PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESERACH

Mohamed Boudiaf University – M'sila-

Faculty of Mathematics and Informatics

Department of Mathematics

Master of Mathematics

Specialty : Mathematics

Option : Fuctional Analysis

Entitled

Variable local Herz-type Hardy spaces and applications

Persented by: **Azeibar Rabab**

Publicly presented on **/07/2023**

in front of the jury :

Dahmane Achour	**Prof.** University of M'sila	**Chairperson**
Douadi Drihem	**Prof.** University of M'sila	**Sypervisor**
Abdelhamid Tallab	**Prof.** University of M'sila	**Examinator**

University years**: 2022/2023**

DEDICATION

I dedicate this work to my dear parents :
Fatima and Djamel
for their endless love , support and encouragement throughout my pursuit for education . I hope this achievement will fulfill the dream they envisioned in me
.To my beloved siblings
Khaoula , Wafa , Hadjer, and Taha ...
for the shine and sincere love they surrounded me with, to my little nephew Yakin wishing him to accomplish all his precious big dreams

.

Acknowledgments

Above all, I would thank Allah, the almighty for providing me with patience and will to study and reach this level and finish my dissertation ; all great praise for him. This thesis could not have been completed without the help of my teacher and supervisor **Prof Douadi Drihem** to whom I would like to express my sincere thanks , I am grateful for his encouragement , insight advice ,support and patience .My gratitude also goes to the members of the jury : **Prof Madani Moussai** and **Prof Dahmane Achour** who accepted to examine my work In addition My special thanks also go to all the teachers of the department of mathemathics particularly those who taught me before.

Contents

Notation

- $\mathbb{R}^n$ we denote the n-dimensional real Euclidean space.
- $\mathbb{N}$ we denote the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- $\mathbb{Z}$ we denote the set of all integer numbers.
- For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + ... + \alpha_n$.
- The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by $xy = x_1 y_1 + ... + x_n y_n$.
- The expression $f \lesssim g$ means that $f \leqslant cg$ for some independent constant c (and non-negative functions f and g).
- $f \approx g$ means $f \lesssim g \lesssim f$.
- As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.
- supp$f$ is the support of the function f , i.e., the closure of its non-zero set.
- Let $E \subset \mathbb{R}^n$ be a measurable set. $|E|$ stands for the (Lebesgue) measure of E. χ_E denotes its characteristic function.
- $\mathcal{S}(\mathbb{R}^n)$ is used in place of set of all Schwartz functions on $\mathbb{R}^n$.
- $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$.

- Given a function $f \in L^1_{loc}(\mathbb{R}^n)$; the Hardy-Littlewood maximal operator is defined by

  $$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \forall x \in \mathbb{R}^n.$$

  The Fourier transform of a function $f \in \mathcal{S}\left(\mathbb{R}^n\right)$ is defined by

  $$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx, \quad x \in \mathbb{R}^n$$

  Its inverse is denoted by $\mathcal{F}^{-1}(f)$ or $\check{f}$.

- The convolution $f * g$ is is defined by

  $$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy \quad f, g \in \mathcal{S}\left(\mathbb{R}^n\right)$$

Introduction

The history of Herz space goes back to the authors Beurling and Herz in the sixty‘s of the last century [18]. In 1968, Herz and Beurling presented some fundamental form of Herz spaces to study some convolution algebras. Later, Herz generalized these space to convergence for Hardy space.

Recently, as a generalization of Lebesgue space with variable exponent, Herz space with variable exponents are introduced. In fact, in 2010 Izuki proved the boundedness of sublinear operators on Herz space with variable exponents $\dot{K}^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ and $K^{\alpha}_{p(\cdot),q}(\mathbb{R}^n)$ in [20]. In 2012, Almeida and Drihem obtained boundedness results for a wide class of classical operators on Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ and $K^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ in [2]. In 2016, Drihem and Seghiri gave a new norm equivalents of the variable Herz spaces $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and they proved the atomic decomposition for Herz-type Hardy spaces of variable smoothness and integrability.

Our work is divided in to three chapters.

In the first one, we collects fundamental notation and concepts. We also give some key lemmas needed in the proofs of main statements.

In the second chapter we define Herz type-Hardy space with variable smoothness and integrability and present a few aspects of their properties. We present the boundedness of pseudo-differential operators on local variable Herz-type Hardy spaces

In the last chapter, we presente the molecular decomposition of Herz type Hardy space and the boundedness of singular integral operator of convolution type on such spaces.

CHAPTER 1

THE MIXED LEBESGUE-SEQEUNCE SPACES

In this chapter, we expose the concepts and results used throughout this theses. We recall some fundamental proprieties on modular space, variable Lebesgue spaces and mixed Lebesgue-sequence spaces. We also give some key technical lemmas that we will use later.

1.1 Definition and basic properties of modular space

We refer to the monographs [10] and [11] for an exposition on semi-modular spaces. We start by recalling about the semi-modular functional space.

Definition 1.1 *Let X be a (real or complex) vector space. A function $\varrho : X \longrightarrow [0, +\infty]$ is called a semi-modular on X if it satisfies the following condition:*

1. $\varrho(0) = 0$.
2. $\varrho(\lambda x) = \varrho(x)$ *for all* $x \in X$, *and all* λ *with* $|\lambda| = 1$.
3. $\varrho(\lambda x) = 0$ *for all* $\lambda > 0$ *implies* $x = 0$.
4. ϱ *is left-continuous on* $[0, +\infty)$ *for every* $x \in X$.

The function ϱ is called a modular if, in addition,

5. $\varrho(x) = 0$ *implies* $x = 0$.

   We say that ϱ *is continuous if*

6. *the mapping* $\lambda \longrightarrow \varrho(\lambda x)$ *is continuous on* $[0, +\infty)$ *for all* $x \in X$.

   A semi-modular ϱ *is said to be quasi-convex if there exist* $L \geqslant 1$ *such that*

$$\varrho(\delta x + (1 - \delta)y) \leq L(\delta \varrho(x) + (1 \quad \delta)\varrho(y))$$

for all $x, y \in X$ *and* $0 \leq \delta \leq 1$, *if* $L = 1$ *it's said to be convex.*

Now we give examples of modular functions :

Example 1.2 *Let* Ω *is a Lebesgue measurable subset of* $\mathbb{R}^n$.

(a) If $1 \leqslant p < \infty$, *then*

$$\varrho_p(g) := \int_\Omega |g(x)|^p dx.$$

defines a continuous modular on the space of all measurable functions on Ω.

(b) If $1 \leqslant p < \infty$, *then*

$$\varrho_p\left((x_j)\right) := \sum_{j=0}^{\infty} |x_j|^p \,.$$

defines a continuous modular on $\mathbb{R}^n$.

(c) Let $\varphi_\infty(t) := \infty \cdot \chi_{(1,\infty)}(t)$. *Then*

$$\varrho_\infty(f) := \int_\Omega \varphi_\infty(|f(x)|)dx.$$

defines a semi-modular on on the space of all measurable functions on Ω *which is not continuous.*

(d) Let $\omega \in L^1_{loc}(\mathbb{R}^n)$ *with* $\omega > 0$ *almost everywhere and* $1 \leqslant p < \infty$. *Then*

$$\varrho(f) := \int_\Omega |f(x)|\omega(x)dx.$$

defines a continuous modular on the space of all measurable functions on Ω.

Definition 1.3 *If* ϱ *be a semimodular or modular on* X, *then*

$$X_\varrho = \{x \in X : \lim_{\lambda \longrightarrow 0} \varrho(\lambda x) = 0\}$$

is called a semimodular space or modular space, respectively.

Proposition 1.4 *If* ϱ *is a semimodular or modular on* X, *then*

$$X_\varrho = \{x \in X, \exists \lambda > 0 : \varrho(\lambda x) < \infty\}$$

is called a semimodular space or modular space, respectively.

Theorem 1.5 *Let* ϱ *be a semimodular on* X. *Then* X_ϱ *is a quasi-normed* R*-vector space. The quasi-norm, called the Luxemburg quasi-norm, is defined by*

$$\|x\|_\varrho := \inf\left\{\lambda > 0 : \varrho(\frac{x}{\lambda}) \leqslant 1\right\}.$$

Lemma 1.6 *(Norm-modular unit ball property). Let* ϱ *be semi-modular on* X. *Then*

$$\|x\|_\varrho \leq 1 \Leftrightarrow \varrho(x) \leq 1.$$

If ϱ *is continuous, then also*

$\|x\|_\varrho < 1 \Leftrightarrow \varrho(x) < 1$, *and :* $\|x\|_\varrho = 1 \Leftrightarrow \varrho(x) = 1$.

Corollary 1.7 *Let* ϱ *be a semi-modular on* X *and* $x \in X_\varrho$.

(a) *If* $\|x\|_\varrho \leq 1$, *then* $\varrho(x) \leq \|x\|_\varrho$.

(b) *If* $1 < \|x\|_\varrho$, *then* $\|x\|_\varrho \leq \varrho(x)$.

(c) $\|x\|_\varrho \leq \varrho(x) + 1$.

Remark 1.8 *The proof of the above results can be found in [7].*

1.2 Variable Lebesgue spaces

In this section we recall and present some properties of variable Lebesgue spaces. Given an open set $\Omega \subset \mathbb{R}^n$. We put

$$\mathcal{P}_0(\Omega) := \{p : \text{ measurable} : p(\cdot) : \Omega \longrightarrow [c, \infty[: \text{for some } c > 0\}.$$

The elements of $\mathcal{P}_0(\Omega)$ are called exponent functions. In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p(\cdot)$.

We denote by

$$\mathcal{P}(\Omega) := \{p : \text{ measurable} : p(\cdot) : \Omega \subset \mathbb{R}^n \longrightarrow [1, \infty[\}.$$

Given $p \in \mathcal{P}_0(\Omega)$ and a set $E \subseteq \Omega$, let

$$p^-(E) = \text{ess} - \inf_{x \in E} p(x), \qquad p^+(E) = \text{ess-sup}_{x \in E} p(x).$$

If the domain $E = \Omega = \mathbb{R}^n$, we will simply write

$$p^- = p^-(\mathbb{R}^n), \qquad p^+ = p^+(\mathbb{R}^n).$$

Definition 1.9 *Given Ω and $p \in \mathcal{P}_0(\Omega)$. The variable Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by*

$$L^{p(\cdot)}(\Omega) := \left\{ f \text{ measurable} : \exists \lambda > 0 : \lim_{\lambda \longrightarrow 0} \varrho_{L^{p(\cdot)}(\Omega)}(\lambda f) = 0 \right\},$$

equipped with the following quasi-norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}(f/\lambda) \leqslant 1\}.$$

Definition 1.10 *Given Ω and $p \in P_0(\Omega)$, define $L^{p(\cdot)}_{loc}(\Omega)$ by*

$$L^{p(\cdot)}_{loc}(\Omega) := \{f \text{ measurable} : f \in L^{p(\cdot)}(K), : \text{for every compact set} : K \subset \Omega\}.$$

Lemma 1.11 *If* $p \in \mathcal{P}(\mathbb{R}^n)$, *then* $\|f\|_{p(\cdot)} \leqslant 1$ *and* $\varrho_{p(\cdot)}(f) \leqslant 1$ *are equivalent. For* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *we have:*

(i) If $\|f\|_{p(\cdot)} \leqslant 1$, *then* $\varrho_{p(\cdot)}(f) \leqslant \|f\|_{p(\cdot)}$.

(ii) If $1 < \|f\|_{p(\cdot)}$, *then* $\|f\|_{p(\cdot)} \leqslant \varrho_{p(\cdot)}(f)$.

For the proof of this Lemma, see [10].

Lemma 1.12 *(generalized Hölder's inequality). Let* $p, q, s \in \mathcal{P}(\mathbb{R}^n)$ *such that*

$$\frac{1}{s(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *and* $g \in L^{q(\cdot)}(\mathbb{R}^n)$ *then* $fg \in L^{s(\cdot)}(\mathbb{R}^n)$ *and*

$$\|fg\|_{s(\cdot)} \leqslant c\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

1.2.1 Logarithmic Hölder continuity

We say that a function $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\text{loc}}$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \tag{1.1}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the log-*Hölder continuous at the origin* (or has a log *decay at the origin*), if there exist a constant $c_{\log} > 0$ such that

$$|g(x) - g(0)| \leq \frac{c_{\log}}{\log(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$. We say that g satisfies the log-*Hölder continuous at infinity* (or has a log *decay at infinity*), if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally-log -Hölder continuous*, abbreviated $g \in C^{\log}$, if it is *locally log-Hölder continuous* and satisfies the *log-Hölder decay* condition. The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally log-Hölder constant* and the *log-Hölder decay constant*, respectively. We note that all functions $g \in C^{\log}_{\mathrm{loc}}$ always belong to L^∞. By $\tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$ and $\tilde{\mathcal{P}}_\infty^{\log}(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively. The notation $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for all those exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x|\to\infty} p(x)$. Obviously we have $\tilde{\mathcal{P}}^{\log}(\mathbb{R}^n) \subset \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n) \cap \tilde{\mathcal{P}}_\infty^{\log}(\mathbb{R}^n)$. Note that $p \in \tilde{\mathcal{P}}^{\log}(\mathbb{R}^n)$ if and only if $p' \in \tilde{\mathcal{P}}^{\log}(\mathbb{R}^n)$, and since $(p')_\infty = (p_\infty)'$ we write only p'_∞ for any of these quantities. We define the following class of variable exponents

$$\mathcal{P}^{\log} := \Big\{ p \in \mathcal{P} : \frac{1}{p} \in C^{\log} \Big\},$$

were introduced in [6, Section 2]. We define $\frac{1}{p_\infty} := \lim_{|x|\to\infty} \frac{1}{p(x)}$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded.

1.3 The space $\ell^{q(\cdot)}\big(L^{p(\cdot)}\big)$

The targets of this section are twofold. The first one is to recall the concept of variable mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ originally introduced by Almeida and Hästö in [1]. The second one is to recall some fundamental properties related to $\ell^{q(\cdot)}(L^{p(\cdot)})$.

Definition 1.13 *Let* $p, q \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$. *The* mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ *is defined on sequences of* $L^{p(\cdot)}$ *-functions by the modular*

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf \Big\{ \lambda_v > 0 : \varrho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}} \Big) \le 1 \Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf \Big\{ \mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}(f_v)_v \Big) \le 1 \Big\}. \qquad (1.2)$$

Since $q^+ < \infty$, then we can replace (1.2) *by the simpler expression*

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} .$$

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$.

Observed in [1, Proposition 3.5] that $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a modular if $p^+ < \infty$ and it is continuous if $p^+, q^+ < \infty$. Also in [1, Theorem 3.6] it was shown that (1.2) defines a norm in $\ell^{q(\cdot)}(L^{p(\cdot)})$ if $q(\cdot) \geq 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $p(\cdot) \geq 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$ a.e. More recently, it was observed in [22, Theorem 1] that it also becomes a norm if $1 \leq q(x) \leq p(x) \leq \infty$ a.e. on $\mathbb{R}^n$. Simple calculations show that

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \|(|f_v|^r)_v\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r})}, \ r > 0.$$

It is not difficult to verify that $\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty$ implies $(f_v)_v \in \ell^{q(\cdot)}(L^{p(\cdot)})$, which means $f_v \in L^{p(\cdot)}$ for any $v \in \mathbb{N}_0$. On the other hand we also note that the left-continuity of the semi-modular confirms useful equivalence

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq 1 \quad \text{if and only if} \quad \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) \leq 1 \text{ (unit ball property).}$$

The next lemma is a Hardy-type inequality, see [9, Lemma 2].

Lemma 1.14 *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ and $\left\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ belong to ℓ^q and

$$\left\|\{\delta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} + \left\|\{\eta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = cI,$$

with $c > 0$ only depending on a and q.

The following lemma is from [8, Lemma 2.11].

Lemma 1.15 *Let* $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$*. For any cubes (balls)* P *and* Q*, such that* $P \subset Q$ *we have*

$$C\left(\frac{|Q|}{|P|}\right)^{1/p^+} \leq \frac{\|\chi_Q\|_{p(\cdot)}}{\|\chi_P\|_{p(\cdot)}} \leq c\left(\frac{|Q|}{|P|}\right)^{1/p^-}$$

with $c, C > 0$ *are independent of* $|Q|$ *and* $|P|$*.*

The proof of the following results are given in [2].

Lemma 1.16 *Let* $\alpha \in L^\infty(\mathbb{R}^n)$ *and* $r_1 > 0$*. If* α *is* log-*Hölder continuous both at the origin and at infinity, then*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} \left(\frac{r_1}{r_2}\right)^{\alpha^+} & \text{if} \quad 0 < r_2 \leq \frac{r_1}{2} \\ 1 & \text{if} \quad \frac{r_1}{2} < r_2 \leq 2r_1 \\ \left(\frac{r_1}{r_2}\right)^{\alpha^-} & \text{if} \quad r_2 > 2r_1 \end{cases}$$

for any $x \in B(0, r_1) \setminus B(0, \frac{r_1}{2})$ *and* $y \in B(0, r_2) \setminus B(0, \frac{r_2}{2})$*, with the implicit constant not depending on* x, y, r_1 *and* r_2*.*

Lemma 1.17 *Let* $p \in \tilde{\mathcal{P}}_\infty^{\log}(\mathbb{R}^n)$ *and let* $R = B(0, r) \setminus B(0, \frac{r}{2})$*. If* $|R| \geq 2^{-n}$*, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{\frac{1}{p(x)}} \approx |R|^{\frac{1}{p_\infty}}$$

with the implicit constants independent of r *and* $x \in R$*.*

The left-hand side equivalence remains true for every $|R| > 0$ *if we assume, additionally,* $p \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n) \cap \tilde{\mathcal{P}}_\infty^{\log}(\mathbb{R}^n)$*.*

CHAPTER 2

BOUNDENESS OF PSEUDO-DIFFERENTIAL OPERATORS ON LOCAL VARIABLE HERZ-TYPE HARDY SPACES

2.1 Variable Herz-type Hardy spaces

The targets of this section are twofold. The first one is to recall the concept of variable exponent Herz spaces and variable Herz-type Hardy spaces. The second one is to recall some important results related to these function spaces. Most of the results of this section was proved by Drihem and Seghiri in [9]

First, we present the concept of variable exponent Herz spaces. For convenience, we set

$$B_k := B(0, 2^k)\,, \quad R_k := B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}\,, \quad k \in \mathbb{Z}.$$

Definition 2.1 *Let* $p, q \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$ *and* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha \in L^\infty(\mathbb{R}^n)$*. The* inhomogeneous Herz space $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *consists of all* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *such that*

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|f\,\chi_{B_0}\|_{p(\cdot)} + \big\|\left(2^{k\alpha(\cdot)} f\,\chi_k\right)_{k\geq 1}\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{2.1}$$

Similarly, the homogeneous Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all* $f \in$

$L^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})$ *such that*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \| \left(2^{k\alpha(\cdot)} f\, \chi_k\right)_{k\in\mathbb{Z}} \|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{2.2}$$

If α and p, q are constant, then $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = K^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ are the classical Herz spaces.

Let us denote

$$\|\{g_k\}\|_{\ell^q_>(L^{p(\cdot)})} := \left(\sum_{k=0}^{\infty} \|g_k\|^q_{p(\cdot)}\right)^{1/q} \quad \text{and} \quad \|\{g_k\}\|_{\ell^q_<(L^{p(\cdot)})} := \left(\sum_{k=-\infty}^{-1} \|g_k\|^q_{p(\cdot)}\right)^{1/q}$$

for sequences $\{g_k\}_{k\in\mathbb{Z}}$ of measurable functions (with the usual modification if $q = \infty$).

The following statement plays a crucial role in our work.

Proposition 2.2 *Let $\alpha \in L^\infty(\mathbb{R}^n)$, $p, q \in \tilde{\mathcal{P}}^{\log}_0(\mathbb{R}^n)$. If α and q are* log*-Hölder continuous at infinity, then*

$$K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = K^{\alpha_\infty}_{p(\cdot),q_\infty}(\mathbb{R}^n).$$

Additionally, if α and q have a log *decay at the origin, then*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \left\|\{2^{k\alpha(0)} f\, \chi_k\}\right\|_{\ell^{q(0)}_<(L^{p(\cdot)})} + \left\|\{2^{k\alpha_\infty} f\, \chi_k\}\right\|_{\ell^{q_\infty}_>(L^{p(\cdot)})}. \tag{2.3}$$

Lemma 2.3 *Let $q \in \tilde{\mathcal{P}}^{\log}_0(\mathbb{R}^n)$, $p \in \tilde{\mathcal{P}}^{\log}_\infty(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$, and let α and q be log-Hölder continuous, at infinity, with $\alpha \in L^\infty(\mathbb{R}^n)$ and*

$$-\frac{n}{p_\infty} < \alpha_\infty < \frac{n}{p'_\infty}. \tag{2.4}$$

If a sublinear operator T satisfies

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \quad x \notin supp f, \tag{2.5}$$

for any integrable and compactly supported functions f, and T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then T is bounded on $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Lemma 2.4 *Let* $q \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$, $p \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n) \cap \tilde{\mathcal{P}}_\infty^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$, *and let* α *and* q *are* log*-Hölder continuous, both at the origin and at infinity, such that* $\alpha \in L^\infty(\mathbb{R}^n)$ *and*

$$-\frac{n}{p^+} < \alpha^- \leq \alpha^+ < n\left(1 - \frac{1}{p^-}\right). \tag{2.6}$$

Let T *be as in Lemma 2.3. Then* T *is bounded on* $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Since the Hardy-Littlewood maximal operator $\mathcal{M}$ is sublinear, satisfies the size condition (2.5) and it is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ and $1 < p^- \leq p^+ \leq \infty$ (see [7, Theorem 4.3.8]), we immediately arrive at the following result.

Corollary 2.5 *Let* $0 < q \leq \infty$, $p \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$, *and* $\alpha \in L^\infty(\mathbb{R}^n)$.

(i) If (2.4) *holds and* α *satisfies the* log *decay condition at infinity, then* $\mathcal{M}$ *is bounded on* $K_{p(\cdot),q}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(ii) If (2.6) *holds and* α *has a* log *decay both at the origin and at infinity, then* $\mathcal{M}$ *is bounded on* $\dot{K}_{p(\cdot),q}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Now, we recall the definition of variable Herz-type Hardy spaces. Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) := \sup_{\varphi \in \mathcal{A}_N} |\varphi_N^*(f)(x)|,$$

where $\mathcal{A}_N := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha| \leq N, |\beta| \leq N} |x^\alpha \partial^\beta \varphi(x)| \leq 1\}$ with $\varphi_t := t^{-n}\varphi(\frac{\cdot}{t})$.

Definition 2.6 *Let* $p, q \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$ *and* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha \in L^\infty(\mathbb{R}^n)$ *and* $N > n + 1$. *The* inhomogeneous Herz-type Hardy space $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ *consists of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $G_N f \in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ *and we define*

$$\|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} := \|G_N f\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Similarly, the homogeneous Herz-type Hardy space $H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is defined as the set of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $G_N f \in \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and we define*

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|G_N f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Definition 2.7 *Let* $\alpha \in L^\infty(\mathbb{R}^n)$, $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$ *and* $s \in \mathbb{N}_0$. *A function a is said to be a central* $(\alpha(\cdot), p(\cdot))$*-atom, if*

(i) $\mathrm{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \le r\}, \quad r > 0,$

(ii) $\|a\|_{p(\cdot)} \le |\overline{B(0,r)}|^{-\alpha(0)/n}, \quad 0 < r < 1,$

(iii) $\|a\|_{p(\cdot)} \le |\overline{B(0,r)}|^{-\alpha_\infty/n}, \quad r \ge 1,$

(iv) $\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \quad |\beta| \le s.$

A function a on $\mathbb{R}^n$ *is said to be a central* $(\alpha(\cdot), p(\cdot))$*-atom of restricted type, if it satisfies the conditions* (iii), (iv) *above and* $\mathrm{supp} a \subset B(0,r), r \ge 1$

The following atomic decomposition.

Theorem 2.8 *Let* α *and* q *are* log*-Hölder continuous at infinity and* $p \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \le p^+ < \infty$. *For any* $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, *we have*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \tag{2.7}$$

where the series converges in the sense of distributions, $\lambda_k \ge 0$, *each* a_k *is a central* $(\alpha(\cdot), p(\cdot))$*-atom of restricted type with* $\mathrm{supp} a_k \subset B_k$ *and*

$$\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \le c\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha_\infty \ge n(1 - \frac{1}{p_\infty})$ *and* $s \ge \left\lfloor \alpha_\infty + n(\frac{1}{p_\infty} - 1) \right\rfloor$, *and if* (2.7) *holds, then* $f \in HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, *and*

$$\|f\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\left\{ \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \right\},$$

where the infimum is taken over all the decompositions of f as above.

Theorem 2.9 *Let α and q are* log-*Hölder continuous, both at the origin and at infinity and $p \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. For any $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \tag{2.8}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with $\operatorname{supp} a_k \subset B_k$ *and*

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \leq c\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely, if $\alpha(\cdot) \geq n(1 - \frac{1}{p^-})$ and $s \geq \left\lfloor \alpha^+ + n(\frac{1}{p^-} - 1)\right\rfloor$, and if (2.8) holds, then $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, and

$$\|f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\left\{\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f as above.

2.2 An characterization for variable Herz-type Hardy spaces

In this section, we establish some real-variable characterizations of variable Herz-type Hardy spaces. To begin with, present the concepts of various maximal functions as follows.

Definition 2.10 *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. For $t > 0$, set $\varphi_t = t^{-n}\varphi(\cdot/t)$. For any $x \in \mathbb{R}^n$ we define the maximal functions $\varphi^*_+(f)$, $\varphi^*_N(f)$, $\varphi^*_{\nabla,N}(f)$ and $\varphi^{**}_M(f)$ by*

$$\varphi^*_+(f)(x) = \sup_{t>0} |(f * \varphi_t)(x)|$$

$$\varphi^*_\nabla(f)(x) = \sup_{|x-y|<t} |(f * \varphi_t)(y)|$$

$$\varphi_{\nabla,N}^{*}(f)(x)=\sup_{t>0}\sup_{|x-y|<Nt}|(f*\varphi_t)(y)|,\quad N>1$$

and

$$\varphi_{M}^{**}(f)(x)=\sup_{(y,t)\in\mathbb{R}_{+}^{n+1}}|(f*\varphi_t)(y)|\left(\frac{t}{|x-y|+t}\right)^{M},\quad M\in\mathbb{N}.$$

Next we give the following characterization

Theorem 2.11 *Let $q\in\tilde{\mathcal{P}}_{0}^{\log}(\mathbb{R}^n)$, $p\in\tilde{\mathcal{P}}_{\infty}^{\log}(\mathbb{R}^n)$ with $1<p^-\le p^+<\infty$, and let α and q be log-Hölder continuous, at infinity such that $\alpha\in L^\infty(\mathbb{R}^n)$ and $\alpha^->0$. For $\varphi\in\mathcal{S}(\mathbb{R}^n)$, the following statements are equivalent:*

(a) $f\in HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(b) *For some* $N>1$, $\varphi_{\nabla,N}^{*}(f)\in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(c) $\varphi_{\nabla}^{*}(f)\in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(d) $\varphi_{+}^{*}(f)\in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Moreover,

$$\|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\left\|\varphi_{\nabla,N}^{*}(f)\right\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\|\varphi_{\nabla}^{*}(f)\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\left\|\varphi_{+}^{*}(f)\right\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Theorem 2.12 *Let $q\in\tilde{\mathcal{P}}_{0}^{\log}(\mathbb{R}^n)$, $p\in\tilde{\mathcal{P}}_{0}^{\log}(\mathbb{R}^n)\cap\tilde{\mathcal{P}}_{\infty}^{\log}(\mathbb{R}^n)$ with $1<p^-\le p^+<\infty$, and let α and q be log-Hölder continuous, both at the origin and at infinity such that $\alpha\in L^\infty(\mathbb{R}^n)$ and $\alpha^->0$. For $\varphi\in\mathcal{S}(\mathbb{R}^n)$, the following statements are equivalent:*

(a) $f\in H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(b) *For some* $N>1$, $\varphi_{\nabla,N}^{*}(f)\in\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(c) $\varphi_{\nabla}^{*}(f)\in\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(d) $\varphi_{+}^{*}(f)\in\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Moreover,

$$\|f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\left\|\varphi_{\nabla,N}^{*}(f)\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\|\varphi_{\nabla}^{*}(f)\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\approx\left\|\varphi_{+}^{*}(f)\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Proof. By similarity we only consider the homogeneous case. (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) : This follows from

$$\varphi_{+}^{*}(f)(x)\le\varphi_{\nabla}^{*}(f)(x)\le\varphi_{\nabla,N}^{*}(f)(x).$$

Note that for any $N > n+1$,

$$\varphi_{\nabla}^{*}(f)\,(x) \lesssim G_N\,(f)\,(x)\,. \tag{2.9}$$

(a) $\Longrightarrow$ (c) : This is a consequence of (2.9). To complete the proof, it suffices to show that (d) $\Longrightarrow$ (b) and (d) $\Longrightarrow$ (a).

We first prove (d) $\Longrightarrow$ (b). For $l, N \in \mathbb{N}$, define

$$g_{\varepsilon,l,N}^{*}\,(x) = \sup_{|x-y|<Nt<1/\varepsilon} |(f * \varphi_t)\,(y)|\,\big(\frac{Nt}{Nt+\varepsilon}\big)\,(1+\varepsilon N\,|y|)^{-l}\,.$$

So by the Fatou lemma of series and integration, we only need to prove that

$$\|g_{\varepsilon,l,N}^{*}\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim N^{n/r}\|\varphi_{+}^{*}\,(f)\,\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}, \quad 0<r<1.$$

Now, we further define

$$h_{\varepsilon,l,N}^{*}\,(x) = \sup_{|x-y|<Nt<1/\varepsilon} t\,|\nabla_y\,(f * \varphi_t)\,(y)|\,\big(\frac{Nt}{Nt+\varepsilon}\big)\,(1+\varepsilon N\,|y|)^{-l}\,.$$

As in [28] if l is sufficiently large, then we have

$$h_{\varepsilon,l,N}^{*}\,(x) \le c\left(\mathcal{M}\,\left(g_{\varepsilon,l,N}^{*}\right)^{\delta}(x)\right)^{1/\delta}, \quad 0<\delta<1,$$

where the positive constant c is independent of ε, N and f. Let

$$A_\varepsilon := \left\{x : h_{\varepsilon,l,N}^{*}\,(x) \le \tilde{c}g_{\varepsilon,l,N}^{*}\,(x)\right\}, \qquad \text{and} \quad A_\varepsilon^{c} := \mathbb{R}^n \backslash A_\varepsilon$$

where $\tilde{c}$ will be chosen later. Take $0<\delta<1$ such that $0<\alpha^{-}\le\alpha^{+}<n\big(\frac{1}{\delta}-\frac{1}{p^{-}}\big)$, then by Lemma 2.4, we obtain that

$$\begin{aligned}
\|g_{\varepsilon,l,N}^{*}\chi_{A_\varepsilon}\|_{K_{\ \ p(\cdot),q(\cdot)}^{\alpha(\cdot)}} &\le \frac{1}{\tilde{c}}\|h_{\varepsilon,l,N}^{*}\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\ \alpha(\cdot)}} \\
&\le \frac{c}{\tilde{c}}\|\mathcal{M}\,\left(g_{\varepsilon,l,N}^{*}\right)^{\delta}\|_{\dot{K}_{p(\cdot)/\delta,q(\cdot)/\delta}^{\delta\alpha(\cdot)}}^{1/\delta} \\
&\le \frac{c}{\tilde{c}}\|\left(g_{\varepsilon,l,N}^{*}\right)^{\delta}\|_{\dot{K}_{p(\cdot)/\delta,q(\cdot)/\delta}^{\delta\ \alpha(\cdot)}}^{1/\delta} \\
&= \frac{c}{\tilde{c}}\|g_{\varepsilon,l,N}^{*}\|_{\dot{K}_{\ \ p(\cdot),q(\cdot)}^{\alpha(\cdot)}},
\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|g_{\varepsilon,l,N}^{*}\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} &\leq \left\|g_{\varepsilon,l,N}^{*}\chi_{A_{\varepsilon}}\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}+\left\|g_{\varepsilon,l,N}^{*}\chi_{A_{\varepsilon}}\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\\ &\leq \left\|g_{\varepsilon,l,N}^{*}\chi_{A_{\varepsilon}}\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}+\frac{c}{\tilde{c}}\left\|g_{\varepsilon,l,N}^{*}\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\\ &\leq 2\left\|g_{\varepsilon,l,N}^{*}\chi_{A_{\varepsilon}}\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}},\end{aligned}$$

provided that $\tilde{c}$ is sufficiently large. Thus the proof that (d) $\Longrightarrow$ (b) can be reformulated as showing that

$$\left\|g_{\varepsilon,l,N}^{*}\chi_{A_{\varepsilon}}\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\lesssim N^{n/r}\left\|\varphi_{+}^{*}\left(f\right)\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\quad 0<r<1. \tag{2.10}$$

It will know that, see [28]

$$g_{\varepsilon,l,N}^{*}\left(x\right)\leq cN^{n/r}\left(\mathcal{M}\left(\varphi_{+}^{*}\left(f\right)\right)^{r}\left(x\right)\right)^{1/r},\quad \text{for any } 0<r<1 \quad \text{and } x\in A_{\varepsilon}.$$

Now choosing r such that $0<\alpha^{-}\leq\alpha^{+}<n\left(\frac{1}{r}-\frac{1}{p^{-}}\right)$, then by Lemma 2.4, and the last inequality we get

$$\begin{aligned}\left\|g_{\varepsilon,l,N}^{*}\chi_{A_{\varepsilon}}\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} &\lesssim N^{n/r}\left\|\left(\mathcal{M}\left(\varphi_{+}^{*}\left(f\right)\right)^{r}\right)^{1/r}\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\\ &\lesssim N^{n/r}\left\|\mathcal{M}\left(\varphi_{+}^{*}\left(f\right)\right)^{r}\right\|_{\dot{K}_{p(\cdot)/r,q(\cdot)/r}^{r\alpha(\cdot)}}^{1/r}\\ &\lesssim N^{n/r}\left\|\varphi_{+}^{*}\left(f\right)\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.\end{aligned}$$

This finishes the proof of (d) $\Longrightarrow$ (b). Moreover,

$$\left\|\varphi_{\nabla,N}^{*}(f)\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}\lesssim N^{n/r}\left\|\varphi_{+}^{*}\left(f\right)\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}. \tag{2.11}$$

Now we consider (d) $\Longrightarrow$ (a). It is easy to verify that

$$\varphi_{M}^{**}\left(f\right)\left(x\right)\lesssim\varphi_{\nabla}^{*}(f)\left(x\right)+\sum_{k=0}^{\infty}2^{-kM}\varphi_{\nabla,2^{k+1}}^{*}\left(f\right)\left(x\right), \tag{2.12}$$

and since

$$G_{N}\left(f\right)\left(x\right)\lesssim\varphi_{M}^{**}\left(f\right)\left(x\right) \tag{2.13}$$

for all $N > M+n+1$ see [12]. From (2.12), (2.13) and (2.11), we have gives that if N is large enough, then

$$\begin{aligned}\left\|u^*_{\varepsilon,l,N}\chi_{A_\varepsilon}\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &\lesssim \left\|\varphi^*_{\nabla}(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} + \sum_{k=0}^{\infty} 2^{-kM}\left\|\varphi^*_{\nabla,2^{k+1}}(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \\ &\lesssim \left\|\varphi^*_{+}(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \sum_{k=0}^{\infty} 2^{-k(M-n/r)} \\ &\lesssim \left\|\varphi^*_{+}(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\end{aligned}$$

for any N large enough, where $M > n/r$. Hence the proof of Theorem is completed. ■

Remark 2.13 *If α and q are constants, then the statements corresponding to Theorems 2.11 and 2.12 can be found in [38, Theorem3.3].*

2.3 Local variable Herz-type Hardy spaces

In this section, we give the definition of local variable Herz-type Hardy spaces and the atomic decomposition of these function spaces.

Let $\tilde{G}_N f$ be the maximal function f defined by

$$\tilde{G}_N f(x) = \sup_{\varphi\in\mathcal{A}_N}\ \sup_{0<t<1,|x-y|<Nt} |\left(f*\varphi_t\right)(y)|,$$

where $\mathcal{A}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|\le N,|\beta|\le N} |x^\alpha \partial^\beta \varphi(x)| \le 1\}$ and $\varphi_t = t^{-n}\varphi(\frac{\cdot}{t})$. Then we introduce localized variable Herz-type Hardy spaces as follows.

Definition 2.14 *Let $q \in \tilde{\mathcal{P}}^{\log}_0(\mathbb{R}^n), \alpha \in L^\infty(\mathbb{R}^n)$ and $p \in \tilde{\mathcal{P}}^{\log}_0(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$.*

(i) *If α and q be log-Hölder continuous, both at the origin and at infinity and let $\alpha(\cdot) \ge n\left(1-\frac{1}{p^-}\right)$. Then a function $f \in L^{p(\cdot)}(\mathbb{R}^n\backslash\{0\})$ is said to be in the space $h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if $\tilde{G}_N f$ belongs to the space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ for any $N > \frac{\alpha^+ + p^-}{p^- - 1} + 1$. Moreover, we define that*

$$\|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \left\|\tilde{G}_N(f)\right\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

(ii) *If* α *and* q *be log-Hölder continuous, at infinity and* $\alpha_\infty \geq n\big(1-\frac{1}{p_\infty}\big)$. *Then a function* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *is said to be in the space* $hK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ *if* $\tilde{G}_N f$ *belongs to the space* $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ *for any* $N > \frac{\alpha_\infty p_\infty}{p_\infty - 1} + 1$. *Moreover, we define that*

$$\|f\|_{hK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} = \left\|\tilde{G}_N(f)\right\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Now, we establish the maximal function characterizations of the local variable Herz-type Hardy spaces.

Theorem 2.15 *Let* α *and* q *are* log-*Hölder continuous at infinity and* $p \in \tilde{\mathcal{P}}_\infty^{\log}(\mathbb{R}^n)$ *such that* $1 < p^- \leq p^+ < \infty$, *and* $\alpha_\infty \geq n\big(1-\frac{1}{p_\infty}\big)$. *For* $f \in \mathcal{S}'(\mathbb{R}^n)$, *the following statements are equivalent:*

(a) $f \in hK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(b) $\tilde{\varphi}_\nabla^*(f) = \sup\limits_{0<t<1} \sup\limits_{|x-y|<t} |(f*\varphi_t)(y)| \in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(c) $\tilde{\varphi}_+^*(f) = \sup\limits_{0<t<1} |(f*\varphi_t)(y)| \in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Moreover,

$$\|f\|_{hK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \|\tilde{\varphi}_\nabla^*(f)\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \left\|\tilde{\varphi}_+^*(f)\right\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

Theorem 2.16 *Let* α *and* q *are* log-*Hölder continuous both at the origin and at infinity and* $p \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n) \cap \tilde{\mathcal{P}}_\infty^{\log}(\mathbb{R}^n)$ *such that* $1 < p^- \leq p^+ < \infty$, *and* $\alpha(\cdot) \geq n\big(1-\frac{1}{p^-}\big)$. *For* $f \in \mathcal{S}'(\mathbb{R}^n)$, *the following statements are equivalent:*

(a)$f \in h\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(b) $\tilde{\varphi}_\nabla^*(f) = \sup\limits_{0<t<1} \sup\limits_{|x-y|<t} |(f*\varphi_t)(y)| \in \dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

(c) $\tilde{\varphi}_+^*(f) = \sup\limits_{0<t<1} |(f*\varphi_t)(y)| \in \dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Moreover,

$$\|f\|_{h\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \|\tilde{\varphi}_\nabla^*(f)\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \approx \left\|\tilde{\varphi}_+^*(f)\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

We omit the proofs of Theorems 2.15 and 2.16 since they are essentially is similar to the proof of Theorem 2.12.

On the other hand, we obtain the following relation between the local variable Herz-type Hardy spaces and the variable Herz-type Hardy spaces as follows.

Theorem 2.17 *Let* α *and* q *are* log-*Hölder continuous both at infinity and* $p \in \tilde{\mathcal{P}}_{\infty}^{\log}(\mathbb{R}^n)$ *such that* $1 < p^- \leq p^+ < \infty$, *and* $\alpha_\infty \geq n\big(1 - \frac{1}{p_\infty}\big)$. *Suppose that* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *such that* $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$, *and*

$$\int_{\mathbb{R}^n} x^\beta \varphi(x)\, dx = 0, \quad \textit{for all } \beta \in \mathbb{N}^n, |\beta| \leq N.$$

Then

$$\|f - \varphi * f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \|f\|_{hK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}$$

Moreover, if $f \in hK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, *then* $f - \varphi * f \in HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Theorem 2.18 *Let* α *and* q *are* log-*Hölder continuous both at the origin and at infinity, and* $p \in \tilde{\mathcal{P}}_{0}^{\log}(\mathbb{R}^n) \cap \tilde{\mathcal{P}}_{\infty}^{\log}(\mathbb{R}^n)$ *such that* $1 < p^- \leq p^+ < \infty$, *and* $\alpha(\cdot) \geq n\big(1 - \frac{1}{p^-}\big)$. *Suppose that* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *such that* $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$, *and*

$$\int_{\mathbb{R}^n} x^\beta \varphi(x)\, dx = 0, \quad \textit{for all } \beta \in \mathbb{N}, |\beta| \leq N.$$

Then

$$\|f - \varphi * f\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \|f\|_{h\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}$$

Moreover, if $f \in h\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, *then* $f - \varphi * f \in H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Proof. By similarity we only consider the homogeneous case. Take $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$. Using Theorem 2.12, we have

$$\begin{aligned} \big\|f - \varphi * f\big\|_{H\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} &\approx \big\|\sup_{t>0} |\psi_t * (f - \varphi * f)|\,\big\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \\ &\lesssim c\big\|\sup_{0<t<1} |\psi_t * f|\,\big\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} + c\big\|\sup_{0<t<1} |\psi_t * \varphi * f|\,\big\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \\ &\quad + c\big\|\sup_{t\geq 1} |\varphi_t * (f - \varphi * f)|\,\big\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \end{aligned}$$

On the other hand, it will know that, see [28]

$$\sup_{0<t<1} |\psi_t * f| \lesssim \tilde{G}_N f(x) \tag{2.14}$$

and

$$\sup_{0<t<1} |\psi_t * \varphi * f| \lesssim \tilde{G}_N f(x) \tag{2.15}$$

and

$$\sup_{t\geq 1} |\psi_t * (f - \varphi * f)| \lesssim \tilde{G}_N f(x). \tag{2.16}$$

From (2.14), (2.15) and (2.16), we see that

$$\|f - \varphi * f\|_{H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|\tilde{G}_N f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

■

Now we give the notion of block.

Definition 2.19 *Let* $\alpha \in L^\infty(\mathbb{R}^n), p \in \mathcal{P}(\mathbb{R}^n)$ *and* $q \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$*. A function* a *is said to be a central* $(\alpha(\cdot), p(\cdot))$*-block, if*

(i) $\operatorname{supp} a \subset \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| \leq r\}, \quad r > 0,$

(ii) $\|a\|_{p(\cdot)} \lesssim r^{-\alpha(0)}, \quad 0 < r < 1,$

(iii) $\|a\|_{p(\cdot)} \lesssim r^{-\alpha_\infty}, \quad r \geq 1.$

A function a *is said to be a central* $(\alpha(\cdot), p(\cdot))$*-block of restricted type, if it satisfies the condition* (iii)*, above and* $\operatorname{supp} a \subset B(0,r), r \geq 1$.

Next we come to the atomic decomposition theorems.

Theorem 2.20 *Let* α *and* q *are* log*-Hölder continuous at infinity and* $p \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$*. For any* $f \in hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*, we have*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \quad \textit{in the sense of } \mathcal{S}'(\mathbb{R}^n). \tag{2.17}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$,*each* a_k *is a central* $(\alpha(\cdot), p(\cdot))$*-block of restricted type with* $\operatorname{supp} a_k \subset B_k$ *and*

$$\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \leq c\|f\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely,if $\alpha_\infty \geq n(1 - \frac{1}{p_\infty})$*, and if* (2.17) *holds, then* $f \in hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*, and*

$$\|f\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\left\{\left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f *as above.*

Theorem 2.21 *Let* α *and* q *are* log*-Hölder continuous, both at the origin and at infinity and* $p \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$*. For any* $f \in h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*, we have*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n), \tag{2.18}$$

where for $k \leq 0$*,* a_k *is a central* $(\alpha(\cdot), p(\cdot))$*-atom, while for* $k > 0$*,* a_k *is a central* $(\alpha(\cdot), p(\cdot))$*-block, with* $\operatorname{supp} a_k \subset B_k$ *and*

$$\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty} \leq c\|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely,if $\alpha(\cdot) \geq n(1 - \frac{1}{p^-})$ *and* $s \geq \left\lfloor \alpha^+ + n(\frac{1}{p^-} - 1)\right\rfloor$*, and if* (2.18) *holds, then* $f \in H\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*, and*

$$\|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf\left\{\left(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right)^{1/q(0)} + \left(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right)^{1/q_\infty}\right\},$$

where the infimum is taken over all the decompositions of f *as above.*

Remark 2.22 *In the necessity part of the Theorems 2.20 and 2.21, the atoms in the decompositions* (2.17) *and* (2.18) *can be taken to be supported in dyadic annuli.*

2.4 Pseudo-differential operators on local variable Herz-type Hardy spaces

Recently, the boundedness of pseudo-differential operators on variable function spaces was studied by many authors (see [35, 37]).In this section is to show the boundedness of pseudo-differential operators of order zero on local variable Herz-type Hardy spaces $hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Theorem 2.23 *Let* $q \in \tilde{\mathcal{P}}^{\log}_0(\mathbb{R}^n)$ *and* $p \in \tilde{\mathcal{P}}^{\log}_\infty(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$*, and let* α *and* q *are* log*-Hölder continuous, at infinity such that* $\alpha \in L^\infty(\mathbb{R}^n)$ *and* $\alpha_\infty \geq n\big(1 - \frac{1}{p_\infty}\big)$*. If*

$$Tf(x) := \int_{\mathbb{R}^n} \hat{f}(x)\, \sigma(x,\xi)\, e^{2\pi i x\xi} d\xi$$

with $\sigma \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $\left|\partial^\gamma_x \partial^\beta_\xi \sigma(x,\xi)\right| \leq c_{\gamma,\beta}(1+|\xi|)^{-|\beta|}$*, then*

$$\|T(f)\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\, \|f\|_{hK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Theorem 2.24 *Let* $q \in \tilde{\mathcal{P}}^{\log}_0(\mathbb{R}^n)$ *and* $p \in \tilde{\mathcal{P}}^{\log}_0(\mathbb{R}^n) \cap \tilde{\mathcal{P}}^{\log}_\infty(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$*, and let* α *and* q *are* log*-Hölder continuous, both at the origin and at infinity such that* $\alpha \in L^\infty(\mathbb{R}^n)$ *and* $\alpha(\cdot) \geq n\big(1 - \frac{1}{p^-}\big)$*. If* T *be as in theorem 2.23, then*

$$\|T(f)\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\, \|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Proof. By similarity we only consider the homogeneous case. Let $f \in h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, by Theorem 2.21, we have

$$f = \sum_{l=-\infty}^{\infty} \lambda_l a_l, \tag{2.19}$$

in the sense of distributions, where for $l \leq 0, a_l$ is a central $(\alpha(\cdot), p(\cdot))$-atom, while for $l > 0$, a_l is a central $(\alpha(\cdot), p(\cdot))$-block, with supp$a_l \subset B_l$ and

$$\|f\|_{h\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \inf \left(\sum_{l=-\infty}^{-1} |\lambda_l|^{q(0)} \right)^{1/q(0)} + \left(\sum_{l=0}^{\infty} |\lambda_l|^{q_\infty} \right)^{1/q_\infty}.$$

By Proposition 2.2, it suffices to estimate

$$\left\|\left\{2^{k\alpha(0)}\tilde{\varphi}_+^*\left(T\left(f\right)\right)\chi_k\right\}\right\|_{\ell_<^{q(0)}(L^{p(\cdot)})} \quad \text{and} \quad \left\|\left\{2^{k\alpha_\infty}\tilde{\varphi}_+^*\left(T\left(f\right)\right)\chi_k\right\}\right\|_{\ell_>^{q_\infty}(L^{p(\cdot)})}.$$

For $k<0$, we have

$$\left\|\left\{2^{k\alpha(0)}\tilde{\varphi}_+^*\left(T\left(f\right)\right)\chi_k\right\}\right\|_{\ell_<^{q(0)}(L^{p(\cdot)})} \lesssim F_1+F_2+F_3,$$

where

$$F_1 := \left(\sum_{k=-\infty}^{-1}\left(2^{k\alpha(0)}\sum_{l=-\infty}^{k-2}|\lambda_l|\left\|\tilde{\varphi}_+^*\left(Ta_l\right)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right)^{\frac{1}{q(0)}}$$

$$F_2 := \left(\sum_{k=-\infty}^{-1}\left(2^{k\alpha(0)}\sum_{l=k-1}^{-1}|\lambda_l|\left\|\tilde{\varphi}_+^*\left(Ta_l\right)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right)^{\frac{1}{q(0)}}$$

and

$$F_3 := \left(\sum_{k=-\infty}^{-1}\left(2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|\left\|\tilde{\varphi}_+^*\left(Ta_l\right)\chi_k\right\|_{p(\cdot)}\right)^{q(0)}\right)^{\frac{1}{q(0)}}$$

For F_3, by the $L^{p(\cdot)}$-boundedness of $\mathcal{M}$, we get

$$\begin{aligned}
2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|\left\|\tilde{\varphi}_+^*\left(Ta_l\right)\chi_k\right\|_{p(\cdot)} &\lesssim \sum_{l=0}^{\infty}|\lambda_l|\,2^{k\alpha(0)}\left\|\mathcal{M}\left(Ta_l\right)\right\|_{p(\cdot)} \\
&\lesssim \sum_{l=0}^{\infty}|\lambda_l|\,2^{k\alpha(0)}\left\|Ta_l\right\|_{p(\cdot)} \\
&\lesssim \sum_{l=0}^{\infty}|\lambda_l|\,2^{k\alpha(0)}\left\|a_l\right\|_{p(\cdot)} \\
&\lesssim 2^{k\alpha(0)}\sum_{l=0}^{\infty}|\lambda_l|\,2^{-l\alpha^-}\left\|2^{l\alpha(\cdot)}a_l\right\|_{p(\cdot)} \\
&\lesssim 2^{k\alpha(0)}\Big(\sum_{l=0}^{\infty}|\lambda_l|^{q_\infty}\Big)^{1/q_\infty}.
\end{aligned}$$

Therefore, we conclude that

$$F_3 \lesssim \Big(\sum_{l=0}^{\infty}|\lambda_l|^{q_\infty}\Big)^{1/q_\infty}.$$

For F_2, similar to the estimate of F_3 and applying Lemma 1.14, we obtain

$$
\begin{aligned}
F_2 &\lesssim \left(\sum_{k=-\infty}^{-1} \left(\sum_{l=k-1}^{-1} |\lambda_l| \, 2^{(k-l)\alpha(0)} \right)^{q(0)} \right)^{\frac{1}{q(0)}} \\
&\lesssim \Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \Big)^{1/q(0)}.
\end{aligned}
$$

For F_1. By Theorem 4 in [17], we can write

$$
\varphi_t * (Ta_l)(x) = \int_{\mathbb{R}^n} K_t(x; x-y) \, a_l(y) \, dy,
$$

then we expand $K_t(x; x-y)$ in a Taylor series about $y = 0$. By the vanishing moment of a_l, we get that

$$
\varphi_t * (Ta_l)(x) = \sum_{|\beta|=N+1} \int_{\mathbb{R}^n} \partial_y^\beta K_t(x; x-y) \, y^\beta a_l(y) \, dy.
$$

where $0 < \theta < 1$ and $N \in \mathbb{N}$ satisfying that $N + 1 > \alpha^+ + n(\frac{1}{p^-} - 1)$. Noting that $x \in R_k$ with $k \geq l + 2$, by Theorem 4 in [17], we have

$$
\begin{aligned}
|\varphi_t * (Ta_l)(x)| &\lesssim |x|^{-(n+N+1)} \int_{\mathbb{R}^n} |y|^{N+1} a_l(y) \, dy \\
&\lesssim 2^{l(N+1)} 2^{-k(n+N+1)} \|a_l\|_1 \\
&\lesssim 2^{(k-l)\alpha^+} 2^{l(N+1)} 2^{-k(n+N+1)} \left\|2^{l\alpha(\cdot)} a_l\right\|_1.
\end{aligned}
$$

Applying Hölder's inequality and the fact that a_l is a central $(\alpha(\cdot), p(\cdot))$-atom, we obtain

$$
\begin{aligned}
\left\|2^{k\alpha(0)} \tilde{\varphi}_+^* (Ta_l) \chi_k\right\|_{p(\cdot)} &\lesssim 2^{(k-l)\alpha^+} 2^{l(N+1)-k(n+N+1)} \left\|\chi_{B_l}\right\|_{p'(\cdot)} \left\|\chi_{B_k}\right\|_{p(\cdot)} \\
&\lesssim 2^{(k-l)\alpha^+} 2^{l(N+1)-k(N+n+1)} \left\|\chi_{B_l}\right\|_{p'(\cdot)} \left\|\chi_{B_k}\right\|_{p(\cdot)} \\
&\lesssim 2^{(l-k)\left(N+1+n-\alpha^+ - \frac{n}{p^-}\right)},
\end{aligned}
$$

where in the last estimate we have used Lemma 1.15. Then, we obtain

$$
2^{k\alpha(0)} \sum_{l=-\infty}^{k-2} |\lambda_l| \left\|\tilde{\varphi}_+^* (Ta_l) \chi_k\right\|_{p(\cdot)} \lesssim \sum_{l=-\infty}^{k-2} |\lambda_l| \, 2^{(l-k)\left(N+1+n-\alpha^+ - \frac{n}{p^-}\right)}.
$$

Therefore, by Lemma 1.14, we have

$$F_1 \lesssim \Big(\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)} \Big)^{1/q(0)}.$$

For $k \geq 0$, we choose a radial smooth function η such that supp$\eta \subset B\,(0,1)$ and $\eta = 1$ near the origin. We split $T = T_1 + T_2$ by decomposing $K\,(x;y) = K_1\,(x;y) + K_2\,(x;y) = \eta K\,(x;y) + (1-\eta)\,K\,(x;y)$. Then T_1 and T_2 are of order zero. We have

$$\begin{aligned}
\big\|\big\{2^{k\alpha_\infty}\tilde{\varphi}_+^*\,(T\,(f))\,\chi_k\big\}\big\|_{\ell_>^{q_\infty}(L^{p(\cdot)})} &\lesssim \left(\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=-\infty}^{\infty}|\lambda_l|\,\big\|\tilde{\varphi}_+^*\,(T_1a_l)\,\chi_k\big\|_{p(\cdot)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}} \\
&\quad + \left(\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=-\infty}^{-1}|\lambda_l|\,\big\|\tilde{\varphi}_+^*\,(T_2a_l)\,\chi_k\big\|_{p(\cdot)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}} \\
&\quad + \left(\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=0}^{k-2}|\lambda_l|\,\big\|\tilde{\varphi}_+^*\,(T_2a_l)\,\chi_k\big\|_{p(\cdot)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}} \\
&\quad + \left(\sum_{k=0}^{\infty}\left(2^{k\alpha_\infty}\sum_{l=k-1}^{\infty}|\lambda_l|\,\big\|\tilde{\varphi}_+^*\,(T_2a_l)\,\chi_k\big\|_{p(\cdot)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}} \\
&:= H_1 + H_2 + H_3 + H_4.
\end{aligned}$$

To estimate H_1, since supp$\tilde{\varphi}_+^*\,(T_1a_l) \subset B_{l+1}$ and $L^{p(\cdot)}$ boundedness of $\mathcal{M}$, we deduce that

$$\begin{aligned}
\sum_{l=-\infty}^{\infty}|\lambda_l|2^{k\alpha_\infty}\big\|\tilde{\varphi}_+^*\,(T_1a_l)\,\chi_k\big\|_{p(\cdot)} &\lesssim \sum_{l=k-1}^{\infty}|\lambda_l|2^{k\alpha_\infty}\big\|\|\mathcal{M}\,(T_1a_l)\,\big\|_{p(\cdot)} \\
&\lesssim \sum_{l=k-1}^{\infty}|\lambda_l|2^{k\alpha_\infty}\big\|\|T_1a_l\big\|_{p(\cdot)} \\
&\lesssim \sum_{l=k-1}^{\infty}|\lambda_l|2^{k\alpha_\infty}\big\|\|a_l\big\|_{p(\cdot)} \\
&\lesssim \sum_{l=k-1}^{\infty}|\lambda_l|2^{(k-l)\alpha_\infty}.
\end{aligned}$$

Therefore, by Lemma 1.14, we obtain

$$H_1 \lesssim \Big(\sum_{k=0}^{\infty}|\lambda_k|^{q_\infty}\Big)^{1/q_\infty}.$$

For H_4, it is obvious that

$$\sum_{l=k-1}^{\infty} |\lambda_l| \left\|\tilde{\varphi}_+^* \left(T_2 a_l\right) \chi_k\right\|_{p(\cdot)} \lesssim \sum_{l=k-1}^{\infty} |\lambda_l| 2^{(k-l)\alpha_\infty}.$$

Again by Lemma 1.14, we get

$$H_4 \lesssim \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\Big)^{1/q_\infty}.$$

For H_2, it will know that

$$\left|\left(K_2\right)_t (x, y)\right| \lesssim c_m \left(1+|y|\right)^{-m} \tag{2.20}$$

for any $m \geq n$ see ([12], Theorem 4). If $x \in R_k$ and $l < 0 < k$, by (2.12) we obtain that

$$\begin{aligned}
\varphi_t * \left(T_2 a_j\right)(x) &= \left|\int_{\mathbb{R}^n} \left(K_2\right)_t (x, x-y)\, a_l(y)\, dy\right| \\
&\lesssim \int_{\mathbb{R}^n} \frac{1}{\left(1+|x-y|\right)^m} \left|a_l(y)\right| dy \\
&\lesssim 2^{l(N+1)} |x|^{-(N+n+1)} \int_{\mathbb{R}^n} \left|a_l(y)\right| dy \\
&\lesssim 2^{l(N+1)} 2^{-k(N+n+1)} \left\|a_l\right\|_1
\end{aligned}$$

So by the fact that $2^{l\alpha(0)} \approx 2^{l\alpha(y)}$ $\left(y \in B_l \text{ and } l < 0\right)$, we get

$$\begin{aligned}
2^{k\alpha_\infty} \varphi_t * \left(T_2 a_l\right)(x) &\lesssim 2^{l(N+1)+k\alpha_\infty} |x|^{-(n+N+1)} \left\|a_l\right\|_1 \\
&\lesssim 2^{(k-l)\alpha^+} 2^{l(N+1)-k(n+N+1)} \left\|2^{l\alpha(\cdot)} a_l\right\|_1, \quad k \geq 0 > l.
\end{aligned}$$

Applying Hölder's inequality and the fact that a_l is a dyadic central $(\alpha(\cdot), p(\cdot))$-atom, we obtain

$$\begin{aligned}
\left\|2^{k\alpha_\infty} \varphi_t * \left(T_2 a_l\right) \chi_k\right\|_{p(\cdot)} &\lesssim 2^{(k-l)\alpha^+} 2^{l(N+1)-k(n+N+1)} \left\|\chi_{B_l}\right\|_{p'(\cdot)} \left\|\chi_{B_k}\right\|_{p(\cdot)} \\
&\lesssim 2^{(k-l)\alpha^+} 2^{l(N+1)-k(n+N+1)} \left\|\chi_{B_l}\right\|_{p'(\cdot)} \left\|\chi_{B_k}\right\|_{p(\cdot)} \\
&\lesssim 2^{(l-k)\left(N+1+n-\alpha^+-\frac{n}{p^-}\right)},
\end{aligned}$$

where in the last estimate we have used Lemma 1.15. We take $N \in \mathbb{N}$ satisfying that $N+1 > \alpha^{+} + n(\frac{1}{p^{-}} - 1)$, then

$$2^{k\alpha_\infty} \sum_{l=-\infty}^{-1} |\lambda_l| \left\| \tilde{\varphi}_+^* \left(T_2 a_l\right) \chi_k \right\|_{p(\cdot)} \lesssim c 2^{-k\left(N+1+n-\alpha^{+}-\frac{n}{p^{-}}\right)} \sum_{l=-\infty}^{-1} |\lambda_l| \, 2^{l\left(N+1+n-\alpha^{+}-\frac{n}{p^{-}}\right)}$$

$$\lesssim 2^{-k\left(N+1+n-\alpha^{+}-\frac{n}{p^{-}}\right)} \Big(\sum_{l=-\infty}^{-1} |\lambda_l|^{q(0)} \Big)^{1/q(0)}.$$

Therefore,

$$H_2 \lesssim \Big(\sum_{l=-\infty}^{-1} |\lambda_l|^{q(0)} \Big)^{1/q(0)}.$$

For H_3, by similar argument in the estimate H_2, we obtain

$$2^{k\alpha_\infty} \sum_{l=0}^{k-2} |\lambda_l| \left\| \tilde{\varphi}_+^* \left(T_2 a_l\right) \chi_k \right\|_{p(\cdot)} \lesssim \sum_{l=0}^{k-2} 2^{(l-k)\left(N+1+n-\alpha^{+}-\frac{n}{p^{-}}\right)}.$$

Again by Lemma 1.14, we get

$$H_4 \lesssim \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \Big)^{1/q_\infty}.$$

The proof is complete. ■

Remark 2.25 *Corresponding statements to Theorems 2.23 and 2.24, with α, p and q constants, can be found in [26], while with α and q constants Theorems 2.23 and 2.24 are proved in [37], under the assumption that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Here we are requiring the log-Hölder continuity at two points only (zero and infinity).*

CHAPTER 3

BOUNDEDNESS OF SINGULAR INTEGRAL OPERATOR OF CONVOLUTION TYPE ON VARIABLE HERZ-TYPE HARDY SPACES

In recent years, it turned out that atomic and molecular decomposition of some function spaces are extremely useful in many aspects. This concerns, for instance, the investigation of (compact) embeddings between function spaces. But this applies equally to questions of mapping properties of some operators, such as Calderón-Zygmund operators, the commutator of Calderón-Zygmund operator with a BMO function and to trace problems, where arguments can be equivalently transferred to the sequence space, which is often more convenient to handle. In this chapter, we shall give the boundedness of singular integral operators of convolution type on $HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. To do this, we establish the molecular decompositions of inhomogeneous Herz-type Hardy spaces.

3.1 Molecular decompositions of variable inhomogeneous Herz-type Hardy spaces

The main goal of this section is to prove an molecular decomposition result for $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$. First we give the notation of molecule

Definition 3.1 *Let* $p \in \mathcal{P}(\mathbb{R}^n), \alpha : \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha \in L^\infty(\mathbb{R}^n)$ *and* $\alpha \geq n\big(1 - \frac{1}{p_\infty}\big)$. *Let* $l \in \mathbb{N}_0$,

$$s \geq \left\lfloor \alpha_\infty + n(\frac{1}{p_\infty} - 1) \right\rfloor .$$

Let

$$\varepsilon > \max\Big(\frac{s}{n}, \frac{\alpha_\infty}{n} + \frac{1}{p_\infty} - 1\Big), \quad b_\infty = 1 - \frac{1}{p_\infty} + \varepsilon$$

and

$$a_\infty = 1 - \frac{1}{p_\infty} - \frac{\alpha_\infty}{n} + \varepsilon.$$

A function $M_l \in L^{p(\cdot)}(\mathbb{R}^n)$ *is said to be a dyadic central* $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$*-molecule of restricted type if it satisfies*

(i) $\|M_l\|_{p(\cdot)} \leq 1$,

(ii) $\mathcal{R}_{p(\cdot)}(M_l) = \|M_l\|_{p(\cdot)}^{a_\infty/b_\infty} \left\| |\cdot|^{nb_\infty} M_l \right\|_{p(\cdot)}^{1-a_\infty/b_\infty} < \infty$,

(iii) $\int_{\mathbb{R}^n} M_l(x) x^\beta dx = 0$ *for any* β *with* $|\beta| \leq s$.

In the next statement we shows that the molecular is a generalization of atom.

Lemma 3.2 *Let* $\alpha, p, s, \varepsilon, a_\infty, b_\infty$ *be as in Definition 3.1. Let* $l \in \mathbb{N}_0$. *If* M_l *is central* $(\alpha(\cdot), p(\cdot))$*-atom of restricted type supported on* $B(0, 2^l)$, *then* M_l *is a central* $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$*-molecule of restricted type.*

We immediately arrive at the following result.

Lemma 3.3 *Let* $\alpha, s, \varepsilon, a_l, b_l, l \in \mathbb{N}_0$ *be as in Definition 3.1, and* $p \in \tilde{\mathcal{P}}_\infty^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$. *Let* α *and* q *be log-Hölder continuous, both at the origin and at* *infinity. Then there exists a constant C such that for any dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$-molecule of restricted type M_l,*

$$\|M_l\|_{HK^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C,$$

where $C > 0$ is independent of l.

Proof. We follows the idea of [30]. Let M_l be a dyadic central $(\alpha(\cdot), p(\cdot); s, \varepsilon)_l$-molecule. Assume that $2^{v_l} < \|M_l\|_{p(\cdot)}^{-\frac{1}{\alpha_\infty}} \leq 2^{v_l+1}, v_l \geq 0$. Recall that

$$B_{v_l} = \left\{x \in \mathbb{R}^n : |x| < 2^{v_l-1}\right\} \quad \text{and} \quad R_{k+v_l} = \left\{x \in \mathbb{R}^n : 2^{k+v_l-1} \leq |x| < 2^{k+v_l}\right\}$$

for any $k \in \mathbb{N}_0$. Therefore

$$M_l(x) = \sum_{k=0}^{\infty} M_l(x)\chi_{k+v_l}(x), \quad x \in \mathbb{R}^n,$$

where χ_{k+v_l} and χ_{v_l} are respectively the characteristic function of R_{k+v_l} and $B_{v_l}, k > 0$. Let $M_{l,k} = M_l\chi_{k+v_l}$. We denote by $\mathcal{P}_s$ the class of all real polynomials of degree less than s. Let $P_{R_k}M_{l,k} \in \mathcal{P}_s$ be the unique polynomial satisfying

$$\int_{R_k} \big(M_{l,k}(x) - P_{R_k}M_{l,k}(x)\big)x^\beta = 0, \quad |\beta| \leq s.$$

Let $\mathrm{Q}_{l,k} = \big(P_{R_k}M_{l,k}\big)\chi_{k+v_l}, k > 0$.

Step 1. In this step we prove that there is a positive constant C and sequences of numbers $\{\lambda_{l,k}\}_{k\in\mathbb{N}_0}$ such that

$$\Big(\sum_{k=0}^{\infty} |\lambda_{l,k}|^{q_\infty}\Big)^{\frac{1}{q_\infty}} < C, \tag{3.1}$$

and

$$M_{l,k} = \lambda_{l,k}a_{l,k} + \mathrm{Q}_{l,k},$$

where each $a_{l,k}$ is a $(\alpha(\cdot), p(\cdot))$-atom and the constant C is independent of l. Without loss of generality, assume that $\mathcal{R}_{p(\cdot)}(M_l) = 1$, which leads to

$$\big\| |\cdot|^{nb_\infty} M_l\big\|_{p(\cdot)} = \big\|M_l\big\|_{p(\cdot)}^{-\frac{a_\infty}{b_\infty - a_\infty}} \leq c2^{v_l a_\infty n},$$

where $c > 0$ is independent of l. Let $\{\varphi_j^{l,k} : |j| \leq s\} \in \mathcal{P}_s(\mathbb{R}^n)$ be such that

$$\langle \varphi_\mu^{l,k}, \varphi_v^{l,k} \rangle_{R_{k+v_l}} = \frac{1}{|R_{k+v_l}|} \int_{R_{k+v_l}} \varphi_\mu^{l,k}(x) \varphi_v^{l,k}(x) dx = \delta_{\mu,\nu}.$$

Then

$$\mathrm{Q}_{l,k}(x) = \sum_{|\beta| \leq s} \langle M_l, \varphi_\beta^{l,k} \rangle_{R_{k+v_l}} \varphi_\beta^{l,k}(x), \quad x \in R_k. \tag{3.2}$$

In addition by Hölder's inequality

$$\begin{aligned} |\mathrm{Q}_{l,k}(x)| &\lesssim \frac{1}{|R_{k+v_l}|} \int_{R_{k+v_l}} |M_{l,k}(x)| \, dx \\ &\lesssim \frac{\|\chi_{R_{k+v_l}}\|_{p'(\cdot)}}{|R_{k+v_l}|} \|M_{l,k}\|_{p(\cdot)} \\ &\lesssim \frac{\|M_{l,k}\|_{p(\cdot)}}{\|\chi_{R_{k+v_l}}\|_{p(\cdot)}} \end{aligned}$$

for any $x \in R_k$, where the last estimate follows by Lemma 1.17. Therefore

$$\begin{aligned} \|M_{l,k} - \mathrm{Q}_{l,k}\|_{p(\cdot)} &\lesssim \|M_{l,k}\|_{p(\cdot)} \\ &\lesssim \| \, |\cdot|^{nb_\infty} M_{l,k}\|_{p(\cdot)} 2^{-(k+v_l)nb_\infty} \\ &\lesssim 2^{v_l a_\infty n} 2^{-(k+v_l)nb_\infty} \\ &= c 2^{-nka_\infty} |B_{k+v_l}|^{-\alpha_\infty/n}, \end{aligned}$$

Consequently $M_{l,k} - \mathrm{Q}_{l,k} = \gamma_k a_{l,k}$, where $\gamma_k = c2^{-nka_\infty}$ and $a_{l,k}$ is a central $(\alpha(\cdot), p(\cdot))$-atom of support contained in B_{k+v_l}, $k \geq 0$. The constant c is independent of k and l. Let

$$\lambda_k = \begin{cases} \gamma_k, & \text{if} \quad k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore we obtain the estimate (3.1).

Step 2. In this step we prove that $\sum_{k=0}^{\infty} \mathrm{Q}_{l,k}$ has a $(\alpha(\cdot), p(\cdot))$-atom decomposition. Let $\{\varphi_j^{l,k} : |j| \leq s\} \in \mathcal{P}_s(\mathbb{R}^n)$ be the dual basis of $\{x^\gamma : |\gamma| \leq s\}$ with respect to the weight

$\frac{1}{|R_{k+v_l}|}$ on R_{k+v_l}, that is

$$\left\langle \varphi_j^{l,k}, x^\gamma \right\rangle = \frac{1}{|R_{k+v_l}|} \int_{R_{k+v_l}} \varphi_j^{l,k}(x)\, x^\gamma dx = \delta_{j,\gamma}.$$

If set $\varphi_j^{l,k}(x) = \sum\limits_{|v|\leq s} \beta_{vj}^{l,k} x^v$ and $\psi_j^{l,k}(x) = \sum\limits_{|v|\leq s} \tau_{vj}^{l,k} \varphi_v^{l,k}(x)$, then we have

$$\tau_{vj}^{l,k} = \left\langle \psi_j^{l,k}, \varphi_v^{l,k} \right\rangle = \sum_{|\gamma|\leq s} \beta_{v\gamma}^{l,k} \left\langle \varphi_j^{l,k}, x^\gamma \right\rangle = \sum_{|\gamma|\leq s} \beta_{v\gamma}^{l,k} \delta_{j\gamma} = \beta_{vj}^{l,k}.$$

So $\varphi_j^{l,k}(x) = \sum\limits_{|v|\leq s} \beta_{vj}^{l,k} \varphi_v^{l,k}(x)$. For $x \in R_{k+v_l}$, we have

$$\begin{aligned} \left\langle M_{l,k}, \varphi_j^{l,k} \right\rangle_{R_{k+v_l}} \varphi_j^{l,k}(x) &= \left\langle M_{l,k}, \sum_{|v|\leq s} \beta_{vj}^{l,k} x^v \right\rangle_{R_{k+v_l}} \varphi_j^{l,k}(x) \\ &= \sum_{|v|\leq s} \left\langle M_{l,k}, x^v \right\rangle_{R_{k+v_l}} \beta_{vj}^{l,k} \varphi_j^{l,k}(x), \end{aligned}$$

which together with (3.2) implies that

$$\mathrm{Q}_{l,k}(x) = \sum_{|j|\leq s} \left\langle M_{l,k}, x^j \right\rangle_{R_{k+v_l}} \varphi_j^{l,k}(x), \quad \text{if} \quad x \in R_{k+v_l}. \tag{3.3}$$

We set $E := \{x \in \mathbb{R}^n : 1 \leq |x| \leq 2\}$, $F := \{x \in \mathbb{R}^n : |x| \leq 1\}$, $\{e_j : |j| \leq s\} \in \mathcal{P}_s(\mathbb{R}^n)$ satisfying

$$\frac{1}{|E|} \int_E e_j(x)\, x^\gamma dx = \delta_{j,\gamma},$$

and $\{\tilde{e}_j : |j| \leq s\} \in \mathcal{P}_s(\mathbb{R}^n)$ satisfying

$$\frac{1}{|F|} \int_F \tilde{e}_j(x)\, x^\gamma dx = \delta_{j,\gamma}.$$

Noting that

$$\delta_{j,\gamma} = \frac{1}{|R_{k+v_l}|} \int_{R_{k+v_l}} \psi_j^{l,k}(x)\, x^\gamma dx = \frac{1}{|E|} \int_E 2^{(k+v_l)|\gamma|} \psi_j^{l,k}\left(2^{k+v_l} y\right) y^\gamma dy,$$

we get $e_j(y) = 2^{k|\gamma|} \psi_j^{l,k}\left(2^k y\right)$. Thus in turn leads to that for $k > 0$,

$$\psi_j^{l,k}(x) = 2^{-(k+v_l)|j|} e_j\left(2^{-(k+v_l)} x\right),\ x \in R_{k+v_l}.$$

Similarly, we have

$$\psi_j^{l,k}(y) = 2^{-v_l|j|}\tilde{e}_j\left(2^{-v_l-1}x\right),\, x \in F.$$

Let $C := \sup\limits_{j:|j|\leq s}\{\|e_j\|_\infty, \|\tilde{e}_j\|_\infty\}$. Then we have

$$|\psi_j^{l,k}(x)| \leq C2^{-(k+v_l)|j|}, \quad \text{for } k \geq 0, \tag{3.4}$$

where the constant C is independent of l. Let

$$N_j^{l,k} = \sum_{i=k}^{\infty} |\tilde{R}_{i+v_l}|\langle M_{l,i}, x^j\rangle_{\tilde{R}_{i+v_l}}, \quad k \geq 0.$$

First observe that

$$N_j^{l,l} = \sum_{i=0}^{\infty} |\tilde{R}_{i+v_l}|\langle M_{l,i}, x^j\rangle_{\tilde{R}_i} = \sum_{i=0}^{\infty}\int_{\tilde{R}_{i+v_l}} M_l(x)\, x^j dx = \int_{\mathbb{R}^n} M_l(x)\, x^j dx = 0.$$

Now by Hölder's inequality and Lemma 1.17 it follows

$$\begin{aligned}|N_j^{l,k}| &\leq \sum_{i=k}^{\infty}\int_{R_{i+v_l}} \left|M_{l,i}(x)\, x^j\right| dx \\ &\leq \sum_{i=k}^{\infty} \left\| |\cdot|^j M_{l,i}\right\|_{p(\cdot)} \left\|\chi_{R_{i+v_l}}\right\|_{p'(\cdot)}.\end{aligned}$$

By Lemma 1.17 we get

$$\left\|\chi_{R_{i+v_l}}\right\|_{p'(\cdot)} \approx |R_{i+v_l}|^{1-\frac{1}{p_\infty}}$$

which yields that

$$\begin{aligned}|N_j^{l,k}| &\leq c2^{v_l(|j|-nb_\infty-\frac{n}{p_\infty}+n)}\sum_{i=k}^{\infty} 2^{i(|j|-nb_\infty-\frac{n}{p_\infty}+n)}\left\| |\cdot|^{nb_\infty} M_l\right\|_{p(\cdot)} \\ &\lesssim 2^{v_l(|j|-nb_\infty-\frac{n}{p_\infty}+n+a_\infty n)}\sum_{i=k}^{\infty} 2^{i(|j|-n\varepsilon)} \\ &\lesssim 2^{v_l(|j|-nb_\infty-\frac{n}{p_\infty}+n+a_\infty n)}2^{k(|j|-\varepsilon)},\end{aligned}$$

with the implicit constant not depending on i and l, which yields that

$$|N_j^{l,k}| \lesssim 2^{v_l(|j|-nb_\infty-\frac{n}{p_\infty}+n)}2^{k(|j|-n\varepsilon)}.$$

Consequently, with the help of (3.4),

$$\frac{\left|N_j^{l,k}\psi_j^{l,k}\left(x\right)\chi_k\left(x\right)\right|}{\left|R_{k+v_l}\right|}\lesssim 2^{v_l\left(-nb_\infty-\frac{n}{p_\infty}+a_\infty n\right)}2^{-kn(\varepsilon+1)}, \tag{3.5}$$

which tends to zero if k tends to infinity. Using Abel's transform and (3.5) we obtain

$$\sum_{k=0}^{\infty}\mathrm{Q}_{l,k}(x)$$

can be rewritten as

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{|j|\leq s}\left\langle M_k,x^j\right\rangle_{\tilde{R}_{k+v_l}}\psi_j^{l,k}\left(x\right)\\
&=\sum_{|j|\leq s}\sum_{k=0}^{\infty}\Big(\sum_{i=0}^{k}\left|\tilde{R}_{k+v_l}\right|\left\langle M_{l,i},x^j\right\rangle_{\tilde{R}_{k+v_l}}\Big)\Big(\frac{\psi_j^{l,k}\left(x\right)\chi_{k+v_l}\left(x\right)}{|\tilde{R}_{k+v_l}|}-\frac{\psi_j^{l,k+1}\left(x\right)\chi_{k+v_l+1}\left(x\right)}{|\tilde{R}_{k+v_l+1}|}\Big)\\
&=-\sum_{|j|\leq s}\sum_{k=0}^{\infty}N_j^{l,k+1}\Big(\frac{\psi_j^{l,k}\left(x\right)\chi_{k+v_l}\left(x\right)}{|\tilde{R}_{k+v_l}|}-\frac{\psi_j^{l,k+1}\left(x\right)\chi_{k+v_l+1}\left(x\right)}{|\tilde{R}_{k+v_l+1}|}\Big).
\end{aligned}$$

On the other hand, we have

$$\Big|N_j^{l,k+1}\Big(\frac{\psi_j^{l,k}\left(x\right)\chi_{k+v_l}\left(x\right)}{|\tilde{R}_{k+v_l}|}-\frac{\psi_j^{l,k+1}\left(x\right)\chi_{k+v_l+1}\left(x\right)}{|\tilde{R}_{k+v_l+1}|}\Big)\Big|$$

can be estimated by

$$\begin{aligned}
&C|N_j^{l,k+1}||\psi_j^{l,k+1}\left(x\right)||\tilde{R}_{k+v_l}|^{-1}\\
&\leq C2^{v_l\left(-nb_\infty-\frac{n}{p_\infty}+a_\infty n\right)}2^{-k(n\varepsilon+n)}\\
&\leq C2^{-kna_\infty}|\tilde{R}_{k+v_l}|^{-\frac{1}{p_\infty}-\frac{\alpha_\infty}{n}}
\end{aligned}$$

for some positive constant c independent of l and k. Let

$$\lambda_{l,k}=\begin{cases} C2^{-kna_\infty}, & \text{if} \qquad k\geq 0,\\ 0, & \text{otherwise}\end{cases} \tag{3.6}$$

and

$$a_{l,k}^{(j)}=\begin{cases}\frac{(-N_j^{l,k+1})}{\lambda_{l,k}}\Big(\frac{\psi_j^{l,k}(x)\chi_{k+v_l}(x)}{|\tilde{R}_{k+v_l}|}-\frac{\psi_j^{l,k+1}(x)\chi_{k+v_l+1}(x)}{|\tilde{R}_{k+v_l+1}|}\Big), & \text{if} \qquad k\geq 0 \text{ and } |j|\leq s,\\ 0, & \text{otherwise}\end{cases}$$

Then we have

$$\sum_{k=0}^{\infty} \mathrm{Q}_{l,k}(x) = \sum_{|j|\leq s} \sum_{k=0}^{\infty} \lambda_{l,k} a_{l,k}^{(j)},$$

where $a_{l,k}^{(j)}$ is an $(\alpha(\cdot), p(\cdot))$-atom supported on B_{k+v_l}. In addition we obtain the estimate (3.1). ■

Now we come to the molecular decomposition theorems.

Theorem 3.4 *Let $\alpha, p, q, s, \varepsilon, a, b$ be as in Lemma 3.3. Then $f \in HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ if and only if f can be represented as*

$$f = \sum_{k=0}^{\infty} \lambda_k M_k,$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each M_k is a dyadic central $(\alpha(.), p(.); s, \varepsilon)$-molecule of restricted type, and

$$\Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \Big)^{\frac{1}{q_\infty}} \lesssim \|f\|_{HK_{p(.),q(.)}^{\alpha(.)}}.$$

Moreover,

$$\|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)} \approx \inf \Big(\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty} \Big)^{\frac{1}{q_\infty}},$$

where the infimum is taken over all the decompositions of f as above.

Remark 3.5 *Corresponding statement to Theorem 3.4 were proved by Liu and Wang [36], with α and q constants, under the assumption that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (both in the homogeneous and the inhomogeneous situation). Also Here we are requiring the log-Hölder continuity at two points only (zero and infinity).*

3.2 Singular integral operator on $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$

The target of this section is to show the boundedness of a convolution type singular integral operator on $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$. The main theorem of this chapter is the following.

Theorem 3.6 *Let* $q \in \tilde{\mathcal{P}}_0^{\log}(\mathbb{R}^n)$ *and* $p \in \tilde{\mathcal{P}}_\infty^{\log}(\mathbb{R}^n)$ *with* $1 < p^- \leq p^+ < \infty$*, let*

$$Tf(x) = \int_{\mathbb{R}^n} K(x;y) f(y)\,dy, \quad x \notin suppf, \tag{3.8}$$

for any central $(\alpha(\cdot), p(\cdot))$*-atom* f *with the kernel* K *satisfies*

$$\left|k(x;y) - k(x;0)\right| \leq c\frac{|y|^\delta}{|x|^{n+\delta}}, \quad |x| \geq 2|y|,$$

where c *is a positive constant and* $0 < \delta \leq 1$*. Let* α *and* q *be log-Hölder continuous, at infinity, with* $\alpha \in L^\infty(\mathbb{R}^n)$ *and*

$$n\left(1 - \frac{1}{p_\infty}\right) \leq \alpha_\infty < n\left(1 - \frac{1}{p_\infty}\right) + \delta.$$

If T *satisfying* $\int_{\mathbb{R}^n} Tf(x)\,dx = 0$*, and bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$*, then* T *is also bounded on* $HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$*.*

Proof. Let $f \in HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$, by Theorem 2.8, we have

$$f = \sum_{l=0}^{\infty} \lambda_l a_l,$$

in the sense of distributions, where each a_l is a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type with supp$a_l \subset B_l$ and

$$\left(\sum_{l=0}^{\infty} |\lambda_l|^{q_\infty}\right)^{1/q_\infty} \leq c\|f\|_{HK_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$

By Theorem it suffices to show Tf is a central $(\alpha(\cdot), p(\cdot); 0, \varepsilon)_l$-molecule of restricted type where $\left(\frac{\alpha_\infty}{n} - 1 + \frac{1}{p_\infty}\right) \leq \varepsilon < \frac{\delta}{n} + \frac{1}{p_\infty}$, to be determined later, $a_\infty = 1 - \frac{1}{p_\infty} - \frac{\alpha_\infty}{n} + \varepsilon$, $\quad b_\infty = 1 - \frac{1}{p_\infty} + \varepsilon$. Obviously, we only need to verify the size condition for molecules, that is

$$\mathcal{R}_{p(\cdot)}(T(f)) = \|T(f)\|_{p(\cdot)}^{a_\infty/b_\infty} \left\||\cdot|^{nb_\infty} T(f)\right\|_{p(\cdot)}^{1-a_\infty/b_\infty} < \infty.$$

We first estimate $\left\||.|^{nb} T(f)(.)\right\|_{L^{p(.)}}$. In fact, we have

$$\left\||\cdot|^{nb_\infty} T(f)\right\|_{L^{p(\cdot)}(B(0,2r))} \lesssim r^{nb_\infty} \|T(f)\|_{p(.)} \lesssim r^{nb_\infty} \|f\|_{p(.)} \lesssim r^{nb_\infty - \alpha_\infty}.$$

On the other hand, for any x with $|x| > 2r$ the vanishing moment of f and the regularity of K, we have

$$\begin{aligned}
|T(f)(x)| &= \Big|\int_{\mathbb{R}^n} K(x;y) f(y)\, dy\Big| \\
&= \Big|\int_{\mathbb{R}^n} (k(x;y) - k(x;0)) f(y)\, dy\Big| \\
&\lesssim \int_{\mathbb{R}^n} \frac{|y|^{\delta}}{|x-y|^{n+\delta}} |f(y)|\, dy \\
&\lesssim r^{n+\delta} |x|^{-(n+\delta)} \mathcal{M}f(x)
\end{aligned}$$

and since $nb_\infty - n - \delta < 0$, we have

$$\begin{aligned}
\Big\| |\cdot|^{nb_\infty} T(f) \Big\|_{L^{p(\cdot)}(\mathbb{R}^n / B(0,2r))} &\lesssim r^{n+\delta} \Big\| |\cdot|^{nb_\infty - (n+\delta)} \mathcal{M}(f) \Big\|_{L^{p(\cdot)}(\mathbb{R}^n / B(0,2r))} \\
&\lesssim r^{nb_\infty} \|\mathcal{M}(f)\|_{p(\cdot)} \\
&\lesssim r^{nb_\infty} \|f\|_{p(\cdot)} \lesssim r^{nb_\infty - \alpha_\infty}.
\end{aligned}$$

Thus, we get

$$\mathcal{R}_{p(\cdot)}(T(f)) \lesssim r^{-\alpha_\infty a_\infty / b_\infty} r^{(nb_\infty - \alpha_\infty)(1 - a_\infty / b_\infty)} \lesssim 1.$$

This finishes the proof. ■

Remark 3.7 *Corresponding statement to Theorem 3.6 , with s, p and q constants, can be found in [30, Theorem 6.2.3], while with s and q constants Theorem 3.6 is proved in [36, Theorem 3.3].*

Bibliography

[1] Almeida, A., Hästö, P.: Besov spaces with variable smoothness and integrability. J. Funct. Anal. **258** (2010), 1628-1655.

[2] Almeida, A., Drihem, D.: Maximal, potential and singular type operators on Herz spaces with variable exponts. J. Math. Anal. Appl. **394** (2012), 781-795.

[3] Cruz-Uribe, D., Fiorenza, A., Martell, J.M. Pérez, C.: The boundedness of classical operators on variable $L^{p(\cdot)}$ spaces. Ann. Acad. Sci. Fenn. Math . **31** (2006), 239-264.

[4] Diening, L.: Maximal function on generalized Lebesque spaces $L^{p(\cdot)}$. Math. Inequal. Appl. **7** (2004), no. 2, 245–253.

[5] Diening, L., Hästö, P., Roudenko, S.: Function spaces of variable smoothness and integrability. J. Funct. Anal. **256** (2009), no. 6, 1731–1768.

[6] Diening, L., Harjulehton, P., Hästö, P., Mizuta, Y., Shimomura, T..: Maximal functions in variable exponent spaces: limiting cases of the exponent. *Ann. Acad. Sci. Fenn. Math.* **34** (2009), *no.* 2, 503–522

[7] Diening, L., Harjulehton, P., Hästö, P., Růžička, M.: Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics. vol 2017, Springer-Verlag, Berlin, 2011.

[8] Drihem, D.: Some properties of variable Besov-type spaces. Funct. Approx. Comment. Math. **52** (2015), *no.* 2, 193-221.

[9] Drihem, D., Seghiri, F.: Notes on the Herz-type Hardy spaces of variable smoothness and inegrability, Math. Ineq and Appl. **19** (2016), 145–165.

[10] Drihem, D., Zeghad, Z.: Variable local Herz-type Hardy spaces and applications, in preparation.

[11] Fan, D., Yang, D.: The weighted Herz-type Hardy spaces $h\dot{K}^s_{p,q}(w_1, w_2)$. Approx. Theory Appl. (N.S.) **13** (1997), no. 4, 19-41.

[12] Fefferman, C., Stein, E.M.: H^p spaces of several variables, Acta. Math. **129** (1973), 137–193.

[13] Frazier, M., Jawerth, B.: A discrete transform and decomposition of distribution spaces. J. Funct. Anal. **93** (1990), *no.* 1, 34–170.

[14] García-Cuerva, J.: Hardy spaces and Beurling algebras. J. Lond. Math. Soc. **39** (1989), 499–513.

[15] García-Cuerva, J., Herrero, M.-J.L.: A theory of Hardy spaces associated to Herz spaces. Proc. Lond. Math. Soc. **69**(1994), 605–628.

[16] Gonçalves, H. F.: Non-smooth atomic decomposition of variable 2-microlocal Besov-type and Triebel-Lizorkin-type spaces, arXiv:2011.08490.

[17] Goldberg, D.: A Local version of real Hardy spaces. Duke Math. J. **46** (1979), 27-42.

[18] Herz, C.: Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms. J. Math. Mech. **18** (1968), 283–324.

[19] Izuki, M.: Herz and amalgam spaces with variable exponent, the Haar wavelets and greediness of the wavelet system. East. J. Approx. **15** (2009), 87-109.

[20] Izuki, M.: Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. Anal. Math. **36** (2010), 33-50.

[21] Izuki, M., Noi, T.: Boundedness of some integral operators and commutators on generalized Herz spaces with variable exponents. http://www.sci.osaka-cu.ac.jp/math/OCAMI/preprint/2011/11-15.pdf.

[22] Kempka, H., Vybíral, J.: A note on the spaces of variable integrability and summability of Almeida and Hästö. Proc. Amer. Math. Soc. **141** (2013), 3207-3212.

[23] Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Math. J. **116** (1991), 592-618.

[24] Liang,Y., Yang, D., Zhuo, C.: Intrinsic Square Function Characterizations of Hardy Spaces with Variable Exponents. Bull. Malays. Math. Sci. Soc. 39 (2016), no. 4, 1541–1577.

[25] Liu, Z., Wang, H.: The Herz-type Hardy spaces with variable exponent and thier applications. Taiwanese. J. Math. **16** (2012), 1363-1389.

[26] Lu, S., Yang, D.: The weighted Herz-type Hardy spaces and its applications. Sci. China (Ser. A) **38**,662–673 (1995)

[27] Lu, S., Yang, D.: The local versions of $H^p(\mathbb{R}^n)$ spaces at the origin. Studia Math. **116** (1995), 103-131.

[28] Lu, S., Yang, D.: Some characterizations of weighted Herz-type Hardy spaces and their applications. Acta Math. Sin. (Engl. Ser.) **13** (1997), 45–58.

[29] Lu, S., Yang, D.: Boundedness of sublinear operators in Herz spaces on Vilenkin groups and its application. Math. Nachr. **191** (1998), 229–246.

[30] Lu, S., Yang, D., Hu, G.: Herz type spaces and their applications, Beijing: Science Press, 2008.

[31] Li, X., Yang, D.: Boundedness of some sublinear operators on Herz spaces . Illinois J. Math. **40** (1996), 484-501.

[32] Růžička, M.: Electrorheological fluids: modeling and mathematical theory. Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin, (2000).

[33] Samko, S.: Variable exponent Herz spaces. Mediterr. J. Math. **10** (2013), 2007-2025.

[34] Soria, F., Weiss, G.: A remark on singular integrals and power weights. Indiana Univ. Math. J. **43** (1994), 187-204.

[35] Tsutsui, Y.: Pseudo-differential operators of class $S^m_{0,0}$ on the Herz-type spaces. Hokkaido Math. J. **38** (2009), 283-302.

[36] Wang, H., Liu, Z.: The Herz-type Hardy spaces with variable exponent and thier applications. Taiwanese. J. Math. **16** (2012), 1363-1389.

[37] Wang, H., Liu, Z.: Local Herz-type Hardy spaces with variable exponent. Banach J. Math. Anal. **9** (2015) 359–378.

[38] Wang, H., Liu, Z.: Some characterizations of Herz-type Hardy spaces with variable exponent. Ann. Funct. Anal. **6** (2015), 224-233.

ملخص

في هذه الأطروحة ، ندرس استمرارية العوامل التكاملية الفردية والمشغلين تفاضلات زائفة على مساحات هاردي المحلية من نوع هرتس ذات الأسس المتغيرة. نقدم بعض خصائصها مثل التحلل الذري و التركيب الجزيئي لهذه المساحات

كلمات مفتاحية

مساحات هيرز ، مساحات هيرز من نوع هاردي ، دالة قصوى ، أس متغير ، العوامل التفاضلية الزائفة ، التكامل الفردي ، الذرة ، الجزيء

Abstract

In this thesis, we study the boundedness of singular integral operators and pseudo-diferential operators on variable local Herz-type Hardy spaces. To do these, we present some their properties such us the atomic and molecular decomposition of such spaces.

Key words

Herz spaces, Herz-type Hardy spaces, maximal function, variable exponent, pseudo-differential operators, singular integral operators, atom, molecule.

Résumé

Dans cette thèse, nous étudions la continuité d'opérateurs intégraux singuliers et d'opérateurs pseudo-différentiels sur des espaces de Herz-type Hardy locaux à exposant variables. Pour ce faire, nous présentons certaines leurs propriétés telles que la décomposition atomique et moléculaire de tels espaces.

Mot-clés

Espaces de Herz, espaces de Herz-type Hardy, fonction maximale, exposant variable, opérateurs pseudo-différentiels, intégrale singuliers, atome, moléculaire