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On the Applications of Lebesgue Spaces to the Study of Partial Differential Equations

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Derri Sara

Dedication

I dedicate this work:

to the invisible hand

and the constant support

whose prayers are the reason I am here today,

to my dearest **mother**

Derri Sara

Contents

Notation

We introduce the necessary notations and definition which are used in the sequel..

H	Hilbert space.
E	Banach space.
X'	Topological dual of X.
Ω	Open set of $\mathbb{R}$.
$\partial\Omega$	Border of Ω.
$L^p(\Omega)$	$= \Big\{ u : \Omega \to \mathbb{R}$ measurable : $\int_\Omega \lvert u\rvert^p dx \leq \infty$ with $1 \leq p < \infty. \Big\}$
a.e.	Almost everywhere.
$\lVert u\rVert_{L^p}$	$= \Big(\int_\Omega \lvert u(x)\rvert^p dx \Big)^{\frac{1}{p}}$.
$L^\infty(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable and $\exists C : \lvert u(x)\rvert \leq C\ a.e\ in\ \Omega\}$.
$\lVert u\rVert_{L^\infty}$	$= \inf\{C : \lvert u(x)\rvert \leq C$ a.e in $\Omega\}$.
$C(\Omega)$	Continuous function space in Ω.
$D(A)$	Domain of definition of a bounded operator A.
$< .,. >$	Define a scalar product.
$\mathcal{L}(X, Y)$	Set of continuous linear applications.
$\mathbb{R}^N$	Euclidean space of dimension N, where N is a nonzero natural number.
x	Vector in $\mathbb{R}^N, x = (x_1, x_2, ..., x_N), x_i \in \mathbb{R}, 1 \leq i \leq N$.
dx	Lebesgue measure in N-dimensional space.
$\partial\Omega = \Gamma$	boundary of Ω.
$\lvert E\rvert$	$mes(E)$ measure of the set E.
$\lvert.\rvert$	Hilbert norm .
χ_E	Characteristic function of set E.

$\frac{\partial u}{\partial t}$	Outward normal derivative.
$f_n \to f$	Denotes that the sequence $\{f_n\}$ converge to f .
$f_n \rightharpoonup f$	Denotes that the sequence $\{f_n\}$ converge weakly to f.
Supp u	Support of the function u.
$\int_\Omega f(x)dx$	Integral of f in Ω with respect to the Lebesgue measure .
$\mathcal{D}(\Omega)$	Space of infinitely differentiable functions on Ω with compact support in Ω.
$\mathcal{M}$	be a measurable space.
$\mathcal{W}^{k,p}(\xi)$	Sobolev space of functions $u \in L^p(\xi)$ whose weak derivatives $D^\alpha u$ up to order k belong to $L^p(\xi)$.

Introduction

Partial differential equations, which will be abbreviated as ”PDE” in the following, constitute an important branch of applied mathematics and this field is becoming increasingly important in modern times. EDPs have great utility in mod- delusion of many phenomena of different natures such as physics, chemistry, biology and other sciences. In other words, EDPs intervene in many other fields: in chemistry to model reactions, in economics to study market behavior, in finance to study derivative products and in image processing to restore damage. These problems boil down to mathematical models usually written in the form L(u) = f , (1) where L is a definite operator from a function space E to a function space F, u is the unknown function and f a given function. PDEs probably first appeared during the 17th century. In- the field of PDEs has been enriched as the sciences have developed and especially physics. However, the systematic study of PDEs is much more recent and it is only during of the 20th century that mathematicians began to develop and indeed this theory has recently experienced a great theoretical and practical advancement. The mathematical analysis of these partial differential equations requires an appropriate choice of functional spaces and a clear definition of the notion of solution (the existence and sometimes uniqueness). One of the things to keep in mind about PDEs is to ask the question: can we get solutions explicitly?. So what mathematics can do on the other hand, is to say if one or more solutions exist, and to describe sometimes very precisely- ment some properties of these solutions. So usually an exact solution of problem (1) is not easy to find and sometimes we cannot even find the explicit solution. This leads to the introduction of the notion of the weak solution which appeared in 1934 in the work of Jean Leray. therefore, in most cases it is very difficult, if not impossible, to exhibit the solutions of a PDE. In some cases we manage to try to show that the problem admits a unique solution (it is said to be well-posed). But sometimes we can calculate numerical approximations of the solutions

Moreover,the work presented in this dissertation concerns the study of partial differential equations in Lebesgue spaces.

specifically,in this work we study the applications of Lebesgue spacesto the study of partial differential .

1^{st} chapter

In this chapter we present the basic theoretical background required for the rest of the work. This includes fundamental definitions and essential concepts related to functional analysis, Lebesgue spaces, and weak formulations.

2^{nd} chapter

We focus on the variational approach to partial differential equations. In particular, we study the Lax–Milgram theorem and its application to proving the existence and uniqueness of weak solutions for a class of boundary value problems.

3^{rd} chapter

, In the third chapter, we investigate existence results using compactness methods. These techniques, based on compactness properties in functional spaces, allow us to obtain convergence results and establish the existence of solutions under weaker assumptions.

Chapter 1

Preliminaries and basic tools

1.1 Measurable function

Definition 1.1. *[8] Let $(X, \mathcal{M})$ be a measurable space. A function $f : X \to \mathbb{R}$ is said to be measurable if for every $\alpha \in \mathbb{R}$, the set*

$$\{x \in X \mid f(x) > \alpha\}$$

belongs to $\mathcal{M}$.

Equivalently, f is measurable if for every Borel set $B \subset \mathbb{R}$, one has

$$f^{-1}(B) \in \mathcal{M}.$$

1.2 Lebesgue spaces

Definition 1.2. *[5] Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p < \infty$. We define*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \ (or\ \mathbb{C}) \ \middle| \ f \text{ is measurable and } \int_\Omega |f(x)|^p \, d\mu(x) < \infty \right\}.$$

For $p = \infty$, we define

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R}(or\ \mathbb{C}) | \ f \text{ is measurable and essentially bounded} \}.$$

The associated norm is defined by

$$|f|p = \left(\int |f(x)|^p, d\mu(x) \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|f\|\infty = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

Definition 1.3. *For $u \in L^p(\Omega)$,the L^p norm is defined by*

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}}.$$

The space $(L^p(\Omega), \|\cdot\|_{L^p})$ is a normed vector space

Proposition 1.1. *For $1 \leq p < +\infty$,The functional*

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}$$

defines a norm on $L^p(\Omega)$.

1.2.1 Hölder inequality

Definition 1.4. *[6] [7]*

Let$(X, \mathcal{M}, \mu)$ be a measure space, and let $p, q \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then

$$\int_X |f(x)g(x)| \, d\mu(x) \leq \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \left(\int_X |g(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}.$$

1.2.2 Young's inequality

Definition 1.5. *[6] [7]*

Let $a, b \geq 0$ and let $p, q \in (1, \infty)$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Remark 1.1. *Holder inequality is fundamental in the study of Lebesgue spaces . It ensures that the product of two functions in dual spaces belongs to* $L^1(\Omega)$.

1.2.3 Minkowski inequality

Definition 1.6. *[6] [7] Let* $(X, \mathcal{M}, \mu)$ *be a measure space and let* $p \in [1, \infty)$. *If* $f, g \in L^p(\mu)$, *then*

$$\left(\int_X |f(x) + g(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \leq \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} + \left(\int_X |g(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}.$$

Remark 1.2. *Minkowski inequality shows that the* L^p *norm satisfies the triangale inequality . Therefore ,* $L^p(\Omega)$ *is a normed vector space .*

1.2.4 Completeness

For every $1 \leq p \leq +\infty$,the space $L^p(\Omega)$ is a Banach space.

1.2.5 The case of $L^2(\Omega)$

The space $L^2(\Omega)$ is a Hilbert space endowed with the inner product :

$$(u, v)_{L^2} = \int_\Omega u(x)v(x)dx.$$

1.2.6 A Random elliptic problem

Definition 1.7. *A random elliptic problem is an elliptic partial differential equation involving random parameters, random coefficients, or stochastic data. Such problems are used to model physical phenomena with uncertainty.*

$$-\nabla \cdot \Big(a(x, \varphi)\nabla u(x, \varphi)\Big) = f(x, \varphi) \quad \textit{in } \Omega$$

where $a(x,\varphi)$ and $f(x,\varphi)$ are random data.

example 1.1. *Let us consider a one-dimensional random elliptic problem on the domain $\xi = (0,1)$. We define the stochastic equation with homogeneous Dirichlet boundary conditions as follows:*

$$-\frac{d}{dx}\left(a(x,\varphi)\frac{du}{dx}(x,\varphi)\right) = f(x,\varphi), \quad x \in \xi \tag{1.1}$$

$$u(0,\varphi) = 0, \quad u(1,\varphi) = 0 \tag{1.2}$$

Where $\varphi \in \mathcal{W}$ represents the random outcomes from a sample space $\mathcal{W}$. For a concrete application, let the random coefficient $a(x,\varphi)$ and the source term $f(x,\varphi)$ be given by:

- $a(x,\varphi) = 2 + Y(\varphi)\sin(x)$
- $f(x,\varphi) = 10 \cdot Z(\varphi)$

Here, $Y(\varphi)$ and $Z(\varphi)$ are independent random variables (for instance, uniformly distributed), ensuring that the coefficient $a(x,\varphi)$ remains strictly positive to preserve the ellipticity of the problem.

1.2.7 Banach–Alaoglu theorem

The Banach –Alaoglu theorem is a fundamental theorem is functional analysis which states that the closed unit ball of the dual space of a normed vector space is compact in the weak -*topology

Theorem 1.1 (Banach–Alaoglu)**.** *[9] [2] [3]*

Let X be a normed vector space and let X^* be its dual space. Then the closed unit ball

$$B_{X^*} = \{f \in X^* : \|f\| \leq 1\}$$

is compact in the weak- topology $\sigma(X^*, X)$*

1.2.8 The Rellich–kondrachov theorem

The Rellich–kondrachov theorem is a fundamental compacteness result in Sobolev spaces .It states that bounded sequences in certin Sobolov spaces admit strongly convergent subsequences in

Lebesgue spaces

Theorem 1.2. *[2] [4] [1] Let Ω be a bounded open subset of $\mathbb{R}^n$ with sufficiently smooth boundary. Let $1 \leq p < n$.*

Then the embedding

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact for every

$$1 \leq q < p^*$$

where

$$p^* = \frac{np}{n-p}$$

is the Sobolev critical exponent.

example 1.2. *Let the spatial dimension be $n = 3$ and consider the Sobolev space $W^{1,2}(\xi)$ (commonly denoted as $H^1(\xi)$), so we have $p = 2$. Since $1 \leq p < n$ $(2 < 3)$, we can directly compute the Sobolev critical exponent p^* as follows:*

$$p^* = \frac{np}{n-p} = \frac{32}{3-2} = \frac{6}{1} = 6 \tag{1.3}$$

According to the Rellich-Kondrachov theorem, the embedding:

$$W^{1,2}(\xi) \hookrightarrow L^q(\xi) \tag{1.4}$$

is compact for any exponent q satisfying $1 \leq q < 6$.

This implies that from any bounded sequence of functions $\{u_k\}$ in $W^{1,2}(\xi)$, one can extract a subsequence that converges strongly in $L^5(\xi)$ or $L^{5.9}(\xi)$. However, this strong convergence property is generally lost at the critical limit $q = 6$.

Chapter 2

Weak formulation of elliptic equations using the Lax–Milgram theorem

The Lax–Milgram theorem is a central result in functional analysis and the theory of partial differential equations. It provides sufficient conditions ensuring the existence and uniqueness of a solution to variational problems formulated on Hilbert spaces.

Let V be a real Hilbert space equipped with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$.

2.1 The Lax-Miligram theorem

2.1.1 Bilinear form

A mapping

$$a : VV \to \mathbb{R}$$

is called a bilinear form if it is linear with respect to each argument, i.e., for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{R}$:

$$a(\alpha u + \beta v, w) = \alpha a(u, w) + \beta a(v, w),$$

$$a(u, \alpha v + \beta w) = \alpha a(u, v) + \beta a(u, w).$$

2.1.2 Continuity

A bilinear form $a(\cdot,\cdot)$ is said to be continuous on VV if there exists a constant $C > 0$ such that:

$$|a(u, v)| \le C\|u\|\,\|v\| \quad \forall u, v \in V.$$

This inequality expresses that the bilinear form is bounded with respect to the norm topology of V.

2.1.3 Coercivity

A bilinear form $a(\cdot,\cdot)$ is called coercive if there exists a constant $\alpha > 0$ such that:

$$a(u,u) \geq \alpha \|u\|^2 \quad \forall u \in V.$$

Coercivity implies that the form controls the norm from below and plays a crucial role in ensuring uniqueness and stability of solutions.

2.1.4 Continuous linear functional

A mapping $L : V \to \mathbb{R}$ is called a linear functional if:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v), \quad \forall u, v \in V,\ \ \alpha, \beta \in \mathbb{R}.$$

It is said to be continuous if there exists a constant $C > 0$ such that:

$$|L(v)| \leq C\|v\| \quad \forall v \in V.$$

Concrete Numerical Application of Lax-Milgram Theorem

To provide a rigorous verification of the Lax-Milgram theorem, we consider a concrete numerical example in a one-dimensional setting. Let the domain be the open interval $\xi = (0, \pi)$. We define the following boundary value problem:

$$\begin{cases} -u''(x) + 3u(x) = \sin(x), \quad x \in (0,\pi), \\ u(0) = 0, \quad u(\pi) = 0. \end{cases} \tag{2.1}$$

This problem corresponds to the general model with the constant coefficient $c(x) = 3$ and the data source $f(x) = \sin(x)$.

example 2.1. *We verify the existence and uniqueness of the solution using the Hilbert space $X = H_0^1(0, \pi)$, equipped with the inner product norm $\|u\|_{H_0^1} = (\int_0^\pi |u'(x)|^2 dx)^{1/2}$.*

The continuous bilinear form $a(\cdot,\cdot) : H_0^1 H_0^1 \to \mathbb{R}$ and the linear functional $F(\cdot) : H_0^1 \to \mathbb{R}$ for this specific problem are:

$$a(u, v) = \int_0^\pi u'(x)v'(x)dx + 3\int_0^\pi u(x)v(x)dx \tag{2.2}$$

$$F(v) = \int_0^\pi \sin(x)v(x)dx \tag{2.3}$$

We compute the exact numerical constants to satisfy the Lax-Milgram conditions:

1. ***Coercivity:*** *Since $3\int_0^\pi |u(x)|^2 dx \geq 0$, we directly evaluate:*

$$a(u, u) = \int_0^\pi |u'(x)|^2 dx + 3\int_0^\pi |u(x)|^2 dx \geq \int_0^\pi |u'(x)|^2 dx = \|u\|_{H_0^1}^2 \tag{2.4}$$

Thus, $a(\cdot,\cdot)$ is coercive with the exact numerical constant $\alpha = 1$.

2. ***Continuity (Boundedness):*** *Applying Hölder's inequality and noting that $\|c\|_{L^\infty} = 3$, we have:*

$$|a(u, v)| \leq \|u'\|_{L^2}\|v'\|_{L^2} + 3\|u\|_{L^2}\|v\|_{L^2} \tag{2.5}$$

By Poincaré's inequality on $(0, \pi)$, the embedding constant is $C_P = 1$ (since the first eigenvalue of the Laplacian on this domain is $\lambda_1 = 1$), meaning $\|u\|_{L^2} \leq \|u'\|_{L^2} = \|u\|_{H_0^1}$. Therefore:

$$|a(u, v)| \leq \|u\|_{H_0^1}\|v\|_{H_0^1} + 3\|u\|_{H_0^1}\|v\|_{H_0^1} = 4\|u\|_{H_0^1}\|v\|_{H_0^1} \tag{2.6}$$

Hence, the continuity condition holds with the sharp numerical bound $M = 4$.

3. ***Boundedness of F:*** *We compute the L^2-norm of our specific function $f(x) = \sin(x)$:*

$$\|f\|_{L^2}^2 = \int_0^\pi \sin^2(x)dx = \frac{\pi}{2} \implies \|f\|_{L^2} = \sqrt{\frac{\pi}{2}} \approx 1.253 \tag{2.7}$$

Using the Cauchy-Schwarz inequality, the linear functional is bounded by:

$$|F(v)| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \sqrt{\frac{\pi}{2}}\|v\|_{H_0^1} \tag{2.8}$$

Since $\alpha = 1 > 0$*,* $M = 4 < +\infty$*, and* $\|f\|_{L^2} = \sqrt{\frac{\pi}{2}}$ *is finite, the Lax-Milgram theorem guarantees that the problem* (2.1) *has a unique weak solution* $u \in H_0^1(0, \pi)$*.*

Exact Analytical Verification: *To demonstrate the absolute accuracy of the weak formulation framework, we can find the exact analytical solution by assuming* $u(x) = A\sin(x)$*. Substituting this into* (2.1)*:*

$$-(-A\sin(x)) + 3A\sin(x) = \sin(x) \implies 4A\sin(x) = \sin(x) \implies A = \frac{1}{4} \tag{2.9}$$

The unique exact solution is $u(x) = \frac{1}{4}\sin(x)$*, which perfectly belongs to* $H_0^1(0, \pi)$*.*

Theorem 2.1. *Let* V *be a Hilbert space,* $a(u, v)$ *a bilinear form that is continuous and coercive, and* L *a continuous linear functional on* V*.*

Then there exists a unique element $u \in V$ *such that:*

$$a(u, v) = L(v), \quad \forall v \in V$$

2.2 Application of the Lax-Milgram theorem

2.2.1 Model problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in L^2(\Omega)$. We consider the Poisson problem:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

example 2.2. *Consider the one-dimensional Poisson problem*

$$\begin{cases} -u''(x) = 1, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{2.10}$$

Let

$$V = H_0^1(0,1),$$

and define the bilinear form

$$a(u,v) = \int_0^1 u'(x)v'(x)\,dx,$$

and the linear functional

$$L(v) = \int_0^1 v(x)\,dx.$$

The weak formulation of the problem is:

Find $u \in H_0^1(0,1)$ such that

$$a(u,v) = L(v), \qquad \forall v \in H_0^1(0,1),$$

that is,

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 v(x)\,dx, \qquad \forall v \in H_0^1(0,1).$$

We verify the assumptions of the Lax-Milgram theorem:

1. *The bilinear form $a(\cdot,\cdot)$ is continuous:*

$$|a(u,v)| \le \|u\|_{H_0^1(0,1)}\|v\|_{H_0^1(0,1)}.$$

2. *The bilinear form $a(\cdot,\cdot)$ is coercive:*

$$a(v,v) = \int_0^1 |v'(x)|^2\,dx = \|v\|^2_{H_0^1(0,1)}.$$

3. The linear functional L is continuous:

$$|L(v)| = \left|\int_0^1 v(x)\,dx\right| \le C\|v\|_{H_0^1(0,1)}.$$

Therefore, by the Lax-Milgram theorem, there exists a unique weak solution

$$u \in H_0^1(0,1).$$

Moreover, the exact solution is

$$u(x) = \frac{x(1-x)}{2},$$

since

$$-u''(x) = 1, \qquad u(0) = u(1) = 0.$$

2.2.2 Weak formulation

Find $u \in H_0^1(\Omega)$ such that:

$$a(u,v) = L(v), \quad \forall v \in H_0^1(\Omega),$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx, \quad L(v) = \int_\Omega fv\,dx.$$

Verification of the hypotheses

Continuity of $a(u,v)$

Using the Cauchy-Schwarz inequality:

$$|a(u,v)| = \left|\int_\Omega \nabla u \cdot \nabla v\,dx\right| \le \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}.$$

Since $\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$, we obtain:

$$|a(u,v)| \le \|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}.$$

Thus, $a(u,v)$ is continuous.

Coercivity of $a(u,u)$

We have:

$$a(u,u) = \int_\Omega |\nabla u|^2 dx = \|\nabla u\|^2_{L^2(\Omega)}.$$

By Poincaré inequality, there exists $C > 0$ such that:

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

Hence, the norm $\|u\|_{H_0^1(\Omega)}$ is equivalent to $\|\nabla u\|_{L^2(\Omega)}$, and therefore:

$$a(u,u) \geq \alpha\|u\|^2_{H_0^1(\Omega)} \quad \text{for some } \alpha > 0.$$

Thus, $a(u,u)$ is coercive.

Continuity of $L(v)$

Using the Cauchy-Schwarz inequality:

$$|L(v)| = \left|\int_\Omega fv\, dx\right| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}.$$

Using Poincaré inequality:

$$\|v\|_{L^2(\Omega)} \leq C\|v\|_{H_0^1(\Omega)}.$$

Thus:

$$|L(v)| \leq C\|f\|_{L^2(\Omega)}\|v\|_{H_0^1(\Omega)}.$$

So L is continuous.

All the assumptions of the Lax-Milgram theorem are satisfied. Therefore, there exists a unique solution $u \in H_0^1(\Omega)$.

2.3 Applications of the Lax–Milgram theorem to random linear elliptic equations

The Lax–Milgram theorem plays a fundamental role in the study of random linear elliptic problems. It provides a powerful variational framework to prove the existence and uniqueness of weak solutions for equations involving random coefficients.

2.3.1 Random elliptic problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with sufficiently smooth boundary $\partial\Omega$.

Consider the following random elliptic problem:

$$-\nabla \cdot \Big(a(x,\varphi).\nabla u(x,\varphi)\Big) = f(x) \quad \text{in } \Omega,$$

with homogeneous Dirichlet boundary condition:

$$u(x,\varphi) = 0 \quad \text{on } \partial\Omega,$$

where:

- $a(x,\varphi)$ is a random coefficient,
- $\varphi \in \mathcal{D}$ represents the random parameter,
- $f \in L^2(\Omega)$ is a given function.

We assume that the random coefficient satisfies:

$$0 < \alpha \leq a(x,\varphi) \leq \beta < +\infty,$$

for almost every $x \in \Omega$ and every $\varphi \in \mathcal{D}$.

2.3.2 Weak formulation

We seek a weak solution in the Sobolev space:

$$V = H_0^1(\Omega).$$

Multiplying the equation by a test function $v \in V$ and integrating over Ω, we obtain:

$$\int_\Omega a(x, \varphi)\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx.$$

We define the bilinear form:

$$A_\varphi(u, v) = \int_\Omega a(x, \varphi)\nabla u \cdot \nabla v \, dx$$

and the linear functional:

$$L(v) = \int_\Omega f v \, dx.$$

Thus, the weak formulation becomes:

Find $u(\varphi) \in V$ such that:

$$A_\varphi(u, v) = L(v), \quad \forall v \in V.$$

2.3.3 Continuity of the bilinear form

Using the boundedness of $a(x, \varphi)$, we have:

$$|A_{\varphi(u,v)}| \leq \beta \int_\Omega |\nabla u||\nabla v| dx.$$

Applying the Cauchy–Schwarz inequality:

$$|A_{\varphi(u,v)}| \leq \beta \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}.$$

Hence:

$$|A_{\varphi(u,v)}| \leq \beta \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

Therefore, the bilinear form A_φ is continuous.

2.3.4 Coercivity

From the positivity assumption:

$$a(x, \varphi) \geq \alpha > 0,$$

we obtain:

$$A_{\varphi(u,u)} = \int_\Omega a(x, \varphi) |\nabla u|^2 dx \geq \alpha \int_\Omega |\nabla u|^2 dx.$$

Thus:

$$A_{\varphi(u,u)} \geq \alpha \|u\|^2_{H_0^1(\Omega)}.$$

Therefore, the bilinear form is coercive.

2.3.5 Existence and uniqueness of the solution

Since:

- $A_{\varphi(\cdot,\cdot)}$ is bilinear,
- continuous,
- coercive,

and L is a continuous linear functional on $H_0^1(\Omega)$, all assumptions of the Lax–Milgram theorem are satisfied.

Consequently, for almost every $\varphi \in \mathcal{D}$, there exists a unique weak solution:

$$u(\varphi) \in H_0^1(\Omega)$$

such that:

$$A_{\varphi(u,v)} = L(v), \quad \forall v \in H_0^1(\Omega).$$

We conclused that

the Lax-Miligram theorem provides an afficient variational method for proving the well-posedness of random linear elliptic equations. This approach constitutes a fundamental tool in stochastic partial differential equations and uncertainty quantification theory.

Chapter 3

Weak formulation of elliptic equations using compactness theorems

The Compactness Method is one of the most important techniques used in the study of nonlinear partial differential equations. Its main goal is to prove the existence of a weak solution when solving the problem directly is difficult.

3.0.1 Basic idea

Suppose we want to solve a nonlinear problem of the form:

$$Au = f$$

or in weak form:

$$\int_\Omega a(x,u)\nabla u \cdot \nabla v \, dx = \langle f, v\rangle.$$

Because the operator is nonlinear, classical linear methods cannot always be applied.

The compactness method proceeds through the following steps.

Step 1: Construct an approximate sequence

Instead of solving the original problem directly, we construct a sequence of approximate solutions:

$$(u_n)_{n\in\mathbb{N}}.$$

Each problem is easier to solve than the original one.

Step 2: Obtain a priori estimates

We prove that the sequence is bounded in a suitable functional space such as:

$$H_0^1(\Omega) \quad \text{or} \quad W^{1,p}(\Omega).$$

For example:

$$\|u_n\|_{H_0^1(\Omega)} \leq C.$$

This boundedness is fundamental.

Step 3: Use compactness

By compactness theorems, every bounded sequence admits a convergent subsequence.

Thus, there exists a subsequence such that:

$$u_n \rightharpoonup u \quad \text{weakly},$$

and sometimes:

$$u_n \to u \quad \text{strongly in } L^2(\Omega).$$

This is the key idea of compactness.

Step 4: Pass to the Limit

Finally, we pass to the limit in the weak formulation and prove that the limit function:

$$u$$

is a weak solution of the original problem.

3.1 Application to the problem in the image

Consider the nonlinear elliptic problem:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a(x,u)\frac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega,$$

with boundary condition:

$$u = 0 \quad \text{on } \partial\Omega.$$

Its weak formulation is:

$$\sum_{i=1}^{n} \int_{\Omega} a(x,u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx = \langle f, v\rangle, \quad \forall v \in H_0^1(\Omega).$$

The compactness method is applied as follows:

1. Construct approximate solutions (u_n).
2. Derive uniform estimates.
3. Extract a convergent subsequence using compactness.
4. Pass to the limit in the weak formulation.
5. Conclude that the limit is a weak solution.

Concrete Numerical Example of the Compactness Method

To provide a clear and structured application of the 5-step compactness method described in Section 3.1, we consider a concrete non-linear elliptic problem on the one-dimensional domain $\xi = (0, \pi)$ with $n = 1$.

We define the non-linear boundary value problem as follows:

$$\begin{cases} -\frac{d}{dx}\left(\left(1 + u(x)^2\right)\frac{du}{dx}(x)\right) = \sin(x), \quad x \in (0,\pi), \\ u(0) = 0, \quad u(\pi) = 0. \end{cases} \tag{3.1}$$

This corresponds to the non-linear coefficient $a(x,u) = 1 + u^2$ and the source data $f(x) = \sin(x)$.

example 3.1. *We work within the Hilbert space $X = H_0^1(0,\pi)$, equipped with the standard norm $\|u\|_{H_0^1} = (\int_0^\pi |u'(x)|^2 dx)^{1/2}$. The weak formulation consists of finding $u \in H_0^1(0,\pi)$ such that:*

$$\int_0^\pi \Big(1 + u(x)^2\Big) u'(x) v'(x) dx = \int_0^\pi \sin(x) v(x) dx, \quad \forall v \in H_0^1(0,\pi). \tag{3.2}$$

We apply the five structural steps directly to this problem:

1. ***Construct approximate solutions (u_m)***

   Let $\{\phi_k\}_{k=1}^\infty$ be the classical Galerkin basis functions for $H_0^1(0,\pi)$, given explicitly by $\phi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx)$. For each finite dimension $m \geq 1$, we define the approximate subspace $X_m =$ span$\{\phi_1, \ldots, \phi_m\}$ and seek an approximate solution $u_m \in X_m$ satisfying the weak equation restricted to X_m.

2. ***Derive uniform estimates***

   We test the approximate weak relation by choosing the test function $v = u_m$. This yields:

$$\int_0^\pi \Big(1 + u_m(x)^2\Big) |u_m'(x)|^2 dx = \int_0^\pi \sin(x) u_m(x) dx. \tag{3.3}$$

   Expanding the left-hand side, we note that $\int_0^\pi u_m^2 |u_m'|^2 dx \geq 0$. Therefore, we establish the lower bound:

$$\int_0^\pi \Big(1 + u_m(x)^2\Big) |u_m'(x)|^2 dx \geq \int_0^\pi |u_m'(x)|^2 dx = \|u_m\|_{H_0^1}^2. \tag{3.4}$$

   For the right-hand side, we apply Cauchy-Schwarz and Poincaré inequalities (where $C_P = 1$ on this domain):

$$\int_0^\pi \sin(x) u_m(x) dx \leq \|\sin\|_{L^2} \|u_m\|_{L^2} \leq \sqrt{\frac{\pi}{2}} \|u_m\|_{H_0^1}. \tag{3.5}$$

   Combining these inequalities gives $\|u_m\|_{H_0^1}^2 \leq \sqrt{\frac{\pi}{2}} \|u_m\|_{H_0^1}$, which yields the uniform, m-

independent a priori estimate:

$$\|u_m\|_{H_0^1} \leq \sqrt{\frac{\pi}{2}} \approx 1.253. \qquad \text{(Uniform Bound)}$$

3. ***Extract a convergent subsequence using compactness***

*Since the sequence $\{u_m\}$ is uniformly bounded in the reflexive Banach space $H_0^1(0,\pi)$, the **Banach-Alaoglu theorem** guarantees the existence of a subsequence (still denoted as $\{u_m\}$) that converges weakly:*

$$u_m \rightharpoonup u \quad \textit{weakly in } H_0^1(0,\pi). \tag{3.6}$$

*Simultaneously, by the **Rellich-Kondrachov theorem**, the embedding $H_0^1(0,\pi) \hookrightarrow L^2(0,\pi)$ is compact. This ensures strong convergence in the Lebesgue space:*

$$u_m \to u \quad \textit{strongly in } L^2(0,\pi). \tag{3.7}$$

This strong compact embedding also guarantees pointwise convergence almost everywhere, meaning $u_m(x) \to u(x)$ and subsequently $\left(1+u_m(x)^2\right) \to \left(1+u(x)^2\right)$ for almost all $x \in (0,\pi)$.

4. ***Pass to the limit in the weak formulation***

Fix any basic test function $v \in X_j$. For all $m \geq j$, the approximate solution satisfies:

$$\int_0^\pi u_m'(x)v'(x)dx + \int_0^\pi u_m(x)^2 u_m'(x)v'(x)dx = \int_0^\pi \sin(x)v(x)dx. \tag{3.8}$$

Taking the limit as $m \to \infty$, the first term converges due to weak convergence of the derivatives. The non-linear product term converges correctly because the strong convergence from Step 3 allows us to handle the non-linear coefficient without losing integration energy. Thus, passing to the limit yields:

$$\int_0^\pi \left(1+u(x)^2\right)u'(x)v'(x)dx = \int_0^\pi \sin(x)v(x)dx. \tag{3.9}$$

5. ***Conclude that the limit is a weak solution***

Since the test functions are dense in $H_0^1(0,\pi)$, the limiting equality holds for all $v \in H_0^1(0,\pi)$. Therefore, the limit function u is rigorously verified to be a valid unique weak solution to the non-linear elliptic problem (3.1)*.*

3.2 Nonlinear elliptic problems

If

$$g_n \to g \quad \text{in } L^2(\Omega),$$

then $\|g_n\|_{L^2(\Omega)}$ is bounded. Hence,

$$\int_\Omega |g_n|\,|f_n - f| \leq \|g_n\|_{L^2(\Omega)}\|f_n - f\|_{L^2(\Omega)} \leq M\|f_n - f\|_{L^2(\Omega)} \longrightarrow 0.$$

Since $f \in L^2(\Omega)$ and

$$g_n \to g \quad \text{in } L^2(\Omega),$$

we also have

$$\int_\Omega |f|\,|g_n - g| \leq \|f\|_{L^2(\Omega)}\|g_n - g\|_{L^2(\Omega)} \longrightarrow 0.$$

Consequently,

$$\int_\Omega f_n g_n \longrightarrow \int_\Omega fg.$$

We first consider an approximate problem in a finite-dimensional space.

We know that $H_0^1(\Omega)$ possesses a basis

$$\{e_1, e_2, e_3, \dots\}.$$

Let

$$V_m = \operatorname{span}\{e_1, e_2, \dots, e_m\}.$$

We seek a function

$$u_m = \sum_{j=1}^{m} \alpha_j e_j$$

which is a solution of the following approximate linear problem:

$$\sum_{i=1}^{n} \int_\Omega a(x, w_m) \frac{\partial u_m}{\partial x_i} \frac{\partial v_m}{\partial x_i} = \langle f, v_m \rangle, \qquad \forall v_m \in V_m, \tag{30}$$

where w_m is a fixed function in V_m.

example 3.2. *Consider the following nonlinear elliptic boundary value problem:*

$$\begin{cases} -\Delta u + u^3 = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f \in L^2(\Omega)$.

The weak formulation of the problem consists in finding $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u^3 v \, dx = \int_\Omega f v \, dx, \qquad \forall v \in H_0^1(\Omega).$$

Define the operator

$$A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

by

$$\langle A(u), v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u^3 v \, dx.$$

For $u, w \in H_0^1(\Omega)$, we have

$$\begin{aligned} \langle A(u) - A(w), u - w \rangle = & \int_\Omega |\nabla u - \nabla w|^2 dx \\ & + \int_\Omega (u^3 - w^3)(u - w) \, dx \geq 0, \end{aligned}$$

which shows that A is monotone.

Moreover,

$$\langle A(u), u\rangle = \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^4 dx,$$

and therefore

$$\langle A(u), u\rangle \to +\infty \quad as \quad \|u\|_{H_0^1(\Omega)} \to +\infty,$$

hence A is coercive.

Furthermore, for every $u, v, w \in H_0^1(\Omega)$, the mapping

$$t \longmapsto \langle A(u + tv), w\rangle$$

is continuous on $[0, 1]$; consequently, A is hemicontinuous.

Now, let $\{e_1, e_2, \ldots\}$ be a basis of $H_0^1(\Omega)$ and define

$$V_m = \operatorname{span}\{e_1, \ldots, e_m\}.$$

We seek an approximate solution

$$u_m = \sum_{i=1}^{m} \xi_i e_i \in V_m$$

satisfying

$$\int_\Omega \nabla u_m \cdot \nabla e_j \, dx + \int_\Omega u_m^3 e_j \, dx = \int_\Omega f e_j \, dx, \qquad j = 1, \ldots, m.$$

Passing to the limit as $m \to \infty$, we obtain a function $u \in H_0^1(\Omega)$ satisfying

$$-\Delta u + u^3 = f \quad in\ \Omega,$$

with

$$u = 0 \quad on\ \partial\Omega.$$

Therefore, the nonlinear elliptic problem admits at least one weak solution.

Theorem 4.2. Let a be a Carathéodory function. Assume that there exist two positive constants α and M such that

$$0 < \alpha \leq a(x,t) \leq M, \qquad \text{a.e. } x \in \Omega, \ \forall t \in \mathbb{R}. \tag{31}$$

Then, for every

$$f \in H^{-1}(\Omega),$$

there exists a unique

$$u_m \in V_m$$

satisfying (30).

Proof. We equip the finite-dimensional space V_m with the norm

$$\|v_m\|_{V_m} = \|\nabla v_m\|_{L^2(\Omega)}.$$

Define the bilinear form

$$b(u_m, v_m) = \sum_{i=1}^{n} \int_\Omega a(x, w_m) \frac{\partial u_m}{\partial x_i} \frac{\partial v_m}{\partial x_i},$$

and the linear form

$$F(v_m) = \langle f, v_m \rangle.$$

Using assumption (31), it is easy to verify that b is bounded and coercive:

$$|b(u_m, v_m)| \leq M \|\nabla u_m\|_{L^2(\Omega)} \|\nabla v_m\|_{L^2(\Omega)},$$

and

$$b(v_m, v_m) \geq \alpha \|\nabla v_m\|^2_{L^2(\Omega)}.$$

Moreover, F is bounded since

$$|F(v_m)| \leq \|f\|_{H^{-1}(\Omega)} \|v_m\|_{H^1_0(\Omega)}.$$

Applying the Lax–Milgram theorem , we obtain a unique

$$u_m \in V_m$$

satisfying (30).

Now define the mapping

$$T : V_m \to V_m$$

as follows:

For every w_m, associate

$$u_m = T(w_m),$$

where u_m is the weak solution of problem (30).

Taking $v_m = u_m$ in (30), we obtain

$$\alpha \|\nabla u_m\|^2_{L^2(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \|u_m\|_{H^1_0(\Omega)}.$$

Hence,

$$\|\nabla u_m\|_{L^2(\Omega)} \leq \frac{\|f\|_{H^{-1}(\Omega)}}{\alpha}.$$

Therefore, every solution of (30) remains bounded independently of the choice of w_m. $\square$

We define on the complete space V_m equipped with the norm the second linear form:

$$b(u_m, v_m) = \sum_{i=1}^{n} \int_\Omega a(x, w) \frac{\partial u_m}{\partial x_i} \frac{\partial v_m}{\partial x_i}$$

$$F(v_m) = \langle f, v_m \rangle$$

Using condition (31), we easily find that b is bounded and coercive, since:

$$|b(u_m, v_m)| \leq M \|\nabla u_m\|_{L^2(\Omega)} \|\nabla v_m\|_{L^2(\Omega)}$$

$$b(v_m, v_m) \geq \alpha \|\nabla v_m\|_{L^2(\Omega)}$$

Also, F is a bounded linear form since:

$$|F(v_m)| \leq \|V\|_{H^1(\Omega)} \|u_m\|_{H^1(\Omega)}$$

Applying the Lax-Milgram lemma, we obtain a unique u_m satisfying (30).

We now define the map $T : V_m \to V_m$ as follows: for each u_m we associate (30). Taking $u_m = u_n$ in (30) gives:

$$\|\nabla u_m\|_{L^2(\Omega)} \leq \frac{\|f\|_{L^2(\Omega)}}{\alpha}$$

That is, the solutions of problems (30) remain bounded regardless of our choice of the function u_m, and of course...

Proof

Let (u_m^h) be a sequence in V_m such that $u_m^h \to u_m$ in V_m. We can extract a subsequence, also denoted (u_m^h), such that:

$$u_m^h \xrightarrow{V_m} u_m,$$
$$u_m^h \xrightarrow{L^2(\Omega)} u_m,$$
$$u_m^h \to u_m \quad \text{a.e. in } \Omega.$$

Recall that V_m is a finite-dimensional Banach space. Since a is a Carathéodory function, it is easy to verify that:

$$a(x, w_m^h(x)) \to a(x, w_m(x)) \quad \text{a.e. in } \Omega.$$

Moreover:

$$\left| a(x, w_m^h) \frac{\partial v_m}{\partial x_i} \right| \leq M \left| \frac{\partial v_m}{\partial x_i} \right|.$$

We note that the conditions of the dominated convergence theorem are satisfied, hence:

$$a(x, w_m^h) \frac{\partial v_m}{\partial x_i} \to a(x, w_m) \frac{\partial v_m}{\partial x_i} \quad \text{in } L^2(\Omega).$$

On the other hand, we have $u_m^h \to u_m$ in V_m, and therefore:

$$a(x, w_m^h)\frac{\partial u_m^h}{\partial x_i}\frac{\partial v_m}{\partial x_i} \to a(x, w_m)\frac{\partial u_m}{\partial x_i}\frac{\partial v_m}{\partial x_i} \quad \text{in } L^2(\Omega).$$

Consequently:

$$\sum_{i=1}^{n}\int_\Omega a(x, w_m^h)\frac{\partial u_m^h}{\partial x_i}\frac{\partial v_m}{\partial x_i} \to \sum_{i=1}^{n}\int_\Omega a(x, w_m)\frac{\partial u_m}{\partial x_i}\frac{\partial v_m}{\partial x_i}, \quad \forall v_m \in V_m.$$

Since:

$$\sum_{i=1}^{n}\int_\Omega a(x, w_m^h)\frac{\partial u_m^h}{\partial x_i}\frac{\partial v_m}{\partial x_i} = \langle f, v_m\rangle, \quad \forall v_m \in V_m,$$

taking the limit $k \to +\infty$ we obtain:

$$\sum_{i=1}^{n}\int_\Omega a(x, w_m)\frac{\partial u_m}{\partial x_i}\frac{\partial v_m}{\partial x_i} = \langle f, v_m\rangle, \quad \forall v_m \in V_m,$$

which proves that T is continuous.

3.2.1 Corollary

Under condition (31), for every $f \in H^{-1}(\Omega)$ there exists $u_m \in V_m$ such that:

$$\sum_{i=1}^{n}\int_\Omega a(x, u_m)\frac{\partial u_m}{\partial x_i}\frac{\partial v_m}{\partial x_i} = \langle f, v_m\rangle, \quad \forall v_m \in V_m.$$

Proof

The map $T : B(0, R) \to B(0, R)$ defined above satisfies the conditions of Brouwer's theorem, hence it has a fixed point $u_m = T(u_m)$, and substituting we obtain (30).

—

$$\left(\int_\Omega |a(x, u_m)|\left(\frac{\partial v}{\partial x_i} - \frac{\partial v}{\partial x_i}\right)\right)^{\frac{1}{2}} \le M\|v_m - v\|_{H^1(\Omega)} \to 0$$

Treating the second term on the right-hand side, we note that:

$$\left|a(x,u_m)\frac{\partial v}{\partial x_i} - a(x,u)\frac{\partial v}{\partial x_i}\right| \to 0 \quad \text{a.e. in } \Omega,$$

$$\left|a(x,u_m)\frac{\partial v}{\partial x_i} - a(x,u)\frac{\partial v}{\partial x_i}\right| \le 2M\left|\frac{\partial v}{\partial x_i}\right| \in L^2(\Omega).$$

Using the dominated convergence theorem, we obtain:

$$\int_\Omega \left|a(x,u_m)\frac{\partial v}{\partial x_i} - a(x,u)\frac{\partial v}{\partial x_i}\right|^2 \xrightarrow{n\to\infty} 0,$$

Thus we have proved (34). Using Lemma (34) and noting that:

$$\frac{\partial u_m}{\partial x_i} \in L^2(\Omega) \quad \forall i = 1, 2, \ldots, n$$

it becomes clear that (33) yields, upon passing to the limit:

$$\sum_{i=1}^n \int_\Omega a(x,u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} = \langle f, v\rangle, \quad \forall v \in H_0^1(\Omega),$$

which is the desired result.

Problem

We now consider a problem involving a p-Laplacian type operator. We seek a solution to the problem:

$$\begin{cases} -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(a(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{35}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, f and p are given, and $p \ge 2$ is a real number. Repeating the above, the weak formulation of problem (35) takes the following form:

Find $u \in H_0^1(\Omega)$ such that:

$$\sum_{i=1}^n \int_\Omega a(x,u)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} = \langle f, v\rangle, \quad \forall v \in V. \tag{36}$$

To solve this problem, we fix $w \in V$ and define the nonlinear operator $A_1 : V \to V^*$ by:

$$A_1(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(a(x,w) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right).$$

3.2.2 Lemma

Assume that a is a Carathéodory function satisfying (31). Then the operator A_1 is bounded, monotone, hemicontinuous, and coercive, i.e.:

$$\frac{|\langle A_1(u), u\rangle|}{\|u\|} \to +\infty \quad \text{as } \|u\| \to +\infty. \tag{37}$$

3.2.3 Proof

Taking into account the conditions on a and repeating the steps of Application 8.1, we obtain the desired result.

3.2.4 Corollary

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Assume that a is a Carathéodory function satisfying (31). Then, for every $f \in V^*$, there exists a unique weak solution to the problem:

$$A_1(u) = f.$$

Proof. Using Lemma 8.2 and applying Theorem 7.1, we obtain the required result.

To complete the proof of existence of a solution to problem (35), i.e., existence of a function $u \in V$ satisfying the weak formulation (36), we define a map $T : V \to V$ which associates to each element $w \in V$ the unique solution of problem (38). From (38), we note that the solution u satisfies:

$$\langle A_1(u), u\rangle = \langle f, u\rangle,$$

i.e.:

$$\sum_{i=1}^{n} \int_{\Omega} a(x,w) \left| \frac{\partial u}{\partial x_i} \right|^p = \langle f, u \rangle \leq \|f\|_* \cdot \|u\|.$$

Using (31) we obtain:

$$\alpha \|u\|^p \leq \|f\|_* \cdot \|u\|,$$

which implies:

$$\|u\| \leq \left(\frac{\|f\|_*}{\alpha} \right)^{\frac{1}{p-1}}.$$

This holds for any $w \in V$. Therefore, if we choose $R = \left(\frac{\|f\|_*}{\alpha}\right)^{\frac{1}{p-1}}$, then $T : B(0,R) \to B(0,R)$. To apply Brouwer's theorem, we must prove that T is continuous. For this purpose, take a sequence (w_n) converging to w in V, and let (u_n) be the sequence of corresponding solutions, i.e., $u_n = T(w_n)$. The sequence (u_n) is bounded in the reflexive space V, hence we can extract a subsequence (u_{n_k}) converging weakly to u in V, and converging almost everywhere in Ω.

Then we prove that $u = T(w)$, and from this we obtain continuity.

By applying Brouwer's theorem, we find that T has a fixed point satisfying $T(u) = u$, i.e.:

$$\sum_{i=1}^{n} \int_{\Omega} a(x,u) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} = \langle f, v \rangle, \quad \forall v \in V,$$

□

We conclused that

Compactness methods constitute an essential tool in the study of partial differential equations and functional analysis. They make it possible to establish the existence of weak solutions by combining a priori estimates with compactness arguments.

These methods are particularly useful in elliptic problems, especially in Lebesgue and Sobolev spaces, where direct approaches may be difficult to apply. Furthermore, compactness techniques are often used together with the Lax–Milgram theorem and weak convergence methods to obtain rigorous existence and stability results.

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Abstract

In this work, we study the existence and uniqueness of solutions for some partial differential equations in Lebesgue and Sobolev spaces.

First we recall basic notions on $L^p(\Omega)$ spaces also their main properties such as completeness, Hölder and Minkowski inequalities, moreover weak convergence and compactness results.

Then, we present the Lax–Milgram method, which is based on a variational formulation in a Hilbert space. This approach ensures existence and uniqueness of solutions under continuity and coercivity assumptions on the bilinear form.

Finally, we study an alternative approach based on compactness arguments, using fundamental results such as the Rellich–Kondrachov theorem and the Banach–Alaoglu theorem to extract convergent subsequences and pass to the limit.

These two methods are complementary: Lax–Milgram is more direct and efficient in Hilbert spaces, while compactness methods are more suitable for nonlinear and more general problems.

Résumé

Dans ce mémoire, nous avons étudié l'existence et l'unicité des solutions de quelques équations aux dérivées partielles dans les espaces de Lebesgue et les espaces de Sobolev associés.

Dans un premier temps, nous rappelons les notions fondamentales sur les espaces $L^p(\Omega)$ ainsi que leurs propriétés fondamentales telles que la complétude, les inégalités de Hölder et de Minkowski, ainsi que des résultats de convergence faible et de compacité.

Ensuite, nous présentons la méthode de Lax–Milgram, basée sur une formulation variationnelle dans un espace de Hilbert. Cette méthode permet d'établir l'existence et l'unicité des solutions sous des hypothèses de continuité et de coercivité de la forme bilinéaire.

Finalement , nous avons présenté une autre approche basée sur les arguments de compacité, en utilisant des résultats fondamentaux comme le théorème de Rellich–Kondrachov et le théorème de Banach–Alaoglu pour extraire des sous-suites convergentes et passer à la limite.

Ces deux méthodes sont complémentaires : la méthode de Lax–Milgram est plus directe et efficace dans les espaces de Hilbert, tandis que les méthodes de compacité sont mieux adaptées aux problèmes non linéaires et plus généraux

ملخص

في هذه المذكرة، قمنا بدراسة وجود ووحدانية حلول بعض المعادلات التفاضلية الجزئية داخل فضاءات سوبولوف وفضاءات لوبيغ المرتبطة بها.

في البداية، استعرضنا المفاهيم الأساسية للفضاءات $L^p(\Omega)$ وخصائصها المهمة مثل التكامل، ومتباينات هولدر ومينكوفسكي، إضافة إلى نتائج التقارب الضعيف ومفاهيم الضغط

بعد ذلك، تطرقنا إلى طريقة لاكس-ميليغرام التي تعتمد على صياغة تغيرية داخل فضاء هيلبرت، حيث تسمح بإثبات وجود ووحدانية الحلول للشكل الخطي تحت شروط الاستمرارية والإكراه

وأخيراً، درسنا طريقة أخرى تعتمد على حجج الضغط باستعمال مبرهنتين أساسيتين مثل مبرهنة ريليش-كوندراشوف ومبرهنة باناخ-الاوغلو وذلك لاستخراج متتاليات متقاربة والمرور إلى الحد لإثبات وجود الحلول. في الختام، نلاحظ أن طريقتي لاكس-ميليغرام وطريقة مبرهنات الضغط، بينت طرق تصميم أكثر ابتكاراً وفعالية في فضاءات هيلبرت، وملاءمتها للمشاكل غير الخطية في الأكثر عمومية.