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Theme

On type-2 and type-3 Fuzzy sets

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Introduction

In classical set theory, an element's membership is strictly binary. It belongs entirely or not at all to a given collection. However, real-world human reasoning contains inherent vagueness. Linguistic expressions and complex data are filled with ambiguity. Binary logic fails to capture these uncertainties adequately.

To address this limitation, Professor Lotfi A. Zadeh introduced Fuzzy Sets in 1965 [67]. He allowed membership values to span continuously between 0 and 1. Zadeh laid the mathematical foundation for approximate reasoning. He built a bridge between math and human intuition.

As engineering and data environments grew more complex, limitations emerged. Precise, crisp membership grades could not handle secondary uncertainties. They failed against conflicting expert opinions and heavy data noise [49]. To model these higher-order uncertainties, the theory naturally evolved.

Zadeh extended his original work in 1975 to introduce Type-2 Fuzzy Sets (T2FS) [69]. In T2FS, the membership grade itself becomes a fuzzy set [81]. This framework creates a deeper Footprint of Uncertainty (FOU) [49]. In recent years, this trajectory culminated in Type-3 Fuzzy Sets (T3FS) [80]. Type-3 sets generalize Type-2 sets by modeling membership grades as Type-2 fuzzy entities. This progression offers unprecedented mathematical flexibility for deep uncertainty representation.

The primary objective of this dissertation is to provide a rigorous mathematical review of these three generations of fuzzy sets. We analyze their definitions, mathematical operations, and foundational algebraic structures. To achieve this, this master dissertation is organized into three core chapters:

- **Chapter 1: Type-1 Fuzzy Sets.** This chapter reviews foundational concepts of classical fuzzy sets and relations. It details their operational characteristics, projections, and Cartesian products. It also introduces the crucial role of triangular norms (t-norms) and t-conorms [34].

- **Chapter 2: Type-2 Fuzzy Sets.** This chapter shifts focus to membership functions within Type-2 spaces [49]. We analyze geometric representations, including triangular, trapezoidal, and Gaussian membership functions [36]. It introduces new methods of partitioning the closure of support (CoS).

- **Chapter 3: Type-3 Fuzzy Sets.** The final chapter explores advanced mathematical definitions and set-theoretic operations of Type-3 systems [80]. Particular emphasis is placed on the hierarchical interpretation of alpha-cuts (α-cuts). We analyze their crucial role in type-reduction and uncertainty handling.

Chapter 1

Fuzzy Sets

Fuzzy set theory was proposed by Zadeh[67] in 1965 as an extension of the classical notion of a set. With the proposed methodology, Zadeh introduced a mathematic method with which decision making using fuzzy descriptions of some information becomes possible. The basis of this theory is the fuzzy set, which is a set that does not have clearly defined limits and can contain elements only at some degree. In other words, elements can have a certain degree of membership. Hence, suitable functions are used namely membership functions that determine the membership degree of each element in a fuzzy set. If we consider an input variable x with a field of definition X, the fuzzy set A in X is defined as $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$, where $\mu_A(x)$ is the membership function of X in fuzzy set A and may range from 0 to 1.

1.1 Fuzzy sets

We begin by recalling fuzzy sets and exploring their fundamental properties and operations. The concept of fuzzy set was first introduced by Lotfi A. Zadeh in his paper [67].

Definition 1. [67] *Let X be a nonempty set.*
A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the

element x in the fuzzy subset A for $x \in X$.

Notation 2. *The set of all fuzzy subsets of X will be denoted by $F(X)$.*

Example 3. *Let $X = \{x, y, z\}$ be a set. $F_1 = \{(x, 0.2), (y, 1.0), (z, 0.9)\}$ and $F_2 = \{(x, 0.1), (y, 0.8), (z, 0.5)\}$ are two fuzzy subsets on X.*

Example 4. *We can define the membership function $\mu_A(x)$ for the fuzzy set "Numbers Close to 5" as follows:*

$$\mu_A(x) = \begin{cases} 0 & \textit{if } x \leq 2 \\ \frac{x-2}{3} & \textit{if } 2 < x \leq 5 \\ \frac{8-x}{3} & \textit{if } 5 < x < 8 \\ 0 & \textit{if } x \geq 8 \end{cases}$$

1.2 Operations of fuzzy sets

For two fuzzy sets A and B on a set X, several operations are defined in the following way (see [67]).

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, for any $x \in X$;

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$, for any $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x) \rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x) \rangle \mid x \in X\}$;

(v) $\bar{A} = \{\langle x, 1 - \mu_A(x) \rangle \mid x \in X\}$.

Example 5. *If we consider the fuzzy sets*

$$A_1(x) = \begin{cases} 1, \quad if \quad 40 \leq x < 50, \\ 1 - \frac{x-50}{10}, \quad if \quad 50 \leq x < 60, \\ 0, \quad if \quad 60 \leq x \leq 100. \end{cases}$$

$$A_2(x) = \begin{cases} 0, & if \quad 40 \leq x < 50, \\ \frac{x-50}{10}, & if \quad 50 \leq x < 60, \\ 1 - \frac{x-60}{10}, & if \quad 60 \leq x < 70, \\ 0, & if \quad 70 \leq x \leq 100. \end{cases}$$

Then their union is

$$(A_1 \cup A_2)(x) = \begin{cases} 1, & if \quad 40 \leq x < 50, \\ 1 - \frac{x-50}{10}, & if \quad 50 \leq x < 55, \\ \frac{x-50}{10}, & if \quad 55 \leq x \leq 60, \\ 1 - \frac{x-60}{10}, & if \quad 60 \leq x \leq 70, \\ 0, & if \quad 70 \leq x \leq 100. \end{cases}$$

The intersection can be expressed as

$$(A_1 \cap A_2)(x) = \begin{cases} 0, & if \quad 40 \leq x < 50, \\ \frac{x-50}{10}, & if \quad 50 \leq x < 55, \\ 1 - \frac{x-50}{10}, & if \quad 55 \leq x < 60, \\ 0, & if \quad 60 < x \leq 100. \end{cases}$$

The complement of A_1 can be written

$$\overline{A_1}(x) = \begin{cases} 0, & if \quad 40 \leq x < 50, \\ \frac{x-50}{10}, & if \quad 40 \leq x < 60, \\ 1, & if \quad 60 \leq x \leq 100. \end{cases}$$

Example 6. *Let $X = \mathbb{R}$ and let A be the set of reals greater than 10 and B the set of reals close to 1 are characterized respectively by its membership functions*

$$\mu_A(x) = \begin{cases} 0, & if \quad x \leq 10, \\ (1 + (x-10)^{-2})^{-1}, & if \quad x > 10, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & if \quad x \leq 10, \\ (1 + (x-10)^{4})^{-1}, & if \quad x > 10. \end{cases}$$

So, we get $A \cap B$ set of reals greater than 10 and close to 11 given by its membership *function*

$$\mu_{A\cap B}(x) = \begin{cases} 0, & if \quad x \leq 10, \\ min[(1+(x-10)^{-2})^{-1}, (1+(x-10)^4)^{-1}], & if \quad x > 10. \end{cases}$$

And $A \cup B$ *the set of real numbers greater than 10 or close to 11 given by its membership function*

$$\mu_{A\cup B}(x) = max[(1+(x-10)^{-2})^{-1}, (1+(x-10)^4)^{-1}], x \in X.$$

1.3 Membership Functions in Type-1 Fuzzy Sets

This section provides the rigorous analytical definitions for the membership function typologies discussed previously, utilizing the specific numerical parameters illustrated in the provided graphical representations.

1.3.1 Triangular Membership Function

The Triangular Membership Function is defined by a triplet (a, b, c). According to the graphical instance, these parameters are assigned as $a = 2$, $b = 5$, and $c = 8$. The piecewise linear function is expressed as:

$$\mu_A(x) = \begin{cases} 0, & x \leq 2 \\ \frac{x-2}{3}, & 2 < x \leq 5 \\ \frac{8-x}{3}, & 5 < x < 8 \\ 0, & x \geq 8 \end{cases} \tag{1.1}$$

Example Analysis: This function is ideally suited for modeling the linguistic variable **"Average Height"**. As shown in the diagram, an individual with a value of 5 units represents the absolute prototype of the set ($\mu_A = 1$), while the membership degree decreases linearly as the value deviates toward the boundaries of 2 and 8.

1.3.2 Trapezoidal Membership Function (TrapMF)

The Trapezoidal form expands the modal peak into an interval $[b, c]$. Based on the visual data, the parameters are defined as $a = 2$, $b = 4$, $c = 7$, and $d = 9$. The mathematical formulation is:

$$\mu_A(x) = \begin{cases} 0, & x \leq 2 \\ \frac{x-2}{2}, & 2 < x \leq 4 \\ 1, & 4 < x``` \leq 7 \\ \frac{9-x}{2}, & 7 < x < 9 \\ 0, & x \geq 9 \end{cases} \tag{1.2}$$

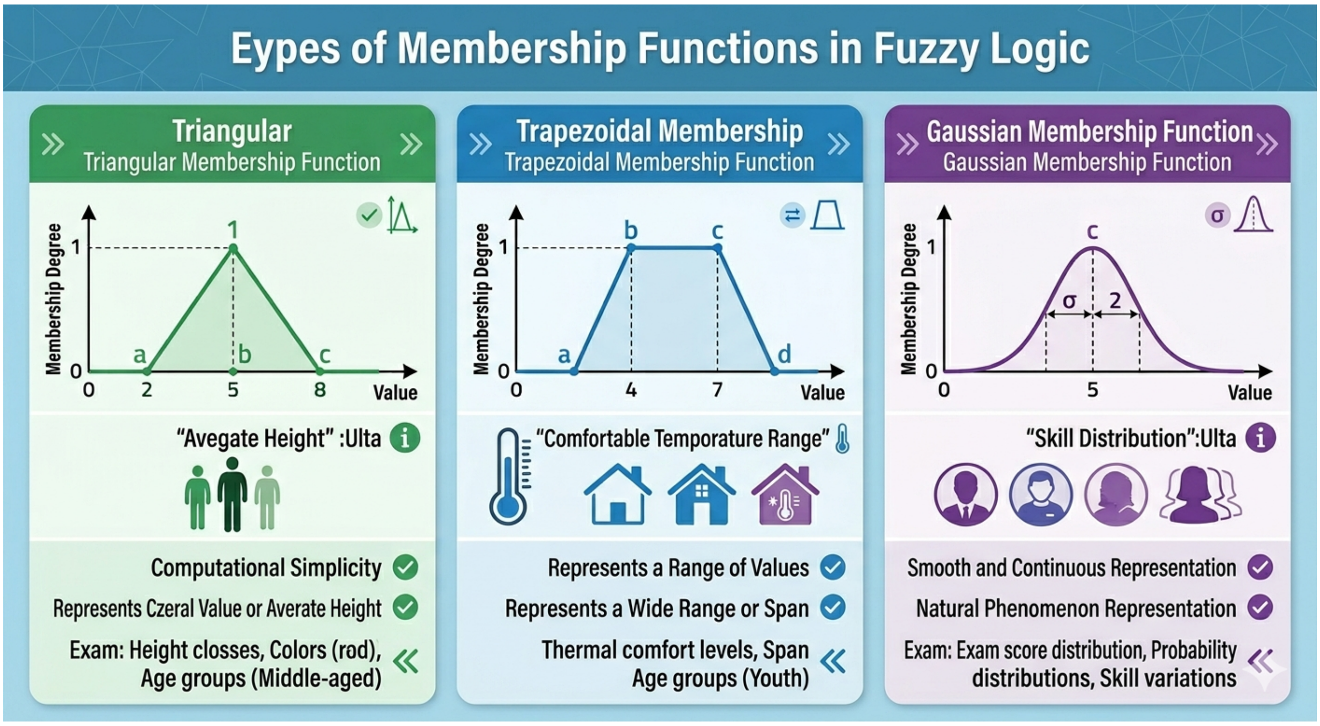
Example Analysis: This profile represents a **"Comfortable Temperature Range"**. Any temperature within. the plateau $[4, 7]$ maintains a full membership of 1, providing a stable region of "comfort" before the gradual transition into "uncomfortable" zones (Average Height, Comfortable Temperature, Skill Distribution) . (slopes between $2 - 4$ and $7 - 9$).

1.3.3 Gaussian Membership Function (GaussMF)

The Gaussian function provides a smooth bell curve determined by the center $c = 5$ and width $\sigma = 2$. Its analytical form, which avoids the sharp vertices of linear models, is defined as:

$$\mu_A(x) = \exp\left(-\frac{1}{2}\left(\frac{x-5}{2}\right)^2\right) \tag{1.3}$$

Example Analysis: This curve is employed for modeling **"Skill Distribution"** or natural phenomena. Unlike linear functions, the Gaussian model never reaches absolute zero, reflecting the probabilistic nature of human skills where even extreme outliers retain a minimal, non-zero degree of membership in the distribution.



1.4 $\alpha-$cuts

Definition 7. ***(The*** α***-cut of a fuzzy set)*** [37] *Let A be a fuzzy set on a set X. The α-cut of A is the crisp subset*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\} \text{ where } \alpha \in [0,1].$$

Particular cases:

1. *If $\alpha = 0$, then $A_0 = X$.*

2. *If $\alpha = 1$, then $A_1 = Ker(A)$.*

Example 8. *Let $X = \{L, M, N, O, P, K\}$, and A be a fuzzy subset of X given by $A = \{< L, 0.1 >, < M, 0.3 >, < N, 1.0 >, < O, 0.7 >, < P, 0.5 >, < K, 0.0 >\}$. Then, the α-cut of A is given by:*

$A_0 = \{x \in X, A(x) > 0\} = X;$

$A_{0.1} = \{x \in X, A(x) \geq 0.1\} = \{L, M, N, O, P\};$

$A_{0.2} = \{x \in X, A(x) \geq 0.2\} = \{M, N, O, P\};$

$A_{0.3} = \{x \in X, A(x) \geq 0.3\} = \{M, N, O, P\};$

$A_{0.4} = \{x \in X, A(x) \geq 0.4\} = \{N, O, P\};$

$A_{0.5} = \{x \in X, A(x) \geq 0.5\} = \{N, O, P\};$

$A_{0.6} = \{x \in X, A(x) \geq 0.6\} = \{N, O\};$

$A_{0.7} = \{x \in X, A(x) \geq 0.7\} = \{N, O\};$

$A_{0.8} = \{x \in X, A(x) \geq 0.8\} = \{N\};$

$A_{0.9} = \{x \in X, A(x) \geq 0.9\} = \{N\};$

$A_1 = \{x \in X, A(x) \geq 1\} = \{N\}.$

1.5 T-norms and t-conorms

The history of triangular-norms (t-norms) started with Menger [41]. His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements.

1.5.1 T-norms

Definition 9. [35] *A t-norm* T *on* $[0,1]$ *is a function* $T : [0,1]^2 \rightarrow [0,1]$ *satisfies the following four axioms:*

(T1) Commutativity: $(\forall x, y \in [0,1])(T(x,y) = T(y,x))$.

(T2) Associativity: $(\forall x, y, z \in [0,1])(T(x, T(y,z)) = T(T(x,y), z)))$.

(T3) Monotonicity: $(\forall x, y, z \in [0,1])(x \leq y \Rightarrow T(x,z) \leq T(y,z))$.

(T4) Boundary condition: $(\forall x \in [0,1])(T(x,1) = x)$.

Conditions $(T4)$ *and* $(T3)$ *imply that for any t-norm* T *it holds that* $T(x,y) \leq x, T(x,y) \leq y, T(x,y) \leq Min(x,y)$ *and* $T(x,0) = 0$.

Example 10. *The following four operations are the most common t-norms:*

(T5) Minimum: $T_M(x, y) = min\{x, y\}$.

(T6) Product: $T_P(x, y) = x \cdot y$.

(T7) Lukasiewicz: $T_L(x, y) = max\{x + y - 1, 0\}$.

(T8) Drastic product:

$$T_D(x, y) = \begin{cases} x, & if \;\; y = 1 \\ y, & if \;\; x = 1 \\ 0, & if \;\; x, y < 1. \end{cases}$$

Let T be a t-norm on $[0, 1]$. An element $\alpha \in]0, 1[$ is called a zero divisor of T if there exists some $b > 0$ such that $T(a, b) = 0$. An element $\alpha \in [0, 1]$ is called an idempotent element of T if $T(a, a) = a$. T is called Archimedean if $T(x, x) < x$, for any $x \in [0, 1]$. Each $\alpha \in [a, b]$ is an idempotent element of the Minimum t-norm T_M(Actually T_M is the only t-norm whose set of idempotent is equal $[0, 1]$), T_M has no zero divisor. Each $\alpha \in]0, 1[$ is a zero divisor of the Lukasiewicz t-norm T_L as well of the Drastic product t-norm T_D. For two t-norms T_1 and T_2 on $[0, 1]$, we define:

$$T_1 \leq T_2 \Leftrightarrow (\forall x, y \in [0, 1])(T_1(x, y) \leq T_2(x, y)).$$

Let T_1 and T_2 be two t-norms. If $T_1 \leq T_2$, then T_1 is called weaker than T_2 (or equivalently, T_2 is called stronger than T_1). Note that T_D is the weakest t-norm, and T_M is the strongest t-norm, i.e., for any t-norm it holds: $(T9)$ $T_D \leq T \leq T_M$. Since, $T_L \leq T_P$, it obviously holds: $(T10)$ $T_D \leq T_L \leq T_P \leq T_M$.

Example 11.

1. $T_0(x, y) = \begin{cases} 0, & if \;\; (x, y) \in [0, 1[^2 \\ min(x, y), & otherwise; \end{cases}$

2. $T_1(x, y) = max(x + y - 1, 0)$;

3. $T_{1.5}(x, y) = \frac{xy}{2 - x - y + xy}$;

4. $T_2(x, y) = xy$;

5. $T_{2.5}(x,y) = \frac{xy}{x+y-xy}$;

6. $T_3(x,y) = min(x,y)$.

We have $T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$.

Proof. 1. $T_0(x,y) = \begin{cases} 0, & if \quad (x,y) \in [0,1[^2; \\ min(x,y), & otherwise. \end{cases}$

If $(x,y) \in [0,1[^2$ then, $T_0 \leq T_1$.

If $(x,y) \notin [0,1[^2$, i.e., $(x,y) \in \{1\} \times [0,1]$ or $[0,1] \times \{1\}$.

If $(x,y) \in \{1\} \times [0,1]$: $T_0(x,y) = T_0(1,y) = y$ and $T_1(x,y) = T_1(1,y) = y$ then, $T_0 \leq T_1$.

If $(x,y) \in [0,1] \times \{1\}$: $T_0(x,y) = T_0(x,1) = x$ and $T_1(x,y) = T_1(x,1) = x$ then, $T_0 \leq T_1$. So, $T_0(x,y) \leq T_1(x,y)$.

Then, $T_0 \leq T_1$.

2. $T_1(x,y) = max(x+y-1,0)$ there are two cases:

(1) $x+y-1 \leq 0 \Rightarrow T_1(x,y) = max(x+y-1,0) = 0 \leq T_{1.5}(x,y)$;

(2) $x+y-1 > 0 \Rightarrow T_1(x,y) = max(x+y-1,0) = x+y-1$.

$$\begin{aligned} T_{1.5}(x,y) - T_1(x,y) &= \frac{xy}{2-x-y+xy} - (x+y-1) \\ &= \frac{(xy-(x+y-1)(2-(x+y)+xy))}{2-x-y+xy}. \end{aligned}$$

Since $(2-x-y+xy) > 0$, it is suffices to determine the sign of the numinator $[xy+(x+y-1)(x+y-xy-2)]$

$(x+y-1)(x+y-xy-2)+xy$

$= (x+y-1)((x+y-1)-(xy+1))+xy$

$= (x+y-1)^2 - (x+y-1)(xy+1)+xy$

$= (x+y-1)^2 - x^2y - xy^2 + 2xy$

$= (x+y-1)^2 + (xy-x^2y) + (xy-xy^2) \geq 0$.

Therefore, $T_{1.5}(x,y) - T_1(x,y) \geq 0$.

Then, $T_1(x,y) \leq T_{1.5}(x,y)$.

(3) $T_2(x,y) = xy$.

$$\begin{aligned} T_{1.5}(x,y) - T_2(x,y) &= \frac{xy}{2-x-y+xy} - xy \\ &= \frac{xy - xy(2-(x+y)+xy)}{2-x-y+xy}. \end{aligned}$$

Since $2-(x+y)+xy > 0$, thus it is enough to determine the sign of the numinator

$$\begin{aligned} xy - xy(2-(x+y)+xy) &= xy + xy(x+y-xy-2) \\ &= xy(x+y-xy-1) \\ &= xy((x-1)+y(1-x)) \\ &= xy(x-1)(1-y) \leq 0. \end{aligned}$$

Thus, $T_{1.5}(x,y) - T_2(x,y) \leq 0$. Then, $T_{1.5}(x,y) \leq T_2(x,y)$.

(4) $T_{2.5}(x,y) = \frac{xy}{x+y-xy}$.

$$\begin{aligned} T_2(x,y) - T_{2.5}(x,y) &= xy - \frac{xy}{x+y-xy} \\ &= \frac{xy(x+y-xy) - xy}{x+y-xy} \\ &= \frac{xy(x+y-xy-1)}{x+y-xy}. \end{aligned}$$

The denominator is positive ($x+y-xy > 0$), the numerator sign should be studied

$$\begin{aligned} xy(x+y-xy-1) &= xy(x(1-y)+(y-1)) \\ &= xy((1-y)(1-x)) \leq 0. \end{aligned}$$

Thus, $T_2(x,y) - T_{2.5}(x,y) \leq 0$. Then, $T_2(x,y) \leq T_{2.5}(x,y)$.

3. Finally,

$$T_{2.5}(x,y) - T_3(x,y) = \begin{cases} \frac{xy}{x+y-xy} - x, & if \quad x \leq y; \\ \frac{xy}{x+y-xy} - y, & otherwise. \end{cases}$$

If $x \leq y$: $T_{2.5}(x, y) \leq min(x, y)$.

If $x > y$: $T_{2.5}(x, y) \leq min(x, y)$.

Thus, for all $(x, y) \in [0, 1[^2$ $T_{2.5}(x, y) \leq T_3(x, y)$.

Consequently,

$$T_0(x, y) \leq T_1(x, y) \leq T_{1.5}(x, y) \leq T_2(x, y) \leq T_{2.5}(x, y) \leq T_3(x, y).$$

□

Definition 12. *A binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *is a continuous t-norm if* $([0, 1], *)$, *is a topological monoid with unit* 1 *such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and*
$b \leq d$ $(a, b, c, d \in [0, 1])$.

1.5.2 T-conorms

Definition 13. [35] *A t-conorm is a function* $S : [0, 1]^2 \to [0, 1]$ *that for any* $x, y, z \in [0, 1]$ *satisfies* $(T1) - (T3)$ *of Definition 9 and the following boundary condition* $S(x, 0) = S(0, x) = x,\ S(x, 1) = S(1, x) = 0.$

Remark 14. *Given a t-norm* T, *we find the associated dual t-conorm* S *by* $S(x, y) = 1 - T(1 - x, 1 - y)$. *The dual t-conorms w.r.t.* T_M, T_P, T_L *and* T_D *are given by:*

(S1) Maximum: $S_M(x, y) = max\{x, y\}$.

(S2) Probabilistic sum: $S_P(x, y) = x + y - x \cdot y$.

(T7) Lukasiewicz: $S_L(x, y) = min\{x + y, 1\}$.

(T8) Drastic sum:

$$S_D(x, y) = \begin{cases} 1, & if \quad (x, y) \in [0, 1]^2 \\ max\{x, y\}, & otherwise. \end{cases}$$

Definition 15. ***(Duality between operation)*** [42] *A t-norm* T *and a t-conorm* S *are said to be dual for the strict negation* n *if they satisfy the following formulas* *for all* $x, y \in [0, 1]$:

$S(x, y) = N(T(N(x), N(y)))$;

$T(x, y) = N(S(N(x), N(y)))$.

Example 16. *Let* $X = \{a, b, c\}$, *let* A *and* B *be two fuzzy subsets of* X *such that* $A = \{\langle a, 0.2\rangle, \langle b, 0.4\rangle, \langle c, 0.8\rangle\}$, $B = \{\langle a, 0.9\rangle, \langle b, 0.1\rangle, \langle c, 0.5\rangle\}$. *We can use the operators of Lukasiewicz to define the union and the intersection by*

1. $\mu_{A\cap_T B}(x) = T(\mu_A, \mu_B) = max(\mu_A(x) + \mu_B(x) - 1, 0),\ for\ any\ x \in X;$

2. $\mu_{A\cup_S B}(x) = S(\mu_A, \mu_B) = min(\mu_A(x) + \mu_B(x), 1),\ for\ any\ x \in X.$

Then, we get

1. $A \cap_T B = \{\langle a, 0.1\rangle, \langle b, 0\rangle, \langle c, 0.3\rangle\}$;

2. $A \cup_S B = \{\langle a, 1\rangle, \langle b, 0.5\rangle, \langle c, 1\rangle\}$.

Chapter 2

Type-2 Fuzzy Sets

In 1975, Zadeh presented the definition of type-2 fuzzy set(T2 FS)[1]. How to provide the membership function of a fuzzy set, different people have different means. T2 FSs can give people much greater freedom to achievement of membership grade, can put out a better solution for the problems of linguistic ambiguity and data noise, thus the study of T2 FS becomes a hot topic in artificial intelligence.

Mendel and his collaborators have done great works on type-2 fuzzy logic systems, type-2 fuzzy control and application[2], especially converted the discuss of interval type-2 fuzzy sets(IT2 FS) to the upper membership function (UMF) and lower membership function(LMF) of the corresponding footprint of uncertainty (FOU) (i.e., two type-1 fuzzy sets), and the study of IT2 FS are widely used in neural network[3], mobile robot [4] and the analysis of linguistic dynamic orbits based on IT2 FSs [5].

By the Zadeh's definition, for a given type-1 fuzzy set, it is not difficult for us to present the corresponding analysis representation and its geometry figure. In 2014, Mo and Wang presented the method of partition of FOU to represent a given IT2 FS. However, it is not easy to represent a general type-2 fuzzy sets[6], and in 2017, Wang and Mo sorted general T2 FSs into four classes: discrete, partially connected, connected and compounded T2 FSs[7], and then used partition of CoS to substituted partition of FOU, and second/third partition of CoS is presented to represent single-connected, and complex-connected general T2 FSs.

In the paper, the classification and the method of partition of CoS is introduced, and for aT2 FS, its representation is also given, and the properties of T2 FSs are also discussed.

2.1 Definitions

Definition 17. *A fuzzy set is of type-2, if its membership function is a type-1 fuzzy set.*

Definition 18. *A type-2 fuzzy set* ω *can be defined equivalently by a type- 2 membership function* $\mu_\omega^2(x,u)$*, where* $x \in X, u \in J_x \subseteq I$*, i.e.,*

$$\omega = \left\{ \left((x,u), \mu_\omega^2(x,u) \right) \mid x \in X, u \in J_x \subseteq [0,1] \right\}$$

or

$$\omega = \int_{x \in X} \int_{u \in J_x} \frac{\mu_\omega^2(x,u)}{(x,u)}$$

where x *(or* u *) is the primary (secondary) variable,* $J_x \subseteq I$ *is the primary membership,* $0 \leq \mu_\omega^2(x,u) \leq 1$ *is the secondary membership function,* $\iint$ *is the union of all the admission* x *and* u*. For the discrete case,* $\int$ *is replaced by* $\sum$*.*

In 2016, Mendel provided primary membership J_x another definition, $J_x \subseteq X \times I$ instead of $J_x \subseteq I$, so the symbol L_x which is a nonempty close subset is used to represented primary membership grade

By using multi-mapping, Mo and Wang presented new definition for type-2 fuzzy set as follows:

Definition 5. Let $C\left(2^I\right)$ be the set of all nonempty close subset of $I = [0,1]$, a type-2 fuzzy set ω on X is defined as

$$\omega = \left\{ (x,y,z) \mid x \in X, u \in L_x \in C\left(2^I\right), z = \mu_\omega^2(x,u) \in I \right\}$$

where x, u, z are the primary/secondary/third variables respectively, L_x is the primary membership grade defined by a multi-mapping

$$\mu_\omega^1 : X \to C\left(2^I\right)$$

i.e., for every $x \in X$, there is $L_x \in C\left(2^I\right)$, such that

$$\mu_\omega^1(x) = L_x$$

μ_ω^1 is called the primary membership function, and μ_ω^2 is the secondary membership function defined as follows:

$$\mu_\omega^2 : \bigcup_{x \in X} x \times L_x \to I$$

Where the secondary membership function can be seen as the membership function of a type-1 fuzzy set on $\bigcup_{x \in X} x \times L_x$.

Mo and Wang also provided the two-segment definition for type-2 fuzzy sets.

Definition 6. A type-2 fuzzy set ω on X is defined as

$$\mu_\omega : X \to \bigcup_{x \in X} I^{L_x}$$

where μ_ω is the membership function of ω, I^{L_x} is the set of all the fuzzy set on $L_x \in C\left(2^I\right)$, i.e.,

$$I^{L_x} = \{f_x \mid f_x : L_x \to I\}$$

2.2 Operations of fuzzy sets type-2

Definition 19 (Mathematical Operations)**.** *Operations on general Type-2 fuzzy sets are governed by Zadeh's Extension Principle. Let $\tilde{A}$ and $\tilde{B}$ be two Type-2 fuzzy sets,* *their Union (∪), Intersection (∩), and Complement (¬) are defined as:*

$$\mu_{\tilde{A}\cup\tilde{B}}(x,w) = \sup_{w=\max(u,v)} \min\left(\mu_{\tilde{A}}(x,u), \mu_{\tilde{B}}(x,v)\right) \tag{2.1}$$

$$\mu_{\tilde{A}\cap\tilde{B}}(x,w) = \sup_{w=\min(u,v)} \min\left(\mu_{\tilde{A}}(x,u), \mu_{\tilde{B}}(x,v)\right) \tag{2.2}$$

$$\mu_{\neg\tilde{A}}(x,w) = \mu_{\tilde{A}}(x, 1-u) \quad (\textit{where } w = 1-u) \tag{2.3}$$

Example 20. *Let two discrete secondary memberships at element x be $\mu_{\tilde{A}}(x) = 0.5/0.6 + 1.0/0.7$ and $\mu_{\tilde{B}}(x) = 1.0/0.5 + 0.4/0.8$. Evaluating the intersection via the minimum pairwise combination yields:*

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = 1.0/0.5 + 0.4/0.6 + 0.4/0.7 \tag{2.4}$$

2.3 Membership Functions in Type-2 Fuzzy Sets

2.3.1 Triangular Membership Function:

Practical Example: "Warm Temperature"

In this model, we represent the concept of "Warm Temperature" using a *Triangular Interval Type-2 Fuzzy Set.* Due to varying expert opinions, a single crisp membership value is insufficient. Thus, we utilize:

- **Upper Membership Function (UMF)**: The maximum degree of membership.
- **Lower Membership Function (LMF)**: The minimum guaranteed degree of membership.
- **Footprint of Uncertainty (FOU)**: The region between UMF and LMF.

Definition 21. *The UMF and LMF are defined as follows:*

Upper Membership Function (UMF):

$$\mu_U(x) = \begin{cases} 0, & x \leq 20 \\ \frac{x-20}{10}, & 20 < x \leq 30 \\ \frac{40-x}{10}, & 30 < x < 40 \\ 0, & x \geq 40 \end{cases}$$

Lower Membership Function (LMF):

$$\mu_L(x) = \begin{cases} 0, & x \leq 22 \\ \frac{x-22}{8}, & 22 < x \leq 30 \\ \frac{38-x}{8}, & 30 < x < 38 \\ 0, & x \geq 38 \end{cases}$$

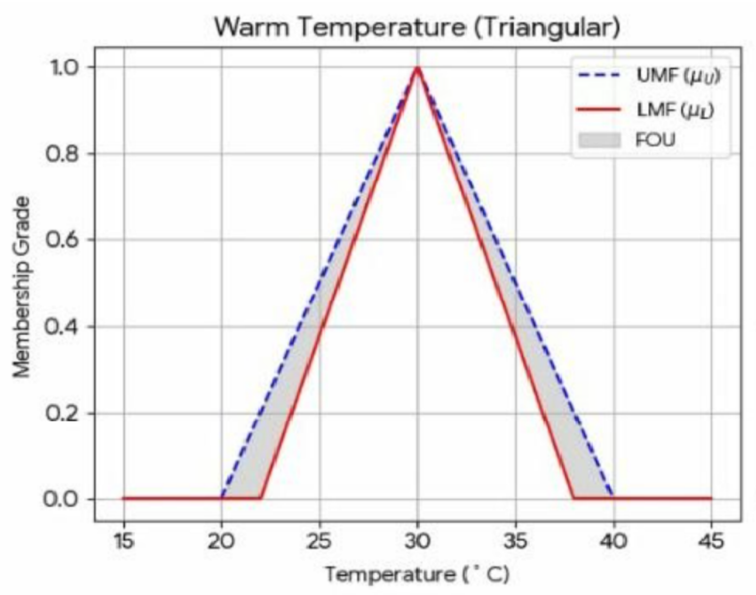
Temperature (°C)	UMF	LMF
20	0	0
25	0.5	0.375
30	1.0	1.0
35	0.5	0.375
40	0	0

Table 2.1: Membership values for Warm Temperature.

Example 22.

Graph Explanation:

The blue dashed line represents the UMF, while the solid red line represents the LMF. The shaded area denotes the FOU. This is widely used in intelligent control, HVAC systems, and robotics.



2.3.2 Trapezoidal Membership Function:

Practical Example: "Comfortable Speed"

We represent "Comfortable Speed" for a vehicle where drivers have different perceptions of comfort.

Definition 23. ***Upper Membership Function (UMF):***

$$\mu_U(x) = \begin{cases} 0, & x \leq 40 \\ \frac{x-40}{20}, & 40 < x \leq 60 \\ 1, & 60 < x \leq 100 \\ \frac{120-x}{20}, & 100 < x < 120 \\ 0, & x \geq 120 \end{cases}$$

Lower Membership Function (LMF):

$$\mu_L(x) = \begin{cases} 0, & x \leq 50 \\ \frac{x-50}{15}, & 50 < x \leq 65 \\ 0.7, & 65 < x \leq 95 \\ \frac{110-x}{15}, & 95 < x < 110 \\ 0, & x \geq 110 \end{cases}$$

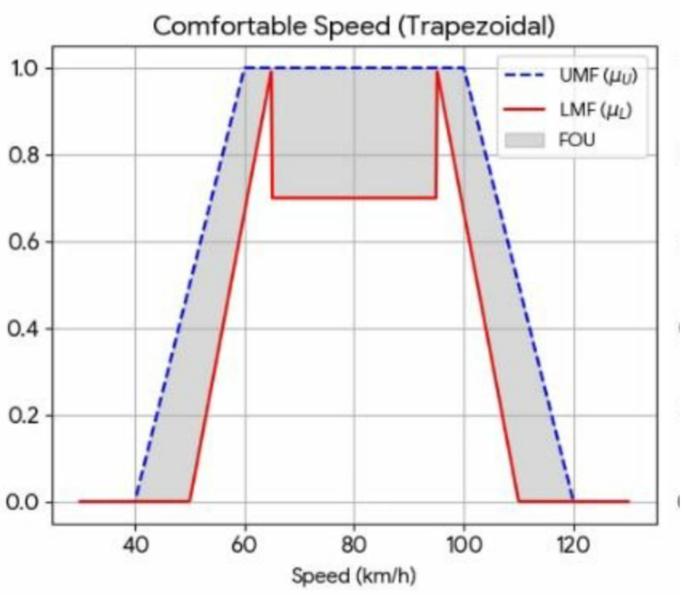
Speed (km/h)	UMF	LMF
40	0	0
60	1.0	0.67
80	1.0	0.70
100	1.0	0.67
120	0	0

Table 2.2: Membership values for Comfortable Speed.

Example 24.

Interpretation:

Speeds between 60 and 100 km/h are highly comfortable according to the UMF, but the LMF remains lower (0.7) to account for stricter driver preferences. This model is ideal for stable states like safety levels or quality of service.



2.3.3 Gaussian Membership Function:

Practical Example: "High Blood Pressure"

Medical diagnosis often involves uncertainty. We use a *Gaussian Interval Type-2 Fuzzy Set* to model "High Blood Pressure."

Definition 25. ***Upper Membership Function (UMF):***

$$\mu_U(x) = e^{-\frac{(x-140)^2}{2(20)^2}}$$

Lower Membership Function (LMF):

$$\mu_L(x) = 0.7 \cdot e^{-\frac{(x-140)^2}{2(15)^2}}$$

Example 26.

Why Gaussian?

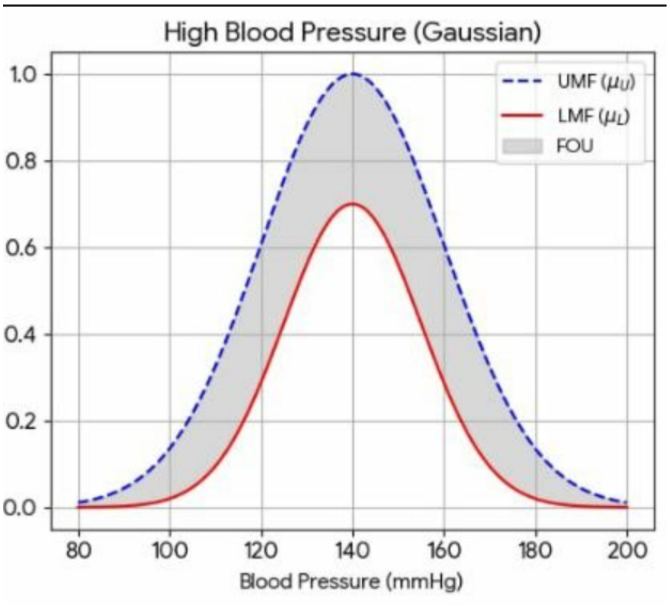
The Gaussian function is smooth and mimics natural data distribution perfectly. It is highly effective in medical fields, image processing, and machine learning because it provides a more realistic, gradual transition than triangular or trapezoidal functions.

Blood Pressure (mmHg)	UMF	LMF
100	0.14	0.02
120	0.61	0.29
140	1.00	0.70
160	0.61	0.29
180	0.14	0.02

Table 2.3: Membership values for High Blood Pressure.

Graph Explanation:

The blue dashed curve (UMF) and the solid red curve (LMF) enclose the FOU, representing the medical uncertainty around the central value of 140 mmHg.



2.4 Advanced Representations: α-planes

Definition 27 (α-plane Decomposition)**.** *An α-plane slices a general Type-2 fuzzy set horizontally at a specific secondary membership level $\alpha \in$. It decomposes a three-dimensional general Type-2 system into a series of computable Interval Type-2 fuzzy* *sets:*

$$\tilde{A}_\alpha = \{((x,u), \mu_{\tilde{A}}(x,u) \geq \alpha) \mid \forall x \in X\} \tag{2.5}$$

Example 28. *In a complex control loop, computing mathematical operations directly on a continuous 3D function is intensive. By choosing* $\alpha = 0.5$*, we slice the set horizontally, reducing the representation into a simple 2D interval bounded boundary at that exact elevation.*

2.5 Algebraic Framework: t-norms and t-conorms

Definition 29 (Type-2 Triangular Norms)**.** *In Type-2 fuzzy logic, t-norms and t-conorms extend point-value logical operations to functional combinations. A Type-2 t-norm models generalized intersection (AND), while a Type-2 t-conorm models generalized union (OR).*

Example 30. *Using the Product t-norm framework to calculate the intersection of two Interval Type-2 sets* $\tilde{A}$ *and* $\tilde{B}$ *yields boundary formulas determined by direct multiplication:*

$$\bar{\mu}_{\tilde{A}\cap\tilde{B}}(x) = \bar{\mu}_{\tilde{A}}(x) \times \bar{\mu}_{\tilde{B}}(x), \quad \underline{\mu}_{\tilde{A}\cap\tilde{B}}(x) = \underline{\mu}_{\tilde{A}}(x) \times \underline{\mu}_{\tilde{B}}(x) \tag{2.6}$$

If $\bar{\mu}_{\tilde{A}}(x) = 0.8$ *and* $\bar{\mu}_{\tilde{B}}(x) = 0.9$*, the combined upper boundary evaluates to* $0.8 \times 0.9 = 0.72$.

Chapter 3

Type-3 Fuzzy Sets: Operations and Properties

The final chapter investigates the mathematical definitions and set-theoretic operations governing general Type-3 fuzzy systems. It provides a detailed study of the hierarchical interpretation of alpha-cuts (α-cuts) and α-planes [49]. This chapter demonstrates how continuous, multi-layered volumetric structures can be systematically decomposed into crisp, computable intervals, drastically reducing mathematical complexity and facilitating efficient type-reduction.

3.1 Definitions

Definition 31. *Let X be a universal set. A* ***Type-3 fuzzy set*** *T is defined as:*

$$T = \Big\{ \Big(x, \mu_T(x)\Big) \mid x \in X \Big\}$$

where $\mu_T(x)$ is a Type-2 fuzzy set on $[0,1]$. Equivalently:

$$\mu_T(x) = \int_{u\in[0,1]} \frac{f_x(u)}{u}, \quad f_x(u) = \int_{v\in[0,1]} \frac{\tau_x(u,v)}{v}$$

Thus, the ***tertiary membership function*** *is:*

$$\tau_x : [0,1] \times [0,1] \to [0,1]$$

with $\tau_x(u, v)$ *being the tertiary membership grade of* u *at secondary level* v.

3.2 Operations of fuzzy sets type-3

Let A and B be two Type-3 fuzzy sets on X, with tertiary membership functions $\tau_A(x, u, v)$ and $\tau_B(x, u, v)$.

Union

$$\tau_{A \cup B}(x, u, v) = \tau_A(x, u, v) \oplus \tau_B(x, u, v)$$

where $\oplus$ is a t-conorm (e.g., maximum or probabilistic sum).

Intersection

$$\tau_{A \cap B}(x, u, v) = \tau_A(x, u, v) \otimes \tau_B(x, u, v)$$

where $\otimes$ is a t-norm (e.g., minimum or product).

Complement

$$\tau_{A^c}(x, u, v) = 1 - \tau_A(x, u, v)$$

3.3 Membership Functions in Type-3 Fuzzy Sets

Several membership function shapes are used.

3.3.1 Triangular Type-3 Membership Function

Simple and computationally efficient.

$$f(x) = \max\left(1 - \frac{|x - c|}{a}, 0\right)$$

3.3.2 Trapezoidal Membership Function

Mathematical DefinitionAn Interval Type-3 (IT3) Trapezoidal Membership Function is characterized by a three-dimensional volumetric space formed by vertical or horizontal slice decompositions across secondary membership levels $\alpha \in [0,1]$ [1.2].At each specific α-plane, the continuous IT3 fuzzy set collapses into a baseline Interval Type-2 fuzzy set bounded by a parameterized Upper Membership Function (UMF) and a Lower Membership Function (LMF) [1.1, 1.2]:Upper Membership Function (UMF): Denoted as $\bar{\mu}_{\tilde{A}_\alpha}(x)$, defines the wider outer boundary of the fuzzy volume [1.1, 1.2].Lower Membership Function (LMF): Denoted as $\underline{\mu}_{\tilde{A}_\alpha}(x)$, defines the narrower inner core of the fuzzy volume [1.1, 1.2].The mathematical piecewise formulations at any given α-plane are expressed as follows [1.1]:

$$\mu_{A_\alpha}(x) = \max\left(0, \min\left(\frac{x - a_1(\alpha)}{b_1(\alpha) - a_1(\alpha)}, 1, \frac{d_1(\alpha) - x}{d_1(\alpha) - c_1(\alpha)}\right)\right)$$

$$\underline{\mu}_{\tilde{A}_\alpha}(x) = \max\left(0, \min\left(\frac{x - a_2(\alpha)}{b_2(\alpha) - a_2(\alpha)}, 1, \frac{d_2(\alpha) - x}{d_2(\alpha) - c_2(\alpha)}\right)\right)$$

Where the parameters $a(\alpha), b(\alpha), c(\alpha)$, and $d(\alpha)$ are functions that vary dynamically (linearly or non-linearly) with respect to the α-level, representing the contraction and shifting of the trapezoidal footprint across the hierarchical secondary axis [1.1].2. Numerical Example To model a safe "Comfortable Speed" for an autonomous train under highly dynamic weather conditions, let us evaluate the system parameters at a specific horizontal slice $\alpha = 0.5$ [1.1]:UMF Parameters (Outer Trapezoid): $[a_1 = 20,\ b_1 = 40,\ c_1 = 60,\ d_1 = 80]$ LMF Parameters (Inner Trapezoid): $[a_2 = 30,\ b_2 = 45,\ c_2 = 55,\ d_2 = 70]$ Evaluation at a current velocity of $x = 35$ km/h: UMF Boundary

$$(= \mu) := \mu(35) = \max\left(0, \min\left(\frac{35-20}{40-20}, 1, \frac{80-35}{80-60}\right)\right) = \min\left(\frac{15}{20}, 1, \frac{45}{20}\right) = 0.75$$

LMF Boundary ($\underline{\mu}$):$\underline{\mu}(35) = \max\left(0, \min\left(\frac{35-30}{45-30}, 1, \frac{70-35}{70-55}\right)\right) = \min\left(\frac{5}{15}, 1, \frac{35}{15}\right) = 0.33$Result: At $x = 35$ km/h and $\alpha = 0.5$, the resulting fuzzy primary membership grade is precisely bounded within the interval $[0.33, 0.75]$ [1.1, 1.2].3.

3.3.3 Gaussian Membership Function

Smooth and widely used in control systems.

$$f(x) = e^{-\frac{(x-c)^2}{2\sigma^2}}$$

Remark 32. *In fuzzy sets of the third type (Type-3 Fuzzy Sets), the representation of the belonging function geometrically requires a three-dimensional space (3D); because we are not dealing with a simple curve, but with a pyramidal figure in which the levels of certainty overlap. Here is a graphical representation of a triangular affiliation function of the third type in which the uncertainty imprint (FOU) appears and the horizontal cut planes take triangular and intersecting three-dimensional shapes.*

3.4 Alpha-Cut in Type-3 Fuzzy Sets

The α-cut of a Type-3 fuzzy set is the collection of all tuples whose tertiary membership value is at least α.

Mathematically:

$$\tilde{\tilde{A}}_\alpha = \{(x, u, v) \mid \mu_{\tilde{\tilde{A}}}(x, u, v) \geq \alpha\}$$

Example of Alpha-Cut in Type-3 Fuzzy Sets Suppose:

$$\mu_{\tilde{\tilde{A}}}(x, u, v) = 0.85$$

and:

$$\alpha = 0.7$$

Since:

$$0.85 \geq 0.7$$

the tuple (x, u, v) belongs to the α-cut set.

If:

$$\mu_{\tilde{\tilde{A}}}(x, u, v) = 0.4$$

then:

- It is excluded from the α-cut.

Role of Alpha-Cuts in Type-3 Fuzzy Systems Alpha-cuts are essential because they simplify higher-order fuzzy structures.

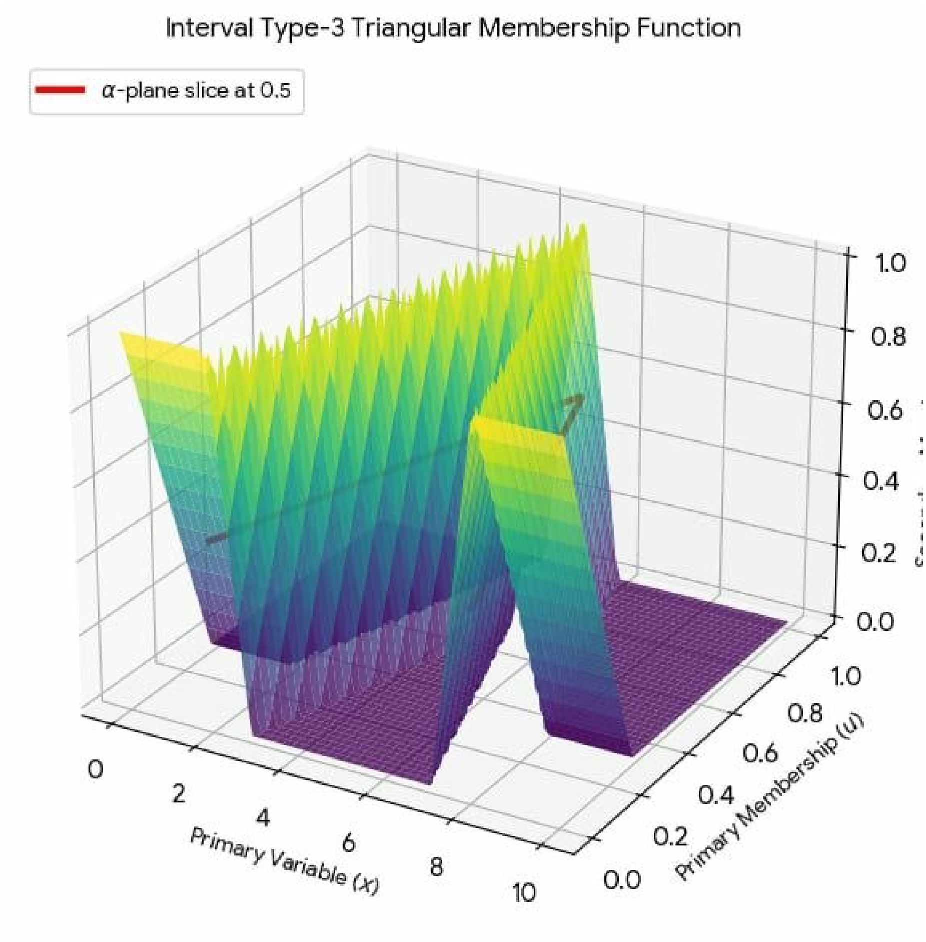
They are used in:

- Type reduction
- Defuzzification
- Rule evaluation
- Optimization
- Approximate reasoning

Type Reduction Using Alpha-Cuts Type reduction in Type-3 systems is extremely difficult.

Alpha-cuts help by:

- Decomposing fuzzy structures
- Converting higher-order fuzzy sets into intervals
- Simplifying computations



3.5 T-Norm and T-Conorm in Type-3 Fuzzy Sets

In fuzzy logic systems, **T-norms** and **T-conorms** are fundamental mathematical operators used to generalize logical AND and OR operations.

They are essential in:

- Fuzzy inference systems
- Decision making
- Fuzzy control
- Approximate reasoning

- Aggregation operations

In **Type-3 Fuzzy Sets (T3FS)**, these operators become more sophisticated because they must handle tertiary uncertainty.

3.5.1 T-Norm

Definition 33. *A **T-norm** is a function used to model fuzzy intersection (AND operation).*

$$T : [0,1] \times [0,1] \to [0,1]$$

For two membership values a and b:

$$T(a,b)$$

represents:

$$a\ AND\ b$$

Properties of T-Norms A valid T-norm must satisfy:

Commutativity

$$T(a,b) = T(b,a)$$

Associativity

$$T(a, T(b,c)) = T(T(a,b), c)$$

Monotonicity

If:

$$a_1 \leq a_2$$

then:

$$T(a_1, b) \leq T(a_2, b)$$

Boundary Condition

$$T(a, 1) = a$$

Common T-Norms **Minimum T-Norm**

$$T(a, b) = \min(a, b)$$

Algebraic Product

$$T(a, b) = ab$$

Bounded Product

$$T(a, b) = \max(0, a + b - 1)$$

3.5.2 T-Conorm

Definition 34. *A **T-conorm** (or S-norm) models fuzzy union (OR operation).*

$$S : [0, 1] \times [0, 1] \to [0, 1]$$

Where:

$$S(a, b)$$

represents:

$$a \; OR \; b$$

Properties of T-Conorms A valid T-conorm satisfies:

1. Commutativity
2. Associativity
3. Monotonicity
4. Boundary condition:

$$S(a, 0) = a$$

Common T-Conorms **Maximum T-Conorm**

$$S(a, b) = \max(a, b)$$

Algebraic Sum

$$S(a, b) = a + b - ab$$

Bounded Sum

$$S(a, b) = \min(1, a + b)$$

Example Using Minimum T-Norm Suppose:

$$\mu_A(x, u, v) = 0.7$$

$$\mu_B(x, u, v) = 0.5$$

Using minimum T-norm:

$$T(0.7, 0.5) = 0.5$$

Thus:

$$\mu_{A\cap B} = 0.5$$

Example Using Maximum T-Conorm Given:

$$\mu_A(x, u, v) = 0.7$$

$$\mu_B(x, u, v) = 0.5$$

Using maximum T-conorm:

$$S(0.7, 0.5) = 0.7$$

Thus:

$$\mu_{A\cup B} = 0.7$$

Applications T-norms and T-conorms in Type-3 fuzzy systems are used in:

- Artificial Intelligence
- Robotics
- Medical diagnosis
- Intelligent control systems
- Pattern recognition

T-norms and T-conorms are core operators in Type-3 fuzzy logic systems. They extend classical fuzzy intersection and union into higher-order uncertainty environments.

Although computationally complex, they provide powerful tools for modeling highly uncertain real-world systems.

Conclusion

Throughout this dissertation, we have systematically traced the mathematical evolution of fuzzy set theory from its classic binary-defying origins to its modern, high-dimensional extensions [67, 69]. By examining Type-1, Type-2, and Type-3 fuzzy sets, we highlighted how each successive generation absorbs a deeper layer of environmental and linguistic uncertainty [49]. Type-1 sets provided the fundamental bridge from crisp logic to gradual membership; Type-2 sets successfully captured the vagueness inherent in defining membership boundaries themselves [81]; and Type-3 sets established an advanced hierarchical framework capable of modeling multi-layered, compound uncertainty [80].

Our rigorous review of the algebraic properties demonstrated that tools like t-norms, t-conorms, and decomposition theorems remain vital across all types, ensuring mathematical consistency during system transitions [34]. Furthermore, the structural analysis of alpha-cuts (α-cuts) highlighted in Type-3 systems revealed how complex, continuous fuzzy structures can be systematically reduced into crisp, computable intervals, making intricate operations mathematically tractable [49].

In conclusion, while higher-type fuzzy sets introduce significant computational complexity and algebraic difficulty, their superior robustness and fidelity in mimicking human decision-making make them an indispensable asset for the future of artificial intelligence, control systems, and automated medical diagnosis. Future research paths stemming from this work could explore optimizing the algorithmic performance of Type-3 type-reduction methods or integrating these multi-layered sets into real-time industrial applications.

Bibliography

[1] M. Akram, S. Shahzadi, and A.B. Saeid, Single valued neutrosophic hypergraphs, TWMS J. App. Eng. Math, 8 (1), (2018), 122-135.

[2] K. Atanassov, Intuitionistic fuzzy sets, VII ITKRs scientic session, Sofia, (1983).

[3] K.Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems, 20, (1986), 87-96.

[4] K. Atanassov, Intuitionistic fuzzy sets, Springer-Verlag, Heidelberg/NewYork, (1999).

[5] K. Atanassov, On Intuitionistic fuzzy sets, Springer, Berlin,(2012).

[6] B. Bede, Mathematics of fuzzy sets and fuzzy logic, STUDFUZZ, 295.

[7] R. Bélàohlvek, Fuzzy relational systems: Foundations and principles, kluwer academic publishers/plenum publishers, New York.

[8] A. Bennoui, L. Zedam, S. Milles, Several types of single-valued neutrosophic ideals and filters on a lattice Isik University Press, 13 (1), (2023), 175-188.

[9] R. Biswas, On fuzzy sets and intuitionistic fuzzy sets, NIFS, 3, (1997), 3-11.

[10] J. J. Buckley, E. Eslami, An introduction to fuzzy logicand fuzzy sets, Springer-Verlag Brlin Heidelbeg Gmbh,(2002).

[11] P. Burillo, H. Bustince, Intuitionistic fuzzy relations(PartI), Mathware and soft computing magasine, 2, (1995), 5-38.

[12] P. Burillo, H. Bustince, Intuitionistic fuzzy relations(PartII), Effect of Atanassov's operators on the properties of the intuitionistic fuzzy relations, Mathware and soft computing magasine, 2, (1995), 117-148.

[13] H. Bustince, P. Burillo, Structures on intuitionistic fuzzy relations, Fuzzy Sets and Systems, 3, (1996), 293–303.

[14] H. Bustince, Construction of intuitionistic fuzzy relations with predetermined properties, Fuzzy sets and systems, 3, (2003), 79-403.

[15] N. Çagman, S. Enginoglu and F. Çitak, Fuzzy soft sets theory and its applications, Iranian journal of fuzzy systems, 8, 3, (2011), 137-147.

[16] B.Y. Cao, Z.L. Liu, Z. Yu-Bin, H. Mi, Fuzzy systems operations research and management, Springer cham heidelberg new york dordrecht, London, (2016).

[17] C. Cornelis and G. Deschrijver, The compositional rule of inference in an intuitionistic fuzzy logic setting, K. Striegnitz (Ed.), Proceedings of student session, Kluwer academic publishers, (2001), 83-94.

[18] B. C. Cuong, Picture fuzzy sets, Journal of computer science and cybernetics, 30, (2014), 409-420.

[19] B. C. Cuong and P.V. Hai, Some fuzzy logic operators for picture fuzzy sets, In proceedings of the IEEE seventh international conference on knowledge and systeurs engineering, Ho Chi Minh, Vietnam, (2015), 132-137.

[20] C. B. Cuong, R. T.Ngan and B. D. Hai, An involutive picture fuzzy negator on picture fuzzy sets and some de morgan triples, Seventh international conference on knowledge and systems engineering (KSE), IEEE, (2015).

[21] G. Des chrijver, E.E. Kerre, On the composition of intuitionistic fuzzy relations, Fuzzy sets and systems, 136, (2003), 333-361.

[22] G. Des chrijver and E.E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy sets and systems, 133, (2003), 227-235.

[23] S. Díaz, E. Induráin, V. Janis and S. Montes, Aggregation of convex intuitionistic fuzzy sets, 308 (1), (2015), 61-71.

[24] N.V. Dinh, N.X. Thao, N.M. Chau, and J. Sci. Devel, On the picture fuzzy database: Theories and application, Faculty of information technology, Viet Nam national University of agriculture, 13, (2015), 1028-1035.

[25] P.A. Ejegwa, Pythagorean fuzzy set and its application in career placements based on academic performance using max-min-max composition, Springer nature remains neutral, 5, (2019), 165-175.

[26] J.A. Goguen, L-fuzzy sets, Journal of mathematical analysis and applications, 18 (1), (1967), 145-174.

[27] S. Gottwald, A treatise on many-valued logics, Research studies press, baldock, Hertfordshire, England, (2001).

[28] K. Gündogdu, C. Kahraman, Spherical fuzzy sets and spherical fuzzy TOPSIS method, Journal of intelligent and fuzzy systems, (2018), 1–16.

[29] B. Hassane, Mémoire de majister, Sur la relation d'ordre flou de ponsard, Université de M'sila, (2010).

[30] M. Jezewski, R. Czabanski and J. Leski, Introduction to fuzzy sets, Silesian University of technelogy, Akademicka Str, Gliwice, Poland, 16, 44-100.

[31] C. Kahraman, I. Otay, Fuzzy multi-criteria decision-making using neutrosophic sets, Springer nature switzerland AG, (2019).

[32] J. Kim, P.K. Lim, J.G. Lee, K. Hur, Single valued neutrosophic relation, Annals of fuzzy mathematics and informatics, 16 (2), (2018), 201–221.

[33] E.P. Klement, R. Mesiar and A. Stupnanova, Picture fuzzy sets and 3-fuzzy sets, IEEE international conference on fuzzy systems (FUZZ-IEEE), (2018).

[34] E.P. Klement and R. Mesiar, Intervals and more: Aggregation functions for picture fuzzy sets, Beyond traditional probabilistic data processing techniques: Interval, fuzzy etc. Methods and their applications, 835, (2020), 179-194.

[35] E.P. Klement, R. Mesiar and E. Pap, Triangular Norms,Springer Science and Business Media, 8 (2000).

[36] Liang, Q., and Mendel, J. M. (2000). "Interval type-2 fuzzy logic systems: theory and design." *IEEE Transactions on Fuzzy Systems*, 8(5), pp. 535–550.

[37] G.J. Klir, B. Yuan, Fuzzy sets and logic, Theory and applications, Prenties hall PTR, (1995).

[38] F. Kutlu Gundogdu, C. Kahraman, Extension of WASPAS with spherical fuzzy sets, 30, (2019), 269-292.

[39] A. Latreche, O. Barkat, S. Milles, F. Ismail, Single valued neutrosophic mappings defined by single valued neutrosophic relations with applications, Neutrosophic sets and systems 1, (2020), 203-220.

[40] D. Liu, Y. Luo and Z. Liu, The linguistic picture fuzzy set and its application in multi-criteria decision-making: An illustration to the TOPSIS and TODIM methods based on entropy weight, (2020).

[41] K. Menger, Statistical metrics, Proceedings of the national academy of sciences, USA, (1942).

[42] B.B. Meunier, La logique floue et ses application, Addison wesley, Paris, (1995).

[43] S. Milles, Sur les ensembles ordonnés flous intuitionnistes, Université de M'sila, Doctoral thesis, University of M'sila, (2017).

[44] S. Milles, L. Zedam, E. Rak, Characterizations of intuitionistic fuzzy ideals and filters based on lattice operations, Journal of fuzzy set valued analysis, 3, (2017), 143-159.

[45] S. Milles, The lattice of intuitionistic fuzzy topologies generated by intuitionistic fuzzy relations, Applications and applied mathematics: An international journal (AAM), 15 (2), (2020), 942-956.

[46] S. Milles, E. Nart, F. Ismail, A. Latreche, Construction of intuitionistic fuzzy mappings with applications, Universal journal of mathematics and applications, 3 (4), (2020), 144-155.

[47] S. Milles, O. Barkat, A. Latreche, Completeness and compactness in standard single valued neutrosophic metric spaces, International journal of neutrosophic science, 12, (2021).

[48] S. Milles, S. Boudaoud, L. Zedam, Principal intuitionistic fuzzy ideals and filters on a lattice, University of M'sila, 40 (1), (2021), 96-104.

[49] Mendel, J. M. (2001). *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. Prentice Hall, Upper Saddle River, NJ.

[50] D. Saadi, A. Abdelaziz, More on picture fuzzy sets and their properties, TWMS j. App and Eng. Math. Accepted to appear.

[51] A. Saha, F. Smarandache, J. Baidya and D. Dutta, MADM using m-generalized q-neutrosophic sets neutrosophic sets and systems, 35, (2016).

[52] S. Sahoo, M. Pal, Different types of products on intuitionistic fuzzy graphs, Pacific science review A: Natural science and engineering 17, (2015), 87-96.

[53] A.A. Salama, F. Smarandache, Neutrosophic crisp set theory, The educational publisher columbus. Ohio, (2015).

[54] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific journal of mathematics, 1, (1960), 313-334.

[55] S. Sebastian, A study on multi-fuzziness, Doctor of philosophy in mathematics, Department of mathematical sciences, Kannur University, (2011).

[56] F. Smarandache, In: Neutrosophy.neutrisophic property sets and logic, American research press, Rehoboth, USA, (1998).

[57] F. Smarandache, Neutrosophic probability and statistics, In: A unifying field in logics: Neutrosophic logic. Neutrosophy, Neutrosophic net, Info learn quest, USA, (2007).

[58] F. Smarandache, n-valued refined neutrosophic logic and its applications to physics, Prog. Phys. 8, (2013), 143–146.

[59] R. Srivastava, S.N. Lal, A.k. Srivastava, Fuzzy T1-topological spaces. Journal of mathematical analysis and applications, 102(2), (1984), 442–448.

[60] B. K. Tripathy, M. K. Satapathy, P. K. Choudhury, Intuitionistic fuzzy lattices and intuitionistic fuzzy boolean algebras, International journal of engineering and technology, 5, (2013), 2352-2361.

[61] V. Veeramani, R. Batulan, Some characterisations of α-Cuts in intuitionistic fuzzy set theory, AMS, Subject classification, (2008).

[62] G. J. Wang, Y. Yutte, Intuitionistic fuzzy sets and L-fuzzy sets, Fuzzy sets and systems, 110, (2000), 271-274.

[63] H. Wang, F. Smarandache, Y.Q. Zhang and R. Sunderraman, Single valued neutrosophic sets, Multispace multistructure, Neutrosophic transdisciplinarity, 4, (2010), 410-413.

[64] R. R. Yager, Level sets and the representation theorem for intuitionistic fuzzy sets, Soft computing, 14, (2010), 1-7.

[65] Y. Yang, X. Tan, C. Meng, The multi-fuzzy soft set and its application in decision making, Applied mathematical modelling 37, (2013), 4915-4923.

[66] H.L. Yang, J. Intell, Z.L. Guo and X. Liao, On single valued neutrosophic relations, Fuzzy sets and systems, 30, (2016), 1045–1056.

[67] L.A. Zadeh, Fuzzy sets, Information and control, 8, (1965), 338-353.

[68] L. A. Zadeh, Similarity relations and fuzzy orderings, Information sciences, 3, (1971), 177-200.

[69] Zadeh, L. A. (1975). "The concept of a linguistic variable and its application to approximate reasoning—I." *Information Sciences*, 8(3), pp. 199–249.

[70] L. Zedam, A. Amroune, B. Davvaz, Szpilrajn theorem for intuitionistic fuzzy orderings, Annals of fuzzy mathematics and informatics, 9, (2015), 703-718.

[71] L. Zedam, S. Milles, A. Bennoui, Ideals and filters on a lattice in neutrosophic setting, Applications, Applied mathematics, 16 (2), (2021), 1140.

[72] H. J. Zimmerman, Fuzzy sets theory and its application, Third edition, Kluwer academic publishers, Boston, Dordrehlt, London, (1996).

[73] L A Zadeh. The concept of a linguistic variable and its application to approximate reasoning-I. Information Sciences, 1975,8(3):199-249.

[74] J M Mendel, R I John, F L Liu. Interval type-2 fuzzy logic systems made simple. IEEE Transactions on Fuzzy Systems, 2006,14(6):808-821.

[75] Aliev R A, Pedrycz W, Guirimov B G, Aliev R R, Ilhan U, Babagil M, Mammadli. Type-2 fuzzy neural networks with fuzzy clustering and differential evolution optimization. Information Sciences, 2011, 181(9):1591-1608.

[76] C.D.Li, J.Y. Yi, Y. Yu, D B Zhao. Inverse control of cable-driven parallel mechanism using type-2 neural network. Acta Automatica Sinica, 2010, 36(3):459-464.

[77] H Mo, F.-Y Wang, Z.Q.Xiao, etal. Stabilities of linguistic dynamic systems based on interval type-2 fuzzy sets. Acta Automatica Sinica, 2011,37(8):1018-1024.

[78] H.Mo, F.-Y Wang, M.Zhou, etal. Footprint of uncertainty for type-2 fuzzy sets. Information Sciences, 2014, 272: 96-110.

[79] F.-Y Wang, H. Mo. Some fundamental issues of type-2 fuzzy sets. 43(7): 1114-1140, 2017

[80] Castillo, O., and Melin, P. (2022). *Interval Type-3 Fuzzy Systems: Theory and Design*. Springer, Cham.

[81] Karnik, N. N., and Mendel, J. M. (2001). “Operations on type-2 fuzzy sets.” *Fuzzy Sets and Systems*, 122(2), pp. 327–348.

[82] F.-Y Wang. On the abstraction of conventional dynamic systems: From numerical analysis to linguistic analysis. Information Sciences, 171(1-3):233-259,2005.

[83] Willian Zhu, F.-Y Wang. Reduction and axiomization of covering generalized rough sets. Information Sciences, 2003, 152:217-230.

[84] H. Mo, F.-Y Wang. Linguistic dynamic systems based on computing with words and their stabilities. Science China Series F: Information Sciences, 2009, 52(5):780-796.

[85] J Zhou, C L P Chen, L Chen, etal. A collaborative fuzzy clustering algorithm in distributed network environments. IEEE Transactions on Fuzzy Systems, 2014, 22(6):1443-1456.

[86] C L P Chen, J Wang, C H Wang, L Chen. A new learning algorithm for a fully connected neuro-fuzzy inference systems. IEEE Transactions on Neural Networks and Learning Systems,2014, 25(10):1741-1757.

[87] L Li, Y S Lv, F.-Y Wang. Traffic signal timing via deep reinforcement learning. IEEE/CAA Journal Automatica Sinica, 2016, 3(3):247-254.

[88] L Li, D Wen. Parallel systems for traffic control: A rethinking. IEEE Transactions on Intelligent Transportation Systems, 2016, 17(4):1179-1182.

ملخص

تُقدّم هذه الأطروحة دراسةً رياضية شاملة للتطور الهرمي للمجموعات الضبابية عبر أجيالها الثلاثة، بهدف تبسيط التعقيد الحسابي والاستعصاء الجبري للأنظمة ثلاثية الأبعاد وجعلها قابلة للحساب الفعلي. ويتناول الفصل الأول العلاقات ومؤثرات المنطق الجبري (t-نرمس) للنوع الأول، بينما يركز الفصل الثاني على النمذجة الهندسية لدوال الانتماء الفترية من النوع الثاني وتحليل بصمة عدم اليقين (سصو). وينِتهي البحث في الفصل الثالث بدراسة أنظمة النوع الثالث واستخدام تفكيك مستويات ألفا (α-لنس) لتحويل البنى الحجمية المعقدة إلى فترات حاسمة، مما يجعل هذه الأنظمة ركيزة أساسية لتطوير خوارزميات الذكاء الاصطناعي، والتحكم الذكي، وأنظمة التشخيص الطبي.

كلمات مفتاحية

المجموعات الضبابية من النوع الثالث، مستويات ألفا، بصمة عدم اليقين، الموحدات المثلثية، تقليص النوع، الذكاء الاصطناعي.

Abstract

This thesis presents a comprehensive mathematical review of the hierarchical evolution of fuzzy sets across their three generations, aiming to simplify the computational complexity and algebraic intractability of three-dimensional systems to make them practically computable. The first chapter addresses fuzzy relations and algebraic triangular norms (t-norms) for Type-1 systems, while the second chapter focuses on the geometric modeling of Interval Type-2 membership functions and the analysis of the Footprint of Uncertainty (FOU). Finally, the research concludes in the third chapter by investigating general Type-3 fuzzy hierarchies and employing alpha-plane (α-planes) decompositions to transform complex volumetric structures into crisp, actionable intervals, establishing these advanced systems as a vital cornerstone for next-generation artificial intelligence, intelligent control, and automated medical diagnosis.

Key words

Type-3 Fuzzy Sets, α-planes, Footprint of Uncertainty, Triangular Norms, Type-Reduction, Artificial Intelligence.

Résumé

Cette thèse présente une étude mathématique approfondie de l'évolution hiérarchique des ensembles flous à travers leurs trois générations, visant à simplifier la complexité computationnelle et l'intraitabilité algébrique des systèmes tridimensionnels pour les rendre calculables en pratique. Le premier chapitre aborde les relations floues et les normes triangulaires (t-normes) pour le Type-1, tandis que le deuxième chapitre se concentre sur la modélisation géométrique des fonctions d'appartenance de Type-2 prévalant sur des intervalles et l'analyse de l'empreinte d'incertitude (FOU). Enfin, la recherche se conclut au troisième chapitre par l'étude des hiérarchies floues de Type-3 et l'utilisation de la décomposition en plans alpha (α-planes) pour transformer les structures volumétriques complexes en intervalles nets, faisant de ces systèmes avancés un pilier essentiel pour l'intelligence artificielle de nouvelle génération, le contrôle intelligent et le diagnostic médical automatisé.

Mot-clés

Ensembles flous de Type-3, Plans alpha,Empreinte d'incertitude, Réduction de type, Intelligence artificielle.