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A study of Deformable and Conformable Laplace Transform and its applications

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Dedications

I dedicate this modest memoir.
To my dearest parents.
To my sisters, my brothers, And to all who encouraged and supported me to reach this level of study.
I may this work be the fulfillment of your much-alleged wishes, and the result of your unfailing support.
Thank you for always being there for me.

Contents

Notations

- Γ: Denotes Gamma function.
- β: Denotes Beta function.
- $\mathcal{I}$: Fractional integral in the sense of Riemann-Liouville.
- $\mathcal{D}$: Fractional derivatives in the Riemann-Liouville sense.
- ${}^{C}\mathcal{D}$: Fractional derivatives in the sense of Caputo.
- $\mathcal{T}_n(x)$: The Chebyshev Polynomials.
- $\mathcal{P}_n(x)$: The Legendre Polynomials.
- C_L: Denotes functions that are causal, piecewise continuous, and of exponential order.
- $F(s)$: Denotes Laplace transform of f.
- DLT: Deformable Laplace Transform.
- DILT: Deformable Inverse Laplace Transform .
- $\langle .,. \rangle$: Scalar product.
- ∂: The partial derivative operator.
- Δ: The laplace operator.
- ∇: The gradient operator .
- For $a \in \mathbb{R}^n$, $\mid x \mid$: Denotes the Euclidean norm in $\mathbb{R}^n$.
- If $f : \mathbb{R}^n \to \mathbb{C}$, the support of f is denoted by $suppf$.
- $\mathcal{D}(\mathbb{R}^n)$: Is the space of functions $\mathcal{C}^\infty(\mathbb{R}^n)$ with compact support, $\mathcal{D}'(\mathbb{R}^n)$ is the dual space of $\mathcal{D}(\mathbb{R}^n)$, is called also the space of distributions on $\mathbb{R}^n$.
- p': Is the conjugate exponent of p, $\frac{1}{p} + \frac{1}{p'} = 1$ where $p \in [1, +\infty[$.
- $L^p(\mathbb{R}^n)$: Is the space of measurable functions f on $\mathbb{R}^n$ such that

$$\parallel f \parallel_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} \mid f(x) \mid^p dx)^{\frac{1}{p}} < \infty.$$

Introduction

On 30 Sep. 1695, the fractional calculus was brought up as a question by Hopital, in a letter to Leibniz. On that day, Hopital made an inquiry about the result of $\frac{d^n x}{dx^n}$ in case $n = \frac{1}{2}$ Leibniz responded that it would be "an apparent paradox, from which one day useful consequences will be drawn". Since then, fractional calculus is the main field of interest to mathematicians. Many famous mathematicians participated to the theory of the fractional calculus over the past years, among them Liouville, Riemann, Abel, Weyl, Fourier, Lacroix, Leibniz and Letnikov, Grunwald. In 1819, Lacroix was the first mathematician published a paper mentioning a fractional derivative. It is noteworthy that the most popular definition in the fractional calculus is the Riemann-Liouville (R-L) definition. However, the fractional calculus is considered unpopular in comparison with the ordinary calculus. One likely reason is the presence of various nonequivalent definitions of fractional derivatives such as: R-L fractional derivative, Riesz derivative, Caputo derivative, and Grunwald-Letnikov fractional derivative. Nowadays, an increasing number of researchers are paying attentions to the fractional calculus, since they found that the fractional order derivatives and fractional order integrals are more fitting for the description of many physical phenomena in the real world. For example, the nonlinear oscillation of earthquake can be modeled better with fractional derivatives, also modeling the fluid-dynamic by fractional derivatives can eliminate the deficiency emerging from the occurrence of continuum traffic flow.

Orthogonal polynomials constitute a vast branch of mathematics, theoretical physics, and mathematical physics. These polynomials have been a subject of study for a long time. In the early $19th$ century, more precisely in 1939, orthogonal polynomials received their first detailed study by [12]. Sequences of orthogonal polynomials emerged as solution to equations in mathematical physics, particularly partial differential equations. Among the most important classical orthogonal polynomials, we mention the Jacobi, Chebyshev, Legendre, and Hermite polynomials,... Recently, the deformable derivative and its properties has been introduced. In this work, we have investigated the concept of Deformable Laplace Transform (DLT) in more detail. Also, some classical properties of the DLT are included. Furthermore, we use this transform to find exact solutions to differential equations of arbitrary order. Also, we study the Conformable Laplace Transform of the Fractional chebyshev and Legendre Polynomials.

Our dissertation is organized into three chapters:

- In the first chapter, we have discussed some basic theories of fractional calculation. We give definitions of some special function (Gamma, Beta), then we present fractional integrals and derivatives fractional (in the sense of Riemann-Liouville and Caputo), many of the properties of these concepts will be used in the next chapters.
- In the second chapter, we give a definition of the Laplace transform in terms of the used(deformable fractional derivative), where we use this transformation to find exact solutions to arbitrary order differential equations.
- In the last chapter, we will propose the generating functions for the fractional Chebyshev and Legendre polynomials, we take a look over the definition of the conformable fractional Laplace transform of order α and try to investigate some of its properties.
- Finally, we general conclusions and properties are drawn.

CHAPTER 1

PRELIMINARIES ON FRACTIONAL CALCULATION

In this chapter, we have discussed some basic theories of fractional calculation. We give definitions of some special function (Gamma,Beta), then we present fractional integrals and derivatives fractional (in the sense of Riemann-Liouville and Caputo).

1.1 specials functions

1.1.1 Gamma function

Definition 1.1. [3] The Gamma function is called the function defined by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt \quad then\ z \in \mathbb{C}\ and\ Re(z) > 0.$$

properties[3] We have the following properties

1. $\Gamma(z+1) = z\Gamma(z)$.
2. $\Gamma(n+1) = (n)!\ n \in \mathbb{N}$.
3. $\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$,$n \in \mathbb{N}$

Remark 1.1. [3] Determination of the Gamma function for negative values not integer by the formula $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ and the transition from one interval to another $(-1,0), (-2,-1)$.....
The Gamma function does not exist for integer negative value.

Example 1.1. [3] Let $z = \frac{1}{2}$ to calculate $\Gamma(\frac{1}{2})$, we use a change of variable, we pose that $t = \tau^2$, we obtain

$$\begin{aligned}\Gamma(\frac{1}{2}) &= \int_0^{+\infty} t^{\frac{1}{2}-1} e^{-t}\, dt \\ &= \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t}\, dt \\ &=2 \int_0^{+\infty} e^{-\tau^2}\, d\tau \quad \text{(from the Gauss integral)} \\ &=2\left(\frac{\sqrt{\pi}}{2}\right) \\ &=\sqrt{\pi}\end{aligned}$$

1.1.2 Beta Function

Definition 1.2. [3] Beta function is a type of Euler integral defined for all complexes z and w by

$$\beta(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt \quad , Re(z) > 0 \; and \; Re(w) > 0.$$

Proposition 1.1. *[3] The relation between the Gamma and Beta function given by for all $z, w \in \mathbb{C}$ with $Re(z) > 0; Re(w) > 0$, we have*

$$\beta(z,w) = \frac{\Gamma(z)\cdot\Gamma(w)}{\Gamma(z+w)}.$$

Proof *Let $D =]0,+\infty[\times]0,+\infty[$, we have*

$$\begin{aligned}\Gamma(z)\Gamma(w) &= \left(\int_0^{+\infty} x^{z-1}e^{-x}\, dx\right)\left(\int_0^{+\infty} y^{w-1}e^{-y}\, dy\right) \\ &= \int_0^{+\infty}\int_0^{+\infty} x^{z-1}y^{w-1}e^{-(x+y)}\, dx\, dy.\end{aligned}$$

Using the following change of variable

$$\begin{cases} u = x+y \\ v = \frac{x}{x+y} \end{cases} \Longrightarrow \begin{cases} x = uv \\ y = u(1-v) \end{cases}$$

and

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ (1-v) & -u \end{vmatrix} = -uv - u(1-v) = -u.$$

Similarly, the domain $D^{'}$ corresponding to D in the coordinates u, v is

$$D' = \{(u, v)/u \geq 0, 0 \leq v \leq 1\}$$

So

$$\begin{aligned}\iint_D x^{z-1} y^{w-1} e^{-(x+y)}\, dx\, dy =& \iint_{D'} (uv)^{z-1} (u(1-v))^{w-1} e^{-u} |-u|\, du\, dv \\ =& \iint_{D'} (u)^{z+w-1} v^{z-1} (1-v)^{w-1} e^{-u}\, du\, dv \\ =& \int_0^{+\infty} \int_0^1 (u)^{z+w-1} v^{z-1} (1-v)^{w-1} e^{-u}\, du\, dv \\ =& \left(\int_0^{+\infty} (u)^{z+w-1} e^{-u}\, du \right) \left(\int_0^1 v^{z-1} (1-v)^{w-1}\, dv \right) \\ =& \Gamma(z+w) \cdot \beta(z, w).\end{aligned}$$

Therefore, we have

$$\beta(z, w) = \frac{\Gamma(z) \cdot \Gamma(w)}{\Gamma(z+w)}$$

Proposition 1.2. *[3]*

1. $\beta(z, w) = \beta(w, z)$,*(symmetrical).*
2. $\beta(z, 1) = \frac{1}{z}$.

1.2 Fractional integral in the sense of Riemann-Liouville

Let be $\Omega = [a, b]$ with $(-\infty < a < b < +\infty)$ a finite interval on $\mathbb{R}$ and $f \in L^1([a, b])$ an integrable function on Ω, we have.

$$\mathcal{I}^1_{a^+} = \int_a^x f(t)\, dt.$$

The iteration of $(\mathcal{I}^1_{a^+} f)$ allows us to obtain the primitive of f with vanishes at a and whose derivative vanishes at a, in addition, we have

$$\begin{aligned}\mathcal{I}^2_{a^+} f(x) =& \mathcal{I}^1_{a^+} (\mathcal{I}^1_{a^+} f(x)) \\ =& \int_a^x \mathcal{I}^1_{a^+} f(t)\, dt \\ =& \int_a^x \left(\int_a^t f(s)\, ds \right) dt\end{aligned}$$

We put that $g(t)=\int_a^t f(s)\,ds$, from the integral by part. We have

$$\begin{aligned}\mathcal{I}_{a^+}^2 f(x) &= \left[t\int_a^t f(s)\,ds\right]_a^x - \int_a^x tf(t)\,dt\\ &= x\int_a^x f(s)\,ds - \int_a^x tf(t)\,dt\\ &= x\int_a^x f(t)\,dt - \int_a^x tf(t)\,dt\\ &= \int_a^x (x-t)\,f(t)\,dt\end{aligned}$$

Therefore by n^{th} iteration, we get

$$\mathcal{I}_{a^+}^n f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt$$

this formula is called Cauchy formula, and according to the property of Gamma $\Gamma(n)=(n-1)!$,$\forall n\in\mathbb{N}^*$, we have

$$\mathcal{I}_{a^+}^n f(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)\,dt$$

Definition 1.3. [3] Let be $\Omega=[a,b]$ with $(-\infty<a<b<+\infty)$ an interval finished on $\mathbb{R}$, and $f\in L^1([a,b])$ an integrable function on $\mathbb{R}$

The integrals

$$\mathcal{I}_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt \quad ,x>a, Re(\alpha)>0. \tag{1.1}$$

$$\mathcal{I}_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\,dt \quad ,x<b, Re(\alpha)>0. \tag{1.2}$$

Went the left(right)fractional Riemann-Liouville order
$\alpha\in\mathbb{C}$ $(Re(\alpha)>0)$ respectively.

Example 1.2. [3] Let $f(x)=(x-a)^{\beta-1}$, We have

$$\mathcal{I}_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\,dt$$

With the change of variable $t=a+s(x-a)$, we have

$$\begin{cases} t=a \Longleftrightarrow s=0\\ t=x \Longleftrightarrow s=1\\ dt=(x-a)ds\end{cases}$$

So, with the definition of the Beta function, we get

$$\begin{aligned}\mathcal{I}_{a^+}^{\alpha} f(x) =& \frac{1}{\Gamma(\alpha)}(x-a)^{\alpha+\beta-1}\int_0^1 s^{\beta-1}(1-s)^{\alpha-1}\,ds\\ =& \frac{B(\beta,\alpha)}{\Gamma(\alpha)}(x-a)^{\alpha+\beta-1}\end{aligned}$$

Using gets Proposition 1.1, we obtain

$$\mathcal{I}_{a^+}^{\alpha} f(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\alpha+\beta-1}$$

Remark 1.2. [3] The integral of constant function in the Riemann-Liouville sense of order $\alpha > 0$ is given by

$$\mathcal{I}_{a^+}^{\alpha} f(x) = \frac{C}{\Gamma(\alpha+1)}(x-a)^{\alpha} \quad and \quad \mathcal{I}_{b^-}^{\alpha} f(x) = \frac{C}{\Gamma(\alpha+1)}(b-x)^{\alpha}$$

$f(x) = C \in \mathbb{R}$

Proposition 1.3. *[3] Let be $f \in \mathbb{C}([a,b])$, and $Re(\alpha) > 0$, $Re(\beta) > 0$, the Riemann-Liouville integrals have the following properties*

1. $\mathcal{I}_{a^+}^{\alpha}\mathcal{I}_{a^+}^{\beta} f(x) = \mathcal{I}_{a^+}^{\beta}\mathcal{I}_{a^+}^{\alpha} f(x) = \mathcal{I}_{a^+}^{\alpha+\beta} f(x)$.
2. $\mathcal{I}_{b^-}^{\alpha}\mathcal{I}_{b^-}^{\beta} f(x) = \mathcal{I}_{b^-}^{\beta}\mathcal{I}_{b^-}^{\alpha} f(x) = \mathcal{I}_{b^-}^{\alpha+\beta} f(x)$.

1.3 Fractional derivatives in the Riemann-Liouville sense

Definition 1.4. [3] Let $f \in L^1([a,b])$ an integral function on $[a,b]$. The fractional derivatives in the Riemann-Liouville sense of $\mathcal{D}_{a^+}^{\alpha} f(x)$ and $\mathcal{D}_{b^-}^{\alpha} f(x)$ orderly $\alpha \in \mathbb{C}$,$Re(\alpha) > 0$, Are defined by

$$\begin{aligned}\mathcal{D}_{a^+}^{\alpha} f(x) =& \left(\frac{d}{dx}\right)^n (\mathcal{I}_{a^+}^{n-\alpha} f(x))\\ =& \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\,dt, n = [Re(\alpha)]+1, x > a.\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\mathcal{D}_{b^-}^{\alpha} f(x) =& \left(-\frac{d}{dx}\right)^n (\mathcal{I}_{b^-}^{n-\alpha} f(x))\\ =& \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}}\,dt, n = [Re(\alpha)]+1, x < b.\end{aligned} \tag{1.4}$$

Respectively $[Re(\alpha)]$ is the integer part of $Re(\alpha)$.

properties[3] Let be $\alpha, \beta \in \mathbb{C}$ with $Re(\alpha) > 0$, $Re(\beta) > 0$ and $a, b \in \mathbb{R}$. We have

1. $\mathcal{D}^{\alpha}_{a^+}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}$.
2. $\mathcal{D}^{\alpha}_{b^-}(b-x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1}$.

properties[3]

1. All these derivatives coincide with the usual derivatives for integer orders

$$\forall m \in \mathbb{N}^*, \begin{cases} \mathcal{D}^{m}_{a^+}f(x) = f^{(m)}(x) \\ \mathcal{D}^{m}_{b^-}f(x) = (-1)^m f^{(m)}(x) \end{cases}$$

2. If $0 < \alpha < 1$,So $n = [\alpha] + 1$. Then (1.3) and (1.4) becomes

$$\mathcal{D}^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}\,dt, x > a.$$

$$\mathcal{D}^{\alpha}_{b^-}f(x) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)}{(t-x)^{\alpha}}\,dt, x < b.$$

3. If $\beta = 1$ and $0 < Re(\alpha) < 1$, then the Riemann-Liouville derivative of a constant function in general is not zero

$$\mathcal{D}^{\alpha}_{a^+}1 = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \quad and \quad \mathcal{D}^{\alpha}_{b^-}1 = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)}$$

4. For all $j = 1, 2, ..., n = [Re(\alpha)] + 1$ with $Re(\alpha) \geq 0$, we have

$$\mathcal{D}^{\alpha}_{a^+}(x-a)^{\alpha-j} = \mathcal{D}^{\alpha}_{b^-}(b-x)^{\alpha-j} = 0$$

1.4 Fractional derivatives in the sense of Caputo

Definition 1.5. [3] Let be $\alpha \in \mathbb{C}$ with $n = [Re(\alpha)] + 1$ and $f\ [a,b] \longrightarrow \mathbb{C}$ a function such that $f^{(n)} \in L^1([a,b])$. The fractional derivatives of order α of f in the Caputo sense are defined by

$$^{C}\mathcal{D}^{\alpha}_{a^+}f(x) = \mathcal{I}^{n-\alpha}_{a^+}f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,dt \tag{1.5}$$

and

$$^{C}\mathcal{D}^{\alpha}_{b^-}f(x) = (-1)^n\mathcal{I}^{n-\alpha}_{b^-}f^{(n)}(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}\,dt \tag{1.6}$$

Proposition 1.4. *[3] Let* $\alpha > 0$,$\beta 0$,$n = [\alpha] + 1$, *we have the following properties iff*$(t) \in \mathbb{C}^m([a, b])$, *so*

$$(\mathcal{I}^{\alpha}_{a^+} {}^{C}\mathcal{D}^{\alpha}_{a^+} f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(f)^k(a)}{k!}(t-a)^k$$

and

$$(\mathcal{I}^{\alpha}_{b^-} {}^{C}\mathcal{D}^{\alpha}_{b^-} f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^{n-k}(f)^k(b)}{k!}(b-t)^k$$

from particular if $0 < \alpha \leq 1$

$$(\mathcal{I}^{\alpha}_{a^+} {}^{C}\mathcal{D}^{\alpha}_{a^+} f)(t) = f(t) - f(a).$$

$$(\mathcal{I}^{\alpha}_{b^-} {}^{C}\mathcal{D}^{\alpha}_{b^-} f)(t) = f(t) - f(b).$$

Proposition 1.5. *[3] The relation between the derivatives in the sense of Caputo* $(1.5), (1.6)$ *and the derivatives in the sense of Riemann-Liouville* $(1.3), (1.4)$ *are given by*

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+} f(x) = \mathcal{D}^{\alpha}_{a^+} \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right) \tag{1.7}$$

and

$$ {}^{C}\mathcal{D}^{\alpha}_{b^-} f(x) = \mathcal{D}^{\alpha}_{b^-} \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-x)^k \right) \tag{1.8}$$

We note that if $f^{(k)}(a) = 0$ *therefore* $k = 0, 1, ..., n-1$ *,So*

$$ {}^{C}\mathcal{D}^{\alpha}_{a^+} f(x) = \mathcal{D}^{\alpha}_{a^+} f(x) \quad and \quad {}^{C}D^{\alpha}_{b^-} f(x) = \mathcal{D}^{\alpha}_{b^-} f(x)$$

1.5 The Chebyshev Polynomials

The number of subjects in which chebyshev polynomials take an important value cannot be underestimated. These polynomials are widely used in numerical analysis including: least-squares approximation. polynomial approximation, numerical integration, and spectral methods for partial differential equations. There are several kinds of Chebyshev polynomials. In this thesis the researchers only consider the first kind of the Chebyshev polynomials which is a solution of the equation below

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0 \tag{1.9}$$

Chebyshev polynomials of the first are denoted by $T_n(x)$ and considered the most important of the Chebyshev polynomials.

Figure 1 shows the first few Chebyshev polynomials $T_n(x)$.

Definition 1.6. [1](The Chebyshev polynomials $T_n(x)$)
The Chebyshev polynomial of the first kind $T_n(x)$ is a polynomial in x of degree n, defined by

$$T_n(x) = \cos(n\theta)\ when\ x = \cos(\theta) \tag{1.10}$$

If $x \in [-1, 1]$ then $\theta \in [0, \pi]$. The ranges are traversed in opposite directions, since $x = 1$ corresponds to $\theta = 0$, And $x = -1$ corresponds to $\theta = \pi$.
Because of the close relationship of $T_n(x)$ with the trigonometric function $'cosine'$, we can conclude the first few Chebychev polynomials as follows
$\cos((0)\theta) = 1 \longrightarrow T_0(x) = 1$
$\cos((1)\theta) = \cos(\theta) \longrightarrow T_1(x) = x$
$\cos((2)\theta) = \cos(2\theta) = 2\cos^2(\theta) - 1 \longrightarrow T_2(x) = 2x^2 - 1$
$\cos((3)\theta) = \cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta) \longrightarrow T_3(x) = 4x^3 - 3x$
In practice it not convenient to work out each $T_n(x)$. So by making use of the relation of the trigonometric functions and their sum identities with $T_n(x)$, We can derive a Recurrence Formula for $T_n(x)$.
Consider:

$$T_{n+1}(x) = \cos((n+1)\theta) = \cos(n\theta + \theta) = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)$$

$$T_{n-1}(x) = \cos((n-1)\theta) = \cos(n\theta - \theta) = \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta)$$

Adding the above two equations, we get

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad , n = 1, 2, 3... \tag{1.11}$$

With the initial condition $T_0(x) = 1$ and $T_1(x) = x$, is called a Recurrence Formula for $T_n(x)$.
We can deduce from (1.11) that the leading coefficient in $T_n(x)$ is equal to 2^{n-1} for any $n > 1$.
The first five Chebyshev polynomials $T_n(x)$ are listed in Table 1. The Chebyshev polynomials $T_n(x)$ can also be defined in terms of the sums as follows

$$T_n(x) = \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!} \cdot (2x)^{n-2k} \ , n > 0 \tag{1.12}$$

Where $\left[\frac{n}{2}\right]$ is the floor function denoting largest integer less than $\frac{n}{2}$

n	$T_n(x)$
0	1
1	x
2	$2x^2-1$
3	$4x^3-3x$
4	$8x^4-8x^2-1$
5	$16x^5-20x^3+5x$

Table 1.1: The Chebyshev polynomials $T_n(x)$

1.5.1 The Legendre Polynomials

The Legendre polynomials arise mostly in problems involving spheres or spherical coordinates. In the spherical polar coordinates, the Legendre polynomials accomplish a major function, since the angular dependence is best handled by spherical harmonics that are defined in terms of Legendre polynomials. That's, why Legendre polynomials are also called spherical harmonics. There are two kinds of Legendre polynomials. In this thesis the researchers only consider the first kind of Legendre polynomials which is a solution of the equation below

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \tag{1.13}$$

Legendre polynomials of the first are denoted by $P_n(x)$.

Definition 1.7. [1](The Legendre polynomials $P_n(x)$)

The Legendre polynomials of the first kind $P_n(x)$ is a polynomials in x of degree n, defined by the Rodrigues 'formula'

$$P_n(x)=\frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}[(x^2-1)^n] \tag{1.14}$$

for arbitrary real (or complex) values of the variable x.

The Legendre Polynomials $P_n(x)$ can also be defined in terms of the sums as follows

$$P_n(x)=\frac{1}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}\cdot x^{n-2k}, n=0,1,2,... \tag{1.15}$$

The series representation of $P_n(x)$ is as important as the series representation of $T_n(x)$ in our thesis. It can be derived by using the Rodrigues 'formula' (1.14) as shown below.

Consider the Rodrigues 'formula' (1.14). Then apply the binomial theorem to $[(x^2-1)^n]$, So

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}\left[\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot (x^2)^{n-k}(-1)^k\right]$$

Since we have a power series, we can insert $\frac{d^n}{dx^n}$ into the series

$$\begin{aligned}
P_n(x) =& \frac{1}{2^n \cdot n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{d^n}{dx^n}[(x^2)^{n-k}](-1)^k \\
=& \frac{1}{2^n}\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \cdot \frac{d^n}{dx^n}[x^{2n-2k}] \\
=& \frac{1}{2^n}\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \cdot \frac{(2n-2k)!}{(n-2k)!}x^{n-2k} \\
=& \frac{1}{2^n}\sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} \cdot \frac{(2n-2k)!}{n!(n-2k)!}x^{n-2k} \\
=& \frac{1}{2^n}\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2n-2k}{n} x^{n-2k}
\end{aligned}$$

Since $2n-2k$ in the combination should be greater than or equal to n, we have $0 \le k \le \left[\frac{n}{2}\right]$, So

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k}\binom{2n-2k}{n} x^{n-2k}$$

The Legendre polynomials $P_n(x)$ can also be defined recursively using Recurrence Formulas. These formulas are particularly helpful for computer evaluation of Legendre polynomials of higher order and their derivatives. We will use the Generating Function of the Legendre polynomials in order to obtain some Recurrence Formulas. The Generating Function of $P_n(x)$ is

$$\sum_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}} \tag{1.10}$$

(i) Consider (1.10),and differentiate it with respect to t, we get

$$\begin{aligned}
n\sum_{n=0}^{\infty} t^{n-1} P_n(x) =& -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) \\
=& (1-2xt+t^2)^{-\frac{3}{2}}(x-t)
\end{aligned}$$

Multiply both sides by $(1-2xt+t^2)$, we get

$$\begin{aligned}(1-2xt+t^2)n\sum_{n=0}^{\infty}t^{n-1}P_n(x) =&(1-2xt+t^2)^{-\frac{1}{2}}(x-t)\\ =&\sum_{n=0}^{\infty}t^nP_n(x)(x-t)\end{aligned}$$

Let each t be raised to the power of n
$(n+1)\sum_{n=-1}^{\infty}t^nP_{n+1}(x)-2xn\sum_{n=0}^{\infty}t^nP_n(x)+(n-1)\sum_{n=1}^{\infty}t^nP_{n-1}(x)$

$$=x\sum_{n=0}^{\infty}t^nP_n(x)-\sum_{n=1}^{\infty}t^nP_{n-1}(x)$$

Now equating the coefficient of t^n, we get

$$(n+1)P_{n+1}(x)-2xnP_n(x)+(n-1)P_{n-1}(x)=xP_n(x)-P_{n-1}(x)$$

Combining like terms

$$(n+1)P_{n+1}(x)=(1+2n)xP_n(x)-nP_{n-1}(x) \tag{1.8}$$

(1.6) is a Recurrence Formula for the Legendre polynomials.
(ii) Consider(1.10), And differentiate it with respect to x, we get

$$\begin{aligned}\sum_{n=0}^{\infty}t^nP_n^{'}(x)=&-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2t)\\ =&(1-2xt+t^2)^{-\frac{3}{2}}(t)\end{aligned}$$

Multiply both sides by $(t-x)$, we get

$$(t-x)\sum_{n=0}^{\infty}t^nP_n^{'}(x)=(t-x)(1-2xt+t^2)^{-\frac{3}{2}}(t)$$

According to (i), we have

$$(t-x)\sum_{n=0}^{\infty}t^nP_n^{'}(x)=-t\sum_{n=0}^{\infty}nt^{n-1}P_n(x)$$

Let each t be raised to the power n.Then equating the coefficient of t^n, we get

$$nP_n(x)=xP_n^{'}(x)-P_{n-1}^{'}(x) \tag{1.6}$$

Which is another Recurrence Formula for $P_n(x)$ including the derivatives.
The Legendre polynomials $P_n(x)$ from an orthogonal set over the interval $[-1, 1]$,with respect to a weight function equals 1. Specifically this orthogonal set satisfies the Legendre differential equation (1.9) which is sometimes useful to be written in the following form

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

$$< P_n, P_m >= \int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & , n \neq m \\ \frac{2}{2n+1} & , n = m \end{cases}$$

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$

Table 1.2: The Legendre polynomials $P_n(x)$

CHAPTER 2

DEFORMABLE LAPLACE TRANSFORM AND ITS APPLICATIONS

In this chapter, we give a definition of the Laplace transform in terms of the used(deformable fractional derivative), we use this transformation in order to find exact solutions to several deformable fractional differential equations.

2.1 function C_L

The class of real functions C_L is formed by piecewise continuous causal functions of exponential order, a function is causal if it is zero for $x < 0$, $f(x) = 0$. If $x < 0$. It continues piecewise if it only admits points of discontinuities of first space (admitting a limit on the left and a limit on the right). it is of exponential order if it is bounded by an exponential .i.e if there exist real constants, $M \geq 0$ and α such that

$$|f(x)| \leq Me^{\alpha x}, \forall x \geq x_0.$$

2.1.1 definition of Laplace transform

Definition 2.1. The Laplace transform of a function of C_L is defined by

$$\mathcal{L}(f(x))(s) = F(s) = \int_0^{+\infty} e^{-sx} f(x)dx$$

s is a complex variable (frequency) and $F(s)$ a complex function.

Remark 2.1.

1. F is defined by an improper integral which does not always converge if $f \notin C_L$.
2. if f is discontinuous at 0, the lower bound of the integral should be noted 0^+.

Example 2.1. Calculate the Laplace transform of the function $f(x) = H(x)e^{2x}$.
We have

$$f(x) = \begin{cases} e^{2x}, x \geq 0 \\ 0, x < 0 \end{cases}$$

$$\begin{aligned} F(s) = \int_0^{+\infty} e^{-sx} f(x)dx &= \int_0^{+\infty} e^{-sx} e^{2x} dx \\ &= \int_0^{+\infty} e^{(2-s)x} dx = \frac{1}{2-s} \left[e^{(2-s)x}\right]_0^{+\infty} \end{aligned}$$

Let $s = \alpha + i\beta$, Then
$e^{(2-s)x} = e^{(2-\alpha)x} e^{-i\beta x}$,So
$\lim_{x\to+\infty} |e^{(2-s)x}| = \lim_{x\to+\infty} e^{(2-\alpha)x} = 0$ if $\alpha > 2$, therefor
$\lim_{x\to+\infty} e^{(2-s)x} = 0$ $F(s) = \frac{1}{2-s}\left[e^{(2-s)x}\right]_0^{+\infty} = \frac{1}{2-s}(0-1) = \frac{1}{s-2}$ Then, $F(s) = \frac{1}{s-2}$,$Re(s) > 2$

Example 2.2.

$$H(x) = \begin{cases} 1, x \geq 0 \\ 0, x < 0 \end{cases}$$

$$F(s) = \int_0^{+\infty} e^{-sx} H(x)dx = \int_0^{+\infty} e^{-sx} dx \tag{2.1}$$

$$= \frac{-1}{s}\left[e^{-sx}\right]_0^{+\infty}. \tag{2.2}$$

Let $s = \alpha + i\beta$, then
$e^{-sx} = e^{-\alpha x} e^{-i\beta x}$,So
$\lim_{x\to+\infty} |e^{-sx}| = \lim_{x\to+\infty} e^{-\alpha x} = 0$ if $\alpha > 0$,i.e:$\lim_{x\to+\infty} e^{-sx} = 0$,if$R(s) > 0$
then $F(s) = \frac{-1}{s}(0-1) = \frac{1}{s}$,if $Re(s) > 0$
i.e.$\mathcal{L}(H(s)) = \frac{1}{s}, Re(s) > 0.$

Theorem 2.1. *Let f a function define on $\mathbb{R}$ such that*

(i) $f(x) = 0, \forall x < 0$

(ii) *f is piecewise continuous from* $[0, +\infty[$.

(iii) *there exist a constants $M \geq 0$ such that $\forall x \geq x_0, |f(x)| \leq Me^{rx}$ then the Laplace transform of f exist, for all $Re(s) > r$.*

Proof we have

$$\int_0^{+\infty} e^{-sx} f(x)dx = \int_0^{x_0} e^{-sx} f(x)dx + \int_{x_0}^{+\infty} f(x)dx$$

, the integral
$\int_0^{x_0} e^{-sx}f(x)dx$ exists because f is piecewise continuous.
Concerning the other integral, note that
$|e^{-sx}f(x)| = |e^{-(\alpha+i\beta)x}f(x)| = e^{-\alpha x}|f(x)| \leq Me^{-(\alpha-r)x}$
or $\int_0^{+\infty} Me^{-(\alpha-r)x}dx$ is converged because $Re(s) = \alpha > r$. So
according to the criterion of comparison of generalized integrals, the integrals
$\int_{x_0}^{+\infty} |e^{-sx}f(x)|dx$ is converged
Also, we result $\int_{x_0}^{+\infty} e^{-sx}f(x)dx$ exists for subsequent
$\int_0^{+\infty} e^{-sx}f(x)dx$ is exists in the half-plan $\{s \in \mathbb{C}, s > r\}$.

2.2 some Basic definition and Tools

2.2.1 deformable derivative

Definition 2.2. [6] Given a real-valued function $h(t)$ defined on interval (a, b), then for arbitrary order α deformable derivative of a function h is defined as

$$\mathcal{D}^{\alpha}h(t) = \lim_{\epsilon \to 0} \frac{(1+\epsilon\beta)h(t+\epsilon\alpha) - h(t)}{\epsilon} \tag{2.3}$$

Where $\alpha + \beta = 1$ if the aforementioned limit exists $\forall\alpha$ such that $0 \leq \alpha \leq 1$, then deformable derivative of the real-valued function $h(t)$ is denoted by the symbol $\mathcal{D}^{\alpha}h(x)$.

Remark 2.2. [6] in this definition $(2, 2)$ we note

1. for $\alpha = 0 \Longrightarrow \mathcal{D}^0 h(t) = h(t)$
2. for $\alpha = 1 \Longrightarrow \mathcal{D}^1 h(t) = \frac{dh}{dt} = h^{'}(t)$

The deformable derivative can be considered as α derivative as well.
The following theorem are useful of the aforementioned definition .

Theorem 2.2. *[6] Let h be α-differentiable at a point t_0 for some α then h is continuous at t_0.*
it can be easily shown that operators $\mathcal{D}^{\alpha}$ has the following properties.

Theorem 2.3. *[6] Let h_1 and h_2 be α-differentiable at t. Then*

1. *Linearity: $\mathcal{D}^{\alpha}(ah_1 + bh_2) = a\mathcal{D}^{\alpha}h_1 + b\mathcal{D}^{\alpha}h_2$,$\forall a, b \in \mathbb{R}$.*
2. *Commutativity: $\mathcal{D}^{\alpha_1} \cdot \mathcal{D}^{\alpha_2} = \mathcal{D}^{\alpha_2} \cdot \mathcal{D}^{\alpha_1}$.*
3. *$\mathcal{D}^{\alpha}(s) = \beta s$,$\forall$ constant functions $h(t) = s$.*
4. *$\mathcal{D}^{\alpha}(h_1 \cdot h_2) = (\mathcal{D}^{\alpha}h_1) \cdot h_2 + \alpha h_1 \mathcal{D}h_2$,hence $\mathcal{D}^{\alpha}$ does not follow the Leibniz rule.*

5. $\mathcal{D}^{\alpha}h(t) = \beta h(t) + \alpha \mathcal{D}h(t)$.

Following are the arbitrary order deformable derivative of well-known elementary functions
$\mathcal{D}^{\alpha}(t^k) = \beta t^k + k\alpha t^{k-1}, k \in \mathbb{R}$.
$\mathcal{D}^{\alpha}(e^t) = \beta e^t + \alpha e^t$.
$\mathcal{D}^{\alpha}(\sin wt) = \beta \sin wt + \alpha w \cos wt$.

2.2.2 Deformable integral

This section introduced deformable integral,the inverse operator for deformable integral, the inverse operator for deformable derivative. Besides discussing some basic properties of this deformable fractional integral, we list out deformable fractional integral of some elementary functions
Also, We introduced deformable from exponential function .

Definition 2.3. [6] Let h be a continuous function defined on $[a, b]$, we define α-fractional integral of h.
Denoted by $\mathcal{I}_a^{\alpha}h$, by the integral

$$\mathcal{I}_a^{\alpha}h(t) = \int_a^t e^{-\frac{\beta}{\alpha}(t-x)}h(x)d_{\alpha}x \tag{2.4}$$

Where $\alpha + \beta = 1, \alpha \in]0, 1]$ and $d_{\alpha}x = \frac{1}{\alpha}dx$.
Next theorem explains some basic properties of this fractional integral.

Theorem 2.4. *[6] The operator $\mathcal{I}_a^{\alpha}$ possesses the following properties*

1. *Linearity: $\mathcal{I}_a^{\alpha}(bh_1 + ch_2) = b\mathcal{I}_a^{\alpha}h_1 + c\mathcal{I}_a^{\alpha}h_2$.*
2. *Commutativity: $\mathcal{I}_a^{\alpha_1}\mathcal{I}_a^{\alpha_2} = \mathcal{I}_a^{\alpha_2}\mathcal{I}_a^{\alpha_1}$*
   Where $\alpha_i + \beta_i = 1, i = 1, 2$

Following are the functional integral of some well-known elementary functions

Proposition 2.1. *[6]*

1. $\mathcal{I}_a^{\alpha} \sin t = \frac{1}{\alpha^2+\beta^2}(\beta \sin t - \alpha \cos t + e^{\frac{\beta}{\alpha}(a-t)}(\alpha \cos a - \beta \sin a))$.
2. $\mathcal{I}_a^{\alpha} e^t = (e^t - e^{\frac{(a-\beta t)}{\alpha}})$.
3. $\mathcal{I}_a^{\alpha}\lambda = \frac{\lambda}{\beta}(1 - e^{\frac{\beta}{\alpha}(a-t)})$, *Where λ is a constant.*
4. $\mathcal{I}_0^{\alpha} t^m = \frac{1}{\beta}\left(\sum_{r=0}^{m} \frac{(-1)^r(m)!}{(m-r)!}(\frac{\alpha}{\beta})^r t^{m-r} + (-1)^{m+1}m!(\frac{\alpha}{\beta})^m e^{-\frac{\beta}{\alpha}t}\right)$.

Furthermore the deformable exponential function is defined as follows

Definition 2.4. [6]**(Deformable exponential function)** For some point $s, t \in \mathbb{R}$ with $s \leq t$, the exponential function with respect to $\mathcal{D}^\alpha$ in $(2,1)$ is defined as:

$$e_\alpha(t,s) = e^{-\int_s^t \beta d_\alpha u}. \tag{2.5}$$

Where $\alpha + \beta = 1$,$\alpha \in]0,1]$and $d_\alpha u = \frac{1}{\alpha} du$.

Theorem 2.5. *[6] A differentiable function f at a point $t \in (a,b)$ is always α-differentiable at that point for any α.*
Moreover, We have $\mathcal{D}^\alpha f(t) = \beta f(t) + \alpha \mathcal{D} f(t)$
Where $\alpha + \beta = 1$ and $\mathcal{D} = \frac{d}{dt}$ is the usual derivative .

Theorem 2.6. *[6]Let f be differentiable at a point t for some α, then it is continuous there.*

Theorem 2.7. *Let f be defined in (a,b), for any α, f is α-differentiable if and only if it is differentiable. The operators $\mathcal{D}_\alpha$ and $\mathcal{I}_a^\alpha$ possess the following properties.*

Theorem 2.8. *[6] (Taylor's theorem) Suppose f is n-times α-differentiable and such that all α-derivatives are continuous on $[a, a+h]$.*
Then, $f(a+h) = \sum_{k=0}^{n-1} \frac{h^k}{k! a^k} \left[\mathcal{D}_k^\alpha f(a) - \beta \frac{(1-\theta)^{(k-n+1)} h}{\alpha n} \mathcal{D}_k^\alpha f(a+\theta h) \right] + \frac{h^n}{n! \alpha} \mathcal{D}_n^\alpha f(a+\theta h)$,
Where $\mathcal{D}_k^\alpha = \mathcal{D}^\alpha \mathcal{D}^\alpha ... \mathcal{D}^\alpha$ (k times) and $0 < \theta < 1$.
We state and prove the following result.

Theorem 2.9. *[6] The operator $\mathcal{D}^\alpha$ possesses also the following property*

$$\mathcal{D}^\alpha(fg^{-1}) = \frac{g\mathcal{D}^\alpha(f) - \alpha f}{g^2}.$$

Proof We have

$$\begin{aligned}
\mathcal{D}^\alpha(fg^{-1}) =& \beta(fg^{-1}) + \alpha \mathcal{D}(fg^{-1}) \\
=& \beta(fg^{-1}) + \alpha[\mathcal{D} f g^{-1} + f \mathcal{D} g^{-1}] \\
=& \beta f g^{-1} + \alpha \mathcal{D} f g^{-1} + \alpha f \mathcal{D} g^{-1} \\
=& [\beta f + \alpha \mathcal{D} f] g^{-1} + \alpha f \mathcal{D} g^{-1} \\
=& \frac{g\mathcal{D}^\alpha(f) - \alpha f}{g^2}.
\end{aligned}$$

The proof is complete.

Theorem 2.10. *[6] Suppose f and g are α-differentiable then*

$$\mathcal{D}^\alpha(f \circ g)(t) = \beta(f \circ g)(t) + \alpha \mathcal{D}(f \circ g)(t) = \beta(f \circ g)(t) + \alpha f^{'}(g(t)) g^{'}(t)$$

proof Since

$$\mathcal{D}^{\alpha} f(t)=\lim_{\epsilon \longrightarrow 0} \frac{(1+\epsilon \beta) f(t+\epsilon \alpha)-f(t)}{\epsilon}=\lim_{\epsilon \longrightarrow 0}\left[\frac{f(t+\epsilon \alpha)-f(t)}{\epsilon}+\beta f(t+\epsilon \alpha)\right] \tag{2.6}$$

We have

$$\begin{aligned}
\mathcal{D}^{\alpha} f(g(t)) &=\lim_{\epsilon \longrightarrow 0}\left[\frac{f(g(t+\epsilon \alpha))-f(g(t))}{\epsilon}+\beta f(g(t+\epsilon \alpha))\right] \\
&=\lim_{\epsilon \longrightarrow 0}\left[\frac{f(g(t+\epsilon \alpha))-f(g(t))}{g(t+\epsilon \alpha)-g(t)} \cdot \frac{g(t+\epsilon \alpha)-g(t)}{\epsilon}+\beta f(g(t+\epsilon \alpha))\right] \\
&=\lim_{\epsilon \longrightarrow 0}\left[\frac{f(g(t)+\epsilon_0)-f(g(t))}{\epsilon_0} \cdot \frac{g(t+\epsilon \alpha)-g(t)}{\epsilon}+\beta f(g(t+\epsilon \alpha))\right]
\end{aligned}$$

Where $\epsilon_0 \longrightarrow 0$ as $\epsilon \longrightarrow 0$. We obtain

$$\begin{aligned}
\mathcal{D}^{\alpha}(f g(t)) = & \lim_{\epsilon_0 \longrightarrow 0} \frac{f(g(t)+\epsilon_0)-f(g(t))}{\epsilon_0} \cdot \lim_{\epsilon \longrightarrow 0} \frac{g(t+\epsilon \alpha)-g(t)}{\epsilon}+\lim_{\epsilon \longrightarrow 0} \beta f(g(t+\epsilon \alpha)) \\
& =f^{'}(g(t)) \alpha g^{'}(t)+\beta f(g(t))=\alpha \mathcal{D}[f(g(t))]+\beta f(g(t)) \\
& =\beta f(g(t))+\alpha \mathcal{D}[f(g(t))].
\end{aligned}$$

the proof is now complete.

Theorem 2.11. *[6] Let f be continuous on $[a, b]$. Then $\mathcal{I}_a^{\alpha} f$ is α-differentiable in (a, b)*
And we have
$\mathcal{D}^{\alpha}(\mathcal{I}_a^{\alpha} f)(t)=f(t)$
$\mathcal{I}_a^{\alpha}(\mathcal{D}^{\alpha} f)(t)=f(t)-e^{\frac{\beta}{\alpha}(a-t)} f(a)$.

2.3 Exact Solution Of deformable fractional equation using the Laplace transform

2.3.1 Deformable Laplace transform

We introduce the Laplace transform and the deformable Laplace transform (DLT)in very natural way.We also discuss existence theorem of DLT.

Definition 2.5. [6] if f is a function (locally integrable). Defined on $\mathbb{R}_{+}$,with values in $\mathbb{C}$, we call the **classical Laplace transform** of f the function

$$\mathcal{L}(f)(s)=\int_0^{+\infty} e^{-s t} f(t)\, d t=\lim_{x \longrightarrow+\infty} \int_0^x e^{-s t} f(t)\, d t.$$

For the values of s for which this integral converges.

Definition 2.6. [10] Let $f\ [0,+\infty[\longrightarrow \mathbb{R}$.For $\alpha \in]0,1]$, We define DLT of order α denoted by $L^\alpha f$, by the integral

$$L^\alpha f(t) = F^\alpha(p) = \int_0^{+\infty} e^{-st} f(t) e_\alpha(t,0) d_\alpha t. \tag{2.7}$$

Where $\alpha + \beta = 1$, $p = s + \frac{\beta}{\alpha}$ and $d_\alpha t = \frac{1}{\alpha} dt$

Theorem 2.12. *[10] A continuous function f is of exponential order a, there exists a positive real number C , s.t. $|f(t)| \leq Ce^{at}$.*
Where $a,\ t \geq 0$, then DLT of f exists for $s > a$.
Proof

$$\begin{aligned} |L^\alpha f(t)| =& \left| \int_0^{+\infty} e^{-st} f(t) e_\alpha(t,0) d_\alpha t \right| \\ \leq& \int_0^{+\infty} \left| e^{-st} f(t) e_\alpha(t,0) \right| d_\alpha t \\ \leq& \int_0^{+\infty} e^{-st} \left| f(t) \right| e_\alpha(t,0) d_\alpha t \\ \leq& \int_0^{+\infty} e^{-st} C e^{at} e_\alpha(t,0) d_\alpha t \ \ , s > a \\ =& \frac{C}{\alpha} \left(\frac{1}{p-a} \right). \end{aligned} \tag{2.8}$$

Remark 2.3. [6] It should however be kept in mind that the aforementioned condition is sufficient but not necessary.
For example ,$L^\alpha\left(\frac{1}{\sqrt{t}}\right)$ exists though $\frac{1}{\sqrt{t}}$ is discontinuous at $t = 0$.
Now, We list out DLT of some certain function in the following proposition

Proposition 2.2. *[6] this table*

	The function	Deformable Laplace Transform
1.	1	$\frac{1}{\alpha}\left(\frac{1}{p}\right)$
2.	t^m	$\frac{1}{\alpha}\left(\frac{m!}{p^{m+1}}\right), p > 0, m = 0,1,2,...$
3.	e^{at}	$\frac{1}{\alpha}\left(\frac{1}{p-a}\right), p > a.$
4.	$\sin at$	$\frac{1}{\alpha}\left(\frac{a}{p^2+a^2}\right)$
5.	$\sinh at$	$\frac{1}{\alpha}\left(\frac{a}{p^2-a^2}\right)$
6.	$\cos at$	$\frac{1}{\alpha}\left(\frac{a}{p^2+a^2}\right)$
7.	$\cosh at$	$\frac{1}{\alpha}\left(\frac{a}{p^2-a^2}\right)$

Table 2.1: Deformable Laplace Transform

2.3.2 Basic properties of deformable Laplace transform

A part from discussing fundamental properties of DLT like linearity and change of scale property, the section deals with fundamental theorems, first shifting and second shifting property.

Theorem 2.13. *[6] The operator L^α possesses the following properties*

1. *Linearity* $L^\alpha(af+bg) = aL^\alpha f + aL^\alpha g$
2. *First translation or shifting property if* $L^\alpha g(t) = G^\alpha(p)$
   Then $L^\alpha(e^{at}g(t)) = G^\alpha(p-a)$.
3. *Second translation or shifting property* $L^\alpha g(t) = G^\alpha(p)$ *and*

$$F(t) = \begin{cases} g(t-a), t > a \\ 0, t < a \end{cases}$$

   Then,$L^\alpha F(t) = e^{-ap}G^\alpha(p)$
4. *Change of scale property if* $L^\alpha g(t) = G^\alpha(p)$
   Then $L^\alpha g(at) = \frac{1}{a}G^\alpha\left(\frac{p}{a}\right)$
5. *DLT of successive derivative if* $L^\alpha g(t) = G^\alpha(p)$, *then*
   $L^\alpha(\underbrace{D^\alpha D^\alpha ... D^\alpha}_{n\ times})g(t) = (\alpha p+\beta)^n G^\alpha(p) - (\alpha p+\beta)^{n-1}g(0) - (\alpha p+\beta)^{n-2}\mathcal{D}^\alpha g(0) - (\alpha p+\beta)^{n-3}$
   $\mathcal{D}^\alpha\mathcal{D}^\alpha g(0) - ... - (\underbrace{\mathcal{D}^\alpha\mathcal{D}^\alpha ... \mathcal{D}^\alpha}_{n\text{-}1\ times})g(0)$
6. *DLT of* $tg(t)$
   If $L^\alpha g(t) = G^\alpha(p)$
   Then $L^\alpha(t^n g(t)) = (-1)^n \frac{d^n}{dp^n}G^\alpha(p)$
7. *DLT of* $\frac{g(t)}{t}$
   if $L^\alpha(g(t)) = G^\alpha(p)$
   Then $L^\alpha(\frac{g}{t}) = \int_p^{+\infty} G^\alpha(p)dp$.

Proof Linearity is obvious from definition, parts $(2), (4), (8)$ can be established easily, for part(3)
We have

$$\begin{aligned} L^{\alpha}F(t) =& \int_0^{+\infty} e^{-st}F(t)e_a(t,0)d_{\alpha}t \\ =& \frac{1}{\alpha}\left(\int_0^{a} e^{-pt}F(t)dt + \int_a^{+\infty} e^{-pt}F(t)dt\right) \\ =& \frac{1}{\alpha}\left(0 + \int_a^{+\infty} e^{-pt}g(t-a)dt\right) \\ =& \frac{1}{\alpha}\int_a^{+\infty} e^{-pt}g(t-a)dt \\ =& e^{-ap}\frac{1}{\alpha}\int_0^{+\infty} e^{-pu}g(u)du \\ =& e^{-ap}G^{\alpha}(p). \end{aligned}$$

To prove part(5), we have

$$\begin{aligned} L^{\alpha}(\mathcal{D}^{\alpha}g(t)) =& \int_0^{+\infty} e^{-st}\mathcal{D}^{\alpha}(g(t))e_{\alpha}(t,0)d_{\alpha}t \\ =& \frac{1}{\alpha}\int_0^{+\infty} e^{-st}[\beta g + \alpha g]e_{\alpha}(t,0)dt \end{aligned} \tag{2.9}$$

Therefore,

$$L^{\alpha}(\mathcal{D}^{\alpha}g(t)) = (\alpha p + \beta)G^{\alpha}(p) - g(0). \tag{2.10}$$

Now replacing g by $\mathcal{D}^{\alpha}g$ and $\mathcal{D}^{\alpha}g$ by $\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}g$ in (5) , we obtain

$$L^{\alpha}(\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}g(t)) = (\alpha p + \beta)^2G^{\alpha}(p) - (\alpha p + \beta)g(0) - \mathcal{D}^{\alpha}g(0). \tag{2.11}$$

With the same process we have

$$L^{\alpha}(\underbrace{\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}...\mathcal{D}^{\alpha}}_{\text{n times}})g(t) = (\alpha p + \beta)^nG^{\alpha}(p) - (\alpha p + \beta)^{n-1}g(0) - (\alpha p + \beta)^{n-2}\mathcal{D}^{\alpha}g(0) - (\alpha p + \beta)^{n-3}$$

$$\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}g(0) - ... - (\underbrace{\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}...\mathcal{D}^{\alpha}}_{\text{n-1 times}})g(0).$$

For part(7), We have

$$
\begin{aligned}
L^{\alpha}(g(t)) =& G^{\alpha}(p) = \frac{1}{\alpha}\int_{0}^{+\infty} e^{-pt}g(t)dt \\
=& \frac{d}{dp}G^{\alpha}(p) = \frac{d}{dp}\left(\frac{1}{\alpha}\int_{0}^{+\infty} e^{-pt}g(t)dt\right) \\
=& \frac{1}{\alpha}\int_{0}^{+\infty} \frac{\partial}{\partial p}(e^{-pt})g(t)dt \\
=& \frac{1}{\alpha}\int_{0}^{+\infty} e^{-pt}(-tg(t))dt \\
=& L^{\alpha}(-tg(t)) \longrightarrow L^{\alpha}(tg(t)) = (-1)^{1}\frac{d}{dp}G^{\alpha}(p)
\end{aligned}
$$

similarly
$L^{\alpha}(t^{2}g(t)) = (-1)^{2}\frac{d^{2}}{dp^{2}}(G^{\alpha}(p))$.
$L^{\alpha}(t^{3}g(t)) = (-1)^{3}\frac{d^{3}}{dp^{3}}(G^{\alpha}(p))$.
$L^{\alpha}(t^{n}g(t)) = (-1)^{n}\frac{d^{n}}{dp^{n}}(G^{\alpha}(p))$.

2.3.3 Deformable Inverse Laplace Transform

This section explains about DILT and further Heaviside expansion formula and convolution theorem for DILT are also discussed DILT is process for determining the function which generates given DLT. If $G^{\alpha}(p)$ is the DLT of $g(t)$, then $g(t)$ is called DILT of $G^{\alpha}(p)$. The operator for DILT is $(L^{\alpha})^{-1}$.

Theorem 2.14. *[6] (Heaviside expansion formula for DILT) if $G_1(p)$ and$G_2(p)$ are two polynomials and $G_2(p)$ has degree less then degree of $G_1(p)$, Then*

$$(L^{\alpha})^{-1}\left(\frac{G_2(p)}{G_1(p)}\right) = \alpha\sum_{r=1}^{m}\frac{G_2(a_r)}{G_1(a_r)}e^{a_r t}$$

Where $G_1(p)$ has distinct zeroes $a_1, a_2, ..., a_m$.
Proof We have

$$
\begin{aligned}
\frac{G_2(p)}{G_1(p)} =& \frac{G_2(p)}{(p-a_1)\cdot(p-a_2)...(p-a_m)} \\
=& \frac{A_1}{(p-a_1)} + \frac{A_2}{(p-a_2)} + ... + \frac{A_m}{(p-a_m)}.
\end{aligned}
$$

Multiplying both sides by $(p-a_r)$ and taking the limit as $p \longrightarrow a_r$

$$A_r = \lim_{p\longrightarrow a_r}\frac{G_2(p)(p-a_r)}{G_1(S)} = \frac{G_2(a_r)}{G_1'(a_r)}$$

i.e

$$\frac{G_2(p)}{G_1(p)} = \frac{G_2(a_1)}{G_1'(a_1)}\left(\frac{1}{p-a_1}\right) + \frac{G_2(a_2)}{G_1'(a_2)}\left(\frac{1}{p-a_2}\right) + ... + \frac{G_2(a_m)}{G_1'(a_m)}\left(\frac{1}{p-a_m}\right).$$

Taking DILT on both sides, We have

$$(L^\alpha)^{-1}\left(\frac{G_2(p)}{G_1(p)}\right) = \alpha \sum_{r=1}^{m} \frac{G_2(a_r)}{G_1'(a_r)} e^{a_r t}.$$

Theorem 2.15. *[6](convolution theorem) If* $(L^\alpha)^{-1}(G_1^\alpha(p)) = g_1(t)$ *and* $(L^\alpha)^{-1}(G_2^\alpha(p)) = g_2(t)$ *then*

$$(L^\alpha)^{-1}(G_1^\alpha(p)G_2^\alpha(p)) = g_1 * g_2 = \int_0^t g_1(x)g_2(t-x)d_\alpha x. \tag{2.12}$$

2.3.4 Applications of deformable Laplace transform to arbitrary order deferential equations

In first example, we discuss method of solving homogeneous linear differential equations of arbitrary order, whereas in second and third examples non-homogeneous linear arbitrary order differential equations are solved.

Example 2.3. [6] Let a homogeneous deformable fractional differential equation

$$\begin{cases} \mathcal{D}^\alpha y(t) - \lambda y(t) = 0, \\ y(0) = y_0. \end{cases}$$

with λ is a constant.
We Take the DLT on both the sides using equation (7).

$$\mathcal{L}_\alpha(\mathcal{D}^\alpha y(t) - \lambda y(t)) = 0 \Leftrightarrow (\alpha p + \beta)Y^\alpha(p) - y(0) - \lambda Y^\alpha(p) = 0. \tag{2.13}$$

$$\Longrightarrow Y^\alpha(p) = \frac{y_0}{\alpha p + \beta - \lambda}. \tag{2.14}$$

Using DILT, general solution is given by

$$y(t) = y_0 e^{(\frac{\lambda-\beta}{\alpha})t}.$$

Example 2.4. [6] Let non-homogeneous deformable fractional equation

$$\begin{cases} \mathcal{D}^\alpha y(t) = \sin t, \\ y(0) = 0. \end{cases}$$

We take the DLT on both the sides, the we have the expression

$$(\alpha p+\beta)Y^{\alpha}(p)=\frac{1}{\alpha(p^2+1)}\Longrightarrow Y^{\alpha}(p)=\frac{1}{\alpha(p^2+1)(\alpha p+\beta)}$$

by the inverse deformable Laplace transform, we have

$$L_{\alpha}^{-1}(Y^{\alpha}(p))=L_{\alpha}^{-1}\left(\frac{1}{\alpha(p^2+1)(\alpha p+\beta)}\right) \tag{2.15}$$

And the general solution is given by

$$y(t)=\left(\frac{\beta}{\alpha^2+\beta^2}\right)\sin t-\left(\frac{\alpha}{\alpha^2+\beta^2}\right)\cos t+\left(\frac{\alpha}{\alpha^2+\beta^2}\right)e^{-\frac{\beta}{\alpha}t} \tag{2.16}$$

Example 2.5. [6] Consider the problem

$$\begin{cases}(\mathcal{D}^{\frac{1}{4}}\mathcal{D}^{\frac{1}{4}})u(t)=e^{3t}\\ u(0)=1\\ \mathcal{D}^{\alpha}u(0)=0\end{cases}$$

Taking DLT both by using Eq, we obtain

$$(\alpha p+\beta)L^{\alpha}(u)-(\alpha p+\beta)u(0)-\mathcal{D}^{\alpha}u(0)=\frac{4}{p-3}\Longleftrightarrow L^{\alpha}(u)=\frac{4}{(p-3).(\alpha p+\beta)^2}+\frac{1}{\alpha p+\beta}. \tag{2.17}$$

$$\Longrightarrow u=(L^{\alpha})^{-1}\left[\frac{4}{(p-3)(\alpha p+\beta)^2}+\frac{1}{\alpha p+\beta}\right]. \tag{2.18}$$

Here $\alpha=\frac{1}{4}$,$\beta=\frac{3}{4}$, therefore

$$u=(L^{\frac{1}{4}})^{-1}\left[\frac{64}{(p-3)(p+3)^2}+\frac{4}{p+3}\right]$$

Using convolution theorem for DILT
$(L^{\frac{1}{4}})^{-1}\left[\frac{64}{(p-3)(p+3)^2}\right]=4\int_0^t xe^{3t-6x}d_{\alpha}x=16e^{3t}\int_0^t xe^{-6x}dx=-\frac{8}{3}te^{-3t}-\frac{4}{9}e^{-3t}+\frac{4}{9}e^{3t}$
and $(L^{\frac{1}{4}})^{-1}\left[\frac{1}{p-3}\right]=\frac{1}{4}e^{-3t}$. Therefore, complete solution is

$$u(t)=\frac{5}{9}e^{-3t}-\frac{8}{3}te^{-3t}+\frac{4}{9}e^{3t} \tag{2.19}$$

Remark 2.4. [6] It should be noted that all the aforementioned solutions coincide with $\alpha=1$

CHAPTER 3

THE CONFORMABLE FRACTIONAL LAPLACE TRANSFORM OF THE FRACTIONAL CHEBYSHEV AND LEGENDRE POLYNOMIALS

In this chapter, we propose Generating Functions for the fractional Chebyshev and Legendre polynomials. we take a look over the definition of the conformable fractional Laplace transform of order α and try to investigate some of its properties.

3.1 The Generating Functions of the Fractional Chebyshev and Legendre Polynomials

Before starting, we may notice that the fractional Chebyshev and Legendre polynomials of any fractional order is continuous on $\mathbb{R}$, and hence is piecewise continuous.
Also any polynomials is of fractional exponential order. This follows easily from the fact.

$$e^{\mu\frac{t^\alpha}{\alpha}} = \sum_{n=0}^{\infty} \frac{\left(\mu\frac{t^\alpha}{\alpha}\right)^n}{n!} \geq \frac{\left(\mu\frac{t^\alpha}{\alpha}\right)^n}{n!} \to t^{n\alpha} \leq \frac{n!\alpha^n}{\mu^n} e^{\mu\frac{t^\alpha}{\alpha}}$$

So the existence of the conformable fractional Laplace transform of the fractional Chebyshev and Legendre Polynomials is ensured

3.1.1 The Generating Function of the Fractional Chebyshev polynomials

We remember the first few fractional Chbyshev polynomials, and then apply the conformable fractional Laplace transform operator to each of which

$$T_{0\alpha}(x) = 1 \to \mathcal{T}_{0\alpha}(s) = \mathcal{L}_\alpha\left\{T_{0\alpha}(x)\right\} = \frac{1}{s} \tag{3.1a}$$

$$T_{1\alpha}(x) = x^\alpha \to \mathcal{T}_{1\alpha}(s) = \mathcal{L}_\alpha\left\{T_{1\alpha}(x)\right\} = \alpha\frac{\Gamma(2)}{s^2} \tag{3.1b}$$

$$T_{2\alpha}(x) = 2x^{2\alpha} - 1 \to \mathcal{T}_{2\alpha}(s) = \mathcal{L}_\alpha\left\{T_{2\alpha}(x)\right\} = 2\alpha^2\frac{\Gamma(3)}{s^3} - \frac{1}{s} \tag{3.1c}$$

$$T_{3\alpha}(x) = 4x^{3\alpha} - 3x^\alpha \to \mathcal{T}_{3\alpha}(s) = \mathcal{L}_\alpha\left\{T_{3\alpha}(x)\right\} = 4\alpha^3\frac{\Gamma(4)}{s^4} - 3\alpha\frac{\Gamma(2)}{s^2} \tag{3.1d}$$

$$T_{4\alpha}(x) = 8x^{4\alpha} - 8x^{2\alpha} + 1 \to \mathcal{T}_{4\alpha}(s) = \mathcal{L}_\alpha\left\{T_{4\alpha}(x)\right\} = 8\alpha^4\frac{\Gamma(5)}{s^5} - 8\alpha^2\frac{\Gamma(3)}{s^3} + \frac{1}{s} \tag{3.1e}$$

$$T_{5\alpha}(x) = 16x^{5\alpha} - 20x^{3\alpha} + 5x^\alpha \to \mathcal{T}_{5\alpha}(s) = \mathcal{L}_\alpha\left\{T_{5\alpha}(x)\right\} = 16\alpha^5\frac{\Gamma(6)}{s^6} - 20\alpha^3\frac{\Gamma(4)}{s^4} + 5\alpha\frac{\Gamma(2)}{s^2} \tag{3.1f}$$

Now, we tend to find a series representation of $T_{n\alpha}(x)$ similar to those of the classical Chebyshev and Legendre polynomials.

In order to derive an analytic expression for the conformable fractional Laplace transform of $T_{n\alpha}(x)$,

we notice that the only difference is the exponent of x in the fractional Chebyshev polynomials, which comes multiplied by α.

So by considering the series representation of $T_n(x)$.

We can deduce a series representation representation of $T_{n\alpha}(x)$.

The series representation of $T_{n\alpha}(x)$ is

$$T_{n\alpha}(x) = n \cdot 2^{n-1} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k-1)!}{2^{2k} k!(n-2k)!} \cdot x^{(n-2k)\alpha} \qquad : n > 0 \tag{3.2}$$

By applying the conformable fractional Laplace transform operator to the above series (3.2), We get

$$\begin{aligned}
\mathcal{T}_{n\alpha}(s) = \mathcal{L}_\alpha\left\{T_{n\alpha}(x)\right\} =&\mathcal{L}_\alpha\left\{n\cdot 2^{n-1}\sum_{k=0}^{\left|\frac{n}{2}\right|}\frac{(-1)^k(n-k-1)!}{2^{2k}k!(n-2k)!}\cdot x^{(n-2k)\alpha}\right\} \quad : n>0 \\
=&n\cdot 2^{n-1}\sum_{k=0}^{\left|\frac{n}{2}\right|}\frac{(-1)^k(n-k-1)!}{2^{2k}k!(n-2k)!}\cdot\mathcal{L}_\alpha\left\{x^{(n-2k)\alpha}\right\} \\
=&n\cdot 2^{n-1}\sum_{k=0}^{\left|\frac{n}{2}\right|}\frac{(-1)^k(n-k-1)!}{2^{2k}k!(n-2k)!}\cdot\alpha^{n-2k}\frac{\Gamma(1+n-2k)}{s^{1+n-2k}} \\
=&n\cdot 2^{n-1}\sum_{k=0}^{\left|\frac{n}{2}\right|}\frac{(-1)^k(n-k-1)!}{2^{2k}k!}\cdot\frac{\alpha^{n-2k}}{s^{1+n-2k}}
\end{aligned}$$

$$\bullet\ \mathcal{T}_{n\alpha}(s) = \mathcal{L}_\alpha\left\{T_{n\alpha}(x)\right\} = n\cdot 2^{n-1}\sum_{k=0}^{\left|\frac{n}{2}\right|}\frac{(-1)^k(n-k-1)!}{2^{2k}k!}\cdot\frac{\alpha^{n-2k}}{s^{1+n-2k}} \quad ,n>0 \tag{3.3}$$

By expanding the series (3.3), we get

$$n=1\rightarrow\mathcal{T}_{1\alpha}(s) = 1(2^0)\left(\frac{0!}{2^0 0!}\cdot\frac{\alpha}{s^2}\right) = \frac{\alpha}{s^2} \tag{3.4a}$$

$$n=2\rightarrow\mathcal{T}_{2\alpha}(s) = 2(2^1)\left(\frac{1!}{2^0 0!}\cdot\frac{\alpha^2}{s^3}-\frac{0!}{2^2 1!}\cdot\frac{\alpha^0}{s}\right) = 2\alpha^2\frac{\Gamma(3)}{s^3}-\frac{1}{s} \tag{3.4b}$$

$$n=3\rightarrow\mathcal{T}_{3\alpha}(s) = 3(2^2)\left(\frac{2!}{2^0 0!}\cdot\frac{\alpha^3}{s^4}-\frac{1!}{2^2 1!}\cdot\frac{\alpha}{s^2}\right) = 4\alpha^3\frac{\Gamma(4)}{s^4}-\frac{3\alpha}{s^2} \tag{3.4c}$$

$$n=4\rightarrow\mathcal{T}_{4\alpha}(s) = 4(2^3)\left(\frac{3!}{2^0 0!}\cdot\frac{\alpha^4}{s^5}-\frac{2!}{2^2 1!}\cdot\frac{\alpha^2}{s^3}+\frac{1!}{2^4 2!}\cdot\frac{\alpha^0}{s}\right) = 8\alpha^4\frac{\Gamma(5)}{s^5}-8\alpha^2\frac{\Gamma(3)}{s^3}+\frac{1}{s} \tag{3.4d}$$

$$n=5\rightarrow\mathcal{T}_{5\alpha}(s) = 5(2^4)\left(\frac{4!}{2^0 0!}\cdot\frac{\alpha^5}{s^6}-\frac{3!}{2^2 1!}\cdot\frac{\alpha^3}{s^4}+\frac{2!}{2^4 2!}\cdot\frac{\alpha}{s^2}\right) = 16\alpha^5\frac{\Gamma(6)}{s^6}-20\alpha^3\frac{\Gamma(4)}{s^4}+\frac{5\alpha}{s^2} \tag{3.4e}$$

And so on.

By comparing the values that we have in (3.1) with the calculations for (3.4) $n>0$, we deduce that (3.3) is a valid analytic expression for computing the conformable fractional Laplace transform of the fractional Chebyshev polynomials. Furthermore, if we substitute $\alpha=1$, then (3.3) is the Laplace transform of the Chebyshev polynomials.

3.1.2 The Conformable Fractional Laplace Transform of Fractional Legendre polynomials

We remember the first few fractional Legendre polynomials, and then apply the conformable fractional Laplace transform operator to each of which

$$P_{0\alpha}(x) = 1 \to \mathcal{P}_{0\alpha}(s) = \mathcal{L}_\alpha\left\{P_{0\alpha}(x)\right\} = \frac{1}{s} \tag{3.5a}$$

$$P_{1\alpha}(x) = x^\alpha \to \mathcal{P}_{1\alpha}(s) = \mathcal{L}_\alpha\left\{P_{1\alpha}(x)\right\} = \alpha\frac{\Gamma(2)}{s^2} \tag{3.5b}$$

$$P_{2\alpha}(x) = \frac{1}{2}(3x^\alpha - 1) \to \mathcal{P}_{2\alpha}(s) = \mathcal{L}_\alpha\left\{P_{2\alpha}(x)\right\} = \frac{1}{2}\left(3\alpha^2\frac{\Gamma(3)}{s^3} - \frac{1}{s}\right) \tag{3.5c}$$

$$P_{3\alpha}(x) = \frac{1}{2}(5x^{3\alpha} - 3x^\alpha) \to \mathcal{P}_{3\alpha}(s) = \mathcal{L}_\alpha\left\{P_{3\alpha}(x)\right\} = \frac{1}{2}\left(5\alpha^3\frac{\Gamma(4)}{s^4} - 3\alpha\frac{\Gamma(2)}{s^2}\right) \tag{3.5d}$$

$$P_{4\alpha}(x) = \frac{1}{8}(35x^{4\alpha} - 30x^{2\alpha} + 3) \to \mathcal{P}_{4\alpha}(s) = \mathcal{L}_\alpha\left\{P_{4\alpha}(x)\right\} = \frac{1}{8}\left(35\alpha^4\frac{\Gamma(5)}{s^5} - 30\alpha^2\frac{\Gamma(3)}{s^3} + \frac{3}{s}\right) \tag{3.5e}$$

$$P_{5\alpha}(x) = \frac{1}{8}(63x^{5\alpha} - 70x^{3\alpha} + 15x^\alpha) \to \mathcal{P}_{5\alpha}(s) = \mathcal{L}_\alpha(s)\left\{P_{5\alpha}(x)\right\} = \frac{1}{8}\left(63\alpha^5\frac{\Gamma(6)}{s^6} - 70\alpha^3\frac{\Gamma(4)}{s^4} + 15\alpha\frac{\Gamma(2)}{s^2}\right) \tag{3.5f}$$

And so on

we will look for a series representation of $P_{n\alpha}(x)$ by considering the series representation of $P_n(x)$. So the series representation of $P_{n\alpha}(x)$ is

$$\mathcal{P}_{n\alpha}(x) = \frac{1}{2^n}\sum_{k=0}^{|\frac{1}{2}|}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}\cdot x^{(n-2k)\alpha} \quad , n = 0, 1, 2, ... \tag{3.6}$$

Apply the conformable fractional Laplace transform operator to the above series (3.6), to get

$$\begin{aligned}
\mathcal{P}_{n\alpha}(s) =& \mathcal{L}_\alpha\left\{P_{n\alpha}(x)\right\} = \mathcal{L}_\alpha\left\{\frac{1}{2^n}\sum_{k=0}^{|\frac{n}{2}|}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}\cdot x^{(n-2k)\alpha}\right\}, n = 0, 1, 2, ... \\
=& \frac{1}{2^n}\sum_{k=0}^{|\frac{n}{2}|}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}\cdot \mathcal{L}_\alpha\left\{x^{(n-2k)\alpha}\right\} \\
=& \frac{1}{2^n}\sum_{k=0}^{|\frac{n}{2}|}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}\cdot \alpha^{n-2k}\frac{\Gamma(1+n-2k)}{s^{1+n-2k}} \\
=& \frac{1}{2^n}\sum_{k=0}^{|\frac{n}{2}|}\frac{(-1)^k(2n-2k)!}{k!(n-k)!}\cdot \frac{\alpha^{n-2k}}{s^{1+n-2k}}
\end{aligned}$$

$$\bullet\ \mathcal{P}_{n\alpha}(s) = \mathcal{L}_\alpha\left\{P_{n\alpha}(x)\right\} = \frac{1}{2^n}\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} \frac{(-1)^k(2n-2k)!}{k!(n-k)!}\cdot\frac{\alpha^{n-2k}}{s^{1+n-2k}} \quad, n = 0, 1, 2, ... \tag{3.7}$$

Expand the series(3.7), to get

$$n = 0 \to \mathcal{P}_{0\alpha}(s) = \frac{1}{2^0}\left(\frac{0!}{0!0!}\cdot\frac{\alpha^0}{s^1}\right) = \frac{1}{s} \tag{3.8a}$$

$$n = 1 \to \mathcal{P}_{1\alpha}(s) = \frac{1}{2^1}\left(\frac{2!}{0!1!}\cdot\frac{\alpha^1}{s^2}\right) = \frac{\alpha}{s^2} \tag{3.8b}$$

$$n = 2 \to \mathcal{P}_{2\alpha}(s) = \frac{1}{2^2}\left(\frac{4!}{0!2!}\cdot\frac{\alpha^2}{s^3} - \frac{2!}{1!1!}\cdot\frac{\alpha^0}{s}\right) = \frac{1}{2}\left(3\alpha^2\frac{\Gamma(3)}{s^3} - \frac{1}{s}\right) \tag{3.8c}$$

$$n = 3 \to \mathcal{P}_{3\alpha}(s) = \frac{1}{2^3}\left(\frac{6!}{0!3!}\cdot\frac{\alpha^3}{s^4} - \frac{4!}{1!2!}\cdot\frac{\alpha}{s^2}\right) = \frac{1}{2}\left(5\alpha^3\frac{\Gamma(4)}{s^4} - \frac{3\alpha}{s^2}\right) \tag{3.8d}$$

$$n = 4 \to \mathcal{P}_{4\alpha}(s) = \frac{1}{2^4}\left(\frac{8!}{0!4!}\cdot\frac{\alpha^4}{s^5} - \frac{6!}{1!3!}\cdot\frac{\alpha^2}{s^3} + \frac{4!}{2!2!}\cdot\frac{\alpha^0}{s}\right) = \frac{1}{8}\left(35\alpha^4\frac{\Gamma(5)}{s^5} - 30\alpha^2\frac{\Gamma(3)}{s^3} + \frac{3}{s}\right) \tag{3.8e}$$

$$n = 5 \to \mathcal{P}_{5\alpha}(s) = \frac{1}{2^5}\left(\frac{10!}{0!5!}\cdot\frac{\alpha^5}{s^6} - \frac{8!}{1!4!}\cdot\frac{\alpha^3}{s^4} + \frac{6!}{2!3!}\cdot\frac{\alpha}{s^2}\right) = \frac{1}{8}\left(63\alpha^5\frac{\Gamma(6)}{s^6} - 70\alpha^3\frac{\Gamma(4)}{s^4} + \frac{15\alpha}{s^2}\right) \tag{3.8f}$$

And so on.

Again, compare the values that we have in (3.5) with the calculations of (3.7) for $n \geq 0$. we deduce that (3.7) is a valid analytic expression for computing the conformable fractional Laplace transform of the fractional Legendre polynomials .

3.2 The Orthogonality of the Fractional Chebyshev and Legendre Polynomials

In the classical sense, any two function $f(x)$ and $g(x)$ are said to be orthogonal over the interval $[a, b]$ with respect to the weighting function $w(x)$, if $\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx = 0$.
In the case of $T_n(x)$ and $P_n(x)$, the interval is $[-1, 1]$. So if we want to investigate the orthogonality of $T_{n\alpha}(x)$and $P_{n\alpha}(x)$ on such interval, then we have to extend the definition of $T_a(f)(t)$ to include negative values of t. So to avoid the trouble of being undefined on $[-1, 0]$, we assume α throughout this section to be of the from $\alpha = \frac{1}{2j+1}$ $j \in \mathbb{N}$. In such a case $t^{1-\alpha}$ is defined for all $t \in \mathbb{R}$, and $x^{n\alpha}$ is defined for all $x \in \mathbb{R}$, and for all n. So we can put the following definition.

Definition 3.1. [1] For $0 < \alpha = \frac{1}{2j+1} \leq 1 \;\; j \in \mathbb{N}$, we define:

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \qquad t \neq 0$$

If $t = 0$, then we define

$$T_\alpha(f)(0) = \lim_{t \to 0} T_\alpha(f)(t).$$

3.2.1 The Orthogonality of the Fractional Chebyshev Polynomials

Theorem 3.1. *[1] For $n \neq m$, we have:*

$$\int_{-1}^{1} \frac{T_{n\alpha}(x) T_{m\alpha}(x)}{\sqrt{1 - x^{2\alpha}}} d_\alpha(x, -1) = 0 \tag{3.9}$$

Proof *The fractional equation of the Chebyshev Polynomial*

$$(1 - x^{2\alpha}) D^{2\alpha} y - \alpha x^\alpha D^\alpha y + \alpha^2 k^2 y = 0$$

$|x| < 1, \;\; \alpha \in (0, 1], \;\; k \in \mathbb{R}$

Since $T_{n\alpha}$, is a solution for the fractional equation of the fractional equation, then

$$\sqrt{1 - x^{2\alpha}} D^\alpha \left[\sqrt{1 - x^{2\alpha}} D^\alpha T_{n\alpha}(x) \right] + \alpha^2 n^2 T_{n\alpha}(x) = 0 \tag{3.10}$$

Similarly for $T_{m\alpha}$

$$\sqrt{1 - x^{2\alpha}} D^\alpha \left[\sqrt{1 - x^{2\alpha}} D^\alpha T_{m\alpha}(x) \right] + \alpha^2 m^2 T_{m\alpha}(x) = 0 \tag{3.11}$$

Multiply by $T_{m\alpha}(x)$ and by $T_{n\alpha}(x)$ and subtract the resulting equation to get

$$\begin{aligned} j =& T_{m\alpha}(x) \left(\sqrt{1 - x^{2\alpha}} D^\alpha \left[\sqrt{1 - x^{2\alpha}} D^\alpha T_{n\alpha}(x) \right] + \alpha^2 n^2 T_{n\alpha}(x) \right) \\ &- T_{n\alpha}(x) \left(\sqrt{1 - x^{2\alpha}} D^\alpha \left[\sqrt{1 - x^{2\alpha}} D^\alpha T_{m\alpha}(x) \right] + \alpha^2 m^2 T_{m\alpha}(x) \right) \\ =& 0 \end{aligned} \tag{3.12}$$

Divide both sides of by $\sqrt{1 - x^{2\alpha}}$, and then apply the fractional integral I_α^{-1} to get

$$\begin{aligned} \int_{-1}^{1} \frac{j(x)}{\sqrt{1 - x^{2\alpha}}} d_\alpha(x, -1) = & \int_{-1}^{1} \frac{T_{m\alpha}(x) (\sqrt{1 - x^{2\alpha}} D^\alpha [\sqrt{1 - x^{2\alpha}} D^\alpha T_{n\alpha}(x)])}{\sqrt{1 - x^{2\alpha}}} d_\alpha(x, -1) \\ & - \int_{-1}^{1} \frac{T_{n\alpha}(x) (\sqrt{1 - x^{2\alpha}} D^\alpha [\sqrt{1 - x^{2\alpha}} D^\alpha T_{m\alpha}(x)])}{\sqrt{1 - x^{2\alpha}}} d_\alpha(x, -1) \\ & + \int_{-1}^{1} \frac{\alpha^2 (n^2 - m^2) T_{n\alpha}(x) T_{m\alpha}(x)}{\sqrt{1 - x^{2\alpha}}} d_\alpha(x, -1) = 0 \end{aligned} \tag{3.13}$$

Apply theorem of the integration by parts for fractional derivatives to. So the first term integral will be

$$\begin{aligned}\int_{-1}^{1}\frac{T_{m\alpha}(x)(\sqrt{1-x^{2\alpha}}D^{\alpha}[\sqrt{1-x^{2\alpha}}D^{\alpha}T_{n\alpha}(x)])}{\sqrt{1-x^{2\alpha}}}d_{\alpha}(x,-1) \qquad (3.14)\\ =\int_{-1}^{1}T_{m\alpha}(x)\left(D^{\alpha}\left[\sqrt{1-x^{2\alpha}}D^{\alpha}T_{n\alpha}\right]\right)d_{\alpha}(x,-1)\\ =T_{m\alpha}\sqrt{1-x^{2\alpha}}D^{\alpha}T_{n\alpha}(x)|_{-1}^{1}-\int_{-1}^{1}\sqrt{1-x^{2\alpha}}\\ D^{\alpha}T_{n\alpha}(x)D^{\alpha}T_{m\alpha}(x)d_{\alpha}(x,-1)\\ =-\int_{-1}^{1}\sqrt{1-x^{2\alpha}}D^{\alpha}T_{n\alpha}D^{\alpha}T_{m\alpha}(x)d_{\alpha}(x,-1)\end{aligned}$$

Similarly for the second term integral of (3.14)

$$\begin{aligned}\int_{-1}^{1}\frac{T_{n\alpha}(x)(\sqrt{1-x^{2\alpha}}D^{\alpha}[\sqrt{1-x^{2\alpha}}D^{\alpha}T_{m\alpha}(x)])}{\sqrt{1-x^{2\alpha}}}d_{\alpha}(x,-1) \qquad (3.15)\\ =-\int_{-1}^{1}\sqrt{1-x^{2\alpha}}D^{\alpha}T_{n\alpha}(x)D^{\alpha}T_{m\alpha}(x)d_{\alpha}(x,-1)\end{aligned}$$

Substitute (3.14) and (3.13) in (3.12) , So

$$\begin{aligned}\int_{-1}^{1}\frac{j(x)}{\sqrt{1-x^{2\alpha}}}d_{\alpha}(x,-1)=&-\int_{-1}^{1}\sqrt{1-x^{2\alpha}}D^{\alpha}T_{n\alpha}(x)d_{\alpha}(x,-1)\\ &+\int_{-1}^{1}\sqrt{1-x^{2\alpha}}D^{\alpha}T_{n\alpha}(x)D^{\alpha}T_{m\alpha}(x)d_{\alpha}(x,-1)\\ &+\int_{-1}^{1}\frac{\alpha^{2}(n^{2}-m^{2})T_{n\alpha}(x)T_{m\alpha}(x)}{\sqrt{1-x^{2\alpha}}}d_{\alpha}(x,-1)=0\end{aligned}$$

So

$$\int_{-1}^{1}\frac{\alpha^{2}(n^{2}-m^{2})T_{n\alpha}(x)T_{m\alpha}(x)}{\sqrt{1-x^{2\alpha}}}d_{\alpha}(x,-1)=0$$

Since $n\neq m$, Divide both sides of the above equation by $\alpha^{2}(n^{2}-m^{2})$

$$\int_{-1}^{1}\frac{T_{n\alpha}(x)T_{m\alpha}(x)}{\sqrt{1-x^{2\alpha}}}d_{\alpha}(x,-1)=0 \qquad (3.16)$$

3.2.2 The Orthogonality of the Fractional Legendre Polynomials

Theorem 3.2. *[1] For $n \neq m$, we have*

$$\int_{-1}^{1} P_{n\alpha}(x)P_{m\alpha}(x)d_{\alpha}(x,-1) = 0 \tag{3.17}$$

Proof *The fractional equation of the Legendre polynomial*

$$(1-x^{2\alpha})D^{2\alpha}y - 2\alpha x^{\alpha}D^{\alpha}y + \alpha^2 k(k+1)y = 0 \tag{3.18}$$

$|x| < 1, \alpha \in (0,1], k \in \mathbb{R}$
Since $P_{n\alpha}$ is a solution for the fractional equation, then

$$D^{\alpha}[(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)] + \alpha^2 n(n+1)P_{n\alpha}(x) = 0 \tag{3.19}$$

Similarly for $P_{m\alpha}$

$$D^{\alpha}[(1-x^{2\alpha})D^{\alpha}P_{m\alpha}(x)] + \alpha^2 m(m+1)P_{m\alpha}(x) = 0 \tag{3.20}$$

Multiply (3.19) by $P_{m\alpha}(x)$ and (3.2) by $P_{n\alpha}(x)$ and subtract the resulting equation to get

$$\begin{aligned} j = P_{m\alpha}(x)(D^{\alpha}[(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)] + \alpha^2 n(n+1)P_{n\alpha}(x)) \\ - P_{n\alpha}(x)(D^{\alpha}[(1-x^{2\alpha})D^{\alpha}P_{m\alpha}(x)] + \alpha^2 m(m+1)P_{m\alpha}(x)) = 0 \end{aligned} \tag{3.21}$$

Apply the fractional integral I_{α}^{1} to (3.21) , to get

$$\begin{aligned} \int_{-1}^{1} j(x)d_{\alpha}(x,-1)\alpha = \int_{-1}^{1} P_{m\alpha}(x)D^{\alpha}[(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)]d_{\alpha}(x,-1) \\ - \int_{-1}^{1} P_{n\alpha}(x)D^{\alpha}[(1-x^{2\alpha})D^{\alpha}P_{m\alpha}(x)]d_{\alpha}(x,-1) \\ + \int_{-1}^{1} \alpha^2(n(n+1) - m(m+1))P_{n\alpha}(x)P_{m\alpha}(x)d_{\alpha}(x,-1) = 0 \end{aligned} \tag{3.22}$$

Apply theorem of the integration by parts for fractional derivatives to (3.22). So the first term integral will be

$$\begin{aligned}\int_{-1}^{1} P_{m\alpha}(x)D^{\alpha}[(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)]d_{\alpha}(x,-1) \\ =&P_{m\alpha}(x)(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)|_{-1}^{1} \\ &-\int_{-1}^{1}(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)D^{\alpha}d_{\alpha}(x,-1) \\ =&-\int_{-1}^{1}(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)D^{\alpha}P_{m\alpha}(x)]d_{\alpha}(x,-1)\end{aligned} \tag{3.23}$$

Similarly for the second term integral of(3.22).

$$\begin{aligned}\int_{-1}^{1} P_{n\alpha}(x)D^{\alpha}[(1-x^{2\alpha})D^{\alpha}P_{m\alpha}(x)]d_{\alpha}(x,-1) \\ =-\int_{-1}^{1}(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)D^{\alpha}P_{m\alpha}(x)]d_{\alpha}(x,-1)\end{aligned} \tag{3.24}$$

Substitute (3.23) and(3.24)in(3.22), So

$$\begin{aligned}\int_{-1}^{1} j(x)d_{\alpha}(x,-1)\alpha =& -\int_{-1}^{1}(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)D^{\alpha}P_{m\alpha}(x)d_{\alpha}(x,-1) \\ &+\int_{-1}^{1}(1-x^{2\alpha})D^{\alpha}P_{n\alpha}(x)D^{\alpha}P_{m\alpha}(x)d_{\alpha}(x,-1) \\ &+\int_{-1}^{1}\alpha^{2}(n(n+1)-m(m+1))P_{n\alpha}(x)P_{m\alpha}(x)d_{\alpha}(x,-1)=0\end{aligned} \tag{3.25}$$

So

$$\int_{-1}^{1}\alpha^{2}(n(n+1)-m(m+1))P_{n\alpha}(x)P_{m\alpha}(x)d_{\alpha}(x,-1)=0 \tag{3.26}$$

Since $n \neq m$ divide both sides of the above equation by $\alpha^{2}(n(n+1)-m(m+1))$

$$\int_{-1}^{1} P_{n\alpha}(x)P_{m\alpha}(x)d_{\alpha}(x,-1)=0 \tag{3.27}$$

General Conclusions and perspectives

In this, work we studied the concept of the concept of the classical Laplace transform. we tried to produce Laplace transform via the deformable integral approach with its corresponding properties and observed that these all coincide with classical Laplace transform for $\alpha = 1$.
Also, we studied the conformable fractional Laplace transform of the fractional Chebyshev and Legendre polynomials.
In future work, we will try to answer the following two questions:
Question 1: What is geometric interpretation and physical significance of DLT ?
Question 2: Is there any similarity between the classical fractional Laplace transform corresponding to various fractional derivative and DLT.

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ملخص: في هذه المذكرة قمنا بإيجاد الحلول الدقيقة للمعادلات التفاضلية الكسرية ذات مشتق كسري، يسمى المشتق الكسري المشوه من الرتبة α حيث

$$0 < \alpha \leq 1$$

أيضا قمنا بدراسة تحويل لابلاس المطابق ذو الرتبة الكسرية لبعض كثيرات الحدود من نوع تشيبيشاف وكثيرات الحدود من نوع لوجندر ذات الرتبة الكسرية .

الكلمات المفتاحية : تحويل لابلاس ، المشتق الكسري المشوه ، كثيرات الحدود من نوع تشيبيشاف كثيرات الحدود من نوع لوجندر .

Dans cette thèse, nous avons étudié la transformée de Laplace déformable et conformable, ou nous avons donné des solutions exactes d’équations différentielles fractionnaires avec une dérivée fractionnaire appelée dérivée fracionnaire déformable d’order α, ou $0 < \alpha \leq 1$. Nous avons égalment étudié la transformée de Laplace d’ordre fractionnaire conformable des polynomes fractionnaire de Chebyshev et Legendre.

Mots-Clés: transformée de Laplace, Dérivée fractionnaire déformmable, polynomes de Chebyshev, polynomes de Legendre.

In this dissertation, we studied the deformable and conformable Laplace transform, where we gave exact solutions of fractional differential equations with a fractional derivative called the deformable fractional derivative of order α, where $0 < \alpha \leq 1$. Also we studied the conformable fractional Laplace transform of the fractional Chebyshev and Legendre polynomials.

Keywords: Laplace transform, Deformable fractional derivative, Chebyshev polynomials, Legendre polynomials .