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Theme

On Boolean Lattices
(Sur les Treillis Booléens)

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إهداء

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إلى اللجنة المكونة من اساتذة رافقونا طوال العامين السابقين الأستاذ **عمرون عبد العزيز**والاستاذ **لجلط لحسن** والاستاذ **غضبان ناصر** جزاكم الله خيرا
رحاب وسهام.

Contents

Introduction

The concept of Boolean lattices is introduced by the mathematician Georges Boole in 1847. A Boolean lattice is a mathematical structure that is a partially ordered set (poset) where every pair of elements has a unique least upper bound (join) and greatest lower bound (meet). This lattice structure arises from the interplay between set operations, particularly union (join) and intersection (meet), and the containment relationship between sets. Those meet and join operations are distributive to each other, and they have a smallest element and a greatest element denoted respectively 0 and 1.

One of the defining properties of Boolean lattices is the presence of complementation, which allows for the existence of a unique complement of each element. Complementation gives rise to an interesting duality property, where the operations of union and intersection interchange with the roles of meet and join, respectively.

Boolean lattices have numerous practical applications in different domains. In computer science, they are extensively employed in the design and analysis of digital circuits which are the heart of any electronic machine. Furthermore, the applications of Boolean lattices extend beyond computer science and mathematics. In information theory, they play a crucial role in the study of data compression, coding theory, and error correction. By employing Boolean lattices, it becomes possible to analyze and optimize the representation and transmission of information.

In addition, Boolean lattices find applications in social network analysis, where they can model and analyze relationships between individuals or groups based on shared characteristics or interests. This facilitates the study of network structures, clustering, and community detection.

In this thesis, we deal with the subject of lattices, which are a particular class of ordered sets and we will accentuate the study on Boolean lattices and their algebraic structures. To that end, we organize our thesis to three chapters as follows:

- **In the first chapter**, we recall the necessary concepts and properties of binary relations, partially orders, ordered sets (posets), particular elements of ordered sets and morphisms of ordered sets.

- **In the second chapter**, we present the concepts of lattices, ideals, filters, sub-lattices and lattice morphisms. Moreover, we provide some classes of lattices are distributive lattices, modular lattices and complemented lattices.

- **In the third chapter**, we study the Boolean lattices and their algebric structures, then we provide the relationship between Boolean rings and Boolean lattices. Finally, we solve some linear Boolean equations and inequalities.

Further information on the concepts of ordered sets, lattices and Boolean lattices can be found in [1, 2, 4, 5, 6, 7, 9, 10].

Chapter 1

Preliminaries on ordered sets

In this chapter, we recall the necessary concepts and properties of binary relations, partially orders and ordered sets.

1.1 Binary relations on sets

In this section, we recall the notion of binary relations on a set and their properties.

1.1.1 Definitions and properties

Definition 1.1 (Cartesian product). *[7]*
Let E and F be two sets, we denote $E \times F$ and we call the Cartesian product of E and F the set of all ordered pairs whose first component belongs to E and the second to F:

$$E \times F = \{(x, y) \mid x \in E \text{ and } y \in F\}.$$

Example 1.1. *Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ be two finite sets then we have the following:*

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\} \text{ and}$$
$$B \times A = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}.$$

Remark 1.1. *As a remark, the Cartesian product does not necessary commutative. As can seen from Example 1.1 that $A \times B \neq B \times A$.*

Definition 1.2 (**Binary relation on a set**). *[7]*
A binary relation R on a set E is a part (sub-set) of $E^2 = E \times E$ ($R \subseteq E^2$), i.e.,

$$R = \{(a,b) \mid (a,b) \in E^2\}.$$

If R is a binary relation on a set E and $(a,b) \in R$, we say that a is related to b according to R and we denote aRb for short.

Definition 1.3 (**Properties of binary relations**). *[2]*
If E a set and R be a binary relation on E. The relation R is said to be:

- *Reflexive if $(a,a) \in R$, for all $a \in E$.*
- *Irreflexive if $(a,a) \notin R$, for all $a \in E$.*
- *Symmetric, if $(a,b) \in R$, then $(b,a) \in R$, for all $a, b \in E$.*
- *Antisymmetric if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$, for all $a, b \in E$.*
- *Transitive if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a, b, c \in E$.*

Example 1.2. *Let $E = \{1,2,3,4\}$ be a finite set. We define on E the following binary relations as follows*

$$R_1 = \{(1,1),(1,2),(2,2),(2,3),(3,3),(4,4)\};$$
$$R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,1),(4,4)\};$$
$$R_3 = \{(1,2),(2,1)\};$$
$$R_4 = \{(1,1),(1,2),(2,1),(2,2)\};$$
$$R_5 = \{(1,1),(1,2),(1,3),(1,4)\};$$
$$R_6 = \{(2,1),(3,1),(3,2),(4,4)\}.$$

Then it holds that

1) *R_1 is reflexive because it contains all pairs of the form (a,a) for every element $a \in E$, i.e., it contains $(1,1),(2,2),(3,3)$ and $(4,4)$. Then it is not irreflexive;*

2) *R_2 is not reflexive because the pair $(3,3) \notin R_2$, also it is not irreflexive because $(1,1) \in R_2$;*

3) *R_3 is irreflexive because $(a,a) \notin R_4$, for any $a \in E$. Then it is not reflexive.*

4) R_4 is symmetric because for every $(a,b) \in R_4$, we have $(b,a) \in R_4$ like $(1,2)$ and $(2,1)$ the both are in R_4;

5) R_5 is antisymmetric, then it not symmetric because we have $(1,2) \in R_5$, but $(2,1) \notin R_5$;

6) R_6 is transitive because $(3,2),(2,1)$ and $(3,1)$ are there in R_6. But R_3 is not transitive, indeed $(1,2) \in R_3$, $(2,1) \in R_3$ and $(1,1) \notin R_3$.

1.1.2 Representations of binary relations

In this subsection, we present some types of representations of a binary relations.

Representation of a binary relation by a matrix:

Let $E = \{a_1, a_2, ..., a_m\}$ and $F = \{b_1, b_2, ..., b_n\}$ are finite sets containing m and n elements respectively. Let R be a binary relation from E to F. Then R can be represented by the matrix $M_R = [m_{ij}]$ of $m \times n$ elements which is defined by

$$m_{ij} = \begin{cases} 1 & if(x_i, x_j) \in R; \\ 0 & if(x_i, x_j) \notin R. \end{cases}$$

Example 1.3. *Let $E = \{1,2,3\}$ be a finite set and $R = \{(a,b) \in E^2 \mid a \text{ devides } b\}$ be a binary relation on E. Then*

$$R = \{(1,1),(1,2),(1,3),(2,2),(3,3)\}.$$

In this case, R can be represented by the following matrix

$$M_R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Representation of a binary relation by a directed graph:

Definition 1.4. *There is another way of picturing a binary relation on a finite set using a direct graph. If a binary relation R is defined on a finite set E then the elements of E are represented by vertices, and the ordered pairs of R are represented by directed edges. As an example, we take the binary relation given in Example 1.3. Its directed graph is shown in the following figure.*

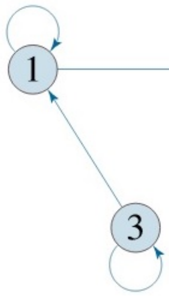


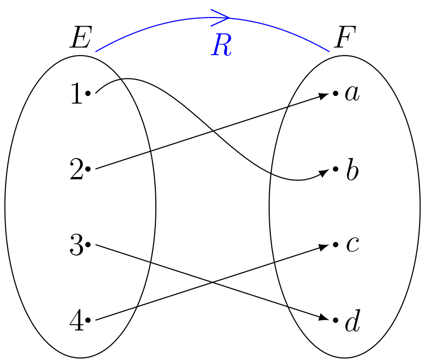
Figure 1.1: The directed graph of a binary relation R.

Representation of a binary relation by Arrow a Diagram:

Definition 1.5. *A binary relation between two finite sets E and F can be represented using an arrow diagram. As an example, let $E = \{1, 2, 3, 4\}$ and $F = \{a, b, c, d\}$ be two finite sets. We define a binary relation R between E and F as*

$$R = \{(1, b), (2, a), (3, d), (4, c)\}.$$

The above diagram is its arrow diagram defined as follow



1.2 Partially ordered sets

In this section, we give the notion of a partially ordered set and its particular elements.

1.2.1 Definitions and examples

Definition 1.6. *[5] A binary relation $\leqslant$ on a set E is called a partial order if that is reflexive, antisymmetric and transitive. The set E equipped with a partial order $\leqslant$ is called a partially ordered set (poset, for short) denoted $(E, \leqslant)$.*

- ***Reflexive:*** *for any* $x \in E$*, we have* $x \leqslant x$*;*
- ***Antisymmetric:*** *for any* $x, y \in E$*, we have (*$x \leqslant y$ *and* $y \leqslant x$*), then* $x = y$*;*
- ***Transitive:*** *for any* $x, y, z \in E$*, we have (* $x \leqslant y$ *and* $y \leqslant z$*), then* $x \leqslant z$*.*

As a remark, the notation $x \leqslant y$ *means that* x *is less than or equal to* y*.*

Example 1.4. *1) The usual inequality* $\leqslant$ *is an order relation on the known sets* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ *and* $\mathbb{R}$*. So* $(\mathbb{N}, \leqslant), (\mathbb{Z}, \leqslant), (\mathbb{Q}, \leqslant)$ *and* $(\mathbb{R}, \leqslant)$ *are posets.*

2) Let E *be a set, then the inclusion relation* $\subseteq$ *on the set of all subsets of* E *is an order relation.*

3) The divisibility relation $|$ *is a partial order on* $\mathbb{N}^*$*. Indeed,*

- $x \mid x$*, for any* $x \in \mathbb{N}^*$*, so* $|$ *is reflexive.*
- *for any* $x, y \in \mathbb{N}^*$*, if* $x \mid y$ *and* $y \mid x$*, then* $x \leqslant y$ *and* $y \leqslant x$*. Then* $x = y$*, thus the relation* $|$ *is antisymmetric.*
- *for any* $x, y, z \in \mathbb{N}^*$*, if* $x \mid y$ *and* $y \mid z$*, then* $x \mid z$*. Therefore* $|$ *is transitive.*

Definition 1.7 (Strict order). *[7]*
A binary relation on a nonempty set is called a strict partial order (or strict order) if it is irreflexive and transitive.

Example 1.5. *Let* $<$ *be the strict inequality on the set of integers between* 1 *and* 10 *inclusive, i.e.,* $E = \{1, 2, ..., 10\}$*. We have already noted that* $<$ *is irreflexive (since for all* $x \in E$ *we have that* $x \nless x$*) and transitive (since for all* $x, y, z \in E$*, we have that if* $x < y$ *and* $y < z$*, then* $x < z$*). Therefore* $<$ *is a strict order on* X*.*

Remark 1.2. *Any strict order* $<$ *on a non-empty set* E *is necessary antisymmetric, because there are not* $x, y \in E$ *such that* $x < y$ *and* $y < x$*.*

Definition 1.8 (Hasse diagram). *A finite poset* $(E, \leqslant)$ *can be represented by a diagram or reflexivity and transitivity are implicit. Each element of* E *is represented by a point, a segment (an arc) joining two points* x *and* y *represent* $x \leqslant y$*, we use the "up" and the "down" to do without an arrowed direction.*

To draw a Hasse diagram of a finite poset $(E, \leqslant)$*, we follow the following steps:*

- *We represent the elements of E by points with consideration if an element x is strictly greater than another element y, we place the representation of x higher than that of y.*

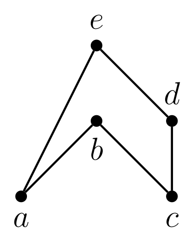
- *In order to not change the diagram, we do not represent the whole order relation $\leqslant$. On the one hand if $x \leqslant y$, but there exists z different from x and from y such that $x \leqslant z$ and $z \leqslant y$, then we do not draw the segment between x and y, on the other hand we do not represent not loops from an element to itself.*

- *We take care as much as possible not to cross the segments.*

Example 1.6. *Let $E = \{a, b, c, d, e\}$ be a finite set and $\leqslant$ be a partially ordered on E defined as follows*

$$\leqslant = \{(a, b), (a, e), (c, b), (c, d), (c, e), (d, e), (a, a), (b, b), (c, c), (d, d), (e, e)\}.$$

Here we present the Hasse diagram of this poset $(E, \leqslant)$.



Remark 1.3. *A finite poset may be has many types of Hasse diagrams, but in fact they are isomorphic.*

1.2.2 Particular elements of ordered sets

Next, we present some particular elements of an ordered set.

Definition 1.9. *Let $(E, \leqslant)$ be a poset and A be a subset of E ($A \subseteq E$). We say that*

(1) an element $x \in E$ is a lower bound of A if $x \leqslant a$, for all $a \in A$.
The set of lower bounds of A, denoted by A^L.

(2) an element $x \in E$ is an upper bound of A if $a \leqslant x$, for all $a \in A$.
The set of upper bounds of A, denoted by A^U.

(3) an element $m \in A$ is the minimum of A, if $m \leqslant a$, for all $a \in A$.

(4) an element $M \in A$ is the maximum of A, if $a \leqslant M$, for all $a \in A$.

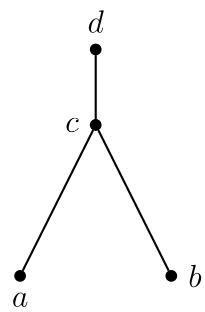
(5) an element $a_1 \in E$ is the least upper bound of A if a_1 is the minimum of A^U.

(6) an element $a_0 \in E$ is the greatest lower bound of A if a_0 is the maximum of A^L.

(7) an element $x \in E$ is said to be minimal, if there is not an other element $y \in E$ such that $y < x$.

(8) an element $x \in E$ is said to be maximal, if there is not an other element $y \in E$ such that $x < y$.

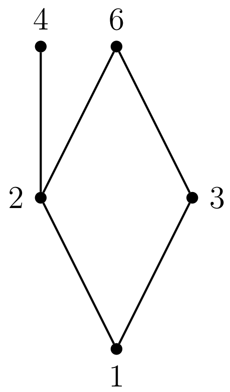
Example 1.7. *Let $(E = \{a, b, c, d\}, \leqslant)$ be a finite poset represented by its following Hasse diagram.*



In this poset, we have the following

- The maximal element of E is d;
- The minimal elements of E are a and b;
- The greatest element of E is d;
- E has not a least element.

Example 1.8. *$(E = \{1, 2, 3, 4, 6\}, \leqslant)$ be a finite poset represented by its following Hasse diagram.*

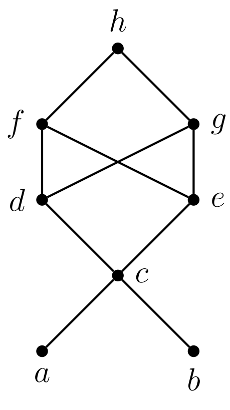


In this poset, we have

- *the maximal elements of E are* 4 *and* 6*;*
- *the minimal element of E is* 1*;*
- *E has not a greatest element;*
- *the least element of E is* 1*.*

Example 1.9. *Let $(E = \{a, b, c, d, e, f, g, h\}, \leqslant)$ be a finite poset given by the following Hasse diagram. Let $B_1 = \{a, b\}$ and $B_2 = \{c, d, e\}$ be two subsets of E. Then the following holds*

- *B_1 has not lower bounds but its upper bounds are c, d, e, f, g and h.*
- *The lower bounds of B_2 are a, b and c. Also, its upper bounds are f, g and h.*



1.2.3 Morphisms of ordered sets

Here, we give the notion of a morphism between two partially ordered sets.

Definition 1.10 (order morphism). *[2] Let $(E, \leq_E)$ and $(F, \leq_F)$ be two posets. An application $f : E \longrightarrow F$ is called an order morphism or an increasing mapping, if for any $x, y \in E$:*

$$x \leq_E y \Rightarrow f(x) \leq_F f(y).$$

We also define:

- *a **decreasing application** if for any $x, y \in E$:*

$$x \leq_E y \Rightarrow f(y) \leq_F f(x).$$

  This is a morphism endowing F with reciprocal order.

- *a **strictly increasing application** if for any $x, y \in E$:*

$$x <_E y \Rightarrow f(x) <_F f(y).$$

- *a **strictly decreasing application** if for any $x, y \in E$:*

$$x <_E y \Rightarrow f(y) <_F f(x).$$

Example 1.10. *1) The identical mapping of an ordered set is an order morphism.*

2) Let $(D(6), |)$ be the poset of the positive divisors of 6 and $(D(30), |)$ be the poset of the positive divisors of 30 ordered by the divisibility order. Let $f : D(6) \longrightarrow D(30)$ be a mapping defined as follows:

$$f(x) = x, \text{ for all } x \in D(6).$$

The mapping f is an order morphism between the two posets $(D(6), |)$ and $(D(30), |)$.

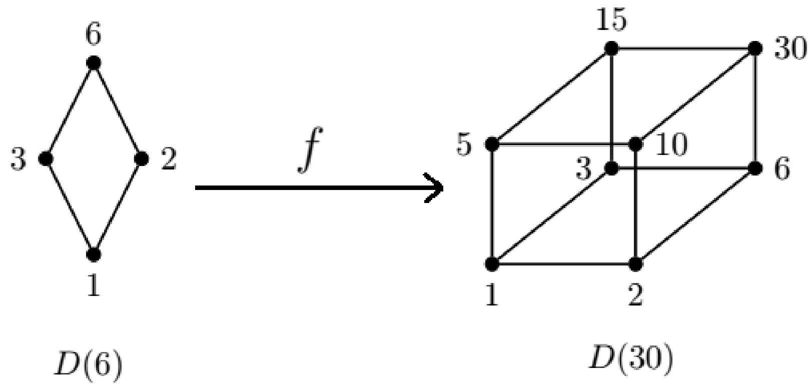


Figure 1.2: The posets $(D(6), |)$ and $(D(30), |)$.

Chapter 2

Lattice structures

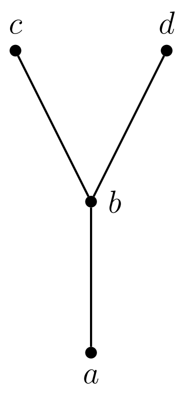
In this chapter, we recall the necessary concepts and properties of lattices, ideals and filters of lattices, sub-lattices and lattice morphisms.

2.1 Generalities on lattices

2.1.1 Lattices as ordered sets

Definition 2.1 (**meet-semilattice**). *[7]*
A poset $(P, \leq_p)$ is said to be a meet-semilattice if for any pair of elements x and y of P, the greatest lower bound (infimum) of x and y exists and denoted $x \wedge y$.

Example 2.1. *Let $(P = \{a, b, c, d\}, \leqslant)$ be a finite poset represented by its following Hasse diagram.*

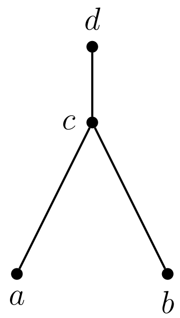


This poset is a meet-semilattice, because the infimum $x \wedge y$ exists in P, for any $x, y \in P$.

Definition 2.2 (**join-semilattice**). *[7]*

A poset $(P, \leq_p)$ is said to be a join-semilattice if for any pair of elements x and y of P, the least upper bound (supermum) of x and y exists and denoted $x \vee y$.

Example 2.2. *Let $(P = \{a, b, c, d\}, \leqslant)$ be a finite poset represented by its following Hasse diagram.*



This poset is a join-semilattice, because the supermum $x \vee y$ exists in P, for any $x, y \in P$.

Definition 2.3 (**lattice**). *[7]*

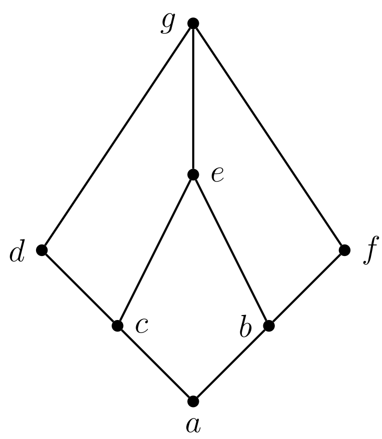
A poset $(P, \leq_p)$ is said to be a lattice if it is a join and meet-semilattice, i.e., for any pair of elements x and y of P, the greatest lower bound (infimum) $x \wedge y$ and the least upper bound (supermum) $x \vee y$ of x and y exist.

Example 2.3. *1) The poset $(\mathbb{N}, \leqslant)$ ordered by the usual order is a lattice, where $x \wedge y = min(x, y)$ and $x \vee y = max(x, y)$, for any $x, y \in \mathbb{N}$.*

2) The poset $(\mathbb{N}^*, |)$ ordered by the divisibility order is a lattice, where $x \wedge y = gcd(x, y)$ and $x \vee y = lcm(x, y)$, for any $x, y \in \mathbb{N}^*$.

3) Let $(P(E); \subseteq)$ be the poset of all parts of the set E. This poset forms a lattice. Indeed, for all $A, B \in P(E)$, we have $A \wedge B = A \cap B$ and $A \vee B = A \cup B$.
The empty set is the smallest element of $P(E)$ and the set E is the greatest element of $P(E)$.

4) Let $P = (\{a, b, c, d, e, f, g\}, \leqslant)$ be the poset given by the bellow Hasse diagram. This poset has the structure of a lattice.



Theorem 2.1. *[5]*

Let $(L, \leqslant, \wedge, \vee)$ be a lattice. The operations $\wedge$ and $\vee$ have the following algebraic properties:

- ***Idempotence:***

  $x \vee x = x$ *and* $x \wedge x = x$.

- ***The commutativity***:

  $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$.

- ***Associativity:***

  $x \vee (y \vee z) = (x \vee y) \vee z$ *and* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

- ***Absorption lows***:

  $x \vee (x \wedge y) = x$ *and* $x \wedge (x \vee y) = x$.

Proof.

- **Idempotence:** $x \wedge x = inf\{x, x\} = x$ and $x \vee x = sup\{x, x\} = x$.

  Then $x \wedge x = x = x \vee x$.

- **The commutativity**:

  $x \wedge y = inf\{x, y\} = inf\{y, x\} = y \wedge x$ and

  $x \vee y = sup\{x, y\} = sup\{y, x\} = y \vee x$.

- **Associativity:**

  We have to prove: $x \vee (y \vee z) = (x \vee y) \vee z$.

  Suppose that $T = x \vee (y \vee z)$ so $x \leq T$ and $y \vee z \leq T$, then $x \leq T, y \leq T$ and $z \leq T$,

  i.e., $x \vee y \leq T$ and $z \leq T$. so T is an upper bound of $\{(x \vee y), z\}$.

Let M an upper bound of $\{(x \vee y), z\}$, so $(x \vee y) \leq M$ and $z \leq M$. Then $x \leq M, y \leq M$ and $z \leq M$. Thus $x \leq M$ and $(y \vee z) \leq M$. Hence, M is an upper bound of $\{x, (y \vee z)\}$. The fact that T is the smallest upper bound of $\{x, (y \vee z)\}$ implies that $T \leq M$. Then T is the smallest upper bound of $\{(x \vee y), z\}$. So $T = (x \vee y) \vee z$, therefore $x \vee (y \vee z) = (x \vee y) \vee z$.

In similar way, we prove that $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

- **Absorption lows**:
  we prove that: $x \wedge (x \vee y) = x$.
  Obvious that $x \wedge (x \vee y) \leq x$ because $x \wedge (x \vee y)$ is a lower bound of $\{x, (x \vee y)\}$. Also, $x \leq x$ and $x \leq x \vee y$, so x is a lower bound of $\{x, (x \vee y)\}$. Since $x \wedge (x \vee y)$ is the greatest lower bound of $\{x, (x \vee y)\}$. Then $x \leq x \wedge (x \vee y)$. Thus $x \wedge (x \vee y) = x$.

  In similar way, we prove that $x \vee (x \wedge y) = x$.

□

2.1.2 Lattices as algebraic structures

Definition 2.4. *[5]*
*An algabraic lattice $(L, \wedge, \vee)$ is a set L equipped with two binary operations $\wedge$ (**meet**) and $\vee$ (**join**) which satisfy the following laws for all $x, y, z \in L$:*

Commutative law:

1 - $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

Associative law:

2- $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$.

Absorption law:

3- $x \wedge (y \vee z) = x$ and $(x \vee y) \wedge z = x$.

Idempotent law:

4- $x \wedge x = x$ and $x \vee x = x$.

In this case, the unique ordered relation with respect to those binary operations "$\wedge$" and "$\vee$" on L is defined as

$$x \leqslant y \text{ if an only if } x \wedge y = x \text{ if an only if } x \vee y = y.$$

2.1.3 Ideals and filters of lattices

Definition 2.5 (**Filters**). *[2]*

Let $(L, \leqslant, \wedge, \vee)$ *be a lattice. We call a filter of* L *any non-empty subset* F *of* L *verifying:*

1. *If* $x \in F$ *and* $x \leqslant y$*, then* $y \in F$*;*
2. *If* $x \in F$ *and* $y \in F$*, then* $x \wedge y \in F$*.*

Definition 2.6 (**Ideals**). *[2]*

Let $(L, \leqslant, \wedge, \vee)$ *be a lattice. We call an ideal of* L *any non-empty subset* I *of* L *verifying:*

1. *If* $x \in I$ *and* $y \leqslant x$*, then* $y \in I$*;*
2. *If* $x \in I$ *and* $y \in I$*, then* $x \vee y \in I$*.*

Example 2.4. *Let* $(D(30), |, gcd, lcm)$ *be the lattice of the positive divisors of* 30 *ordered by the divisibility order. Let* $F_1 = \{2, 6, 10, 30\}$ *and* $F_2 = \{2, 10, 30\}$ *be to subsets of* $D(30)$*. Then it holds that*

- F_1 *is a filter of* $D(30)$*;*
- F_2 *is not a filter of* $D(30)$*. Indeed,* $2 \in F_2$ *and* $2 \mid 6$*, but* $6 \notin F_2$*.*

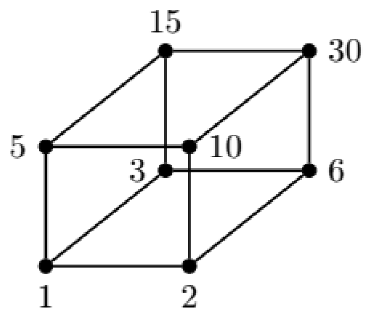


Figure 2.1: The Hasse diagram of the lattice $(D(30), |, gcd, lcm)$.

Example 2.5. *Let* $(D(30), |, gcd, lcm)$ *be the same lattice given in Figure 3.1. Let* $I_1 = \{1, 3, 5, 15\}$ *and* $I_2 = \{5, 10, 15, 30\}$ *be two subsets of* $D(30)$*.*

- $I_1 = \{1, 3, 5, 15\}$ *is an ideal of* (30)*;*

- $I_2 = \{5, 10, 15, 30\}$ *is not ideal of* (30). *Indeed, we have* $10 \in I_2$ *and* $2 \mid 10$, *but* $2 \notin I_2$.

Definition 2.7 (**Principal filters**). *[7]*
Let $(L, \leqslant, \wedge, \vee)$ *be a lattice. A filter* F *of* L *is said to be principal, if there exists an element* $\alpha \in L$ *such that* $F = F_\alpha$ *where*

$$F_\alpha = \{x \in L \mid \alpha \leqslant x\}.$$

Definition 2.8 (**Principal ideals**). *[7]*
Let $(L, \leqslant, \wedge, \vee)$ *be a lattice. An ideal* I *of* L *is said to be principal, if there exists an element* $\beta \in L$ *such that* $I = I_\beta$ *such that*

$$I_\beta = \{x \in L \mid x \leqslant \beta\}.$$

Example 2.6. *Let* $(D(6), |, gcd, lcm)$ *be the lattice of the positive divisors of* 6 *ordered by the divisibility order. We take* $\alpha = 3$ *and* $\beta = 3$. *Then it holds that*

- *The principal filter generated by* 3 *is* $F_3 = \{3, 6\}$;
- *The principal ideal generated by* 3 *is* $I_3 = \{1, 3\}$.

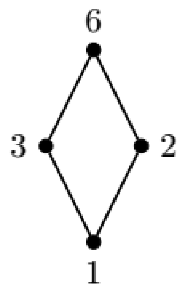


Figure 2.2: The Hasse diagram of the lattice $(D(6), |, gcd, lcm)$.

Definition 2.9 (**maximal filters**). *[7]*
Let $(L, \leqslant, \wedge, \vee)$ *be a lattice. A proper filter* F *of* L *(i.e.,* $F \subsetneq L$*) is said to be maximal (or ultrafilter) if for any proper filter* F' *of* L*, it has* $F \subseteq F'$ *implies* $F' = F$.

Definition 2.10 (**maximal ideals**). *[7]*
Let $(L, \leqslant, \wedge, \vee)$ *be a lattice. A proper ideal* I *of* L *(i.e.,* $I \subsetneq L$*) is said to be maximal if for any proper ideal* I' *of* L*, it has* $I \subseteq I'$ *implies* $I' = I$.

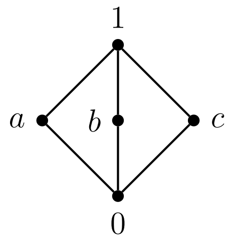
Example 2.7. *Let $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ be the lattice given in Example 2.4 its maximal filters and ideals are:*

- $F_2 = \{x \in D(30) : 2 \mid x\} = \{2, 6, 10, 30\}$;
- $F_3 = \{x \in D(30) : 3 \mid x\} = \{3, 6, 15, 30\}$;
- $F_5 = \{x \in D(30) : 5 \mid x\} = \{5, 10, 15, 30\}$;
- $I_6 = \{x \in D(30) : x \mid 6\} = \{1, 2, 3, 6\}$;
- $I_{10} = \{x \in D(30) : x \mid 10\} = \{1, 2, 5, 10\}$;
- $I_{15} = \{x \in D(30) : x \mid 15\} = \{1, 3, 5, 15\}$.

Definition 2.11. *[1] In a given lattice $(L, \leqslant, \wedge, \vee)$, we say that an element $b \in L$ covers an element $a \in L$ if $b < x < a$ does not satisfy by any $x \in L$.*

Definition 2.12. *[5] Let $(L, \leqslant, \wedge, \vee, 0)$ be a lattice with a smallest element 0. An element $a \in L$ is called atom if it covers 0, i.e., if $0 < b \leqslant a$ implies $b = a$, for any $b \in L$.*

Example 2.8. *Let $L = \{0, a, b, c, 1\}$ be the lattice given by the bellow Hasse diagram. This lattice has three atoms are a, b and c.*



Theorem 2.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $\alpha \in L$ be an atom. Then the principal filter F_α generated by α is maximal.*

Proof. Let α be an atom of L and F_α be the principal filter generated by α. We would like to prove that the filter F_α is maximal. Let F be a filter of L such that $F_\alpha \subsetneq F$. Then there exists $\beta \in F$ and $\beta \notin F_\alpha$. So we have two possible cases are $\beta < \alpha$ implies $\beta = 0$, so $F = L$ or β is also an atom, then $\alpha \wedge \beta = 0 \in F$. Thus $F = L$. Therefore, F_α is maximal. □

Definition 2.13 (Prime filters). *[7] A proper filter F of a lattice $(L, \leqslant, \wedge, \vee)$ is said to be prime if:*

$$a \vee b \in F \text{ implies } a \in F \text{ or } b \in F, \text{ for any } a, b \in L.$$

Definition 2.14 (**Prime ideals**). *[7] A proper ideal I of a lattice $(L, \leqslant, \wedge, \vee)$ is said to be prime if:*

$$a \wedge b \in I \text{ implies } a \in I \text{ or } b \in I, \text{ for any } a, b \in L.$$

Example 2.9. *Let $(D(30), |, gcd, lcm)$ be the lattice given by the Hasse diagram in figure 3.1. Let $F_3 = \{3, 6, 15, 30\}$ and $F_{15} = \{15, 30\}$ be two filters of $D(30)$. Let $I_{15} = \{1, 3, 5, 15\}$ and $I_2 = \{1, 2\}$ be two ideals of $D(30)$. Then it holds that:*

- *F_3 is a prime filter of $D(30)$;*
- *F_{15} is not a prime filter of $D(30)$. Indeed, $3 \vee 5 = lcm(3, 5) = 15 \in F_{15}$, but $3 \notin F_{15}$ and $5 \notin F_{15}$;*
- *I_{15} is a prime ideal of $D(30)$;*
- *I_2 is not a prime ideal of $D(30)$. Because $6 \wedge 10 = gcd(6, 10) = 2 \in I_2$, but $6 \notin I_2$ and $10 \notin I_2$;*

Proposition 2.1 (**Caracterization of ultrafilter**). *[9]*
Let $(L, \leqslant, \wedge, \vee)$ be a lattice with a smallest element 0 and F be a proper filter of L. Then the following two assertions are equivalent:

1. *F is an ultrafilter;*
2. *For all $x \notin F$, there exists $y \in F$ such that $x \wedge y = 0$.*

Proof.

- For the direct implication assume that F is an ultrafilter of L. Suppose that there exists $x \notin F$ such that for all $y \in F, x \wedge y \neq 0$.
  Let $G = F \cup \{x\}$ and $a_1, ..., a_n$ be elements of F. We put $a = x \wedge a_1 \wedge ... \wedge a_n$. Then $a = x \wedge y$ with $y = a_1 \wedge ... \wedge a_n \in F$. Thus $a \notin F$. By assuming a proper filter F_G we have $F \subsetneq G \subseteq F_G$ which contradicts the maximality of F.
- Conversely, we suppose that F is not an ultrafilter and for all $x \notin F$, there exists $y \in F$ such that $x \wedge y = 0$. Then there exists F' a proper filter of L such that $F \subsetneq F'$. Hence there exists $x \in F'$ and $x \notin F$. Therefore there exists $y \in F$ such that $x \wedge y = 0$ with x and

y belong to F'. Thus $0 \in F'$ which contradicts the fact that F is proper. Consequently, F is maximal.

□

Proposition 2.2 (**Caracterization of maximal ideals**). *[9]*
Let $(L, \leqslant, \wedge, \vee)$ be a lattice with a greatest element 1 and I be a proper ideal of L. Then the following two assertions are equivalent:

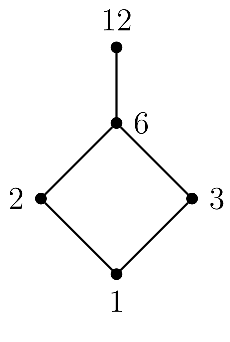
1. *I is a maximal ideal;*

2. *For all $x \notin I$, there exists $y \in I$ such that $x \vee y = 1$.*

2.1.4 Sub-lattices and lattice morphisms

Definition 2.15 (**Sublattices**). *[7]*
A sublattice of a lattice L is a subset S such that $x \wedge y$ and $x \vee y$ are in S for all $x, y \in S$.

Example 2.10. *Let $L = \{1, 2, 3, 6, 12\}$ be the lattice given by the Hasse diagram in the above figure ordered by the divisibility order.*



Let $S_1 = \{1, 2, 3, 6\}$ and $S_2 = \{1, 2, 3, 12\}$ be two subsets of L. Then S_1 is a sublattice of L, but S_2 is not. Because $2, 3 \in S_2$ and $2 \vee 3 = 6 \notin S_2$.

Definition 2.16 (**lattice morphisms**). *[7]*
Let $(L_1, \leqslant_1, \wedge_1, \vee_1)$ and $(L_2, \leqslant_2, \wedge_2, \vee_2)$ be two lattices. A mapping $f : L_1 \longrightarrow L_2$ verifies the two conditions

$$f(a \wedge_1 b) = f(a) \wedge_2 f(b) \text{ and } f(a \vee_1 b) = f(a) \vee_2 f(b), \text{ for any } a, b \in L_1$$

is called a lattice morphism.

Example 2.11. *[3] Let* $D_6 = \{1, 2, 3, 6\}$ *and* $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ *be two lattices defined by ordered by the divisibility order, and* $f : D_6 \longmapsto D_{30}$ *be a mapping defined in the following table.*

x	1	2	3	6
$f(x)$	1	2	5	10

The mapping f *is a lattice-morphism. Indeed, let* $x, y \in L$*, we have:*

(1) *if* $(x, y) = (1, 1)$*,then*

$$f(1 \wedge 1) = f(1) \wedge f(1) = 1 \wedge 1 = 1 \text{ and } f(1 \vee 1) = f(1) \vee f(1) = 1 \vee 1 = 1$$

(2) *if* $(x, y) = (1, 2)$*, then*

$$f(1 \wedge 2) = f(1) = 1 = f(1) \wedge f(2) = 1 \wedge 2 \text{ and } f(1 \vee 2) = f(2) = 2 = f(1) \vee f(2) = 1 \vee 2;$$

(3) *if* $(x, y) = (1, 3)$*, then*

$$f(1 \wedge 3) = f(1) = 1 = f(1) \wedge f(3) = 1 \wedge 5 \text{ and } f(1 \vee 3) = f(3) = 5 = f(1) \vee f(3) = 1 \vee 5;$$

(4) *if* $(x, y) = (1, 6)$*, then*

$$f(1 \wedge 6) = f(1) = 1 = f(1) \wedge f(6) = 1 \wedge 10 \text{ and}$$
$$f(1 \vee 6) = f(6) = 10 = f(1) \vee f(6) = 1 \vee 10;$$

(5) *if* $(x, y) = (2, 2)$*, then*

$$f(2 \wedge 2) = f(2) = 2 = f(2) \wedge f(2) = 2 \wedge \mathit{2} \text{ and } f(2 \vee 2) = f(2) = 2 = f(2) \vee f(2) = 2 \vee 2;$$

(6) *if* $(x, y) = (2, 3)$*, then*

$$f(2 \wedge 3) = f(1) = 1 = f(2) \wedge f(3) = 2 \wedge 5 \text{ and}$$
$$f(2 \vee 2) = f(6) = 10 = f(2) \vee f(3) = 2 \vee 5;$$

(7) *if* $(x, y) = (2, 6)$*, then*

$$f(2 \wedge 6) = f(2) = 2 = f(2) \wedge f(6) = 2 \wedge 10 = 2 \text{ and}$$
$$f(2 \vee 6) = f(6) = 10 = f(2) \vee f(6) = 2 \vee 10;$$

(8) if $(x, y) = (3, 3)$, *then*

$$f(3 \wedge 3) = f(3) = 5 = f(3) \wedge f(3) = 5 \wedge 5 = 5 \text{ and}$$
$$f(3 \vee 3) = f(3) = 5 = f(3) \vee f(3) = 5 \vee 5;$$

(9) if $(x, y) = (3, 6)$, *then*

$$f(3 \wedge 6) = f(3) = 5 = f(3) \wedge f(6) = 5 \wedge 10 = 5 \text{ and}$$
$$f(3 \vee 6) = f(6) = 10 = f(3) \vee f(6) = 5 \vee 10.$$

Example 2.12. *Let* L *and* M *be two lattices defined by the following Hasse diagrams. The mapping* $g : L \longmapsto M$ *defined by* $g(n) = n$ *is not a lattice morphism because:*
$g(2 \vee 3) = g(12)$ *and* $g(2) \vee g(3) = 2 \vee 3 = 6$ *then* $12 \neq 6$.

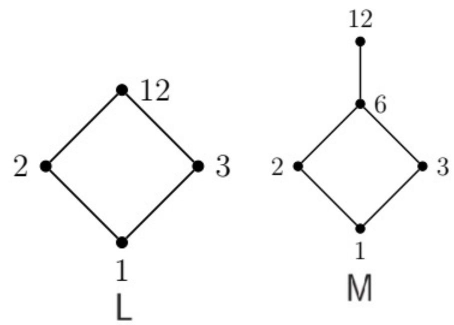


Figure 2.3

2.2 Algebraic properties of some classes of lattices

2.2.1 Distributive lattices

Definition 2.17. *[7]*
A lattice $(L, \wedge, \vee)$ *is distributive if one of the two equivalent conditions is satisfied, for any* $x, y, z \in L$:

(D1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$*;*

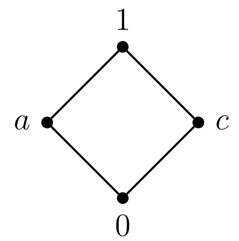
(D2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*.*

Example 2.13. *1) The lattice* $(\mathbb{N}, \leqslant)$ *ordered by the usual order is distributive;*

2) The lattice $(\mathbb{N}^*, |)$ *ordered by the divisibility order is distributive;*

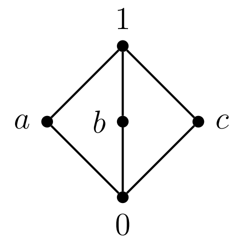
3) Let $(P(E); \subseteq)$ *be the lattice of all parts of the set* E*. This lattice is distributive;*

4) Let $(L = \{0, a, c, 1\}, \leqslant, \wedge, \vee)$ *be a lattice. Then* L *is distributive.*

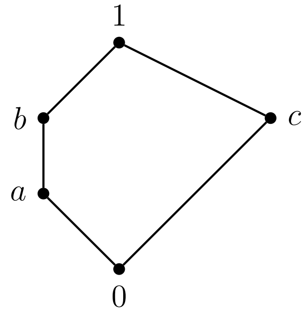


Example 2.14. *Here we give examples of non-distributive lattices.*

1. *Let* $(M_3 = \{0, a, b, c, 1\}, \leqslant, \wedge, \vee)$ *be the diamond lattice, then* L *is not distributive. Indeed,* $a \wedge (b \vee c) = a \wedge 1 = a$*, but* $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$*. So* $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$*;*



2. *Let* $(N_5 = \{0, a, b, c, 1\}, \leqslant, \wedge, \vee)$ *be the pentagon lattice is also not distributive, because* $b \wedge (a \vee c) = b \wedge 1 = b$ *and* $(b \wedge a) \vee (b \wedge c) = a \vee 0 = a$*. Then* $b \wedge (a \vee c) \neq (b \wedge a) \vee (b \wedge c)$*.*



Remark 2.1. *Here, we show that the two conditions* $(D1)$ *and* $(D2)$ *are equivalent. Indeed, suppose that* $(D2)$ *is verified and we prove that* $(D1)$ *is also verified. We can write:*

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= ((x \vee y) \wedge x) \vee ((x \vee y) \wedge z)\\ &= x \vee ((x \vee y) \wedge z) \quad (\textit{absorption laws})\\ &= x \vee (x \wedge z) \vee (y \wedge z)\\ &= x \vee (y \wedge z) \quad (\textit{absorption laws}).\end{aligned}$$

The reciprocal, suppose that (2) *holds, we can write:*

$$\begin{aligned}(x \wedge y) \vee (x \wedge z) &= ((x \wedge y) \vee x) \wedge ((x \wedge y) \vee z)\\ &= x \wedge (z \vee (x \wedge y)) \quad (\textit{absorption law})\\ &= x \wedge (z \vee x) \wedge (z \vee y)\\ &= x \wedge (y \vee z) \quad (\textit{absorption law}).\end{aligned}$$

Theorem 2.3. *[2] Let* $(L, \wedge, \vee)$ *be a lattice and* a, b, c *be three elements of* L*. Then*

(i) $a \wedge (b \vee c) \geqslant (a \wedge b) \vee (a \wedge c)$*;*

(ii) $a \vee (b \wedge c) \leqslant (a \vee b) \wedge (a \vee c)$*;*

(iii) $a \geqslant c$ *implies* $a \wedge (b \vee c) \geqslant (a \wedge b) \vee c$*;*

(iv) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leqslant (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$*.*

Theorem 2.4. *[2] Let* $(L, \wedge, \vee)$ *be a lattice. Then* L *is distributive if and only if it has not a copy of one of the two sub-lattices* M_3 *or* N_5*.*

Theorem 2.5. *[2] Let* $(L, \leqslant, \wedge, \vee)$ *be a distributive lattice with a greatest element* 1*. Then every maximal ideal in* L *is prime. Dually, in a distributive lattice with smallest element* 0 *every ultrafilter is prime.*

2.2.2 Modular lattices

Definition 2.18. *[7] A lattice* $(L, \wedge, \vee)$ *is said to be modular if for all* $x, y, z \in L$*:*

$$x \leqslant z \text{ implies } x \vee (y \wedge z) = (x \vee y) \wedge z.$$

(Note for any lattice, if $x \leqslant z$, then $x \vee (y \wedge z) \leqslant (x \vee y) \wedge z$)).

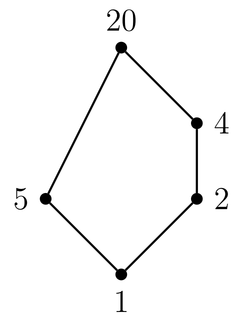
Theorem 2.6. *[7] Every distributive lattice is modular.*

Proof. Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $a, b, c \in L$ such that $a \leqslant c$, we have $a \vee c = c$. Then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c.$$

Hence L is modular. □

Example 2.15. *(1) Let $N_5 = \{1, 2, 4, 5, 20\}$ be the pentagon lattice ordered by the divisibility order. The fact that $2 \leq 4$ but $2 \vee (5 \wedge 4) = 2 \vee 1 = 2$ and $(2 \vee 5) \wedge 4 = 20 \wedge 4 = 4$. Then $2 \vee (5 \wedge 4) \neq (2 \vee 5) \wedge 4$. Thus, N_5 is not modular.*

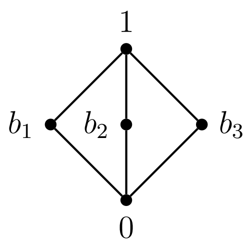


(2) Let $(M_3 = \{0, b_1, b_2, b_3, 1\}, \leqslant, \wedge, \vee)$ be the diamond lattice. Then L is not distributive but it is modular. Indeed, let $a = 0, c = b_1$

Then $a \vee (b \wedge c) = 0 \vee (b \wedge b_1) = b \wedge b_1$

and $(a \vee b) \wedge c = (0 \vee b) \wedge b_1 = b \wedge b_1$

Let $a = b_1, c = 1$, then $a \vee (b \wedge c) = b_1 \vee (b \wedge b1) = b_1 \vee b$ and $(a \vee b) \wedge c = (b_1 \vee b) \wedge c = b_1 \vee b$.



Theorem 2.7. *[2] Let $(L, \wedge, \vee)$ be a lattice. Then L is modular if and only if it has not a copy of the pentagon sub-lattice N_5.*

2.2.3 Complemented lattices

Definition 2.19. *[5] A lattice $(L, \leqslant, \wedge, \vee)$ is called bounded if it has a smallest element 0 and a greatest element 1.*

Theorem 2.8. *Any finite lattice is bounded.*

Proof. Let $(L, \leqslant, \wedge, \vee)$ be a bounded lattice, then $L = \{x_1, x_2, \cdots, x_n\}$. Thus, $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ is the smallest element of L and $x_1 \vee x_2 \vee \cdots \vee x_n$ is the greatest element of L. □

Definition 2.20. *[5] A bounded lattice $(L, \leqslant, \wedge, \vee, 0, 1)$ is called complemented if for each $x \in L$ there is an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. In this case, the element y is called a complement of x.*

Example 2.16. *(1) Let $N_5 = \{1, 2, 4, 5, 20\}$ be the pentagon lattice ordered by the divisibility order and given in Example 2.15. Then 5 has two complements are 2 and 4. Also the smallest element 1 and the greatest element 20 are complemented. Therefore N_5 is a complemented lattice;*

(2) Let E be a non-empty set and $(P(E), \subseteq, \cap, \cup)$ be the lattice of all subsets of E ordered by the inclusion order. The smallest element of $P(E)$ is the empty set $\emptyset$ and its greatest element is the set E. Moreover, this lattice is complemented. Indeed,

$$A \cap A^c = \emptyset \text{ and } A \cup A^c = E \text{ for any } A \in P(E).$$

Theorem 2.9. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a complemented lattice. If L is distributive, then any element $x \in L$ has one complement $y \in L$ denoted $x' = y$.*

Proof. Assume that L is a complemented and distributive. Let $x \in L$, suppose that it has two complement $y_1, y_1 \in L$. Then $x \wedge y_1 = 0, x \wedge y_2 = 0, x \vee y_1 = 1$ and $x \vee y_2 = 1$. Thus

$$\begin{aligned} y_1 = y_1 \vee 0 = y_1 \vee (x \wedge y_2) \\ &= (y_1 \vee x) \wedge (y_1 \vee y_2) \\ &= 1 \wedge (y_1 \vee y_2) \\ &= (y_1 \vee y_2). \end{aligned}$$

and

$$
\begin{aligned}
y_2 = y_2 \vee 0 &= y_2 \vee (x \wedge y_1) \\
&= (y_2 \vee x) \wedge (y_2 \vee y_1) \\
&= 1 \wedge (y_2 \vee y_1) \\
&= (y_1 \vee y_2).
\end{aligned}
$$

Therefore, $y_1 = y_2$. □

Example 2.17. *The lattice* $D(30) = \{1, 2, 3, 6, 10, 15, 30\}$ *ordered by the divisibility order is Complemented. We have* $2' = 15, 15' = 2,\ 3' = 10, 10' = 3,\ 6' = 5, 5' = 6,\ 1' = 30$ *and* $30' = 1$.

Chapter 3

Boolean Lattices

In this chapter, we present the concept of Boolean lattices and their properties.

3.1 Algebraic structure of a Boolean lattice

This section is devoted to give the algebraic structure of a Boolean lattice.

Definition 3.1. *[2]*
A lattice $(B, \leqslant, \wedge, \vee)$ is called Boolean lattice if that is:

(i) Distributive;

(ii) Bounded (it has smallest element 0 and greatest element 1);

(iii) Complemented.

Example 3.1. *1) The lattice $(P(E), \subseteq, \cap, \cup)$ is a Boolean lattice, where $0 = \emptyset, 1 = E$ and $A' = A^c$, for any $A \in P(E)$;*

2) The chain $U = \{0, 1\}$ is a Boolean lattice called the unit Boolean lattice;

3) The lattice $(D(6), |, gcd, lcm)$ is a Boolean lattice;

4) The lattice $(D(8), |, gcd, lcm)$ is not a Boolean lattice, because it does not complemented. Indeed, the element 2 has not a complement in $D(8)$.

3.1.1 Properties of Boolean lattices

Proposition 3.1. *[9] Let* $(B, \leqslant, \wedge, \vee, 0, 1,')$ *be a Boolean lattice. Then the following properties hold*

(1) $0^{'} = 1$ *and* $1^{'} = 0$*;*

(2) $(x^{'})^{'} = x$*, for any* $x \in B$*;*

(3) The Morgan's laws are hold, for any $x, y \in B$*:*

$$(x \wedge y)^{'} = x^{'} \vee y^{'} \text{ and } (x \vee y)^{'} = x^{'} \wedge y^{'};$$

(4) $x \leqslant y$ *if and only* $x \wedge y^{'} = 0$*, for any* $x, y \in B$*;*

(5) $x \leqslant y$ *if and only* $x^{'} \vee y = 1$*, for any* $x, y \in B$*.*

Proof. Let $x, y \in B$.

(3): We prove the Morgan's laws.

- The fact that B is distributive implies that

$$\begin{aligned}(x \wedge y) \vee (x' \vee y') &= (x \vee x' \vee y') \wedge (y \vee x' \vee y') \\ &= (1 \vee y') \wedge (1 \vee x') \\ &= 1 \wedge 1 = 1.\end{aligned}$$

  And

$$\begin{aligned}(x \wedge y) \wedge (x^{'} \vee y^{'}) &= (x \wedge y \wedge x^{'}) \vee (x \wedge y \wedge y^{'}) \\ &= (0 \wedge y) \vee (0 \wedge x) \\ &= 0 \vee 0 = 0.\end{aligned}$$

  Thus, $(x' \vee y')$ is the complement of $(x \wedge y)$. Therefore, $(x \wedge y)' = (x' \vee y')$.

- Daully, since B is distributive, it holds that

$$\begin{aligned}(x \vee y) \wedge (x' \wedge y') &= (x \wedge (x' \wedge y')) \vee (y \wedge (x' \wedge y')) \\ &= ((x \wedge x') \wedge y') \vee ((y \wedge y') \wedge x') \\ &= (0 \wedge y') \vee (0 \wedge x') \\ &= 0 \vee 0 = 0.\end{aligned}$$

And

$$\begin{aligned}(x \vee y) \vee (x' \wedge y') &= (x' \vee (x \vee y)) \wedge (y' \vee (x \vee y)) \\ &= ((x' \vee x) \vee y) \wedge ((y' \vee y) \vee x) \\ &= (1 \vee y) \wedge (1 \vee x) \\ &= 1 \wedge 1 = 1.\end{aligned}$$

Then, $(x' \wedge y')$ is the complement of $(x \vee y)$. Therefore, $(x \vee y)' = (x' \wedge y')$.

(4): For the direct implication we suppose that $x \leqslant y$ and we prove that $x \wedge y' = 0$. We have $x \leqslant y$, then $x \wedge y' \leqslant y \wedge y'$. Thus $x \wedge y' \leqslant 0$. Therefore $x \wedge y' = 0$.
To prove the converse implication we assume that $x \wedge y' = 0$. Then

$$\begin{aligned}(x \wedge y') \vee y = 0 \vee y &\Longrightarrow (x \vee y) \wedge (y' \vee y) = y \\ &\Longrightarrow (x \vee y) \wedge 1 = y \\ &\Longrightarrow x \vee y = y.\end{aligned}$$

Consequently, $x \leqslant y$. □

Definition 3.2 (**Addition operation**). *[9] In a Boolean lattice $(B, \leqslant, \wedge, \vee, 0, 1, ')$, we can define an addition operation for any $x, y \in B$ as follow:*

$$x + y = (x \wedge y') \vee (x' \wedge y).$$

Example 3.2. *(1) We knew the bounded distributive lattice $D(30), |, gcd, lcm, 1, 30, ')$ is complemented, then it is a Boolean lattice. The addition operation in this Boolean lattice is defined for any $x, y \in D(30)$ as follow*

$$x + y = (x \wedge y') \vee (x' \wedge y) = lcm(gcd(x, y'), gcd(x', y));$$

(2) Let E be a non-empty set and $(P(E), \subseteq, \cap, \cup, ^c)$ be the Boolean lattice of all subsets of E ordered by the inclusion order. The addition operation in this Boolean lattice called the symmetric difference denoted $\triangle$. For any $A, B \in P(E)$, we have

$$A + B = A \triangle B = (A \cap B^c) \cup (A^c \cap B).$$

Theorem 3.1. *[9] The addition operation in a Boolean lattice can be transformed as*

$$x + y = (x \vee y) \wedge (x' \vee y').$$

Proof. Let $x, y \in B$, then

$$\begin{aligned} x + y &= (x \wedge y') \vee (x' \wedge y) \\ &= ((x \wedge y') \vee x') \wedge ((x \wedge y') \vee y)) \\ &= ((x \vee x') \wedge (y' \vee x')) \wedge ((x \vee y) \wedge (y' \vee y)) \\ &= (1 \wedge (y' \vee x')) \wedge ((x \vee y) \wedge 1) \\ &= (y' \vee x') \wedge (x \vee y) \\ &= (x \vee y) \wedge (x' \vee y'). \end{aligned}$$

□

Proposition 3.2. *[9] Let $(B, \leqslant, \wedge, \vee, 0, 1,')$ be a Boolean lattice. Then it holds that*

$$(x + y)' = (x' \vee y) \wedge (x \vee y')$$

and

$$(x + y)' = (x \wedge y) \vee (x' \wedge y').$$

Theorem 3.2. *[9] The lattice $D(n)$ ordered by the divisibility order $|$ is a Boolean lattice if and only if n is not divisible by any square of a prime number. It means, n has the form $p_1 \cdot p_2 \cdots p_s$ such that p_i are distinct prime numbers.*

In this case, the operations of the Boolean lattice $D(n)$ are defined as:
$x \cdot y = gcd(x, y)$, $x \vee y = lcm(x, y)$, $x' = \dfrac{n}{x}$ and $x + y = lcm(gcd(x, \dfrac{n}{y}), gcd(\dfrac{n}{x}, y))$.

Example 3.3. *(1) The fact that $6 = 2 \cdot 3$ is not divisible by any square of a prime number implies that $D(6)$ is Boolean lattice;*

(2) These numbers 30 and 210 are not divisible by any square of a prime number. Thus, $D(30)$ and $D(210)$ are Boolean lattices;

(3) Since $12 = 2^2 \cdot 3$ and $60 = 2^2 \cdot 3 \cdot 5$ are divisible by the square of the prime number 2, it holds that $D(12)$ and $D(60)$ are not a Boolean lattices.

3.2 Relationship between Boolean lattices and Boolean rings

In this section, we give the algebraic relationship between Boolean lattices and Boolean rings.

3.2.1 Boolean lattices are Boolean rings

Theorem 3.3. *Let* $(B, \leqslant, \wedge, \vee, 0, 1, ')$ *be a Boolean lattice and* $x, y, z \in B$. *The following properties are verified.*

(i) $x + y = y + x$;

(ii) $x + 0 = x$;

(iii) $x + x = 0$;

(iv) $(x + y) + z = x + (y + z)$.

Proof. Let $x, y, z \in B$. Then

(i)

$$\begin{aligned} x + y &= (x \wedge y') \vee (x' \wedge y) \\ &= (y' \wedge x) \vee (y \wedge x') \\ &= (y \wedge x') \vee (y' \wedge x) \\ &= y + x. \end{aligned}$$

(ii)

$$\begin{aligned} x + 0 &= (x \wedge 0') \vee (x' \wedge 0) \\ &= (x \wedge 1) \vee (x' \wedge 0) \\ &= x \vee 0 \\ &= x. \end{aligned}$$

(iii)

$$\begin{aligned} x + x &= (x \wedge x') \vee (x' \wedge x) \\ &= x \wedge x' \\ &= 0. \end{aligned}$$

(iv) It is not difficult to see that $(x + y) + z = x + (y + z)$.

□

The above Theorem 3.3 guarantees the following corollaries.

Corollary 3.1. *Let $(B, \leqslant, \wedge, \vee, 0, 1,')$ be a Boolean lattice and $x, y, z \in B$. Then the addition operation "$+$" is*

(i) Commutative and associative;

(ii) It has the smallest element 0 of B as a neutral element;

(iii) The opposite of $x \in B$ with respect to "$+$" is itself.

Corollary 3.2. *If $(B, \leqslant, \wedge, \vee, 0, 1,')$ is a Boolean lattice, then the algebraic structure $(B, +)$ is a commutative group.*

Definition 3.3. *Let $(B, \leqslant, \wedge, \vee, 0, 1,')$ be a Boolean lattice. We define a multiplication operation on B as follows*

$$x \cdot y = x \wedge y, \text{ for any } x, y \in B.$$

Remark 3.1. *It is easy to see that this multiplication is commutative, associative, distributive over the addition $+$ of B and it has 1 as a neutral element. Also, it is idempotent, i.e.,*

$$x^2 = x \cdot x = x \wedge x = x, \text{ for any } x \in B.$$

Definition 3.4. *[2] We call Boolean ring any unitary ring whose multiplication is idempotent, i.e., $x^2 = x \cdot x = x$.*

The above Corollary 3.2 and the Remark 3.1 lead to the following theorem.

Theorem 3.4. *If $(B, \leqslant, 0, 1,')$ is a Boolean lattice, then the algebraic structure $(B, +, \cdot)$ is a commutative Boolean ring.*

3.2.2 Boolean rings are Boolean lattices

Theorem 3.5. *[2] Any Boolean ring* $(B, \leqslant, \wedge, \vee, 0, 1, ')$ *is a Boolean lattice by defining the two following binary operations for any* $x, y \in B$*:*

$$x \wedge y = x \cdot y \text{ and } x \vee y = x + y + x \cdot y.$$

Proof. It is not difficult to see that the operation $\wedge$ is commutative, associative and idempotent. Also, the operation $\vee$ is:

- Commutative, $x + y = x + y + xy = y + x + yx = y + x$;
- Associative, in the one hand it holds that

$$\begin{aligned}(x \vee y) \vee z &= (x \vee y) + z + (x \vee y)z \\ &= x + y + xy + z + (x + y + xy)z \\ &= x + y + z + xy + zx + yz + xyz.\end{aligned}$$

  And in the other hand it follows that

$$\begin{aligned}x \vee (y \vee z) &= x + (y \vee z) + x(y \vee z) \\ &= x + y + z + yz + x(y + z + yz) \\ &= x + y + z + xy + xz + yz + xyz;\end{aligned}$$

- Idempotent,

$$\begin{aligned}x \vee x &= x + x + x^2 \\ &= x + x + x \\ &= x + 0 \\ &= x.\end{aligned}$$

- The absorption laws are verified:

$$\begin{aligned}x \wedge (x \vee y) &= x(x + y + xy) \\ &= x^2 + xy + x^2y \\ &= x + xy + xy \\ &= x + 0 = x.\end{aligned}$$

  and $x \vee (x \wedge y) = x + xy + xy = x$;

- The distribution of $\wedge$ over $\vee$:

$$x \wedge (y \vee z) = x(y + z + yz)$$
$$= xy + xz + xyz.$$

and $(x \wedge y) \vee (x \wedge z) = xy + xz + x^2yz = xy + xz + xyz$;

- The smallest and the greatest elements of B: for all $x \in B$, we have $x \wedge 0 = x \cdot 0 = 0$, so $0 \leqslant x$. Moreover, $x \wedge 1 = x \cdot 1 = x$, then $x \leqslant 1$;

- The complement element:
Let $x \in B$ and $y = x + 1 \in B$. Then

$$x \wedge y = x(x + 1) = x^2 + x = x + x = 0$$

and
$$x \vee y = x + x + 1 + x(x + 1) = 1.$$

Thus, y is the complement of x. Hence, $x' = y = x + 1$.

□

Theorem 3.6. *[1] The number of elements of a finite Boolean lattice B is always of the form 2^n, where the natural number n presents the atoms of B. Moreover, Any two Boolean lattices with the same finite number of elements they are isomorphic.*

3.3 Boolean morphisms

Definition 3.5. *[7] Let B_1 and B_2 be two Boolean lattices. A Boolean morphism from B_1 to B_2 is a mapping $f : B_1 \longrightarrow B_2$ with the following properties:*

(1) f preserves the meet and the join operations of B_1 for any $x, y \in B_1$, i.e.,

$$f(x \wedge_1 y) = f(x) \wedge_2 f(y)$$
and
$$f(x \vee_1 y) = f(x) \vee_2 f(y);$$

(2) f preserves the smallest element 0 and the greatest element 1 of B_1, i.e.,

$$f(0) = 0 \text{ and } f(1) = 1;$$

(3) f preserves the complemented property, i.e.,

$$f(x') = (f(x))', \text{ for any } x \in B_1.$$

Example 3.4. *(i) Let $U = \{0, 1\}$ be the unit Boolean lattice, E be a non-empty set and $(P(E), \subseteq, \cap, \cup)$ be its Boolean lattice of all its sub-sets. We take x_0 a fixed element of E. The mapping $f : P(E) \longrightarrow U$ defined by:*

$$f(X) = \begin{cases} 1 & if\ x_0 \in X; \\ 0 & if\ x_0 \notin X. \end{cases}$$

This mapping is a Boolean morphism.

(ii) We define a mapping $f : D(6) \longrightarrow D(30)$ in the following table

x	1	2	3	6
$f(x)$	1	2	15	30

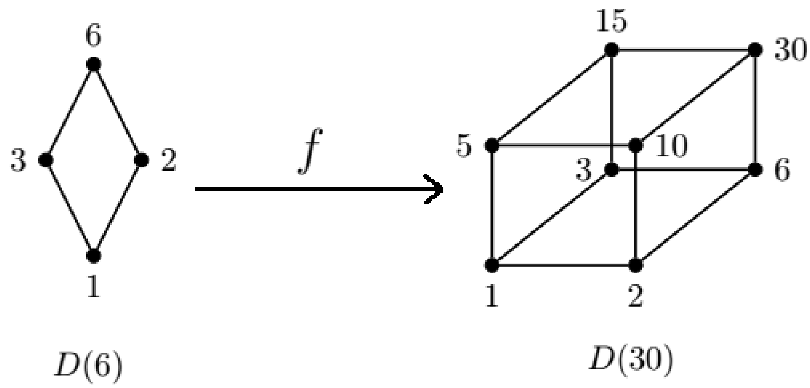


Figure 3.1: The lattices $(D(6), |)$ and $(D(30), |)$.

This mapping f is a Boolean morphism. Indeed,

(1) f preserves the meet and the join operations, i.e.,

- $f(x \vee_1 y) = f(x) \vee_2 f(y)$.

  If $(x, y) = (2, 3)$, *then*
  $$\begin{cases} f(2 \vee_1 3) = f(6) = 30 \\ f(2) \vee_2 f(3) = 2 \vee_2 15 = 30; \end{cases}$$

- $f(x \wedge_1 y) = f(x) \wedge_2 f(y)$.
  $$\begin{cases} f(2 \wedge_1 3) = f(1) = 1 \\ f(2) \wedge_2 f(3) = 2 \wedge_2 15 = 1; \end{cases}$$

(2) f preserves the smallest element and the greatest element, i.e.,

$$f(0_{D(6)}) = f(1) = 1 = 0_{D(30)} \text{ and } f(1_{D(6)}) = f(6) = 30 = 1_{D(30)};$$

(3) f preserves the complement elements, i.e.,

$$f(x') = (f(x))', \text{ for any } x \in B_1.$$

$$\begin{cases} f(1') = f(6) = 30 \\ f(1)' = 1' = 30 \end{cases}$$
$$\begin{cases} f(2') = f(3) = 15 \\ f(2)' = 2' = 15 \end{cases}$$
$$\begin{cases} f(3') = f(2) = 2 \\ f(3)' = 15' = 2 \end{cases}$$
$$\begin{cases} f(6') = f(1) = 1 \\ f(6)' = 30' = 1. \end{cases}$$

3.4 Equations on Boolean lattices

In this section, we give some methods to solve linear Boolean equations and inequalities.

Theorem 3.7. *[9] Let $(B, \leqslant, \wedge, \vee, 0, 1, ')$ be a Boolean lattice and a, b be two fixed elements of B. We consider the following equation in B:*

$$a \cdot x + b = 0. \tag{3.1}$$

(i) The equation (3.1) has a solution if and only if $b \leqslant a$;

(ii) An element $x_0 \in B$ is a solution of (3.1) if and only if $b \leqslant x_0 \leqslant a + b + 1$.

Proof. (i) : To prove the direct implication, assume that the equation (3.1) has a solution $x_0 \in B$. Then $a \cdot x_0 + b = 0$, so $a \cdot x_0 + b + b = 0 + b$. Hence $a \cdot x_0 = b$ such that $a \cdot x_0 = a \wedge x_0$. Thus, $b \leqslant a$. For the converse implication, we consider that $b \leqslant a$. Then $a \wedge b = b$, so $a \wedge b + b = b + b$. Hence $a \wedge b + b = 0$. Therefore, $a \cdot b + b = 0$. Consequently, b is a solution of (3.1);

(ii) : We suppose that $x_0 \in B$ is a solution of (3.1), then $a \cdot x_0 + b = 0$. On the one hand, $a \cdot x_0 = b$ so $b \leqslant x_0$. On the other hand,

$$\begin{aligned}(a + b + 1) \cdot x_0 &= a \cdot x_0 + b \cdot x_0 + 1 \cdot x_0 \\ &= a \cdot x_0 + b + x_0 \quad (\text{because } b \leqslant x_0) \\ &= 0 + x_0 \quad (\text{because } a \cdot x_0 + b = 0) \\ &= x_0.\end{aligned}$$

Thus, $x_0 \leqslant a + b + 1$. Consequently

$$b \leqslant x_0 \leqslant a + b + 1.$$

Conversely, we suppose that $b \leqslant x_0 \leqslant a + b + 1$. Then

$$\begin{aligned}a \cdot b \leqslant a \cdot x_0 \leqslant a \cdot (a + b + 1) &\Longrightarrow a \cdot b \leqslant a \cdot x_0 \leqslant a \cdot a + a \cdot b + a \cdot 1 \\ &\Longrightarrow b \leqslant a \cdot x_0 \leqslant a + b + a \\ &\Longrightarrow b \leqslant a \cdot x_0 \leqslant b.\end{aligned}$$

Hence, $a \cdot x_0 = b$, thus $a \cdot x_0 + b = 0$. Therefore, x_0 is a solution of (3.1). □

Proposition 3.3. *[9] Let $(B, \leqslant, \wedge, \vee, 0, 1, ')$ be a Boolean lattice and a, b, c be three fixed elements of B. We consider the following inequality in B:*

$$a \cdot x + b \leqslant c. \tag{3.2}$$

(i) The inequality (3.2) can be transformed to the form of the equation (3.1) as

$$a \cdot c' \cdot x + b \cdot c' = 0;$$

(ii) The inequality (3.2) has a solution if and only if $b \cdot c' \leqslant a \cdot c'$;

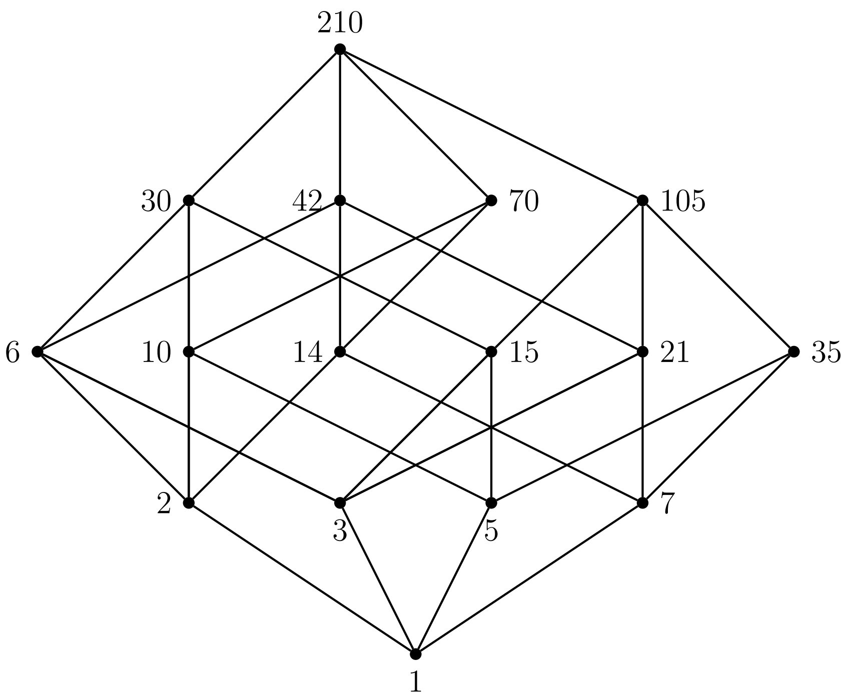
(iii) An element $x_0 \in B$ is a solution of (3.2) if and only if $b \cdot c' \leqslant x_0 \leqslant a \cdot c' + b \cdot c' + 1$.

Proof. (i) : To transform the inequality (3.2) to the form of the equation (3.1), we need to apply the Proposition 3.1 (4). We have $a \cdot x + b \leqslant c$, then $(a \cdot x + b) \wedge c' = 0 \Leftrightarrow a \cdot c' \cdot x + b \cdot c' = 0$. Thus, it has the form of the equation (3.1).

(ii) and (iii) can be proved directly by using the Theorem 3.7. □

Next, we give an illustrative example to solve some Boolean equations and inequalities.

Example 3.5. *Let $D(210)$ be the Boolean lattice of the positive divisors of 210 ordered by the divisibility order and represented in the following Hasse diagram.*



(1) Let $6 \cdot x + 2 = 0_{D(210)}$ be an equation with the form of (3.1) such that $a = 6$ and $b = 2$. The fact that $b \leqslant a$ (i.e., $b \mid a$) implies from Theorem 3.1 that its solutions are given by

$$S = \{x_0 \in D(210) \mid b \leqslant x_0 \leqslant a + b + 1_{D(210)}\}$$
$$= \{x_0 \in D(210) \mid 2 \leqslant x_0 \leqslant 6 + 2 + 1_{D(210)}\}.$$

We need to calculate $6+2+1_{D(210)}$*. Thus*

$$\begin{aligned} 6+2+1_{D(210)} &= 6+2' = 6+105 \\ &= (6 \wedge 105') \vee (6' \wedge 105) \\ &= (6 \wedge 2) \vee (35 \wedge 105) \\ &= 2 \vee 35 \\ &= 70. \end{aligned}$$

Hence,

$$\begin{aligned} S &= \{x_0 \in D(210) \mid 2 \leqslant x_0 \leqslant 6+2+1_{D(210)}\} \\ &= \{x_0 \in D(210) \mid 2 \leqslant x_0 \leqslant 70\} \\ &= \{2, 10, 14, 70\}; \end{aligned}$$

(2) Let $6 \cdot x + 2 \leqslant 3$ *be an inequality with the form of* (3.2) *such that* $a = 6$*,* $b = 2$ *and* $c = 3$*. We have*

$$b \cdot c' = 2 \wedge 3' = 2 \wedge 70 = 2$$

and

$$a \cdot c' = 6 \wedge 3' = 6 \wedge 70 = 2.$$

Therefore, $b \cdot c' \leqslant a \cdot c'$ *(i.e.,* $b \cdot c' \mid a \cdot c'$*). Hence this inequality* $6x + 2 \leqslant 3$ *has solutions in* $D(210)$ *given by*

$$\begin{aligned} S &= \{x_0 \in D(210) \mid b \cdot c' \leqslant x_0 \leqslant a \cdot c' + b \cdot c' + 1_{D(210)}\} \\ &= \{x_0 \in D(210) \mid 2 \leqslant x_0 \leqslant 2+2+1_{D(210)}\} \\ &= \{x_0 \in D(210) \mid 2 \leqslant x_0 \leqslant 1_{D(210)}\} \\ &= \{2, 6, 10, 14, 30, 42, 70, 210\}. \end{aligned}$$

Proposition 3.4. *[9] In a Boolean lattice* B *we consider the system of linear equations* (3.3) *with the form:*

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases} \tag{3.3}$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are fixed elements of B. We take $\Delta = a_1b_2 - a_2b_1 = a_1b_2 + a_2b_1$ the determinant of the associated matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ of this system (3.3). If $\Delta = 1$, then the system (3.3) has a unique solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_2c_1 + b_1c_2 \\ a_2c_1 + a_1c_2 \end{pmatrix}.$$

Proof. Assume that, $\Delta = 1$. Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\Delta} \cdot \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} = \frac{1}{\Delta} \cdot \begin{pmatrix} b_2 & b_1 \\ a_2 & a_1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\Delta} \cdot \begin{pmatrix} b_2c_1 + b_1c_2 \\ a_2c_1 + a_1c_2 \end{pmatrix}.$$

The fact that $\Delta = 1$ implies that $\frac{1}{\Delta} = 1$. Therefore, the unique solution of this system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_2c_1 + b_1c_2 \\ a_2c_1 + a_1c_2 \end{pmatrix}.$$

□

Example 3.6. *We define on the Boolean lattice $D(210)$ the following system*

$$\begin{cases} 2x + 105y = 2 \\ 105x + 6y = 2 \end{cases}.$$

The fact that $\Delta = 2 \cdot 6 + 105 \cdot 105 = 2 + 105 = 1_{D(210)}$, then the unique solution of this system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \cdot 2 + 105 \cdot 2 \\ 105 \cdot 2 + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Remark 3.2. *On a finite Boolean lattice, if the determinant $\Delta \neq 1$, then the system (3.3) has not a solution.*

Conclusion

In this thesis, we have presented the notion of Boolean lattices and their properties. To that end, we have recalled the necessary concepts and properties of binary relations, partially orders and ordered sets. Moreover, we have provided the necessary concepts and properties of lattices, ideals, filters, sub-lattices and lattice morphisms. Further, we have focused our attention on the algebraic structure of Boolean lattices and their characterizations. Finally, we have shown some methods to solve linear equations and inequalities on a Boolean lattice.

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ملخص

في هذه المذكرة نقدم مفهوم المشابك البولية وخصائصها، حيث نعرض المفاهيم والخصائص الضرورية للعلاقات الثنائية، الترتيبات الجزئية والمجموعات المرتبة. علاوة على ذلك، نقدم المفاهيم والخصائص الضرورية للشبكات المرتبة، حيث نركز اهتمامنا على البنية الجبرية للشبكات البولية وتجسيدها. في الأخير، نعطي بعض الطرق لحل المعادلات الخطية والمتراجحات على شبكة بولية.

كلمات مفتاحية:

ترتيب جزئي، مجموعة مرتبة، شبكة، شبكة بولية.

Abstract

In this thesis, we present the notion of Boolean lattices and their properties. To that end, we recall the necessary concepts and properties of binary relations, partially orders and ordered sets. Moreover, we present the necessary concepts and properties of lattices, ideals, filters, sub-lattices and lattice morphisms. Further, we focus our attention on the algebraic structure of Boolean lattices and their characterizations. Finally, we give some methods to solve linear equations and inequalities on a Boolean lattice.

Key words :

Ordered set, lattices, ideals, filters, sub-lattice, lattice morphism, Boolean lattice.

Résumé

Dans ce mémoire, nous présentons la notion de treillis booléens et leurs propriétés. Pour cela, nous rappelons les concepts et les propriétés nécessaires des relations binaires, ordres partiels et ensembles ordonnés. De plus, nous présentons les concepts et les propriétés nécessaires des treillis, idéaux, filtres, sous-treillis et morphismes de treillis. Nous concentrons notre attention sur la structure algébrique des treillis booléens et leurs caractérisations. Enfin, nous donnons quelques méthodes pour résoudre des équations linéaires et des inégalités dans un treillis booléen.

Mots-clés :

Ensemble ordonné, treillis, idéal, filtre, sous-treillis, morphisme de treillis, treillis booléen.