivative I' is continuous from Ω to E', we say that I is of class C^1 on Ω and write $I \in C^1(\Omega)$.

Remark 1.2.

- Notice that $I'(u)$ is defined on E, even if I is defined only on Ω.
- If I is Fréchet differentiable at $u \in \Omega$, then I is continuous at u.
- If I is Fréchet differentiable at $u \in \Omega$, then I is Gâteaux differentiable at $u \in \Omega$ and $I'(u) = I'_G(u)$.

Proposition 1.3. *Suppose that $\Omega \subset E$ is an open set such that I is G-differentiable in Ω and that $I'_G : \Omega \to E'$ is continuous. Then I is also Fréchet differentiable at u, and we have $I'_G(u) = I'(u)$.*

Remark 1.3. The importance of this result resides in the fact that calculating the Gâteaux derivative and then proving its continuity is often technically easier than directly proving the Fréchet differentiability. Also, we deduce that the functional is of class C^1.

1.2.3 Critical Points

Definition 1.10. Let O be an open subset of a Banach space E, and let $I : O \to \mathbb{R}$ be differentiable. We say that $u \in O$ is a *critical point* of I if

$$I'(u) = 0.$$

That is , $\forall\, v \;\; in\, E \;\langle I'(u), v\rangle = 0$

1.3 Operators on Banach Spaces

Definition 1.11. The operator $T : E \to F$ is called *continuous at the point* $x_0 \in E$ if for any sequence $(x_n) \subset E$ we have

$$x_n \to x_0 \text{ in } E \implies T(x_n) \to T(x_0) \text{ in } F$$

or equivalently,

$$\|x_n - x_0\|_E \to 0 \implies \|T(x_n) - T(x_0)\|_F \to 0.$$

This definition is means that:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in E : \|x - x_0\|_E \leq \delta \implies \|T(x) - T(x_0)\|_F \leq \varepsilon.$$

Definition 1.12 (Coercivity)**.** Let $J : E \to \mathbb{R} \cup \{+\infty\}$ be a functional on E. We say that J is *coercive* if and only if

$$\lim_{\|u\|_E \to +\infty} \frac{J(u)}{\|u\|_E} = +\infty.$$

Definition 1.13 (Bounded from Below)**.** Let E be a Banach space and $I : E \to \mathbb{R}$ a functional. We say that I is *bounded from below* if there exists a constant $m \in \mathbb{R}$ such that

$$I(u) \geq m \quad \text{for all } u \in E.$$

Definition 1.14 (Equicontinuous)**.** We say that the sequence (f_n) is equicontinuous if:

- $f_n : K \to \mathbb{R}$, with K compact.
- $\forall \varepsilon > 0$, there exists $\delta > 0$, $\forall \quad x, y \in K$:

$$|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon, \quad \forall n.$$

The δ does not depend on n.

Definition 1.15 (Extremum points)**.** [17] Let E be a Banach space, and let $I : E \to \mathbb{R}$ be a functional. A point $u_0 \in E$ is called an *extremum* of I if there exists a neighborhood V_{u_0} of u_0 such that either

1. $I(u) \leq I(u_0)$ for all $u \in V_{u_0}$, in which case I is said to have a *maximum* at u_0 .
2. $I(u) \geq I(u_0)$ for all $u \in V_{u_0}$, in which case I is said to have a *minimum* at u_0 .

Definition 1.16 (Carathéodory function)**.** Let $f : T \times X \to \mathbb{R}$, where $T \subset \mathbb{R}$ and X is a real Banach space, such that:

- $f(\cdot, x)$ is measurable for every $x \in X$,
- $f(t, \cdot)$ is continuous for almost every $t \in T$.

Chapter 2

Variational Methods for a class of p-Kirchhoff Problem on the half-line

2.1 Introduction

Kirchhoff equation was introduced in 1876 as an extension of the classical **D'Alembert's wave equation**, considering the effects of the changes in the length of a string during vibrations. Due to its importance and applications in physics, electrical engineering, and biology, the existence and multiplicity of **nontrivial solutions** for Kirchhoff type **boundary value problems (BVPs)** have been widely studied.([11] [3]) Many authors have obtained solutions for Kirchhoff type equations using **variational methods**. **Genus theory**, introduced by Krasnoselskii ([8]), has been applied to prove existence and multiplicity results for p**-Kirchhoff type problems**, including problems on bounded domains, fractional cases, and $p(t)$**-Kirchhoff equations**. ([18] [14]) For example, a classical p-Kirchhoff boundary value problem is given by

$$-\left[M\left(\int_\Omega |\nabla u|^p\,dx\right)\right]\Delta_p u = f(x,u) \quad \text{in } \Omega, \quad u=0 \quad \text{on } \partial\Omega$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1<p<N$, and M and f are continuous functions. In this chapter, we establish the **variational framework** for the p-Kirchhoff functional on the **half-line** and study its main properties, such as **Gateaux differentiability** and the **continuity of the derivative**, as a preliminary step for applying genus theory to prove existence and multiplicity of solutions.

In this chapter , we consider the following kirchhoff problem:

$$\begin{cases} (a+b\,\|u\|^p_{W_0^{1,p}})\left(-\left(|u'(t)|^{p-2}\,u'(t)\right)' + |u(t)|^{p-2}\,u(t)\right) = \lambda h(t,u(t)) \\ u(0)=u(\infty)=0 \end{cases} \tag{2.1}$$

where $h:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is a carathéodory function .

Definition 2.1. We say that $u\in W_0^{1,p}(0,\infty)=X$ is a *weak solution* of Problem (2.1) if and only if

$$\left(a+b\,\|u\|^p_{W_0^{1,p}}\right)\int_0^\infty |u'(t)|^{p-2}\,u'(t)v'(t)\,dt+\int_0^\infty |u(t)|^{p-2}\,u(t)v(t)\,dt=\lambda\int_0^\infty h(t,u(t))v(t)\,dt$$

for all $v\in X$.

Denote by $C([0,\infty))$ the space of continuous functions on $[0,\infty)$, and $C_0^\infty([0,\infty))$ the space of indefinitely differentiable functions with compact support in $[0,\infty)$.

Now $W_0^{1,p}(0,\infty)$ denotes the closure of $C_0^\infty(0,\infty)$ in $W^{1,p}(0,\infty)$.

Let $r:[0,\infty)\to(0,\infty)$ be a continuously differentiable and bounded function with

$$M_1=\max\left(\|r\|_{L^q},\|r'\|_{L^q}\right)<\infty.$$

$C_{l,r}(0,\infty)$ is defined by

$$C_{l,r}(0,\infty)=\left\{u\in C([0,\infty);\mathbb{R}):\lim_{t\to\infty} r(t)u(t)\text{ exists}\right\}.$$

endowed with the norm

$$\|u\|_{\infty,r}=\sup_{t\in[0,\infty)} r(t)\,|u(t)|\,.$$

Remark 2.1 (Difference Between Classical p-Laplacian and p-Kirchhoff Problem)**.** In classical p-Laplacian problems, the operator acts *locally*, depending only on u and its derivatives at each point:

$$-\Delta_p u=-\mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

In contrast, the p-Kirchhoff problem includes a *nonlocal term*, where the coefficient depends on the global Sobolev norm of u:

$$M\left(\|u\|^p_{W_0^{1,p}}\right)(-\Delta_p u).$$

On the half-line $[0,\infty)$ nonlocality complicates the problem, motivating the use of variational and genus methods to prove existence and multiplicity of solutions.

Remark 2.2 (Difficulties on the Half-Line)**.** Studying the p-Kirchhoff problem on the half-line $[0,\infty)$ presents specific challenges. Unlike bounded domains, standard compactness results for Sobolev spaces do not hold, making it harder to extract convergent subsequences. Moreover, the nonlocal term, depending on the global Sobolev norm $\|u\|^p_{W_0^{1,p}}$ means that any change in u affects the operator throughout the domain. These factors require careful treatment using weighted functional spaces, variational methods, and Krasnoselskii genus theory to prove existence and multiplicity of solutions.

Definition 2.2 (Krasnoselskii genus)**.** Let $A \in \Sigma$. The Krasnoselskii genus $\gamma(A)$ is defined as beinxg the least positive integer n such that there is an odd mapping $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$. If such n does not exist, we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

Proposition 2.1. *[15] Let $A, B \in \Sigma$. If there exists an odd map $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$. Consequently, if there exists an odd homeomorphism $f : A \to B$, then $\gamma(A) = \gamma(B)$.*

Theorem 2.1. *[15] Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Note that $\gamma(S^{N-1}) = N$. If E is of infinite dimension and separable and S is the unit sphere in E, then $\gamma(S) = \infty$

Definition 2.3 (Ps condition)**.** Let $J \in C^1(E, \mathbb{R})$. If any sequence $(u_n) \subset E$ for which $(J(u_n))$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ in E' possesses a convergent subsequence, then we say that J satisfies the *Palais-Smale condition* (the (PS) condition).

In order to establist the existence and multiplicity of solution of problem (2.1) we need an important result about the existence of critical point of symetric functional . [22] [9]

Theorem 2.2. *Let $J \in C^1(E, \mathbb{R})$ be a functional satisfying the Palais–Smale condition. Also suppose that:*

- *J is bounded from below and even;*
- *there is a compact set $K \in \Sigma$ such that $\gamma(K) = k$ and $\sup_{x \in K} J(x) < J(0)$.*

Then J possesses at least k pairs of distinct critical points and their corresponding critical values are less than $J(0)$.

2.2 EMBEDDING THEOREM

We study the embedding of $W_0^{1,p}(0, \infty)$ into the weighted space $C_{l,r}([0, \infty))$, which is essential for proving the compactness results used later.
Similar reasoning as in [13] yields the following compactness criterion in the space $Cl, r([0, +\infty))$.

Lemma 2.1. *Let $D \subset C_{l,r}([0, \infty), \mathbb{R})$ be a bounded set. Then D is relatively compact if the following conditions hold:*

(a) *D is equicontinuous on any compact sub-interval of $\mathbb{R}^+$, i.e., for all compact $J \subset [0,\infty)$ and for all $\varepsilon > 0$, there exists $\delta > 0$ for all $t_1, t_2 \in J$,*

$$|t_1 - t_2| < \delta \ \Rightarrow \ |r(t_1)u(t_1) - r(t_2)u(t_2)| \leq \varepsilon,$$

for all $u \in D$;

(b) *D is equiconvergent at ∞, i.e., for all $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that for all $t > T(\varepsilon)$,*

$$|r(t)u(t) - (ru)(\infty)| \leq \varepsilon,$$

for all $u \in D$.

Lemma 2.2. *The following embbiding : $W_0^{1,p}(0,\infty) \hookrightarrow C_{l,r}(0,\infty)$ is continuous , that is , there existe $M > 0$ such that :*

$$\|u\|_{C_{l,r}} \leq M \ \|u\|_{W_0^{1,p}}$$

Proof: Let $u \in W_0^{1,p}(0,\infty)$ using Holder inequality and some classical techniques , we set :

$$\begin{aligned}
|r(t)u(t)| &= |r(t)u(t) - r(0)u(0)| \\
&= \left| \int_0^t (ru)'(s)\, ds \right| \\
&\leq \left| \int_0^t \big(r'(s)u(s) + r(s)u'(s)\big)\, ds \right| \\
&\leq \int_0^\infty |r'(s)u(s)| \ ds + \int_0^\infty |r(s)u'(s)| \ ds \\
&\leq \left(\int_0^\infty (r'(s))^q\, ds \right)^{\frac{1}{q}} \left(\int_0^\infty (u(s))^p\, ds \right)^{\frac{1}{p}} \\
&\quad + \left(\int_0^\infty (r(s))^q\, ds \right)^{\frac{1}{q}} \left(\int_0^\infty (u'(s))^p\, ds \right)^{\frac{1}{p}} \\
&\leq |r'|_{L^q} \left(\int_0^\infty (u(s))^p\, ds \right)^{\frac{1}{p}} + |r|_{L^q} \left(\int_0^\infty (u'(s))^p\, ds \right)^{\frac{1}{p}} \\
&\leq \max\big(|r'|_{L^q} \, , |r|_{L^q} \big) \, \|u\|_{W_0^{1,p}} \, .
\end{aligned}$$

Hence, we have

$$\|u\|_{\infty,r} \leq M \, \|u\|_{W_0^{1,p}}, \quad \text{where } M = \max(\|r'\|_{L^q}, \|r\|_{L^q}).$$

This estimate shows that the weighted sup-norm is controlled by the Sobolev norm, hence the embedding is continuous.

Lemma 2.3. *The embedding*

$$W_0^{1,p}(0,\infty) \hookrightarrow\hookrightarrow C_{l,r}[0,\infty)$$

is compact.

Proof: Let D be a bounded subset of $X = W_0^{1,p}(0,\infty)$. Due to the continuous embedding $X \hookrightarrow C_{l,r}[0,\infty)$, it follows that D is also bounded in $C_{l,r}[0,\infty)$.

Consequently, there exists a constant $R > 0$ such that

$$\|u\|_{W_0^{1,p}} \leq R, \quad \forall u \in D.$$

(a) D is equicontinuous on every compact interval of $[0,\infty)$: Indeed, let $u \in D$ and let $t_1, t_2 \in J \subset [0,\infty)$, where J is a compact sub-interval. By using Hölder's inequality, we have:

$$\begin{aligned}
|r(t_1)u(t_1) - r(t_2)u(t_2)| &= \left|\int_{t_1}^{t_2} (ru)'(s)\, ds\right| \\
&= \left|\int_{t_1}^{t_2} \big(r'(s)u(s) + r(s)u'(s)\big)\, ds\right| \\
&\leq \int_{t_1}^{t_2} |r'(s)u(s)|\, ds + \int_{t_1}^{t_2} |r(s)u'(s)|\, ds \\
&\leq \left(\int_{t_1}^{t_2} r'^q(s) ds\right)^{1/q} \left(\int_{t_1}^{t_2} u^p(s) ds\right)^{1/p} \\
&\quad + \left(\int_{t_1}^{t_2} r^q(s) ds\right)^{1/q} \left(\int_{t_1}^{t_2} u'^p(s) ds\right)^{1/p} \\
&\leq \max\left(\int_{t_1}^{t_2} r'^q(s) ds; \int_{t_1}^{t_2} r^q(s) ds\right) \|u\|_X \\
&\leq R \max\left(\int_{t_1}^{t_2} r'^q(s) ds; \int_{t_1}^{t_2} r^q(s) ds\right) \longrightarrow 0
\end{aligned}$$

as $|t_1 - t_2| \to 0$

(b) D is equiconvergent at ∞; for $t \in [0,\infty)$ and $u \in D$

$$\begin{aligned}
|(ru)(t) - (ru)(\infty)| &= \left|\int_t^{\infty} (ru)'(s)\, ds\right| \\
&\leq \left|\int_t^{\infty} (r'(s)u(s) + r(s)u'(s)) ds\right| \\
&\leq R \max\left(\int_t^{\infty} r'^q(s) ds; \int_t^{\infty} r^q(s) ds\right) \longrightarrow 0
\end{aligned}$$

as $t \to \infty$.

Hence, we obtain the compact embedding:

$$X \hookrightarrow\hookrightarrow C_{l,r}[0,\infty).$$

2.3 Main Result

the main result in this chapter is the following :

Theorem 2.3. *Assume that the function $h(t,u)$ satisfies the following conditions:*

(h1) *There exists a constant $\gamma \in [0, 2p-1)$ and two positive functions α, β and*

$$\frac{\alpha}{r}, \frac{\beta}{r^{\gamma+1}} \in L^1(0,\infty),$$

such that

$$|h(t,s)| \leq \alpha(t) + \beta(t)\,|s|^{\gamma} \quad \text{for all } (t,s) \in [0,\infty) \times \mathbb{R}.$$

(h2) *The function $h(t,s)$ is odd with respect to s , and*

$$H(t,s) = \int_0^s h(t,\xi)\,d\xi > 0 \quad \text{for all } (t,s) \in [0,\infty) \times \mathbb{R} \setminus \{0\}.$$

Then, for any $k \in \mathbb{N}$, there exists λ_k such that if $\lambda > \lambda_k$, Problem (2.1) has at least k distinct pairs of nontrivial solutions.

Proof

The corresponding functional associated with Problem (2.1) is defined by

$$\varphi(u) = \frac{a}{p}\,\|u\|^p_{W_0^{1,p}} + \frac{b}{2p}\,\|u\|^{2p}_{W_0^{1,p}} - \lambda \int_0^\infty H(t,u(t))\,dt, \tag{2.2}$$

where

$$H(t,u) = \int_0^u h(t,\xi)\,d\xi.$$

1. Regularity of the Functional ϕ

We show that the functional ϕ belongs to class C^1. The proof is divided into three steps.

Step 1 : **Well-definedness of ϕ.**

Indeed, let $u \in X$. It follows from assumption (h1) and Lemma(2.2) tha

$$\begin{aligned}
\int_0^\infty |H(t,u(t))|\, dt &= \int_0^\infty \left| \int_0^u h(t,\xi)\, d\xi \right| dt \\
&\leq \int_0^\infty \int_0^u |h(t,\xi)|\, d\xi\, dt \\
&\leq \int_0^\infty \int_0^u \left(\alpha(t) + \beta(t)\,|xi|^\gamma\right) d\xi\, dt \\
&\leq \int_0^\infty \left(\alpha(t)u(t) + \frac{\beta(t)}{\gamma+1}\,|u(t)|^{\gamma+1}\right) dt \\
&\leq \int_0^\infty \left(\frac{\alpha(t)}{r(t)}\, r(t)\,|u(t)| + \frac{\beta(t)}{(\gamma+1)(r(t))^{\gamma+1}} \left(r(t)\,|u(t)|\,\right)^{\gamma+1}\right) dt \\
&\leq \|u\|_{\infty,r} \int_0^\infty \frac{\alpha(t)}{r(t)}\, dt + \frac{\|u\|_{\infty,r}^{\gamma+1}}{\gamma+1} \int_0^\infty \frac{\beta(t)}{(r(t))^{\gamma+1}}\, dt \\
&\leq c\,\|u\|_{W_0^{1,p}} \left|\frac{\alpha}{r}\right|_{L^1} + \frac{c^{\gamma+1}}{\gamma+1}\,\|u\|_{W_0^{1,p}}^{\gamma+1} \left|\frac{\beta}{r^{\gamma+1}}\right|_{L^1} \\
&\leq c_1\,\|u\|_{W_0^{1,p}} + c_2\,\|u\|_{W_0^{1,p}}^{\gamma+1} < \infty
\end{aligned}$$

Hence φ is well defined on X

Step 2 : **Gâteaux-differentiability of ϕ**

We consider the functional

$$\phi(u) = \phi_1(u) + \phi_2(u),$$

Where :

$$\phi_1(u) = \frac{a}{p}\|u\|_{W_0^{1,p}}^p + \frac{b}{2p}\|u\|_{W_0^{1,p}}^{2p},$$

$$\phi_2(u) = \int_0^\infty H(t,u(t))\, dt, \quad H(t,u) = \int_0^u h(t,s)\, ds.$$

To prove that ϕ is Gâteaux-differentiable, we divide the proof into two parts corresponding to ϕ_1 and ϕ_2:

1. Differentiability of ϕ_1: We show that ϕ_1 is Gâteaux-differentiable and compute its derivative.[4] ϕ_1 is expressed as a sum of two terms, each involving a power of the norm in $W_0^{1,p}$.

   Since the norm $\|u\|_{W_0^{1,p}}$ is Gâteaux-differentiable, and the sum of differentiable functionals is also differentiable, it follows that ϕ_1 is Gâteaux-differentiable. and its derivative is given by:

   $$\phi_1'(u)\cdot v = \left(a + b\,\|u\|_{W_0^{1,p}}^p\right) \left[\int_0^\infty |u(t)|^{p-2}\, u(t)v(t)\, dt + \int_0^\infty |u'(t)|^{p-2}\, u'(t)v'(t)\, dt\right], \quad \forall v \in W_0^{1,p}.$$

   Proof:

2. Differentiability of ϕ_2

   For $u \in X$ We consider the functional ϕ_2 defined by :

$$\phi_2(u) = \int_0^\infty H(t, u(t))\, dt, \quad H(t,u) = \int_0^u h(t,s)\, ds.$$

   For $u\,, v \in X$ for small $s > 0$, we have:

$$\phi_2(u+sv) - \phi_2(u) = \int_0^\infty \big(H(t, u(t)+sv(t)) - H(t,u(t))\big)\, dt.$$

   The mean value theorem gives $\theta \in (0,1)$ such that

$$H(t, u(t)+sv(t)) - H(t,u(t)) = sv(t)\, h(t, u(t)+s\theta v(t)), \quad \theta \in (0,1).$$

   Then,

$$\phi_2(u+sv) - \phi_2(u) = \int_0^\infty sv(t)\, h(t, u(t)+s\theta v(t))\, dt.$$

   Hence

$$\frac{\phi_2(u+sv) - \phi_2(u)}{s} = \int_0^\infty h(t, u(t)+s\theta v(t)) v(t)\, dt.$$

   We estimate the integrand using the weighted norm $r(t)$:

$$\begin{aligned} |z_{u,v}| &= |u(t)+s\theta v(t)|\, r(t) \\ &\le r(t)|u(t)| + s\theta |v(t)|\, r(t) \\ &\le r(t)|u(t)| + r(t)|v(t)| \\ &\le \|u\|_{\infty,r} + \|v\|_{\infty,r} \\ &\le c\big(\|u\|_{W_0^{1,p}} + \|v\|_{W_0^{1,p}}\big) = R_{u,v}. \end{aligned} \tag{2.3}$$

   By assumption (H1), and using (2.3) , we get :

$$|h(t, u+s\theta v)||v(t)| \le \alpha(t)|v(t)| + \beta(t) R_{u,v}^\gamma |v(t)|,$$

$$\begin{aligned} |h(t, u+s\theta v)||v(t)| &\le \big(\alpha(t) + \beta(t)|u+s\theta v|^\gamma\big)|v(t)| \\ &\le \left(\alpha(t) + \beta(t)(r(t))^\gamma \frac{|u+s\theta v|^\gamma}{(r(t))^\gamma}\right)|v(t)| \\ &\le \alpha(t)|v(t)| + \beta(t)(r(t))^\gamma R_{u,v}^\gamma |v(t)| \\ &\le \frac{\alpha(t)}{r(t)}\, r(t)|v(t)| + \frac{\beta(t)}{(r(t))^{\gamma+1}}\, R_{u,v}^\gamma\, r(t)|v(t)| \\ &\le \frac{\alpha(t)}{r(t)}\|v\|_{\infty,r} + \frac{\beta(t)}{(r(t))^{\gamma+1}} R_{u,v}^\gamma \|v\|_{\infty,r} \\ &\le c\left(\frac{\alpha(t)}{r(t)} + \frac{\beta(t)}{(r(t))^{\gamma+1}} R_{u,v}^\gamma\right)\|v\|_{W_0^{1,p}} \qquad \in L^1(0,\infty). \end{aligned}$$

The Lebesgue Dominated Convergence Theorem implies that :

$$\lim_{s\to 0}\int_0^\infty h(t,u(t)+s\theta v(t))v(t)\,dt=\int_0^\infty h(t,u(t))v(t)\,dt. \tag{2.4}$$

Hence : ϕ_2 is Gâteaux-differentiable, and its derivative is given by :

$$\phi_2'(u)\cdot v=\int_0^\infty h(t,u(t))\,v(t)\,dt,\quad \forall v\in X.$$

Step 3: **Continuity of ϕ'**

Let $\{u_n\}\subset X$ be a sequence such that $u_n\to u$ in X as $n\to\infty$. Since X embeds continuously into $C_{l,r}$, we also have $u_n\to u$ in $C_{l,r}$. Consequently, the sequence (u_n) is bounded in $C_{l,r}$, so there exists a constant $R>0$ such that

$$\|u_n\|_{\infty,r}\le R\quad \text{for all } n.$$

For all $v\in X$, such that $\|v\|\le 1$ we have :

$$(\varphi_2'(u_n)-\varphi_2'(u),v)=\int_0^\infty \big(h(t,u_n(t))-h(t,u(t))\big)v(t)\,dt.$$

Hence,

$$|(\varphi_2'(u_n)-\varphi_2'(u),v)|\le\int_0^\infty |h(t,u_n(t))-h(t,u(t))||v(t)|\,dt\le\|v\|_{\infty,r}\int_0^\infty |h(t,u_n(t))-h(t,u(t))|r(t)\,dt.$$

Using the continuous embedding of X into $C_{l,r}$ again, there exists a constant $c>0$ such that

$$|(\varphi_2'(u_n)-\varphi_2'(u),v)|\le c\|v\|_{W_0^{1,p}}\int_0^\infty |h(t,u_n(t))-h(t,u(t))|r(t)\,dt.$$

Taking the supremum over all $v\in X$ with $\|v\|_{W_0^{1,p}}\le 1$, we obtain

$$\|\varphi_2'(u_n)-\varphi_2'(u)\|_{X'}\le c\int_0^\infty |h(t,u_n(t))-h(t,u(t))|r(t)\,dt.$$

Passing to the limit as $n\to\infty$ and using the Lebesgue dominated convergence theorem, we deduce that

$$\varphi_2'(u_n)\to\varphi_2'(u)\quad \text{in } X'.$$

Therefore, φ_2' is continuous, and hence $\varphi\in C^1(X,\mathbb{R})$, with

$$\langle\varphi'(u),v\rangle=(a+b\|u\|^p_{W_0^{1,p}})\left[\int_0^\infty |u(t)|^{p-2}u(t)v(t)\,dt+\int_0^\infty |u'(t)|^{p-2}u'(t)v'(t)\,dt\right]-\lambda\int_0^\infty h(t,u(t))v(t)\,dt. \tag{2.5}$$

Thus, the critical points of φ correspond exactly to the weak solutions of (2.1)

2. ϕ is bounded from below

Since the space X is continuously embedded into $C_{l,r}([0,\infty))$, there exists a positive constant c such that, for every $u \in X$, the following estimate holds

$$r(t)|u(t)| \leq \|u\|_{\infty,r} \leq c\|u\|_{W_0^{1,p}}. \tag{2.6}$$

Using assumptions (h_1) and (h_2) together with inequality (2.5), we estimate the functional ϕ as follows

$$\begin{aligned}
\phi(u) &= \frac{a}{p}\|u\|^p_{W_0^{1,p}} + \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda \int_0^\infty H(t,u(t))\,dt \\
&\geq \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda \int_0^\infty H(t,u(t))\,dt \\
&\geq \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda \int_0^\infty \left(\alpha(t)u(t) + \frac{\beta(t)}{\gamma+1}|u(t)|^{\gamma+1}\right)dt \\
&\geq \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda \int_0^\infty \left(\frac{\alpha(t)}{r(t)}\,r(t)|u(t)| + \frac{\beta(t)}{(\gamma+1)(r(t))^{\gamma+1}}(r(t)|u(t)|)^{\gamma+1}\right)dt \\
&\geq \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda\|u\|_{\infty,r}\int_0^\infty \frac{\alpha(t)}{r(t)}dt - \frac{\lambda}{\gamma+1}\|u\|^{\gamma+1}_{\infty,r}\int_0^\infty \frac{\beta(t)}{(r(t))^{\gamma+1}}dt \\
&\geq \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda c\|u\|_{W_0^{1,p}}\left|\frac{\alpha}{r}\right|_{L^1} - \frac{\lambda c^{\gamma+1}}{\gamma+1}\|u\|^{\gamma+1}_{W_0^{1,p}}\left|\frac{\beta}{r^{\gamma+1}}\right|_{L^1}.
\end{aligned} \tag{2.7}$$

Since $2p > \gamma+1 > 1$, we obtain

$$\lim_{\|u\|_{W_0^{1,p}}\to\infty} \phi(u) = +\infty.$$

Hence, the functional ϕ is coercive , consequently , since of C^1 we set ϕ bounded from below.

3. The functional ϕ satisfies the Palais–Smale condition

We verify that any sequence $(u_n) \subset X$ satisfying : $\phi(u_n)$ is bounded and $\phi'(u_n) \to 0$

contains a convergent subsequence in X, ensuring the compactness requirement of the Palais–Smale condition.

Let $(u_n) \subset X$ be a sequence such that

$$|\varphi(u_n)| \leq B \tag{2.8}$$

$$\varphi'(u_n) \to 0 \quad \text{in } X'.$$

First, we prove that (u_n) is bounded in X. Assume by contradiction that (u_n) is unbounded, Then there exists a subsequence (u_{n_k}) of (u_n) such that

$$\|u_{n_k}\| \to \infty \quad \text{as } k \to \infty.$$

Using inequalities (2.7) and (2.8), we obtain

$$B \geq \varphi(u_n) \geq C_1\|u_n\|^{2p} - C_2\|u_n\|^{\gamma+1} - C_3\|u_n\|.$$

Then

$$C_1\|u_n\|^{2p} - C_2\|u_n\|^{\gamma+1} - C_3\|u_n\| \leq B \tag{2.9}$$

which contradicts the boundedness of $(\varphi(u_n))$. Therefore, (u_n) is bounded in X .

Since (u_n) is bounded in X and $X = W_0^{1,p}(0,\infty)$ is a reflexive Banach space, there exist $u \in X$ and a subsequence, still denoted by (u_n), such that

$$u_n \rightharpoonup u \quad \text{weakly in } X.$$

Moreover, since the embedding

$$X = W_0^{1,p}(0,\infty) \hookrightarrow C_{l,r}([0,\infty))$$

is compact, we have

$$u_n \to u \quad \text{in } C_{l,r}([0,\infty)).$$

Next, by direct calculation , we obtain :

$$\begin{aligned}
\langle \varphi'(u_n) - \varphi'(u),\, u_n - u\rangle =&(a + b\|u_n\|^p)\left(\int_0^\infty |u_n(t)|^{p-2}u_n(t)\,(u_n(t) - u(t))\,dt\right.\\
&\left.+ \int_0^\infty |u_n'(t)|^{p-2}u_n'(t)\,(u_n'(t) - u'(t))\,dt\right)\\
&- \lambda\int_0^\infty h(t, u_n(t))\,(u_n(t) - u(t))\,dt\\
&- (a + b\|u\|^p)\left(\int_0^\infty |u(t)|^{p-2}u(t)\,(u_n(t) - u(t))\,dt\right.\\
&\left.+ \int_0^\infty |u'(t)|^{p-2}u'(t)\,(u_n'(t) - u'(t))\,dt\right)\\
&+ \lambda\int_0^\infty h(t, u(t))\,(u_n(t) - u(t))\,dt.
\end{aligned}$$

We have

$$\|u_n - u\|_{\infty,r} < M_1 \quad \text{and} \quad \|u_n\|_{\infty,r} < M_2. \tag{2.10}$$

Now, by using (h_1), we have

$$
\begin{aligned}
&|h(t,u_n(t)) - h(t,u(t))|\,|u_n(t)-u(t)| \\
&\le |h(t,u_n(t)) - h(t,u(t))| \cdot \frac{1}{r(t)}\, r(t)|u_n(t)-u(t)| \\
&\le \frac{1}{r(t)}\|u_n-u\|_{\infty,r}\,|h(t,u_n(t)) - h(t,u(t))| \\
&\le \frac{M_1}{r(t)}\big(|h(t,u_n(t))| + |h(t,u(t))|\big) \\
&\le \frac{M_1}{r(t)}\big(\alpha(t)+\beta(t)|u_n(t)|^\gamma + \alpha(t)+\beta(t)|u(t)|^\gamma\big) \\
&\le 2M_1\frac{\alpha(t)}{r(t)} + M_1\frac{\beta(t)}{r(t)}\big(|u_n(t)|^\gamma + |u(t)|^\gamma\big) \\
&\le 2M_1\frac{\alpha(t)}{r(t)} + M_1\frac{\beta(t)}{r(t)^{\gamma+1}}\big(r(t)^\gamma\big(|u_n(t)|^\gamma + |u(t)|^\gamma\big) \\
&\le 2M_1\frac{\alpha(t)}{r(t)} + M_1\frac{\beta(t)}{r(t)^{\gamma+1}}\big(\|u_n\|^\gamma_{\infty,r} + \|u\|^\gamma_{\infty,r}\big) \\
&\le 2M_1\frac{\alpha(t)}{r(t)} + M_1\frac{\beta(t)}{r(t)^{\gamma+1}}(M_2^\gamma + \|u\|^\gamma_{\infty,r}) \in L^1(0,\infty).
\end{aligned}
$$

From the Lebesgue Dominated Convergence Theorem, we have

$$
\lambda\int_0^\infty (h(t,u_n(t)) - h(t,u(t)))(u_n(t)-u(t))\,dt \to 0. \tag{2.11}
$$

Since $\varphi'(u_n)\to 0$ and $u_n \rightharpoonup u$ in X, we have

$$
\langle \varphi'(u_n) - \varphi'(u),\, u_n - u\rangle \to 0 \quad \text{as } n\to\infty.
$$

Consequently, from (2.11), we have $S_n \to 0$ as $n\to\infty$, where

$$
\begin{aligned}
S_n =&(a+b\|u_n\|^p_{W_0^{1,p}})\int_0^\infty |u_n'(t)|^{p-2}u_n'(t)(u_n'(t)-u'(t))\,dt \\
&+ (a+b\|u_n\|^p_{W_0^{1,p}})\int_0^\infty |u_n(t)|^{p-2}u_n(t)(u_n(t)-u(t))\,dt \\
&- (a+b\|u\|^p_{W_0^{1,p}})\int_0^\infty |u'(t)|^{p-2}u'(t)(u_n'(t)-u'(t))\,dt \\
&- (a+b\|u\|^p_{W_0^{1,p}})\int_0^\infty |u(t)|^{p-2}u(t)(u_n(t)-u(t))\,dt.
\end{aligned}
$$

We can rewrite S_n as

$$\begin{aligned}
S_n =&(a+b\|u_n\|^p_{W_0^{1,p}})\int_0^\infty (|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t))(u_n'(t)-u'(t))\,dt\\
&+(a+b\|u_n\|^p_{W_0^{1,p}})\int_0^\infty (|u_n(t)|^{p-2}u_n(t)-|u(t)|^{p-2}u(t))(u_n(t)-u(t))\,dt\\
&+(a+b\|u_n\|^p_{W_0^{1,p}})\int_0^\infty |u'(t)|^{p-2}u'(t)(u_n'(t)-u'(t))\,dt\\
&+(a+b\|u_n\|^p_{W_0^{1,p}})\int_0^\infty |u(t)|^{p-2}u(t)(u_n(t)-u(t))\,dt\\
&-(a+b\|u\|^p_{W_0^{1,p}})\int_0^\infty |u(t)|^{p-2}u(t)(u_n(t)-u(t))\,dt\\
&-(a+b\|u\|^p_{W_0^{1,p}})\int_0^\infty |u'(t)|^{p-2}u'(t)(u_n'(t)-u'(t))\,dt.
\end{aligned}$$

From the weak convergence of (u_n) in X and since

$$|u'|^{p-2}u',\ |u|^{p-2}u\in L^{\frac{p}{p-1}}(0,\infty),$$

we have

$$\int_0^\infty |u'(t)|^{p-2}u'(t)(u_n'(t)-u'(t))\,dt+\int_0^\infty |u(t)|^{p-2}u(t)(u_n(t)-u(t))\,dt\to 0\quad \text{as } n\to\infty. \tag{2.12}$$

Hence, from (2.12) and the fact that $S_n\to 0$, we obtain

$$\begin{aligned}
&(a+b\|u_n\|^p_{W_0^{1,p}})\\
&\times\Bigg[\int_0^\infty (|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t))(u_n'(t)-u'(t))\,dt\\
&\qquad+\int_0^\infty (|u_n(t)|^{p-2}u_n(t)-|u(t)|^{p-2}u(t))(u_n(t)-u(t))\,dt\Bigg]\to 0.
\end{aligned}$$

Since (u_n) is bounded in $W_0^{1,p}(0,\infty)$, we deduce that

$$\begin{aligned}
&\int_0^\infty (|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t))(u_n'(t)-u'(t))\,dt\\
&+\int_0^\infty (|u_n(t)|^{p-2}u_n(t)-|u(t)|^{p-2}u(t))(u_n(t)-u(t))\,dt\to 0.
\end{aligned}$$

We need to use the following inequality :

$$(|x|^{p-2}x-|y|^{p-2}y,\,x-y)\geq\begin{cases} C_p|x-y|^p, & \text{if } p\geq 2,\\ C_p\dfrac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1<p<2.\end{cases}$$

$\forall x,y\in\mathbb{R}$ ([20]) To complete the proof of the Palais–Smale condition, we follow the arguments used in [20]. We distinguish two cases: $p\geq 2$ and $1<p<2$.

First case: $p \geq 2$.

By the inequality , we have

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \geq \frac{1}{2^p}|x-y|^p.$$

we obtain

$$\begin{aligned}&\int_0^\infty (|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t))(u_n'(t)-u'(t))\,dt \\ &+ \int_0^\infty (|u_n(t)|^{p-2}u_n(t) - |u(t)|^{p-2}u(t))(u_n(t)-u(t))\,dt \\ &\geq \frac{1}{2^p}\left(\int_0^\infty |u_n'(t)-u'(t)|^p dt + \int_0^\infty |u_n(t)-u(t)|^p dt\right).\end{aligned}$$

Since

$$\begin{aligned}&\int_0^\infty (|u_n'|^{p-2}u_n' - |u'|^{p-2}u')(u_n'-u')\,dt \\ &+ \int_0^\infty (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\,dt \to 0,\end{aligned}$$

$asn \to \infty$

and

$$\int_0^\infty |u_n'(t)-u'(t)|^p dt + \int_0^\infty |u_n(t)-u(t)|^p dt \to 0$$

Second case: $1 < p < 2$.

From the inequality

$$\big(|x|^{p-2}x - |y|^{p-2}y\big)(x-y) \geq \frac{p-1}{\sqrt{2}}\,\frac{|x-y|^p}{(|x|^p|y|^p)^{(2-p)/p}} \quad \text{if } 1<p<2,$$

we obtain

$$\int_0^\infty |u_n'(t)-u'(t)|^p\,dt \leq \frac{\sqrt{2}}{p-1}\int_0^\infty \big(|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t)\big)(u_n'(t)-u'(t))\,(|u_n'(t)|^p+|u'(t)|^p)^{(2-p)/p}dt.$$

By applying Hölder's inequality, it follows that

$$\begin{aligned}&\int_0^\infty |u_n'(t)-u'(t)|^p dt \leq \frac{\sqrt{2}}{p-1}\big\|((|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t))(u_n'(t)-u'(t))^{p/2}\big\|_{2/p} \\ &\times \big\|(|u_n'(t)|^p + |u'(t)|^p)^{(2-p)/p}\big\|_{p/(2-p)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_0^\infty |u_n(t)-u(t)|^p dt \leq \frac{\sqrt{2}}{p-1}\big\|((|u_n(t)|^{p-2}u_n(t) - |u(t)|^{p-2}u(t))(u_n(t)-u(t))^{p/2}\big\|_{p/2} \\ &\times \big\|(|u_n(t)|^p + |u(t)|^p)^{(2-p)/p}\big\|_{p/(2-p)}.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^\infty |u_n'(t)-u'(t)|^p dt + \int_0^\infty |u_n(t)-u(t)|^p dt \\
&\le \frac{\sqrt{2}}{p-1}\Big[\big\|((|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t))(u_n'(t)-u'(t))^{p/2}\big\|_{2/p}\big\|(|u_n'(t)|^p+|u'(t)|^p)^{(2-p)/p}\big\|_{p/(2-p)} \\
&\quad + \big\|(|u_n(t)|^{p-2}u_n(t)-|u(t)|^{p-2}u)(u_n-u))^{p/2}\big\|_{2/p}\big\|(|u_n(t)|^p+|u(t)|^p)^{(2-p)/p}\big\|_{p/(2-p)}\Big].
\end{aligned}$$

Since (u_n) is bounded in $W_0^{1,p}$, we can choose constants M_1' and M_2' such that

$$\big\|(|u_n'(t)|^p+|u'(t)|^p)^{(2-p)/p}\big\|_{p/(2-p)} \le M_1', \quad \big\|(|u_n(t)|^p+|u(t)|^p)^{(2-p)/p}\big\|_{p/(2-p)} \le M_2'.$$

Hence,

$$\begin{aligned}
\|u_n-u\|^p \le \frac{\sqrt{2}}{p-1}\max(M_1',M_2')\Big[&\big\|((|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t))(u_n'-u'))^{p/2}\big\|_{p/2} \\
&+ \big\|(|u_n(t)|^{p-2}u_n(t)-|u(t)|^{p-2}u(t))(u_n-u))^{p/2}\big\|_{2/p}\Big] \to 0,
\end{aligned}$$

as $n\to\infty$.

Therefore, we conclude that

$$\|u_n-u\|_{W_0^{1,p}} \to 0 \quad \text{as } n\to\infty,$$

which shows that ϕ satisfies the Palais-Smale condition.

4.The Genus condition :

Consider $\{e_1,e_2,\dots\}$, a Schauder basis of the space $W_0^{1,p}(0,\infty)$ see [23] [7], and for each $k\in\mathbb{N}$, let X_k be the subspace of $W_0^{1,p}(0,\infty)$ generated by the vectors $\{e_1,e_2,\dots,e_k\}$. Clearly, X_k is a finite-dimensional subspace of $W_0^{1,p}(0,\infty)$.

For $\rho>0$, define

$$K_k(\rho)=\left\{u\in X_k : \|u\|^2_{W_0^{1,p}}=\sum_{i=1}^k \xi_i^2=\rho^2\right\}.$$

For any $\rho>0$, we consider the odd homeomorphism $\chi : K_k(\rho)\to S^{k-1}$ defined by

$$\chi(u)=(\xi_1,\xi_2,\dots,\xi_k),$$

where S^{k-1} is the unit sphere in $\mathbb{R}^k$. From Theorem (2.1) and Proposition (2.1) , we deduce that

$$\gamma(K_k(\rho))=k.$$

It follows from hypothesis (h_2) that

$$\int_0^\infty H(t,u(t))\,dt > 0 \quad \text{for any } u \in K_k(\rho).$$

Then , we have

$$\mu_k = \inf_{u\in K_k(\rho)} \int_0^\infty H(t,u(t))\,dt$$

is strictly positive.

Let

$$\lambda_k = \frac{(a+b\rho^p)\rho^p}{\mu_k},$$

and note that $\lambda_k > 0$. Then, for $\lambda > \lambda_k$ and for any $u \in K_k(\rho)$, we have

$$\begin{aligned}
\varphi(u) &= \frac{a}{p}\|u\|^p_{W_0^{1,p}} + \frac{b}{2p}\|u\|^{2p}_{W_0^{1,p}} - \lambda \int_0^\infty H(t,u(t))\,dt \\
&\leq (a + b\|u\|^p_{W_0^{1,p}})\|u\|^p_{W_0^{1,p}} - \lambda\mu_k \\
&\leq (a+b\rho^p)\rho^p - \lambda\mu_k \\
&< (a+b\rho^p)\rho^p - \lambda_k\mu_k = 0.
\end{aligned}$$

Therefore, $\varphi(u) < 0$ for all $u \in K_k(\rho)$. By Theorem(2.2), the functional φ has at least k pairs of distinct critical points. Hence, Problem (2.1) admits at least k distinct pairs of nontrivial solutions.

2.4 Example

The following (B.V.P)

$$\left(1+\|u\|^4_{W_0^{1,4}}\right)\Big(-\left(|u'(t)|^2u'(t)\right)' + |u(t)|^2u(t)\Big) = e^{-t}u|u|, \quad t\in[0,\infty) \tag{2.13}$$

With

$$u(0) = u(\infty) = 0.$$

Satisfies our main result obviously , the nonlinear term is defined by

$$h(t,x) = e^{-t}x|x|, \quad a = b = 1, \quad \lambda = 1.$$

In addition, we choose

$$r(t) = \frac{1}{1+t}, \quad \alpha(t) = \frac{1}{1+t^4}, \quad \beta(t) = e^{-t}, \quad \gamma = 2.$$

Then , we can seet that

$$|h(t,u)| = e^{-t}|u|^2 \leq \alpha(t) + \beta(t)|u|^{\gamma}.$$

Hence, condition (h_1) is satisfied.

On other hand , we have :

$$H(t,u) = \int_0^u h(t,\xi)\, d\xi.$$

$$H(t,u) = \int_0^u e^{-t}\xi|\xi|\, d\xi \;\; > 0 \quad \forall\, u \neq 0.$$

Therefore, condition (h_2) is satisfied.

A Double phase problem on the half-line

In this chapter, we study a class of nonlinear double phase problems in the framework of Musielak–Orlicz spaces. Our aim is to establish suitable embedding results and to investigate the existence of infinitely many weak solutions by variational methods :

$$\begin{cases} -\left(|u'(t)|^{p-2}\,u'(t)+a(t)\;|u'(t)|^{q-2}\,u'(t)\right)'+|u(t)|^{p-2}\,u(t)+a(t)\;|u(t)|^{q-2}\,u(t)=\lambda\,h(t,u(t)),\\ u(0)=u(+\infty)=0. \end{cases} \tag{3.1}$$

3.1 Musielak-Orlicz space

Definition 3.1. ([6]) Let Ω be an open subset of $\mathbb{R}^d$. A function $G:\Omega\times\Omega\times\mathbb{R}^+\to\mathbb{R}$ is called a generalized N-function if it satisfies the following conditions:

1. $G_{x,y}(t):=G(x,y,t)$ is even, continuous, increasing and convex in t, and for each $t\in\mathbb{R}$, $G(x,y,t)$ is measurable in (x,y);
2. $\lim\limits_{t\to 0}\dfrac{G_{x,y}(t)}{t}=0$, for a.a. $(x,y)\in\Omega\times\Omega$;
3. $\lim\limits_{t\to \infty}\dfrac{G_{x,y}(t)}{t}=\infty$, for a.a. $(x,y)\in\Omega\times\Omega$;
4. $G_{x,y}(t)>0$, for all $t>0$ and $(x,y)\in\Omega\times\Omega$.

The function $H:\Omega\times[0,+\infty)\to[0,+\infty)$ defined as

$$H(t,s):=s^p+a(t)s^q,\quad \text{for a.e. } t\in\Omega \text{ and for any } s\in[0,+\infty),$$

with $1<p<q$ and $0\le a(\cdot)\in L^1(\Omega)$, is a generalized N-function (N stands for nice), according to the definition (3.1), and satisfies the so-called (Δ_2) condition, that is

$$H(t,2s)\le 2^qH(t,s),\quad \text{for a.e. } t\in\Omega \text{ and for any } s\in[0,+\infty).$$

Therefore, by [19] we can define the Musielak–Orlicz space $L^H(\Omega)$ as

$$L^H(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable} : \rho_H(u) < \infty\},$$

endowed with the Luxemburg norm

$$\|u\|_H := \inf\left\{\lambda > 0 : \rho_H\Big(\frac{u}{\lambda}\Big) \leq 1\right\}.$$

where ρ_H denotes the H-modular function, defined by

$$\rho_H(u) := \int_\Omega H(t, |u|)\, dx = \int_\Omega \big(|u|^p + a(t)|u|^q\big)\, dt. \tag{3.2}$$

By [[10],[12]], the space $L^H(\Omega)$ is a separable, uniformly convex Banach space. Moreover, from [16], we have the following relations between the norm $\|\cdot\|_H$ and the H-modular.

Proposition 3.1. *Assume that $u \in L^H(\Omega)$, $\{u_j\} \subset L^H(\Omega)$ and $c > 0$. Then:*

(i) for $u \neq 0$, $\|u\|_H = c \iff \rho_H\left(\frac{u}{c}\right) = 1$;

(ii) $\|u\|_H < 1$ (resp. $= 1, > 1$) $\iff \rho_H(u) < 1$ (resp. $= 1, > 1$);

(iii) $\|u\|_H < 1 \Rightarrow \|u\|_H^q \leq \rho_H(u) \leq \|u\|_H^p$;

(iv) $\|u\|_H > 1 \Rightarrow \|u\|_H^p \leq \rho_H(u) \leq \|u\|_H^q$;

(v) $\lim \|u_j\|_H = 0\ (\infty) \iff \lim \rho_H(u_j) = 0\ (\infty)$, as $j \to \infty$.

The associated Sobolev space $W^{1,H}(\Omega)$ is defined by

$$W^{1,H}(\Omega) := \{u \in L^H(\Omega) : |\nabla u| \in L^H(\Omega)\},$$

endowed with the norm

$$\|u\|_{1,H} := \|u\|_H + \|\nabla u\|_H, \tag{3.3}$$

where we write $\|\nabla u\|_H = \||\nabla u|\|_H$ to simplify the notation. We denote by $W_0^{1,H}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,H}(\Omega)$, which can be endowed with the norm

$$\|u\| := \|\nabla u\|_H,$$

which is equivalent to the norm defined in (3.3), thanks to [10], whenever condition (1.2) holds. For any $m \in [1, \infty)$, we denote by $L^m(\Omega)$ the usual Lebesgue space equipped with the norm $\|\cdot\|_m$. Then, by [10], we have the following embeddings .

Definition 3.2. We say that $u \in W^{1,H}(0,\infty) = X$ is a weak solution of Problem (3.1) if and only if :

$$\int_0^{+\infty} (|u'(t)|^{p-2}u'(t)\, v'(t) + a(t)\, |u'(t)|^{q-2}u'(t)\, v'(t)\, dt) + \int_0^{+\infty} (|u(t)|^{p-2}u(t)\, v(t)\, dt + a(t)\, |u(t)|^{q-2}u(t)\, v(t)\, dt)$$

$$= \lambda \int_0^{+\infty} h(t, u(t))\, v(t)\, dt$$

. for all $v \in W^{1,H}$.

3.2 Embedding

Lemma 3.1. *The following embedding holds:*

$$W^{1,H}(0,+\infty) \hookrightarrow C_{l,r}(0,+\infty).$$

Proof. Obviously, we can see that the space $W^{1,H}(0,+\infty)$ is continuously embedded in the space $W^{1,p}(0,+\infty)$. From Lemma (2.2), we have that $W^{1,p}(0,+\infty)$ is continuously embedded in $C^1_{l,r}(0,+\infty)$, that is :

$$W^{1,H}(0,+\infty) \hookrightarrow W^{1,p}(0,+\infty) \hookrightarrow C^1_{l,r}(0,+\infty).$$

Therefore, the space $W^{1,H}(0,+\infty)$ is continuously embedded in $C_{l,r}(0,+\infty)$. □

Lemma 3.2. *The following compact embedding holds:*

$$W^{1,H}(0,+\infty) \hookrightarrow\hookrightarrow C_{l,r}(0,+\infty).$$

Proof. Since

$$W^{1,H}(0,+\infty) \hookrightarrow W^{1,p}(0,+\infty) \hookrightarrow\hookrightarrow C_{l,r}(0,+\infty).$$

the last compact embedding holds thanks to Lemma (2.3) then , we deduse that :

$$W^{1,H}(0,+\infty) \hookrightarrow\hookrightarrow C_{l,r}(0,+\infty).$$

□

3.3 Main Result

The following theorem gives the existence of infinitely many weak solutions for problem (3.1 :

Theorem 3.1. *Assume that the function $h(t,u)$ satisfies the following conditions:*

(h1) *There exist a constant $\gamma \in [0, p-1)$ and two positive functions α, β such that :*

$$\frac{\alpha}{r}, \frac{\beta}{r^{\gamma+1}} \in L^1(0,\infty),$$

and

$$|h(t,s)| \leq \alpha(t) + \beta(t)\,|s|^{\gamma} \quad \textit{for all } (t,s) \in [0,\infty) \times \mathbb{R}.$$

(h2) *The function $h(t,s)$ is odd with respect to s and*

$$H(t,s) = \int_0^s h(t,\xi)\,d\xi > 0 \quad \textit{for all } (t,s) \in [0,\infty) \times \mathbb{R} \setminus \{0\}.$$

Then, for any $k \in \mathbb{N}$, there exists λ_k such that if $\lambda > \lambda_k$, Problem (3.1) has at least k distinct pairs of nontrivial solutions.

Proof. The corresponding functional to Proplem (3.1) is defined as :

$$J(u) = \frac{1}{p}\int_0^{+\infty} |u'(t)|^p\,dt + \frac{1}{q}\int_0^{+\infty} a(t)|u'(t)|^q\,dt + \frac{1}{p}\int_0^{+\infty} |u(t)|^p\,dt + \frac{1}{q}\int_0^{+\infty} a(t)|u(t)|^q\,dt$$
$$-\lambda\int_0^{+\infty} F(t,u(t))\,dt$$

With : $F(t,s) = \int_0^{+\infty} f(t,\tau)\,d\tau$

claim 1 : $J \in C^1(X,\mathbb{R})$

- The functionale J is well defined because $u \in W^{1,H}(0,+\infty)$. It follows from (h1) in theorem

(3.1) and Lemma (3.1) that :

$$\begin{aligned}
\int_0^\infty |H(t,u(t))|\,dt &= \int_0^\infty \left|\int_0^u h(t,\xi)\,d\xi\right| dt\\
&\leq \int_0^\infty \int_0^u |h(t,\xi)|\,d\xi\,dt\\
&\leq \int_0^\infty \int_0^u \big(\alpha(t)+\beta(t)\,|xi|^\gamma\big)\,d\xi\,dt\\
&\leq \int_0^\infty \left(\alpha(t)u(t)+\frac{\beta(t)}{\gamma+1}\,|u(t)|^{\gamma+1}\right) dt\\
&\leq \int_0^\infty \left(\frac{\alpha(t)}{r(t)}\,r(t)\,|u(t)| + \frac{\beta(t)}{(\gamma+1)(r(t))^{\gamma+1}}\big(r(t)\,|u(t)|\,\big)^{\gamma+1}\right) dt\\
&\leq \|u\|_{\infty,r}\int_0^\infty \frac{\alpha(t)}{r(t)}\,dt + \frac{\|u\|_{\infty,r}^{\gamma+1}}{\gamma+1}\int_0^\infty \frac{\beta(t)}{(r(t))^{\gamma+1}}\,dt\\
&\leq c\,\|u\|_{W_0^{1,p}}\left|\frac{\alpha}{r}\right|_{L^1} + \frac{c^{\gamma+1}}{\gamma+1}\,\|u\|_{W_0^{1,p}}^{\gamma+1}\left|\frac{\beta}{r^{\gamma+1}}\right|_{L^1}\\
&\leq c_1\,\|u\|_{W^{1,H}} + c_2\,\|u\|_{W^{1,H}}^{\gamma+1} < \infty
\end{aligned}$$

Then J is well defined on $W^{1,H}$

- J is Gâteaux-differentiable,

  We define the functional : $J_1 : W^{1,H}(0,+\infty) \longrightarrow \mathbb{R}$

  be defined by :

$$J_1(u) = \frac{1}{p}\int_0^{+\infty} |u'(t)|^p\,dt + \frac{1}{q}\int_0^{+\infty} a(t)|u'(t)|^q\,dt + \frac{1}{p}\int_0^{+\infty} |u(t)|^p\,dt + \frac{1}{q}\int_0^{+\infty} a(t)|u(t)|^q\,dt.$$

Obviously, J_1 is Gâteaux -differentiable on $W^{1,H}(0,+\infty)$.

Moreover, we have

$$\langle J_1'(u),v\rangle = \int_0^{+\infty} |u'(t)|^{p-2}u'(t)v'(t)\,dt + \int_0^{+\infty} a(t)|u'(t)|^{q-2}u'(t)v'(t)\,dt$$

$$+\int_0^{+\infty} |u(t)|^{p-2}u(t)v(t)\,dt + \int_0^{+\infty} a(t)|u(t)|^{q-2}u(t)v(t)\,dt.$$

For $u \in W^{1,H}$, we consider the functional J_2 defined by :

$$J_2 = \int_0^{+\infty} F(t,u(t))\,dt$$

Proceeding as in Chapter 2, we prove that J_2 is Gâteaux differentiable on $W^{1,H}(0,+\infty)$

- Using the same arguments as in Chapter 2, we deduce that the derivative J' is continuous on $W^{1,H}(0,+\infty)$. and

$$\begin{aligned}\langle J'(u),v\rangle &= \int_0^{+\infty} |u'(t)|^{p-2}u'(t)v'(t)\,dt + \int_0^{+\infty} a(t)|u'(t)|^{q-2}u'(t)v'(t)\,dt \\ &+ \int_0^{+\infty} |u(t)|^{p-2}u(t)v(t)\,dt + \int_0^{+\infty} a(t)|u(t)|^{q-2}u(t)v(t)\,dt - \lambda\int_0^{+\infty} f(t,u(t))v(t)\,dt.\end{aligned} \tag{3.4}$$

Therefore, $J \in C^1\big(W^{1,H}(0,+\infty),\mathbb{R}\big)$.

claim 2 : J is bounded from below

For every $u \in W^{1,H}(0,+\infty)$, we have

$$\begin{aligned}J(u) = &\int_0^{+\infty}\left(\frac{1}{p}|u'(t)|^p + \frac{1}{q}a(t)|u'(t)|^q\right)\,dt \\ &+ \int_0^{+\infty}\left(\frac{1}{p}|u(t)|^p + \frac{1}{q}a(t)|u(t)|^q\right)\,dt - \lambda\int_0^{+\infty} F(t,u(t))\,dt\end{aligned} \tag{3.5}$$

Since $p<q$, we obtain

$$\frac{1}{p} \geq \frac{1}{q},$$

hence

$$\begin{aligned}J(u) \geq &\frac{1}{q}\int_0^{+\infty}\left(|u'(t)|^p + a(t)|u'(t)|^q\right)\,dt \\ &+ \frac{1}{q}\int_0^{+\infty}\left(|u(t)|^p + a(t)|u(t)|^q\right)\,dt - \lambda\int_0^{+\infty} F(t,u(t))\,dt.\end{aligned}$$

Using the definition of the modular ρ_H, we get

$$J(u) \geq \frac{1}{q}\rho_H(u') + \frac{1}{q}\rho_H(u) - \lambda\int_0^{+\infty} F(t,u(t))\,dt.$$

By Proposition (3.1), for $\|u\|_{W^{1,H}}$ sufficiently large, we have

$$\rho_H(u) \geq \|u\|_H^p, \qquad \rho_H(u') \geq \|u'\|_H^p.$$

Therefore,

$$J(u) \geq \frac{1}{q}\|u'\|_H^p + \frac{1}{q}\|u\|_H^p - \lambda\int_0^{+\infty} F(t,u(t))\,dt.$$

Using the inequality

$$a^p + b^p \leq (a+b)^p \leq 2^{p-1}(a^p+b^p)$$

we deduce that

$$J(u) \geq \frac{1}{q\,2^{p-1}}\left(\|u\|_H + \|u'\|_H\right)^p - \lambda\int_0^{+\infty} F(t,u(t))\,dt.$$

Since

$$\|u\|_{W^{1,H}} = \|u\|_H + \|u'\|_H,$$

it follows that

$$J(u) \geq \frac{1}{q\,2^{p-1}} \|u\|^p_{W^{1,H}} - \lambda \int_0^{+\infty} F(t,u(t))\,dt.$$

Using the growth assumption on F, we obtain

$$J(u) \geq \frac{1}{q\,2^{p-1}} \|u\|^p_{W^{1,H}} - \lambda c \|u\|_{W^{1,H}} \left|\frac{\alpha}{r}\right|_{L^1} - \frac{\lambda c^{\gamma+1}}{\gamma+1} \|u\|^{\gamma+1}_{W^{1,H}} \left|\frac{\beta}{r^{\gamma+1}}\right|_{L^1}.$$

Since $0 < \gamma + 1 < p$, we conclude tha t : $J(u)$ is bounded from below .

claim 3 : J satisfies the (PS) condition

Let $(u_n) \subset W^{1,H}(0,+\infty)$ be a sequence such that

$$|J(u_n)| \leq C \tag{3.6}$$

and

$$J'(u_n) \to 0 \quad \text{in } \left(W^{1,H}(0,+\infty)\right)^*.$$

From (3.5) , (3.6) and since $\gamma \leq p-1$ in all cases we deduce that the sequence (u_n) is bounded in $W^{1,H}$. Thus passing to a subsequence if necessary , there exists $u \in W^{1,H}$ such that $u_n \rightharpoonup u$ weakly in $W^{1,H}$.

Moreover by (3.4) :

$$\begin{aligned}
\langle J'(u_n) - J'(u), u_n - u\rangle = &\int_0^{+\infty} \left(|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t)\right)(u_n'(t) - u'(t))\,dt \\
&+ \int_0^{+\infty} a(t)\left(|u_n'(t)|^{q-2}u_n'(t) - |u'(t)|^{q-2}u'(t)\right)(u_n'(t) - u'(t))\,dt \\
&+ \int_0^{+\infty} \left(|u_n(t)|^{p-2}u_n(t) - |u(t)|^{p-2}u(t)\right)(u_n(t) - u(t))\,dt \\
&+ \int_0^{+\infty} a(t)\left(|u_n(t)|^{q-2}u_n(t) - |u(t)|^{q-2}u(t)\right)(u_n(t) - u(t))\,dt \\
&- \lambda \int_0^{+\infty} \left(f(t,u_n(t)) - f(t,u(t))\right)(u_n(t) - u(t))\,dt.
\end{aligned}$$

Since the embedding

$$W^{1,H}(0,+\infty) \hookrightarrow\hookrightarrow C_{l,r}(0,+\infty)$$

is compact, it follows that

$$u_n \to u \quad \text{in } C_{l,r}(0,+\infty).$$

Hence, there exist two positive constants $M_1, M_2 > 0$ such that

$$\|u_n - u\|_{\infty,r} < M_1$$

and

$$\|u_n\|_{\infty,r} < M_2.$$

Using assumption (h_1), we obtain

$$\begin{aligned}
&|f(t,u_n(t)) - f(t,u(t))|\,|u_n(t)-u(t)|\\
\le&\frac{1}{r(t)}\, r(t)|u_n(t)-u(t)|\,|f(t,u_n(t)) - f(t,u(t))|\\
\le&\frac{1}{r(t)}\|u_n - u\|_{\infty,r}|f(t,u_n(t)) - f(t,u(t))|\\
\le&\frac{M_1}{r(t)}\Big(|f(t,u_n(t))| + |f(t,u(t))|\Big)\\
\le&\frac{M_1}{r(t)}\Big(\alpha(t) + \beta(t)|u_n(t)|^\gamma + \alpha(t) + \beta(t)|u(t)|^\gamma\Big)
\end{aligned}$$

hence

$$\begin{aligned}
\le&\frac{2M_1\alpha(t)}{r(t)} + \frac{M_1\beta(t)}{r(t)}\Big(|u_n(t)|^\gamma + |u(t)|^\gamma\Big)\\
\le&\frac{2M_1\alpha(t)}{r(t)} + \frac{M_1\beta(t)}{(r(t))^{\gamma+1}}(r(t))^\gamma\Big(|u_n(t)|^\gamma + |u(t)|^\gamma\Big)\\
\le&\frac{2M_1\alpha(t)}{r(t)} + \frac{M_1\beta(t)}{(r(t))^{\gamma+1}}\Big(\|u_n\|_{\infty,r}^\gamma + \|u\|_{\infty,r}^\gamma\Big)\\
\le&\frac{2M_1\alpha(t)}{r(t)} + \frac{M_1\beta(t)}{(r(t))^{\gamma+1}}\Big(M_2^\gamma + \|u\|_{\infty,r}^\gamma\Big) \in L^1(0,+\infty).
\end{aligned}$$

Therefore, by the Lebesgue Dominated Convergence Theorem, we get

$$\lambda\int_0^{+\infty}(f(t,u_n(t)) - f(t,u(t)))(u_n(t)-u(t))\,dt \to 0.$$

Since

$$J'(u_n) \to 0$$

and

$$u_n \rightharpoonup u \quad \text{in } W^{1,H}(0,+\infty),$$

we have

$$\langle J'(u_n) - J'(u), u_n - u\rangle \to 0.$$

Consequently,

$$S_n \to 0,$$

where

$$S_n = \int_0^{+\infty} \Big(|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t)\Big)(u_n'(t) - u'(t))\,dt$$
$$+ \int_0^{+\infty} a(t)\Big(|u_n'(t)|^{q-2}u_n'(t) - |u'(t)|^{q-2}u'(t)\Big)(u_n'(t) - u'(t))\,dt.$$

We can rewrite S_n as follows

$$S_n = \int_0^{+\infty} \Big(|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t)\Big)(u_n'(t) - u'(t))\,dt$$
$$+ \int_0^{+\infty} a(t)\Big(|u_n'(t)|^{q-2}u_n'(t) - |u'(t)|^{q-2}u'(t)\Big)(u_n'(t) - u'(t))\,dt$$
$$+ \int_0^{+\infty} |u'(t)|^{p-2}u'(t)(u_n'(t) - u'(t))\,dt$$
$$+ \int_0^{+\infty} a(t)|u'(t)|^{q-2}u'(t)(u_n'(t) - u'(t))\,dt.$$

Since

$$u_n \rightharpoonup u \quad \text{in } W^{1,H}(0,+\infty),$$

and

$$|u'|^{p-2}u' \in L^{\frac{p}{p-1}}(0,+\infty),$$

$$a(t)|u'|^{q-2}u' \in L^{\frac{q}{q-1}}(0,+\infty),$$

we obtain

$$\int_0^{+\infty} |u'(t)|^{p-2}u'(t)(u_n'(t) - u'(t))\,dt \to 0,$$

and

$$\int_0^{+\infty} a(t)|u'(t)|^{q-2}u'(t)(u_n'(t) - u'(t))\,dt \to 0,$$

as $n \to +\infty$.

Hence, from $S_n \to 0$, we deduce that

$$\int_0^{+\infty} \Big(|u_n'(t)|^{p-2}u_n'(t) - |u'(t)|^{p-2}u'(t)\Big)(u_n'(t) - u'(t))\,dt$$
$$+ \int_0^{+\infty} a(t)\Big(|u_n'(t)|^{q-2}u_n'(t) - |u'(t)|^{q-2}u'(t)\Big)(u_n'(t) - u'(t))\,dt \to 0.$$

To complete the proof of the Palais–Smale condition, we distinguish two cases.

Case 1: $p > 2$ and $q > 2$.

Using the inequalities

$$\Big(|x|^{p-2}x - |y|^{p-2}y\Big)(x - y) \geq \frac{1}{2^p}|x - y|^p,$$

and

$$\Big(|x|^{q-2}x-|y|^{q-2}y\Big)(x-y)\geq \frac{1}{2^p}|x-y|^q,$$

for all $x,y\in\mathbb{R}$, we obtain

$$\begin{aligned}&\int_0^{+\infty}|u_n'(t)-u'(t)|^p\,dt\\ \leq&2^p\int_0^{+\infty}\Big(|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t)\Big)(u_n'(t)-u'(t))\,dt,\end{aligned}$$

and

$$\begin{aligned}&\int_0^{+\infty}a(t)|u_n'(t)-u'(t)|^q\,dt\\ \leq&2^q\int_0^{+\infty}a(t)\Big(|u_n'(t)|^{q-2}u_n'(t)-|u'(t)|^{q-2}u'(t)\Big)\\ &\times(u_n'(t)-u'(t))\,dt.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_0^{+\infty}|u_n'(t)-u'(t)|^p\,dt+\int_0^{+\infty}a(t)|u_n'(t)-u'(t)|^q\,dt\\ \leq&2^p\int_0^{+\infty}\Big(|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t)\Big)(u_n'(t)-u'(t))\,dt\\ &+2^q\int_0^{+\infty}a(t)\Big(|u_n'(t)|^{q-2}u_n'(t)-|u'(t)|^{q-2}u'(t)\Big)\\ &\times(u_n'(t)-u'(t))\,dt\to 0.\end{aligned}$$

Therefore,

$$u_n\to u\quad\text{strongly in } W^{1,H}(0,+\infty).$$

Case 2: 1<p<2 and 1<q<2 .

Using the inequalities

$$\Big[\big(|x|^{p-2}x-|y|^{p-2}y\big)(x-y)\Big]^{\frac{p}{2}}\geq\frac{p-1}{\sqrt{2}}\frac{|x-y|^p}{(|x|^p+|y|^p)^{\frac{2-p}{p}}},$$

and

$$\Big[\big(|x|^{q-2}x-|y|^{q-2}y\big)(x-y)\Big]^{\frac{q}{2}}\geq\frac{q-1}{\sqrt{2}}\frac{|x-y|^q}{(|x|^q+|y|^q)^{\frac{2-q}{q}}},$$

we obtain

$$\begin{aligned}&\int_0^{+\infty}|u_n'(t)-u'(t)|^p\,dt\\ \leq&C\left(\int_0^{+\infty}\Big(|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t)\Big)(u_n'(t)-u'(t))\,dt\right)^{\frac{p}{2}}\\ &\times\left(\int_0^{+\infty}\Big(|u_n'(t)|^p+|u'(t)|^p\Big)dt\right)^{\frac{2-p}{p}}.\end{aligned}$$

Similarly, for the double phase term, we have

$$\begin{aligned}
&\int_0^{+\infty} a(t)|u_n'(t)-u'(t)|^q\,dt\\
\leq& C\Bigg(\int_0^{+\infty} a(t)\Big(|u_n'(t)|^{q-2}u_n'(t)-|u'(t)|^{q-2}u'(t)\Big)\\
&\times(u_n'(t)-u'(t))\,dt\Bigg)^{\frac{q}{2}}\\
&\times\left(\int_0^{+\infty} a(t)\Big(|u_n'(t)|^q+|u'(t)|^q\Big)dt\right)^{\frac{2-q}{q}}.
\end{aligned}$$

By using Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^{+\infty} |u_n'(t)-u'(t)|^p\,dt\\
\leq& C\left\|\Big[\Big(|u_n'(t)|^{p-2}u_n'(t)-|u'(t)|^{p-2}u'(t)\Big)(u_n'(t)-u'(t))\Big]^{\frac{p}{2}}\right\|_{\frac{2}{p}}\\
&\times\left\|\Big(|u_n'(t)|^p+|u'(t)|^p\Big)^{\frac{2-p}{p}}\right\|_{\frac{p}{2-p}}
\end{aligned}$$

Similarly, for the double phase term, we have

$$\begin{aligned}
&\int_0^{+\infty} a(t)|u_n'(t)-u'(t)|^q\,dt\\
\leq& C\left\|\Big[a(t)\Big(|u_n'(t)|^{q-2}u_n'(t)-|u'(t)|^{q-2}u'(t)\Big)(u_n'(t)-u'(t))\Big]^{\frac{q}{2}}\right\|_{\frac{2}{q}}\\
&\times\left\|\Big(a(t)|u_n'(t)|^q+a(t)|u'(t)|^q\Big)^{\frac{2-q}{2}}dt\right\|_{\frac{q}{2-q}}
\end{aligned}$$

In the same way,

$$\begin{aligned}
\int_0^{+\infty}|u_n(t)-u(t)|^p\,dt \leq& C\left\|\left(\Big(|u_n(t)|^{p-2}u_n(t)-|u(t)|^{p-2}u(t)\Big)(u_n(t)-u(t))\right)^{\frac{p}{2}}\right\|_{\frac{2}{p}}\\
&\times\left\|\left(\Big(|u_n(t)|^p+|u(t)|^p\Big)\right)^{\frac{2-p}{p}}\right\|_{\frac{p}{2-p}},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{+\infty}a(t)|u_n(t)-u(t)|^q\,dt \leq& C\left\|\left(a(t)\Big(|u_n(t)|^{q-2}u_n(t)-|u(t)|^{q-2}u(t)\Big)(u_n(t)-u(t))\right)^{\frac{q}{2}}\right\|_{\frac{2}{q}}\\
&\times\left\|\left(a(t)\Big(|u_n(t)|^q+|u(t)|^q\Big)\right)^{\frac{2-q}{q}}.\right\|_{\frac{q}{2-q}}
\end{aligned}$$

Hence,

$$\rho_H(u_n - u) + \rho_H(u_n' - u') \longrightarrow 0.$$

By Proposition (3.1 (v)), it follows that

$$\|u_n - u\|_{L^H(0,+\infty)} \longrightarrow 0$$

and

$$\|u_n' - u'\|_{L^H(0,+\infty)} \longrightarrow 0.$$

Therefore,

$$\|u_n - u\|_{W^{1,H}(0,+\infty)} = \|u_n - u\|_{L^H(0,+\infty)} + \|u_n' - u'\|_{L^H(0,+\infty)} \longrightarrow 0.$$

Hence,

$$u_n \to u \qquad \text{strongly in } W^{1,H}(0,+\infty).$$

Consequently, the functional J satisfies the Palais–Smale condition.

Remark 3.1. For $1 < p < 2$ and $q > 2$, the proof follows the same lines as in Case 2.

Claim 4. Consider $\{e_1, e_2, \ldots\}$, a Schauder basis of the see ([7]) $W^{1,H}(0,+\infty)$, and for each $k \in \mathbb{N}$, let X_k be the subspace of $W^{1,H}(0,+\infty)$ generated by the vectors $\{e_1, e_2, \ldots, e_k\}$. For $\rho > 0$, define

$$K_k(\rho) = \left\{u \in X_k : \|u\|_{W^{1,H}} = \rho\right\}.$$

For any $\rho > 0$, we consider the odd homeomorphism

$$\chi : K_k(\rho) \to S^{k-1}$$

defined by

$$\chi(u) = (\xi_1, \xi_2, \ldots, \xi_k),$$

where S^{k-1} denotes the unit sphere in $\mathbb{R}^k$.

From Theorem (2.1) and Proposition (2.1), we deduce that

$$\gamma(K_k(\rho)) = k.$$

It follows from assumption (h_2) that

$$\int_0^{+\infty} F(t, u(t))\, dt > 0, \qquad \forall u \in K_k(\rho).$$

Then

$$\mu_k = \inf_{u \in K_k(\rho)} \int_0^{+\infty} F(t, u(t))\, dt$$

is strictly positive.

Let

$$\lambda_k = \mu_k^{-1} \frac{1}{p} \rho^p$$

and note that $\lambda_k > 0$.

Thus, for every $\lambda > \lambda_k$ and every $u \in K_k(\rho)$, we have

$$\begin{aligned}
J(u) =& \frac{1}{p} \int_0^{+\infty} |u'(t)|^p\, dt + \frac{1}{q} \int_0^{+\infty} a(t)|u'(t)|^q\, dt \\
&+ \frac{1}{p} \int_0^{+\infty} |u(t)|^p\, dt + \frac{1}{q} \int_0^{+\infty} a(t)|u(t)|^q\, dt \\
&- \lambda \int_0^{+\infty} F(t, u(t))\, dt \\
\leq& \frac{1}{p} \Big[\rho_h(u') + \rho_h(u) \Big] - \lambda \mu_k \\
&\leq \frac{1}{p} \Big[\|u'\|_{L^H}^p + \|u\|_{L^H}^p \Big]^p - \lambda \mu_k \\
&\leq \frac{1}{p} \|u\|_{W^{1,H}}^p - \lambda \mu_k \\
&\leq \frac{1}{p} \rho^p - \lambda \mu_k \\
&\leq \frac{1}{p} \rho^p - \lambda_k \mu_k = 0
\end{aligned}$$

Therefore,

$$J(u) < 0 \Rightarrow \sup J(u) < 0 \qquad \forall u \in K_k(\rho).$$

By Theorem (2.2), the functional J possesses at least k pairs of distinct critical points.

Consequently, the considered double phase problem (3.1) admits at least k distinct pairs of nontrivial solutions.

$\square$

3.4 Example

We consider the following double phase problem

$$\begin{cases} -\left(|u'(t)|u'(t)+\dfrac{1}{1+t^2}|u'(t)|^2u'(t)\right)' \\ \qquad\qquad +|u(t)|u(t)+\dfrac{1}{1+t^2}|u(t)|^2u(t)=e^{-t}u|u| \quad t\in[0,+\infty), \\ u(0)=u(+\infty)=0. \end{cases}$$

We choose

$$p=3, \qquad q=4, \qquad a(t)=\frac{1}{1+t^2},$$

and $h(t,u)=e^{-t}u|u|$, $\lambda=1$

Then

$$F(t,u)=\int_0^u h(t,s)\,ds$$

Moreover, we take

$$r(t)=\frac{1}{1+t}, \qquad \alpha(t)=e^{-t}, \qquad \beta(t)=e^{-t}, \qquad \gamma=2.$$

It is easy to verify that assumption (h_1) holds. Indeed,

$$|h(t,u)|=|e^{-t}u|u||=e^{-t}|u|^2=\beta(t)|u|^\gamma,$$

where

$$\beta(t)=e^{-t}, \qquad \gamma=2. \qquad \alpha(t)>0$$

Since

$$|h(t,s)|\leq \alpha(t)+\beta(t)\,|s|^\gamma$$

condition (h_1) is satisfied.

Moreover,

$$F(t,u)=\int_0^u h(t,\xi)\,d\xi=\int_0^u e^{-t}\xi|\xi|\,ds\ >0 \qquad \forall u\neq 0$$

which implies that assumption (h_2) holds.

Therefore, by the main result established in Chapter 3, the above double phase problem admits infinitely many pairs of nontrivial solutions.

Conclusion

In this memoir, we studied the existence and multiplicity of solutions for a class of nonlinear boundary value problems defined on the half-line $[0, +\infty)$, using variational methods and tools from critical point theory.

In the first chapter, we presented the main preliminary results concerning Sobolev spaces, Musielak–Orlicz and Musielak–Sobolev spaces, as well as Double Phase spaces, together with the main tools of critical point theory such as the Palais–Smale condition, Krasnoselskii genus theory, and Clark's theorem.

In the second chapter, we investigated a nonlinear Kirchhoff problem in $W_0^{1,p}(0, +\infty)$. By applying variational methods and verifying the assumptions of Clark's theorem, we proved the existence of infinitely many pairs of nontrivial solutions.

In the third chapter, we extended these results to a Double Phase Kirchhoff problem in $W^{1,H}(0, +\infty)$, and obtained similar multiplicity results using the same variational framework.

These results show the effectiveness of variational methods and critical point theory in studying nonlinear problems on unbounded domains, and they open the way for further extensions to more general operators and growth conditions.

Abstract

In this memoir, we study the existence and multiplicity of infinitely many pairs of nontrivial solutions for two classes of nonlinear problems posed on the half-line. The first one concerns a Kirchhoff type problem studied in the classical Sobolev space. while the second one deals with a double phase problem investigated in the Musielak–Sobolev space . Our approach is based on variational methods, Krasnoselskii genus theory and the Palais–Smale condition, which are important tools in critical point theory.

Keywords: variational methods; genus theory; half-line; Kirchhoff problem; double phase problem; Palais–Smale condition

Résumé

Dans ce mémoire, nous étudions l'existence et la multiplicité d'une infinité de paires de solutions non triviales pour deux classes de problèmes non linéaires définis sur la demi-droite. Le premier concerne un problème de type Kirchhoff étudié dans l'espace de Sobolev classique . tandis que le second traite un problème de type Double Phase étudié dans l'espace de Musielak–Sobolev . Notre approche repose sur les méthodes variationnelles, la théorie du genre de Krasnoselskii ainsi que la condition de Palais–Smale, qui constituent des outils importants de la théorie des points critiques.

Mots-clés : méthodes variationnelles ; théorie du genre ; demi-droite ; problème de Kirchhoff ; problème Double Phase ; condition de Palais–Smale.

الملخص

تتمحور هذه المذكرة حول دراسة **وجود وتعددية عدد غير نهائي من أزواج الحلول غير التافهة** لصنفين من المسائل غير الخطية المعرفة على **نصف المستقيم**. وتتوزع هذه الدراسة على محورين آساسيين:

المسألة الأولى: تتناول مسألة من نوع **كيرشوف (Kirchhoff)**، تمت دراستها في إطار **فضاء سوبوليف** **الكلاسيكي**.

المسألة الثانية: تتناول مسألة من نوع **ثنائي الطور (Phase-field)**، وتمت دراستها في إطار **فضاء موسيلاب (Musielak)**.

وقد اعتمدنا في إنجاز هذا العمل على مقاربة منهجية قائمة على **الطرق التغايرية (Variational methods)**، مع توظيف **نظرية الجنس لكراسنوسيلسكي (Krasnoselskii's genus theory)** وتحقيق **شرط بالي-سمايل (-Palais Smale condition)** لإثبات وجود الحلول وتعددها.

الكلمات المفتاحية:

الطرق التغايرية؛ نظرية الجنس لكراسنوسيلسكي؛ نصف المستقيم؛ مسألة كيرشوف؛ مسائل ثنائية الطور؛ شرط بالي-سمايل.

Liste des termes

FRANÇAIS	ENGLISH
Fonctionnelle d'énergie	Energy functional
Condition de Palais–Smale	Palais–Smale condition
Genre de Krasnoselskii	Krasnoselskii genus
Point critique	Critical point
Théorie des points critiques	Critical point theory
Méthodes variationnelles	Variational methods
Fonction N généralisée	Generalized N-function
Espace de Musielak–Orlicz	Musielak–Orlicz space
Espace double phase	Double phase space
Suite de Palais–Smale	Palais–Smale sequence
Convergence faible	Weak convergence
Convergence forte	Strong convergence
Injection compacte	Compact embedding
Injection continue	Continuous embedding
Base de Schauder	Schauder basis
Fonction de Carathéodory	Carathéodory function
Théorème de convergence dominée	Dominated convergence theorem
Inégalité de Hölder	Hölder inequality
Inégalité de Young	Young inequality
Solution non triviale	Nontrivial solution
Multiplicité des solutions	Multiplicity of solutions

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