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Numerical Methods for Solving Linear Integro-Differential Equations

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Méthodes numériques pour la résolution des équations intégro-différentielles linéaires

Soutenue le 25/02/2025 devant le jury composé de :

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Dedication

To my mother and my older brother Youcef.

To my husband Merzougui Amar.

To my children Mariya Lina, Sirine, Meriam Amina, and Youcef Abdelrahman for having accepted so many sacrifices over the past few years.

To my brothers Elyezid, Oussama, Ilyas, and their wives.

To my best friend Amina for her encouragement.

I dedicate this work.

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Notations

LFIDEs	Linear Fredholm integro-differential equations
LVIDEs	Linear Volterra integro-differential equations
HP.M	Homotopy perturbation method
ES.M	Exponential spline method
SB.M	Schauder bases method
DT.M	Differential transformation method
MVI.M	Modified variational iteration method
AD.M	Adomian decomposition method
MD.M	Modified decomposition method
HermiteW.M	Hermite wavelet method
Haar W.B.M	Haar wavelet bases method
CAS-W.M	CAS wavelet method
TSBPF.M	Hybrid of Taylor series and Block pulse functions
B	Banach space
$L^2(\Omega)$	The set of square integrable functions on Ω
$\langle , \rangle$	Scalar product
$\|\tau\|$	Norm of τ
$C[0,1]$	Set of continues functions on the interval $[a,b]$
$N(\tau,\zeta)$	Kernel of the integro-differential equation
$D_n(\tau)$	Genocchi polynomials
$E_n(\tau)$	Euler polynomials
$B_n(\tau)$	Bernoulli polynomials
$\phi(\tau)$	Exact solution of IDE
$\phi_N(\tau)$	Approximate solution of $\phi(\tau)$
$h(\tau)$	Free term in the IDE

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Introduction

Linear integro-differential equations (LIDEs) play a crucial role in modeling various physical and engineering systems, encompassing fields such as biology and physics, Their applications extend to electric circuit analysis, control theory, signal processing, and neural networks [1]. They are also essential in describing physical phenomena, such as heat transfer, wind ripple formation in deserts, nanohydrodynamics, the glass-forming process, and elasticity theory in physics [2].

LIDEs are significant in formulating optimal control problems, fluid dynamics, boundary value problems in gravitation theory, and electrostatics $[3-6]$.

These types of equations were first introduced by Volterra in the early 1900s and can take various forms. A common form involves the derivative of the unknown function appearing outside the integral. The combination of both differential and integral operators in these equations often makes their analytical solutions challenging or sometimes impossible to obtain, so numerical methods are required.

Recently, there has been growing interest in developing numerical methods to solve LIDEs. Prominent methods include the B-spline method [7], Adomian decomposition method [8], modified Adomian decomposition method [9], Hermite wavelet method [10], CAS wavelet method [11], Homotopy perturbation method [12], modified variational iteration method [13], differential transformation method[14], exponential spline method [15], Haar wavelet bases method [16], Tau method [17], Schauder bases method [18], Legendre Galerkin method [19], Bernoulli matrix method [20], Taylor collocation method [21], Bessel collocation method [22], Euler matrix method [23], and shifted Chebyshev polynomials [24].

In this thesis, we aim to find approximate solutions to high-order linear integro-differential equations using various numerical methods, with a particular focus on collocation methods. Notably, we applied Genocchi polynomials to high-order LFIDEs for the first time [25], building on their successful use in solving fractional calculus problems in 2016 [26], generalized fractional pantograph equations [27], fractional diffusion-wave equations and fractional Klein–Gordon equation [28], and fractional partial differential equations [29].

The general form of the equation to be solved is given by

$$\sum_{k=0}^{m} Q_k(\tau)\phi^{(k)}(\tau) - \lambda \int_{\alpha(\tau)}^{\beta(\tau)} N(\tau,\zeta)\phi(\zeta)d\zeta = h(\tau),\ 0 \leqslant a \leqslant \tau,\zeta \leqslant b,$$

subject to the initial boundary conditions

$$\sum_{k=0}^{m-1} (\theta_{jk}\phi^{(k)}(a) + \delta_{jk}\phi^{(k)}(b)) = \eta_j\ ,\quad j = 0\ ,\ 1,\ ...,\ m-1,$$

where $\alpha(\tau)$ and $\beta(\tau)$ are the limits of integration, $\phi(\tau)$ is the unknown function to be determined, while the kernel $N(\tau,\zeta)$ and the free term h are provided.

This thesis is divided into four well-organized chapters, as outlined below:

The first chapter: We begin by reviewing some fundamental concepts of functional and numerical analysis that are essential for this work, including the definitions of normed space, Banach space, and Hilbert space. We also explore Appell polynomials, with a specific focus on Genocchi polynomials.

The second chapter: We present the concept of integro-differential equations by outlining their forms and classifications. We also discuss their applications in various fields, such as physics, biology, and engineering. Finally, we review some analytical methods for solving these equations.

The third chapter: In this chapter, some numerical methods for solving integro-differential equations are presented, including the variational iteration method, the Adomian and modified decomposition method, and the Taylor collocation method. Additionally, We present, for the first time, the application of Genocchi polynomials in solving high-order linear Fredholm integro-differential equations (LFIDEs) [25]. This approach transforms the LFIDE into a matrix equation, which is then solved to determine the unknown Genocchi coefficients.

The fourth chapter: To demonstrate the effectiveness and efficiency of current method described in the previous chapter, we solved several different examples using the Matlab program. The results are displayed in tables and figures, which include exact solutions, approximate solutions, and absolut errors. Furthermore, we have compared these results with some existing methods.

We conclude our thesis with a general conclusion and perspectives.

Chapter 1

Preliminaries and concepts of polynomials

In this introductory chapter, we review some fundamental concepts of functional and numerical analysis that are essential for this work, including the definition of normed space, Banach space, and Hilbert space. We also explore Appell polynomials, with a specific focus on Genocchi polynomials.

1.1 Concepts of functional analysis

We begin by reviewing some fundamental concepts of functional analysis that are essential for this work.

Definition 1.1.1 *Let X be a real vector space. A mapping $\|.\| : X \to \mathbb{R}$ is called a norm on X if it satisfies the following conditions for all vectors τ, $\zeta \in X$ and scalar $\alpha \in \mathbb{R}$*

(i) $\|\tau\| = 0 \Longleftrightarrow \tau = 0$,

(ii) $\|\alpha\tau\| = |\alpha| \, \|\tau\|$,

(iii) $\|\tau + \zeta\| \leq \|\tau\| + \|\zeta\|$,

the pair $(X, \|.\|)$ is referred to as a normed space.

Definition 1.1.2 *A normed space* $(X, \|.\|)$ *is said to be complete if every Cauchy sequence in* X *converges within* X.

Definition 1.1.3 *A complete normed space is also known as a Banach space.*

Definition 1.1.4 *Let* $(X, \|.\|)$ *be a normed space. A mapping* $\phi : X \to \mathbb{R}$ *is called linear if it satisfies the following condition for all vectors* $\tau,\ \zeta \in X$ *and scalars* $\alpha_1,\ \alpha_2 \in \mathbb{R}$

$$\phi(\alpha_1\tau + \alpha_2\zeta) = \alpha_1\phi(\tau) + \alpha_2\phi(\zeta).$$

A linear mapping $\phi : X \to \mathbb{R}$ *is continuous if there exists a constant* C *satisfying*

$$|\phi(\tau)| \leq C \|\tau\|_X .$$

Let Ω be an open and bounded subset of $\mathbb{R}^n$ with $n \in \mathbb{N}$.

Definition 1.1.5 *For a function* $\phi : \Omega \to \mathbb{R}$ *and* $p \in [1, \infty[$ *we define the norm*

$$\|\phi(\tau)\|_{L^p(\Omega)} = \left(\int_\Omega |\phi(\tau)|^p \, d\Omega \right)^{\frac{1}{p}} .$$

Lemma 1.1.1 *(Minkowski's inequality)* [30] *For* $\phi_1, \phi_2 \in L^p(\Omega)$ *and* $p \in [1, \infty]$, *we have*

$$\|\phi_1 + \phi_2\|_{L^p(\Omega)} \leq \|\phi_1\|_{L^p(\Omega)} + \|\phi_2\|_{L^p(\Omega)} .$$

Theorem 1.1.1 [30] *The space* $L^p(\Omega)$, $p \in [1, \infty]$ *is a Banach space.*

Definition 1.1.6 *Let* X *be a vector space. A mapping* $\langle ., . \rangle : X \times X \to \mathbb{R}$ *is called an inner product on* X *if it satisfies the following conditions for all vectors* $\phi_1,\ \phi_2,\ \phi_3 \in X$ *and scalars* $\alpha_1,\ \alpha_2 \in \mathbb{R}$

(i) $\langle \phi_1, \phi_2 \rangle = \langle \phi_2, \phi_1 \rangle$,

(ii) $\langle \phi_1, \phi_1 \rangle > 0$, *for all* $\phi_1 \neq 0$,

(iii) $\langle \alpha_1\phi_1 + \alpha_2\phi_2, \phi_3 \rangle = \alpha_1 \langle \phi_1, \phi_3 \rangle + \alpha_2 \langle \phi_2, \phi_3 \rangle$.

Definition 1.1.7 *A norm on X is defined as $\|\phi\| = \sqrt{\langle \phi, \phi \rangle}$. If the space $(X, \|.\|)$ is complete, it is called a Hilbert space.*

Theorem 1.1.2 [30] *The space $L^2(\Omega)$ is a Hilbert space equipped with the inner product*

$$\langle \phi_1, \phi_2 \rangle_{L^2(\Omega)} = \int_\Omega \phi_1(\tau)\phi_2(\tau)d\tau.$$

Lemma 1.1.2 *(Hölder's inequality for L^p spaces)* [30] *For $\phi_1 \in L^q(\Omega)$ and $\phi_2 \in L^p(\Omega)$ with $p, q \in [1, \infty]$, and*
$\frac{1}{q} + \frac{1}{p} = 1$, we have $\phi_1\phi_2 \in L^1(\Omega)$ and

$$\|\phi_1\phi_2\|_{L^1(\Omega)} \leq \|\phi_1\|_{L^q(\Omega)} \|\phi_2\|_{L^p(\Omega)}.$$

1.2 Appell polynomials

Appel polynomials are a special sequence of polynomials defined by a generating function approach. A sequence of polynomials $\{P_n(\tau)\}_{n=0}^{\infty}$ is called an Appel sequence if it satisfies the following two key properties:

1- Generating function: There exists a generating function of the form

$$f(\zeta)e^{\tau\zeta} = \sum_{n=0}^{\infty} P_n(\tau)\frac{\zeta^n}{n!},$$

where $f(\zeta)$ is an analytic function. The function $f(\zeta)$ uniquely determines the Appell sequence $\{P_n(\tau)\}$.

2- Differentiation property: Appel polynomials satisfy the differentiation property

$$\frac{d}{d\tau}P_n(\tau) = nP_{n-1}(\tau), \quad n \geq 1,$$

with $P_0(\tau)$ being a constant.

Examples of Appell polynomials:

- Bernoulli polynomials: Generated by the function $\frac{\zeta e^{\zeta\tau}}{e^\zeta - 1}$.
- Euler polynomials: Generated by the function $\frac{2e^{\zeta\tau}}{e^\zeta + 1}$.
- Hermite polynomials: Generated by the function $e^{2\zeta\tau - \zeta^2}$.
- Genocchi polynomials: Also a member of the Appell family, with their own specific properties and generating function.

1.2.1 Bernoulli polynomials

Definition 1.2.1 *The Bernoulli polynomials $B_n(\tau)$ are defined by the generating function*

$$\Omega(\zeta,\tau)=\frac{\zeta e^{\tau\zeta}}{e^{\zeta}-1}=\sum_{n=0}^{\infty}B_n(\tau)\frac{\zeta^n}{n!},$$

where $B_n(\tau)$ denotes the Bernoulli polynomials of order n. Additionally, the Bernoulli polynomials can be expressed as follows

$$B_n(\tau)=\sum_{m=0}^{n}\binom{n}{m}B_{n-m}\tau^m,$$

where $B_n=B_n(0)$ are the Bernoulli numbers.

We present the first few Bernoulli polynomials, which are as follows

$B_0(\tau)=1.$

$B_1(\tau)=\tau-\frac{1}{2}.$

$B_2(\tau)=\tau^2-\tau+\frac{1}{6}.$

$B_3(\tau)=\tau^3-\frac{3}{2}\tau^2+\frac{1}{2}\tau.$

$B_4(\tau)=\tau^4-2\tau^3+\tau^2-\frac{1}{30}.$

$B_5(\tau)=\tau^5-\frac{5}{2}\tau^4+\frac{5}{3}\tau^3-\frac{1}{6}\tau.$

$B_6(\tau)=\tau^6-3\tau^5+\frac{5}{2}\tau^4-\frac{1}{2}\tau^2+\frac{1}{42}.$

Derivatives and integrals of Bernoulli polynomials

Bernoulli polynomials $B_n(\tau)$ possess elegant and useful properties related to their derivatives and integrals

- $\dfrac{dB_n(\tau)}{d\tau}=nB_{n-1}(\tau), \quad n\geq 1,$
- $\displaystyle\int_a^{\tau}B_n(\zeta)d\zeta=\frac{1}{n+1}(B_{n+1}(\tau)-B_{n+1}(a)).$

1.2.2 Euler polynomials

Definition 1.2.2 *Euler polynomials $E_n(\tau)$ are defined by the generating function*

$$\Omega(\zeta,\tau)=\frac{2e^{\tau\zeta}}{e^{\zeta}+1}=\sum_{n=0}^{\infty}E_n(\tau)\frac{\zeta^n}{n!},$$

where $E_n(\tau)$ denotes the Euler polynomials of order n. Additionally, the Euler polynomials can be expressed as follows

$$E_n(\tau) = \tau^n - \frac{1}{2}\sum_{m=0}^{n-1}\binom{n}{m}E_m(\tau).$$

We present the first few Euler polynomials, which are as follows

$E_0(\tau) = 1.$
$E_1(\tau) = \tau - \frac{1}{2}.$
$E_2(\tau) = \tau^2 - \tau.$
$E_3(\tau) = \tau^3 - \frac{3}{2}\tau^2 + \frac{1}{4}.$
$E_4(\tau) = \tau^4 - 2\tau^3 + \tau.$

1.2.3 Genocchi polynomials

During 1817–1889, Italian mathematician Angelo Genocchi first introduced Genocchi numbers and polynomials. Genocchi numbers have been widely studied across various fields of mathematics.

Definition 1.2.3 *The Genocchi numbers D_n can be defined via the generating function*

$$\Omega(\zeta) = \sum_{n=1}^{\infty} D_n \frac{\zeta^n}{n!} = \frac{2\zeta}{e^\zeta + 1}, \qquad (|\zeta| < \pi).$$

These numbers are integers and are related to the Bernoulli numbers through the following formula

$$D_n = 2(1 - 2^n)B_n.$$

The initial Genocchi numbers are

D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{14}	D_{16}
1	-1	0	1	0	-3	0	17	0	-155	0	2073	-38227	929569

We must also note that $D_{2n+1} = 0$, for $n = 1, 2, 3, ...$, and the signs of D_n alternate for even values of n.

In general, it holds that $D_3 = D_5 = ... = 0$, while the even coefficients are given by

$$D_{2n} = 2(1 - 2^{2n})B_{2n} = 2nE_{2n-1},$$

where B_n and E_n represent the well-known Bernoulli and Euler numbers, respectively.

Definition 1.2.4 *The Genocchi polynomials $D_n(\tau)$ can be defined via the generating function*

$$\Omega(\zeta,\tau)=\frac{2\zeta e^{\tau\zeta}}{e^{\zeta}+1}=\sum_{n=0}^{\infty}D_n(\tau)\frac{\zeta^n}{n!},\qquad (|\zeta|<\pi),$$

where $D_n(\tau)$ denotes the Genocchi polynomials of order n. Additionally, the Genocchi polynomials can be expressed as follows

$$D_n(\tau)=\sum_{m=1}^{n}\binom{n}{m-1}D_{n-m+1}\tau^{m-1},\ n=1,\ 2,\ \ldots,\ N$$

or

$$D_n(\tau)=n\tau^{n-1}-\frac{1}{2}\sum_{m=0}^{n-1}\binom{n}{m}D_m(\tau),\ D_0(\tau)=0.$$

We present the first few Genocchi polynomials, which are as follows

$D_1(\tau)=1.$

$D_2(\tau)=2\tau-1.$

$D_3(\tau)=3\tau^2-3\tau.$

$D_4(\tau)=4\tau^3-6\tau^2+1.$

$D_5(\tau)=5\tau^4-10\tau^3+5\tau.$

$D_6(\tau)=6\tau^5-15\tau^4+15\tau^2-3.$

In the special case when $\tau=0$, the Genocchi polynomials yield the Genocchi numbers: $D_n(0)=D_n$.

Some of the essential basic properties of the Genocchi polynomials are as follows

1. $D_n(1)+D_n(0)=0,\ n>1,$
2. $\dfrac{dD_n(\tau)}{d\tau}=nD_{n-1}(\tau),\ n\geq 1,$
3. $\displaystyle\int_0^1 D_j(\tau)D_k(\tau)d\tau=\frac{2(-1)^j j!k!}{(j+k)!}D_{j+k},\quad j,k\geq 1,$
4. $\displaystyle\int_a^b D_n(\tau)d\tau=\frac{1}{n+1}(D_{n+1}(b)-D_{n+1}(a)).$

This polynomial was chosen to find an approximate solution to the high-order linear Fredholm integro-differential equation in Chapter 3 due to its advantages, including:

- The coefficients of each term in Genocchi polynomials are integers, which eliminates the possibility of computational errors. In contrast, most polynomials, such as Euler and Bernoulli polynomials, have non-integer coefficients.

- Genocchi polynomials have fewer terms compared to other polynomials. For instance, $D_6(\tau)$ contains 4 terms, whereas $B_6(\tau)$ (Bernoulli polynomials) has 5 terms, and both shifted Chebyshev polynomials $T_6(\tau)$ and shifted Legendre polynomials $L_6(\tau)$ have 7 terms. As a result, when approximating arbitrary functions, Genocchi polynomials require less CPU time compared to Bernoulli, shifted Chebyshev, and shifted Legendre polynomials.

For additional information, please refer to references [31] and [32], which provide an in-depth discussion of the Genocchi polynomials.

Chapter 2

Introduction to integro-differential equations

The aim of this chapter is to familiarize the reader of this thesis with the concept of integro-differential equations. We will explore their forms and classifications, as well as discuss their applications across various fields, including physics, biology, and engineering.

Finally, we will review some analytical methods for solving these types of equations.

2.1 Definition and classification of integro-differential equations

In the early 1900s, Vito Volterra studied population growth, leading to the development of a new type of equation, which was termed integro-differential equations.

An integro-differential equation is a type of equation that involves both integrals and derivatives of an unknown function, typically depending on one or more variables. In this type of equation, the unknown function $\phi(\tau)$ appears as an ordinary derivative on one side and under the integral sign on the other. Integro-differential equations commonly appear in fields such as physics, engineering, biology, and economics.

The general structure of an integro-differential equation is

$$F(\tau, \phi(\tau), \phi^{'}(\tau), \phi^{''}(\tau), ..., \phi^{(n)}(\tau), \lambda \int_{\alpha(\tau)}^{\beta(\tau)} N(\tau, \zeta, \phi(\zeta), \phi^{'}(\zeta), ..., \phi^{(n-1)}(\zeta)) d\zeta) = 0,$$

where $\phi(\tau)$ is the unknown function, $\phi^{(n)}(\tau)$ denotes the $n-th$ derivative, $N(\tau, \zeta, \phi(\zeta))$ is the kernel function, and $\alpha(\tau)$, $\beta(\tau)$ are the limits of integration.

Integro-differential equations (IDEs) can appear in various forms, but one of the most commonly encountered types is when the derivative of the unknown function appears outside the integral sign.

Our primary focus will be on linear IDEs, with the most well-known form being

$$\sum_{k=0}^{m} Q_k(\tau)\phi^{(k)}(\tau) - \lambda \int_{\alpha(\tau)}^{\beta(\tau)} N(\tau, \zeta)\phi(\zeta)d\zeta = h(\tau). \tag{2.1}$$

Where $\phi(\tau)$ is the unknown function to be solved for, while the kernel $N(\tau, \zeta)$ and the free term $h(\tau)$ are provided. These equations are subject to the initial boundary conditions

$$\sum_{k=0}^{m-1} (\theta_{jk}\phi^{(k)}(a) + \delta_{jk}\phi^{(k)}(b)) = \eta_j \ , \quad j = 0, 1, ..., m-1. \tag{2.2}$$

2.1.1 Classification and terminology

Integro-differential equations can be classified based on several criteria, including the types of differential and integral operators involved, the order of derivatives, etc.

Here are the main classifications:

1. Based on the order of the differential equation

First-order integro-differential equations: Involve first derivatives of the unknown function.

Higher-order integro-differential equations: Involve second or higher derivatives of the unknown function.

2. Based on the type of integral operator

Integro-differential equations can include different types of integrals:

- Volterra integro-differential equations (VIDEs): The integral bounds depend on the variable of differentiation, usually with a fixed lower limit and a variable upper limit. These are common in systems with memory, such as viscoelastic materials.

- Fredholm integro-differential equations (FIDEs): The integral bounds are fixed. These are often used in boundary value problems and mathematical physics.

- Volterra-Fredholm integro-differential equations (VFIDEs): These equations include both Fredholm and Volterra integral operators.

3. Based on linearity

If the exponent of the unknown function $\phi(\tau)$ within the integral is one, the integro-differential equation is classified as linear. However, if the exponent of $\phi(\tau)$ is not one, or if the equation includes nonlinear functions of $\phi(\tau)$, such as $\ln(1+\phi)$, e^{ϕ}, or $\cos(\phi)$, the equation is considered nonlinear.

4. Based on the type of differential equation

- Ordinary integro-differential equations (OIDEs): Involve derivatives with respect to a single variable.

- Partial integro-differential equations (PIDEs): Involve partial derivatives with respect to more than one variable, typically appearing in multi-dimensional problems like fluid dynamics or financial modeling.

5. Based on singularities

An integro-differential equation (IDE) is deemed singular if one or both of the following conditions are satisfied:

- One or both limits of integration extend to infinity.

- Singular kernel: The kernel $N(\tau,\zeta)$ becomes unbounded or undefined at certain points within the integration domain. For instance, if $N(\tau,\zeta)=\dfrac{1}{(\tau-\zeta)^2}$, it is singular at $\tau=\zeta$, which can lead to divergence in the integral.

2.1.2 Examples of integro-differential equations

1- Third-order linear Volterra integro-differential equation

$$\phi^{'''}(\tau)=-1+\tau-\int_0^{\tau}(\tau-\zeta)\phi(\zeta)d\zeta,\ \phi_0=1,\ \phi_0^{'}=-1,\ \phi_0^{''}=1.$$

2- Nonlinear Fredholm integro-differential equation

$$\phi^{'''}(\tau)=-e^{\tau}+\int_{-1}^{1}e^{(\tau-2\zeta)}\phi^2(\zeta)d\zeta,\ \phi(0)=\ \phi^{'}(0)=1,\ \phi(1)=e.$$

3- Second-order linear Volterra-Fredholm integro-differential equation

$$\phi^{''}(\tau)=2\cos(\tau)-1+\int_0^{\tau}\phi(\zeta)d\zeta+\int_0^{\frac{\pi}{2}}\phi(\zeta)d\zeta,\ \phi(0)=1,\ \phi^{'}(0)=1.$$

Another form

$$\phi^{''}(\tau) = -\tau^2 - \sin\tau - \cos\tau + \int_0^{\tau}\int_0^{\pi} s\phi(\zeta)d\zeta ds,\ \phi(0) = 1,\ \phi^{'}(0) = 1.$$

4- Singular integro-differential equation

$$\sum_{m=0}^{n} a_m \phi^{(m)}(\tau) = \frac{1}{\pi}\int_0^{+\infty} \frac{\varphi(\zeta)}{\zeta - \tau} d\zeta, \quad (0 \leq \tau < \infty),$$

with

$$\varphi(\tau) = \sum_{m=0}^{n} b_m \phi^{(m)}(\tau).$$

2.2 Genesis of integro-differential equations

Integro-differential equations are essential tools for modeling a wide range of phenomena across different disciplines. Below are some examples:

1. Biology: The Volterra population model (VPM)

The Volterra model describing the population growth of a species in a closed system is given by

$$\frac{d}{d\tau}P(\tau) = \alpha P(\tau) - \beta P^{2}(\tau) - \gamma P(\tau)\int_0^{\tau} P(\zeta)d\zeta,\ P(0) = P_0.$$

In this model, $P(\tau)$ represents the population at time τ, with α, β, and γ being positive constants. Here, α is the birth rate coefficient, γ is the toxicity coefficient, and β is the crowding coefficient, while P_0 denotes the initial population. The coefficient γ reflects the critical behavior of population dynamics until it eventually declines to zero over time.

When $\beta = 0$ and $\gamma = 0$, the previous equation reduces to the Malthusian differential equation

$$\frac{d}{d\tau}P(\tau) = \alpha P(\tau).$$

2. Physics: Viscoelastic materials

In viscoelastic materials, the stress depends not only on the current strain but also on the history of strain, making integro-differential equations essential for modeling. For example, the relationship between stress $S(\tau)$ and strain $\phi(\tau)$ in a viscoelastic material is described by the integro-differential equation

$$S(\tau) = E\phi(\tau) + \alpha\frac{d}{d\tau}\phi(\tau) + \int_{0}^{\tau}\beta(\tau-\zeta)\phi(\zeta)d\zeta,$$

where

- E is the elastic modulus.

- α is the viscosity.

- β is a relaxation function describing the memory effect.

This equation accounts for both the instantaneous elastic response and the time-dependent viscous response of the material.

3. Electrical engineering: The LRC series circuit

Resistance (R), capacitance (C), and inductance (I), are fundamental components of basic electrical circuits. When a voltage (the input) is applied to a circuit with these elements, it results in a current flow (the output) and a change in the capacitor's charge. The current $L(\tau)$, flows through the circuit in response to the applied voltage $V(\tau)$, both of which are time-dependent variables. However, this doesn't rule out the possibility of a constant voltage (e.g. the voltage source is a battery).

The voltage drop across the resistor is RL (according to Ohm's Law), across the inductor it is $I\frac{d}{d\tau}L(\tau)$, and across the capacitor it is $\frac{1}{C}\int_0^\tau L(\zeta)d\zeta$. According to Kirchhoff's 2 nd Law, the sum of the voltage drops across all the non-supply elements in the circuit must equal the applied voltage from the power source. Thus, an RLC circuit follows

$$I\frac{d}{d\tau}L(\tau) + RL(\tau) + \frac{1}{C}\int_0^\tau L(\zeta)d\zeta = V(\tau),$$

and this is an integro-differential equation.

4. Problems in heat conduction and diffusion

Consider a scenario in nuclear reactor dynamics where the complex relationship between reactor temperature, denoted $T(\zeta,\tau)$, and generated power $\phi(\tau)$ is represented by a system of partial integro-differential equations

$$\begin{cases} \dfrac{d}{d\tau}\phi(\tau) = \displaystyle\int_{-\infty}^{+\infty} \varphi(\zeta)T(\zeta,\tau)d\zeta, \\ \dfrac{\partial}{\partial\tau}T(\zeta,\tau) = \dfrac{\partial^2}{\partial\zeta^2}T(\zeta,\tau) + n(\zeta)\phi(\tau), \quad -\infty < \zeta < \infty, \ \tau > 0, \end{cases}$$

subject to the following conditions

$$\phi(0) = 0, T(\zeta,0) = h_0(\zeta),$$
$$\lim_{\zeta\to\pm\infty} T(\zeta,\tau) = \lim_{\zeta\to\pm\infty} \frac{\partial}{\partial\zeta}T(\zeta,\tau) = 0.$$

In this system, the first equation describes power production as a function of temperature, while the second equation is essentially a diffusion equation with an added source term that arises from the power generated by the reactor.

5. Initial value problem of the third order

Assume ϕ satisfies

$$\begin{cases} \phi^{'''}(\tau) = u(\tau,\phi(\tau)), \quad 0 < \tau < 1, \\ \phi(0) = \phi_0, \ \phi^{'}(0) = \phi_1 \ , \phi^{''}(0) = \phi_2. \end{cases}$$

A first integration yields

$$\phi^{''}(\tau) = \int_0^\tau u(\zeta,\phi(\zeta))d\zeta + \phi_2, \quad 0 < \tau < 1.$$

A second integration yields

$$\phi^{'}(\tau) = \int_0^\tau \left(\int_0^s u(\zeta,\phi(\zeta))d\zeta \right) ds + \phi_2\tau + \phi_1, \quad 0 < \tau < 1.$$

Assuming that u is a continuous function of the two variables, the previous relation becomes

$$\int_0^\tau \int_0^s u(\zeta,\phi(\zeta))d\zeta ds = \int_0^\tau (\tau-\zeta)u(\zeta,\phi(\zeta))d\zeta.$$

Therefore, the integro-differential equation equivalent to the initial third-order differential equation is

$$\phi^{'}(\tau) = \int_0^\tau (\tau-\zeta)u(\zeta,\phi(\zeta))d\zeta + \phi_2\tau + \phi_1, \quad 0 < \tau < 1.$$

This equation is a first-order Volterra integro-differential equation.

2.3 Converting Volterra integro-differential equation to Volterra integral equation

It is clear that the Volterra integro-differential equation contains derivatives on the left-hand side and an integral on the right-hand side. To convert it into a standard Volterra integral equation, we need to integrate both sides n times. In support of this conversion process, it is helpful to review some relevant formulas.

1. Integration of derivatives: From calculus, we observe the following relationships

$$\int_0^\tau \phi^{'}(\zeta)d\zeta = \phi(\tau) - \phi(0),$$

$$\int_0^\tau \int_0^{\tau_1} \phi^{''}(\zeta)d\zeta d\tau_1 = \phi(\tau) - \tau\phi^{'}(0) - \phi(0),$$

$$\int_0^\tau \int_0^{\tau_1} \int_0^{\tau_2} \phi^{'''}(\zeta)d\zeta d\tau_2 d\tau_1 = \phi(\tau) - \frac{1}{2!}\tau^2\phi^{''}(0) - \tau\phi^{'}(0) - \phi(0),$$

and similarly for the derivative of order n.

2. Reducing several integrals into one integral according to the following relations

$$\int_0^\tau \int_0^{\tau_1} \phi(\zeta)d\zeta d\tau_1 = \int_0^\tau (\tau - \zeta)\phi(\zeta)d\zeta,$$

$$\int_0^\tau \int_0^{\tau_1} (\tau - \zeta)\phi(\zeta)d\zeta d\tau_1 = \frac{1}{2}\int_0^\tau (\tau - \zeta)^2\phi(\zeta)d\zeta,$$

$$\int_0^\tau \int_0^{\tau_1} (\tau - \zeta)^2\phi(\zeta)d\zeta d\tau_1 = \frac{1}{3}\int_0^\tau (\tau - \zeta)^3\phi(\zeta)d\zeta,$$

$$\int_0^\tau \int_0^{\tau_1} (\tau - \zeta)^3\phi(\zeta)d\zeta d\tau_1 = \frac{1}{4}\int_0^\tau (\tau - \zeta)^4\phi(\zeta)d\zeta,$$

and so forth. This can be generalized in the following form

$$\int_0^\tau \int_0^{\tau_1} \int_0^{\tau_2} ... \int_0^{\tau_{n-1}} (\tau - \zeta)\phi(\zeta)d\zeta d\tau_{n-1}...d\tau_1 = \frac{1}{n!}\int_0^\tau (\tau - \zeta)^n\phi(\zeta)d\zeta.$$

The following example will demonstrate the conversion to an equivalent Volterra integral equation.

Example 2.3.1 *Consider the first order VIDE*

$$\phi^{'}(\tau) = 1 + \tau - \tau^2 + \int_0^\tau (\tau - \zeta)\phi(\zeta)d\zeta, \ \phi(0) = 3,$$

by integrating both sides once from 0 *to* τ *and applying the formulas provided above, we obtain*

$$\phi(\tau) - \phi(0) = \tau + \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3 + \frac{1}{2!}\int_0^\tau (\tau - \zeta)^2\phi(\zeta)d\zeta,$$
$$\phi(\tau) = 3 + \tau + \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3 + \frac{1}{2!}\int_0^\tau (\tau - \zeta)^2\phi(\zeta)d\zeta.$$

2.4 Converting Fredholm integro-differential equation to Fredholm integral equation

The Fredholm integro-differential equation can be easily converted into the Fredholm integral equation by integrating both sides of the integro-differential equation n times from 0 to τ (or from a to τ if the limits are defined differently), while also applying the given initial conditions with each integration.

Example 2.4.1 *Consider the second order FIDE*

$$\phi^{''}(\tau) = e^{\tau} - \tau + \tau\int_0^1 \zeta\phi(\zeta)d\zeta, \ \phi(0) = 1, \ \phi^{'}(0) = 1.$$

Integrating both sides of this equation twice from 0 *to* τ *and applying the initial conditions yields*

$$\phi(\tau) = e^{\tau} - \frac{1}{6}\tau^3 + \frac{1}{6}\tau^3\int_0^1 \zeta\phi(\zeta)d\zeta.$$

2.5 Analytical resolution of integro-differential equations

Solving integro-differential equations analytically can be challenging, but there are several methods designed for finding exact solutions. Here are some common analytical methods used for solving integro-differential equations.

2.5.1 The direct computation method

Consider a Fredholm integro-differential equation given by

$$\phi^{(n)}(\tau) = h(\tau) + \int_0^1 N(\tau,\zeta)\phi(\zeta)d\zeta,\ \phi^{(k)}(0) = \alpha_k,\quad 0 \leq k \leq n-1, \tag{2.3}$$

withe

$$N(\tau,\zeta) = \varphi(\tau)\psi(\zeta), \tag{2.4}$$

by substituting (2.4) into (2.3), we obtain

$$\phi^{(n)}(\tau) = h(\tau) + \varphi(\tau)\int_0^1 \psi(\zeta)\phi(\zeta)d\zeta,\quad \phi^{(k)}(0) = \alpha_k,\quad 0 \leq k \leq n-1, \tag{2.5}$$

we set

$$\alpha = \int_0^1 \psi(\zeta)\phi(\zeta)d\zeta, \tag{2.6}$$

thus, equation (2.5) can be written as

$$\phi^{(n)}(\tau) = h(\tau) + \alpha\varphi(\tau). \tag{2.7}$$

The next step is to determine the constant α in order to evaluate the exact solution $\phi(\tau)$. Then to find α, we first need to derive an expression for $\phi(\tau)$ using equation (2.7), and then substitute this expression into equation (2.6). To do this, we integrate both sides of equation (2.7) n times from 0 to τ, and by applying the initial conditions $\phi^{(k)}(0) = \alpha_k$, $0 \leq k \leq n-1$, we obtain the following expression for the form

$$\phi(\tau) = f(\tau,\alpha),$$

where $f(\tau,\alpha)$ is the result obtained by integrating equation (2.7) and applying the given initial conditions.

To provide a clear understanding of the technique, we will demonstrate the method through the following example.

Example 2.5.1 *Consider the second order FIDE*

$$\begin{cases} \phi''(\tau) = 2 - \frac{2}{3}\tau - \frac{16}{15}\tau^2 + \int\limits_{-1}^{1} (\tau\zeta + \tau^2\zeta^2)\phi(\zeta)d\zeta \\ \phi(0) = 1,\ \phi'(0) = 1. \end{cases}$$

This equation can be expressed in the form

$$\phi^{''}(\tau) = 2 + (\alpha - \frac{2}{3})\tau + (\beta - \frac{16}{15})\tau^2, \ \phi(0) = 1, \ \phi^{'}(0) = 1, \tag{2.8}$$

where

$$\begin{aligned} \alpha &= \int_{-1}^{1} \zeta\phi(\zeta)d\zeta, \\ \beta &= \int_{-1}^{1} \zeta^2\phi(\zeta)d\zeta, \end{aligned} \tag{2.9}$$

by integrating equation (2.8) *twice from* 0 *to* τ *and utilizing the initial conditions, we obtain*

$$\phi(\tau) = 1 + \tau + \tau^2 + (\alpha - \frac{2}{3})\frac{\tau^3}{6} + (\beta - \frac{16}{15})\frac{\tau^4}{12}, \tag{2.10}$$

by substituting equation (2.10) *into equation* $(2-9)$ *and solving for* α *and* β*, we obtain*

$$\alpha = \frac{2}{3}, \ \beta = \frac{16}{15},$$

which yields the exact solution

$$\phi(\tau) = 1 + \tau + \tau^2.$$

2.5.2 The series solution method

The general form of the Taylor series for an analytic solution $\phi(\tau)$ around $\tau = 0$ is given by

$$\phi(\tau) = \sum_{n=0}^{\infty} \alpha_n \tau^n. \tag{2.11}$$

We will assume that the solution $\phi(\tau)$ of the Volterra integro-differential equation

$$\phi^n(\tau) = h(\tau) + \lambda \int_0^{\tau} N(\tau, \zeta)\phi(\zeta)d\zeta, \ \phi^m(0) = m!\alpha_m, \ 0 \leq m \leq (n-1), \tag{2.12}$$

is analytic, which means it has a Taylor series of the form presented in (2.11), with the coefficients α_n determined recursively. The initial conditions allow us to find the first few coefficients α_n as follows

$$\alpha_0 = \phi(0), \ \alpha_1 = \phi^{'}(0), \ \alpha_2 = \frac{1}{2!}\phi^{''}(0), \ \alpha_3 = \frac{1}{3!}\phi^{'''}(0), ..., \tag{2.13}$$

and so forth. The remaining coefficients α_n in (2.11) will be determined by applying the series solution method to the VIDE (2.12). Substituting (2.11) into both sides of (2.12) yields

$$\left(\sum_{n=0}^{\infty}\alpha_n\tau^n\right)^{(n)}=T\left(h(\tau)\right)+\lambda\int_0^{\tau}N(\tau,\zeta)\left(\sum_{n=0}^{\infty}\alpha_n\zeta^n\right)d\zeta, \tag{2.14}$$

where $T\left(h(\tau)\right)$ represents the Taylor series for $h(\tau)$.

We start by integrating the right-hand side of the integral in (2.14) and collecting terms with like powers of τ. Next, we equate the coefficients of like powers of τ on both sides of the resulting equation to derive a recurrence relation for α_n, where $n \geq 0$. Solving the recurrence relation will fully determine the coefficients α_n, with some of these coefficients specified by the initial conditions. Once we have determined the coefficients α_n, the series solution can be constructed by substituting these coefficients back into (2.11). If an exact solution exists, it can be obtained, if not, the obtained series can be used for numerical purposes. In this case, evaluating more terms will lead to higher accuracy.

Example 2.5.2 *Apply the series solution method to solve the Volterra integro-differential equation*

$$\phi^{'''}(\tau)=1-\tau+2\sin(\tau)-\int_0^{\tau}(\tau-\zeta)\phi(\zeta)d\zeta,\ \phi(0)=1,\ \phi^{'}(0)=\phi^{''}(0)=-1,$$

we use the series solution method by substituting $\phi(\tau)$ *as a power series*

$$\phi(\tau)=\sum_{n=0}^{\infty}\alpha_n\tau^n.$$

Substituting this series into both sides of the equation and expanding $\sin(\tau)$ *using its Taylor series, we have*

$$\left(\sum_{n=0}^{\infty}\alpha_n\tau^n\right)^{'''}=1-\tau+2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\tau^{2n+1}-\int_0^{\tau}(\tau-\zeta)\left(\sum_{n=0}^{\infty}\alpha_n\zeta^n\right)d\zeta.$$

After differentiating the left side three times and evaluating the integral on the right side, we obtain

$$6\alpha_3+24\alpha_4\tau+60\alpha_5\tau^2+...=1+\tau-\frac{1}{2}\alpha_0\tau^2-\frac{1}{3}(1+\frac{1}{2}\alpha_1)\tau^3+...,$$

by applying the initial conditions and equating the coefficients of like powers of τ *on both sides of the previous equation, we obtain*

$$\alpha_0 = 1,\ \alpha_1 = -1, \alpha_2 = \frac{-1}{2},$$

$$\alpha_3 = \frac{1}{3!}, \alpha_4 = \frac{1}{4!}, \alpha_5 = \frac{-1}{5!}.$$

Thus, the series solution is

$$\phi(\tau) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\tau^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\tau^{2n+1},$$

which converges to the exact solution

$$\phi(\tau) = \cos(\tau) - \sin(\tau).$$

It is well-known that many real-world problems, particularly those involving complex equations, do not have an exact analytical solution. Integro-differential equations, in particular, often involve both integrals and derivatives that are difficult to solve symbolically. In such cases, numerical methods provide a practical approach to approximating solutions when analytical methods fail.

Even when analytical solutions exist, they can be highly complex and computationally expensive to implement, especially for large systems or when extreme precision is unnecessary. Numerical methods offer faster and more efficient alternatives, making them particularly useful in applications such as engineering simulations and scientific computing.

In the next chapter, we will explore some of these numerical methods in detail.

Chapter 3

Numerical methods for solving integro-differential equations

In this chapter, we present several numerical methods for solving integro-differential equations, including the variational iteration method (VIM), the Adomian and modified decomposition method, and the Taylor collocation method. Additionally, we present, for the first time, the application of Genocchi polynomials in solving high-order linear Fredholm integro-differential equations (LFIDEs).[25]. This approach transforms the LFIDE into a matrix equation, which is then solved to determine the unknown Genocchi coefficients.

3.1 The variational iteration method (VIM)

Consider a Volterra integro-differential equation given by [33]

$$\phi^{(m)}(\tau) = h(\tau) + \int_0^{\tau} N(\tau,\zeta)\phi(\zeta)d\zeta, \tag{3.1}$$

The correction functional for the integro-differential equation (3.1) is given by

$$\phi_{n+1}(\tau) = \phi_n(\tau) + \int_0^{\tau} \lambda(t)\left(\phi_n^{(m)}(t) - h(t) - \int_0^{t} N(t,r)\phi_n(r)dr\right)dt.$$

where $\lambda(t)$ is the Lagrange multiplier, which is determined based on variational theory. $\phi_n(\tau)$ represents the n-th iteration of the approximate solution.

Determining the Lagrange multiplier is crucial for applying the correction functional. Below, we provide a summary of several iterative formulas that outline the ordinary differential equation (ODE) and its corresponding Lagrange multipliers:

$$\phi' + h(\phi(t), \phi'(t)) = 0,\ \lambda = -1,$$

$$\phi'' + h(\phi(t), \phi'(t), \phi''(t)) = 0,\ \lambda = t - \tau,$$

$$\phi''' + h(\phi(t), \phi'(t), \phi''(t), \phi'''(t)) = 0, \lambda = -\frac{1}{2!}(t-\tau)^2,$$

$$\phi^{(n)} + h(\phi(t), \phi'(t), \phi''(t), ..., \phi^{(n)}(t)) = 0,\ \lambda = (-1)^n \frac{1}{(n-1)!}(t-\tau)^{n-1}.$$

As a result, the solution is given by

$$\phi(\tau) = \lim_{n\to\infty} \phi_n(\tau).$$

The VIM will be demonstrated through the following example

Example 3.1.1 *Apply the VIM to solve the Volterra integro-differential equation*

$$\phi''(\tau) = 1 + \int_0^\tau (\tau - \zeta)\phi(\zeta)d\zeta, \quad \phi(0) = 1, \quad \phi'(0) = 0.$$

The correction functional for this equation is expressed as

$$\phi_{n+1}(\tau) = \phi_n(\tau) + \int_0^\tau (t-\tau)\left(\phi_n''(t) - 1 - \int_0^t (t-r)\phi_n(r)dr\right)dt,$$

where $\lambda = t - \tau$, as presented in the second-order integro-differential equations above.

We can use the initial conditions to choose $\phi_0(\tau) = \phi(0) + \tau\phi'(0) = 1$. Substituting this choice into the correction functional yields the following successive approximations

$$\begin{aligned}
\phi_0(\tau) &= 1,\\
\phi_1(\tau) &= 1 + \frac{1}{2!}\tau^2 + \frac{1}{4!}\tau^4,\\
\phi_2(\tau) &= 1 + \frac{1}{2!}\tau^2 + \frac{1}{4!}\tau^4 + \frac{1}{6!}\tau^6 + \frac{1}{8!}\tau^8,\\
\phi_3(\tau) &= 1 + \frac{1}{2!}\tau^2 + \frac{1}{4!}\tau^4 + \frac{1}{6!}\tau^6 + \frac{1}{8!}\tau^8 + \frac{1}{10!}\tau^{10} + \frac{1}{12!}\tau^{12},
\end{aligned}$$

and so on.

The exact solution is given by

$$\phi(\tau) = \lim_{n\to\infty} \phi_n(\tau) = \cosh(\tau).$$

3.2 The Adomian decomposition method (ADM)

The Adomian decomposition method provides a solution in the form of an infinite series of components that can be determined recursively. If an exact solution exists, the series will represent it; otherwise, it provides an approximation with a high degree of accuracy. Consider a Volterra integro-differential equation of the second kind given by

$$\phi^{''}(\tau) = h(\tau) + \int_0^\tau N(\tau,\zeta)\phi(\zeta)d\zeta,\ \phi(0) = \alpha_0,\ \phi^{'}(0) = \alpha_1. \tag{3.2}$$

Integrating both sides of (3.2) twice from 0 to τ yields

$$\phi(\tau) = \alpha_0 + \alpha_1\tau + L^{-1}(h(\tau)) + L^{-1}\left(\int_0^\tau N(\tau,\zeta)\phi(\zeta)d\zeta\right), \tag{3.3}$$

where the initial conditions ϕ_0 and $\phi^{'}(0)$ are applied, and L^{-1} is a double integral operator. Then we substitute the decomposition series

$$\phi(\tau) = \sum_{n=0}^\infty \phi_n(\tau), \tag{3.4}$$

into both sides of (3.3), we get

$$\sum_{n=0}^\infty \phi_n(\tau) = \alpha_0 + \alpha_1\tau + L^{-1}(h(\tau)) + L^{-1}\left(\int_0^\tau N(\tau,\zeta)\left(\sum_{n=0}^\infty \phi_n(\zeta)\right)d\zeta\right), \tag{3.5}$$

or, correspondingly,

$$\begin{aligned}&\phi_0(\tau) + \phi_1(\tau) + \phi_2(\tau) + ... = \alpha_0 + \alpha_1\tau + L^{-1}(h(\tau))\\ &+L^{-1}\left(\int_0^\tau N(\tau,\zeta)(\phi_0(\zeta))\,d\zeta\right) + L^{-1}\left(\int_0^\tau N(\tau,\zeta)(\phi_1(\zeta))\,d\zeta\right)\\ &+L^{-1}\left(\int_0^\tau N(\tau,\zeta)(\phi_2(\zeta))\,d\zeta\right) + ...\end{aligned} \tag{3.6}$$

To find the components $\phi_0(\tau),\ \phi_1(\tau),\ \phi_2(\tau), ...$ of the solution $\phi_0(\tau)$, we establish the recurrence relation

$$\phi_0(\tau) = \alpha_0 + \alpha_1\tau + L^{-1}(h(\tau)),$$

$$\phi_{m+1}(\tau) = L^{-1}\left(\int_0^{\tau} N(\tau,\zeta)\left(\phi_m(\zeta)\right)d\zeta\right),\ m \geq 0. \tag{3.7}$$

Once the components $\phi_m(\tau)$, $m \geq 0$, have been determined, the solution $\phi(\tau)$ of (3.2) is obtained in the form of a series. According to (3.4), this series converges to the exact solution if it exists. However, for practical applications, a truncated series $\sum_{n=0}^{N}\phi_n(\tau)$, is typically used to approximate $\phi(\tau)$ for numerical purposes.

Example 3.2.1 *Apply the Adomian decomposition method to solve the VIDE*

$$\phi^{'''}(\tau) = -1 + \tau - \int_0^{\tau}(\tau-\zeta)\phi(\zeta)d\zeta,\quad \phi_0 = 1,\ \phi^{'}(0) = -1,\ \phi^{''}(0) = 1, \tag{3.8}$$

by applying the three-fold integral operator L^{-1}, *defined as*

$$L^{-1}(.) = \int_0^{\tau}\int_0^{\tau}\int_0^{\tau}(.)d\tau d\tau d\tau, \tag{3.9}$$

to both sides of (3.8), *and using the given initial conditions, we arrive at the expression*

$$\phi(\tau) = 1 - \tau + \frac{1}{2!}\tau^2 - \frac{1}{3!}\tau^3 + \frac{1}{4!}\tau^4 - L^{-1}\left(\int_0^{\tau}(\tau-\zeta)\phi(\zeta)d\zeta\right), \tag{3.10}$$

by applying (3.4) *and* (3.7), *we derive*

$$\begin{aligned}
\phi_0(\tau) &= 1 - \tau + \frac{1}{2!}\tau^2 - \frac{1}{3!}\tau^3 + \frac{1}{4!}\tau^4,\\
\phi_1(\tau) &= -L^{-1}\left(\int_0^{\tau}(\tau-\zeta)\phi_0(\zeta)d\zeta\right) = \frac{-1}{5!}\tau^5 + \frac{1}{6!}\tau^6 - \frac{1}{7!}\tau^7 + \frac{1}{8!}\tau^8 - \frac{1}{9!}\tau^9.
\end{aligned}$$

Continuing in this manner, we obtain the solution in series form

$$\phi(\tau) = 1 - \tau + \frac{1}{2!}\tau^2 - \frac{1}{3!}\tau + \frac{1}{4!}\tau^4 - \frac{1}{5!}\tau^5 + \frac{1}{6!}\tau^6 - \frac{1}{7!}\tau^7 + ...$$

This series converges to the exact solution

$$\phi(\tau) = e^{-\tau}.$$

Example 3.2.2 *Apply the Adomian decomposition method to solve the FIDE*

$$\begin{cases} \phi'(\tau) = \cos(\tau) + \frac{1}{4}\tau - \frac{1}{4}\int_0^{\frac{\pi}{2}} \tau\zeta\phi(\zeta)d\zeta \\ \phi(0) = 0,\ \phi(\tau) = \sin(\tau), \end{cases}$$

by integrating both sides of the equation from 0 *to* τ, *and by applying the initial conditions, we obtain*

$$\phi(\tau) = \sin(\tau) + \frac{1}{8}\tau^2 - \frac{1}{8}\tau^2 \int_0^{\frac{\pi}{2}} \zeta\phi(\zeta)d\zeta.$$

According to the ADM, we get

$$\begin{aligned}
\phi_0(\tau) &= \sin(\tau) + \frac{1}{8}\tau^2, \\
\phi_1(\tau) &= -\frac{1}{8}\tau^2 \int_0^{\frac{\pi}{2}} \zeta\phi_0(\zeta)d\zeta = -0.1487820\tau^2, \\
\phi_2(\tau) &= -\frac{1}{8}\tau^2 \int_0^{\frac{\pi}{2}} \zeta\phi_1(\zeta)d\zeta = 0.0283060\tau^2, \\
\phi_3(\tau) &= -\frac{1}{8}\tau^2 \int_0^{\frac{\pi}{2}} \zeta\phi_2(\zeta)d\zeta = -0.0053850\tau^2, \\
\phi_4(\tau) &= -\frac{1}{8}\tau^2 \int_0^{\frac{\pi}{2}} \zeta\phi_3(\zeta)d\zeta = 0.00102460\tau^2, \\
\phi_5(\tau) &= -\frac{1}{8}\tau^2 \int_0^{\frac{\pi}{2}} \zeta\phi_4(\zeta)d\zeta = -0.00019490\tau^2.
\end{aligned}$$

This gives

$$\phi(\tau) \simeq \sum_{n=0}^{5} \phi_n(\tau) = \sin(\tau) + 0.0001640\tau^2.$$

3.3 The modified decomposition method (MDM)

The MDM is an improved technique compared to other numerical methods for solving Fredholm integro-differential equations. This method provides a more precise approximate solution while reducing the number of iterative terms. A key requirement for applying this method is that the data function $h(\tau)$ must contain multiple terms. As a result, it can be split into two components ($h(\tau) = h_1(\tau) + h_2(\tau)$).

We examine a Fredholm integro-differential equation expressed as

$$\phi^{(n)}(\tau) = h(\tau) + \varphi(\tau)\int_0^1 \psi(\zeta)\phi(\zeta)d\zeta,\ \phi^{(k)}(0) = \alpha_k,\ 0 \le k \le n-1,$$

by applying the n-fold integral operator L^{-1} to both sides of the above equation and using the given initial conditions, we arrive at the expression

$$\begin{aligned}\phi(\tau) &= \alpha_0+\alpha_1\tau+...+\frac{1}{(n-1)!}\alpha_{n-1}\tau^{n-1}+L^{-1(}h_1(\tau))+\\&+L^{-1}(h_2(\tau))+L^{-1}(\varphi(\tau))\left(\int_0^1\psi(\zeta)\phi_0(\zeta)d\zeta\right)+...\\&+L^{-1}(\varphi(\tau))\left(\int_0^1\psi(\zeta)\phi_{n-1}(\zeta)d\zeta\right).\end{aligned}$$

Then, the iterative solutions are

$$\begin{aligned}\phi_0(\tau) &= \alpha_0+\alpha_1\tau+...+\frac{1}{(n-1)!}\alpha_{n-1}\tau^{n-1}+L^{-1}(h_1(\tau)),\\ \phi_1(\tau) &= L^{-1}(h_2(\tau))+L^{-1}(\varphi(\tau))\left(\int_0^1\psi(\zeta)\phi_0(\zeta)d\zeta\right),\\ \phi_2(\tau) &= L^{-1}(\varphi(\tau))\left(\int_0^1\psi(\zeta)\phi_1(\zeta)d\zeta\right),\\ &.\\ &.\\ \phi_n(\tau) &= L^{-1}(\varphi(\tau))\left(\int_0^1\psi(\zeta)\phi_{n-1}(\zeta)d\zeta\right).\end{aligned}$$

Example 3.3.1 *Apply the (MDM) to solve the FIDE*

$$\begin{cases}\phi'(\tau)=\cos(\tau)+\frac{1}{4}\tau-\frac{1}{4}\int_0^{\frac{\pi}{2}}\tau\zeta\phi(\zeta)d\zeta\\ \phi(0)=0;\ \phi(\tau)=\sin(\tau).\end{cases}$$

To eliminate the first derivative, we integrate both sides once from 0 *to* τ*, and by applying the initial conditions, we obtain*

$$\phi(\tau)=\sin(\tau)+\frac{1}{8}\tau^2-\frac{1}{8}\tau^2\int_0^{\frac{\pi}{2}}\zeta\phi(\zeta)d\zeta.$$

Thus, the approximations are

$$\begin{aligned}\phi_0(\tau) &= \sin(\tau),\\ \phi_1(\tau) &= \frac{1}{8}\tau^2-\frac{1}{8}\tau^2\int_0^{\frac{\pi}{2}}\zeta\sin(\zeta)d\zeta=0.\end{aligned}$$

Since $\phi_1(\tau)=0$*, there is no need to calculate the remaining terms, as they will all be zero. Therefore, the final result is*

$$\phi(\tau)=\sin(\tau).$$

3.4 Collocation method

The collocation method is a widely used numerical technique for solving integro-differential equations. This method is particularly useful because it transforms complex integro-differential problems into systems of algebraic equations, which can be solved more easily. Here's how the method works for integro-differential equations.

3.4.1 Taylor collocation method

Let us consider the Volterra integro-differential equation (VIDE)

$$\sum_{k=0}^{m} Q_k(\tau)\phi^{(k)}(\tau) - \lambda \int_a^\tau N(\tau,\zeta)\phi(\zeta)d\zeta = h(\tau), \ a \leq \tau \leq b. \tag{3.11}$$

Subject to the initial boundary conditions

$$\sum_{k=0}^{m-1}(\theta_{jk}\phi^{(k)}(a) + \beta_{jk}\phi^{(k)}(b) + \delta_{jk}\phi^{(k)}(\alpha)) = \eta_j \ , \quad j = 0, 1, ..., \ m-1. \tag{3.12}$$

The approximate solution is represented by the truncated Taylor series

$$\phi_N(\tau) = \sum_{n=0}^{N} \frac{\phi^{(n)}(\alpha)}{n!}(\tau - \alpha)^n, \tag{3.13}$$

where $\phi^{(n)}(\alpha)$ represent the Taylor coefficients that need to be determined.

Basic matrix relation

To approximate the solution of VIDE using the truncated Taylor series, we proceed as follows:

Step 1: Express the matrix forms of the truncated Taylor series and their derivatives.

$$\phi(\tau) \approx \phi_N(\tau) = \sum_{n=0}^{N} \frac{\phi^{(n)}(\alpha)}{n!}(\tau - \alpha)^n = P(\tau)L_0T,$$

where

$$P(\tau) = [1 \ \ (\tau - \alpha)^1 \ \ (\tau - \alpha)^2 \ \ ... \ \ (\tau - \alpha)^N],$$

and

$$L_0 = \begin{bmatrix} \frac{1}{1} & 0 & 0 & ... & 0 \\ 0 & \frac{1}{1\times1} & 0 & ... & 0 \\ 0 & 0 & \frac{1}{1\times2} & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & ... & \frac{1}{1\times2\times...\times N} \end{bmatrix}, T = \begin{bmatrix} \phi^{(0)}(\alpha) \\ \phi^{(1)}(\alpha) \\ \phi^{(2)}(\alpha) \\ . \\ . \\ \phi^{(N)}(\alpha) \end{bmatrix}.$$

The $k-$th derivative of the function $\phi_N(\tau) = \sum_{n=0}^{N} \frac{\phi^{(n)}(\alpha)}{n!}(\tau-\alpha)^n$ is given by the following expression

$$\phi_N^{(k)}(\tau) = \sum_{n=k}^{N} \frac{\phi^{(n)}(\alpha)}{(n-k)!}(\tau-\alpha)^{n-k} = P(\tau)L_kT,$$

where

$$L_k = \begin{bmatrix} 0 & 0 & ... & \frac{1}{0!} & 0 & ... & 0 \\ 0 & 0 & ... & 0 & \frac{1}{1!} & ... & 0 \\ \vdots & \vdots & ... & \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & 0 & 0 & ... & \frac{1}{(N-k)!} \\ 0 & 0 & ... & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & 0 & 0 & ... & 0 \end{bmatrix}.$$

After substituting the Taylor collocation points defined by $\tau_i = a + i\dfrac{b-a}{N}$, $i = 0, 1, 2, ..., N;\ \tau_0 = a$ and $\tau_N = b$ into the expression for $\phi_N(\tau)$ and $\phi_N^{(k)}(\tau)$, we get

$$\phi_N(\tau_i) = \sum_{n=0}^{N} \frac{\phi^{(n)}(\alpha)}{n!}(\tau_i-\alpha)^n = PL_0T$$

and

$$\phi_N^{(k)}(\tau_i) = \sum_{n=k}^{N} \frac{\phi^{(n)}(\alpha)}{(n-k)!}(\tau_i-\alpha)^{n-k} = PL_kT \tag{3.14}$$

where

$$P=\begin{bmatrix} P(\tau_0)\\ P(\tau_1)\\ .\\ .\\ P(\tau_N)\end{bmatrix}=\begin{bmatrix} (\tau_0-\alpha)^0 & (\tau_0-\alpha)^1 & ... & (\tau_0-\alpha)^N\\ (\tau_1-\alpha)^0 & (\tau_1-\alpha)^1 & ... & (\tau_1-\alpha)^N\\ . & . & & .\\ . & . & & .\\ (\tau_N-\alpha)^0 & (\tau_N-\alpha)^1 & ... & (\tau_N-\alpha)^N\end{bmatrix};$$

then we can express the VIDE as

$$\sum_{k=0}^{m} Q_k(\tau_i)\phi^{(k)}(\tau_i)=\sum_{k=0}^{m}(Q_k P L_k)\,T=H+\lambda I;$$

where

$$Q_k=\begin{bmatrix} Q_k(\tau_0) & 0 & ... & 0\\ 0 & Q_k(\tau_1) & ... & 0\\ . & & & .\\ & . & & \\ . & & ... & .\\ & . & & \\ . & & & .\\ 0 & 0 & ... & Q_k(\tau_N)\end{bmatrix};H=\begin{bmatrix} h(\tau_0)\\ h(\tau_1)\\ .\\ .\\ .\\ h(\tau_N)\end{bmatrix};I=\begin{bmatrix} I(\tau_0)\\ I(\tau_1)\\ .\\ .\\ I(\tau_N)\end{bmatrix}.$$

Step 2: We approximate the kernel $N(\tau,\zeta)$ in terms of Taylor series (in the $\tau=\alpha$ and $\zeta=\alpha$) in the form

$$\begin{aligned} N(\tau,\zeta) &= \sum_{p=0}^{N}\sum_{q=0}^{N} n_{pq}(\tau-\alpha)^p(\zeta-\alpha)^q,\\ n_{pq} &= \frac{1}{p!q!}\frac{\partial^{p+q}N}{\partial\tau^p\partial\zeta^q}\,|_{\tau=\alpha,\zeta=\alpha}\,. \end{aligned}$$

Using the same method, we express the matrix form of $N(\tau,\zeta)$ as follows

$$[N(\tau,\zeta)]=P(\tau)N(P(\xi))^T,$$

where $N=[n_{pq}]$,

$$N = \begin{bmatrix} n_{00} & n_{01} & ... & n_{0N} \\ n_{10} & n_{11} & ... & n_{1N} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ n_{N0} & n_{N1} & ... & n_{NN} \end{bmatrix}, \; P(\xi) = \begin{bmatrix} 1 \\ (\zeta-\alpha) \\ (\zeta-\alpha)^2 \\ . \\ . \\ (\zeta-\alpha)^N \end{bmatrix}^T.$$

For the $(N+1)$ points τ_i, then defining

$$I(\tau_i) = \int_a^{\tau_i} N(\tau_i, \zeta)\phi(\zeta)d\zeta,$$

by substituting the expressions of the kernel matrix $N(\tau_i, \zeta)$ and the approximate solution into the integral $I(\tau_i)$, we obtain

$$\begin{aligned} [I(\tau_i)] &= \int_a^{\tau_i} \left(P(\tau_i)N(P(\xi))^T P(\xi)L_0T\right) d\zeta \\ &= P(\tau_i)N \int_a^{\tau_i} (P(\xi))^T P(\xi)d\zeta L_0 T \\ &= P(\tau_i)NS_{\tau_i}L_0T, \end{aligned}$$

where

$$S_{\tau_i} = [s_{pq}] = \int_a^{\tau_i} (P(\xi))^T P(\xi)d\zeta, \text{ with } S_{\tau_i} = \frac{(\tau_i-\alpha)^{p+q+1} - (a-\alpha)^{p+q+1}}{p+q+1}\big|_{p,q=0,1,\ldots,N}$$

$$S_{\tau_i} = \begin{bmatrix} s_{00}(\tau_i) & s_{01}(\tau_i) & ... & s_{0N}(\tau_i) \\ s_{10}(\tau_i) & s_{11}(\tau_i) & ... & s_{1N}(\tau_i) \\ . & . & . & . \\ . & . & . & . \\ s_{N0}(\tau_i) & s_{N1}(\tau_i) & ... & s_{NN}(\tau_i) \end{bmatrix}.$$

Finally, the Volterra integro-differential equations becomes of the form

$$\left(\sum_{k=0}^{m} Q_k P L_k - \lambda\overline{\Gamma NSL_0}\right) T = H,$$

we can also write this equation as follow

$$\Psi T = H \text{ or } [\Psi; H],$$

where

$$\Psi = [m_{pq}] = \sum_{k=0}^{m} Q_k P L_k - \lambda \overline{PNSL_0}.$$

$\overline{P}$, $\overline{N}$, $\overline{S}$, and $\overline{L_0}$ are matrices of dimensions $((N+1),(N+1)^2)$, $((N+1)^2,(N+1)^2)$, $((N+1)^2,(N+1)^2)$, and $((N+1)^2,(N+1))$, respectively. These matrices can be represented using block matrices as follows

$$\overline{P} = \begin{bmatrix} P(\tau_0) & 0 & ... & 0 \\ 0 & P(\tau_1) & ... & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & ... & P(\tau_N) \end{bmatrix}, \overline{N} = \begin{bmatrix} N & 0 & ... & 0 \\ 0 & N & ... & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & ... & N \end{bmatrix}$$

$$\overline{S} = \begin{bmatrix} S_{\tau_0} & 0 & ... & 0 \\ 0 & S_{\tau_1} & ... & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & ... & S_{\tau_N} \end{bmatrix}, \ \overline{L_0} = \begin{bmatrix} L_0 \\ L_0 \\ . \\ . \\ L_0 \end{bmatrix}.$$

Note that if $Q_k = 0$, $k = 1,\ 2,\ ...,\ N$ and $|\Psi| \neq 0$, the Volterra integral equation has a unique solution. Conversely, if $|\Psi| = 0$, the equation is either unsolvable or has infinitely many solutions.

Step 03: Now let us form the matrix representation of the conditions

$$\sum_{k=0}^{m-1} (\theta_{jk}\phi^{(k)}(a) + \beta_{jk}\phi^{(k)}(b) + \delta_{jk}\phi^{(k)}(\alpha)) = \eta_j \ ,$$

$$j = 0,\ 1\ , ...,\ m-1, \qquad a \leq \alpha \leq b,$$

by applying relation (3.14), we obtain the matrix representations of the functions at points a, b, and α in the following form

$$\left[\phi^{(k)}(a)\right] = AL_kT, \ \left[\phi^{(k)}(b)\right] = BL_kT, \ \left[\phi^{(k)}(\alpha)\right] = CL_kT,$$

where

$$\begin{aligned} A &= [1 \ \ (a-\alpha) \ \ (a-\alpha)^2 \ ... \ (a-\alpha)^N], \\ B &= [1 \ (b-\alpha) \ \ (b-\alpha)^2 \ ... \ (b-\alpha)^N], \\ C &= [1 \ \ 0 \ \ \ 0 \ \ ... \ 0]. \end{aligned}$$

Thus, the matrix form conditions (3.12) become

$$\sum_{k=0}^{m-1}(\theta_{jk}A+\beta_{jk}B+\delta_{jk}C)L_kT=\left[\eta_j\ \right]$$

It can also be expressed by the following formulas

$$E_jT=\left[\eta_j\right] \ \text{ or } \ \left[E_j;\eta_j\right]=\left[e_{j0} \ \ e_{j1} \ \ ... \ \ e_{jN} \ ; \ \eta_j\right]$$

with

$$E_j=\sum_{k=0}^{m-1}(\theta_{jk}A+\beta_{jk}B+\delta_{jk}C)L_k, \quad j=0, \ 1, \ ..., \ m-1.$$

For m initial conditions, the augmented matrix is given by

$$\left[\widetilde{\Psi};\widetilde{H}\right]=\begin{bmatrix} m_{00} & m_{01} & m_{02} & \cdots & m_{0N} & ; & h(\tau_0) \\ m_{10} & m_{11} & m_{12} & \cdots & m_{1N} & ; & h(\tau_1) \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ m_{(N-m)0} & m_{(N-m)1} & m_{(N-m)2} & \cdots & m_{(N-m)N} & ; & h(\tau_{N-m}) \\ e_{00} & e_{01} & e_{02} & \cdots & e_{0N} & ; & \eta_0 \\ e_{10} & e_{11} & e_{12} & \cdots & e_{1N} & ; & \eta_1 \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ e_{(m-1)0} & e_{(m-1)1} & e_{(m-1)2} & \cdots & e_{(m-1)N} & ; & \eta_{m-1} \end{bmatrix}$$

Lastly, if $rank\widetilde{\Psi}=rank\left[\widetilde{\Psi};\widetilde{H}\right]=N+1$, then the unique solution to problem (3.11) can be obtained using the Taylor series solution (3.13).

Example 3.4.1 *Let us consider the LVIDE* [21]

$$\begin{cases} \phi'(\tau) + \phi(\tau) - \int_0^\tau \tau(1+2\tau)e^{\zeta(\tau-\zeta)}\phi(\zeta)d\zeta = 1 + 2\tau, \quad 0 \leq \tau \leq 1, \\ \phi(\tau) = e^{\tau^2}, \ \phi(0) = 1, \end{cases}$$

for $N = 3$, the collocation points are

$$\tau_0 = 0, \ \tau_1 = \frac{1}{3}, \ \tau_2 = \frac{2}{3}, \ \tau_3 = 1.$$

The problem is represented in matrix form by

$$\left(Q_0PL_0 + Q_1PL_1 - \overline{PNSL_0}\right) T = H.$$

After the augmented matrices of the system and applying the given conditions, the resulting solution is

$$\phi(\tau) = 1 + 0,6475\tau^2 + 0,96617\tau^3.$$

3.4.2 Genocchi collocation method and error anlysis

It is noteworthy that Abdulnasir and Phang were the first to apply Genocchi polynomials to solve fractional calculus problems involving differential equations in 2016.

The main objective in this section is to present a new numerical method based on Genocchi polynomials, for solving high order linear Fredholm integro-differential equations (LFIDEs) [25]. The method involves transforming the LFIDE into a matrix equation by approximating the unknown function, its derivatives, and the integral kernel using Genocchi polynomials. By applying equidistant collocation points, we solve the resulting linear system to determine the unknown Genocchi coefficients. To demonstrate the accuracy and efficiency of this approach, we present several numerical examples in the next chapter.

The general form of the LFIDE of order m is given by

$$\sum_{k=0}^{m} Q_k(\tau)\phi^{(k)}(\tau) - \lambda \int_0^1 N(\tau,\zeta)\phi(\zeta)d\zeta = h(\tau), \quad 0 \leqslant a \leqslant \tau, \ \zeta \leqslant b, \tag{3.15}$$

subject to the initial boundary conditions

$$\sum_{k=0}^{m-1}(\theta_{jk}\phi^{(k)}(a) + \delta_{jk}\phi^{(k)}(b)) = \eta_j \ , \quad j = 0, 1, ..., m-1, \tag{3.16}$$

where $\phi(\tau)$ is an unknown function; $Q_k(\tau)$ and $h(\tau)$ are functions defined on $[a,b]$; and the kernel function $N(\tau,\zeta)$ is defined and continuous on $[a,b]\times[a,b]$. The constants θ_{jk}, δ_{jk} and η_j are appropriate values.

Definitions and properties of Genocchi polynomials $D_n(\tau)$ are given in the first chapter.

Function approximation

In this part, we require the following linear independence, upon which the subsequent theoretical results are founded.

Suppose that $\{D_1(\tau),\ D_2(\tau),...,\ D_N(\tau)\}$ be the set of Genocchi polynomials.

Lemma 3.4.1 [34] *The set* $L=\{D_1(\tau),\ D_2(\tau),\ ...,\ D_N(\tau)\}\subset L^2[0,1]$ *is linearly independent in* $L^2[0,1]$.

Proof. To demonstrate that L consists of linearly independent elements in $L^2[0,1]$, it suffices to show that the *Gram* determinant is non-zero. Specifically,

$$Gram(D_1,\ D_2,\ ...,\ D_N)\neq 0.$$

The *Gram* determinant is given by

$$Gram\ (D_1,\ D_2,\ ...,\ D_N)=\begin{vmatrix} \langle D_1,D_1\rangle & \langle D_1,D_2\rangle & ... & \langle D_1,D_N\rangle \\ \langle D_2,D_1\rangle & \langle D_2,D_2\rangle & ... & \langle D_2,D_N\rangle \\ . & . & & . \\ . & . & ... & . \\ . & . & & . \\ \langle D_N,D_1\rangle & \langle D_N,D_2\rangle & ... & \langle D_N,D_N\rangle \end{vmatrix},$$

where $\langle .,.\rangle$ represents the inner product.

To prove that this determinant is non-zero, we first transform the Gram matrix into an upper triangular matrix using Gaussian elimination. It is straightforward to observe that the diagonal elements of the reduced matrix are given by

$$l(m)=\frac{[(m-1)!\times(m!)]^2}{(2m-2)!\times(2m-1)!},\ m\in\mathbb{N}.$$

It is evident that for any $m\in\mathbb{N}$, $l(m)\neq 0$. Therefore, the determinant expressed by

$$\prod_{m=1}^{N} l(m),$$

is non-zero. Thus, the set L comprises linearly independent elements.

Theorem 3.4.1 [34] *let S represent the set Span* $\{D_1(\tau),\ D_2(\tau),\ ...,\ D_N(\tau)\} = Span(L)$, *and let $\phi(\tau)$ be an arbitrary element of $L^2[0,1]$. Since S is a finit-dimensional subspace (vector space) of $L^2[0,1]$, there exists a unique best approximation of $\phi(\tau)$ in S, denoted by $\phi_N(\tau)$, such that*

$$\|\phi(\tau) - \phi_N(\tau)\|_2 \leq \|\phi(\tau) - s(\tau)\|_2, \forall s(\tau) \in S.$$

Since $\phi_N(\tau) \in S$, there exist unique coefficients ω_1, ω_2, ..., ω_N such that

$$\phi(\tau) \simeq \phi_N(\tau) = \sum_{n=1}^{N} \omega_n D_n(\tau) = D(\tau)\varpi. \tag{3.17}$$

$$D(\tau) = [D_1(\tau)\ D_2(\tau)\ ...\ D_N(\tau)], \quad \varpi = [\omega_1\ \omega_2\ ...\ \omega_N]^T.$$

In the following lemma, we demonstrate how the coefficient can be determined. ■

Lemma 3.4.2 [35] *Let $\phi(\tau) \in L^2[0,1]$ be an arbitrary function, approximated by the truncated Genocchi series*

$$\sum_{n=1}^{N} \omega_n D_n(\tau).$$

Then, the coefficients ω_n for all $n = 1, 2, ..., N$ can be determined by the following relation

$$\omega_n = \frac{1}{2n!}(\phi^{(n-1)}(0) + \phi^{(n-1)}(1)),\ n = 1, ...,\ N, \tag{3.18}$$

note that the power of $(n-1)$ in $\phi^{(n-1)}$ represents the $(n-1)$-th derivative of ϕ.

Proof. Suppose $\phi(\tau) \simeq \sum_{n=1}^{N} \omega_n D_n(\tau)$. Using the property $D_n(1) + D_n(0) = 0$ for $n > 1$, we get

$\phi(o) + \phi(1) = \omega_1(D_1(0) + D_1(1)) + \omega_2((D_2(0) + D_2(1)) + ... + \omega_N(D_N(0) + D_N(1)) = 2\omega_1,$

$\Longrightarrow \omega_1 = \frac{1}{2}(\phi(0) + \phi(1)).$

$\phi^{(1)}(0) + \phi^{(1)}(1) = 2\omega_2(D_1(0) + D_1(1)) + 3\omega_3((D_2(0) + D_2(1)) + ... + N\omega_N(D_{N-1}(0)+ D_{N-1}(1)) = 2\omega_2 \times 2,$

$\Longrightarrow \omega_2 = \frac{1}{2(2!)}\left(\phi^{(1)}(0) + \phi^{(1)}(1)\right).$

$\phi^{(2)}(0) + \phi^{(2)}(1) = 3(2)\omega_3(D_1(0) + D_1(1)) + ... + (N-1)N\omega_N(D_{N-2}(0) + D_{N-2}(1)) = 3 \times 2\omega_3 \times 2,$

$\Longrightarrow \omega_3 = \frac{1}{2(3!)}\left(\phi^{(2)}(0) + \phi^{(2)}(1)\right).$

By applying this procedure n times, we get ■

$$\omega_n = \frac{1}{2(n!)}\left(\phi^{(n-1)}(0) + \phi^{(n-1)}(1)\right).$$

Remark 3.4.1 *It is important to note that calculating the approximation coefficient using the Genocchi polynomials for a function that is not $(n-1)$-times differentiable at the points $\tau = 0$ and $\tau = 1$ results in failure. The following example demonstrates this issue.*

Let $\phi(\tau) = 2\tau^{\frac{3}{2}} - 1$, for $N = 3$, we obtain

$$\begin{aligned}
\phi_3(\tau) &= \sum_{n=1}^{3}\omega_n D_n(\tau) = \omega_1 D_1(\tau) + \omega_2 D_2(\tau) + \omega_3 D_3(\tau) \\
\omega_3 &= \frac{1}{2(3!)}\left((2\tau^{\frac{3}{2}} - 1)^{(2)}(0) + (2\tau^{\frac{3}{2}} - 1)^{(2)}(1)\right) \\
&= \frac{1}{2(3!)}\left[\frac{3}{2\sqrt{\tau}}\bigg|_{\tau=0} + \frac{3}{2\sqrt{\tau}}\bigg|_{\tau=1}\right].
\end{aligned}$$

Operational matrices of Genocchi polynomials for solving high-order linear FIDEs

To approximate the solution of equation (3.15) using equation (3.17), we proceed as follows:

Step 1: Express the matrix forms of the truncated Genocchi series and their derivatives.

$$\phi(\tau) \approx \phi_N(\tau) = \sum_{n=1}^{N}\omega_n D_n(\tau) = D(\tau)\varpi,$$

where

$$D(\tau) = [D_1(\tau)\ D_2(\tau)\ ...\ D_N(\tau)],\ \varpi = [\omega_1\ \omega_2\ ...\ \omega_N]^T.$$

Using property (3) of the Genocchi polynomials given in Chapter 1, the relationship between the matrix $D(\tau)$ and its derivatives is expressed by the following equation

$$\begin{aligned}
D^{(1)}(\tau) &= \left[D_1^{(1)}(\tau)\ D_2^{(1)}(\tau)\ ...\ D_N^{(1)}(\tau)\right] \\
&= [0\ \ 2D_1(\tau)\ ...\ \ ND_{N-1}(\tau)]. \\
D^{(1)}(\tau) &= D(\tau)C, \\
D^{(2)}(\tau) &= D^{(1)}(\tau)C = D(\tau)C^2. \\
&. \\
&. \\
D^{(k)}(\tau) &= D(\tau)C^k,
\end{aligned}$$

where

$$C = \begin{bmatrix} 0 & 2 & 0 & 0 & ... & 0 \\ 0 & 0 & 3 & 0 & ... & 0 \\ 0 & 0 & 0 & 4 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & N \\ 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix}.$$

Thus, C is $N \times N$ operational matrix of derivative

$$\begin{aligned} \phi_N^{(k)}(\tau) &= D^{(k)}(\tau)\varpi \\ &= D(\tau)C^k\varpi. \end{aligned}$$

On the other hand we have

$$D(\tau) = P(\tau)G,$$

where

$$G = \begin{bmatrix} \binom{1}{0} D_1 & \binom{2}{0} D_2 & \binom{3}{0} D_3 & ... & \binom{N}{0} D_N \\ 0 & \binom{2}{1} D_1 & \binom{3}{1} D_2 & ... & \binom{N}{1} D_{N-1} \\ 0 & 0 & \binom{3}{2} D_1 & ... & \binom{N}{2} D_{N-2} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \binom{N}{N-1} D_1 \end{bmatrix}, \quad P(\tau) = \begin{bmatrix} 1 \\ \tau \\ \tau^2 \\ . \\ . \\ . \\ \tau^{N-1} \end{bmatrix}^T.$$

Therefore

$$\phi^{(k)}(\tau) \simeq \phi_N^{(k)}(\tau) = P(\tau)GC^k\varpi. \tag{3.19}$$

Based on the above, the matrix relation for the left side of equation (3.15) can be expressed as follows

$$\sum_{k=0}^{m} Q_k(\tau)\phi^{(k)}(\tau) = \sum_{k=0}^{m} Q_k(\tau)P(\tau)GC^k\varpi. \tag{3.20}$$

Step 02: We approximate the integral kernel $N(\tau, \zeta)$ in terms of Genocchi polynomials $D_n(\tau)$ by following theorem.

Theorem 3.4.2 [36] *Let $N(\tau,\zeta) \in C^{N-1}([0,1])$. Then, $N(\tau,\zeta)$ can be approximated using Genocchi polynomials $D_n(\tau)$ up to order N as*

$$N(\tau,\zeta) \simeq \sum_{j=1}^{N}\sum_{k=1}^{N} n_{jk} D_j(\tau) D_k(\zeta) = D(\tau) N_G D^T(\zeta), \tag{3.21}$$

where

$$N_G = [n_{jk}]_{N\times N}\,.$$

$$n_{jk} = \frac{1}{4(k!j!)}(N^{(j-1,k-1)}(0,0) + N^{(j-1,k-1)}(0,1) + N^{(j-1,k-1)}(1,0) + N^{(j-1,k-1)}(1,1)).$$

Proof. Assume that the kernel $N(\tau,\zeta)$ is approximated by N Genocchi polynomials, i.e.

$$N(\tau,\zeta) \simeq \sum_{j=1}^{N}\sum_{k=1}^{N} n_{jk} D_j(\tau) D_k(\zeta) = \sum_{k=1}^{N}\left(\sum_{j=1}^{N} n_{jk} D_j(\tau)\right) D_k(\zeta).$$

Let $\psi_k(\tau) = \sum_{j=1}^{N} n_{jk} D_j(\tau)$. Hence,

$$N(\tau,\zeta) = \sum_{k=1}^{N} \psi_k(\tau) D_k(\zeta).$$

Employing Eq. (3.18) for $N(\tau,\zeta)$, we find

$$\psi_k(\tau) = \frac{1}{2(k!)}\left(\frac{\partial^{k-1}}{\partial\zeta^{k-1}} N(\tau,\zeta)\mid_{\zeta=0} + \frac{\partial^{k-1}}{\partial\zeta^{k-1}} N(\tau,\zeta)\mid_{\zeta=1}\right).$$

Now using the same formula for $\psi_k(\tau)$, we get

$$\begin{aligned}
n_{jk} &= \frac{1}{2(j!)}\left(\frac{\partial^{j-1}}{\partial\tau^{j-1}}\psi_k(\tau)\mid_{\tau=0} + \frac{\partial^{j-1}}{\partial\tau^{j-1}}\psi_k(\tau)\mid_{\tau=1}\right)\\
&= \frac{1}{2(j!)}\left[\frac{\partial^{j-1}}{\partial\tau^{j-1}}\frac{1}{2(k!)}\left(\frac{\partial^{k-1}}{\partial\zeta^{k-1}}N(\tau,\zeta)\mid_{\zeta=0} + \frac{\partial^{k-1}}{\partial\zeta^{k-1}}N(\tau,\zeta)\mid_{\zeta=1}\right)\mid_{\tau=0}\right.\\
&+ \left.\frac{\partial^{j-1}}{\partial\tau^{j-1}}\frac{1}{2(k!)}\left(\frac{\partial^{k-1}}{\partial\zeta^{k-1}}N(\tau,\zeta)\mid_{\zeta=0} + \frac{\partial^{k-1}}{\partial\zeta^{k-1}}N(\tau,\zeta)\mid_{\zeta=1}\right)\mid_{\tau=1}\right]\\
&= \frac{1}{4(k!j!)}\left[\frac{\partial^{j-1}}{\partial\tau^{j-1}}\frac{\partial^{k-1}}{\partial\zeta^{k-1}}N(\tau,\zeta)\mid_{\tau=0,\zeta=0} + \frac{\partial^{j-1}}{\partial\tau^{j-1}}\frac{\partial^{k-1}}{\partial\zeta^{k-1}}N(\tau,\zeta)\mid_{\tau=0,\zeta=1}\right.\\
&+ \left.\frac{\partial^{j-1}}{\partial\tau^{j-1}}\frac{\partial^{k-1}}{\partial\zeta^{k-1}}N(\tau,\zeta)\mid_{\tau=1,\zeta=0} + \frac{\partial^{j-1}}{\partial\tau^{j-1}}\frac{\partial^{k-1}}{\partial\zeta^{k-1}}N(\tau,\zeta)\mid_{\tau=1,\zeta=1}\right].
\end{aligned}$$

■

Matrix relation for the integral part $\mathbf{I}(\tau)$

Substitute the relation (3.17) and (3.20) in the expression of $I(\tau)$, we get

$$\begin{aligned} I(\tau) &= \int_0^1 N(\tau,\zeta)\phi(\zeta)d\zeta = \int_0^1 D(\tau)N_G D^T(\zeta)D(\zeta)\varpi d\zeta \\ &= D(\tau)N_G\left(\int_0^1 D^T(\zeta)D(\zeta)d\zeta\right)\varpi \\ &= P(\tau)GN_G D\varpi. \end{aligned}$$

where

$$D = \int_0^1 D^T(\zeta)D(\zeta)d\zeta = [D_{jk}]_{N\times N},$$

$$D_{jk} = \frac{2(-1)^j j!k!}{(j+k)!}D_{j+k}.$$

Hence, the right side of equation (3.15) is

$$h(\tau) + \lambda\int_0^1 N(\tau,\zeta)\phi(\zeta)d\zeta = h(\tau) + \lambda P(\tau)GN_G D\varpi. \tag{3.22}$$

Next, substituting the matrix relations from formulas (3.20) and (3.22) into equation (3.15) yields the matrix equation

$$\sum_{k=0}^{m} Q_k(\tau)P(\tau)GC^k\varpi = \lambda P(\tau)GN_G D\varpi + h(\tau). \tag{3.23}$$

Step 03: Applying equation (3.19) to condition (3.16) yields

$$\sum_{k=0}^{m-1}(\theta_{jk}P(a) + \delta_{jk}P(b))GC^k\varpi = \eta_j\ , \quad j = 0,\ 1,\ ...,\ m-1. \tag{3.24}$$

Step 04: We define the collocation points

$$\tau_i = \frac{(i-1)}{(N-1)},\ \ i = 1, ...,\ N.$$

Next, we apply these points into equation (3.23), yielding the following result:

$$\left(\sum_{k=0}^{m} Q_k PGC^k - \lambda PGN_G D\right)\varpi = H, \tag{3.25}$$

where

$$Q_k = \begin{bmatrix} Q_k(\tau_1) & 0 & 0 & ... & 0 \\ 0 & Q_k(\tau_2) & 0 & ... & \\ 0 & 0 & Q_k(\tau_3) & ... & 0 \\ . & . & . & & . \\ . & . & . & & . \\ 0 & 0 & 0 & ... & Q_k(\tau_N) \end{bmatrix}.$$

$$P = \begin{bmatrix} P(\tau_1) \\ P(\tau_2) \\ . \\ . \\ P(\tau_N) \end{bmatrix} = \begin{bmatrix} 1 & \tau_1^1 & \tau_1^2 & ... & \tau_1^{N-1} \\ 1 & \tau_2^1 & \tau_2^2 & ... & \tau_2^{N-1} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 1 & \tau_N^1 & \tau_N^2 & ... & \tau_N^{N-1} \end{bmatrix}; H = \begin{bmatrix} h(\tau_1) \\ h(\tau_2) \\ . \\ . \\ h(\tau_N) \end{bmatrix}.$$

The matrix relation above can be written as

$$\digamma \varpi = H \text{ or } [\digamma; H],$$

where

$$\digamma = [F_{jk}] = \sum_{k=0}^{m} Q_k PGC^k - \lambda PGN_G D, \quad j = 0, 1, ..., \ N-1,$$

$$[\digamma; H] = \begin{bmatrix} F_{00} & F_{01} & ... & F_{0(N-1)} & ; & h(\tau_1) \\ F_{10} & F_{11} & ... & F_{1(N-1)} & ; & h(\tau_1) \\ . & . & & . & & . \\ & & ... & & ; & \\ . & . & & . & & . \\ F_{(N-1)0} & F_{(N-1)1} & ... & . & ; & h(\tau_N) \end{bmatrix}.$$

Additionally, the matrix form (3.24) for conditions (3.16) is expressed as

$$E_j \varpi = [\eta_j] \text{ or } [E_j; \eta_j] = [e_{j0} \ e_{j1} \ ... \ e_{j(N-1)} \ ; \ \eta_j],$$

where

$$E_j = \sum_{k=0}^{m-1} (\theta_{jk} P(a) + \delta_{jk} P(b)) GC^k, \quad j = 0, \ 1, ..., \ m-1.$$

For m initial conditions, the augmented matrix is given by

$$\left[\widetilde{F};\widetilde{H}\right]=\begin{bmatrix} F_{00} & F_{01} & F_{02} & \cdots & F_{0(N-1)} & ; & h(\tau_0) \\ F_{10} & F_{11} & F_{12} & \cdots & F_{1(N-1)} & ; & h(\tau_1) \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ F_{(N-1-m)0} & F_{(N-1-m)1} & F_{(N-1-m)2} & \cdots & F_{(N-1-m)(N-1)} & ; & h(\tau_{N-1-m}) \\ e_{00} & e_{01} & e_{02} & \cdots & e_{0(N-1)} & ; & \eta_0 \\ e_{10} & e_{11} & e_{12} & \cdots & e_{1(N-1)} & ; & \eta_1 \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ e_{(m-1)0} & e_{(m-1)1} & e_{(m-1)2} & \cdots & e_{(m-1)(N-1)} & ; & \eta_{m-1} \end{bmatrix}$$

Lastly, if $rank\widetilde{F} = rank\left[\widetilde{F};\widetilde{H}\right] = N$, then the unique solution to problem $(3,15)$ can be obtained using the Genocchi series solution (3.17).

Error anlysis

The following theorem provides the error associated with the function approximation used to solve problem (3.15).

Theorem 3.4.3 [37] *Let* $\phi \in L^2[0,1]$ *be an arbitrary function such that* $\left|\phi^{(n-1)}(\tau)\right| \leq \sigma$ *where* σ *is finite, if* $\phi_N(\tau)$ *denotes the approximation of* $\phi(\tau)$ *using a truncated Genocchi series* $\phi_N(\tau) = \sum_{n=1}^{N} \omega_n D_n(\tau)$ *then the error* $\varrho_N = \|\phi(\tau) - \phi_N(\tau)\|_2$ *is bounded above by*

$$\varrho_N \leq \left(\sum_{n=N+1}^{\infty}\sum_{p=0}^{n} \frac{\sigma^2}{2(n!)^2}\binom{n}{p}^2\frac{D_{n-p}^2}{2p+1}\right)^{\frac{1}{2}},$$

where

$$\varrho_N = \left(\int_0^1 |\phi(\tau) - \phi_N(\tau)|^2\, d\tau\right)^{\frac{1}{2}}$$

$$\begin{aligned}
\varrho_N^2 &= \int_0^1 |\phi(\tau)-\phi_N(\tau)|^2\,d\tau = \int_0^1 \left|\phi(\tau)-\sum_{n=1}^{N}\omega_n D_n(\tau)\right|^2 d\tau \\
&= \int_0^1 \left|\sum_{n=N+1}^{\infty}\omega_n D_n(\tau)\right|^2 d\tau \\
&\leq \sum_{n=N+1}^{\infty}|\omega_n|^2 \int_0^1 |D_n(\tau)|^2\,d\tau,
\end{aligned}$$

we have, $D_n(\tau)=\sum_{p=0}^{n}\binom{n}{p} D_{n-p}\tau^p$. Therefore,

$$\begin{aligned}
\varrho_N^2 &\leq \sum_{n=N+1}^{\infty}|\omega_n|^2\sum_{p=0}^{n}\binom{n}{p}^2 D_{n-p}^2\int_0^1 \tau^{2p}d\tau \\
&= \sum_{n=N+1}^{\infty}|\omega_n|^2\sum_{p=0}^{n}\binom{n}{p}^2 \frac{D_{n-p}^2}{2p+1}.
\end{aligned}$$

Applying formula (3.18), we get

$$\begin{aligned}
\varrho_N^2 &\leq \sum_{n=N+1}^{\infty}\left|\frac{1}{2n!}(\phi^{(n-1)}(0)+\phi^{(n-1)}(1))\right|^2\sum_{p=0}^{n}\binom{n}{p}^2\frac{D_{n-P}^2}{2p+1} \\
&\leq \sum_{n=N+1}^{\infty}\sum_{p=0}^{n}\frac{1}{4(n!)^2}(\left|\phi^{(n-1)}(0)\right|^2+\left|\phi^{(n-1)}(1)\right|^2)\binom{n}{p}^2\frac{D_{n-p}^2}{2p+1} \\
&\leq \sum_{n=N+1}^{\infty}\sum_{p=0}^{n}\frac{1}{4(n!)^2}(2\sigma^2)\binom{n}{p}^2\frac{D_{n-p}^2}{2p+1} \\
&= \sum_{n=N+1}^{\infty}\sum_{p=0}^{n}\frac{\sigma^2}{2(n!)^2}\binom{n}{p}^2\frac{D_{n-p}^2}{2p+1}.
\end{aligned}$$

Therfore,

$$\varrho_N \leq \left(\sum_{n=N+1}^{\infty}\sum_{p=0}^{n}\frac{\sigma^2}{2(n!)^2}\binom{n}{p}^2\frac{D_{n-p}^2}{2p+1}\right)^{\frac{1}{2}}.$$

If the exact solution is not available, the error estimation is provided by this theorem.

Theorem 3.4.4 [38] *Let $\phi(\tau)$ be the unknown solution, and $\phi_N(\tau)$, $\phi_{N+1}(\tau)$ be the approximate solutions of $\phi(\tau)$. The error estimation given by $\Xi_N = \left\|\phi_N(\tau)-\phi_{N+1}(\tau)\right\|_2$ is convergent.*

Proof. *we have*

$$\begin{aligned}\Xi_N &= \left\|\phi_{N+1}(\tau)-\phi_N(\tau)\right\|_2 = \left\|\phi_{N+1}(\tau)-\phi(\tau)+\phi(\tau)-\phi_N(\tau)\right\|_2\\ &\le \left\|\phi_{N+1}(\tau)-\phi(\tau)\right\| + \|\phi(\tau)-\phi_N(\tau)\|\\ &\le \varrho_{N+1}+\varrho_N,\end{aligned}$$

e_N *is convergent because* ϱ_{N+1} *and* ϱ_N *are also convergents.* ■

Example 3.4.2 *Let be the equation* [39]

$$\begin{cases}\phi'(\tau)-2\tau\phi(\tau) = -2\tau^3-2\tau^2+\dfrac{23}{6}\tau+1+2\int_0^1 \tau\zeta\phi(\zeta)d\zeta,\ 0\le\tau,\ \zeta\le 1,\\ \phi(0)=-1,\end{cases}$$

where $N(\tau,\zeta)=\tau\zeta,\ h(\tau)=-2\tau^3-2\tau^2+\dfrac{23}{6}\tau+1,\ \lambda=2,$
and $q_0(\tau)=-2\tau,\ q_1(\tau)=1.$

The Genocchi polynomials provide an approximate solution $\phi(\tau)$ *to this equation, given by*

$$\phi(\tau)\approxeq\phi_3(\tau)=\sum_{n=1}^{3}\omega_n D_n(\tau),\ (0\le\tau\le 1).$$

For $N=3$*, the collocation points are* $\left\{\tau_1=0,\ \tau_2=\dfrac{1}{2},\ \tau_3=1\right\}$*. The fundamental matrix equation from* (3.25) *is*

$$\left(Q_0PGC^0+Q_1PGC^1\ -\lambda PGN_GD\right)\varpi=H;$$

where

$$Q_0=\begin{bmatrix}0&0&0\\0&-1&0\\0&0&-2\end{bmatrix},\ Q_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ C=\begin{bmatrix}0&2&0\\0&0&3\\0&0&0\end{bmatrix},$$

$$N_G=\begin{bmatrix}\frac14&\frac14&0\\\frac14&\frac14&0\\0&0&0\end{bmatrix},\ G=\begin{bmatrix}\binom{1}{0}D_1&0&0\\\binom{2}{0}D_2&\binom{2}{1}D_1&0\\\binom{3}{0}D_3&\binom{3}{1}D_2&\binom{3}{2}D_1\end{bmatrix}^T=\begin{bmatrix}1&0&0\\-1&2&0\\0&-3&3\end{bmatrix}^T,$$

$$D=\begin{bmatrix}1&0&-\frac12\\0&\frac13&0\\-\frac12&0&\frac{3}{10}\end{bmatrix},\ P=\begin{bmatrix}1&1&1\\0&\frac12&1\\0&\frac14&1\end{bmatrix}^T.$$

Example 3.4.3 *From the above we get*

$$[F;H]=\begin{bmatrix} 0 & 2 & -3 & ; & 1 \\ -\frac{3}{2} & \frac{11}{6} & 1 & ; & \frac{13}{6} \\ -3 & -\frac{1}{3} & \frac{7}{2} & ; & \frac{5}{6} \end{bmatrix}.$$

The initial condition is represented by the following matrix

$$[E_0;\eta_0]=[1\ \ -1\quad 0].$$

The updated augmented matrix related to this condition is

$$\left[\widetilde{F};\widetilde{H}\right]=\begin{bmatrix} 0 & 2 & -3 & ; & 1 \\ -\dfrac{3}{2} & \dfrac{11}{6} & 1 & ; & \dfrac{13}{6} \\ 1 & -1 & 0 & ; & -1 \end{bmatrix}.$$

To determine the Genocchi coefficients, we solve the linear system provided above

$$\varpi=\begin{bmatrix} 0 \\ 1 \\ \dfrac{1}{3} \end{bmatrix}.$$

Therefore, the approximate solution is

$$\phi_3(\tau)=\sum_{n=1}^{3}\omega_n D_n(\tau)=\omega_1 D_1(\tau)+\omega_2 D_2(\tau)+\omega_3 D_3(\tau)=\tau^2+\tau-1.$$

Example 3.4.4 *We pose the following problem*

$$\begin{cases} \phi'(\tau)+\phi(\tau)-\int_0^1\phi(\zeta)d\zeta=e^{-1}-1, \quad \phi(0)=1 \\ \phi(\tau)=e^{-\tau} \end{cases}$$

where, $N(\tau,\zeta)=1$, $h(\tau)=e^{-\tau}-1$, $\lambda=1$, *and* $q_0(\tau)=q_1(\tau)=1$.

The Genocchi polynomials provide an approximate solution $\phi(\tau)$ *to this equation, given by*

$$\phi(\tau)\simeq\phi_4(\tau)=\sum_{n=1}^{4}\omega_n D_n(\tau),\ (0\le\tau\le 1).$$

For $N=4$, *the collocation points are* $\left\{\tau_1=0,\ \tau_2=\dfrac{1}{3},\ \tau_3=\dfrac{2}{3},\ \tau_4=1\right\}$. *The fundamental matrix equation from* (3.25) *is*

$$\left(Q_0PGC^0+Q_1PGC^1-\lambda PGN_GD\right)\varpi=H,$$

where

$$Q_0 = Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} \\ 1 & 1 & 1 & 1 \end{bmatrix}, N_G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & \frac{-1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{-2}{5} \\ \frac{-1}{2} & 0 & \frac{3}{10} & 0 \\ 0 & \frac{-2}{5} & 0 & \frac{17}{35} \end{bmatrix}, H = \begin{bmatrix} 0 \\ e^{-1}-1 \\ e^{-1}-1 \\ e^{-1}-1 \end{bmatrix}.$$

From the above, we get

$$[F; H] = \begin{bmatrix} 0 & 1 & \dfrac{-5}{2} & 1 & ; & e^{-1}-1 \\ 0 & \frac{5}{3} & \dfrac{-7}{6} & \frac{-59}{27} & ; & e^{-1}-1 \\ 0 & \frac{7}{3} & \frac{5}{6} & \frac{-85}{27} & ; & e^{-1}-1 \\ 0 & 3 & \frac{7}{2} & -1 & ; & e^{-1}-1 \end{bmatrix}.$$

The initial condition is represented by the following matrix

$$[E_0; \eta_0] = [1 \ -1 \quad 0 \quad 1].$$

The updated augmented matrix related to this condition is

$$\left[\widetilde{F}; \widetilde{H}\right] = \begin{bmatrix} 0 & 1 & \frac{-5}{2} & 1 & ; & 0 \\ 0 & \frac{5}{3} & \frac{-7}{6} & \frac{-59}{27} & ; & e^{-1}-1 \\ 0 & \frac{7}{3} & \frac{5}{6} & \frac{-85}{27} & ; & e^{-1}-1 \\ 1 & -1 & 0 & 1 & ; & 1 \end{bmatrix}.$$

To determine the Genocchi coefficients, we solve the linear system provided above

$$\varpi = \begin{bmatrix} 0.683 \\ -0.347 \\ 0.101 \\ -0.030 \end{bmatrix}.$$

Therefore, the approximate solution is

$$\begin{aligned}\phi_4(\tau) &= \sum_{n=1}^{4} \omega_n D_n(\tau) = \omega_1 D_1(\tau) + \omega_2 D_2(\tau) + \omega_3 D_3(\tau) + \omega_4 D_4(\tau) \\ &= 1 - \tau + 0.48\tau^2 - 0.12\tau^3.\end{aligned}$$

It can be noted that this approximate solution converges to the exact solution $e^{-\tau}$.

Chapter 4

Numerical examples and results

To demonstrate the effectiveness and efficiency of current method described in the previous chapter, we solved several different examples using the Matlab program. The results are displayed in tables and figures, which include exact solutions, approximate solutions, and absolut errors. Furthermore, we have compared these results with some existing methods.

4.1 Approximate solutions of linear Fredholm integro-differential equations using Genocchi polynomials

4.1.1 Example 1

Consider the following equation [10, 12, 13, 20, 40]

$$\begin{cases} \phi'(\tau) - \int_0^1 \tau\phi(\zeta)d\zeta = \tau e^{\tau} + e^{\tau} - \tau, \qquad \phi(0) = 0, \\ \phi(\tau) = \tau e^{\tau}. \end{cases}$$

Table 1 and Figure 1 present the exact and approximate solutions obtained by our method for Example 1. These results are compared with some existing methods [10, 12, 13, 20, 40] in the Table 2.

Figure 2 shows the absolute errors produced by our method (for $N = 4$, 6, 8, and 10). The results indicate that the errors decrease as N increases.

Table 1. Exact and approximate solutions for $N = 8$ and 10 using Genocchi polynomials.

τ	Exact solution	Approximate solution ($N = 8$)	Approximate solution ($N = 10$)
0.1	0.1105170918	0.1105170700	0.1105170918
0.2	0.2442805516	0.2442805290	0.2442805516
0.3	0.4049576423	0.4049576219	0.4049576422
0.4	0.5967298791	0.5967298539	0.5967298790
0.5	0.8243606354	0.8243606093	0.8243606353
0.6	1.0932712802	1.0932712529	1.0932712802
0.7	1.4096268952	1.4096268587	1.4096268952
0.8	1.7804327428	1.7804327156	1.7804327428
0.9	2.2136428000	2.2136427575	2.2136428000

Table 2. Comparison absolute errors of the current method and some numerical methods.

τ	HP.M [12]	MVI.M [13]	Hermit W.M [10]	TSBPF.M [40]	Bernoullin [20]	Current.M
0.1	$0.2e-05$	$3.0e-09$	$4.9e-10$	$3.2e-07$	$1.5e-10$	$5.7e-11$
0.2	$0.9e-05$	$1.0e-09$	$4.2e-10$	$2.0e-07$	$6.0e-10$	$4.9e-11$
0.3	$0.2e-04$	$2.8e-09$	$4.6e-10$	/	$1.4e-9$	$4.7e-11$
0.4	$0.4e-05$	$7.3e-09$	$4.7e-10$	$4.1e-07$	$2.4e-9$	$4.2e-11$
0.5	$0.6e-04$	$9.1e-09$	$4.9e-10$	/	$3.9e-9$	$3.8e-11$
0.6	$0.8e-04$	$1.0e-08$	$5.2e-10$	$1.4e-07$	$6.5e-9$	$3.5e-11$
0.7	$0.1e-03$	$1.1e-07$	$5.5e-10$	/	$1.3e-8$	$3.3e-11$
0.8	$0.2e-03$	$1.7e-07$	$5.9e-10$	$1.0e-06$	$3.5e-8$	$3.8e-11$
0.9	$0.2e-03$	$1.6e-07$	$6.2e-10$	$3.4e-06$	$1.06e-7$	$7.9e-12$

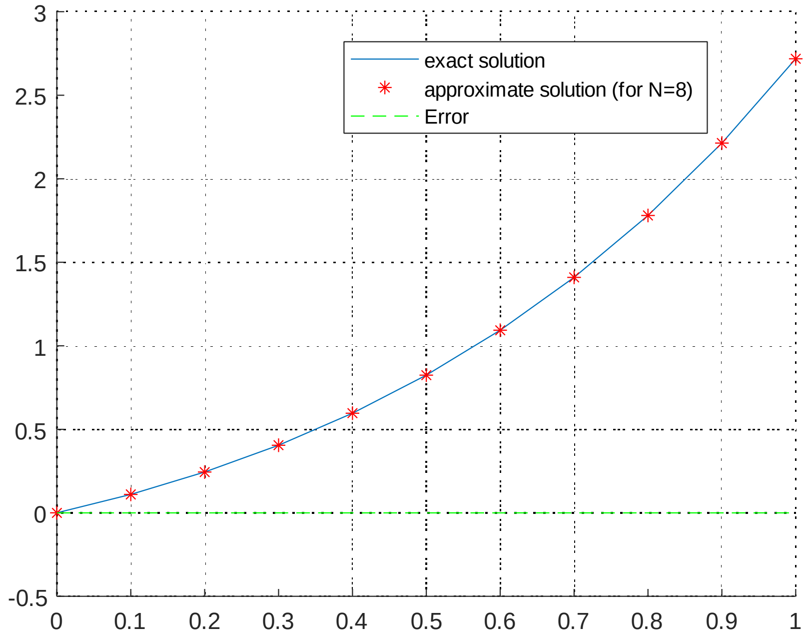

Figure 1. Solutions and absolute errors for $N = 8$ using Genocchi polynomials.

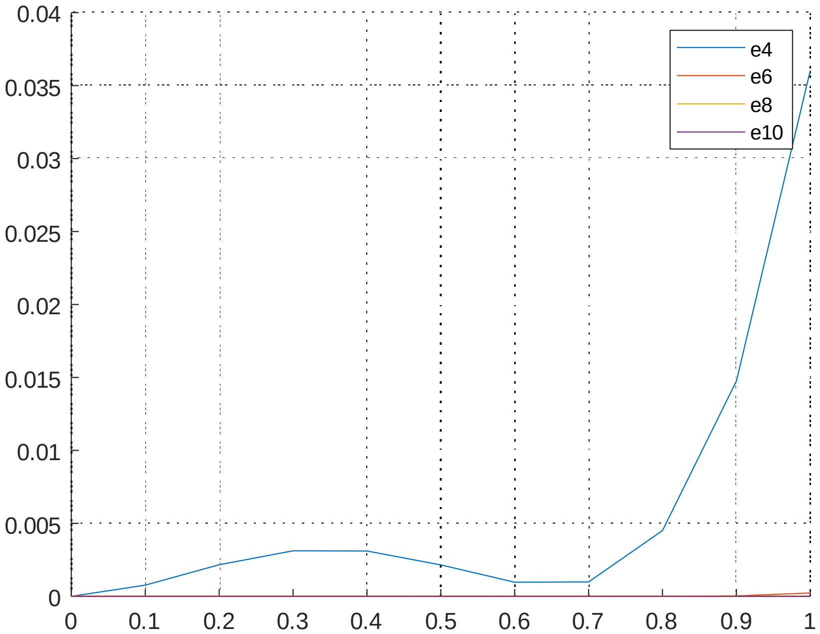

Figure 2. Absolute errors for $N = 4$, 6, 8, and 10 using Genocchi polynomials.

4.1.2 Example 2

Consider the following equation [15]

$$\begin{cases} \phi''(\tau) + \tau\ \phi'(\tau) + \pi^2\phi(\tau) - \int_0^1 (\tau+\zeta)\phi(\zeta)d\zeta = \pi\tau\cos(\pi\tau)\ -\dfrac{2\tau+1}{\pi}, \\ \phi(0) = \phi(1) = 0,\ \phi(\tau) = \sin(\pi\tau). \end{cases}$$

Table 3 and Figure 4 displays the absolute errors obtained by the current method for Example 2. It is observed that the errors decrease as N increases. These results are compared with those from the exponential spline method [15] in Table 4.

Table 3. Absolute errors for $N = 10, 12,$ and 14 using Genocchi polynomials.

τ	Absolute error $(N = 10)$	Absolute error $(N = 12)$	Absolute error $(N = 14)$
0.125	$8.35e-05$	$9.89e-07$	$2.09e-07$
0.250	$1.55e-04$	$1.83e-06$	$1.53e-07$
0375	$2.02e-04$	$2.39e-06$	$7.39e-08$
0.500	$2.19e-04$	$2.59e-06$	$1.63e-08$
0.625	$2.05e-04$	$2.44e-06$	$1.04e-07$
0.750	$1.65e-04$	$1.95e-06$	$1.77e-07$
0.825	$1.06e-04$	$1.27e-06$	$2.23e-07$

Table 4. Comparison of the absolute errors of our method for $N = 14$ with ES.M.

τ	Genocchi polynomials [25]	ES.M [15]
0.1250	$2.1e-07$	$1.4e-06$
0.2500	$1.5e-07$	$2.5e-06$
0.3750	$7.4e-08$	$3.3e-06$
0.5000	$1.6e-08$	$3.7e-06$
0.6250	$1.0e-07$	$3.6e-06$
0.7500	$1.8e-07$	$3.0e-06$
0.8250	$2.2e-07$	$1.8e-06$

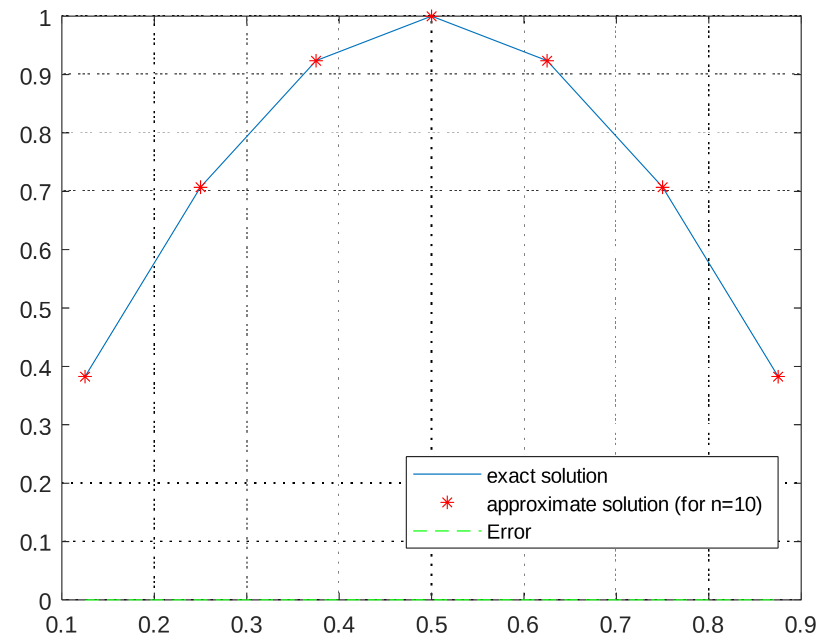

Figure 3. Solutions and absolute errors for $N = 10$ using Genocchi polynomials.

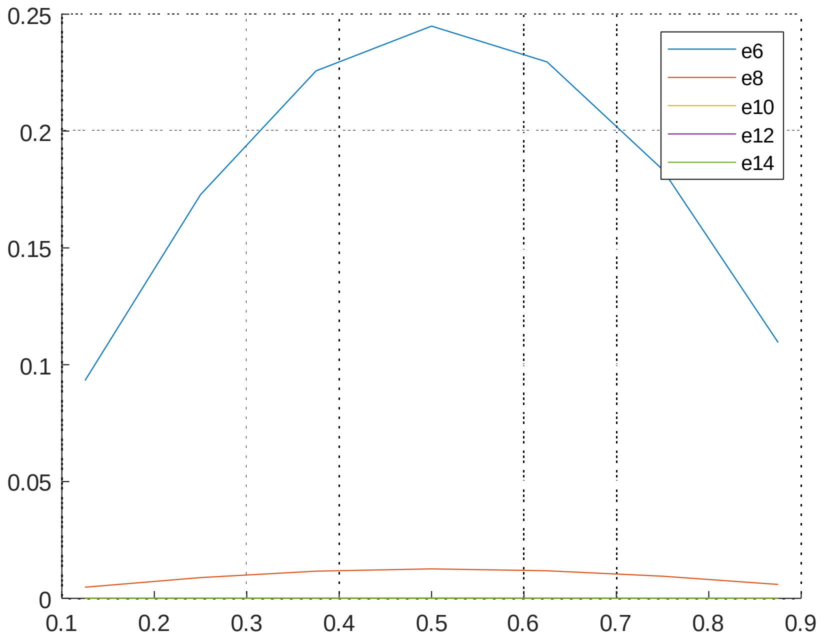

Figure 4. Absolute errors for $N = 4$, 8, 10, 12, and 14 using Genocchi polynomials

4.1.3 Example 3

Consider the following equation [11, 14, 16, 18]

$$\begin{cases} \phi'(\tau) - \int_0^1 \tau\zeta\phi(\zeta)d\zeta = 1 - \frac{1}{3}\tau, \quad \phi(0) = 0, \\ \phi(\tau) = \tau. \end{cases}$$

Table 5 presents a comparison of the absolute errors obtained by the current method with those from existing methods [11, 14, 16, 18] for Example 3. The results show that using Genocchi polynomials provides higher accuracy than the methods in [11, 14, 16, 18]. Solutions and absolute error for $N = 6$ are displayed in Figure 5.

Table 5. Comparison absolute errors of the current method and some numerical methods.

τ	DT.M [14]	CAS-W.M [11]	SB.M [18]	Haar W.B.M [16]	Genocchi polynomials ($N = 2$)
0.1	$1.6e-03$	$2.2e-04$	$3.8e-06$	$1.6e-06$	0
0.2	$6.1e-03$	$6.3e-04$	$1.5e-05$	$2.4e-06$	0
0.3	$1.3e-02$	$7.9e-04$	$3.4e-05$	$2.3e-06$	0
0.4	$2.3e-02$	$2.2e-02$	$6.1e-05$	$1.3e-06$	0
0.5	$3.5e-02$	$5.0e-02$	$9.5e-05$	$4.9e-07$	0
0.6	$6.7e-02$	$2.2e-02$	$1.4e-05$	$9.3e-07$	0
0.7	$7.1e-02$	$1.1e-04$	$1.9e-05$	$1.5e-06$	0
0.8	$8.6e-02$	$1.4e-03$	$2.4e-04$	$1.9e-06$	0
0.9	$1.1e-01$	$2.1e-02$	$3.1e-04$	$5.4e-05$	0

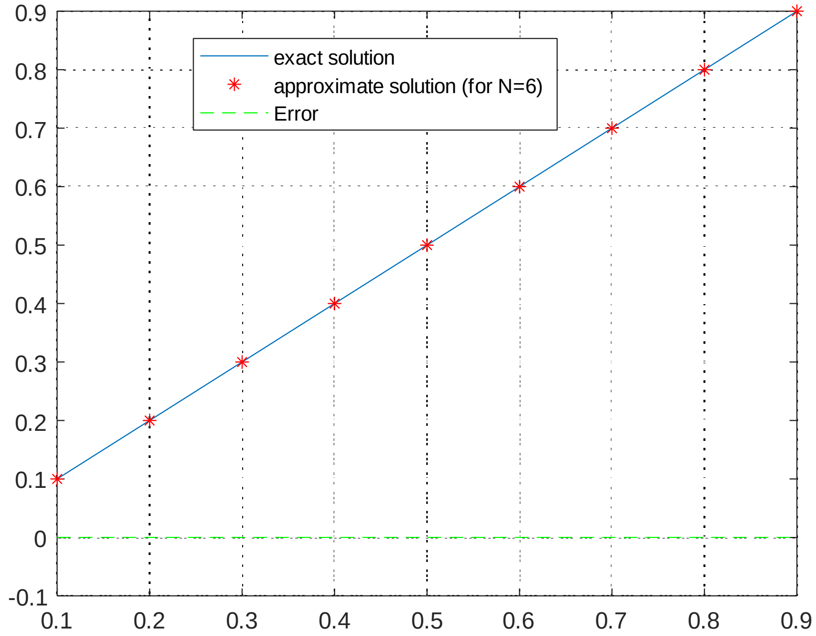

Figure 5. Solutions and absolute errors for $N = 6$ using Genocchi polynomials.

4.1.4 Example 4

Consider the following equation

$$\begin{cases} \phi'(\tau) + \phi(\tau) - \int_0^1 \phi(\zeta)d\zeta = e^{-\tau} - 1, \quad \phi(0) = 1, \\ \phi(\tau) = e^{-\tau}. \end{cases}$$

The exact solution and numerical results obtained by the current method for Example 4 (with $N = 6$, 8, and 10) are presented in Table 6 and Figure 6. The corresponding absolute errors are listed in Table 7, indicating that the errors decrease as N increases.

Table 6. Exact and approximate solutions for $N = 6$, 8, and 10 using Genocchi polynomials.

τ	Exact solution	Approximate. S ($N = 6$)	Approximate. S ($N = 8$)	Approximate. S ($N = 10$)
0	1.00000000000	1.00000000000	0.99999999999	1.00000000004
0.1	0.90483741804	0.90483665130	0.90483741643	0.90483741806
0.2	0.81873075308	0.81872943586	0.81873075106	0.81873075310
0.3	0.74081822068	0.74081685198	0.74081821844	0.74081822069
0.4	0.67032004604	0.67031859904	0.67032004323	0.67032004603
0.5	0.60653065971	0.60652882943	0.60653065657	0.60653065970
0.6	0.54881163609	0.54880940057	0.54881163261	0.54881163607
0.7	0.49658530379	0.49658312686	0.49658529962	0.49658530376
0.8	0.44932896412	0.44932703167	0.44932896014	0.44932896408
0.9	0.40656965974	0.40656559931	0.40656965487	0.40656965970
1	0.36787944117	0.36786402702	0.36787940466	0.36787944108

Table 7. Absolute errors for $N = 6$, 8, and 10, using Genocchi polynomials.

τ	Absolute errors ($N = 6$)	Absolute errors ($N = 8$)	Absolute errors ($N = 10$)
0	$3.3e-17$	$4.9e-14$	$3.5e-11$
0.1	$7.8e-07$	$1.6e-09$	$2.8e-11$
0.2	$1.3e-06$	$2.0e-09$	$2.0e-11$
0.3	$1.4e-06$	$2.2e-09$	$9.7e-12$
0.4	$1.5e-06$	$2.8e-09$	$2.0e-12$
0.5	$1.8e-06$	$3.1e-09$	$1.4e-11$
0.6	$2.2e-06$	$3.5e-09$	$2.5e-11$
0.7	$2.2e-06$	$4.1e-09$	$3.4e-11$
0.8	$1.9e-06$	$3.9e-09$	$4.0e-11$
0.9	$4.1e-06$	$4.9e-09$	$4.1e-11$
1	$1.5e-05$	$3.7e-08$	$9.1e-11$

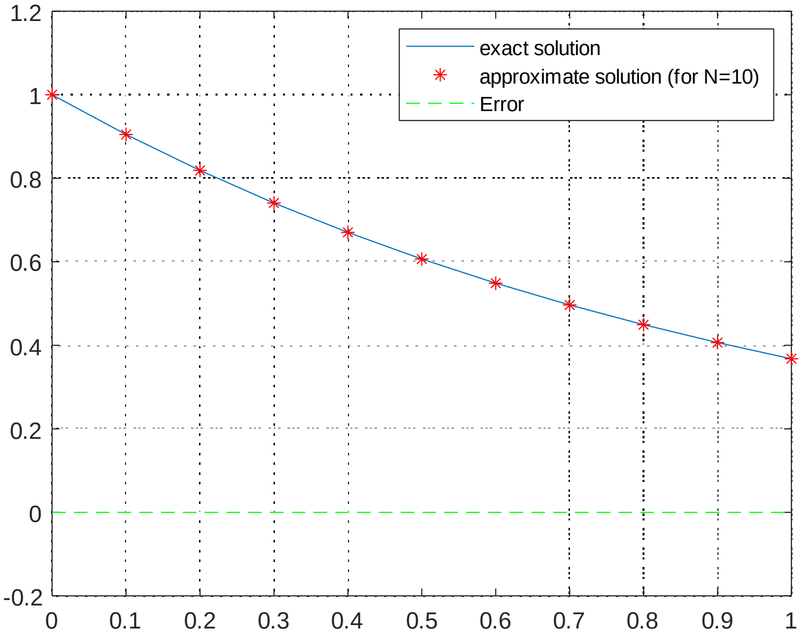

Figure 6. Solutions and absolute errors for $N = 10$ using Genocchi polynomials.

4.1.5 Example 5

Consider the following equation

$$\begin{cases} \phi^{(4)}(\tau) + \phi(\tau) - \int_0^1 (\tau - \zeta)\phi(\zeta)d\zeta = \frac{1}{4} + (1 - 2\ln(2))\tau - \dfrac{6}{(1+\tau)^4}, \\ \phi(\tau) = \ln(\tau + 1). \end{cases}$$

Under the conditions $\phi(0) = \phi^{'}(0) = \phi^{''}(0) = \phi^{'''}(0) = 2$.

Table 8 and Figure 7 present the exact solution, approximate solution, and absolute error obtained using Genocchi polynomials for Example (4.1.5) with $N = 10$. Table 9 provides a comparison of the absolute errors produced by the current method (for $N = 8$) with those from the Power and Chebyshev series [41] for Example 5. The results show that using Genocchi polynomials provides higher accuracy than the approach in [41].

Table 8. Solutions and absolute errors for $N = 10$ using Genocchi polynomials.

τ	Exact solution	Approximate soltion $(N = 10)$	Absolute error $(N = 10)$
0.1	0.095310179	0.095310174	$5.796007e - 09$
0.2	0.182321556	0.182321464	$9.214901e - 08$
0.3	0.262364264	0.262363882	$3.822260e - 07$
0.4	0.336472236	0.336471245	$9.908818e - 07$
0.5	0.405465108	0.405463070	$2.037972e - 06$
0.6	0.470003629	0.469999988	$3.640505e - 06$
0.7	0.530628251	0.530622385	$5.865151e - 06$
0.8	0.587786664	0.587779848	$6.816787e - 06$
0.9	0.641853886	0.641862108	$8.222474e - 06$

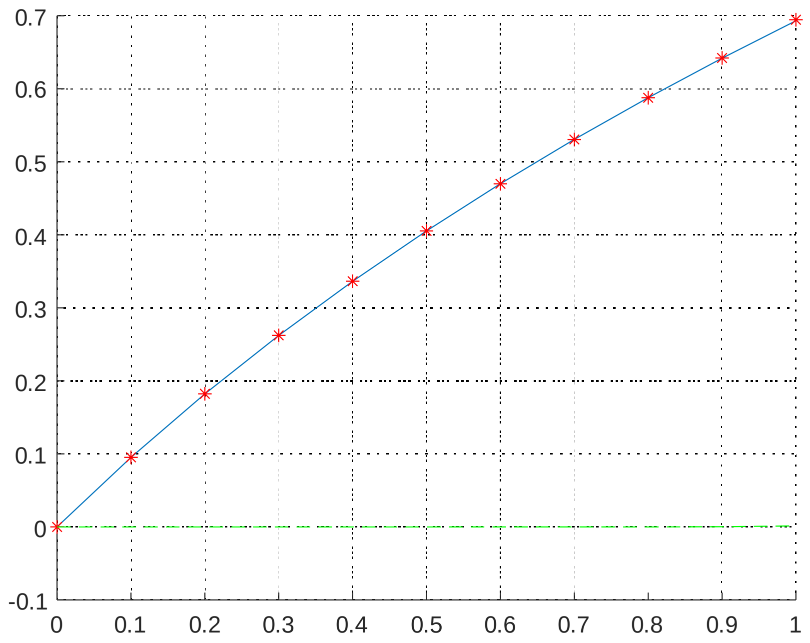

Figure 7. Solutions and absolute errors for $N = 10$ using Genocchi polynomials.

Table 9. Absolute errors for $N = 8$ using Genocchi polynomials and [41].

τ	Power series ($N = 10$)	Chebychev series [41]	Genocchi series ($N = 8$)
0.1	$2.100e-07$	$3.805e-04$	$5.939e-08$
0.2	$1.365e-06$	$3.688e-04$	$1.136e-06$
0.3	$2.265e-05$	$3.449e-04$	$5.185e-06$
0.4	$2.334e-05$	$3.423e-04$	$1.376e-05$
0.5	$2.712e-05$	$4.249e-03$	$2.871e-05$
0.6	$2.837e-05$	$2.495e-03$	$5.123e-05$
0.7	$3.2834e-05$	$2.134e-03$	$6.849e-05$
0.8	$1.781e-04$	$1.553e-03$	$1.324e-05$
0.9	$1.695e-04$	$1.487e-03$	$3.205e-04$

Conclusion

In this thesis, we introduced a new collocation method based on the Genocchi truncated series to solve high-order LFIDEs. The examples and results presented in the last chapter demonstrate the efficiency and effectiveness of the current method, highlighting its superiority over existing methods such as CAS wavelet, Haar wavelet bases, Homotopy perturbation, Schauder bases, Bernoulli polynomials, Power and Chebyshev series,.... Moreover, the method is easy to use and quick to implement using MATLAB, owing to the advantages of Genocchi polynomials, which require fewer terms and smaller coefficients compared to other polynomials. We can conclude that the proposed technique is an excellent mathematical tool for solving such equations. In future research, we plan to extend the application of Genocchi polynomials to more complex problems.

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ملخص:

تلعب المعادلات التكاملية التفاضلية الخطية دورا أساسيا في نمذجة مجموعة واسعة من الظواهر في مجالات مختلفة مثل الهندسة، البيولوجيا والفيزياء، حيث أصبح تطوير الطرق العددية ضروريا لإيجاد حلول تقريبية سريعة ودقيقة لهذه المعادلات.

في هذه الأطروحة نقدم العديد من هذه الطرق حيث نطرح نهجا عدديا جديدا يعتمد على كثيرات حدود جينوتشي.

تتضمن هذه التقنية تحويل المعادلة التكاملية التفاضلية الخطية لفريدهولم الى معادلة مصفوفة، والتي يتم حلها بعد ذلك لتحديد معاملات جينوتشي المجهولة. مع تقديم أمثلة عددية لتوضيح فعالية ودقة هذا الأسلوب.

الكلمات المفتاحية: المعادلات التكاملية التفاضلية الخطية لفريدهولم، المعادلات التكاملية التفاضلية الخطية لفولترا، طريقة التجميع، كثيرات حدود جينوتشي، تحليل الخطأ.

Abstract:

Linear integro-differential equations (IDEs) play a crucial role in modeling a wide range of phenomena across different fields such as engineering, biology, and physic. For this purpose, the development of numerical methods has become necessary to find fast and accurate approximate solutions to such equations. In this thesis we present several of these methods and introduce a new numerical approach based on Genocchi polynomials, this technique involves transforming the LIDE into a matrix equation, which is then solved to determine the unknown Genocchi coefficients. To illustrate the effectiveness and accuracy of numerical methods, we provide several numerical examples.

Keywords: Linear Fredholm integro-differential equations, Linear Volterra integro-differential equations, collocation method, Genocchi polynomials, error analysis.

Résumé :

Les équations intégro-différentielles linéaire (EsIDL) jouent un rôle crucial dans la modélisation d'un large éventail de phénomènes dans différente domaines tels que l'ingénierie, la biologie et la physique.

Pour cela, le développement de méthodes numériques est devenu nécessaire pour trouver des solutions approchées rapides et précises à ces équations. Dans cette thèse, nous présentons plusieurs de ces méthodes et introduisons une nouvelle approche numérique basée sur les polynômes de Genocchi. Cette technique consiste à transformer l'EslDLF en une équation matricielle, qui est ensuite résolue pour déterminer les coefficients de Genocchi inconnus. Pour illustrer l'efficacité et la précision de cette méthode, nous fournissons plusieurs exemples numériques.

Mots-clés: Equations intégro-différentielles linéaires de Fredholm, équations intégro-différentielles linéaire de Volterra, méthode de collocation, polynôme de Genocchi, analyse d'erreur.