

Variable Besov-type Spaces

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Abstract In this paper we introduce Besov-type spaces with variable smoothness and integrability. We show that these spaces are characterized by the φ -transforms in appropriate sequence spaces and we obtain atomic decompositions for these spaces. Moreover the Sobolev embeddings for these function spaces are obtained.

Keywords Atom, embeddings, Besov space, maximal function, variable exponent

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1 Introduction

Besov spaces of variable smoothness and integrability, $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$, initially appeared in the paper of Almeida and Hästö [2]. Several basic properties were established, such as the Fourier analytical characterization and Sobolev embeddings. When p, q, α are constants they coincide with the usual function spaces $B_{p,q}^\alpha$. Later, [9] characterized these spaces by local means and established the atomic characterization. Afterwards, Kempka and Vybíral [20] characterized these spaces by the ball means of differences and also by local means, see [19] for the duality of $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ spaces.

Variable Besov-type spaces have been introduced in [11]–[13], where their basic properties are given, such as the Sobolev type embeddings and that under some conditions these spaces are just the variable Besov spaces. For constant exponents, these spaces unify and generalize many classical function spaces including Besov spaces, Besov–Morrey spaces (see, for example, [35, Corollary 3.3]). Independently, Yang et al. [32] studied these function spaces where several properties are obtained such as atomic decomposition and the boundedness of trace operator. Also, Tyulenev [26, 27] has studied a new function space of variable smoothness. Triebel–Lizorkin spaces with variable smoothness and integrability $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ were introduced in [7], in which Diening et al. proved a discretization by the so-called φ -transform. Also atomic and molecular decompositions of these function spaces are obtained and used to derive trace results. Subsequently, Vybíral [28] established Sobolev–Jawerth embeddings of these spaces. In [33], Triebel–Lizorkin type spaces of variable smoothness and integrability were introduced and

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studied. Their function spaces generalize classical Triebel–Lizorkin-type spaces and Triebel–Lizorkin spaces with variable smoothness and integrability.

The motivation to study such function spaces comes from applications to other fields of applied mathematics, such that fluid dynamics and image processing, see, for example, [22].

The main aim of this paper is to present another Besov-type spaces with variable smoothness and integrability which covers Besov-type spaces with fixed exponents. We establish their φ -transform characterization in the sense of Frazier and Jawerth. We also characterize these spaces by smooth atoms and give some basic properties and Sobolev-type embeddings.

The paper is organized as follows. First we give some preliminaries where we fix some notation and recall some basics facts on function spaces with variable integrability and we give some key technical lemmas needed in the proofs of the main statements. For making the presentation clearer, we give the proof of some lemmas later in Section 6. We then define the Besov-type spaces $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$. In this section several basic properties such as the φ -transform characterization are obtained. In Section 4 we prove elementary embeddings between these functions spaces, as well as Sobolev embeddings. In Section 5, we give the atomic decomposition of $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ spaces.

2 Preliminaries

As usual, we denote by \mathbb{R}^n the n -dimensional real Euclidean space, \mathbb{N} the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter \mathbb{Z} stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq cg$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x .

By $\text{supp } f$ we denote the support of the function f , i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function.

The Hardy–Littlewood maximal operator \mathcal{M} is defined on $L_{\text{loc}}^1(\mathbb{R}^n)$ by

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

and

$$M_B(f) := \frac{1}{|B|} \int_B |f(y)| dy.$$

The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on \mathbb{R}^n . We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on \mathbb{R}^n . The Fourier transform of a tempered distribution f is denoted by $\mathcal{F}f$ while its inverse transform is denoted by $\mathcal{F}^{-1}f$.

For $v \in \mathbb{Z}$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, let $Q_{v,m}$ be the dyadic cube in \mathbb{R}^n , $Q_{v,m} = \{(x_1, \dots, x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, \dots, n\}$. For the collection of all such cubes we use

$$\mathcal{Q} := \{Q_{v,m} : v \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

For each cube Q , we denote its center by c_Q , its lower left-corner by $x_{Q_{v,m}} = 2^{-v}m$ of $Q = Q_{v,m}$ and its side length by $l(Q)$. For $r > 0$, we denote by rQ the cube concentric with Q having the side length $rl(Q)$. Furthermore, we put $v_Q = -\log_2 l(Q)$ and $v_Q^+ = \max(v_Q, 0)$.

For $v \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we set $\tilde{\varphi}(x) := \overline{\varphi(-x)}$, $\varphi_v(x) := 2^{vn}\varphi(2^v x)$, and

$$\varphi_{v,m}(x) := 2^{v\frac{n}{2}}\varphi(2^v x - m) = |Q_{v,m}|^{\frac{1}{2}}\varphi_v(x - x_{Q_{v,m}}) \quad \text{if } Q = Q_{v,m}.$$

By c we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g. $c(p)$ means that c depends on p). Further notation will be properly introduced whenever needed.

The variable exponents that we consider are always measurable functions p on \mathbb{R}^n with range in $[c, \infty[$ for some $c > 0$. We denote the set of such functions by \mathcal{P}_0 . The subset of variable exponents with range $[1, \infty[$ is denoted by \mathcal{P} . We use the standard notation $p^- := \text{ess inf}_{x \in \mathbb{R}^n} p(x)$ and $p^+ := \text{ess sup}_{x \in \mathbb{R}^n} p(x)$.

The variable exponent modular is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \varrho_{p(x)}(|f(x)|) dx,$$

where $\varrho_p(t) = t^p$. The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions f on \mathbb{R}^n such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg (quasi)-norm on this space by the formula

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

A useful property is that $\|f\|_{p(\cdot)} \leq 1$ if and only if $\varrho_{p(\cdot)}(f) \leq 1$, see, for example, [6, Lemma 3.2.4].

Let $p, q \in \mathcal{P}_0$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$ -functions by the semi-modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \inf \left\{ \lambda_v > 0 : \varrho_{p(\cdot)}\left(\frac{f_v}{\lambda_v^{\frac{1}{q(\cdot)}}}\right) \leq 1 \right\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf \left\{ \mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \leq 1 \right\}. \quad (2.1)$$

If $q^+ < \infty$, then we can replace (2.1) by the simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \| |f_v|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. The case $p := \infty$ can be included by replacing the last semi-modular by

$$\varrho_{\ell^{q(\cdot)}(L^\infty)}((f_v)_v) := \sum_v \| |f_v|^{q(\cdot)} \|_\infty.$$

It is known, cf. [2, Theorem 3.6] and [21, Theorem 1], that $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a norm if $q(\cdot) \geq 1$ is constant almost everywhere (a.e.) on \mathbb{R}^n and $p(\cdot) \geq 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$ a.e. on \mathbb{R}^n , or if $1 \leq q(x) \leq p(x) \leq \infty$ a.e. on \mathbb{R}^n .

We say that $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is *locally log-Hölder continuous*, abbreviated $g \in C_{\text{loc}}^{\text{log}}$, if there exists $c_{\text{log}}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\text{log}}(g)}{\log(e + \frac{1}{|x-y|})} \quad (2.2)$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the *log-Hölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally-log-Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants $c_{\log}(g)$ and c_{\log} are called the *locally log-Hölder constant* and the *log-Hölder decay constant*, respectively. We note that all functions $g \in C_{\log}^{\log}$ always belong to L^∞ .

We define the following class of variable exponents

$$\mathcal{P}^{\log} := \left\{ p \in \mathcal{P} : \frac{1}{p} \in C^{\log} \right\},$$

were introduced in [5, Section 2]. We define $\frac{1}{p_\infty} := \lim_{|x| \rightarrow \infty} \frac{1}{p(x)}$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded. It was shown in [6, Theorem 4.3.8] that $\mathcal{M} : L^{p(\cdot)} \rightarrow L^{p'(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$, see also [5, Theorem 1.2]. Also if $p \in \mathcal{P}^{\log}$, then the convolution with a radially decreasing L^1 -function is bounded on $L^{p(\cdot)}$:

$$\|\varphi * f\|_{p(\cdot)} \leq c \|\varphi\|_1 \|f\|_{p(\cdot)}.$$

We also refer to the papers [3] and [4], where various results on maximal function in variable Lebesgue spaces were obtained.

It is known that for $p \in \mathcal{P}^{\log}$ we have

$$\|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \approx |B|, \quad (2.3)$$

with constants only depending on the log-Hölder constant of p (see, for example, [6, Section 4.5]). Here p' denotes the conjugate exponent of p given by $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Recall that $\eta_{v,m}(x) := 2^{nv}(1 + 2^v|x|)^{-m}$, for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$. Note that $\eta_{v,m} \in L^1$ when $m > n$ and that $\|\eta_{v,m}\|_1 = c_m$ is independent of v , where this type of function was introduced in [18] and [6].

2.1 Some Technical Lemmas

In this subsection we present some results which are useful for us. The following lemma is from [20, Lemma 19], see also [7, Lemma 6.1].

Lemma 2.1 *Let $\alpha \in C_{\log}^{\log}$ and let $R \geq c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from (2.2) for α . Then*

$$2^{v\alpha(x)} \eta_{v,m+R}(x-y) \leq c 2^{v\alpha(y)} \eta_{v,m}(x-y)$$

with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $v, m \in \mathbb{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$2^{v\alpha(x)} \eta_{v,m+R} * f(x) \leq c \eta_{v,m} * (2^{v\alpha(\cdot)} f)(x), \quad x \in \mathbb{R}^n,$$

where $c > 0$ is independent of v and m .

Lemma 2.2 *Let $r, R, N > 0$, $m > n$ and $\theta, \omega \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp } \mathcal{F}\omega \subset \overline{B(0,1)}$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$, we have*

$$|\theta_R * \omega_N * g(x)| \leq c A(\eta_{N,m} * |\omega_N * g|^r(x))^{1/r}, \quad x \in \mathbb{R}^n, \quad (2.4)$$

where $\theta_R = R^n \theta(R \cdot)$, $\omega_N = N^n \omega(N \cdot)$, $\eta_{N,m} := N^n (1 + N|\cdot|)^{-m}$ and

$$A = \max(1, (NR^{-1})^m).$$

The proof of this lemma is given in [12, Lemma 2.2].

We will make use of the following statement, see [5, Lemma 3.3].

Lemma 2.3 *Let $p \in \mathcal{P}^{\log}$. Then for every $m > 0$ there exists $\beta \in (0, 1)$ only depending on m and $c_{\log}(p)$ such that*

$$\begin{aligned} & \left(\frac{\beta}{|Q|} \int_Q |f(y)| dy \right)^{p(x)} \\ & \leq \frac{1}{|Q|} \int_Q |f(y)|^{p(y)} dy + \min(|Q|^m, 1) \left(\frac{1}{|Q|} \int_Q ((e + |x|)^{-m} + (e + |y|)^{-m}) dy \right), \end{aligned}$$

for every cube $Q \subset \mathbb{R}^n$, all $x \in Q \subset \mathbb{R}^n$ and all $f \in L^{p(\cdot)} + L^\infty$ such that $\|f\|_{L^{p(\cdot)} + L^\infty} \leq 1$.

Let $p \in \mathcal{P}^{\log}$. In the proof of this lemma we need only that

$$\int_Q |f(y)|^{p(y)} dy \leq 1$$

and/or $\|f\|_\infty \leq 1$. We set

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L_{p(\cdot)}^{\tau(\cdot)})} := \sup_{P \in \mathcal{Q}} \left\| \left(\frac{f_v}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geq v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

where, $v_P = -\log_2 l(P)$ and $v_P^+ = \max(v_P, 0)$.

The following lemma is the $\ell^{q(\cdot)}(L_{p(\cdot)}^{\tau(\cdot)})$ -version of Lemma 4.7 from Almeida and Hästö [2] (we use it, since the maximal operator is in general not bounded on $\ell^{q(\cdot)}(L^{p(\cdot)})$, see [2, Example 4.1]).

Lemma 2.4 *Let $\tau \in C_{\text{loc}}^{\log}$, $\tau^- > 0$, $p \in \mathcal{P}^{\log}$, $q \in \mathcal{P}_0^{\log}$ with $0 < q^- \leq q^+ < \infty$ and $\tau^+ < (\tau p)^-$. For any m large enough, there exists $c > 0$ such that*

$$\|(\eta_{v,m} * f_v)_v\|_{\ell^{q(\cdot)}(L_{p(\cdot)}^{\tau(\cdot)})} \leq c \|(f_v)_v\|_{\ell^{q(\cdot)}(L_{p(\cdot)}^{\tau(\cdot)})}$$

for any $(f_v)_v \in \ell^{q(\cdot)}(L_{p(\cdot)}^{\tau(\cdot)})$.

The proof of this lemma is postponed to the Appendix.

Let $\widetilde{L_{p(\cdot)}^{\tau(\cdot)}}$ be the collection of functions $f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{L_{p(\cdot)}^{\tau(\cdot)}}} := \sup \left\| \left\| \frac{f \chi_P}{|P|^{\tau(\cdot)}} \right\|_{p(\cdot)} \right\|_{p(\cdot)} < \infty, \quad p \in \mathcal{P}_0, \quad \tau : \mathbb{R}^n \rightarrow \mathbb{R}^+,$$

where the supremum is taken over all dyadic cubes P with $|P| \geq 1$. Notice that

$$\|f\|_{\widetilde{L_{p(\cdot)}^{\tau(\cdot)}}} \leq 1 \Leftrightarrow \sup_{P \in \mathcal{Q}, |P| \geq 1} \left\| \left\| \frac{f}{|P|^{\tau(\cdot)}} \right\|^{q(\cdot)} \chi_P \right\|_{p(\cdot)/q(\cdot)} \leq 1. \quad (2.5)$$

Recall that $\theta_v = 2^{vn} \theta(2^v \cdot)$, $v \in \mathbb{Z}$.

Lemma 2.5 *Let $v \in \mathbb{Z}$, $\tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- > 0$, $p \in \mathcal{P}_0^{\text{log}}$ and $\theta, \omega \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp } \mathcal{F}\omega \subset \overline{B(0,1)}$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any dyadic cube P with $|P| \geq 1$, we have*

$$\left\| \frac{\theta_v * \omega_v * f}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leq c \|\omega_v * f\|_{L_{p(\cdot)}^{\tau(\cdot)}},$$

such that the right-hand side is finite, where $c > 0$ is independent of v and $l(P)$.

We will present the proof in the Appendix.

Lemma 2.6 *Let $\alpha, \tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}_0^{\text{log}}$ with $0 < q^- \leq q^+ < \infty$. Let $(f_k)_{k \in \mathbb{N}_0}$ be a sequence of measurable functions on \mathbb{R}^n . For all $v \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, let*

$$g_v(x) := \sum_{k=0}^{\infty} 2^{-|k-v|\delta} f_k(x).$$

Then there exists a positive constant c , independent of $(f_k)_{k \in \mathbb{N}_0}$ such that

$$\|(g_v)_v\|_{\ell^{q(\cdot)}(L_{p(\cdot)}^{\tau(\cdot)})} \leq c \|(f_v)_v\|_{\ell^{q(\cdot)}(L_{p(\cdot)}^{\tau(\cdot)})}, \quad \delta > 0.$$

The proof of Lemma 2.6 can be obtained by the same arguments used in [12, Lemma 2.10] and [20, Lemma 8].

3 The Spaces $\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$

In this section we present the Fourier analytical definition of Besov-type spaces of variable smoothness and integrability and we prove their basic properties in analogy to the Besov-type spaces with fixed exponents. Select a pair of Schwartz functions Φ and φ such that

$$\text{supp } \mathcal{F}\Phi \subset \overline{B(0,2)} \quad \text{and} \quad |\mathcal{F}\Phi(\xi)| \geq c \quad \text{if } |\xi| \leq \frac{5}{3} \quad (3.1)$$

and

$$\text{supp } \mathcal{F}\varphi \subset \overline{B(0,2)} \setminus B(0,1/2) \quad \text{and} \quad |\mathcal{F}\varphi(\xi)| \geq c \quad \text{if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}, \quad (3.2)$$

where $c > 0$. We put $\varphi_v := 2^{vn} \varphi(2^v \cdot)$, $v \in \mathbb{N}$.

Definition 3.1 *Let $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$, $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^+$ and $p, q \in \mathcal{P}_0$. Let Φ and φ satisfy (3.1) and (3.2), respectively. The Besov-type space $\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}} := \sup_{P \in \mathcal{Q}} \left\| \left(\frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geq v_P^+} \right\|_{\ell^{q(\cdot)}(L_{p(\cdot)})} < \infty, \quad (3.3)$$

where φ_0 is replaced by Φ .

Using the system $(\varphi_v)_{v \in \mathbb{N}_0}$ we can define the quasi-norm

$$\|f\|_{B_{p,q}^{\alpha,\tau}} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left(\sum_{v=v_P^+}^{\infty} 2^{v\alpha q} \|(\varphi_v * f)\chi_P\|_p^q \right)^{\frac{1}{q}}$$

for constants α and $p, q \in (0, \infty]$, with the usual modification if $q = \infty$. The Besov-type space $B_{p,q}^{\alpha,\tau}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{B_{p,q}^{\alpha,\tau}} < \infty$. It is well-known that these spaces do not depend on the choice of the system $(\varphi_v)_{v \in \mathbb{N}_0}$ (up to equivalence of quasi-norms). Further details on the classical theory of these spaces can be found in [8, 31] and [35], see also [10] for recent developments. Moreover, $B_{p,q}^{\alpha,0}$ are just the classical Besov spaces, see [23] and [25] for the theory of these function spaces.

One recognizes immediately that if α , τ , p and q are constants, then

$$\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} = B_{p,q}^{\alpha,\tau}$$

in the sense of equivalent quasi-norms. When $q := \infty$ the Besov-type space $\mathfrak{B}_{p(\cdot),\infty}^{\alpha(\cdot),\tau(\cdot)}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\sup_{P \in \mathcal{Q}, v \geq v_P^+} \left\| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} < \infty.$$

Let B_J be any ball of \mathbb{R}^n with radius 2^{-J} , $J \in \mathbb{Z}$. In the definition of the spaces $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ if we replace the dyadic cubes P by the balls B_J , then we obtain equivalent quasi-norms. From these if we replace dyadic cubes P in Definition 3.1 by arbitrary cubes P , we then obtain equivalent quasi-norms.

The Besov space of variable smoothness and integrability $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} := \|(2^{v\alpha(\cdot)} \varphi_v * f)_{v \in \mathbb{N}_0}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty,$$

which introduced and investigated in [2], see [20] for further results. Taking $\alpha \in \mathbb{R}$ and $q \in (0, \infty)$ as constants we derive the spaces $B_{p(\cdot),q}^{\alpha}$ studied by Xu in [30]. Obviously,

$$\mathfrak{B}_{p(\cdot),q}^{\alpha,0} = B_{p(\cdot),q}^{\alpha}$$

in the sense of equivalent quasi-norms. We refer the reader to the recent paper [34] for further details, historical remarks and more references on embeddings of Besov-type spaces with fixed exponents. We mention that the variable Triebel–Lizorkin version of our spaces introduced on this paper is given in [14]. Variable Besov–Morrey spaces are given in [1], see [17] and [29] for the variable 2-microlocal Besov–Triebel–Lizorkin-type spaces.

Sometimes it is of great service if one can restrict $\sup_{P \in \mathcal{Q}}$ in the definition of $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ to a supremum taken with respect to dyadic cubes with side length ≤ 1 . The next lemma can be obtained by an argument similar to that used in the proof of [11, Lemma 3.6].

Lemma 3.2 *Let $\alpha, \tau \in C_{\text{loc}}^{\log}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}_0^{\log}$ with $(\tau p - 1)^- \geq 0$ and $0 < q^+ < \infty$. A tempered distribution f belongs to $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ if and only if*

$$\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^{\#} := \sup_{P \in \mathcal{Q}, |P| \leq 1} \left\| \left(\frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geq v_P} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Furthermore, the quasi-norms $\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}$ and $\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^{\#}$ are equivalent.

Remark 3.3 We like to point out that this result with fixed exponents is given in [35, Lemma 2.2].

The following conclusion implies under some suitable conditions the variable Besov-type spaces $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ are just the Besov spaces $B_{\infty,\infty}^{\alpha(\cdot)+n(\tau(\cdot)-\frac{1}{p(\cdot)})}$, whose proof is similar to that of [11, Theorem 3.8], the details being omitted.

Theorem 3.4 *Let $\alpha, \tau \in C_{\text{loc}}^{\log}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}_0^{\log}$ with $p^+, q^+ < \infty$. If $(\tau p - 1)^- > 0$ or $(\tau p - 1)^- \geq 0$ and $q := \infty$, then*

$$\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} = B_{\infty,\infty}^{\alpha(\cdot)+n(\tau(\cdot)-\frac{1}{p(\cdot)})}$$

with equivalent quasi-norms.

Remark 3.5 From this theorem we obtain

$$2^{v(\alpha(x)+n(\tau(x)-\frac{1}{p(x)}))} |\varphi_v * f(x)| \leq c \|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \quad (3.4)$$

for any $f \in B_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$, $x \in \mathbb{R}^n$, $\alpha, \tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}_0^{\text{log}}$, where $c > 0$ is independent of v and x .

In the following theorem we have the possibility to define these spaces by replacing $v \geq v_p^+$ by $v \in \mathbb{N}_0$ in Definition 3.1, where the main arguments used in its proof rely on [11, Theorem 3.12], so we omit the details and when $\tau := 0$, was obtained by Sickel [24].

Theorem 3.6 Let $\alpha, \tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}_0^{\text{log}}$ with $p^+, q^+ < \infty$. If $(\tau p - 1)^+ < 0$ or $(\tau p - 1)^+ \leq 0$ and $q := \infty$, then

$$\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^{\blacktriangle} := \sup_{P \in \mathcal{Q}} \left\| \left(\frac{2^{\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

is an equivalent quasi-norm in $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$.

Let Φ and φ satisfy, respectively (3.1) and (3.2). By [16, pp. 130–131], there exist functions $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.1) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.2) such that for all $\xi \in \mathbb{R}^n$,

$$\mathcal{F}\tilde{\Phi}(\xi)\mathcal{F}\Psi(\xi) + \sum_{j=1}^{\infty} \mathcal{F}\tilde{\varphi}(2^{-j}\xi)\mathcal{F}\psi(2^{-j}\xi) = 1, \quad (3.5)$$

where $\tilde{\Phi} = \overline{\Phi(-\cdot)}$ and $\tilde{\varphi} = \overline{\varphi(-\cdot)}$. Furthermore, we have the following identity for all $f \in \mathcal{S}'(\mathbb{R}^n)$; see [16, (12.4)]

$$\begin{aligned} f &= \Psi * \tilde{\Phi} * f + \sum_{v=1}^{\infty} \psi_v * \tilde{\varphi}_v * f \\ &= \sum_{m \in \mathbb{Z}^n} \tilde{\Phi} * f(m) \Psi(\cdot - m) + \sum_{v=1}^{\infty} 2^{-vn} \sum_{m \in \mathbb{Z}^n} \tilde{\varphi}_v * f(2^{-v}m) \psi_v(\cdot - 2^{-v}m). \end{aligned}$$

Recall that the φ -transform S_φ is defined by setting $(S_\varphi)_{0,m} = \langle f, \Phi_m \rangle$ where $\Phi_m(x) = \Phi(x-m)$ and $(S_\varphi)_{v,m} = \langle f, \varphi_{v,m} \rangle$ where $\varphi_{v,m}(x) = 2^{v\frac{n}{2}} \varphi(2^v x - m)$ and $v \in \mathbb{N}$. The inverse φ -transform T_ψ is defined by

$$T_\psi \lambda := \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \Psi_m + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \psi_{v,m},$$

where $\lambda = \{\lambda_{v,m}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \subset \mathbb{C}$, see [16].

For any $\gamma \in \mathbb{Z}$, we put

$$\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^* := \sup_{P \in \mathcal{Q}} \left\| \left(\frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geq v_p^+ - \gamma} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty$$

where $\varphi_{-\gamma}$ is replaced by $\Phi_{-\gamma}$.

Lemma 3.7 Let $\alpha, \tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- > 0$, $p, q \in \mathcal{P}_0^{\text{log}}$ and $0 < q^+ < \infty$. The quasi-norms $\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^*$ and $\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}$ are equivalent with equivalent constants depending on γ .

Proof The proof is a straightforward adaptation of [12, Lemma 3.9] and [35, Lemma 2.6]. We will present the proof into two steps.

Step 1 In this step we prove that

$$\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^* \lesssim \|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}.$$

We need only to consider the case $\gamma > 0$. By the scaling argument, it suffices to consider the case

$$\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq 1 \quad (3.6)$$

and show that the modular of f on the left-hand side is bounded. In particular, we will show that

$$\sum_{v=v_P^+-\gamma}^{\infty} \left\| \left\| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \right\|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c$$

for any dyadic cube P . As in [35, Lemma 2.6], it suffices to prove that for all dyadic cube P with $l(P) \geq 1$,

$$I_P := \sum_{v=-\gamma}^0 \left\| \left\| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \right\|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c$$

and for all dyadic cube P with $l(P) < 1$,

$$J_P := \sum_{v=v_P^+-\gamma}^{v_P^--1} \left\| \left\| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \right\|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c.$$

The estimate of I_P , clearly follows from the inequality

$$\left\| \left\| \frac{\varphi_v * f}{|P|^{\tau(\cdot)}} \right\|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c$$

for any $v = -\gamma, \dots, 0$ and any dyadic cube P with $l(P) \geq 1$. This claim can be reformulated as showing that

$$\left\| \left\| \frac{\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \right\| \leq c. \quad (3.7)$$

From (3.1) and (3.2), we find $\omega_v \in \mathcal{S}(\mathbb{R}^n)$, $v = -\gamma, \dots, -1$ and $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_v = \omega_v * \Phi, \quad v = -\gamma, \dots, -1 \quad \text{and} \quad \varphi = \varphi_0 = \eta_1 * \Phi + \eta_2 * \varphi_1.$$

Therefore

$$\varphi_v * f = \omega_v * \Phi * f \quad \text{for } v = -\gamma, \dots, -1$$

and

$$\varphi_0 * f = \eta_1 * \Phi * f + \eta_2 * \varphi_1 * f.$$

Using Lemma 2.5, (2.5) and (3.6) to estimate the left-hand side of (3.7) by

$$C \|\Phi * f\|_{\widetilde{L_{\tau(\cdot)}^{p(\cdot)}}} + C \|\varphi_1 * f\|_{\widetilde{L_{\tau(\cdot)}^{p(\cdot)}}} \leq c.$$

To estimate J_P , denote by $P(\gamma)$ the dyadic cube containing P with $l(P(\gamma)) = 2^\gamma l(P)$. If $v_P \geq \gamma + 1$, applying the fact that $v_{P(\gamma)} = v_P - \gamma$, and $P \subset P(\gamma)$, we then have

$$J_P \leq \sum_{v=v_{P(\gamma)}}^{v_P-1} \left\| \left| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P(\gamma)|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{P(\gamma)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c.$$

If $1 \leq v_P \leq \gamma$, we write

$$J_P = J_P^1 + J_P^2,$$

where

$$J_P^1 := \sum_{v=v_P-\gamma}^{-1} \left\| \left| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}}$$

and

$$J_P^2 := \sum_{v=0}^{v_P-1} \left\| \left| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Let $P(2^{v_P})$ be the dyadic cube containing P with $l(P(2^{v_P})) = 2^{v_P} l(P) = 1$. By the fact that

$$\frac{|P(2^{v_P})|^{\tau(\cdot)}}{|P|^{\tau(\cdot)}} \lesssim 2^{n v_P \tau^+} \lesssim c(\gamma),$$

we have

$$J_P^2 \lesssim \sum_{v=v_{P(2^{v_P})}}^{v_P-1} \left\| \left| \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P(2^{v_P})|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{P(2^{v_P})} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c.$$

By the arguments similar to that used in the estimate for I_P , we obtain $J_P^1 \leq c$.

Step 2 We will prove that

$$\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \lesssim \|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)*}}.$$

It suffices to show that

$$\left\| \left| \frac{\Phi * f}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq c$$

for all $P \in \mathcal{Q}$ with $l(P) \geq 1$ and all $f \in \mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ with

$$\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)*}} \leq 1. \quad (3.8)$$

This claim can be reformulated as showing that

$$\left\| \frac{\Phi * f}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leq c.$$

There exist $\varrho_v \in \mathcal{S}(\mathbb{R}^n)$, $v = -\gamma, \dots, 1$, such that

$$\Phi * f = \varrho_{-\gamma} * \Phi_{-\gamma} * f + \sum_{v=1-\gamma}^1 \varrho_v * \varphi_v * f,$$

see [16, p.130]. Using Lemma 2.5 we get

$$\|\varrho_{-\gamma} * \Phi_{-\gamma} * f\|_{\widetilde{L}_{p(\cdot)}^{\tau(\cdot)}} \lesssim \|\Phi_{-\gamma} * f\|_{\widetilde{L}_{p(\cdot)}^{\tau(\cdot)}} \leq c$$

and

$$\|Q_v * \varphi_v * f\|_{\widetilde{L}_{p(\cdot)}^{\tau(\cdot)}} \lesssim \|\varphi_v * f\|_{\widetilde{L}_{p(\cdot)}^{\tau(\cdot)}} \leq c, \quad v = 1 - \gamma, \dots, 1,$$

by using (2.5) and (3.8). The proof is complete. \square

Definition 3.8 Let $p, q \in \mathcal{P}_0$, $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^+$ and let $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$. Then we define

$$\mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)} := \{\lambda = \{\lambda_{v,m}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \subset \mathbb{C} : \|\lambda\|_{\mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}} < \infty\},$$

where

$$\|\lambda\|_{\mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}} := \sup_{P \in \mathcal{Q}} \left\| \left(\frac{\sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot) + \frac{n}{2})} \lambda_{v,m} \chi_{v,m}}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geq v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

If we replace dyadic cubes P by arbitrary balls B_J of \mathbb{R}^n with $J \in \mathbb{Z}$, we then obtain equivalent quasi-norms, where the supremum is taken over all $J \in \mathbb{Z}$ and all balls B_J of \mathbb{R}^n .

Lemma 3.9 Let $\alpha, \tau \in C_{\text{loc}}^{\log}$, $\tau^- \geq 0$, $p, q \in \mathcal{P}_0^{\log}$, $0 < q^+ < \infty$, $v \in \mathbb{N}_0, m \in \mathbb{Z}^n$, $x \in Q_{v,m}$ and $\lambda \in \mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$. Then there exists $c > 0$ independent of x, v and m such that

$$|\lambda_{v,m}| \leq c 2^{-v(\alpha(x) + \frac{n}{2})} |Q_{v,m}|^{\tau(x)} \|\lambda\|_{\mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}} \|\chi_{v,m}\|_{p(\cdot)}^{-1}.$$

Proof Let $\lambda \in \mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$, $v \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $x \in Q_{v,m}$, with $Q_{v,m} \in \mathcal{Q}$. Then

$$|\lambda_{v,m}|^{p^-} = |Q_{v,m}|^{-1} \int_{Q_{v,m}} |\lambda_{v,m}|^{p^-} \chi_{v,m}(y) dy.$$

Using the fact that $2^{v(\alpha(x) - \alpha(y))} \leq c$ and $2^{v(\tau(x) - \tau(y))} \leq c$ for any $x, y \in Q_{v,m}$, we obtain

$$\frac{2^{v(\alpha(x) + \frac{n}{2})p^-}}{|Q_{v,m}|^{p^- \tau(x)}} |\lambda_{v,m}|^{p^-} \lesssim |Q_{v,m}|^{-1} \int_{Q_{v,m}} \frac{2^{v(\alpha(y) + \frac{n}{2})p^-}}{|Q_{v,m}|^{p^- \tau(y)}} |\lambda_{v,m}|^{p^-} \chi_{v,m}(y) dy.$$

Applying Hölder's inequality to estimate this expression by

$$c |Q_{v,m}|^{-1} \left\| \frac{2^{v(\alpha(\cdot) + \frac{n}{2})p^-}}{|Q_{v,m}|^{p^- \tau(\cdot)}} |\lambda_{v,m}|^{p^-} \chi_{v,m} \right\|_{\frac{p}{p^-}} \|\chi_{v,m}\|_{(\frac{p}{p^-})'} \lesssim \|\lambda\|_{\mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}}^{p^-} \|\chi_{v,m}\|_{\frac{p}{p^-}}^{-1},$$

where we have used (2.3). Therefore for any $x \in Q_{v,m}$,

$$|\lambda_{v,m}| \lesssim 2^{-v(\alpha(x) + \frac{n}{2})} |Q_{v,m}|^{\tau(x)} \|\lambda\|_{\mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}} \|\chi_{v,m}\|_{p(\cdot)}^{-1},$$

which completes the proof. \square

As in [12], and using Lemma 3.9 we obtain the following statement.

Lemma 3.10 Let $\alpha, \tau \in C_{\text{loc}}^{\log}$, $\tau^- \geq 0$, $p, q \in \mathcal{P}_0^{\log}$ and $\Psi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy, respectively, (3.1) and (3.2). Then for all $\lambda \in \mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$

$$T_\psi \lambda := \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \Psi_m + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \psi_{v,m}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$; moreover, $T_\psi : \mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)} \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous.

For a sequence $\lambda = \{\lambda_{v,m}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \subset \mathbb{C}$, $0 < r \leq \infty$ and a fixed $d > 0$, set

$$\lambda_{v,m,r,d}^* := \left(\sum_{h \in \mathbb{Z}^n} \frac{|\lambda_{v,h}|^r}{(1 + 2^v |2^{-v}h - 2^{-v}m|)^d} \right)^{\frac{1}{r}}$$

and $\lambda_{r,d}^* := \{\lambda_{v,m,r,d}^*\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \subset \mathbb{C}$.

The proof of the following lemma is postponed to the Appendix.

Lemma 3.11 *Let $\alpha, \tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- > 0$, $p, q \in \mathcal{P}_0^{\text{log}}$, $0 < q^+ < \infty$ and $0 < r < \frac{(\tau p)^-}{\tau^+}$. Then for d large enough*

$$\|\lambda_{r,d}^*\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \approx \|\lambda\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}.$$

By this result, Lemma 3.7 and by the same arguments given in [12, Theorem 3.14] we obtain the following statement.

Theorem 3.12 *Let $\alpha, \tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- > 0$, $p, q \in \mathcal{P}_0^{\text{log}}$ and $0 < q^+ < \infty$. Suppose that $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.1) and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.2) such that (3.5) holds. The operators $S_\varphi : \mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \rightarrow \mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ and $T_\psi : \mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \rightarrow \mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$.*

From Theorem 3.12, we obtain the next important property of spaces $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$.

Corollary 3.13 *Let $\alpha, \tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- > 0$, $p, q \in \mathcal{P}_0^{\text{log}}$ and $0 < q^+ < \infty$. The definition of the spaces $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ is independent of the choices of Φ and φ .*

4 Embeddings

For the spaces $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ introduced above we want to show some embedding theorems. We say a quasi-Banach space A_1 is continuously embedded in another quasi-Banach space A_2 , $A_1 \hookrightarrow A_2$, if $A_1 \subset A_2$ and there is a $c > 0$ such that $\|f\|_{A_2} \leq c\|f\|_{A_1}$ for all $f \in A_1$. We begin with the following elementary embeddings.

Theorem 4.1 *Let $\alpha, \tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- > 0$ and $p, q, q_0, q_1 \in \mathcal{P}_0^{\text{log}}$.*

(i) *If $q_0 \leq q_1$, then*

$$\mathfrak{B}_{p(\cdot),q_0(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \hookrightarrow \mathfrak{B}_{p(\cdot),q_1(\cdot)}^{\alpha(\cdot),\tau(\cdot)}.$$

(ii) *If $(\alpha_0 - \alpha_1)^- > 0$, then*

$$\mathfrak{B}_{p(\cdot),q_0(\cdot)}^{\alpha_0(\cdot),\tau(\cdot)} \hookrightarrow \mathfrak{B}_{p(\cdot),q_1(\cdot)}^{\alpha_1(\cdot),\tau(\cdot)}.$$

The proof can be obtained by using the same method as in [2, Theorem 6.1]. We next consider embeddings of Sobolev-type. It is well-known that

$$B_{p_0,q}^{\alpha_0,\tau} \hookrightarrow B_{p_1,q}^{\alpha_1,\tau},$$

if $\alpha_0 - \frac{n}{p_0} = \alpha_1 - \frac{n}{p_1}$, where $0 < p_0 < p_1 \leq \infty$, $0 \leq \tau < \infty$ and $0 < q \leq \infty$ (see e.g. [35, Corollary 2.2]). In the following theorem we generalize these embeddings to variable exponent case.

Theorem 4.2 *Let $\alpha_0, \alpha_1, \tau \in C_{\text{loc}}^{\text{log}}$, $\tau^- > 0$ and $p_0, p_1, q \in \mathcal{P}_0^{\text{log}}$ with $q^+ < \infty$. If $\alpha_0(\cdot) > \alpha_1(\cdot)$ and $\alpha_0(\cdot) - \frac{n}{p_0(\cdot)} = \alpha_1(\cdot) - \frac{n}{p_1(\cdot)}$ with $(\frac{p_0}{p_1})^+ < 1$, then*

$$\mathfrak{B}_{p_0(\cdot),q(\cdot)}^{\alpha_0(\cdot),\tau(\cdot)} \hookrightarrow \mathfrak{B}_{p_1(\cdot),q(\cdot)}^{\alpha_1(\cdot),\tau(\cdot)}.$$

Proof Let $f \in \mathfrak{B}_{p_0(\cdot),q(\cdot)}^{\alpha_0(\cdot),\tau(\cdot)}$ with $\|f\|_{\mathfrak{B}_{p_0(\cdot),q(\cdot)}^{\alpha_0(\cdot),\tau(\cdot)}} \leq 1$ and P be any dyadic cube of \mathbb{R}^n .

Case 1 $l(P) > 1$. Let $Q_v \subset P$ be a cube, with $\ell(Q_v) = 2^{-v}$ and $x \in Q_v \subset P$. By Lemma 2.2 we have for any $m > n$, $d > 0$

$$|\varphi_v * f(x)| \leq c(\eta_{v,m} * |\varphi_v * f|^d(x))^{\frac{1}{d}}.$$

We have

$$\begin{aligned} & \eta_{v,m} * |\varphi_v * f|^d(x) \\ &= 2^{vn} \int_{\mathbb{R}^n} \frac{|\varphi_v * f(z)|^d}{(1 + 2^v|x-z|)^m} dz \\ &= \int_{3Q_v} \frac{|\varphi_v * f(z)|^d}{(1 + 2^v|x-z|)^m} dz + \sum_{k \in \mathbb{Z}^n, \|k\|_\infty \geq 2} \int_{Q_v^k} \frac{|\varphi_v * f(z)|^d}{(1 + 2^v|x-z|)^m} dz, \end{aligned}$$

where $Q_v^k := Q_v + k\ell(Q_v)$. Let $z \in Q_v^k$ with $k \in \mathbb{Z}^n$ and $|k| > 4\sqrt{n}$. Then $|x-z| \geq |k|2^{-v-1}$ and the second integral is bounded by

$$c|k|^{-m} M_{Q_v^k}(|\varphi_v * f|^d),$$

where the positive constant c independent of x, k and v . Fix $r, d > 0$ such that

$$0 < r < \min(1, (p_0\tau)^-) \quad \text{and} \quad 0 < d < \min\left(\frac{p_1^-}{2r}, \frac{q^-}{2r}, \frac{r}{\tau^+}\right).$$

Then

$$0 < r < \frac{1}{2} \min\left(\frac{p_1^-}{d}, \frac{q^-}{d}, 2\right).$$

We have

$$\begin{aligned} & \left\| \left(\frac{2^{v\alpha_1(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}^{rd} \\ & \lesssim \left\| \left(\frac{2^{v\alpha_1(\cdot)}}{|P|^{\tau(\cdot)}} (M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}} \chi_P \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}^{rd} \end{aligned} \quad (4.1)$$

$$+ \sum_{k \in \mathbb{Z}^n, \|k\|_\infty \geq 2} |k|^{\sigma rd} \left\| \left(\frac{2^{v\alpha_1(\cdot)} |k|^{b(\cdot)}}{|P|^{\tau(\cdot)}} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}} \chi_P \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}^{rd}, \quad (4.2)$$

where

$$b(\cdot) := -\frac{sn\tau(\cdot)}{dr} - 2c_{\log} \left(\frac{1}{q\tau} \right) \tau(\cdot) - c_{\log} \left(\alpha_1 - \frac{n}{p_1} \right) - \frac{n\tau(\cdot)}{rt^-} - \kappa, \quad \kappa > 0$$

and $\sigma = (-b)^+ - \frac{m}{d}$ with $\frac{r}{p_0(\cdot)\tau(\cdot)} + \frac{1}{t(\cdot)} = 1$, where s will be chosen later.

Estimate of (4.1). We will prove that (4.1), with power $\frac{1}{rd}$, is bounded by

$$c \left\| \left(\frac{2^{v\alpha_1(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_{3P} \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \lesssim \|f\|_{\mathfrak{B}_{p_0(\cdot), q(\cdot)}^{\alpha_0(\cdot), \tau(\cdot)}}. \quad (4.3)$$

By the scaling argument, we see that it suffices to consider the case when the left-hand side of (4.3) is less than or equal 1. Therefore we will prove that

$$\sum_{v=0}^{\infty} \left\| \left| \frac{c 2^{v\alpha_1(\cdot)}}{|P|^{\tau(\cdot)}} (M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}} \chi_P \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leq 1$$

for some positive constant $c > 0$. This clearly follows from the inequality

$$\left\| \left\| \frac{c 2^{v\alpha_1(\cdot)}}{|P|^{\tau(\cdot)}} (M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}} \chi_P \right\|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leq \left\| \left\| \frac{2^{v\alpha_0(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \right\|^{q(\cdot)} \chi_{3P} \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-v} = \delta.$$

This claim can be reformulated as showing that

$$\left\| \left\| \frac{c \delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha_1(\cdot)}}{|P|^{\tau(\cdot)}} (M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}} \right\|^{q(\cdot)} \chi_P \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \leq 1,$$

which is equivalent to

$$\int_P \frac{\delta^{-\frac{p_1(x)}{q(x)}} 2^{v\alpha_1(x)p_1(x)}}{|P|^{\tau(x)p_1(x)}} (M_{3Q_v}(|\varphi_v * f|^d))^{\frac{p_1(x)}{d}} dx \lesssim 1. \quad (4.4)$$

Since α_1 and p_1 are log-Hölder continuous, we can move $2^{v(\alpha_1(x) - \frac{n}{p_1(x)})}$ inside the integral by Lemma 2.1:

$$\delta^{-\frac{1}{q(x)}} \frac{2^{v(\alpha_1(x) - \frac{n}{p_1(x)})}}{|P|^{\tau(x)}} (M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}} \lesssim \frac{\delta^{-\frac{1}{q(x)}}}{|P|^{\tau(x)}} (M_{3Q_v}(2^{v(\alpha_1(\cdot) - \frac{n}{p_1(\cdot)})d} |\varphi_v * f|^d))^{\frac{1}{d}} \quad (4.5)$$

for any $x \in Q_v \subset P$. Observe that $0 < d < \frac{r}{\tau^+}$. The right-hand side of (4.5) can be rewritten as

$$\left(\frac{1}{|P|^r} (\delta^{-\frac{d}{q(x)}} M_{3Q_v}(2^{v(\alpha_1(\cdot) - \frac{n}{p_1(\cdot)})d} |\varphi_v * f|^d))^{\frac{\tau(x)}{d}} \right)^{\frac{r}{\tau(x)}}. \quad (4.6)$$

By Lemma 2.3, Remark 3.5 and since $\frac{1}{q}$ and τ are log-Hölder continuous,

$$\delta^{-\frac{r}{q(x)\tau(x)}} \left(\frac{1}{|3Q_v|} \int_{3Q_v} 2^{v(\alpha_1(y) - \frac{n}{p_1(y)})d} |\varphi_v * f(y)|^d dy \right)^{\frac{r}{d\tau(x)}}$$

can be estimated by

$$\begin{aligned} & \frac{c}{|3Q_v|} \int_{3Q_v} \delta^{-\frac{r}{q(y)\tau(y)}} 2^{\frac{v(\alpha_1(y) - \frac{n}{p_1(y)})r}{\tau(y)}} |\varphi_v * f(y)|^{\frac{r}{\tau(y)}} dy + |Q_v|^s g(x) \\ & \lesssim \int_{3Q_v} 2^{vn} \delta^{-\frac{r}{q(y)\tau(y)}} 2^{\frac{v(\alpha_1(y) - \frac{n}{p_1(y)})r}{\tau(y)}} |\varphi_v * f(y)|^{\frac{r}{\tau(y)}} dy + h(x) \end{aligned}$$

for any $s > 0$ large enough, where

$$g(x) = (e + |x|)^{-s} + M_{3Q_v}((e + |\cdot|)^{-s}), \quad x \in \mathbb{R}^n, s > 0$$

and

$$h(x) = (e + |x|)^{-s} + \mathcal{M}((e + |\cdot|)^{-s})(x), \quad x \in \mathbb{R}^n, s > 0.$$

These two functions will be used throughout the paper. Therefore (4.6), with power $\frac{1}{\tau(x)}$, is bounded by

$$c \left\| \frac{\delta^{-\frac{r}{q(\cdot)\tau(\cdot)}} 2^{v\frac{\alpha_0(\cdot)r}{\tau(\cdot)}} |\varphi_v * f|^{\frac{r}{\tau(\cdot)}}}{|P|^r} \chi_{3P} \right\|_{\frac{p_0(\cdot)\tau(\cdot)}{r}}^{\frac{1}{r}} \left\| 2^{v\frac{n}{\tau(\cdot)}} \chi_{3Q_v} \right\|_{t(\cdot)}^{\frac{1}{r}} + c,$$

by Hölder's inequality, with $\frac{r}{p_0(\cdot)\tau(\cdot)} + \frac{1}{t(\cdot)} = 1$. The second norm is bounded and the first norm is bounded if and only if

$$\left\| \frac{\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha_0(\cdot)} |\varphi_v * f| \chi_{3P}}{|P|^{\tau(\cdot)}} \right\|_{p_0(\cdot)} \lesssim 1,$$

which follows immediately from the definition of δ . Now, we find that the left-hand side of (4.4) can be rewritten as

$$\begin{aligned} & \int_P \left(\frac{\delta^{-\frac{1}{q(x)}} 2^{v(\alpha_1(x) - \frac{n}{p_1(x)})}}{|P|^{\tau(x)}} (M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}} \right)^{p_1(x) - p_0(x)} \\ & \quad \times \left(\frac{\delta^{-\frac{1}{q(x)}} 2^{v(\alpha_1(x) + \frac{n}{p_0(x)} - \frac{n}{p_1(x)})}}{|P|^{\tau(x)}} (M_{3Q_v}(|\varphi_v * f|^d))^{\frac{1}{d}} \right)^{p_0(x)} dx \\ & \lesssim \int_P \frac{1}{|P|^{\tau(x)p_0(x)}} \left(\delta^{-\frac{d}{q(x)}} M_{3Q_v}(2^{v\alpha_0(\cdot)d} |\varphi_v * f|^d) \right)^{\frac{p_0(x)}{d}} dx. \end{aligned}$$

The last expression is bounded if and only if

$$\left\| \frac{1}{|P|^r} \left(\delta^{-\frac{d}{q(\cdot)}} M_{3Q_v}(2^{v\alpha_0(\cdot)d} |\varphi_v * f|^d) \chi_P \right)^{\frac{r}{d\tau(\cdot)}} \right\|_{\frac{p_0(\cdot)\tau(\cdot)}{r}} \lesssim 1.$$

This norm is bounded by

$$\left\| \mathcal{M} \left(\frac{\delta^{-\frac{r}{q(\cdot)\tau(\cdot)}} 2^{\frac{v\alpha_0(\cdot)r}{\tau(\cdot)}} |\varphi_v * f|^{\frac{r}{\tau(\cdot)}}}{|P|^r} \chi_{3P} \right) \right\|_{\frac{p_0(\cdot)\tau(\cdot)}{r}} + c,$$

where we have used again Lemma 2.3 and Remark 3.5. Since the maximal function is bounded in $L^{p(\cdot)}$ when $p \in \mathcal{P}^{\log}$ and $p^- > 1$, this expression is bounded by

$$\left\| \frac{\delta^{-\frac{1}{q(\cdot)\tau(\cdot)}} 2^{\frac{v\alpha_0(\cdot)}{\tau(\cdot)}} |\varphi_v * f|^{\frac{1}{\tau(\cdot)}} \chi_{3P}}{|P|} \right\|_{p_0(\cdot)\tau(\cdot)}^r + c.$$

The last quasi-norm is bounded if and only if

$$\left\| \frac{\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha_0(\cdot)} |\varphi_v * f| \chi_{3P}}{|P|^{\tau(\cdot)}} \right\|_{p_0(\cdot)} \lesssim 1.$$

due to the choice of δ .

Estimate of (4.2). The summation in (4.2) can be rewritten as

$$\sum_{k \in \mathbb{Z}^n, |k| \leq 4\sqrt{n}} \cdots + \sum_{k \in \mathbb{Z}^n, |k| > 4\sqrt{n}} \cdots.$$

The estimation of the first sum follows in the same manner as before. Let us prove that

$$\left\| \left(\frac{2^{v\alpha_1(\cdot)} |k|^{b(\cdot)} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}}}{|\tilde{Q}^k|^{\tau(\cdot)}} \chi_P \right) \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \lesssim 1 \quad (4.7)$$

for any $k \in \mathbb{Z}^n$ with $|k| > 4\sqrt{n}$, where $\tilde{Q}^k := Q(c_P, 2|k|l(P))$. Therefore we will prove that

$$\sum_{v=0}^{\infty} \left\| \left| \frac{2^{v\alpha_1(\cdot)} |k|^{b(\cdot) - n\tau(\cdot)}}{|P|^{\tau(\cdot)}} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}} \chi_P \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \lesssim 1.$$

This clearly follows from the inequality

$$\begin{aligned} & \left\| \left| \frac{c 2^{v\alpha_1(\cdot)} |k|^{b(\cdot) - n\tau(\cdot)}}{|P|^{\tau(\cdot)}} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}} \chi_P \right|^{q(\cdot)} \right\|_{\frac{p_1(\cdot)}{q(\cdot)}} \\ & \leq \left\| \left| \frac{2^{v\alpha_0(\cdot)} \varphi_v * f}{|\tilde{Q}^k|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{\tilde{Q}^k} \right\|_{\frac{p_0(\cdot)}{q(\cdot)}} + 2^{-v} = \delta \end{aligned}$$

for some positive constant c . This claim can be reformulated as showing that

$$\int_P \frac{\delta^{-\frac{p_1(x)}{q(x)}} 2^{v\alpha_1(x)p_1(x)} |k|^{(b(x)-n\tau(x))p_1(x)}}{|P|^{\tau(x)p_1(x)}} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{p_1(x)}{d}} dx \lesssim 1.$$

Since, again, α_1 and p_1 are log-Hölder continuous, we can move $2^{v(\alpha_1(x)-\frac{n}{p_1(x)})}$ inside the integral by Lemma 2.1:

$$\begin{aligned} & \frac{|k|^{-c_{\log}(\alpha_1-\frac{n}{p_1})-\frac{n}{d}} 2^{v(\alpha_1(x)-\frac{n}{p_1(x)})}}{|P|^{\tau(x)}} (M_{Q_v^k}(|\varphi_v * f|^d))^{\frac{1}{d}} \\ & \lesssim \frac{1}{|P|^{\tau(x)}} \left(M_{Q_v^k}(|k|^{-n} 2^{v(\alpha_1(\cdot)-\frac{n}{p_1(\cdot)})d} |\varphi_v * f|^d) \right)^{\frac{1}{d}}, \end{aligned}$$

where the implicit constant is independent of x, v and k . We have

$$\begin{aligned} & \frac{|k|^{b(\cdot)-n\tau(\cdot)} \delta^{-\frac{1}{q(\cdot)}}}{|P|^{\tau(\cdot)}} \left(M_{Q_v^k} \left(\frac{|\varphi_v * f|^d}{|k|^n 2^{-v(\alpha_1(\cdot)-\frac{n}{p_1(\cdot)})d}} \right) \right)^{\frac{1}{d}} \\ & = \left(\frac{|k|^{(b(\cdot)-n\tau(\cdot))\frac{r}{\tau(\cdot)}} \delta^{-\frac{r}{q(\cdot)\tau(\cdot)}}}{|P|^r} \left(M_{Q_v^k} \left(\frac{|\varphi_v * f|^d}{|k|^n 2^{-v(\alpha_1(\cdot)-\frac{n}{p_1(\cdot)})d}} \right) \right)^{\frac{r}{d\tau(\cdot)}} \right)^{\frac{\tau(\cdot)}{r}}. \quad (4.8) \end{aligned}$$

As before, let us prove that this expression, with power $\frac{1}{\tau(\cdot)}$ is bounded. Observe that $Q_v^k \subset Q(x, |k|2^{1-v}) = \tilde{Q}_v^k$. We have

$$\delta^{-\frac{1}{q(x)\tau(x)}} = (2^v \delta)^{-\frac{1}{q(x)\tau(x)} + \frac{1}{q(y)\tau(y)}} (2^v \delta)^{-\frac{1}{q(y)\tau(y)}} 2^{v\frac{1}{q(x)\tau(x)}}, \quad x \in Q_v \subset P, \quad y \in \tilde{Q}_v^k.$$

From Lemma 2.1 it follows that

$$2^{v\frac{1}{q(x)\tau(x)}} \lesssim |k|^{c_{\log}(\frac{1}{q\tau})} 2^{v\frac{1}{q(y)\tau(y)}}$$

and

$$(2^v \delta)^{-\frac{1}{q(x)\tau(x)} + \frac{1}{q(y)\tau(y)}} \lesssim |k|^{c_{\log}(\frac{1}{q\tau})}$$

for any $x \in Q_v, y \in \tilde{Q}_v^k$, where the implicit constant is independent of x, y, k and v . Again by Lemma 2.3 combined with Remark 3.5 and since $\frac{1}{q}$ and τ are log-Hölder continuous,

$$\begin{aligned} & |k|^{(-\frac{sn\tau(x)}{dr} - 2c_{\log}(\frac{1}{q\tau})\tau(x) - \kappa)\frac{r}{\tau(x)} - \frac{n}{t}} \delta^{-\frac{r}{q(x)\tau(x)}} \\ & \quad \times \left(\frac{1}{|\tilde{Q}_v^k|} \int_{\tilde{Q}_v^k} 2^{v(\alpha_1(y)-\frac{n}{p_1(y)})d} |\varphi_v * f(y)|^d dy \right)^{\frac{r}{d\tau(x)}} \\ & \lesssim \frac{|k|^{-\frac{n}{t}}}{|\tilde{Q}_v^k|} \int_{\tilde{Q}_v^k} \delta^{-\frac{r}{q(y)\tau(y)}} 2^{\frac{vr(\alpha_1(y)-\frac{n}{p_1(y)})}{\tau(y)}} |\varphi_v * f(y)|^{\frac{r}{\tau(y)}} dy \\ & \quad + (e + |x|)^{-s} + \frac{1}{|\tilde{Q}_v^k|} \int_{\tilde{Q}_v^k} (e + |y|)^{-s} dy \\ & \lesssim \frac{|k|^{-\frac{n}{t}}}{|\tilde{Q}_v^k|} \int_{\tilde{Q}_v^k} \delta^{-\frac{r}{q(y)\tau(y)}} 2^{\frac{v(\alpha_1(y)-\frac{n}{p_1(y)})r}{\tau(y)}} |\varphi_v * f(y)|^{\frac{r}{\tau(y)}} dy + h(x) \end{aligned}$$

for any $s > 0$ large enough with $\frac{r}{p_0(\cdot)\tau(\cdot)} + \frac{1}{t(\cdot)} = 1$. Therefore the left-hand side of (4.8), with power $\frac{1}{\tau(\cdot)}$, is bounded by

$$c \left\| \frac{|k|^{-nr} \delta^{-\frac{r}{q(\cdot)\tau(\cdot)}} 2^{\frac{v\alpha_0(\cdot)r}{\tau(\cdot)}} |\varphi_v * f|^{\frac{r}{\tau(\cdot)}}}{|P|^r} \chi_{\tilde{Q}_v^k} \right\|_{\frac{p_0(\cdot)\tau(\cdot)}{r}}^{\frac{1}{r}} \|2^{v\frac{n}{t(\cdot)}} \chi_{\tilde{Q}_v^k}\|_{t(\cdot)}^{\frac{1}{t(\cdot)}} |k|^{-\frac{n}{rt}} + c,$$

by Hölder's inequality. As before the second term is bounded and the first norm is bounded if and only if

$$\int_{\tilde{Q}^k} \frac{\delta^{-\frac{p_0(y)}{q(y)}} 2^{v\alpha_0(y)p_0(y)} |\varphi_v * f(y)|^{p_0(y)}}{|\tilde{Q}^k|^{p_0(y)\tau(y)}} dy \lesssim 1,$$

which follows immediately from the definition of δ . The desired estimate, follows using similar arguments as above, with suitable choice of κ , and by taking m large enough.

Case 2 $l(P) \leq 1$. Since τ is log-Hölder continuous, we have

$$|P|^{-\tau(x)} \leq c|P|^{-\tau(y)}(1 + 2^{vP}|x - y|)^{c_{\log}(\tau)} \leq c|P|^{-\tau(y)}(1 + 2^v|x - y|)^{c_{\log}(\tau)}$$

for any $x, y \in \mathbb{R}^n$ and any $v \geq v_P$. Therefore,

$$\frac{1}{|P|^{\tau(\cdot)d}} \eta_{v,m} * (|\varphi_v * f|^d \chi_{3Q_v}) \lesssim \eta_{v,m-c_{\log}(\tau)} * \left(\frac{|\varphi_v * f|^d \chi_{3Q_v}}{|P|^{\tau(\cdot)d}} \right)$$

and

$$\frac{1}{|P|^{\tau(\cdot)d}} \eta_{v,m} * (|\varphi_v * f|^d \chi_{Q_v^k}) \lesssim \eta_{v,m-c_{\log}(\tau)} * \left(\frac{|\varphi_v * f|^d \chi_{Q_v^k}}{|P|^{\tau(\cdot)d}} \right).$$

The arguments here are quite similar to those used in the case $l(P) > 1$, where we did not need to use Lemma 2.3, which could be used only to move $|P|^{\tau(\cdot)}$ inside the convolution and hence the proof is complete. \square

Remark 4.3 We would like to mention that similar arguments give

$$\mathfrak{B}_{p_0(\cdot),q(\cdot)}^{\alpha_0(\cdot),\tau(\cdot)} \hookrightarrow \mathfrak{B}_{\infty,q(\cdot)}^{\alpha_0(\cdot) - \frac{n}{p_0(\cdot)},\tau(\cdot)}$$

if $\alpha_0, \tau \in C_{\text{loc}}^{\log}$, $\tau^- > 0$ and $p_0, q, \tau \in \mathcal{P}_0^{\log}$, with $q^+ < \infty$.

Let $\alpha, \tau \in C_{\text{loc}}^{\log}$, $\tau^- > 0, p, q \in \mathcal{P}_0^{\log}$. From (3.4), we obtain

$$\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \hookrightarrow B_{\infty,\infty}^{\alpha(\cdot)+n\tau(\cdot) - \frac{n}{p(\cdot)}} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Similar arguments of [35, Proposition 2.3] can be used to prove that

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}.$$

Therefore, we obtain the following statement.

Theorem 4.4 *Let $\alpha, \tau \in C_{\text{loc}}^{\log}$, $\tau^- > 0$ and $p, q \in \mathcal{P}_0^{\log}$ with $q^+ < \infty$. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Now we establish some further embedding of the spaces $\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$.

Theorem 4.5 *Let $\alpha, \tau \in C_{\text{loc}}^{\log}$, $\tau^- > 0$ and $p, q \in \mathcal{P}_0^{\log}$ with $q^+ < \infty$. If $(p_2 - p_1)^+ \leq 0$, then*

$$\mathfrak{B}_{p_2(\cdot),q(\cdot)}^{\alpha(\cdot)+n\tau(\cdot) + \frac{n}{p_2(\cdot)} - \frac{n}{p_1(\cdot)},0} \hookrightarrow \mathfrak{B}_{p_1(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}.$$

Proof Recall that

$$\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),0} = B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$$

in the sense of equivalent quasi-norms. Using the Sobolev embeddings

$$B_{p_2(\cdot),q(\cdot)}^{\alpha(\cdot)+n\tau(\cdot) + \frac{n}{p_2(\cdot)} - \frac{n}{p_1(\cdot)}} \hookrightarrow B_{p_1(\cdot),q(\cdot)}^{\alpha(\cdot)+n\tau(\cdot)},$$

(see [2, Theorem 6.4]) it is sufficient to prove that $B_{p_1(\cdot), q(\cdot)}^{\alpha(\cdot)+n\tau(\cdot)} \hookrightarrow \mathfrak{B}_{p_1(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$. We have

$$\sup_{P \in \mathcal{Q}, |P| > 1} \left\| \left(\frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geq v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \leq \|(2^{v\alpha(\cdot)} \varphi_v * f)_{v \in \mathbb{N}_0}\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}.$$

In view of the definition of $B_{p_1(\cdot), q(\cdot)}^{\alpha(\cdot)}$ spaces the last expression is bounded by

$$\|f\|_{B_{p_1(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq \|f\|_{B_{p_1(\cdot), q(\cdot)}^{\alpha(\cdot)+n\tau(\cdot)}}.$$

Now we have the estimates

$$\begin{aligned} & \sup_{P \in \mathcal{Q}, |P| \leq 1} \left\| \left(\frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geq v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \\ & \leq \sup_{P \in \mathcal{Q}, |P| \leq 1} \|(2^{v(\alpha(\cdot)+n\tau(\cdot))+n\tau(\cdot)(v_P-v)} \varphi_v * f)_{v \geq v_P}\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \\ & \leq \sup_{P \in \mathcal{Q}, |P| \leq 1} \|(2^{v(\alpha(\cdot)+n\tau(\cdot))} \varphi_v * f)_{v \in \mathbb{N}_0}\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})} \\ & \leq \|f\|_{B_{p_1(\cdot), q(\cdot)}^{\alpha(\cdot)+n\tau(\cdot)}}, \end{aligned}$$

which completes the proof. \square

5 Atomic Decomposition

The idea of atomic decompositions leads back to M. Frazier and B. Jawerth in their series of papers [15] and [16]. The main goal of this section is to prove an atomic decomposition result for $\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$. We define for $a > 0$, $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, the Peetre maximal function

$$\varphi_v^{*, a} 2^{v\alpha(\cdot)} f(x) := \sup_{y \in \mathbb{R}^n} \frac{2^{v\alpha(y)} |\varphi_v * f(y)|}{(1 + 2^v |x - y|)^a}, \quad v \in \mathbb{N}_0,$$

where φ_0 is replaced by Φ . We now present a fundamental characterization of spaces under consideration.

Theorem 5.1 *Let $\tau, \alpha \in C_{\text{loc}}^{\log}$, $\tau^- > 0$ and $p, q \in \mathcal{P}_0^{\log}$. Let m be as in Lemma 2.4, $a > \frac{m\tau^+}{(\tau p)^-}$ and Φ and φ satisfy (3.1) and (3.2), respectively. Then*

$$\|f\|_{\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}}^{\nabla} := \sup_{P \in \mathcal{Q}} \left\| \left(\frac{\varphi_v^{*, a} 2^{v\alpha(\cdot)} f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geq v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \quad (5.1)$$

is an equivalent quasi-norm in $\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$.

Proof We divide the proof in two steps.

Step 1 It is easy to see that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\|f\|_{\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}}^{\nabla} < \infty$ and any $x \in \mathbb{R}^n$ we have

$$2^{v\alpha(x)} |\varphi_v * f(x)| \leq \varphi_v^{*, a} 2^{v\alpha(\cdot)} f(x).$$

This shows that the right-hand side in (3.3) is less than or equal (5.1).

Step 2 We will prove in this step that there is a constant $C > 0$ such that for every $f \in \mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$

$$\|f\|_{\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}}^{\nabla} \leq C \|f\|_{\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}}. \quad (5.2)$$

We choose $t > 0$ such that $a > \frac{m}{t} > \frac{m}{p}$. By Lemmas 2.2 and 2.1 the estimates

$$\begin{aligned} 2^{v\alpha(y)}|\varphi_v * f(y)| &\leq C_1 2^{v\alpha(y)}(\eta_{v,w} * |\varphi_v * f|^t(y))^{\frac{1}{t}} \\ &\leq C_2 (\eta_{v,w-c_{\log}(\alpha)} * (2^{v\alpha(\cdot)}|\varphi_v * f|^t(y))^{\frac{1}{t}}) \end{aligned} \quad (5.3)$$

are true for any $y \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and any $w > n$. Now divide both sides of (5.3) by $(1+2^v|x-y|)^a$. In the right-hand side we use the inequality

$$(1+2^v|x-y|)^{-a} \leq (1+2^v|x-z|)^{-a}(1+2^v|y-z|)^a, \quad x, y, z \in \mathbb{R}^n;$$

in the left-hand side take the supremum over $y \in \mathbb{R}^n$ and get for all $f \in \mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$, any $x \in P$ any $v \geq v_P^+$ and any $w > \max(n, at + c_{\log}(\alpha))$

$$(\varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x))^t \leq C_2 \eta_{v,at} * (2^{v\alpha(\cdot)}|\varphi_v * f|^t)(x),$$

where $C_2 > 0$ is independent of x, v and f . An application of Lemma 2.4 gives that the left hand side of (5.2) is bounded by

$$\begin{aligned} C \sup_{P \in \mathcal{Q}} \left\| \left(\frac{\eta_{v,at} * (2^{v\alpha(\cdot)}|\varphi_v * f|^t)}{|P|^{\tau(\cdot)t}} \chi_P \right) \right\|_{v \geq v_P^+}^{\frac{1}{t}} &\left\| \ell^{\frac{q(\cdot)}{t}} \left(L^{\frac{p(\cdot)}{t}} \right) \right\| \\ &\lesssim \|(2^{v\alpha(\cdot)}\varphi_v * f)_v\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)})} \\ &= C \|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}. \end{aligned}$$

The proof is complete. \square

Atoms are the building blocks for the atomic decomposition.

Definition 5.2 Let $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$ and let $\gamma > 1$. A K -times continuous differentiable function $a \in C^K(\mathbb{R}^n)$ is called $[K, L]$ -atom centered at $Q_{v,m}$, $v \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if

$$\text{supp } a \subseteq \gamma Q_{v,m}, \quad (5.4)$$

$$|\partial^\beta a(x)| \leq 2^{v(|\beta|+\frac{1}{2})}, \quad \text{for } 0 \leq |\beta| \leq K, \quad x \in \mathbb{R}^n \quad (5.5)$$

and if

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0, \quad \text{for } 0 \leq |\beta| \leq L \text{ and } v \geq 1. \quad (5.6)$$

If the atom a located at $Q_{v,m}$, that means if it fulfills (5.4), then we will denote it by $a_{v,m}$. For $v = 0$ or $L = -1$ there are no moment conditions (5.6) required.

For proving the decomposition by atoms we need the following lemma, see Frazier and Jawerth [15, Lemma 3.3].

Lemma 5.3 Let Φ and φ satisfy, respectively, (3.1) and (3.2) and let $\varrho_{v,m}$ be an $[K, L]$ -atom. Then

$$|\varphi_j * \varrho_{v,m}(x)| \leq c 2^{(v-j)K+v\frac{n}{2}}(1+2^v|x-x_{Q_{v,m}}|)^{-M}$$

if $v \leq j$, and

$$|\varphi_j * \varrho_{v,m}(x)| \leq c 2^{(j-v)(L+n+1)+v\frac{n}{2}}(1+2^j|x-x_{Q_{v,m}}|)^{-M}$$

if $v \geq j$, where M is sufficiently large, $\varphi_j = 2^{jn}\varphi(2^j\cdot)$ and φ_0 is replaced by Φ .

Now we come to the atomic decomposition theorem.

Theorem 5.4 Let $\alpha, \tau \in C_{\text{loc}}^{\log}, \tau^- > 0$ and $p, q \in \mathcal{P}_0^{\log}$ with $0 < q^- \leq q^+ < \infty$. Let $0 < p^- \leq p^+ < \infty$ and let $K, L + 1 \in \mathbb{N}_0$ such that

$$K \geq ([\alpha^+ + n\tau^+] + 1)^+, \quad (5.7)$$

and

$$L \geq \max \left(-1, \left[n \left(\frac{1}{\min(1, \frac{(\tau p)^-}{\tau^+})} - 1 \right) - \alpha^- \right] \right). \quad (5.8)$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$, if and only if it can be represented as

$$f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varrho_{v,m}, \quad \text{converging in } \mathcal{S}'(\mathbb{R}^n), \quad (5.9)$$

where $\varrho_{v,m}$ are $[K, L]$ -atoms and $\lambda = \{\lambda_{v,m}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$. Moreover, $\inf \|\lambda\|_{\mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}}$, where the infimum is taken over admissible representations (5.9), is an equivalent quasi-norm in $\mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$.

If p, q, τ , and α are constants, then the restriction (5.7), and their counterparts, in the atomic decomposition theorem are $K \geq ([\alpha + n\tau] + 1)^+$ and $L \geq \max(-1, [n(\frac{1}{\min(1,p)} - 1) - \alpha])$, which are essentially the restrictions from the works of [10, Theorem 3.12].

Proof The proof follows the ideas in [15, Theorem 6] and [12].

Step 1 Assume that $f \in \mathfrak{B}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)}$ and let Φ and φ satisfy, respectively (3.1) and (3.2). There exist functions $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.1) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.2) such that

$$f = \Psi * \tilde{\Phi} * f + \sum_{v=1}^{\infty} \psi_v * \tilde{\varphi}_v * f,$$

see Section 3. Using the definition of the cubes $Q_{v,m}$ we obtain for all $x \in \mathbb{R}^n$,

$$f(x) = \sum_{m \in \mathbb{Z}^n} \int_{Q_{0,m}} \tilde{\Phi}(x-y) \Psi * f(y) dy + \sum_{v=1}^{\infty} 2^{vn} \sum_{m \in \mathbb{Z}^n} \int_{Q_{v,m}} \tilde{\varphi}(2^v(x-y)) \psi_v * f(y) dy,$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. We define for every $v \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$,

$$\lambda_{v,m} := C_{\tilde{\varphi}} \sup_{y \in Q_{v,m}} |\psi_v * f(y)|, \quad (5.10)$$

where

$$C_{\tilde{\varphi}} := \max \left\{ \sup_{|y| \leq 1} |D^\alpha \tilde{\varphi}(y)| : |\alpha| \leq K \right\}.$$

Define also

$$\varrho_{v,m}(x) := \begin{cases} \frac{1}{\lambda_{v,m}} 2^{vn} \int_{Q_{v,m}} \tilde{\varphi}(2^v(x-y)) \psi_v * f(y) dy & \text{if } \lambda_{v,m} \neq 0, \\ 0 & \text{if } \lambda_{v,m} = 0. \end{cases} \quad (5.11)$$

Similarly we define for every $m \in \mathbb{Z}^n$ the numbers $\lambda_{0,m}$ and the functions $\varrho_{0,m}$ taking in (5.10) and (5.11) $v = 0$ and replacing ψ_v and $\tilde{\varphi}$ by Ψ and $\tilde{\Phi}$, respectively. Let us now check that such $\varrho_{v,m}$ are atoms in the sense of Definition 5.2. Note that the support and moment conditions

are clear by (3.1) and (3.2), respectively. It thus remains to check (5.5) in Definition 5.2. We have

$$\begin{aligned} |D^\beta \varrho_{v,m}(x)| &\leq \frac{2^{v(n+|\beta|)}}{C_{\tilde{\varphi}}} \int_{Q_{v,m}} |(D^\beta \tilde{\varphi})(2^v(x-y))| |\psi_v * f(y)| dy \left(\sup_{y \in Q_{v,m}} |\psi_v * f(y)| \right)^{-1} \\ &\leq \frac{2^{v(n+|\beta|)}}{C_{\tilde{\varphi}}} \int_{Q_{v,m}} |(D^\beta \tilde{\varphi})(2^v(x-y))| dy \\ &\leq 2^{v(n+|\beta|)} |Q_{v,m}| \\ &\leq 2^{v|\beta|}. \end{aligned}$$

The modifications for the terms with $v = 0$ are obvious.

Step 2 Next we show that there is a constant $c > 0$ such that $\|\lambda\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq c \|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}$. For that reason we exploit the equivalent quasi-norms given in Theorem 5.1 involving Peetre's maximal function. Let $v \in \mathbb{N}$. Taking into account that $|x-y| \leq c 2^{-v}$ for $x, y \in Q_{v,m}$ we obtain

$$2^{v(\alpha(x)-\alpha(y))} \leq \frac{c_{\log(\alpha)} v}{\log(e + \frac{1}{|x-y|})} \leq \frac{c_{\log(\alpha)} v}{\log(e + \frac{2^v}{c})} \leq c$$

if $v \geq [\log_2 c] + 2$. If $0 < v < [\log_2 c] + 2$, then $2^{v(\alpha(x)-\alpha(y))} \leq 2^{v(\alpha^+ - \alpha^-)} \leq c$. Therefore,

$$2^{v\alpha(x)} |\psi_v * f(y)| \leq c 2^{v\alpha(y)} |\psi_v * f(y)|$$

for any $x, y \in Q_{v,m}$ and any $v \in \mathbb{N}$. Hence,

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} 2^{v\alpha(x)} \chi_{v,m}(x) &= C_{\tilde{\varphi}} \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(x)} \sup_{y \in Q_{v,m}} |\psi_v * f(y)| \chi_{v,m}(x) \\ &\leq c \sum_{m \in \mathbb{Z}^n} \sup_{|z| \leq c 2^{-v}} \frac{2^{v\alpha(x-z)} |\psi_v * f(x-z)|}{(1 + 2^v |z|)^a} (1 + 2^v |z|)^a \chi_{v,m}(x) \\ &\leq c \psi_v^{*,a} 2^{v\alpha(\cdot)} f(x) \sum_{m \in \mathbb{Z}^n} \chi_{v,m}(x) \\ &= c \psi_v^{*,a} 2^{v\alpha(\cdot)} f(x), \end{aligned}$$

where $a > \frac{m\tau^+}{(\tau p)^+}$ and we have used $\sum_{m \in \mathbb{Z}^n} \chi_{v,m}(x) = 1$. This estimate and its counterpart for $v = 0$ (which can be obtained by a similar calculation) give

$$\|\lambda\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq c \|(\psi_v^{*,a} 2^{v\alpha(\cdot)} f)_v\|_{\ell_{q(\cdot)}(L_{p(\cdot)}^{\tau(\cdot)})} \leq c \|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}},$$

by Theorem 5.1.

Step 3 Assume that f can be represented by (5.9), with K and L satisfying (5.7) and (5.8), respectively. Similar arguments of [12, Theorem 4.3], by using Lemmas 2.4, 2.6, show that $f \in \mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ and that for some $c > 0$,

$$\|f\|_{\mathfrak{B}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq c \|\lambda\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}.$$

Step 4 *The convergence of (5.9).* Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Using (5.4)–(5.5)–(5.6) and the Taylor expansion of φ up to order L with respect to the off-points $x_{Q_{v,m}}$, we get

$$\int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varrho_{v,m}(y) \varphi(y) dy$$

$$= \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varrho_{v,m}(y) \left(\varphi(y) - \sum_{|\beta| \leq L} (y - x_{Q_{v,m}})^\beta \frac{\partial^\beta \varphi(x_{Q_{v,m}})}{\beta!} \right) dy$$

for fixed v . The last factor in the integral can be uniformly estimated from the above by

$$c 2^{-v(L+1)} (1 + |y|^2)^{-\frac{M}{2}} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{\frac{M}{2}} \sum_{|\beta| \leq L+1} |\partial^\beta \varphi(x)|,$$

where $M > 0$ is at our disposal and the positive constant c is independent of x and v . From (5.8) we obtain

$$L + 1 > -\alpha(\cdot) + n \left(\frac{1}{\min(1, \frac{(\tau p)^-}{\tau^+})} - 1 \right).$$

Let

$$\max \left(0, \left(p(\cdot) \left(1 - \frac{1}{\min(1, \frac{(\tau p)^-}{\tau^+})} \right) \right)^+ + 1 \right) < t < 1$$

and $s(\cdot) = \alpha(\cdot) + \frac{n}{p(\cdot)}(t - 1)$. Here if $\frac{(\tau p)^-}{\tau^+} \geq 1$ then we take $t = 1$. Then $L + 1 > -s(\cdot)$. Since $\varrho_{v,m}$ are $[K, L]$ -atoms, then for every $S > 0$, we have

$$|\varrho_{v,m}(y)| \leq c 2^{v\frac{n}{2}} (1 + 2^v |y - x_{Q_{v,m}}|)^{-S}.$$

Therefore,

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varrho_{v,m}(y) \varphi(y) dy \right| \\ & \leq c 2^{-v(L+1)} \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} 2^{v\frac{n}{2}} |\lambda_{v,m}| \frac{(1 + |y|^2)^{-\frac{M}{2}}}{(1 + 2^v |y - x_{Q_{v,m}}|)^S} dy \\ & = c 2^{-v(L+1)} \sum_{h \in \mathbb{Z}^n} \int_{Q_{0,h}} \sum_{m \in \mathbb{Z}^n} 2^{v\frac{n}{2}} |\lambda_{v,m}| \frac{(1 + |y|^2)^{-\frac{M}{2}}}{(1 + 2^v |y - x_{Q_{v,m}}|)^S} dy. \end{aligned} \quad (5.12)$$

Applying [7, Lemma A.2] to obtain

$$\sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| (1 + 2^v |y - x_{Q_{v,m}}|)^{-S} \lesssim \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \eta_{v,S} * \chi_{v,m}(y).$$

Since we have in addition the factor $(1 + |y|^2)^{-\frac{M}{2}}$, Hölder's inequality, the fact that $\|\chi_{Q_{0,h}}\|_{(\frac{p(\cdot)}{t})} \approx 1$, and $(1 + |y|^2)^{-\frac{M}{2}} \lesssim (1 + |h|^2)^{-\frac{M}{2}}$ give that the term (5.12) can be estimated by

$$\begin{aligned} & c 2^{-v(L+1)} \sum_{h \in \mathbb{Z}^n} (1 + |h|^2)^{-\frac{M}{2}} \left\| \frac{1}{|Q_{0,h}|^{\tau(\cdot)}} \eta_{v,S} * \left(\sum_{m \in \mathbb{Z}^n} 2^{v\frac{n}{2}} |\lambda_{v,m}| \chi_{v,m} \right) \chi_{Q_{0,h}} \right\|_{\frac{p(\cdot)}{t}} \\ & \lesssim \sup_{P \in \mathcal{Q}, j \geq j_P^+} \left\| \frac{1}{|P|^{\tau(\cdot)}} 2^{(s(\cdot) + \frac{n}{2})j} \sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}| \chi_{j,m} \chi_P \right\|_{\frac{p(\cdot)}{t}} \\ & \lesssim \|\lambda\|_{\mathfrak{b}_{\frac{p(\cdot)}{t}, \infty}^{s(\cdot), \tau(\cdot)}}, \end{aligned}$$

where the first inequality follows by Lemma 2.4, $L + 1 + s(\cdot) > 0$ and by taking M large enough.

We claim that

$$\mathfrak{b}_{p(\cdot), q(\cdot)}^{\alpha(\cdot), \tau(\cdot)} \hookrightarrow \mathfrak{b}_{p(\cdot), \infty}^{\alpha(\cdot), \tau(\cdot)} \hookrightarrow \mathfrak{b}_{\frac{p(\cdot)}{t}, \infty}^{s(\cdot), \tau(\cdot)}.$$

Then the convergence of (5.9) is now clear. We prove our claim. By the scaling argument, we see that it suffices to consider $\lambda \in \mathfrak{b}_{p(\cdot), \infty}^{\alpha(\cdot), \tau(\cdot)}$ with $\|\lambda\|_{\mathfrak{b}_{p(\cdot), \infty}^{\alpha(\cdot), \tau(\cdot)}} \leq 1$. From Lemma 3.9,

$$|\lambda_{v,m}| \leq c 2^{-v(\alpha(x) - \frac{n}{p(x)} + \frac{n}{2})} |Q_{v,m}|^{\tau(x)}$$

for any $x \in Q_{v,m}$, $v \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$. Let P be a dyadic cube and $v \geq v_P^+$. We have

$$\begin{aligned} & \int_P \left(\frac{1}{|P|^{\tau(x)}} \sum_{m \in \mathbb{Z}^n} 2^{v(s(x) + \frac{n}{2})} |\lambda_{v,m}| \chi_{v,m}(x) \right)^{\frac{p(x)}{t}} dx \\ &= \int_P \left(\frac{1}{|P|^{\tau(x)}} \sum_{m \in \mathbb{Z}^n} 2^{v(s(x) - \frac{nt}{p(x)} + \frac{n}{2})} |\lambda_{v,m}| \chi_{v,m}(x) \right)^{\frac{p(x)}{t} - p(x)} \\ & \quad \times \left(\frac{1}{|P|^{\tau(x)}} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(x) + \frac{n}{2})} |\lambda_{v,m}| \chi_{v,m}(x) \right)^{p(x)} dx \\ & \lesssim \int_P \left(\frac{1}{|P|^{\tau(x)}} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(x) + \frac{n}{2})} |\lambda_{v,m}| \chi_{v,m}(x) \right)^{p(x)} dx \\ & \lesssim 1. \end{aligned}$$

Therefore

$$\|\lambda\|_{\mathfrak{b}_{\frac{p(\cdot)}{t}, \infty}^{s(\cdot), \tau(\cdot)}} \lesssim 1.$$

The proof is complete. \square

6 Appendix

Here we present the proof of Lemmas 2.4, 2.5 and 3.11.

Proof of Lemma 2.4 By the scaling argument, we see that it suffices to consider when

$$\|(f_v)_v\|_{\ell_{q(\cdot)}(L_{p(\cdot)}^{\tau(\cdot)})} \leq 1 \tag{6.1}$$

and show that for any dyadic cube P ,

$$\sum_{v=v_P^+}^{\infty} \left\| \left\| \frac{c \eta_{v,m} * |f_v|}{|P|^{\tau(\cdot)}} \right\|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

for some constant $c > 0$. We distinguish two cases:

Case 1 $l(P) > 1$. Let $Q_v \subset P$ be a cube, with $\ell(Q_v) = 2^{-v}$ and $x \in Q_v \subset P$. We have

$$\begin{aligned} & \eta_{v,m} * |f_v|(x) \\ &= 2^{vn} \int_{\mathbb{R}^n} \frac{|f_v(z)|}{(1 + 2^v|x-z|)^m} dz \\ &= \int_{3Q_v} \frac{|f_v(z)|}{(1 + 2^v|x-z|)^m} dz + \sum_{k \in \mathbb{Z}^n, \|k\|_{\infty} \geq 2} \int_{Q_v^k} \frac{|f_v(z)|}{(1 + 2^v|x-z|)^m} dz \\ &= J_v^1(f_v \chi_{3Q_v})(x) + \sum_{k \in \mathbb{Z}^n, \|k\|_{\infty} \geq 2} J_{v,k}^2(f_v \chi_{Q_v^k})(x), \end{aligned}$$

where $Q_v^k := Q_v + kl(Q_v)$. Let $0 < r < \frac{1}{2} \min(p^-, q^-, 2)$ and define $\tilde{p} = \frac{p}{r}$ and $\tilde{q} = \frac{q}{r}$. Then clearly, $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \leq 1$. Thus we obtain

$$\begin{aligned} & \left\| \left(\frac{\eta_{v,m} * |f_v|}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r \\ & \leq \left\| \left(\frac{J_v^1(f_v \chi_{3Q_v})}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r \end{aligned} \quad (6.2)$$

$$+ \sum_{k \in \mathbb{Z}^n, \|k\|_\infty \geq 2} \left\| \left(\frac{J_{v,k}^2(f_v \chi_{Q_v^k})}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r. \quad (6.3)$$

Estimate of (6.2). We will prove that (6.2) is bounded by a constant independent of P . Clearly, we need to show that

$$\begin{aligned} \left\| \left| \frac{c J_v^1(f_v \chi_{3Q_v})}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} & \leq \left\| \left| \frac{f_v}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{3P} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-v} \\ & = \delta \end{aligned}$$

for some positive constant c . This claim can be reformulated as showing that

$$\left\| \delta^{-\frac{1}{q(\cdot)}} \frac{c J_v^1(f_v \chi_{3Q_v})}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leq 1. \quad (6.4)$$

Let $d > 0$ be such that $\tau^+ < d < (\tau p)^-$. We have

$$\frac{M_{3Q_v}(f_v)}{|P|^{\tau(\cdot)}} = \left(\frac{(M_{3Q_v}(f_v))^{\frac{d}{\tau(\cdot)}}}{|P|^d} \right)^{\frac{\tau(\cdot)}{d}}.$$

Hence, we will prove that

$$\left\| \frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} (M_{3Q_v}(f_v))^{\frac{d}{\tau(\cdot)}}}{|P|^d} \chi_P \right\|_{\frac{p(\cdot)\tau(\cdot)}{d}} \lesssim 1.$$

By Hölder's inequality,

$$|Q_v| M_{3Q_v}(|f_v|^{\frac{d}{\tau(\cdot)}}) \lesssim \left\| \frac{|f_v|^{\frac{1}{\tau(\cdot)}}}{|3Q_v|} \chi_{3Q_v} \right\|_{p(\cdot)\tau(\cdot)}^d \left\| |3Q_v| \chi_{3Q_v} \right\|_{t(\cdot)}^d,$$

where $\frac{1}{d} = \frac{1}{p(\cdot)\tau(\cdot)} + \frac{1}{t(\cdot)}$. The second quasi-norm is bounded, while the first is bounded if and only if

$$\left\| \frac{f_v}{|Q_v|^{\tau(\cdot)}} \chi_{3Q_v} \right\|_{p(\cdot)} \lesssim 1.$$

Notice that $3Q_v \subset \bigcup_{h=1}^{3^n} \check{Q}_v^h$, where $\{\check{Q}_v^h\}_{h=1}^{3^n}$ are disjoint dyadic cubes with side length $l(\check{Q}_v^h) = l(Q_v)$. Therefore $\chi_{3Q_v} \leq \sum_{h=1}^{3^n} \chi_{\check{Q}_v^h}$ and

$$\left\| \frac{f_v}{|Q_v|^{\tau(\cdot)}} \chi_{3Q_v} \right\|_{p(\cdot)} \leq c \sum_{h=1}^{3^n} \left\| \frac{f_v}{|\check{Q}_v^h|^{\tau(\cdot)}} \chi_{\check{Q}_v^h} \right\|_{p(\cdot)} \lesssim 1,$$

where we used (6.1). We can use Lemma 2.3 to obtain that

$$(M_{3Q_v}(f_v))^{\frac{d}{\tau(\cdot)}}$$

can be estimated by

$$M_{3Q_v}(|f_v|^{\frac{d}{\tau(\cdot)}}) + |3Q_v|^s g(x)$$

for any $x \in Q_v$ and any $s > 0$, where g is the same function as in Theorem 4.2. Taking into account that $\frac{1}{q}$ and τ are log-Hölder continuous, $\delta \in [2^{-v}, 1 + 2^{-v}]$, by Lemma 2.1,

$$\delta^{-\frac{d}{q(x)\tau(x)}} (M_{3Q_v}(f_v))^{\frac{d}{\tau(x)}}$$

does not exceed

$$cM_{3Q_v}(|\delta^{-\frac{1}{q(\cdot)}} f_v|^{\frac{d}{\tau(\cdot)}}) + c2^{\frac{vd}{q(x)\tau(x)}} |Q_v|^s g(x) \lesssim M_{3Q_v}(|\delta^{-\frac{1}{q(\cdot)}} f_v|^{\frac{d}{\tau(\cdot)}}) + h(x),$$

where the positive constant c is independent of v and x and we used the fact that

$$\max_{x \in Q_v} 2^{vd/q(x)\tau(x)} |Q_v|^s \leq 1,$$

since $s > 0$ can be taken large enough. Here h is the same function as in Theorem 4.2. Therefore,

$$\begin{aligned} \left\| \frac{(\delta^{-\frac{1}{q(\cdot)}} M_{3Q_v}(f_v))^{\frac{d}{\tau(\cdot)}}}{|P|^d} \right\|_{\frac{p(\cdot)\tau(\cdot)}{d}} &\lesssim \left\| \mathcal{M} \left(\frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} |f_v|^{\frac{d}{\tau(\cdot)}}}{|P|^d} \chi_{3Q_v} \right) \right\|_{\frac{p(\cdot)\tau(\cdot)}{d}} + c \\ &\lesssim \left\| \frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} |f_v|^{\frac{d}{\tau(\cdot)}}}{|P|^d} \chi_{3P} \right\|_{\frac{p(\cdot)\tau(\cdot)}{d}} + c, \end{aligned}$$

since $\frac{p\tau}{d} \in \mathcal{P}^{\log}$, $(\frac{p\tau}{d})^- > 1$ and $\mathcal{M} : L^{\frac{p(\cdot)\tau(\cdot)}{d}} \rightarrow L^{\frac{p(\cdot)\tau(\cdot)}{d}}$ is bounded. The last norm is bounded if and only if

$$\left\| \frac{\delta^{-\frac{1}{q(\cdot)}} |f_v| \chi_{3P}}{|P|^{\tau(\cdot)}} \right\|_{p(\cdot)} \lesssim 1,$$

which follows immediately from the definition of δ .

Estimate of (6.3). We will prove that (6.3) is bounded by a constant independent of P . The summation in (6.3) can be rewritten as

$$\sum_{k \in \mathbb{Z}^n, |k| \leq 4\sqrt{n}} \cdots + \sum_{k \in \mathbb{Z}^n, |k| > 4\sqrt{n}} \cdots. \quad (6.5)$$

The estimation of the first sum follows in the same manner as in the estimate of $J_v^1(f_v \chi_{3Q_v})$, so we need only to estimate the second sum. Let now prove that

$$\left\| \left(|k|^{b(\cdot)} \frac{J_{v,k}^2(f_v \chi_{Q_v^k})}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \left\| \left(\frac{f_v}{|\tilde{Q}^k|^{\tau(\cdot)}} \chi_{\tilde{Q}^k} \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

where $\tilde{Q}^k = Q(c_P, 2|k|l(P))$ and

$$b(\cdot) := m - n \left(1 + \frac{1}{t^-} \right) \tau^+ - 2 \frac{c_{\log}(\frac{d}{q\tau}) \tau(\cdot)}{d} - \frac{s\tau(\cdot)}{d}$$

and s will be chosen later. Again, by the scaling argument, we see that it suffices to consider when the last quasi-norm is less than or equal 1 and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that for any dyadic cube P ,

$$\sum_{v=0}^{\infty} \left\| \frac{c |k|^{b(\cdot)} J_{v,k}^2(f_v \chi_{Q_v^k})}{|P|^{\tau(\cdot)}} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

for some positive constant c . This estimate follows from the inequality

$$\begin{aligned} \left\| \left| \frac{c |k|^{b(\cdot)} J_{v,k}^2(f_v \chi_{Q_v^k})}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} &\leq \left\| \left| \frac{f_v}{|\tilde{Q}^k|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_{\tilde{Q}^k} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-v} \\ &= \delta \end{aligned}$$

for any $v \in \mathbb{N}_0$. This claim can be reformulated as showing that

$$\left\| \delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} \frac{(|k|^{b(\cdot)} J_{v,k}^2(f_v \chi_{Q_v^k}))^{\frac{d}{\tau(\cdot)}}}{|P|^d} \chi_P \right\|_{\frac{p(\cdot)\tau(\cdot)}{d}} \lesssim 1. \quad (6.6)$$

Let $z \in Q_v^k$, $x \in Q_v$ with $k \in \mathbb{Z}^n$ and $|k| > 4\sqrt{n}$. Then $z = h + k2^{-v}$ for some $h \in Q_v$, $|x - z| \geq |k|2^{-v-1}$. Hence

$$\begin{aligned} \delta^{-\frac{1}{q(x)}} |k|^{b(x)} J_{v,k}^2(f_v \chi_{Q_v^k})(x) &\lesssim \delta^{-\frac{1}{q(x)}} |k|^{b(x)-m} M_{Q_v^k}(f_v) \\ &\lesssim \delta^{-\frac{1}{q(x)}} |k|^{b(x)-m+n(1+\frac{1}{t})\tau^+} M_{Q_v^k}(|k|^{-n(1+\frac{1}{t})\tau(\cdot)} f_v) \\ &\lesssim \delta^{-\frac{1}{q(x)}} |k|^{-(2c_{\log}(\frac{d}{q\tau})+s)\frac{\tau(x)}{d}} M_{Q_v^k}(|k|^{-n(1+\frac{1}{t})\tau(\cdot)} f_v) \end{aligned}$$

for any $x \in Q_v$ and any $v \in \mathbb{N}_0$, where $\frac{1}{d} = \frac{1}{p(\cdot)\tau(\cdot)} + \frac{1}{t(\cdot)}$. Observe that $Q_v^k \subset Q(x, |k|2^{1-v}) = \tilde{Q}_v^k$. By Hölder's inequality,

$$|\tilde{Q}_v^k| M_{\tilde{Q}_v^k}(|k|^{-n(1+\frac{1}{t})\tau(\cdot)} |f_v|^{\frac{d}{\tau(\cdot)}}) \lesssim \left\| \frac{|f_v|^{\frac{1}{\tau(\cdot)}}}{|\tilde{Q}_v^k|} \chi_{\tilde{Q}_v^k} \right\|_{p(\cdot)\tau(\cdot)}^d \left\| |\tilde{Q}_v^k| |k|^{-n(1+\frac{1}{t})\tau(\cdot)} \chi_{\tilde{Q}_v^k} \right\|_{t(\cdot)}^d.$$

The second quasi-norm is bounded, while the first is bounded if and only if

$$\left\| \frac{f_v}{|\tilde{Q}_v^k|^{\tau(\cdot)}} \chi_{\tilde{Q}_v^k} \right\|_{p(\cdot)} \lesssim 1, \quad v \in \mathbb{N}_0,$$

which follows by (6.1). Again by Lemma 2.3,

$$(M_{\tilde{Q}_v^k}(|k|^{-n(1+\frac{1}{t})\tau(\cdot)} f_v))^{\frac{d}{\tau(x)}}$$

does not exceed

$$c M_{\tilde{Q}_v^k}(|k|^{-n(1+\frac{1}{t})\tau(\cdot)} |f_v|^{\frac{d}{\tau(\cdot)}}) + c \min(1, |k|^{ns} 2^{(1-v)ns}) ((e + |x|)^{-s} + M_{\tilde{Q}_v^k}((e + |\cdot|)^{-s}))$$

for any $s > 0$ large enough. Hence,

$$\delta^{-\frac{d}{q(x)\tau(x)}} (M_{\tilde{Q}_v^k}(|k|^{-n(1+\frac{1}{t})\tau(\cdot)} f_v))^{\frac{d}{\tau(x)}}$$

is bounded by

$$\begin{aligned} c |k|^{2c_{\log}(\frac{d}{q\tau})} M_{\tilde{Q}_v^k}(\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} |k|^{-n(1+\frac{1}{t})\tau(\cdot)} |f_v|^{\frac{d}{\tau(\cdot)}}) + 2^{\frac{vd}{q\tau}} \min(1, |k|^{ns} 2^{(1-v)ns}) h(x) \\ \lesssim |k|^{2c_{\log}(\frac{d}{q\tau})} M_{\tilde{Q}_v^k}(\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} |k|^{-n(1+\frac{1}{t})\tau(\cdot)} |f_v|^{\frac{d}{\tau(\cdot)}}) + |k|^{ns} h(x), \end{aligned}$$

where $s > 0$ large enough such that $s > \frac{d}{n(q\tau)^-}$. Therefore, the left-hand side of (6.6) is bounded by

$$c \left\| \mathcal{M} \left(|k|^{-nd} \frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} |f_v|^{\frac{d}{\tau(\cdot)}} \chi_{\tilde{Q}^k}}{|P|^d} \right) \right\|_{\frac{p(\cdot)\tau(\cdot)}{d}} + C \lesssim \left\| \frac{\delta^{-\frac{d}{q(\cdot)\tau(\cdot)}} |f_v|^{\frac{d}{\tau(\cdot)}} \chi_{\tilde{Q}^k}}{|\tilde{Q}^k|^d} \right\|_{\frac{p(\cdot)\tau(\cdot)}{d}} + C,$$

after using the fact that $\mathcal{M} : L^{\frac{p(\cdot)\tau(\cdot)}{d}} \rightarrow L^{\frac{p(\cdot)\tau(\cdot)}{d}}$ is bounded. The last norm is bounded if and only if

$$\left\| \frac{\delta^{-\frac{1}{q(\cdot)}} f_v \chi_{\tilde{Q}^k}}{|\tilde{Q}^k|^{\tau(\cdot)}} \right\|_{p(\cdot)} \leq 1,$$

which follows immediately from the definition of δ . Since m can be taken large enough, then the second sum in (6.5) is bounded by

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n, |k| > 4\sqrt{n}} |k|^{-b^- r} \left\| \left(\frac{f_v}{|\tilde{Q}^k|^{\tau(\cdot)}} \chi_{\tilde{Q}^k} \right)_{v \in \mathbb{N}_0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r &\leq \sum_{k \in \mathbb{Z}^n, |k| > 4\sqrt{n}} |k|^{-b^- r} \|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^r \\ &\lesssim 1. \end{aligned}$$

Case 2 $l(P) \leq 1$. As before,

$$\eta_{v,m} * |f_v|(x) \lesssim J_v^1(f_v \chi_{3P})(x) + \sum_{k \in \mathbb{Z}^n, \|k\|_\infty \geq 2} J_{v,k}^2(f_v \chi_{P+kl(P)})(x).$$

We see that

$$J_v^1(f_v \chi_{3P})(x) = \eta_{v,m} * (|f_v| \chi_{3P})(x), \quad x \in P$$

and since τ is log-Hölder continuous, we have

$$|P|^{-\tau(x)} \leq c |P|^{-\tau(y)} (1 + 2^{v_P} |x - y|)^{c_{\log}(\tau)} \leq c |P|^{-\tau(y)}$$

for any $x \in P$ any $y \in 3P$ and any $v \geq v_P$. Hence

$$|P|^{-\tau(x)} J_v^1(f_v \chi_{3P})(x) \lesssim \eta_{v,m-c_{\log}(\tau)} * (|P|^{-\tau(\cdot)} |f_v| \chi_{3P})(x), \quad x \in P.$$

Also, we have

$$|P|^{-\tau(x)} J_{v,k}^2(f_v \chi_{P+kl(P)})(x) \lesssim \eta_{v,m-c_{\log}(\tau)} * (|P|^{-\tau(\cdot)} |f_v| \chi_{P+kl(P)})(x).$$

As before, we obtain

$$\sum_{v=v_P}^{\infty} \left\| \left| \frac{c \eta_{v,m} * f_v}{|P|^{\tau(\cdot)}} \right|^{q(\cdot)} \chi_P \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$

where we did not need to use Lemma 2.3, which could be used only to move $|P|^{\tau(\cdot)}$ inside the convolution. The proof is complete. \square

Proof of Lemma 2.5 We claim that

$$2^{-v \frac{n}{r}} |\omega_v * f(x)| \lesssim \|\omega_v * f\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} \quad (6.7)$$

for any $x \in \mathbb{R}^n$, any $0 < r < p^-$ and any $v \in \mathbb{N}_0$. Indeed, by Lemma 2.2, we have

$$|\omega_v * f(x)| \leq c (\eta_{v,m} * |\omega_v * f|^r(x))^{1/r}$$

for any $x \in \mathbb{R}^n$, any $m > n$, $0 < r < p^-$ and any $v \in \mathbb{N}_0$. We write

$$\eta_{v,m} * |\omega_v * f|^r(x) \lesssim \sum_{i=0}^{\infty} 2^{-i(m-n)} M_{B(x, 2^{i-v})}(|\omega_v * f|^r),$$

where the implicit constant is independent of x and v . Hölder's inequality leads to

$$M_{B(x, 2^{i-v})}(|\omega_v * f|^r) \lesssim 2^{(v-i)n} \|(\omega_v * f) \chi_{B(x, 2^{i-v})}\|_{p(\cdot)}^r \| \chi_{B(x, 2^{i-v})} \|_{h(\cdot)}^r$$

$$\lesssim 2^{(v-i)n+inr\tau^+} \|\omega_v * f\|_{L_{p(\cdot)}^{\tau(\cdot)}}^r \|\chi_{B(x,2^i)}\|_{h(\cdot)}^r,$$

where $\frac{1}{r} = \frac{1}{p(\cdot)} + \frac{1}{h(\cdot)}$. Making m large enough (6.7) follows.

Let P be any dyadic cube. We use again Lemma 2.2, in the form

$$|\theta_v * \omega_v * f(x)| \leq c(\eta_{v,m} * |\omega_v * f|^r(x))^{1/r},$$

where $0 < r < \min(p^-, \frac{(p\tau)^-}{\tau^+})$, $m > n$ large enough and $x \in P$. By the scaling argument, we see that it suffices to prove that

$$\left\| \frac{\theta_v * \omega_v * f}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \lesssim 1$$

for any dyadic cube P , with $l(P) \geq 1$, whenever $\|\omega_v * f\|_{L_{p(\cdot)}^{\tau(\cdot)}} \leq 1$. Let $Q_v \subset P$ be a cube, with $l(Q_v) = 2^{-v}$ and $x \in Q_v \subset P$. As in Lemma 2.4,

$$\eta_{v,m} * |\omega_v * f|^r(x) \leq J_v^1(|\omega_v * f|^r \chi_{3Q_v})(x) + \sum_{k \in \mathbb{Z}^n, \|k\|_\infty \geq 2} J_{v,k}^2(|\omega_v * f|^r \chi_{Q_v^k})(x).$$

Thus we obtain

$$\begin{aligned} \left\| \frac{\theta_v * \omega_v * f}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)}^r &\lesssim \left\| \frac{J_v^1(|\omega_v * f|^r \chi_{3Q_v})}{|P|^{r\tau(\cdot)}} \chi_P \right\|_{\frac{p(\cdot)}{r}} \\ &+ \sum_{k \in \mathbb{Z}^n, \|k\|_\infty \geq 2} \left\| \frac{J_{v,k}^2(|\omega_v * f|^r \chi_{Q_v^k})}{|P|^{r\tau(\cdot)}} \chi_P \right\|_{\frac{p(\cdot)}{r}}. \end{aligned} \quad (6.8)$$

Let us prove that the first norm on the right-hand side is bounded. We have

$$J_v^1(|\omega_v * f|^r \chi_{3Q_v})(x) \lesssim M_{3Q_v}(|\omega_v * f|^r).$$

Let $d > 0$ be such that $\tau^+ < d < \frac{(p\tau)^-}{r}$. We have

$$\frac{M_{3Q_v}(|\omega_v * f|^r)}{|P|^{r\tau(\cdot)}} = \left(2^{v \frac{nd}{\tau(\cdot)}} \frac{M_{3Q_v}(2^{-vn} |\omega_v * f|^r)}{|P|^{dr}} \right)^{\frac{\tau(\cdot)}{d}}.$$

By (6.7), Lemma 2.3 and the fact that $2^{-\frac{vd}{\tau(x)}} \approx 2^{-\frac{vd}{\tau(y)}}$, $x, y \in 3Q_v$,

$$2^{v \frac{nd}{\tau(x)}} (M_{3Q_v}(2^{-vn} |\omega_v * f|^r))^{\frac{d}{\tau(x)}} \lesssim M_{3Q_v}(|\omega_v * f|^{\frac{rd}{\tau(\cdot)}}) + 2^{\frac{vnd}{\tau^-}} 2^{-snv} h(x)$$

for any $s > 0$ large enough and any $x \in Q_v$, where the implicit constant is independent of x and v . Hence

$$\begin{aligned} \left\| \left(\frac{J_v^1(|\omega_v * f|^r \chi_{3Q_v})}{|P|^{r\tau(\cdot)}} \chi_P \right)^{\frac{d}{\tau(\cdot)}} \right\|_{\frac{p(\cdot)\tau(\cdot)}{dr}} &\lesssim \left\| \mathcal{M} \left(\frac{|\omega_v * f|^{\frac{dr}{\tau(\cdot)}} \chi_{3Q_v}}{|P|^{rd}} \right) \right\|_{\frac{p(\cdot)\tau(\cdot)}{dr}} + c \\ &\lesssim \left\| \frac{|\omega_v * f|^{\frac{dr}{\tau(\cdot)}}}{|P|^{rd}} \chi_{3P} \right\|_{\frac{p(\cdot)\tau(\cdot)}{dr}} + c, \end{aligned}$$

after using the fact that $\mathcal{M} : L^{\frac{p(\cdot)\tau(\cdot)}{rd}} \rightarrow L^{\frac{p(\cdot)\tau(\cdot)}{rd}}$ is bounded. The last norm is bounded by 1 if and only if

$$\left\| \frac{\omega_v * f}{|P|^{\tau(\cdot)}} \chi_{3P} \right\|_{p(\cdot)} \lesssim 1.$$

Notice that $3P = \bigcup_{h=1}^{3^n} P_h$, where $\{P_h\}_{h=1}^{3^n}$ are disjoint dyadic cubes with side length $l(P_h) = l(P)$. Therefore $\chi_{3P} = \sum_{h=1}^{3^n} \chi_{P_h}$ and

$$\begin{aligned} \left\| \frac{\omega_v * f}{|P|^{\tau(\cdot)}} \chi_{3P} \right\|_{p(\cdot)} &\leq c \sum_{h=1}^{3^n} \left\| \frac{\omega_v * f}{|P_h|^{\tau(\cdot)}} \chi_{P_h} \right\|_{p(\cdot)} \\ &\lesssim \|\omega_v * f\|_{\widetilde{L_{\tau(\cdot)}^{p(\cdot)}}} \\ &\lesssim 1. \end{aligned}$$

Using a combination of the arguments used in the corresponding case of the proof of Lemma 2.4 and those used in the estimate of J_v^1 above, we arrive at the desired estimate. \square

Proof of Lemma 3.11 Obviously,

$$\|\lambda\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq \|\lambda_{r,d}^*\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}.$$

We will prove that

$$\|\lambda_{r,d}^*\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \lesssim \|\lambda\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}.$$

For each $k \in \mathbb{N}_0$ define

$$\Omega_k := \{h \in \mathbb{Z}^n : 2^{k-1} < 2^v |2^{-v}h - 2^{-v}m| \leq 2^k\}$$

and

$$\Omega_0 := \{h \in \mathbb{Z}^n : 2^v |2^{-v}h - 2^{-v}m| \leq 1\}.$$

Then for any $x \in Q_{v,m} \cap P$,

$$\sum_{h \in \mathbb{Z}^n} \frac{2^{vr(\alpha(x) + \frac{n}{2})} |\lambda_{v,h}|^r}{(1 + 2^v |2^{-v}h - 2^{-v}m|)^d} \quad (6.9)$$

can be rewritten as

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{h \in \Omega_k} \frac{2^{vr(\alpha(x) + \frac{n}{2})} |\lambda_{v,h}|^r}{(1 + 2^v |2^{-v}h - 2^{-v}m|)^d} \\ &\lesssim \sum_{k=0}^{\infty} 2^{-dk} \sum_{h \in \Omega_k} 2^{vr(\alpha(x) + \frac{n}{2})} |\lambda_{v,h}|^r \\ &= c \sum_{k=0}^{\infty} 2^{(n-d)k + (v-k)n + vr(\alpha(x) + \frac{n}{2})} \int_{\bigcup_{z \in \Omega_k} Q_{v,z}} \sum_{h \in \Omega_k} |\lambda_{v,h}|^r \chi_{v,h}(y) dy. \end{aligned} \quad (6.10)$$

Let $x \in Q_{v,m} \cap P$ and $y \in \bigcup_{z \in \Omega_k} Q_{v,z}$. Then $y \in Q_{v,z}$ for some $z \in \Omega_k$ and $2^{k-1} < 2^v |2^{-v}z - 2^{-v}m| \leq 2^k$. From this it follows that y is located in the cube $Q(x, 2^{k-v+3})$. Therefore, (6.10) does not exceed

$$\begin{aligned} &c \sum_{k=0}^{\infty} 2^{(n-d+a)k + (v-k)n} \int_{Q(x, 2^{k-v+3})} 2^{v(\alpha(y) + \frac{n}{2})r} \sum_{h \in \Omega_k} |\lambda_{v,h}|^r \chi_{v,h}(y) dy \\ &= c \sum_{k=0}^{\infty} 2^{(n-d+a)k} M_{Q(x, 2^{k-v+3})}(g_v) \end{aligned}$$

for some positive constant c independent of v and k and

$$g_v = 2^{v(\alpha(\cdot) + \frac{n}{2})r} \sum_{h \in \mathbb{Z}^n} |\lambda_{v,h}|^r \chi_{v,h}, \quad v \geq v_P^\dagger.$$

Observe that

$$M_{Q(x, 2^{k-v+3})}(g_v) \lesssim 2^{kL} \eta_{v,L} * g_v(x)$$

for any $x \in Q_{v,m} \cap P$ and any $L > n$ large enough, where the implicit constant is independent of x, k and v . Therefore (6.9) is bounded by

$$c \eta_{v,L} * g_v(x), \quad x \in Q_{v,m} \cap P.$$

Thanks to Lemma 2.4, we have

$$\begin{aligned} \|\lambda_{r,d}^*\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} &\lesssim \|(\eta_{v,L} * g_v)_v\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{r\tau(\cdot)}{p(\cdot)}})}^{\frac{1}{r}} \\ &\lesssim \|(g_v)_v\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{r\tau(\cdot)}{p(\cdot)}})}^{\frac{1}{r}} \\ &\lesssim \|\lambda\|_{\mathfrak{b}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}, \end{aligned}$$

provided that d is sufficiently large such that $d > n + a + L$. The proof of the lemma is thus complete. \square

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