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# Filters and ideals in a fuzzy lattice

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# Filters and ideals in a fuzzy lattice

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02 July 2019

بسم الله الرحمن الرحيم

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# Introduction

The main objective of this memory is to improve our knowledge about the notions of filters and ideals in a fuzzy lattice.

Some specific objectives are:

- Study fuzzy lattices as fuzzy relations and give some proof of their basic properties.

- Extends or translates some notions concerning fuzzy filters in a fuzzy lattice to fuzzy ideals in a fuzzy lattice.

- Recall the notion of isomorphism between fuzzy lattices, also characterize the converse image of a fuzzy filters and ideals via an isomorphism.

Fuzzy sets and fuzzy relations have been introduced by Zadeh [17, 18]. In 1990, Yuan and Wu [16] introduced the concepts of fuzzy sublattices and fuzzy ideals of a lattice. Ajmal and Thomas [1] defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. Chon in [5] characterized a fuzzy partial order relation using their $\alpha$-cuts and defined a fuzzy lattice as a fuzzy relation, developed some basic properties of fuzzy lattices, characterized a fuzzy lattice using their $\alpha$-cuts, and showed that a fuzzy totally ordered set is a distributive fuzzy lattice. Mezzomo [9, 10, 11] characterized the fuzzy lattice through a fuzzy partial order relation, he defined ideals, filters, $\alpha$-ideals, $\alpha$-filters, fuzzy ideals and fuzzy filters of fuzzy lattices and he characterized theme using their $\alpha$-cuts and show that a subset of a fuzzy lattice is a fuzzy ideal if and only if its $\alpha$-cuts are a crisp ideals. In [8], Mezzomo define some kinds of ideals and filters of fuzzy lattice. More recently, Amroune, Oumhani and Davvaz [2, 3] characterized fuzzy filters and fuzzy t-filters using their $\alpha$-cuts, and they introduced the notion of fuzzy lattices isomorphism (notion can be found also in [15]).

This memory is organized as follows:

In the first chapter, we recall some definitions and well-known results on crisp ordered sets and crisp lattices, this chapter also focuses on fuzzy sets and fuzzy relations, fuzzy lattices and related concepts.

The second chapter, we study and characterize diverse types of filters, fuzzy filters, fuzzy prime filters, $\alpha$-filters and fuzzy $\alpha$-filters. Also, we do the same study for ideals and we study the converse image of fuzzy prime filters and ideals via fuzzy lattices isomorphism.

# Chapter 1

# Generalities on Orders and Lattices

In this chapter, we recall some basic notions and definitions of orders, lattices, fuzzy relations, some of their characteristics and related concepts.

Basic references for this chapter are [5, 6, 9, 17].

## 1.1 Partial Orders and Homomorphisms

### 1.1.1 Partial Orders

**Definition 1.1.1** *Let $X$ be a non-empty set. A binary relation $R$ on $X$ is any subset of $X \times X$.*

*A binary relation will be denoted by $\leq$ .*

**Definition 1.1.2** *A binary relation is said to be*

**(i)** *Reflexive if $x \leq x$, for all $x \in X$.*

**(ii)** *Symmetric if $x \leq y$ implies $y \leq x$, for all $x, y \in X$.*

**(iii)** *Antisymmetric if $x \leq y$ and $y \leq x$ imply $x = y$, for all $x, y \in X$.*

**(iv)** *Transitive if $x \leq y$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in X$.*

**Definition 1.1.3 (Partial order)** *[6] Let $X$ be a non-empty set. An* ***order*** *(or partial order) on $X$ is a binary relation $\leq$ on $X$ which is reflexive, antisymmetric and transitive.*

**Definition 1.1.4** *A set $X$ equipped with an order relation $\leq$ is said to be an* ***ordered set*** *(or partially ordered set). Some authors use the shorthand* ***poset*** *and denoted by $(X, \leq)$.*

**Example 1.1.1**

1. *The relation of divisibility ($x \Re y \Leftrightarrow x \mid y$) is a partial order on $\mathbb{N}^*$, then the couple $(\mathbb{N}^*, |)$ is a poset.*
2. *For all a set $X$, $(\wp(X), \subseteq)$ is a poset, with $\wp(X)$ is the family of parts of $X$.*

**Definition 1.1.5 (Chains and antichains)** *[6] Let $(X, \leq)$ be an ordered set. $X$ is a* ***chain*** *if, for all $x, y \in X$, either $x \leq y$ or $y \leq x$ (that is, if any two elements of $X$ are comparable). Alternative names for a chain are linearly ordered set and totally ordered set. At the opposite extreme from a chain is an antichain. The ordered set $X$ is an* ***antichain*** *if $x \leq y$ in $X$, this implies $x = y$ (that is, if any two elements of $X$ are incomparable).*

**Example 1.1.2**

1. *The sets* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ *equipped with the usual order* $\leqslant$, *are a chains.*
2. *The set* $D(8) = \{1, 2, 4, 8\}$ *of all divisors of the integer* $8$ *equipped with the relation divide* $|$ *is a chain.*

**Remark 1.1.1**

- *The only ordered sets that are both chain and antichain are the singletons.*
- *In a chain:* $x \nleq y \Leftrightarrow y < x$.

**Definition 1.1.6 (Bottom and top)** *[6] Let* $X$ *be an ordered set.*

- *We say* $X$ *has a* ***bottom element*** *if there exists* $\mathbf{0} \in X$ *(called bottom) with the property that* $\mathbf{0} \leq x$ *for all* $x \in X$.
- *Dually,* $X$ *has a* ***top element*** *if there exists* $\mathbf{1} \in X$ *such that* $x \leq \mathbf{1}$ *for all* $x \in X$.

*Note that* $\mathbf{0}$ *and* $\mathbf{1}$ *are unique when they exist. (The uniqueness comes from the antisymmetry of* $\leq$*).*

**Example 1.1.3** *In the ordered set* $(\wp(X), \subseteq)$, *we have* $\mathbf{0} = \emptyset$ *and* $\mathbf{1} = X$.

**Definition 1.1.7** *[6] Let* $X$ *be an ordered set and let* $A \subseteq X$.

- *An element* $u \in X$ *is an* ***upper bound*** *of* $A$ *if* $a \leq u$ *for all* $a \in A$.
- *An element* $l \in X$ *is a* ***lower bound*** *of* $A$ *if* $l \leq a$ *for all* $a \in A$.

  *The set of all upper bounds of* $A$ *is denoted by* $A^u$ *and the set of all lower bounds by* $A^l$ $A^u = \{u \in X / (\forall a \in A) a \leq u\}$ *and* $A^l = \{l \in X / (\forall a \in A) a \geq l\}$.

- $u_0$ *is the* ***least upper bound*** *of* $A$ *if*

  **(i)** $u_0$ *is an upper bound of* $A$*, that is,* $u_0 \in A^u$ *and*

  **(ii)** $u_0 \leq u$ *for all upper bounds* $u \in A^u$.

- $l_0$ *is the* ***greatest lower bound*** *of* $A$ *if*

  **(i)** $l_0$ *is a lower bound of* $A$, *that is,* $l_0 \in A^l$ *and*

  **(ii)** $l \leq l_0$ *for all lower bounds* $l \in A^l$.

*A least upper bound of* $A$ *will be denoted by* $\sup A$ *and a greatest lower bound by* $\inf A$. *We denote the least upper bound of the set* $\{x, y\}$ *by* $x \vee y$ *and denote the greatest lower bound of the set* $\{x, y\}$ *by* $x \wedge\ y$.

**Example 1.1.4** *Consider the ordered set* $(\mathbb{N}^*, |)$ *and Let* $A = \{2, 3, 4, 9\} \subset \mathbb{N}^*$. *We have*
$A^u = \{36z, z \in \mathbb{N}^*\}$, $A^l = \{1\}$ . *Hence*
$\sup A = 36$, $\inf A = 1$.

**Definition 1.1.8** *[6, 13, 14] Let* $X$ *and* $Y$ *be ordered sets, a map* $h : X \longrightarrow Y$ *is said to be*

**(i)** *an order-preserving (or, alternatively, monotone) if for all* $x, y \in X$, $x \leq y$ *in* $X$ *implies* $h(x) \leq h(y)$ *in* $Y$*;*

**(ii)** *an order-embedding (and we write* $h : X \hookrightarrow Y$*) if for all* $x, y \in X$, $x \leq y$ *in* $X$ *if and only if* $h(x) \leq h(y)$ *in* $Y$*;*

**(iii)** *an order-isomorphism if it is an order-embedding which maps* $X$ *onto* $Y$.

**Example 1.1.5** *The map* $h : (D(6), |) \longrightarrow (D(30), |)$ *defined by* $h(x) = x$, *for all* $x \in D(6)$ *is an order-preserving.*

### 1.1.2 Lattices and Homomorphisms

**Definition 1.1.9** *[6] Let* $X$ *be a non-empty ordered set.*

**(i)** $(X, \leq)$ *is called a* ***lattice*** *if* $x \vee y$ *and* $x \wedge\ y$ *exist, for all* $x, y \in X$;

**(ii)** $(X, \leq)$ *is called a* ***complete lattice*** *if* $\sup A$ *and* $\inf A$ *exist, for all* $A \subseteq X$.

**Example 1.1.6**

**(a)** *For any set* $X$*, the ordered set* $(\wp(X), \subseteq)$ *is a complete lattice in which*

$\sup\{A_i / i \in I\} = \cup\{A_i / i \in I\}$,

$\inf\{A_i / i \in I\} = \cap\{A_i / i \in I\}$.

**(b)** $([0,1], \leq, \max, \min)$ *is a complete lattice.*

**Remark 1.1.2** *[6] Let* $L$ *be a lattice. For all* $x, y, z, t \in L$*,*

**(i)** $x \leq y$ *implies* $x \vee z \leq y \vee z$ *and* $x \wedge z \leq y \wedge z$*,*

**(ii)** $x \leq y$ *and* $z \leq t$ *imply* $x \vee z \leq y \vee t$ *and* $x \wedge z \leq y \wedge t$*.*

**Lemma 1.1.1** *[6] Let* $L$ *be a lattice and let* $x, y \in L$*. The following are equivalent:*

**(i)** $x \leq y$;

**(ii)** $x \vee y = y$;

**(iii)** $x \wedge \ y = x$.

**Theorem 1.1.1** *[6] Let* $L$ *be a lattice. Then* $\vee$ *and* $\wedge$ *satisfy, for all* $x, y, z \in L$*,*

**(L1)** $(x \vee y) \vee z = x \vee (y \vee z)$; *(associative laws )*

**(L1)**$^\partial$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;

**(L2)** $x \vee y = y \vee x$; *(commutative laws )*

**(L2)**$^\partial$ $x \wedge y = y \wedge x$;

**(L3)** $x \vee x = x$; *(idempotency laws )*

**(L3)**$^\partial$ $x \wedge x = x$;

**(L4)** $x \vee (x \wedge y) = x$; *(absorption laws )*

**(L4)**$^\partial$ $x \wedge (x \vee y) = x$.

**Proof.**

**(L1)** It suffices to prove that $(x \vee y) \vee z = \sup\{x, y, z\}$. This is the case if $\{x \vee y, z\}^u = \{x, y, z\}^u$. But
$$\begin{aligned} t \in \{x, y, z\}^u &\Leftrightarrow t \in \{x, y\}^u \text{ and } t \geq z \\ &\Leftrightarrow t \geq x \vee y \text{ and } t \geq z \\ &\Leftrightarrow t \in \{x \vee y, z\}^u. \end{aligned}$$

**(L2)** Immediate since, for any set $S$, $\sup S$ is independent of the order in which the elements of $S$ are listed.

**(L3)** Straightforward.

**(L4)** Easily from Lemma 1.1.1, since $x \wedge y \leq x$ and $x \leq x \vee y$.

The same proof for $(L1)^\partial, (L2)^\partial, (L3)^\partial$ and $(L4)^\partial$. ■

**Definition 1.1.10** *A lattice $L$ which has a top and a bottom elements is called* ***bounded lattice****.*

**Definition 1.1.11 (Sublattices)** *Let $L$ be a lattice and let $M$ be a non-empty subset of $L$. $M$ is a* ***sublattice*** *of $L$ if for all $x, y \in L, x, y \in M$ implies $x \vee y \in M$ and $x \wedge y \in M$.*

**Example 1.1.7** *Any one-element subset of a lattice is a sublattice. More generally, any non-empty chain in a lattice is a sublattice.*

**Definition 1.1.12 (Homomorphisms)** *[6] Let $L$ and $K$ be lattices. A map $h : L \to K$ is said to be a* ***homomorphism*** *(or, for emphasis, lattice homomorphism) if, for all $x, y \in L$, it follows that*

**(1)** $h(x \vee y) = h(x) \vee h(y)$;

**(2)** $h(x \wedge y) = h(x) \wedge h(y)$.

*A bijective homomorphism is a lattice* ***isomorphism****.*

### 1.1.3 Filters and Ideals in a Lattice

Hereunder, we recall some definitions and notions of filters and ideals in a lattice.

**Definition 1.1.13** *Let $L$ be a lattice. A non-empty subset $F$ of $L$ is called a* ***filter*** *if for all $x, y \in L$*

**(i)** *If $y \in F$ and $y \leq x$, then $x \in F$,*

**(ii)** *$x, y \in F$ implies $x \wedge y \in F$.*

Using the duality principle we define an ideal as:

**Definition 1.1.14** *Let $L$ be a lattice. A non-empty subset $I$ of $L$ is called an* ***ideal*** *if for all $x, y \in L$*

**(i)** *If $y \in I$ and $x \leq y$, then $x \in I$,*

**(ii)** *$x, y \in I$ implies $x \vee y \in I$.*

**Example 1.1.8** *In $(D(30), |)$ where $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$, the subset $F = \{2, 6, 10, 30\}$ is a filter whereas $I = \{1, 3, 5, 15\}$ is an ideal.*

**Definition 1.1.15** *A filter or ideal is called* ***proper*** *if it does not coincide with $L$.*

**Remark 1.1.3** *Let $L$ be a bounded lattice. It is easy to show that a filter $F$ of $L$ is proper if and only if $\mathbf{0} \notin F$, and dually, an ideal $I$ of $L$ is proper if and only if $\mathbf{1} \notin I$.*

**Definition 1.1.16** *Let $L$ be a lattice. A proper filter $F$ of $L$ is* ***prime*** *if and only if for all $x, y \in L$, $x \vee y \in F$ imply that $x \in F$ or $y \in F$.*

*Dually, A proper ideal $I$ of $L$ is* ***prime*** *if and only if $x, y \in L$ and $x \wedge y \in I$ imply that $x \in I$ or $y \in I$.*

**Definition 1.1.17** *For each $x \in L$, the set $\uparrow x = \{y \in L / x \leq y\}$ is a filter, it is known as the* ***principal filter*** *generated by $x$. Dually, $\downarrow x = \{y \in L / y \leq x\}$ is a* ***principal ideal****.*

**Definition 1.1.18** *A proper filter $F$ of $L$ is said to be a* ***maximal filter*** *if the only filter properly containing $F$ is $L$. In other words, $F$ is a* ***maximal filter*** *if for each filter $X$ of $L$, $F \subseteq X \subseteq L$ imply that $X = F$ or $X = L$. A* ***maximal ideal*** *is defined dually.*

## 1.2 Fuzzy Orders

### 1.2.1 Fuzzy Sets

**Definition 1.2.1** *[17] A **fuzzy subset** $A$ in $X$ is characterized by an application $\mu_A$ : $X \longrightarrow [0,1]$, where $[0,1]$ means real numbers between $0$ and $1$.*

*If $x$ an element of $X$, $\mu_A(x)$ is the degree of membership of $x$.*

*The fuzzy set $A$ in $X$ may be represented as a set of ordered pairs of generic element $x \in X$ and its grade of membership, i.e., $A = \{(x, \mu_A(x)) \diagup x \in X\}$.*

**Example 1.2.1** *Let $X = \{a, b, c\}$ be a universal set. $A_1 = \{(a, 0.3), (b, 1.0), (c, 0.7)\}$ and $A_2 = \{(a, 0.0), (b, 0.9), (c, 0.7)\}$ be a fuzzy subsets in $X$.*

**Notation 1.2.1** *Let $X$ a non-empty set. The set of all fuzzy subset of $X$ will be denoted by $F(X)$.*

**Definition 1.2.2** *[18] The **support** of a fuzzy set $A$, denoted by $S(A)$, we mean all elements of $X$ that belong to $A$ to a nonzero degree. That is, $S(A)$ is a classical set defined by*

$$S(A) = \{x \in X : \mu_A(x) > 0\}.$$

**Example 1.2.2** *$S(A_1) = \{a, b, c\}$ and $S(A_2) = \{b, c\}$.*

### 1.2.2 Fuzzy Sets Operations

The notion related by the operations on fuzzy sets are defined as follows [1, 17].

**Definition 1.2.3 (Equality and inclusion of a fuzzy sets)** **(a)** *Let $X$ be a non-empty set and let $A$, $B \in F(X)$, we say that $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all $x$ in $X$.*

**(b)** *We say that $A$ is included in $B$ if and only if $\mu_A(x) \leq \mu_B(x)$. In symbols $A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x)$, for all $x \in X$.*

*When a fuzzy set $A$ is included in $B$, then $A$ is called a fuzzy subset of $B$.*

**Definition 1.2.4 (Intersection and union of a fuzzy sets)** *Let $X$ be a non-empty set and let $A,\ B \in F\left(X\right)$. For all $x \in X$, the operations of union and intersection are defined as:*

$$\begin{array}{lrcl} \textit{Union:} & \mu_{A\cup B}(x) & = & \max\left\{\mu_A(x), \mu_B(x)\right\} \\ & & = & \mu_A(x) \vee \mu_B(x); \\ \textit{Intersection:} & \mu_{A\cap B}(x) & = & \min\left\{\mu_A(x), \mu_B(x)\right\} \\ & & = & \mu_A(x) \wedge \mu_B(x); \end{array}$$

**Proposition 1.2.1** *Let $X$ be a non-empty set and let $A, B, C \in F(X)$, the following properties hold:*

- $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cup C) = (A \cup B) \cup C$.
- $A \cap B = B \cap A$, $A \cup B = B \cup A$.
- $A \cup \emptyset = A$, $A \cup X = X$.
- $A \cap \emptyset = \emptyset$, $A \cap X = A$.
- $A \cup B \supseteq A \supseteq A \cap B$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

**Definition 1.2.5 (Complement of a fuzzy set)** *The complement of a fuzzy set $A$ is denoted by $CA$ and is defined by $\mu_{CA} = 1 - \mu_A$, For all $x \in X$.*

**Proposition 1.2.2** *Let $X$ be a non-empty set and let $A, B, \in F(X)$, the complement of fuzzy sets verify the following properties:*

- $C\left(A \cap B\right) = CA \cup CB$.
- $C\left(A \cup B\right) = CA \cap CB$.
- $C\left(CA\right) = A$.

**Definition 1.2.6 (Cartesian product on fuzzy set)** *[7] The cartesian product applied to n fuzzy sets can be defined as follows: Let* $\mu_{A_1}, \mu_{A_2}, ..., \mu_{A_n}$ *be membership functions of* $A_1, ..., A_n$. *Then, the membership degree of* $(x_1, ..., x_n) \in X_1 \times ... \times X_n$ *on the fuzzy set* $A_1 \times ... \times A_n$ *is,*

$$\mu_{A_1 \times ... \times A_n}(x_1, ..., x_n) = \min \left\{ \mu_{A_1}(x_1), ..., \mu_{A_n}(x_n) \right\}.$$

**Definition 1.2.7 (The $\alpha$-cuts of a fuzzy set)** *Let* $A$ *be a fuzzy set in* $X$ *and let* $\alpha \in ]0,1]$*,The* $\alpha$*-cut of* $A$*, denoted by* $A_\alpha$*, we mean all elements of* $X$ *that belong to* $A$ *to a degree of at least* $\alpha$. *That is,* $A_\alpha$ *is a classical set defined by*

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}.$$

*If* $\alpha \leq \beta$*, then* $A_\beta \subseteq A_\alpha$.

**Proposition 1.2.3** *Let* $X$ *be a non-empty set and let* $A, B, \in F(X)$*,the* $\alpha$*-cuts of fuzzy sets verify:*

- $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.
- $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.
- $A \subset B \Longleftrightarrow A_\alpha \subset B_\alpha$*, for all* $\alpha \in ]0,1]$.

### 1.2.3 Fuzzy Relations

**Definition 1.2.8** *[18] Let* $X$ *be non-empty set, a fuzzy relation on* $X$*, denoted by* $R$*, is defined as the fuzzy set*

$$R = \left\{ ((x,y), \mu_R(x,y)) : (x,y) \in X^2 \right\},$$

*where the function* $\mu_R : X \times X \longrightarrow [0,1]$ *is called membership function. It gives the degree of membership of the ordered pair* $(x,y)$ *in* $R$ *associating with each pair* $(x,y)$ *in* $X^2$ *a real number in interval* $[0,1]$.

*Two fuzzy relations* $R$ *and* $\Re$ *are equal if and only if, for every pair* $(x,y) \in X^2$,

$$R(x,y) = \Re(x,y).$$

**Remark 1.2.1** *Since a fuzzy relations is a fuzzy set, then the $\alpha$-cut sets and support of fuzzy relations is defined as in fuzzy sets, i.e., the $\alpha$-cut of a fuzzy relation $R : X \times X \longrightarrow [0,1]$ is defined as, for all $x, y \in X$,*

$$R_\alpha = \left\{ (x,y) \in X^2 : R(x,y) \geq \alpha \right\}.$$

*In the same way, we define the support of a fuzzy relation $S(R)$ as*

$$S(R) = \left\{ (x,y) \in X^2 : R(x,y) > 0 \right\}.$$

**Definition 1.2.9** *Let $X$ be a non-empty set, a mapping $R : X \times X \longrightarrow [0,1]$ is a fuzzy binary relation. We say*

**(i)** $R$ is a **fuzzy reflexive** relation if $R(x,x) = 1$, for all $x \in X$;

**(ii)** $R$ is a **fuzzy symmetric** relation if $R(x,y) = R(y,x)$, for all $x, y \in X$;

**(iii)** $R$ is a **fuzzy antisymmetric** relation if $R(x,y) > 0$ and $R(y,x) > 0$ implies $x = y$, for all $x, y \in X$;

**(iv)** $R$ is a **fuzzy transitive** relation if $R(x,z) \geq \sup\limits_{y \in X} \min \{R(x,y), R(y,z)\}$, for all $x, y \in X$.

**Remark 1.2.2** *[9] When R is reflexive, then the transitivity can be rewritten by*

*$R$ is transitive if and only if $R(x,z) = \sup\limits_{y \in X} \min \{R(x,y), R(y,z)\}$, for all $x, y, z \in X$.*

*First, we know that $R(x,z) \geq \sup\limits_{y \in X} \min \{R(x,y), R(y,z)\}$, for all $x, y, z \in X$. On the other hand,*

$$\begin{aligned} \sup_{y \in X} \min \{R(x,y), R(y,z)\} &\geq \min \{R(x,x), R(x,z)\} \\ &= \min \{1, R(x,z)\} \\ &= R(x,z). \end{aligned}$$

*Therefore, we have that $R(x,z) = \sup\limits_{y \in X} \min \{R(x,y), R(y,z)\}$.*

**Definition 1.2.10** *[5] Let $X$ be a non-empty set.*

*A binary fuzzy relation $R : X \times X \longrightarrow [0,1]$ is called a* ***fuzzy equivalence relation*** *in $X$ if $R$ is reflexive, transitive and symmetric.*

*A fuzzy relation $R$ is a* ***fuzzy partial order relation*** *if $R$ is reflexive, antisymmetric and transitive.*

*A fuzzy partial order relation $R$ is a* ***fuzzy total order relation*** *iff either $R(x,y) > 0$ or $R(y,x) > 0$ for all $x, y \in X$.*

*If $R$ is a fuzzy partial order relation on a set $X$, then $(X, R)$ is called a* ***fuzzy partially ordered set*** *or* ***fuzzy poset****. If $R$ is a fuzzy total order relation in a set $X$, then $(X, R)$ is called* ***fuzzy totally ordered set*** *or a* ***fuzzy chain****.*

**Example 1.2.3** *Let $X = \{x, y, z\}$ and let $R : X \times X \longrightarrow [0, 1]$ be a fuzzy relation such that $R(x, x) = R(y, y) = R(z, z) = 1$, $R(x, y) = R(x, z) = R(y, z) = 0$, $R(y, x) = 0.3$, $R(z, x) = 0.2$, and $R(z, y) = 0.6$. Then it is easily checked that $R$ is a fuzzy total order relation.*

**Proposition 1.2.4** *Let $(X, R)$ be a fuzzy poset, $\alpha \in ]0, 1]$ and $x, y, z \in X$. If $R(x, y) > \alpha$ and $R(y, z) > \alpha$, then $R(x, z) > \alpha$.*

**Proof.** Suppose $\alpha \in ]0, 1]$ such that $R(x, y) > \alpha$ and $R(y, z) > \alpha$.

Then, $\min\{R(x,y), R(y,z)\} \geq \alpha$.

So, $\sup_{y \in X} \min\{R(x,y), R(y,z)\} \geq \min\{R(x,y), R(y,z)\} > \alpha$. Therefore, by definition of fuzzy transitivity, $R(x, z) > \alpha$. ■

**Corollary 1.2.1** *[11] Let $(X, R)$ be a fuzzy poset and $x, y, z \in X$. If $R(x, y) > 0$ and $R(y, z) > 0$, then $R(x, z) > 0$.*

## 1.3 Fuzzy Lattices

### 1.3.1 Notions and Definitions

In the following, we give some elementary notions on fuzzy lattices which we need in the sequel.

**Definition 1.3.1** *[5] Let $(X, R)$ be a fuzzy poset, and let $A$ be a non-empty subset of $X$. An element $u \in X$ is said to be an* ***upper bound*** *of a subset $A$ if and only if $R(a, u) > 0$, for all*

*$a \in A$. An upper bound $u_0$ of $A$ is the* ***least upper bound*** *of $A$ if and only if $R(u_0, u) > 0$, for every upper bound $u$ of $A$. An element $l \in X$ is said to be a* ***lower bound*** *of a subset $A$ if and only if $R(l, a) > 0$, for all $a \in A$. A lower bound $l_0$ of $A$ is the* ***greatest lower bound*** *of $A$ if and only if $R(l, l_0) > 0$, for every lower bound $l$ of $A$.*

*A least upper bound of $A$ will be denoted by $supA$ and a greatest lower bound by $infA$. We denote the least upper bound of the set $\{x, y\}$ by $x \sqcup y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \sqcap y$.*

**Remark 1.3.1** *[9] Since $R$ is antisymmetric, then the least upper (greatest lower) bound of $A \subseteq X$, if it exists, is unique.*

*Indeed, suppose that $u_0$ and $u_1$ be two least upper bounds of $A$. Then $R(u_0, u_1) > 0$ and $R(u_1, u_0) > 0$. Therefore, by the antisymmetry, $u_0 = u_1$. Similarly, we prove that $infA$ is unique.*

**Definition 1.3.2** *[5] A fuzzy poset $(X, R)$ is a* ***fuzzy lattice*** *if and only if $x \sqcup y$ and $x \sqcap y$ exist for all $x, y \in X$.*

**Example 1.3.1** *Let $X = \{x, y, z\}$ and let $R : X \times X \to [0, 1]$ be a fuzzy relation such that $R(x, x) = R(y, y) = R(z, z) = 1, R(x, y) = R(x, z) = R(y, z) = 0, R(y, x) = 0.5, R(z, x) = 0.3$, and $R(z, y) = 0.2$. It is easy to check that $R$ is a fuzzy partial order relation. Also, we can calculate the calley tables*

| $\sqcup$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | $x$ | $x$ | $x$ |
| $y$ | $x$ | $y$ | $y$ |
| $z$ | $x$ | $y$ | $z$ |

,

| $\sqcap$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | $x$ | $y$ | $z$ |
| $y$ | $y$ | $y$ | $z$ |
| $z$ | $z$ | $z$ | $z$ |

.

*Thus, $(X, R)$ is a fuzzy lattice.*

### 1.3.2 Fuzzy Lattices Properties

**Proposition 1.3.1** *[9] Let $(X, R)$ be a fuzzy poset and $Y \subseteq X$. If $\Re = R|_{Y \times Y}$, that is, $\Re$ is a fuzzy relation on $Y$ such that for all $x, y \in Y$, $\Re(x, y) = R(x, y)$, then $(Y, \Re)$ is a fuzzy poset.*

**Lemma 1.3.1** *[9] Let $R : X \times X \to [0,1]$ be a fuzzy relation. If $R$ is a fuzzy partial order relation on $X$, then $S(R)$ is a partial order relation on $X$.*

**Proof.** Let $R$ be a fuzzy partial order relation on $X$.

Since $R(x,x) = 1$ for all $x \in X$, $(x,x) \in S(R)$.

Then, $S(R)$ is reflexive.

Suppose $(x,y) \in S(R)$ and $(y,x) \in S(R)$. Then, $R(x,y) > 0$ and $R(y,x) > 0$, hence $x = y$.

Then, $S(R)$ is antisymmetric.

Suppose $(x,y) \in S(R)$ and $(y,z) \in S(R)$. Then, $R(x,y) > 0$ and $R(y,z) > 0$. Since

$$\begin{aligned} R(x,z) &\geq \sup_{y \in X} \min\left[R(x,y), R(y,z)\right] \\ &\geq \min\left[R(x,y), R(y,z)\right] > 0. \end{aligned}$$

That is, $(x,z) \in S(R)$.

Then, $S(R)$ is transitive.

Hence, $S(R)$ is a partial order relation on $X$. ■

**Proposition 1.3.2** *[9] Let $R : X \times X \to [0,1]$ be a fuzzy relation. $R$ is a fuzzy partial order relation on $X$ if and only if $R_\alpha$ are partial order relations on $X$, for all $\alpha \in \left]0,1\right]$.*

**Proof.** For the direct implication. Let $R$ be a fuzzy partial order relation on $X$.

Since $R(x,x) = 1$ for all $x \in X$, then $(x,x) \in R_\alpha$ for all $\alpha \in \left]0,1\right]$. Then, $R_\alpha$ is reflexive.

Suppose $(x,y) \in R_\alpha$ and $(y,x) \in R_\alpha$. Then, $R(x,y) \geq \alpha > 0$ and $R(y,x) \geq \alpha > 0$, by the fact that $R$ is fuzzy antisymmetric, $x = y$. Then, $R_\alpha$ is antisymmetric.

Suppose $(x,y) \in R_\alpha$ and $(y,z) \in R_\alpha$. Then, $R(x,y) \geq \alpha$ and $R(y,z) \geq \alpha$. Since $R(x,z) \geq \sup_{y \in X} \min\{R(x,y), R(y,z)\}$, $R(x,z) \geq \min\{R(x,y), R(y,z)\} \geq \alpha$, that is, $(x,z) \in R_\alpha$. Then, $R_\alpha$ is transitive.

Hence, $R_\alpha$ are partial order relations on $X$.

For the converse implication. Suppose that $R_\alpha$ is a partial order for all $\alpha \in \left]0,1\right]$. Then, $(x,x) \in R_\alpha$ for all $\alpha \in \left]0,1\right]$. Thus $(x,x) \in R_1$, that is, $R(x,x) = 1$. Then, $R$ is reflexive.

Suppose that $R(x,y) > 0$ and $R(y,x) > 0$. Then, $R(x,y) > v > 0$ for some $v \in \left]0,1\right]$ and $R(y,x) > w > 0$ for some $w \in \left]0,1\right]$. Take $u = \min(v,w)$. Then, $R(x,y) > u > 0$ and

$R(y,x) > u > 0$. Thus, $(x,y),(y,x) \in R_u$. Since $R_u$ is antisymmetric, $x = y$. Then, $R$ is antisymmetric.

Let $x,y,z \in X$ and $\alpha = \min(R(x,y),R(y,z))$. So, if $\alpha = 0$ the problem is solved. If $\alpha \neq 0$, we have $(x,y),(y,z) \in R_\alpha$, since $R_\alpha$ is a partial order, then $(x,z) \in R_\alpha$. Therefore, $R(x,z) \geq \alpha = \min\{R(x,y),R(y,z)\}$, for all $y \in X$ and Therefore, $R(x,z) \geq \sup\limits_{y \in X}\min\{R(x,y),R(y,z)\}$, that is, $R$ is fuzzy transitive relation.

Hence, $R_\alpha$ are partial order relations on $X$. ■

**Proposition 1.3.3** *[5] Let $(X,R)$ be a fuzzy poset. If $(X,R_\alpha)$ are lattices for all $\alpha \in ]0,1]$, then $(X,R)$ is a fuzzy lattice.*

**Proof.** Let $(X,R)$ is a fuzzy poset. Suppose that $(X,R_\alpha)$ is a crisp lattice, for all $\alpha \in ]0,1]$.

For all $x,y \in X$, if $R(x,y) > 0$ or $R(y,x) > 0$ the problem is solved i.e., $x \vee y$ exists and $x \vee y = x$ or $y$.

Otherwise $R(x,y) = 0$ and $R(y,x) = 0$. Since $(X,R_\alpha)$ is a lattice for all $\alpha \in ]0,1]$, then there exists $\alpha \in ]0,1]$ and an element $r \in X$ such that $(x,r) \in R_\alpha, (y,r) \in R_\alpha$, and $(r,u) \in R_\alpha$ for every upper bound $u$ of $\{x,y\}$.

Then there exists $r \in X$ such that $R(x,r) \geq \alpha > 0, R(y,r) \geq \alpha > 0$ and $R(r,u) \geq \alpha > 0$ for every upper bound $u$ of $\{x,y\}$.

So, there exists a least upper bound $r$ of $\{x,y\}$ on $(X,R)$.

Similarly, we may show that there exists a greatest lower bound $l$ of $\{x,y\}$ on $(X,R)$. For more detail see [5]. ■

**Remark 1.3.2** *If $(X,R)$ is a fuzzy lattice, $(X,R_\alpha)$ may not be a crisp lattice as seen in the following example.*

**Example 1.3.2** *Let $X = \{a,b,c,d\}$ and let $R : X \times X \to [0,1]$ be a fuzzy relation defined by the table*

| $R$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0 |
| $b$ | 0.3 | 1 | 0 | 0 |
| $c$ | 0.5 | 0 | 1 | 0 |
| $d$ | 0.7 | 0.4 | 0.1 | 1 |

.

*It is easy to see that* $(X, R)$ *is a fuzzy lattice.*

*Consider the relation* $R_{0.5}$

| $R_{0.5}$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0 |
| $b$ | 0 | 1 | 0 | 0 |
| $c$ | 1 | 0 | 1 | 0 |
| $d$ | 1 | 0 | 0 | 1 |

*It is not difficult to see that* $(X, R_{0.5})$ *is a poset but not a crisp lattice.*

**Proposition 1.3.4** *[11] Let* $(X, R)$ *be a fuzzy lattice,* $(X, S(R))$ *be a bounded lattice and* $x, y \in X$*. Then* $x \sqcap y$ *and* $x \sqcup y$ *in terms of* $(X, R)$ *coincide with* $x \wedge y$ *and* $x \vee y$ *in terms of* $(X, S(R))$*.*

**Proof.** Consider $x, y \in X$. Suppose $R(x, y) > 0$, then $x \sqcap y = x$. Since $R(x, y) > 0$, then $(x, y) \in S(R)$. So, $x \wedge y = x$. Now, suppose that $(x, y) \in S(R)$, then $R(x, y) > 0$ and $x \wedge y = x$. Therefore, $x \sqcap y = x$. If $R(y, x) > 0$, is analogously the previous. If the conditions $R(x, y) = 0$ and $R(y, x) = 0$ occur simultaneously, then we suppose $x \sqcap y = z$. So, since $R(x, y) = 0$ and $R(y, x) = 0$, then $(x, y) \notin S(R)$ and $(y, x) \notin S(R)$. Hence, by hypothesis, $x \wedge y = z$. For other hand, if $(x, y) \notin S(R)$ and $(y, x) \notin S(R)$, then $R(x, y) = 0$ and $R(y, x) = 0$. Hence, by hypothesis, $x \sqcap y = z$. The proof for $x \sqcup y$ with $x \vee y$ is analogous.

■

**Proposition 1.3.5** *[5] Let* $(X, R)$ *be a fuzzy lattice and let* $x, y, z \in X$*. Then*

1. $R(x, x \sqcup y) > 0$, $R(y, x \sqcup y) > 0$, $R(x \sqcap y, x) > 0$ *and* $R(x \sqcap y, y) > 0$.
2. $R(x, z) > 0$ *and* $R(y, z) > 0$ *implies* $R(x \sqcup y, z) > 0$.
3. $R(z, x) > 0$ *and* $R(z, y) > 0$ *implies* $R(z, x \sqcap y) > 0$.
4. $R(x, y) > 0$ *if and only if* $x \sqcup y = y$.
5. $R(x, y) > 0$ *if and only if* $x \sqcap y = x$.
6. *If* $R(y, z) > 0$, *then* $R(x \sqcap y, x \sqcap z) > 0$ *and* $R(x \sqcup y, x \sqcup z) > 0$.

**Proof.** Let $(X, R)$ be a fuzzy lattice and let $x, y, z \in X$.

1. $x \sqcup y$ is the least upper bound of $\{x, y\}$, then $x \sqcup y$ is an upper bound of $x$ and $y$. So $R(x, x \sqcup y) > 0, R(y, x \sqcup y) > 0$. Similarly, we prove that $R(x \sqcap y, x) > 0$, $R(x \sqcap y, y) > 0$.
2. $R(x, z) > 0$ and $R(y, z) > 0$, mean that $z$ is an upper bound of $\{x, y\}$ , since $x \sqcup y$ is the least upper bound. So, $R(x \sqcup y, z) > 0$.
3. $R(z, x) > 0$ and $R(z, y) > 0$, prove that $z$ is a lower bound of $\{x, y\}$ , since $x \sqcap y$ is the greatest lower bound. So $R(z, x \sqcap y) > 0$.
4. For the direct implication, suppose $R(x, y) > 0$. Since $R(y, y) = 1 > 0$, $R(x \sqcup y, y) > 0$. And since $R(y, x \sqcup y) > 0$. So, by the antisymmetric, $x \sqcup y = y$.

   For the converse implication suppose $x \sqcup y = y$. Then $R(x, y) = R(x, x \sqcup y) > 0$.
5. The proof is similar to that of (4).
6. Suppose $R(y, z) > 0$. Then

   $$\begin{aligned} R(x \sqcap y, z) &\geq \sup_{t \in X} \min\left[R(x \sqcap y, t), R(t, z)\right], \\ &\geq \min\left[R(x \sqcap y, y), R(y, z)\right] > 0. \end{aligned}$$

   Since $R(x \sqcap y, x) > 0$, $x \sqcap y$ is a lower bound of $\{x, z\}$. Since $x \sqcap z$ is the greatest lower bound of $\{x, z\}$, $R(x \sqcap y, x \sqcap z) > 0$.

   $$\begin{aligned} R(y, x \sqcup z) &\geq \sup_{t \in X} \min\left[R(y, t), R(t, x \sqcup z)\right] \\ &\geq \min\left[R(y, z), R(z, x \sqcup z)\right] > 0. \end{aligned}$$

   Since $R(x, x \sqcup z) > 0$, $x \sqcup z$ is an upper bound of $\{x, y\}$ . Since $x \sqcup y$ is the least upper bound, so $R(x \sqcup y, x \sqcup z) > 0$.

■

**Proposition 1.3.6** *[5] Let $(X, R)$ be a fuzzy lattice and let $x, y, z \in X$. Then*

**(1)** $x \sqcup x = x$; $x \sqcap x = x$.

**(2)** $x \sqcup y = y \sqcup x$; $x \sqcap y = y \sqcap x$.

**(3)** $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$; $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$.

**(4)** $(x \sqcup y) \sqcap x = x$; $(x \sqcap y) \sqcup x = x$.

**Proof.** (1) and (2) are straightforward.

**(3)** Since $R(x, x \sqcup (y \sqcup z)) > 0$ and

$$\begin{aligned} R(y, x \sqcup (y \sqcup z)) &\geq \sup_{t \in X} \min\left[R(y,t), R(t, x \sqcup (y \sqcup z))\right] \\ &\geq \min\left[R(y, y \sqcup z), R(y \sqcup z, x \sqcup (y \sqcup z))\right] > 0, \end{aligned}$$

$R(x \sqcup y, x \sqcup (y \sqcup z)) > 0$ by (2) of Proposition 1.3.5.

Since

$$\begin{aligned} R(z, x \sqcup (y \sqcup z)) &\geq \sup_{t \in X} \min\left[R(z,t), R(t, x \sqcup (y \sqcup z))\right] \\ &\geq \min\left[R(z, y \sqcup z), R(y \sqcup z, x \sqcup (y \sqcup z))\right] > 0, \end{aligned}$$

$R((x \sqcup y) \sqcup z, x \sqcup (y \sqcup z)) > 0$ by (2) of Proposition 1.3.5.

Similarly, we may show $R(x \sqcup (y \sqcup z), (x \sqcup y) \sqcup z) > 0$. By the antisymmetric, $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$.

Similarly, we prove that $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$.

**(4)** Since $R(x, x \sqcup y) > 0$ and $R(x, x) = 1 > 0$, $x$ is a lower bound of $\{x \sqcup y, x\}$. If $z$ is a lower bound of $\{x \sqcup y, x\}$, then $R(z, x) > 0$. Thus $x$ is the greatest lower bound of $\{x \sqcup y, x\}$. Hence $(x \sqcup y) \sqcap x = x$. Similarly, we may show $(x \sqcap y) \sqcup x = x$.

■

**Definition 1.3.3** *[5] Let $(X, R)$ be a fuzzy lattice. $(X, R)$ is distributive if and only if $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ and $(x \sqcup y) \sqcap (x \sqcup z) = x \sqcup (y \sqcap z)$.*

**Theorem 1.3.1** *Let $(X, R)$ be a fuzzy totally ordered set. Then $(X, R)$ is a distributive fuzzy lattice .*

**Proof.** See [5]. ■

# Chapter 2

# Filters and Ideals in a Fuzzy Lattice

The main goal of this chapter is to study and characterize diverse types of filters, fuzzy filters, fuzzy prime filters, $\alpha$-filters and fuzzy $\alpha$-filters in a Fuzzy Lattice. Also,, we do the same study for ideals and we study the converse image of fuzzy prime filters and ideals via fuzzy lattices isomorphism.

## 2.1 Classical Filters and Classical Ideals in a Fuzzy Lattice

In this section, we will study classical filters and ideals of fuzzy lattice $(X,R)$ and prove So,me properties.

### 2.1.1 Definitions and Properties

**Definition 2.1.1** *[8] Let* $(X,R)$ *be a fuzzy lattice, and* $F$ *be a non-empty subset of* $X$*.* $F$ *is a* ***filter*** *of* $(X,R)$ *if it satisfies the following conditions:*

**(i)** *If* $x \in X, y \in F$ *and* $R(y,x) > 0$*, then* $x \in F$*.*

**(ii)** *If* $x, y \in F$*, then* $x \sqcap y \in F$*.*

**Definition 2.1.2** *[8] Let* $(X,R)$ *be a fuzzy lattice, and* $I$ *be a non-empty subset of* $X$*.* $I$ *is an* ***ideal*** *of* $(X,R)$ *if it satisfies the following conditions:*

**(i)** *If* $x \in X, y \in I$ *and* $R(x,y) > 0$*, then* $x \in I$*.*

**(ii)** *If* $x, y \in I$*, then* $x \sqcup y \in I$*.*

**Example 2.1.1** *Let* $(X,R)$ *be the fuzzy lattice defined in Example 1.3.2, and let* $F = \{a,b\}$ *,* $I = \{c,d\}$ *be a non-empty subsets of* $X$*. We can then see easily that* $F$ *is a filter and* $I$ *is an ideal of* $(X,R)$*.*

**Proposition 2.1.1** *[9] Let* $(X,R)$ *be a fuzzy lattice,* $Y \subseteq X$ *is a filter (ideal) of* $(X,R)$ *if and only if* $Y$ *is a filter (ideal) of* $(X,S(R))$*.*

**Proof.** Suppose that $Y$ is a filter of $(X,R)$. Then,

**(i)** $x \in X, y \in Y$, if $(y,x) \in S(R)$,then $R(y,x) > 0$. So,, $x \in Y$.

**(ii)** $x, y \in Y$, then $x \wedge y \in Y$. Straightforward from that; $x \sqcap y$ coincides with $x \wedge y$.

Hence, $Y$ is a filter of $(X,S(R))$.

Conversely, let $Y$ be a filter of $(X,S(R))$.

**(i)** $x \in X, y \in Y$, if $R(y,x) > 0$, then $(y,x) \in S(R)$ and $x \in Y$.

**(ii)** $x, y \in Y$, then $x \sqcap y \in Y$. Straightforward from the fact that $x \wedge y$ coincides with $x \sqcap y$.

Similarly, we can prove that $Y$ is an ideal of $(X,R)$ iff $Y$ is an ideal of $(X, S(R))$.

Hence, $Y$ is a filter of $(X,R)$. ■

### 2.1.2 Filters and Ideals Generated by a non-emty subset

Let $Y$ be a non-empty subset of $X$. A filter (ideal) $W$ in $X$ is said to be generated by $Y$, if $Y \subseteq W$ and for any filter (ideal) $Z$ in $X$, $Y \subseteq Z$ implies $W \subseteq Z$. The filter (ideal) generated by $Y$ will be denoted by $\uparrow Y$ $(\downarrow Y)$.

**Lemma 2.1.1** *[3] If $Y$ be a non-empty subset of a fuzzy lattice $(X,R)$, then*

$$\uparrow Y = \{x \in X / R\left(a_1 \sqcap ... \sqcap a_n, x\right) > 0, \ \textit{for So,me } a_1, .., a_n \in Y\}$$

*is the filter generated by $Y$, and*

$$\downarrow Y = \{x \in X / R\left(x, a_1 \sqcup ... \sqcup a_n\right) > 0, \ \textit{for some } a_1, .., a_n \in Y\}$$

*is the ideal generated by $Y$.*

**Proof.** Let $\uparrow Y = \{x \in X / R\left(\sqcap_{i=1}^n a_i, x\right) > 0, a_1, .., a_n \in Y\}$.

Firstly, we prove that $\uparrow Y$ is non-empty. Let $a \in Y$, since $R(a,a) > 0$, then $a \in \uparrow Y$, hence $\uparrow Y \neq \emptyset$.

To prove that $\uparrow Y$ is a filter, let $x \in \uparrow Y$ and $y \in X$ such that $R(x,y) > 0$, there exist $a_1, a_2, .., a_n \in Y$ such that $R\left(\sqcap_{i=1}^n a_i, x\right) > 0$. Then, $R\left(\sqcap_{i=1}^n a_i, y\right) > 0$, then $y \in \uparrow Y$.

On the other hand, let $x, y \in \uparrow Y$, then there exist $a_1, a_2, .., a_n, b_1, b_2, .., b_m \in Y$ such that $R\left(\sqcap_{i=1}^n a_i, x\right) > 0$ and $R\left(\sqcap_{j=1}^m b_j, y\right) > 0$. It follows that, $R\left(\left(\sqcap_{i=1}^n a_i\right) \sqcap \left(\sqcap_{j=1}^m b_j\right), x\right) > 0$ and $R\left(\left(\sqcap_{i=1}^n a_i\right) \sqcap \left(\sqcap_{j=1}^m b_j\right), y\right) > 0$. Hence, $R\left(\left(\sqcap_{i=1}^n a_i\right) \sqcap \left(\sqcap_{j=1}^m b_j\right), x \sqcap y\right) > 0$. Therefore, $x \sqcap y \in \uparrow Y$.

Next, let $a \in Y$. Since $R(a,a) > 0$, we have $a \in \uparrow Y$. Then, $Y \subseteq \uparrow Y$.

Finally, suppose that $F$ is a filter with $Y \subseteq F$. Then for any $x \in \uparrow Y$, then there exist $a_1, a_2, .., a_n \in Y$ such that : $R\left(\sqcap_{i=1}^n a_i, x\right) > 0$, then $x \in F$. Therefore, $\uparrow Y \subseteq F$. Analogously for ideal. ■

**Proposition 2.1.2** *[8] Let* $(X, R)$ *be a fuzzy lattice and* $Y \subseteq X$ *such that* $(Y, R)$ *is a complete fuzzy lattice. Then* $\uparrow Y$ *and* $\downarrow Y$ *satisfy the following properties:*

**(i)** $Y \subseteq \uparrow Y$;

**(ii)** $Y \subseteq W \Rightarrow \uparrow Y \subseteq \uparrow W$;

**(iii)** $\uparrow\uparrow Y = \uparrow Y$.

**(iv)** $Y \subseteq \downarrow Y$;

**(v)** $Y \subseteq W \Rightarrow \downarrow Y \subseteq \downarrow W$;

**(vi)** $\downarrow\downarrow Y = \downarrow Y$.

**Proof.**

**(i)** Let $y \in Y$, since $R(y, y) > 0$, then $y \in \uparrow Y$.

**(ii)** Suppose that $Y \subseteq W$ and $x \in \uparrow Y$, then there exists $a_1, .., a_n \in Y$ such that $R(a_1 \sqcap ... \sqcap a_n, x) > 0$. As $Y \subseteq W$, and $R(a_1 \sqcap ... \sqcap a_n, x) > 0$. So, $x \in \uparrow W$.

**(iii)** It suffices to prove that $\uparrow\uparrow Y \subseteq \uparrow Y$.

Suppose that $x \in \uparrow\uparrow Y$, then there exists $a_1, .., a_n \in \uparrow Y$ such that $R(a_1 \sqcap ... \sqcap a_n, x) > 0$.

Since $a_1, .., a_n \in \uparrow Y$ and $\uparrow Y$ is a filter, then $a_1 \sqcap ... \sqcap a_n \in \uparrow Y$.

So, there exists $b_1, .., b_m \in Y$ such that $R(b_1 \sqcap ... \sqcap b_m, a_1 \sqcap ... \sqcap a_n) > 0$. By the transitivity, there exists $b_1, .., b_m \in Y$ such that $R(b_1 \sqcap ... \sqcap b_m, x) > 0$. Therefore, $x \in \uparrow Y$.

In the same manner, we prove the properties $(iv), (v)$ and $(vi)$. ■

**Proposition 2.1.3** *[8] Let* $(X, R)$ *be a fuzzy lattice. Then,*

**(i)** $X \in F(X)$ *and* $X \in I(X)$;

**(ii)** $\cap W \in F(X)$, *for all* $W \in F(X)$;

**(iii)** $\cap Z \in I(X)$, *for all* $Z \in I(X)$;

**Proof.**

**(i)** Evident

**(ii)** Suppose $y \in \cap W$, then $y \in W$ for all $W \in F(X)$. If $R(y, x) > 0$, for some $x \in X$, then $x \in W$, for all $W \in F(X)$, since $W$ is a filter, and hence $x \in \cap W$. Thus, $\cap W \in F(X)$.

If $x, y \in \cap W$, then $x, y \in W$, for all $W \in F(X)$, since each $W$ is a filter, then $x \sqcap y \in W$. Therefore, $x \sqcap y \in W$.

**(iii)** Similar to $(ii)$.

■

### 2.1.3 Types of Classical Filters and Ideals in a Fuzzy Lattice

**Definition 2.1.3** *A filter $Y$ of $(X, R)$ such that $Y \neq X$ is called proper filter of $(X, R)$. Duality, an ideal $Y$ of $(X, R)$ such that $Y \neq X$ is called proper ideal of $(X, R)$.*

**Definition 2.1.4** *[12] Let $(X, R)$ be a fuzzy lattice and $Y \subseteq X$ be a filter of $(X, R)$. Then $Y$ is a prime filter if $Y$ is proper $(Y \neq X)$ and for all $x, y \in X, x \sqcup y \in Y$ imply $x \in Y$ or $y \in Y$.*

*Dually, $Y$ is a prime ideal if $Y$ is proper $(Y \neq X)$ and for all $x, y \in X, x \sqcap y \in Y$ imply $x \in Y$ or $y \in Y$.*

**Definition 2.1.5** *[8] Let $Y$ be a proper filter (ideal) of $(X, R)$. We say that the filter (ideal) $Y$ is a **maximal filter (ideal)** of $(X, R)$, if $Y \subseteq Z \subseteq X$, then either $Z = Y$ or $Z = X$, for any filter (ideal) $Z$.*

**Definition 2.1.6** *[8] Let $(X, R)$ be a fuzzy lattice and $x \in X$. Then, the set defined by: $\uparrow x = \{y \in X : R(x, y) > 0\}$ is called **principal filter** of $(X, R)$ generated by $x$.*

*Dually, the set defined by: $\downarrow x = \{y \in X : R(y, x) > 0\}$ is called **principal ideal** of $(X, R)$ generated by $x$.*

We denote the set of all principal filters generated by elements of $X$ by $PF(X)$ whereas $PI(X)$ denote the set of all principal ideals generated by elements of $X$.

**Proposition 2.1.4** *[3]* $(PF(X), \subseteq, \curlywedge, \curlyvee)$ *is a lattice with* $(\uparrow x) \curlywedge (\uparrow y) = (\uparrow x) \cap (\uparrow y) = \uparrow (x \sqcup y)$ *and* $(\uparrow x) \curlyvee (\uparrow y) = \uparrow ((\uparrow x) \cup (\uparrow y)) = \uparrow (x \sqcap y)$*, for any* $\uparrow x, \uparrow y \in PF(X)$*.*

**Proof.** Let $a \in X$,

$$\begin{aligned} a \in \uparrow (x \sqcup y) &\Leftrightarrow R(x \sqcup y, a) > 0 \\ &\Leftrightarrow R(x, a) > 0 \text{ and } R(y, a) > 0 \\ &\Leftrightarrow a \in (\uparrow x) \text{ and } a \in (\uparrow y) \\ &\Leftrightarrow a \in (\uparrow x) \cap (\uparrow y) \\ &\Leftrightarrow a \in (\uparrow x) \curlywedge (\uparrow y). \end{aligned}$$

We have $R(x \sqcap y, x) > 0$ and $R(x \sqcap y, y) > 0$, then $\uparrow x \subseteq \uparrow (x \sqcap y)$ and $\uparrow y \subseteq \uparrow (x \sqcap y)$, hence $(\uparrow x) \cup (\uparrow y) \subseteq \uparrow (x \sqcap y)$. Thus $\langle (\uparrow x) \cup (\uparrow y) \rangle \subseteq \uparrow (x \sqcap y)$.

On the other hand, if $a \in \uparrow (x \sqcap y)$, hence $R(x \sqcap y, a) > 0$. then $x \in (\uparrow x)$ or $y \in (\uparrow y)$, hence $x \in \uparrow ((\uparrow x) \cup (\uparrow y))$ and $y \in \uparrow ((\uparrow x) \cup (\uparrow y))$ so, $x \sqcap y \in \uparrow ((\uparrow x) \cup (\uparrow y))$ and then $a \in \uparrow ((\uparrow x) \cup (\uparrow y))$. Thus $\uparrow (x \sqcap y) \subseteq \uparrow ((\uparrow x) \cup (\uparrow y))$, which complete the proof. ■

**Proposition 2.1.5** $(PI(X), \subseteq, \curlywedge, \curlyvee)$ *is a lattice with* $(\downarrow x) \curlywedge (\downarrow y) = (\downarrow x) \cap (\downarrow y) = \downarrow (x \sqcap y)$ *and* $(\downarrow x) \curlyvee (\downarrow y) = \langle (\downarrow x) \cup (\downarrow y) \rangle = \downarrow (x \sqcup y)$*, for any* $\downarrow x, \downarrow y \in PI(X)$*.*

### 2.1.4 Classical $\alpha$-Filters and $\alpha$-Ideals in a Fuzzy Lattice

**Definition 2.1.7** *[10] Let* $(X, R)$ *be a fuzzy lattice,* $\alpha \in ]0, 1]$ *and* $Y \subseteq X$*.* $Y$ *is an* $\alpha$*-filter of* $(X, R)$*,if it satisfies*

**(i)** If $x \in X$, $y \in Y$ and $R(y, x) \geq \alpha$, then $x \in Y$;

**(ii)** If $x, y \in Y$, then $x \sqcap y \in Y$.

**Definition 2.1.8** *Let* $(X, R)$ *be a fuzzy lattice,* $\alpha \in ]0, 1]$ *and* $Y \subseteq X$*.* $Y$ *is an* $\alpha$*-ideal of* $(X, R)$*, if it satisfies*

**(i)** If $x \in X$, $y \in Y$ and $R(x, y) \geq \alpha$, then $x \in Y$;

**(ii)** If $x, y \in Y$, then $x \sqcup y \in Y$.

**Proposition 2.1.6** *[10] If* $\alpha \leq \beta$*, then any* $\alpha$*-filter is a* $\beta$*-filter.*

**Proof.** Let $Y$ be an $\alpha$-filter and $\alpha \leq \beta$. Then for any $x \in X$, if $R(y,x) \geq \beta$, then $R(y,x) \geq \alpha$, so, $x \in Y$.

Therefore, $Y$ is a $\beta$-filter of $(X,R)$. ■

**Proposition 2.1.7** *If $\alpha \leq \beta$, then any $\alpha$-ideal is a $\beta$-ideal.*

**Corollary 2.1.1** *[10] Let $(X,R)$ be a fuzzy lattice, $\alpha \in \left]0,1\right]$. All filter of $(X,R)$ is an $\alpha$-filter. Dually, all ideal of $(X,R)$ is an $\alpha$-ideal.*

**Example 2.1.2** *Let $X = \{a,b,c,d\}$ and let $R : X \times X \longrightarrow [0,1]$ be a fuzzy relation defined by*

| $R$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 1 | 0.3 | 0.7 | 0.4 |
| $b$ | 0 | 1 | 0.1 | 0 |
| $c$ | 0 | 0 | 1 | 0 |
| $d$ | 0 | 0 | 0.2 | 1 |

.

*Let $F = \{b,c\}$, $I = \{a,d\}$ be a filter, an ideal respectively of $(X,R)$.*

*$F$ is a 0.1-filter and $I$ is a 0.4-ideal.*

## 2.2 Fuzzy Filters and Fuzzy Ideals in a Fuzzy Lattice

In what follows $(X,R)$ denotes a fuzzy lattice.

### 2.2.1 Definitions and Properties

**Definition 2.2.1** *[3] A fuzzy subset $f$ in $(X,R)$ is called a fuzzy filter of $X$ if it satisfies*

**(F1)** *for all $x,y \in X$, $f(x \sqcap y) \geq \min\{f(x), f(y)\}$;*

**(F2)** *$f$ is an order-preserving, that is, for all $x,y \in X$, $R(x,y) > 0$ implies $f(x) \leq f(y)$.*

**Definition 2.2.2** *A fuzzy subset $g$ in $(X, R)$ is called a fuzzy ideal of $X$ if it satisfies*

**(I1)** *for all* $x, y \in X, g(x \sqcup y) \leq \max\{g(x), g(y)\}$;

**(I2)** *For all* $x, y \in X, R(x, y) > 0$ *implies* $g(y) \leq g(x)$.

**Example 2.2.1** *Let $(X, R)$ be a fuzzy lattice where $X = \{x, y, z, t\}$ and $R : X \times X \to [0, 1]$ is a fuzzy relation given by*

| $R$ | $x$ | $y$ | $z$ | $w$ |
|---|---|---|---|---|
| $x$ | 1 | 0 | 0 | 0 |
| $y$ | 0.3 | 1 | 0 | 0 |
| $z$ | 0.5 | 0.2 | 1 | 0 |
| $w$ | 0.8 | 0.4 | 0.1 | 1 |

.

*The fuzzy set* $f = \{(x, 0.7), (y, 0.4), (z, 0.2), (w, 0.0)\}$ *is a fuzzy filter of* $(X, R)$.

*The fuzzy set* $g = \{(x, 0.1), (y, 0.3), (z, 0.4), (w, 0.7)\}$ *is a fuzzy ideal of* $(X, R)$.

**Theorem 2.2.1** *[12] Let $(X, R)$ be a fuzzy lattice. A non-constant fuzzy subset $f$ is a fuzzy filter of $(X, R)$ if and only if a non-empty subsets $f_\alpha$ are filters for every $\alpha \in ]0, 1]$.*

**Proof.** For the direct implication. Let $f$ be a fuzzy filter of $(X, R)$ such that $f_\alpha \neq \emptyset$.

If $x \in f_\alpha$ and $R(x, y) > 0$, then $\alpha \leq f(x) \leq f(y)$, it follows that $y \in f_\alpha$.

On the other hand, suppose that $x, y \in f_\alpha$. It follows that $f(x) \geq \alpha$ and $f(y) \geq \alpha$. This implies that $\min\{f(x), f(y)\} \geq \alpha$. Hence, $f(x \sqcap y) \geq \alpha$, i.e., $x \sqcap y \in f_\alpha$.

Therefore, $f_\alpha$ are filters for every $\alpha \in ]0, 1]$.

For the converse implication. Let $f$ be a non-constant fuzzy subset of $X$ such that for every $\alpha \in ]0, 1]$, $f_\alpha \neq \emptyset$ and $f_\alpha \neq X$, suppose that the cut $f_\alpha$ is a crisp filter of $(X, R)$.

For any $x, y \in X$, if $\min\{f(x), f(y)\} \neq 0$, we denote $\min\{f(x), f(y)\} = \alpha$, then $f(x) \geq \alpha$ and $f(y) \geq \alpha$, it follows that $x, y \in f_\alpha$ which implies that $x \sqcap y \in f_\alpha$, i.e., $f(x \sqcap y) \geq \alpha$. Hence, $f(x \sqcap y) \geq \min\{f(x), f(y)\}$. If $\min\{f(x), f(y)\} = 0$, then the relation $f(x \sqcap y) \geq \min\{f(x), f(y)\}$ is trivially.

On the other hand, setting $f(x) = \alpha$. If $R(x, y) > 0$ and $x \in f_\alpha$, then $y \in f_\alpha$, it follows that $\alpha = f(x) \leq f(y)$, it is easy to see that if $f(x) = 0$, we obtain $f(x) \leq f(y)$.

Therefore, $f$ is a fuzzy filter of $(X, R)$. ■

**Theorem 2.2.2** *Let $(X, R)$ be a fuzzy lattice. A non-constant fuzzy subset $g$ is a fuzzy ideal of $(X, R)$ if and only if a non-empty subsets $g_\alpha$ are ideals for every $\alpha \in\ ]0, 1]$.*

**Proposition 2.2.1** *[12] Let $(X, R)$ be a fuzzy lattice in which $(X, R_\alpha)$ are crisp lattices, for every $\alpha \in\ ]0, 1]$. In this case, if $f$ is a fuzzy filter of $(X, R)$, then the non-empty subsets $f_\alpha$ are crisp filters of $(X, R_\alpha)$ for every $\alpha \in\ ]0, 1]$.*

**Proof.** Suppose that $(X, R_\alpha)$ is a crisp lattice for all $\alpha \in\ ]0, 1]$, and let $f$ be a fuzzy filter of $(X, R)$ such that $f_\alpha \neq \emptyset$ for every $\alpha \in\ ]0, 1]$.

Let $x, y \in f_\alpha$, from $f(x) \geq \alpha$ and $f(y) \geq \alpha$, it follows that $f(x \sqcap y) \geq \min\{f(x), f(y)\} \geq \alpha$. Therefore, $x \sqcap y \in f_\alpha$.

On the other hand, let $x \in f_\alpha$ and let $(x, y) \in R_\alpha$. It follows that $f(x) \geq \alpha$ and $R(x, y) \geq \alpha > 0$, i.e., $\alpha \leq f(x) \leq f(y)$, this implies that $y \in f_\alpha$. ∎

**Proposition 2.2.2** *Let $(X, R)$ be a fuzzy lattice in which $(X, R_\alpha)$ is a crisp lattice, for every $\alpha \in\ ]0, 1]$. In this case, if $g$ is a fuzzy ideal of $(X, R)$, then the non-empty subsets $g_\alpha$ are crisp ideals of $(X, R_\alpha)$ for every $\alpha \in\ ]0, 1]$.*

**Example 2.2.2** *Let $X = \{a, b, c, d, e, f\}$ and let $R : X \times X \to [0, 1]$ be a fuzzy relation defined by*

| $R$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $b$ | 0 | 1 | 0.4 | 0.4 | 0.4 | 1 |
| $c$ | 0 | 0 | 1 | 0 | 0.4 | 1 |
| $d$ | 0 | 0 | 0 | 1 | 1 | 1 |
| $e$ | 0 | 0 | 0 | 0 | 1 | 1 |
| $f$ | 0 | 0 | 0 | 0 | 0 | 1 |

*It is easy to see that $(X, R)$ is a fuzzy lattice and for all $\alpha \in\ ]0, 1]$ $(X, R_\alpha)$ is a crisp lattice.*

*Note that if* $\alpha \leq 0.4$, *then the relation* $R_\alpha$ *given by*

| $R_\alpha$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $b$ | 0 | 1 | 1 | 1 | 1 | 1 |
| $c$ | 0 | 0 | 1 | 0 | 1 | 1 |
| $d$ | 0 | 0 | 0 | 1 | 1 | 1 |
| $e$ | 0 | 0 | 0 | 0 | 1 | 1 |
| $f$ | 0 | 0 | 0 | 0 | 0 | 1 |

.

*And if* $\alpha > 0.4$, *then the relation* $R_\alpha$ *given by*

| $R_\alpha$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $b$ | 0 | 1 | 0 | 0 | 0 | 1 |
| $c$ | 0 | 0 | 1 | 0 | 0 | 1 |
| $d$ | 0 | 0 | 0 | 1 | 1 | 1 |
| $e$ | 0 | 0 | 0 | 0 | 1 | 1 |
| $f$ | 0 | 0 | 0 | 0 | 0 | 1 |

.

*Let* $f$ *and* $g$ *be the fuzzy sets defined on* $X$ *by*

$f = \{(a, 0.1), (b, 0.2), (c, 0.2), (d, 0.5), (e, 0.6), (f, 0.6)\}$,

$g = \{(a, 0.7), (b, 0.4), (c, 0.4), (d, 0.2), (e, 0.2), (f, 0.1)\}$.

$f$ *is a fuzzy filter and* $g$ *is a fuzzy ideal of* $(X, R)$.

*It is easy to verify that* $f_{0.1} = X$ *is a crisp filter of* $(X, R_{0.1})$, $f_{0.2} = \{b, c, d, e, f\}$ *is a crisp filter of* $(X, R_{0.2})$, $f_{0.5} = \{d, e, f\}$ *is a crisp filter of* $(X, R_{0.5})$ *and* $f_{0.6} = \{e, f\}$ *is a crisp filter of* $(X, R_{0.6})$.

*And that* $g_{0.1} = X$ *is a crisp ideal of* $(X, R_{0.1})$, $g_{0.2} = \{a, b, c, d, e\}$ *is a crisp ideal of* $(X, R_{0.2})$, $g_{0.4} = \{a, b, c\}$ *is a crisp ideal of* $(X, R_{0.4})$ *and* $g_{0.7} = \{a\}$ *is a crisp ideal of* $(X, R_{0.7})$.

### 2.2.2 Characterizations of Fuzzy Filters and Fuzzy Ideals in a Fuzzy Lattice

We give further some characterizations of the fuzzy filters and the fuzzy ideals.

**Theorem 2.2.3** *[3] Let* $(X, R)$ *be a fuzzy lattice. A fuzzy set* $f$ *on* $X$ *is a fuzzy filter in* $(X, R)$*, if and only if it satisfies* $(F1)$ *and*

**(F3)** *for all* $x, y \in X,\ f(x) \geq \min\{f(x \sqcap y), f(y)\}$.

**Proof.** Suppose that $(F3)$ is satisfied, i.e., for all $x, y \in X,\ f(x) \geq \min\{f(x \sqcap y), f(y)\}$. If $R(x, y) > 0$ it follows that $x \sqcap y = x$, i.e., $f(y) \geq \min\{f(x \sqcap y), f(x)\} = f(x)$. Hence, $(F2)$ satisfied.

On the other hand, if $(F2)$ is satisfied, then $R(x \sqcap y, y) > 0$ and $R(x \sqcap y, x) > 0$. Hence, $\min\{f(x \sqcap y), f(y)\} \leq \min\{f(x), f(y)\} \leq f(x)$, hence $(F3)$ is satisfied. ■

**Theorem 2.2.4** *Let* $(X, R)$ *be a fuzzy lattice. A fuzzy set* $g$ *on* $X$ *is a fuzzy ideal in* $(X, R)$*, if and only if it satisfies* $(I1)$ *and*

**(I3)** *for all* $x, y \in X,\ g(x) \leq \max\{g(x \sqcup y), g(y)\}$.

**Theorem 2.2.5** *[3] A fuzzy set* $f$ *in a fuzzy lattice* $(X, R)$ *is a fuzzy filter of* $X$ *if and only if it satisfies*

**(F4)** *for all* $x, y, z \in X,\ R(x \sqcap y, z) > 0$ *implies* $f(z) \geq \min\{f(x), f(y)\}$.

**Proof.** Let $x, y, z \in X$. We have $R(x \sqcap y, x) > 0 \Rightarrow f(x) \geq \min\{f(x), f(y)\}$. Then,

$$\begin{aligned} R(x, y) > 0 &\Rightarrow R(x \sqcap x, y) > 0 \\ &\Rightarrow f(y) \geq \min\{f(x), f(x)\} \\ &\Rightarrow f(y) \geq f(x). \end{aligned}$$

Hence, $(F2)$ is satisfied.

On the other hand, since $R(x \sqcap y, x \sqcap y) > 0$, it follows that $f(x \sqcap y) \geq \min\{f(x), f(y)\}$. Hence, $(F1)$ is satisfied.

Conversely, suppose that $f$ is a fuzzy filter of $X$. If $R(x \sqcap y, z) > 0$, then $f(z) \geq f(x \sqcap y) \geq \min\{f(x), f(y)\}$. ■

**Theorem 2.2.6** *A fuzzy set* $g$ *in a fuzzy lattice* $(X, R)$ *is a fuzzy ideal of* $X$ *if and only if it satisfies*

**(I4)** *for all* $x, y, z \in X,\ R(z, x \sqcup y) > 0$ *implies* $g(z) \leq \max\{g(x), g(y)\}$.

**Theorem 2.2.7** *[3] A fuzzy set $f$ in $(X, R)$ is a fuzzy filter of $X$ if and only if it satisfies $(F1)$ and*

**(F5)** *for all $x, y \in X$, $f(x \sqcup y) \geq f(x)$.*

**Proof.** Suppose that $f$ is a fuzzy filter of $(X, R)$. Let $x, y \in X$, $R(x, x \sqcup y) > 0$ implies $f(x \sqcup y) \geq f(x)$. Hence, $(F5)$ is satisfied.

Conversely, suppose that $(F5)$ is satisfied. Let $x, y \in X$ such that $R(x, y) > 0$, since $f(x \sqcup y) \geq f(x)$, it follows that $f(y) \geq f(x)$. ■

**Theorem 2.2.8** *A fuzzy set $g$ in $(X, R)$ is a fuzzy ideal of $X$ if and only if it satisfies $(I1)$ and*

**(I5)** *for all $x, y \in X$, $g(x \sqcap y) \geq g(x)$.*

**Corollary 2.2.1** *[12] A fuzzy set $f$ in $X$ is a fuzzy filter of a fuzzy lattice $(X, R)$ if and only if for any $x, a_1, ..., a_n \in X$,*

$$R(a_1 \sqcap ... \sqcap a_n, x) > 0 \text{ implies } f(x) \geq \min\{f(a_1), ..., f(a_n)\}.$$

**Proof.** Let $f$ be a fuzzy filter in $X$. Suppose that $R(a_1 \sqcap ... \sqcap a_n, x) > 0$, it follows that $f(x) \geq f(a_1 \sqcap ... \sqcap a_n) \geq \min\{f(a_1), ..., f(a_n)\}$.

Conversely, for any $a_1, a_2 \in X$, $R(a_1 \sqcap a_2, a_1 \sqcap a_2) > 0$ implies $f(a_1 \sqcap a_2) \geq \min\{f(a_1), f(a_2)\}$. Hence, $(F1)$ is satisfied.

On the other hand, since $R(a_1, a_2) > 0$ implies that $R(a_1 \sqcap a_2, a_2) > 0$, it follows that $f(a_2) \geq \min\{f(a_1), f(a_2)\}$. Hence, $f(a_2) \geq f(a_1)$. ■

**Corollary 2.2.2** *A fuzzy set $g$ in $X$ is a fuzzy ideal of a fuzzy lattice $(X, R)$ if and only if for any $x, a_1, ..., a_n \in X$,*

$$R(x, a_1 \sqcup ... \sqcup a_n) > 0 \text{ implies } g(x) \leq \max\{g(a_1), ..., g(a_n)\}.$$

Let $Y$ be a crisp set of the fuzzy lattice $(X, R)$, and $\Psi_Y$ be a fuzzy set in $X$ defined by

$$\Psi_Y(x) = \begin{cases} \alpha, & \text{if } x \in Y, \\ \beta, & \text{otherwise.} \end{cases}$$

For all $x \in X$ and $\alpha, \beta \in [0, 1]$ such that $\alpha > \beta$.

**Theorem 2.2.9** *[12] Let $F$ be a non-empty crisp set of $X$. Then $F$ is a filter if and only if $\Psi_F$ is a fuzzy filter of $X$. Moreover $(\Psi_F)_\alpha = F$.*

**Proof.** Let $F$ be a filter of $X$ and $x, y \in X$ :

**1)** If $x \sqcap y \in F$. Then, $\Psi_F(x \sqcap y) \geq \min\{\Psi_F(x), \Psi_F(y)\}$.

**2)** If $x \sqcap y \notin F$ then $x \notin F$ or $y \notin F$. Hence, $\Psi_F(x \sqcap y) = \min\{\Psi_F(x), \Psi_F(y)\}$.

On the other hand, let $x \in X$ and $R(x, y) > 0$.

**1)** If $x \in F$ then $y \in F$. Hence, $\Psi_F(x) \leq \Psi_F(y) = \alpha$.

**2)** If $x \notin F$, then $\beta = \Psi_F(x) \leq \Psi_F(y)$.

Conversely, suppose that $\Psi_F$ is a fuzzy filter of $X$.

If $x, y \in F$, then $\alpha = \min\{\Psi_F(x), \Psi_F(y)\} \leq \Psi_F(x \sqcap y)$. Hence, $x \sqcap y \in F$.

On the other hand let $x \in F$, if $R(x, y) > 0$ then, $\Psi_F(y) \geq \Psi_F(x) = \alpha$. Therefore, $y \in F$.

Finally, let $x \in X$

$$\begin{aligned} x \in (\Psi_F)_\alpha &\Leftrightarrow \Psi_F(x) = \alpha \\ &\Leftrightarrow x \in F. \end{aligned}$$

Which complete the proof. ■

**Theorem 2.2.10** *Let $I$ be a non-empty crisp set of $X$. Then $I$ is an ideal if and only if $\Psi_I$ is a fuzzy ideal of $X$. Moreover $(\Psi_I)_\alpha = I$.*

**Example 2.2.3** *Let $(X, R)$ be the fuzzy lattice defined in Example 2.2.2, the crisp set $F = \{e, f\}$ is a filter of $X$, and the crisp set $I = \{a, b\}$ is an ideal of $X$.*

*We can, for all $x \in X$ and $\alpha, \beta \in [0, 1]$ such that $\alpha > \beta$, verify that $\Psi_F$ $(\Psi_I)$ is a fuzzy filter (ideal) of $X$, where*

$$\Psi_F(x) = \begin{cases} \alpha, & \text{if } x \in F, \\ \beta, & \text{otherwise.} \end{cases} \quad \text{And } \Psi_I(x) = \begin{cases} \alpha, & \text{if } x \in I, \\ \beta, & \text{otherwise.} \end{cases}$$

**Corollary 2.2.3** *[12] Let $F$ be a non-empty crisp set of $X$ such that $(X, R_\alpha)$ is a crisp lattice for all $\alpha \in ]0, 1]$. If $\Psi_F$ is a fuzzy filter of $X$, then $F$ is a crisp filter on $(X, R_\alpha)$. Moreover $(\Psi_F)_\alpha = F$.*

**Proof.** Suppose that $\Psi_F$ is a fuzzy filter of $X$.

**1)** If $x, y \in F$, then $\alpha = \min\{\Psi_F(x), \Psi_F(y)\} \leq \Psi_F(x \sqcap y)$. Hence, $x \sqcap y \in F$.

**2)** If $x \in F$ and $R(x, y) > 0$, therefore $\Psi_F(y) \geq \Psi_F(x) = \alpha$. Hence, $y \in F$.

Finally, let $x \in X$

$$\begin{aligned} x \in (\Psi_F)_\alpha &\Leftrightarrow \Psi_F(x) = \alpha \\ &\Leftrightarrow x \in F. \end{aligned}$$

■

**Corollary 2.2.4** *Let $I$ be a non-empty crisp set of $X$ such that $(X, R_\alpha)$ is a crisp lattice for all $\alpha \in ]0, 1]$. If $\Psi_I$ is a fuzzy ideal of $X$, then $I$ is a crisp ideal on $(X, R_\alpha)$. Moreover $(\Psi_I)_\alpha = I$.*

For any family of fuzzy sets $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ in $X$ we define the fuzzy set $\cap_{i\in I} f_i$ and $\cap_{i\in I} g_i$ in $X$ as follows:

$\cap_{i\in I} f_i = \{(x, \inf_{i\in I} f_i(x)) \diagup x \in X\}$.

$\cap_{i\in I} g_i = \{(x, \sup_{i\in I} g_i(x)) \diagup x \in X\}$.

**Theorem 2.2.11** *[3] Let $(f_i)_{i\in I}$ be a family of fuzzy filters of $(X, R)$. Then $\cap_{i\in I} f_i$ is also a fuzzy filter of $(X, R)$.*

**Proof.** Let $(f_i)_{i\in I}$ be a family of fuzzy filters of $X$. If $x, y \in X$, we have $f_i(x \sqcap y) \geq \min\{f_i(x), f_i(y)\}$ for all $i \in I$, then,

$\inf_{i\in I} f_i(x \sqcap y) \geq \inf_{i\in I} \min\{f_i(x), f_i(y)\} = \min\{\inf_{i\in I}(f_i(x)), \inf_{i\in I}(f_i(y))\}$.

On the other hand let $x, y \in X$, if $R(x, y) > 0$, then it holds that $f_i(y) \geq f_i(x)$ for all $i \in I$. Therefore, $\inf_{i\in I} f_i(y) \geq \inf_{i\in I} f_i(x)$. ■

**Theorem 2.2.12** *Let $(g_i)_{i\in I}$ be a family of fuzzy ideals of $(X, R)$. Then $\cap_{i\in I} g_i$ is also a fuzzy ideal of $(X, R)$.*

### 2.2.3 Fuzzy Filters and Fuzzy Ideals Generated by a non-emty subset

**Definition 2.2.3** *Let $\delta$ be a fuzzy set in $(X, R)$. A fuzzy filter (fuzzy ideal) $\mu$ in $X$ is said to be generated by $\delta$, if $\delta \subseteq \mu$ and for any fuzzy filter (fuzzy ideal) $\psi$ in $X$, $\delta \subseteq \psi$ imply $\mu \subseteq \psi$. The fuzzy filter (fuzzy ideal) generated by $\delta$ will be denoted by $\uparrow \delta$ $(\downarrow \delta)$.*

**Theorem 2.2.13** *[3] If $\delta$ is a fuzzy set, then*

$$\uparrow \delta\left(x\right) = \max\left\{\min_{i \in \{1,..,n\}} \delta\left(a_i\right)/R\left(\sqcap_{i=1}^{n} a_i, x\right) > 0, a_1, a_2, ..., a_n \in X\right\}$$

*is the fuzzy filter generated by $\delta$, and*

$$\downarrow \delta\left(x\right) = \max\left\{\min_{i \in \{1,..,n\}} \delta\left(a_i\right)/R\left(x, \sqcup_{i=1}^{n} a_i\right) > 0, a_1, a_2, ..., a_n \in X\right\}$$

*is the fuzzy ideal generated by $\delta$.*

**Proof.** Let $\uparrow \delta\left(x\right) = \max\left\{\min_{i \in \{1,..,n\}} \delta\left(a_i\right)/R\left(\sqcap_{i=1}^{n} a_i, x\right) > 0, a_1, a_2, ..., a_n \in X\right\}.$

First, we prove that $\uparrow \delta$ is a fuzzy filter.

Let $x, y \in X$ such that $R\left(x, y\right) > 0$, if there exit $a_1, a_2, ....., a_n$ such that $R\left(\sqcap_{i=1}^{n} a_i, x\right) > 0$. Then, $R\left(\sqcap_{i=1}^{n} a_i, y\right) > 0$ and

$$\begin{aligned} \uparrow \delta\left(x\right) &= \max\left\{\min_{i \in \{1,..,n\}} \delta\left(a_i\right)/R\left(\sqcap_{i=1}^{n} a_i, x\right) > 0,\ a_1, a_2, ..., a_n \in X\right\} \\ &\leq \max\left\{\min_{i \in \{1,..,n\}} \delta\left(a_i\right)/R\left(\sqcap_{i=1}^{n} a_i, y\right) > 0,\ a_1, a_2, ..., a_n \in X\right\} \\ &= \uparrow \delta\left(y\right). \end{aligned}$$

On the other hand, let $x, y \in X$. If there exist $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m$ such that $R\left(\sqcap_{i=1}^{n} a_i, x\right) > 0$ and $R\left(\sqcap_{k=1}^{m} b_k, y\right) > 0$, then $R\left(\left(\sqcap_{i=1}^{n} a_i\right) \sqcap \left(\sqcap_{k=1}^{m} b_k\right), x \sqcap y\right) > 0$.

Hence, $\uparrow \delta\left(x \sqcap y\right) \geq \min\left\{\min_{i \in \{1,..,n\}} \delta\left(a_i\right), \min_{k \in \{1,..,m\}} \delta\left(b_k\right)\right\}.$

Since,

$\min\left\{\uparrow \delta\left(x\right), \uparrow \delta\left(y\right)\right\}$

$= \min\left\{\max\left\{\min_{i \in \{1,..,n\}} \delta\left(a_i\right)/R\left(\sqcap_{i=1}^{n} a_i, x\right) > 0, a_1, a_2, ..., a_n \in X\right\},\right.$

$\left.\max\left\{\min_{k \in \{1,..,m\}} \delta\left(b_k\right)/R\left(\sqcap_{k=1}^{m} b_k, y\right) > 0, b_1, b_2, ..., b_m \in X\right\}\right\}$

$= \max\left\{\min\left\{\min_{i \in \{1,..,n\}} \delta\left(a_i\right), \min_{k \in \{1,..,m\}} \delta\left(b_k\right)\right\}/R\left(\sqcap_{i=1}^{n} a_i, x\right) > 0,\right.$ and

$\left.R\left(\sqcap_{k=1}^{m} b_k, y\right) > 0, a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in X\right\},$

it follows that $\uparrow \delta\left(x \sqcap y\right) \geq \min\left\{\uparrow \delta\left(x\right), \uparrow \delta\left(y\right)\right\}$. Hence, $\uparrow \delta$ is a fuzzy filter.

To show that $\delta \subseteq \uparrow \delta$, since $R(x \sqcap x, x) > 0$, it follows that $\uparrow \delta(x) \geq \min\{\delta(x), \delta(x)\} = \delta(x)$.

Finally, suppose that $\psi$ is a fuzzy filter with $\delta \subseteq \psi$. Then for any $x \in X$,

$$\begin{aligned} \uparrow \delta(x) &= \max\left\{\min_{i\in\{1,..,n\}} \delta(a_i) / R\left(\sqcap_{i=1}^{n} a_i, x\right) > 0, a_1, a_2, ..., a_n \in X\right\} \\ &\leq \max\left\{\min_{i\in\{1,..,n\}} \psi(a_i) / R\left(\sqcap_{i=1}^{n} a_i, x\right) > 0, a_1, a_2, ..., a_n \in X\right\} \\ &\leq \max\{\psi(x)\} \\ &= \psi(x). \end{aligned}$$

Analogously for ideal. ■

We denote the set of all filters of $X$ by $FF(X)$ whereas $FI(X)$ denote the set of all ideals of $X$.

**Proposition 2.2.3** *[12]* $(FF(X), \curlywedge, \curlyvee, \mathbf{0}, \mathbf{1})$ *is a closed lattice where for any* $F, G \in FF(X)$ $F \curlyvee G = \uparrow (F \cup G)$, $F \curlywedge G = F \cap G$, $\mathbf{0}$ *is defined by* $\mathbf{0}(x) = 0$ *and* $\mathbf{1}$ *is defined by* $\mathbf{1}(x) = 1$.

**Proposition 2.2.4** $(FI(X), \curlywedge, \curlyvee, \mathbf{0}, \mathbf{1})$ *is a closed lattice where for any* $F, G \in FI(X)$ $F \curlyvee G = \downarrow (F \cup G)$, $F \curlywedge G = F \cap G$, $\mathbf{0}$ *is defined by* $\mathbf{0}(x) = 0$ *and* $\mathbf{1}$ *is defined by* $\mathbf{1}(x) = 1$.

### 2.2.4 Fuzzy $\alpha$-Filters and Fuzzy $\alpha$-Ideals in a Fuzzy Lattice

**Definition 2.2.4** *[3] Let* $(X, R)$ *be a fuzzy lattice,* $\alpha \in ]0, 1]$. *A fuzzy set* $f$ *in* $(X, R)$ *is called a fuzzy* $\alpha$*-filter of* $X$ *if it satisfies.*

**(F1)** *for all* $x, y \in X$, $f(x \sqcap y) \geq \min\{f(x), f(y)\}$,

**(FA)** *for all* $x, y \in X$, $R(x, y) \geq \alpha$ *implies* $f(x) \leq f(y)$.

**Definition 2.2.5** *Let* $(X, R)$ *be a fuzzy lattice,* $\alpha \in ]0, 1]$. *A fuzzy set* $g$ *in* $(X, R)$ *is called a fuzzy* $\alpha$*-ideal of* $X$ *if it satisfies.*

**(I1)** *for all* $x, y \in X$, $g(x \sqcup y) \leq \max\{g(x), g(y)\}$,

**(IA)** *for all* $x, y \in X$, $R(y, x) \geq \alpha \Rightarrow g(x) \leq g(y)$.

**Example 2.2.4** *The fuzzy filter* $f$ *(ideal* $g$*) defined in Example 2.2.2 is a fuzzy* $0.4$*-filter (ideal).*

**Proposition 2.2.5** *[12] If $\alpha \leq \beta$, then any fuzzy $\alpha$-filter (fuzzy $\alpha$-ideal) is a fuzzy $\beta$-filter (fuzzy $\beta$-ideal).*

**Proof.** Let $f$ be a fuzzy $\alpha$-filter of $(X, R)$ and $\alpha \leq \beta$. Then, for any $x, y \in X$, it holds that

$R(x, y) \geq \beta \Rightarrow R(x, y) \geq \alpha \Rightarrow f(x) \leq f(y)$.

Analogously for ideal. ■

**Theorem 2.2.14** *[3] Let $(X, R)$ be a fuzzy lattice and $(f_i)_{i \in I}$ be a family of filters such that $f_i$ is a fuzzy $\alpha_i$-filters of $(X, R)$. Then $\cap_{i \in I} f_i$ is a fuzzy $\sup_{i \in I} \alpha_i$-filter of $(X, R)$.*

**Proof.** Let $(X, R)$ be a fuzzy lattice and suppose that for all $i \in I$, $f_i$ is a fuzzy $\alpha_i -$filter of $(X, R)$.

For all $x, y \in X$,

$$\begin{aligned} R(x, y) \geq \sup_{i \in I} \alpha_i \quad &\Rightarrow \quad f_i(x) \geq f_i(y), \forall i \in I \\ &\Rightarrow \quad \inf_{i \in I} f_i(x) \geq \inf_{i \in I} f_i(y) \qquad \blacksquare \\ &\Rightarrow \quad (\cap_{i \in I} f_i)(x) \geq (\cap_{i \in I} f_i)(y). \end{aligned}$$

**Theorem 2.2.15** *Let $(X, R)$ be a fuzzy lattice and $(g_i)_{i \in I}$ be a family of ideals such that $g_i$ is a fuzzy $\alpha_i$-ideals of $(X, R)$. Then $\cap_{i \in I} g_i$ is a fuzzy $\inf_{i \in I} \alpha_i$-ideal of $(X, R)$.*

**Theorem 2.2.16** *[12] Let $(X, R)$ be a fuzzy lattice and $\alpha \in ]0, 1]$. A mapping $f : X \rightarrow [0, 1]$ is a fuzzy $\alpha$-filter of $(X, R)$ if and only if $f_\alpha = \emptyset$ or $f_\alpha = X$ or $f_\alpha$ is an $\alpha$-filter of $(X, R)$.*

**Proof.** Let $f : X \rightarrow [0, 1]$ be a fuzzy $\alpha$-filter for all $\alpha \in ]0, 1]$. Suppose that $f_\alpha \neq \phi$ and $f_\alpha \neq X$, To prove that the cut $f_\alpha$ is an $\alpha$-filter.

Let $x, y \in f_\alpha$, i.e., $f(x) \geq \alpha$ and $f(y) \geq \alpha$, then $f(x \sqcap y) \geq \alpha$. Therefore, $x \sqcap y \in f_\alpha$.

Let $x \in f_\alpha$ and $R(x, y) \geq \alpha$ then $f(x) \geq \alpha$, since $f(y) \geq f(x) \geq \alpha$. Therefore, $y \in f_\alpha$.

Conversely, suppose that $f_\alpha$ is an $\alpha$-filter of $(X, R)$ for all $\alpha \in ]0, 1]$. Let $x, y \in X$.

If $\min \{f(x), f(y)\} = 0$, then we have $f(x \sqcap y) \geq \min \{f(x), f(y)\}$. If $\min \{f(x), f(y)\} = \lambda$, then $f(x) \geq \lambda$ and $f(y) \geq \lambda$, then $x, y \in f_\lambda$ this implies that $x \sqcap y \in f_\lambda$, hence $f(x \sqcap y) \geq \lambda$. Consequently, $f(x \sqcap y) \geq \min \{f(x), f(y)\}$.

On the other hand, if $f(x) = 0$, it holds that $f(x) \leq f(y)$. Otherwise we put $f(x) = \lambda$, we have $R(x, y) \geq \alpha$ and $x \in f_\lambda$, hence $y \in f_\lambda$. Therefore, $\lambda = f(x) \leq f(y)$ which complete the proof. ■

**Theorem 2.2.17** *Let $(X,R)$ be a fuzzy lattice and $\alpha \in\, ]0,1]$. A mapping $g : X \to [0,1]$ is a fuzzy $\alpha$-ideal of $(X,R)$ if and only if $g_\alpha = \emptyset$ or $g_\alpha = X$ or $g_\alpha$ is an $\alpha$-ideal of $(X,R)$.*

**Theorem 2.2.18** *[12] Let $(X,R)$ be a fuzzy lattice such that $(X,R_\alpha)$ is a crisp lattice for every $\alpha \in\, ]0,1]$ and $f$ be a fuzzy subset of $(X,R)$. $f$ is a fuzzy $\alpha$-filter of $(X,R)$ if and only if $f_\alpha = \phi$ or $f_\alpha = X$ or $f_\alpha$ is a crisp filter of $(X,R_\alpha)$.*

**Proof.** Suppose that $(X,R_\alpha)$ is a crisp lattice for every $\alpha \in\, ]0,1]$. Let $f$ be a fuzzy $\alpha$-filter of $(X,R)$. Suppose that $f_\alpha \neq \phi$ and $f_\alpha \neq X$.

If $x,y \in f_\alpha$ we have $f(x) \geq \alpha$ and $f(y) \geq \alpha$, hence $f(x \sqcap y) \geq \min\{f(x), f(y)\} \geq \alpha$. Therefore, $x \sqcap y \in f_\alpha$.

On the other hand, let $x \in f_\alpha$ and $(x,y) \in R_\alpha$. Hence, $f(x) \geq \alpha$ and $R(x,y) \geq \alpha$, then $\alpha \leq f(x) \leq f(y)$, which implies that $y \in f_\alpha$.

Conversely, suppose that $f_\alpha$ is a crisp filter of $(X,R_\alpha)$ for all $\alpha \in\, ]0,1]$. Let $x,y \in X$, if $\min\{f(x), f(y)\} = 0$, we have immediately $f(x \sqcap y) \geq \min\{f(x), f(y)\}$, if $\min\{f(x), f(y)\} \neq 0$, we put $\min\{f(x), f(y)\} = \lambda$, $\lambda \in\, ]0,1]$, then $x,y \in f_\lambda$ which implies that $x \sqcap y \in f_\lambda$, hence $f(x \sqcap y) \geq \lambda$, and then $f(x \sqcap y) \geq \min\{f(x), f(y)\}$.

On the other hand, if $f(x) = 0$, we have $f(x) \leq f(y)$. If $f(x) \neq 0$ we put $f(x) = \lambda$, $\lambda \in\, ]0,1]$ then $R(x,y) \geq \lambda$ and $f(x) \geq \lambda$, this implies that $x \in f_\lambda$ and $(x,y) \in R_\lambda$, hence $y \in f_\lambda$. Therefore, $\lambda = f(x) \leq f(y)$ which complete the proof. ■

**Theorem 2.2.19** *Let $(X,R)$ be a fuzzy lattice such that $(X,R_\alpha)$ is a crisp lattice for every $\alpha \in\, ]0,1]$ and $g$ be a fuzzy subset of $(X,R)$. $g$ is a fuzzy $\alpha$-ideal of $(X,R)$ if and only if $g_\alpha = \phi$ or $g_\alpha = X$ or $g_\alpha$ is a crisp ideal of $(X,R_\alpha)$.*

## 2.3 Fuzzy Prime Filters and Fuzzy Prime Ideals In a Fuzzy Lattice

In this section, we will focus on prime filters and fuzzy prime filters of a fuzzy lattice $(X,R)$.

## 2.3.1 Definitions and Properties

We recall that $F \subseteq X$ is a prime filter of a fuzzy lattice $(X, R)$ if $F$ is proper $(F \neq X)$ and for all $x, y \in X$, $x \sqcup y \in F$ imply $x \in F$ or $y \in F$.

Dually, $I$ is a prime ideal if $I$ is proper $(I \neq X)$ and for all $x, y \in X$, $x \sqcap y \in I$ imply $x \in I$ or $y \in I$.

**Definition 2.3.1** *[12] Let $(X, R)$ be a fuzzy lattice and $f$ be a fuzzy filter of $(X, R)$. Then $f$ is called a fuzzy prime filter if for any $x, y \in X$, $f(x \sqcup y) = \max\{f(x), f(y)\}$.*

**Definition 2.3.2** *Let $(X, R)$ be a fuzzy lattice and $g$ be a fuzzy filter of $(X, R)$. Then $g$ is called a fuzzy prime ideal if for any $x, y \in X$, $g(x \sqcap y) = \max\{g(x), g(y)\}$.*

**Example 2.3.1** *Let $X = [0, 1]$ and $R$ be the fuzzy relation on $X$ defined by*

$$R(x, y) = \begin{cases} 1 & \text{if } x = y, \\ \lambda & (\lambda \in ]0, 1]) \ \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

*$R$ is a fuzzy ordering relation on $[0, 1]$.*

*Since $R(x, y) > 0$ or $R(y, x) > 0$, for every $x, y \in [0, 1]$, then $R$ is a fuzzy total ordering relation. Hence, $([0, 1], R)$ is a fuzzy lattice.*

*$f$, $g$ are a fuzzy sets defined by*

$f : [0, 1] \to [0, 1], f(x) = x,$

$g : [0, 1] \to [0, 1], f(x) = 1 - x.$

*$f$ is a fuzzy prime filter, and $g$ is a fuzzy prime ideal.*

**Theorem 2.3.1** *[12] Let $(X, R)$ be a fuzzy lattice. If $f$ is a non-constant fuzzy filter of $(X, R)$. $f$ is a fuzzy prime filter of $X$ if and only if $f_\alpha = \phi$ or $f_\alpha = X$ or $f_\alpha$ is a prime filter of $X$ for all $\alpha \in [0, 1]$.*

**Proof.** Let $f$ be a fuzzy prime filter of $(X, R)$ with $f_\alpha \neq \phi$ and $f_\alpha \neq X$. If $x \sqcup y \in f_\alpha$ implies that $f(x \sqcup y) \geq \alpha$ it follows that $\max\{f(x), f(y)\} \geq \alpha$, i.e., $x \in f_\alpha$ or $y \in f_\alpha$. Hence, $f_\alpha$ is a prime filter of $X$.

Conversely, let $f$ be a non-constant fuzzy subset of $X$ such that for every $\alpha \in [0,1]$ $f_\alpha$ is a prime filter of $X$.

Let $x, y \in X$, setting $f(x \sqcup y) = \lambda$ it follows that $x \sqcup y \in f_\lambda$, which implies $x \in f_\lambda$ or $y \in f_\lambda$, i.e., $\max\{f(x), f(y)\} \geq \lambda$, which implies $\max\{f(x), f(y)\} \geq f(x \sqcup y)$.

In other hand, since $f$ is a filter, $R(x, (x \sqcup y)) > 0$ and $R(y, (x \sqcup y)) > 0$ this imply $f(x \sqcup y) \geq f(x)$ and $f(x \sqcup y) \geq f(y)$, it follows that $f(x \sqcup y) = \max\{f(x), f(y)\}$. Hence, $f$ is a prime filter of $X$. ■

**Theorem 2.3.2** *Let $(X, R)$ be a fuzzy lattice. If $g$ is a non-constant fuzzy ideal of $(X, R)$. $g$ is a fuzzy prime ideal of $X$ if and only if $g_\alpha = \phi$ or $g_\alpha = X$ or $g_\alpha$ is a prime ideal of $X$ for all $\alpha \in [0,1]$.*

**Example 2.3.2** *Let $(X, R)$ be the fuzzy lattice defined in Example 2.2.2.*

*Let $f = \{(a, 0.0), (b, 0.3), (c, 0.3), (d, 0.5), (e, 0.5), (f, 0.6)\}$. be the fuzzy filter of $(X, R)$.*

*$f$ is a fuzzy prime filter of $X$.*

*It is easy checked that $f_{0.3} = \{b, c, d, e, f\}$, $f_{0.5} = \{d, e, f\}$, $f_{0.6} = \{f\}$ are a prime filters of $X$.*

*Let $g = \{(a, 0.8), (b, 0.5), (c, 0.3), (d, 0.5), (e, 0.3), (f, 0.0)\}$. be the fuzzy ideal of $(X, R)$.*

*$g$ is a fuzzy prime ideal of $X$.*

*It is easy checked that $f_{0.3} = \{a, b, c, d, e\}$, $f_{0.5} = \{a, b, d\}$, $f_{0.8} = \{a\}$ are a prime ideals of $X$.*

**Remark 2.3.1** *Let $(Y_i)_{i \in I}$ be a family of fuzzy prime filters (ideals) of $(X, R)$. $\cap_{i \in I} Y_i$ is not a fuzzy prime filter (ideal).*

**Example 2.3.3** *Let $(X, R)$ be a fuzzy lattice where $X = \{0, a, b, 1\}$ and $R$ is a fuzzy relation given by*

| $R$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 0.2 | 0.3 | 0.7 |
| $a$ | 0 | 1 | 0 | 0.6 |
| $b$ | 0 | 0 | 1 | 0.5 |
| 1 | 0 | 0 | 0 | 1 |

.

*Let* $F_1 = \{(0, 0.0), (a, 0.7), (b, 0.0), (1, 0.7)\}$, $F_2 = \{(0, 0.0), (a, 0.0), (b, 0.4), (1, 0.8)\}$ *be two fuzzy prime filters of* $(X, R)$*.*

*So,* $F_1 \cap F_2 = \{(0, 0.0), (a, 0.0), (b, 0.0), (1, 0.7)\}$.

*It is easily to checked that* $F_1 \cap F_2$ *is a fuzzy filter, but* $(F_1 \cap F_2)(a \sqcup b) = (F_1 \cap F_2)(1) = \min(F_1(1), F(1)) = 0.7$.

*In other hand,* $\max\{(F_1 \cap F_2)(a), (F_1 \cap F_2)(b)\} = \max(0, 0) = 0 \neq 0.7$. *Hence* $F_1 \cap F_2$ *is not a fuzzy prime filter.*

*Let* $I_1 = \{(0, 0.4), (a, 0.2), (b, 0.0), (1, 0.0)\}$, $I_2 = \{(0, 0.6), (a, 0.0), (b, 0.5), (1, 0.0)\}$ *be two fuzzy prime ideals of* $(X, R)$*.*

*So,* $I_1 \cap I_2 = \{(0, 0.4), (a, 0.0), (b, 0.0), (1, 0.0)\}$.

*It is easily to checked that* $I_1 \cap I_2$ *is a fuzzy ideal, but* $(I_1 \cap I_2)(a \sqcap b) = (I_1 \cap I_2)(0) = 0.6$.

*In other hand,* $\max\{(I_1 \cap I_2)(a), (I_1 \cap I_2)(b)\} = \max(0, 0) = 0 \neq 0.6$. *Hence* $I_1 \cap I_2$ *is not a fuzzy prime ideal.*

**Theorem 2.3.3** *[3] Let* $F$ *be a proper filter of* $(X, R)$*.* $F$ *is prime filter if and only if for any filters* $H$ *and* $G$*,* $H \cap G \subseteq F$ *implies* $H \subseteq F$ *or* $G \subseteq F$*.*

**Proof.** Suppose that $F$ is a prime filter and $H, G$ are two filters satisfying $H \cap G \subseteq F$. If $H \nsubseteq F$, then there exists $x \in H$, and $x \notin F$. Let $y \in G$. Then $x \sqcup y \in H$ and $x \sqcup y \in G$ as $H, G$ are filters, $R(x, x \sqcup y) > 0$ and $R(x, x \sqcup y) > 0$, and so, $x \sqcup y \in H \cap G$. From $H \cap G \subseteq F$ we have $x \sqcup y \in F$. Since $F$ is a prime filter, it holds that $y \in F$. Therefore, $G \subseteq F$.

Conversely, let $F$ be a proper filter of $X$ verifying for any filters $H$ and $G$, $H \cap G \subseteq F$ implies $H \subseteq F$ or $G \subseteq F$. Let $x, y \in X$ such that $x \sqcup y \in F$. $\langle\{x\}\rangle$ and $\langle\{y\}\rangle$ are filters by Lemma 2.1.1. Since $\langle\{x\}\rangle \cap \langle\{y\}\rangle = \langle\{x \sqcup y\}\rangle \subseteq F$, it follows that $\langle\{x\}\rangle \subseteq F$ or $\langle\{y\}\rangle \subseteq F$. Thus, $x \in F$ or $y \in F$. This shows that $F$ is a prime filter. ■

**Theorem 2.3.4** *Let* $I$ *be a proper ideal of* $(X, R)$*.* $I$ *is a prime ideal if and only if for any ideals* $H$ *and* $G$*,* $H \cap G \subseteq I$ *implies* $H \subseteq I$ *or* $G \subseteq I$*.*

**Proof.** Consider two ideals $H$ and $G$ verifying $H \cap G \subseteq I$ and suppose that $H \subsetneq I$ and $G \subsetneq I$, then there exist $x, y$ in $X$ such that $x \in H$, $x \notin I$ and $y \in G$, $y \notin I$. Since $H, G$ are ideals, $R(x \sqcap y, x) > 0$ and $R(x \sqcap y, y) > 0$, hence $x \sqcap y \in H$ and $x \sqcap y \in G$, so, $x \sqcap y \in H \cap G$. From $H \cap G \subseteq I$ we have $x \sqcap y \in I$. Since $I$ is a prime ideal, it holds that $x \in I$ or $y \in I$, this mean $H \subseteq I$ or $G \subseteq I$, contradiction.

So, $I$ is a prime ideal implies then, for any ideals $H$ and $G$ such that $H \cup G \subseteq I$, implies $H \subseteq I$ or $G \subseteq I$.

On the other hand, let $I$ be an ideal of $X$ verifying for any ideals $H$ and $G$, $H \cap G \subseteq I$ implies $H \subseteq I$ or $G \subseteq I$, and show that $I$ is a prime ideal. For this, let $x, y$ in $X$ with $x \sqcap y \in I$.

$x \sqcap y \in I$, this implies $\downarrow x \cap \downarrow y = \downarrow (x \sqcap y) \subseteq I$, this implies $\downarrow x \in I$ or $\downarrow y \in I$, i.e., $x \in I$ or $y \in I$.

Hence $I$ is prime. ■

**Theorem 2.3.5** *[12] Let $(X, R)$ be a fuzzy lattice. If $F$ is a filter of $(X, R)$, then $F$ is a prime filter of $X$ if and only if $\Psi_F$ is a fuzzy prime filter of $X$.*

**Proof.** Let $F$ is a prime filter of $X$ and $x, y \in X$.

If $x \in F$ or $y \in F$, then $\Psi_F(x \sqcup y) = \alpha = \max\{\Psi_F(x), \Psi_F(y)\}$.

If $x \notin F$ and $y \notin F$, then $\Psi_F(x \sqcup y) = \beta = \max\{\Psi_F(x), \Psi_F(y)\}$.

Conversely, if $\Psi_F$ is a fuzzy prime fuzzy filter of $X$, then $\Psi_F(x \sqcup y) = \max\{\Psi_F(x), \Psi_F(y)\}$.

If $x \sqcup y \in F$ then $\Psi_F(x \sqcup y) = \alpha$, hence $\max\{\Psi_F(x), \Psi_F(y)\} = \alpha$, then $\Psi_F(x) = \alpha$ or $\Psi_F(y) = \alpha$, which implies $x \in F$ or $y \in F$. ■

**Theorem 2.3.6** *Let $(X, R)$ be a fuzzy lattice. If G is an ideal of $(X, R)$, then G is a prime ideal of $X$ if and only if $\Psi_G$ is a fuzzy prime ideal of $X$.*

**Theorem 2.3.7** *[12] Let $(X, R)$ be a fuzzy lattice and $f$ be a non-constant fuzzy filter satisfying the following condition: for any fuzzy filters $h$ and $g$: $h \cap g \subseteq f$ implies $h \subseteq f$ or $g \subseteq f$. Then $f$ is fuzzy prime filter.*

**Theorem 2.3.8** *Let $(X, R)$ be a fuzzy lattice and $g$ be a non-constant fuzzy ideal satisfying the following condition: for any fuzzy ideals $h$ and $k$: $h \cap k \subseteq g$ implies $h \subseteq g$ or $k \subseteq g$. Then $g$ is fuzzy prime filter.*

### 2.3.2 Lattices Homomorphisms, Filters and Ideals

Next, we will define the notion of homomorphism between fuzzy ordered structures.

**Definition 2.3.3** *[12] Let $(X, R)$ and $(Y, \Re)$ two fuzzy lattices. The function $h : X \to Y$ is said to be an homomorphism if it satisfies*

**(1)** $h(x \sqcup y) = h(x) \sqcup h(y)$ *and* $h(x \sqcap y) = h(x) \sqcap h(y)$*, for all* $x, y \in X$.

**(2)** $R(x, y) \leq \Re(h(x), h(y))$*, for all* $x, y \in X$.

*If $h$ is a bijection, then $h$ is said to be an isomorphism.*

**Example 2.3.4** *Let $(X, R)$ be the fuzzy lattice defined in Example 2.2.1.*

*Let $Y = \{x', y', z', v', w'\}$ and let $\Re : Y \times Y \to [0, 1]$ is a fuzzy relation defined by*

| $\Re$ | $x'$ | $y'$ | $z'$ | $v'$ | $w'$ |
|---|---|---|---|---|---|
| $x'$ | 1 | 0 | 0 | 0 | 0 |
| $y'$ | 0.3 | 1 | 0 | 0 | 0 |
| $z'$ | 0.6 | 0.5 | 1 | 0 | 0 |
| $v'$ | 0.7 | 0.5 | 0 | 1 | 0 |
| $w'$ | 0.9 | 0.7 | 0.4 | 1 | 1 |

.

*It is easy to prove that $(Y, \Re)$ is a fuzzy lattice.*

*The function $h : X \longrightarrow Y$, where $h(x) = x'$, $h(y) = y'$, $h(z) = z'$ and $h(w) = w'$, is an homomorphism.*

Let $(X, R)$ and $(Y, \Re)$ two fuzzy lattices and $h : X \to Y$ be an homomorphism. Let $\delta$ be a fuzzy set in $Y$. We denote by $h^{-1}(\delta)$ the set $h^{-1}(\delta) = \{(x, \delta(h(x))) / x \in X\}$.

**Definition 2.3.4** *[15] Let $(X, R)$, $(Y, \Re)$ be L-fuzzy posets. A map $h : X \longrightarrow Y$ is said to be*

**(1)** *an L-fuzzy order-embedding if* $R(x, y) = \Re(h(x), h(y))$ *for all* $x, y \in X$;

**(2)** *an L-fuzzy order-isomorphism if it is an L-fuzzy order-embedding which maps $X$ onto $Y$.*

**Theorem 2.3.9** *[12] Let $(X,R)$ and $(Y,\Re)$ be two fuzzy lattices and let $h:X\to Y$ be an isomorphism. $f$ is a fuzzy filter of $(Y,\Re)$ if and only if $h^{-1}(f)$ is a fuzzy filter of the lattice $(X,R)$. In addition, $h^{-1}(f)$ is isomorphic with the fuzzy filter $f$.*

**Proof.** Let $h:X\to Y$ be an isomorphism and $f$ a fuzzy filter of $Y$.

Let $x,y\in X$, then

$$\begin{aligned} h^{-1}(f)(x\sqcap y) &= f(h(x\sqcap y)) \\ &= f(h(x)\sqcap h(y)) \\ &\geq \min\{f(h(x),f(h(y))\} \\ &= \min\{h^{-1}(f)(x),h^{-1}(f)(y)\}. \end{aligned}$$

Hence $h^{-1}(f)(x\sqcap y)\geq \min\{h^{-1}(f)(x),h^{-1}(f)(y)\}.$

On the other hand, let $x,y\in X$, $R(x,y)>0$ implies $h^{-1}(f)(x)\leq h^{-1}(f)(y).$

Since $h$ is an isomorphism and $R(x,y)>0$ implies $\Re(h(x),h(y))>0$.

Using the fact that $f$ is a filter $\Re(h(x),h(y))>0$, this implies that $f(h(x))\leq f(h(y)).$

Which equivalently $h^{-1}(f)(x)\leq h^{-1}(f)(y).$

Consequently, $h^{-1}(f)$ is a fuzzy filter of the lattice $(X,R)$.

Conversely, let $h^{-1}(f)$ be a fuzzy filter and let $a,b\in Y$, then there exists a unique pair $(x,y)\in X^2$ such that $a=h(x)$ and $b=h(y)$. We have

$$\begin{aligned} f(a\sqcap b) &= f(h(x)\sqcap h(y)) \\ &= f(h(x\sqcap y)) \\ &= h^{-1}(f)(x\sqcap y) \\ &\geq \min\{h^{-1}(f)(x),h^{-1}(f)(y)\} \\ &= \min\{f(h(x)),f(h(y))\} \\ &= \min\{f(a),f(b)\}. \end{aligned}$$

Hence $f(a\sqcap b)\geq \min\{f(a),f(b)\}.$

On the other hand, for $a,b\in Y$, $\Re(a,b)>0$, we have to prove that $f(a)\leq f(b)$.

$a,b\in Y$, since $h$ is an isomorphism, then there exists a pair $(x,y)\in X^2$ such that $a=h(x)$ $b=h(y)$. $\Re(a,b)=\Re(h(x),h(y))=R(x,y)>0$ [4, 15], this implies $h^{-1}(f)(x)\leq h^{-1}(f)(y)$ which means that $f(h(x))\leq f(h(y))$, i.e., $f(a)\leq f(b)$.

Consequently, $f$ is a fuzzy filter. ■

**Theorem 2.3.10** *Let* $(X,R)$ *and* $(Y,\Re)$ *be two fuzzy lattices and let* $h : X \to Y$ *be an isomorphism.* $g$ *is a fuzzy ideal of* $(Y,\Re)$ *if and only if* $h^{-1}(g)$ *is a fuzzy ideal of the lattice* $(X,R)$*. In addition,* $h^{-1}(g)$ *is isomorphic with the fuzzy ideal* $g$*.*

**Theorem 2.3.11** *[12] Let* $(X,R)$ *and* $(Y,\Re)$ *be two fuzzy lattices and let* $h : X \to Y$ *be an isomorphism. If* $h^{-1}(f)$ *is a fuzzy* $\alpha$*-filter of the lattice* $(X,R)$*, then* $f$ *is a fuzzy* $\alpha$*-filter of* $(Y,\Re)$*.*

**Proof.** Let $h : X \to Y$ be an isomorphism and $h^{-1}(f)$ be a fuzzy $\alpha$-filter of $Y$.

The verification of the axiom $(F1)$ is straightforward.

For $a,b \in Y$, $\Re(a,b) \geq \alpha$, we have to prove that $f(a) \leq f(b)$.

$$\begin{aligned} \Re(a,b) \geq \alpha &\Rightarrow \Re(h(x),h(y)) \geq \alpha \\ &\Rightarrow R(x,y) \geq \alpha \\ &\Rightarrow h^{-1}(f)(x) \leq h^{-1}(f)(y) \\ &\Rightarrow f(h(x)) \leq f(h(y)) \\ &\Rightarrow f(a) \leq f(b). \end{aligned}$$

As desired. ■

**Theorem 2.3.12** *Let* $(X,R)$ *and* $(Y,\Re)$ *be two fuzzy lattices and* $h : X \to Y$ *be an isomorphism. If* $h^{-1}(g)$ *is a fuzzy* $\alpha$*-ideal of the lattice* $(X,R)$*, then* $g$ *is a fuzzy* $\alpha$*-ideal of* $(Y,\Re)$*.*

**Theorem 2.3.13** *[12] Let* $(X,R)$ *and* $(Y,\Re)$ *be two fuzzy lattices and let* $h : X \to Y$ *be an isomorphism. Then* $f$ *is a fuzzy prime filter of* $(Y,\Re)$ *if and only if* $h^{-1}(f)$ *is a fuzzy prime filter of the fuzzy lattice* $(X,R)$*.*

**Proof.** Let $h : X \to Y$ be an isomorphism and let $f$ be a fuzzy prime filter of $Y$.

Suffices to prove that $h^{-1}(f)(x \sqcup y) = \max\{h^{-1}(f)(x), h^{-1}(f)(y)\}$.

Let $x,y \in X$, then

$$\begin{aligned} h^{-1}(f)(x \sqcup y) &= f(h(x \sqcup y)) \\ &= f(h(x) \sqcup h(y)) \\ &= f(h(x)) \sqcup f(h(y)) \\ &= \max\{h^{-1}(f)(x), h^{-1}(f)(y)\}. \end{aligned}$$

Hence $h^{-1}(f)$ is a fuzzy prime filter.

Conversely, let $h^{-1}(f)$ be a fuzzy prime filter and let $a, b \in Y$, then there exists a unique pair $(x, y) \in X^2$ such that $a = h(x)$ and $b = h(y)$. We have

$$\begin{aligned} f(a \sqcup b) &= f(h(x) \sqcup h(y)) \\ &= f(h(x \sqcup y)) \\ &= (h^{-1}f)(x \sqcup y) \\ &= \max\{(h^{-1}f)(x), (h^{-1}f)(y)\} \\ &= \max\{f(a), f(b)\}. \end{aligned}$$

Hence, $f$ is a fuzzy prime filter of $Y$. ■

**Theorem 2.3.14** *Let $(X, R)$ and $(Y, \Re)$ two fuzzy lattices and let $h : X \to Y$ be an isomorphism. Then $g$ is a fuzzy prime ideal of $(Y, \Re)$ if and only if $h^{-1}(g)$ is a fuzzy prime ideal of the fuzzy lattice $(X, R)$.*

# Conclusion

In this memory, we have study some characterizations of filters, ideals, fuzzy filters, fuzzy ideals, $\alpha$-filters, $\alpha$-ideals, prime filters and prime ideals of a fuzzy lattice.

The notion of isomorphism between fuzzy lattices was recalled, also the converse image of a fuzzy filter and fuzzy ideal via an isomorphism has been characterized.

# Bibliography

[1] N. Ajmal and K.V. Thomas, Fuzzy lattices, Information sciences, 79 (3-4) (1994) 271-291.

[2] A. Amroune and B. Davvaz, Fuzzy ordered sets and duality for finite fuzzy distributive lattices, Iranian Journal of Fuzzy Systems, 8 (5) (2011) 1-12.

[3] A. Amroune, A. Oumhani and B. Davvaz, Kinds of t-fuzzy Filters of Fuzzy Lattices, Fuzzy Information and Engineering, 9 (3) (2017) 325-343.

[4] A. Amroune and A. Oumhani, A representation theorem for infinite fuzzy distributive lattices, Journal of Intelligent and Fuzzy Systems, 32 (1) (2017) 35-42.

[5] I. Chon, Fuzzy partial order relations and fuzzy lattices, Korean Journal Mathematics, 17 (4) (2009) 361-374.

[6] B.A. Davey and H.A. Priestley, Introduction to lattices and order, Cambridge University Press, New York, (2002).

[7] K.H. Lee, First Course on Fuzzy Theory and Applications, Advances in Intelligent and Soft Computing, Springer-Verlag Berlin Heidelberg, 27 (2004).

[8] I. Mezzomo, B.C. Bedregal and R.H.N. Santiago, Kinds of Ideals of Fuzzy Lattice, Second Brasilian Congress on Fuzzy Systems, (2012) 657-671.

[9] I. Mezzomo, B.C. Bedregal and R.H.N. Santiago, On fuzzy ideals of fuzzy lattice, IEEE International Conference on Fuzzy Systems, (2012) 1-5.

[10] I. Mezzomo, B.C. Bedregal and R.H.N. Santiago, $\alpha$-ideals of fuzzy lattice, In IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), (2013) 157-162.

[11] I. Mezzomo, B.C. Bedregal and R.H.N. Santiago, Types of fuzzy ideals in fuzzy lattices, Journal of Intelligent and Fuzzy Systems, 28 (2015) 929-945.

[12] A. Oumhani and A. Amroune, More on fuzzy lattices and their filters, Analele Universităţii Oradea Fasc. Matematica, Tom XXV (4) (2018) 143-155.

[13] D. Ponasse and J.C. Carrega, Algèbre et topologie booléennes, Masson, Paris, (1979).

[14] B. Schröder, Ordered sets: An introduction with connections from combinatorics to topology, Birkhäuser, (2016).

[15] W. Xie, Z. Qiye and F. Lei, The Dedekind–MacNeille completions for fuzzy posets, Fuzzy Sets and Systems, 160 (16) (2009) 2292-2316.

[16] B. Yuan and W. Wu, Fuzzy ideals on a distributive lattice, Fuzzy Sets Syst, 35 (1990) 231-240.

[17] L.A. Zadeh, Fuzzy Sets, Information and Control, 8 (1965) 338-353.

[18] L.A. Zadeh, Similarity relations and fuzzy orderings, Information sciences, 3 (1971) 177-200.

تم بحمد الله

# خلاصة

في هذه المذكرة، سنقوم بدراسة بعض خصائص المرشحات والمثاليات، والمرشحات الضبابية، والمثاليات الضبابية، وα-مرشحات، وα-مثاليات، والمرشحات الأولية، والمثاليات الأولية لشبكة ضبابية.

نذكر بمفهوم التماثل بين الشبكات الضبابية، كما سيتم تخصيص الصور العكسية لمرشح أولي ضبابي عبر إيزومورفيسم (تماثل).

**كلمات مفتاحية:** شبكة ضبابية، مرشح ضبابي، مثالي ضبابي، α-مرشح، α-مثالي، تماثل شبكات.

# Abstract

In this memory, we will study some characterizations of filters, ideals, fuzzy filters, fuzzy ideals, α-filters, α-ideals, prime filters and prime ideals of a fuzzy lattice.

We will recall the notion of isomorphism between fuzzy lattices, also the converse image of a fuzzy prime filter will be characterized via an isomorphism.

**Keywords:** Fuzzy lattice, fuzzy filter, fuzzy ideal, α-filter, α-ideal, lattices homomorphism.

# Résumé

Dans ce mémoire, nous étudierons quelques caractéristiques de filtres, idéaux, filtres flous, idéaux flous, α -filtre, α -idéal, filtres et idéaux premiers d’un treillis flou.

Nous rappellerons La notion d'isomorphisme entre treillis flous, de même que l'image inverse d'un filtre premier flou seront caractérisée via un isomorphisme.

**Mots-clés:** Treillis flou, filtre flou, idéal flou, α -filtre, α -idéal, homomorphisme de Treillis.