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Thème

Contributions aux équations aux dérivées fractionnaires conformables

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Dedications

To my family, thank you for your assistance and encouragement over the years.

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Resume

In this thesis, we have studied four problems of three types of fractional differential equations with integral boundary conditions:

- ✓ At first, their solutions are presented in form of nonlinear integral equations by using the Green function, which her properties are analyzed to determine the appropriate way to solve the proposed problems.
- ✓ Secondly, we proved the positivity, existence and uniqueness of solutions to the proposed problems by using some fixed theorems.
- ✓ Next, we give a new results concerning the stability of fractional equations in Ulam-Hyres sense to prove the stability of solutions for proposed problems.
- ✓ Finally, several examples are presented to confirm the effectiveness of some utilized theorems.

Keywords: Fractional differential equations, Integral boundary conditions, Fixed point theorems, Ulam-Hyers stability.

Dans cette thèse, nous avons étudié quatre problèmes de trois types d'équations différentielles fractionnaires avec des conditions aux limites intégrales:

- ✓ Premièrement, nous représentons leurs solutions qui sont en forme d'équations intégrales non linéaires utilisant la fonction de Green, dont les propriétés sont étudiées pour déterminer la manière appropriée pour résoudre les problèmes proposés.
- ✓ Deuxièmement, nous avons prouvé la positivité, l'existence et l'unicité des solutions aux problèmes proposés basée à certains théorèmes de point fixe.
- ✓ Ensuite, nous avons donné de nouveaux résultats concernant la stabilité des équations fractionnaires au sens d'Ulam-Hyers pour prouver la stabilité des solutions aux problèmes proposés.
- ✓ Enfin, plusieurs exemples sont donnés pour confirmer l'efficacité de certains théorèmes utilisés.

Mots-Clés: Équation différentielles fractionnaires, Conditions aux limites intégrales, Théorèmes de point fixe, Stabilité d'Ulam-Hyers.

Abbreviations and notations

$\Gamma(.)$	The Gamma function	12
$\mathcal{D}^{\alpha}_{a^+}x$	The left conformable fractional derivative of order α of a function x	15
$\mathcal{D}^{\alpha}_{b^-}x$	The right conformable fractional derivative of order α of a function x	15
$\mathcal{I}^{\alpha}_{a^+}x$	The left conformable fractional integral of order α of a function x	16
$\mathcal{I}^{\alpha}_{b^-}x$	The right conformable fractional integral of order α of a function x	16
Ω_i	$:= \{x \in E,\ \Vert x\Vert < r_i,\ r_i > 0,\ i = 0, 1, \cdots\}$ is the bounded open subset of E	18
$\overline{\Omega_i}$	the closure of subset Ω_i	18
$\varphi_p(t)$	$= t\,\vert t\vert^{p-2},\ p > 1$ The p-Laplacian operator where $(\varphi_p)^{-1} = \varphi_q,\ \frac{1}{p} + \frac{1}{q} = 1$	9
${}^{c}D^{\alpha}_{a^+}x$	The left Caputo fractional derivative of order α of a function x	14
${}^{c}D^{\alpha}_{a^+}x$	The right Caputo fractional derivative of order α of a function x	14
$AC^n(a, b)$	Space of differentiable functions of order $n-1$ and the $(n-1)^{th}$ derivative is absolutely continuous on (a, b)	15
$B(.,.)$	The Beta function	13
$C(a, b)$	Space of continuous functions (a, b)	15
$C^n(a, b)$	Space of n-times differentiable functions on (a, b) and the n^{th} derivative is continuous	15
$D^{\alpha}_{a^+}x$	The left Riemann-Liouville fractional derivative of order α of a function x	14
$D^{\alpha}_{b^-}x$	The right Riemann-Liouville fractional derivative of order α of a function x	14
E	$= C([0, 1], [0, +\infty))$ the Banach space of all continuous functions x from $[0, 1]$ into $[0, +\infty)$ with the norm $\Vert x\Vert = \max_{t\in[0,1]} \vert x(t)\vert$	18
f^0	$= \lim_{x\to 0} \max_{t\in[0,1]} \frac{f(t,x)}{x}$	31

f^∞	$= \lim_{x\to+\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x}$	33
f_0	$= \lim_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x}$	33
f_∞	$= \lim_{x\to+\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x}$	31
$I^\alpha_{a^+}x$	The left Riemann-Liouville fractional integral of order α of a function x	14
$I^\alpha_{a^+}x$	The right Riemann-Liouville fractional integral of order α of a function x	14
$L\left(a,b\right)$	Space of measurable functions, integrable over $L\left(a,b\right)$.	15
CFBVP	Conformable fractional boundary value problem	10
CFPs	Conformable fractional problems	11
FBVP	Fractional boundary value problem	10

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General introduction

Fractional calculus was begun from a letter written by L’Hôspital to Leibniz in 1695 asking him if $n = \frac{1}{2}$, what does it mean $\frac{d^n f}{dx^n}$. Leibniz then responded saying “An apparent paradox, from which one day useful consequences will be drawn”(Oldham et Spanier, see [31], which is the first monograph devoted to Fractional Calculus in 1974). A lot of contributions to the theory of fractional calculus take up by famous mathematicians. In June of same year, the ”First Conference on Fractional Calculus and its Applications” has organized by Ross at the University of New Haven, and edited its proceedings [34]. Since then the application of fractional calculus is extended to many fields of science and engineering (mathematical physics, biology, bioengineering, control theory, hydrology, thermodynamics, mechanic and finance; see the books [13, 24, 32, 37]).

Various types of fractional derivative definitions were introduced in history (the most popular ones are Riemann-Liouville and Caputo). Unfortunately, they do not inherit all the operational behaviors from the typical first derivative, such as the rules of product, quotient, composition and chain and semigroup property. A new type of fractional derivative was introduced by Khalil et al. in [20] and developed by Abdeljawad [1] called "conformable fractional derivative", its has a limit form as in the classical derivative and its properties are almost similar than the classical one, which has made it an interesting subject of several topics.

As same as the fractional calculus, the development of the Ulam-Hyers stability also has a long history. In 1940, Ulam asked the following question in a seminar at Wisconsin University about stability of group homomorphism: does there exists a relationship between the exact and approximate homomorphisms, from a group Θ_1 to a metric group Θ_2 (for more details see [42, 43]). After one year, Hyers gave The first reply to Ulam’s question in the context of Banach spaces in the case of additive mappings, this type of stability is called Ulam-Hyers stability. In 1978, Rassias in [33] gave more extension to Hyers’s idea, where the bound for the norm of Cauchy difference was setting in more general form. This concept of stability is termed as Ulam-Hyers-Rassias stability. The classical concept of Ulam-Hyers stability is required to find a function which satisfies a suitable approximation inequality. This approach can guarantee that there exists a close exact solution useful in many applications such as numerical analysis and optimization, where finding the exact solution is impossible. Recently, the stability research is also be a key for developing the fractional calculus, for the Ulam-Hyers stability of fractional differential equations, see [5,9,10,19,23,28,46] and for conformable fractional differential equations, see [8,25,53].

Differential equations with integral boundary conditions are an important class of problems which arise in the fields of electro chemistry, chemical engineering, cellular systems, heat transmission, plasma physics, thermo elasticity, heat conduction, underground water, population dynamics, blood fow models, ect. They spring up when values of the function on the boundary are related to its values inside the domain, they have physical significations (total mass, moments, $\cdots$) and with the fact that fractional-order models are more degrees of freedom than integer-order models, that is, we refer the reader to studying the following fractional problems with integral boundary conditions [3,6,7,10,11,18,26,27,36,38–40,44,46–49,51,52] and the references cited therein, for specifically; the authors in [50] concerned with the p-Laplacian fractional differential equations involving left Riemann-Liouville derivatives and the Riemann-Stieltjes integral boundary condition:

$$\begin{aligned} &-D_{0^+}^{\beta}\left(\varphi_p\left(D_{0^+}^{\alpha}x\right)\right)(t)=\lambda f\left(t,x\left(t\right)\right),\ \beta\in(0,1],\ \alpha\in(1,2],\ t\in(0,1),\\ &x\left(0\right)=0,\ D_{0^+}^{\alpha}x\left(0\right)=0,\ x\left(1\right)=\int_0^1 x\left(t\right)dA\left(t\right). \end{aligned}$$

The p-Laplacian operator φ_p is defined as $\varphi_p\left(t\right)=t\left|t\right|^{p-2},\ p>1$, where $\left(\varphi_p\right)^{-1}=\varphi_q,\ \frac{1}{p}+\frac{1}{q}=1$, the function $f:(0,1)\times(0,+\infty)\rightarrow(0,+\infty)$ is continuous. A is a function of bounded variation and $\int_0^1 x\left(t\right)dA\left(t\right)$ denotes the Riemann-Stieltjes integral of x with respect to A, dA can be a signed measure. The main results are established by using the method of upper and lower solutions and the Schauder fixed point theorem. Only a few authors have discussed the conformable fractional problems with integral boundary conditions, see [17,29,30,41,54]. Among them some work is devoted to the Ulam-Hyers stability of nonlinear fractional differential equations with integral boundary conditions as well, see [4,6], for instance; M. Houas et al. in [19] studied the The main results by applying the contraction mapping, O'Regan fixed point theorem and theory of Ulam-Hyers stability to the following fractional boundary value problem involving left Caputo fractional derivatives with nonlocal boundary conditions:

$$\begin{aligned} &{}^{c}D_{0^+}^{\alpha}\left({}^{c}D_{0^+}^{\beta}+\lambda\right)x\left(t\right)=f\left(t,x(t)\right)+\int_0^t\frac{\left(t-s\right)^{\sigma-1}}{\Gamma\left(\sigma\right)}f\left(s,x(s)\right)ds,\ t\in[0,T],\\ &x\left(0\right)=x_0+g\left(x\right),\ x\left(T\right)=\theta\int_0^{\eta}\frac{\left(\eta-s\right)^{p-1}}{\Gamma\left(p\right)}x\left(s\right)ds,\ \eta\in(0,T), \end{aligned}$$

where $\alpha,\ \beta\in(0,1],\ \alpha+\beta\in(1,2]$, f and g are continuous functions, and $\sigma, p, \lambda,\ x_0,\ \theta$ are real constants.

The main contribution of this thesis is the study of the existence, uniqueness and stability of solutions of certain fractional differential equations with integral boundary conditions. This thesis consists of 4 chapters as follows:

We begin by recalling, in Chapter 1, the notions of gamma and Beta functions, which play an important role in the theory of fractional differential equations. Then, we give the concepts of derivation and fractional integration in the sense of Riemann Liouville, Caputo and conformable and their properties. Next, we cite some useful fixed point theorems. We finish it by giving an overview on the Ulam-Hyers stability definitions for fractional differential equations considered.

In the second chapter, we use the Guo-Krasnoselskii fixed point theorem, the Banach contraction mapping principle and the Ulam-Hyers stability to prove the existence, uniqueness and stability of positive solution for the following mixed fractional boundary value problem (mixed-FBVP) with integral boundary conditions involving both right conformable and left Caputo fractional derivatives:

$$\begin{aligned} &\mathcal{D}_{1^-}^{\beta}\left({}^{c}D_{0^+}^{\alpha}x\right)(t)=f\left(t,x\left(t\right)\right),\ t\in\left(0,1\right),\\ &x\left(0\right)=\gamma\int_0^1 x\left(t\right)dt,\ {}^{c}D_{0^+}^{\alpha}x\left(1\right)=0,\end{aligned} \tag{P1}$$

where $\alpha,\ \beta\in\left(0,1\right],\ \gamma\in\left(0,1\right)$ and the function $f:[0,1]\times\mathbb{R}_+\to\mathbb{R}_+$ is continuous. Finally, the main result is strengthened through examples. The results in this chapter are published in "*Moroccan J. of Pure and Appl. Anal.*" in title "*A new class of mixed fractional differential equations with integral boundary conditions*" , see [15].

In the third chapter, we concerning by the the following two conformable fractional boundary value problem (two-CFBVP) with integral boundary conditions:

$$\begin{aligned} &\mathcal{D}_{0^+}^{\beta}\mathcal{D}_{0^+}^{\alpha}x\left(t\right)+\lambda f\left(t,x\left(t\right)\right)=0,\ t\in\left(0,1\right),\\ &\mathcal{D}_{0^+}^{\alpha}x\left(0\right)=0,\ x\left(1\right)=\gamma\int_0^1 x\left(t\right)dt,\end{aligned} \tag{P2}$$

where $\alpha,\ \beta\in\left(0,1\right],\ \lambda>0,\ \gamma\geq0$ and the function $f:[0,1]\times\mathbb{R}_+\to\mathbb{R}_+$ is continuous. We establish the existence, uniqueness of positive solution by using the Guo-Krasnoselskii fixed point theorem and the Banach contraction principle theorem. Also, we give the theory of Ulam-Hyers stability of solutions for the problem considered. We end this chapter with two illustrative examples. The results in this chapter are accepted for publication in "*Stud. Univ. Babeş-Bolyai Math.*" in title "*Nonlinear two conformable fractional differential equation with integral boundary condition*" , see [14].

In the last chapter, we used Guo-Krasnoselskii fixed point theorem and Banach contraction mapping principle combined with Ulam-Hyers stability to study the existence, uniqueness and stability of positive solution for the following two p-Laplacian problems involving conformable fractional derivatives (p-Laplacian CFPs) with integral boundary conditions:

$$\begin{gathered}\mathcal{D}_{0^+}^{\beta}\left(\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)\right)(t)=-f\left(t,x\left(t\right)\right),\ t\in\left(0,1\right),\\ \varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(0)=0,\ \varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(1)=\lambda\int_0^1\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(t)\,dt,\\ x\left(0\right)=0,\ x\left(1\right)=\gamma\int_0^1 x\left(t\right)dt,\end{gathered}\tag{P3}$$

and

$$\begin{aligned}&\mathcal{D}_{0^+}^{\beta}\left(\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)\right)(t)=f\left(t,x\left(t\right)\right),\ t\in\left(0,1\right),\\ &\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(0)=0,\ \left(\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)\right)'(0)=\mu\int_0^1\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(t)\,dt,\\ &x\left(0\right)=0,\ x'\left(0\right)=\lambda\int_0^1 x\left(t\right)dt,\end{aligned}\tag{P4}$$

where $\alpha,\ \beta\in(1,2]$, the function $f:[0,1]\times\mathbb{R}_+\to\mathbb{R}_+$ is continuous, the parameter λ and γ belong to $[1,2)$ and the p-Laplacian operator $\varphi_p\left(t\right)=t\left|t\right|^{p-2},\ p>1$, where $\left(\varphi_p\right)^{-1}=\varphi_q,\ \frac{1}{p}+\frac{1}{q}=1$. One example is also given to show the applicability of our results.

We end this thesis with a general conclusion and some perspectives.

CHAPTER 1

PRELIMINARY CONCEPTS OF FRACTIONAL CALCULUS

This chapter is concerned with basic results, functions, definitions, lemmas, theorems, notations, which will be used throughout the thesis.

1.1 Some special functions

We give the special functions Gamma and Beta, which use in the formation of fractional order derivatives and integration, by:

1.1.1 Gamma function

The Gamma function [32] was introduced by the Swiss mathematician Leonhard Euler in 1729, by

$$\Gamma\left(\rho\right)=\int_{0}^{\infty}t^{\rho-1}e^{-t}dt,\ Re\left(\rho\right)>0.$$

Γ is a extension of the factorial function to positive real values and complex argument with positive real part. Some of the most important properties are

$$\begin{aligned}&\Gamma\left(1\right)=\Gamma\left(2\right)=1,\\&\Gamma\left(\rho+1\right)=\rho\Gamma\left(\rho\right),\\&\Gamma\left(1/2\right)=\sqrt{\pi},\\&\Gamma\left(n\right)=(n-1)!,\ n\in\mathbb{N},\\&\Gamma\left(n+1/2\right)=\frac{\sqrt{\pi}}{2^{n}}\left(2n-1\right)!,\ n\in\mathbb{N}.\end{aligned}$$

1.1.2 Beta function

The Beta function [32] is studied by Euler and Legendre, which is a kind of Euler integral. For complex numbers p and q, the function is defined by :

$$B\left(p,q\right)=\int_0^1 s^{p-1}\left(1-s\right)^{q-1}ds,\ Re\left(p\right)>0,\ Re\left(q\right)>0,$$

which is symmetric function. The relationship between beta function and gamma function

$$B\left(p,q\right)=\frac{\Gamma\left(p\right)\Gamma\left(q\right)}{\Gamma\left(p+q\right)}.$$

1.2 Fractional calculus

1.2.1 The Riemann-Liouville fractional integrals and derivatives

The starting point of the Riemann-Liouville definition was Cauchy's integral formula:
Let $f:[a,b]\to\mathbb{R}$ is a continuous function. The left primitive of f is given by

$$J_{a^+}^1 f\left(t\right)=\int_a^t f\left(\tau_1\right)d\tau_1.$$

For the primitive second of f and according to Fubini's theorem, we have

$$J_{a^+}^2 f\left(t\right)=\int_a^t\left(\int_a^{\tau_1} f\left(\tau_2\right)d\tau_2\right)d\tau_1=\int_a^t\left(t-\tau\right)f\left(\tau\right)d\tau.$$

Cauchy's formula for repeated integral ([32], p.64) is

$$J_{a^+}^n f\left(t\right)=\int_a^t\int_a^{\tau_1}\cdots\int_a^{\tau_{n-1}} f\left(\tau_n\right)d\tau_n\cdots d\tau_2 d\tau_1=\frac{1}{(n-1)!}\int_a^t\left(t-\tau\right)^{n-1}f\left(\tau\right)d\tau,\ t>0,\ n\in\mathbb{N}^*.$$

Moreover, the right primitive of f is given by

$$J_{b^-}^1 f\left(t\right)=\int_b^t f\left(s\right)ds=-\int_t^b f\left(s\right)ds.$$

We could define in the same way the right-sided integral of order n^{th} of f by

$$J_{b^-}^n f\left(t\right)=\frac{\left(-1\right)^n}{(n-1)!}\int_t^b\left(s-t\right)^{n-1}f\left(s\right)ds.$$

If n is substituted by a positive real number and the Euler gamma function is used instead of the factorial, a new notions of fractional integration is obtained:

Definition 1.1. The left and right Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x \in L(a,b)$ are defined respectively by

$$I_{a^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\, ds,$$
$$I_{b^-}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} x(s)\, ds.$$

Further, the definitions of Riemann-Liouville and Caputo fractional derivatives are giving by:

Definition 1.2. [24] The left Riemann-Liouville's fractional derivative of order $n-1 \leq \alpha < n,\ n \in \mathbb{N}^*$ of a continuous function $x : [a,b] \to \mathbb{R}$ is given by

$$D_{a^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} x(s)\, ds.$$

The right Riemann-Liouville's fractional derivative of order $n-1 \leq \alpha < n,\ n \in \mathbb{N}^*$ terminating at b of x is defined by

$$D_{b^-}^{\alpha} x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} x(s)\, ds.$$

Definition 1.3. [24] The left Caputo fractional derivative of order $n-1 \leq \alpha < n,\ n \in \mathbb{N}^*$ of a function $x \in AC^n([a,b], \mathbb{R})$ is given by

$${}^{c}D_{a^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n x(s)\, ds.$$

The right Caputo fractional derivative of order $n-1 \leq \alpha < n,\ n \in \mathbb{N}^*$ terminating at b of $x \in AC^n([a,b], \mathbb{R})$ is defined by

$${}^{c}D_{b^-}^{\alpha} x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n x(s)\, ds.$$

Remark 1.1. When $a = 0$, we write D^{α}, ${}^{c}D^{\alpha}$ and I^{α}.

Next Lemma present some properties between the fractional integrals and derivatives:

Lemma 1.1. *[24] For $n-1 \leq \alpha < n,\ n \in \mathbb{N}^*$ and $x \in L\left(a,b\right)$*

(i)

$$D_{b^-}^{\alpha}\left(I_{b^-}^{\alpha}x\left(t\right)\right) = D_{a^+}^{\alpha}\left(I_{a^+}^{\alpha}x\left(t\right)\right) = {}^{c}D_{b^-}^{\alpha}\left(I_{b^-}^{\alpha}x\left(t\right)\right) = {}^{c}D_{a^+}^{\alpha}\left(I_{a^+}^{\alpha}x\left(t\right)\right) = x\left(t\right).$$

(ii) *If $x \in C\left(a,b\right)$ with a fractional derivative belongs to $C\left(a,b\right) \cap L\left(a,b\right)$, then*

$$I_{a^+}^{\alpha}\left(D_{a^+}^{\alpha}x\left(t\right)\right) = x\left(t\right) - \sum_{k=1}^{n}\frac{x^{(k)}\left(a\right)}{k!}\left(t-a\right)^{\alpha-k}.$$

$$I_{b^-}^{\alpha}\left(D_{b^-}^{\alpha}x\left(t\right)\right) = x\left(t\right) - \sum_{k=1}^{n}\frac{\left(-1\right)^{k}x^{(k)}\left(b\right)}{k!}\left(b-t\right)^{\alpha-k}.$$

(iii) *If $x \in AC^{n}\left(a,b\right)$ or $x \in C^{n}\left(a,b\right)$, then*

$$I_{a^+}^{\alpha}\left({}^{c}D_{a^+}^{\alpha}x\left(t\right)\right) = x\left(t\right) - \sum_{k=0}^{n-1}\frac{x^{(k)}\left(a\right)}{k!}\left(t-a\right)^{k}.$$

$$I_{b^-}^{\alpha}\left({}^{c}D_{b^-}^{\alpha}x\left(t\right)\right) = x\left(t\right) - \sum_{k=0}^{n-1}\frac{\left(-1\right)^{k}x^{(k)}\left(b\right)}{k!}\left(b-t\right)^{k}.$$

1.2.2 Conformable fractional calculus

The conformable fractional derivative is a new well behaved definition of fractional derivative, based on a simple limit definition. The original definition of the conformable derivatives of order $0 < \alpha \leq 1$ is defined in [20] and generalized in [1].

Definition 1.4. [20] The left conformable fractional derivative starting from a of a function $x : [a, +\infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by

$$\mathcal{D}_{a^+}^{\alpha}x\left(t\right) = \lim_{\varepsilon \to 0}\frac{x\left(t+\varepsilon\left(t-a\right)^{1-\alpha}\right) - x\left(t\right)}{\varepsilon},\ for\ all\ t > a.$$

If $\mathcal{D}_{a^+}^{\alpha}x$ exists on (a,b) then $\mathcal{D}_{a^+}^{\alpha}x\left(a\right) = \lim_{t \to a^+}\mathcal{D}_{a^+}^{\alpha}x\left(t\right)$. If $\mathcal{D}_{a^+}^{\alpha}x\left(t_0\right)$ exists and is finite, then we say that x is left-differentiable at t_0.

The right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at b of x is defined by

$$\mathcal{D}_{b^-}^{\alpha}x\left(t\right) = -\lim_{\varepsilon \to 0}\frac{x\left(t+\varepsilon\left(b-t\right)^{1-\alpha}\right) - x\left(t\right)}{\varepsilon},\ for\ all\ t > a.$$

If $\mathcal{D}_{b^-}^{\alpha}x\left(t\right)$ exists on (a,b) then $\mathcal{D}_{b^-}^{\alpha}x\left(b\right) = \lim_{t \to b^-}\mathcal{D}_{b^-}^{\alpha}x\left(t\right)$. If $\mathcal{D}_{b^-}^{\alpha}x\left(t_0\right)$ exists and is finite, then we say that x is right-differentiable at t_0.

Remark 1.2. [20]

(i) Note that if x is differentiable, then

$$\mathcal{D}_{a^+}^{\alpha} x\left(t\right) = \left(t-a\right)^{1-\alpha} x'\left(t\right), \;\; \mathcal{D}_{b^-}^{\alpha} x\left(t\right) = -\left(b-t\right)^{1-\alpha} x'\left(t\right),$$

where $x'\left(t\right) = \lim_{\varepsilon \to 0} \left[x\left(t+\varepsilon\right) - x\left(t\right)\right]/\varepsilon$.

(ii) If the conformable fractional derivative of x of order α exists, then we simply say x is α-differentiable.

(iii) Differentiability implies α-differentiability, but the contrary is not true, that is, a non-differentiable function can be α-differentiable.

Definition 1.5. [20] The left and right conformable fractional integral of a function x of order $0 < \alpha \leq 1$ are defined respectively by

$$\mathcal{I}_{a^+}^{\alpha} x\left(t\right) = \int_a^t \left(s-a\right)^{\alpha-1} x\left(s\right) ds,$$

$$\mathcal{I}_{b^-}^{\alpha} x\left(t\right) = \int_t^b \left(b-s\right)^{\alpha-1} x\left(s\right) ds.$$

Remark 1.3. [20] When $a = 0$ we write $\mathcal{D}^{\alpha}$ and $\mathcal{I}^{\alpha}$.

The next lemma introduce some conformable fractional integrals and derivatives properties.

Lemma 1.2. *[20] For $0 < \alpha \leq 1$, we have*

(i) *If x is a continuous function on (a,b) then*

$$\mathcal{D}_{a^+}^{\alpha}\left(\mathcal{I}_{a^+}^{\alpha} x\left(t\right)\right) = \mathcal{D}_{b^-}^{\alpha}\left(\mathcal{I}_{b^-}^{\alpha} x\left(t\right)\right) = x\left(t\right).$$

(ii) *If $\mathcal{D}_{a^+}^{\alpha} x, \; \mathcal{D}_{b^-}^{\alpha} x$ are continuous on (a,b) then*

$$\mathcal{I}_{a^+}^{\alpha}\left(\mathcal{D}_{a^+}^{\alpha} x\left(t\right)\right) = x\left(t\right) - x\left(a\right), \;\; \mathcal{I}_{b^-}^{\alpha}\left(\mathcal{D}_{b^-}^{\alpha} x\left(t\right)\right) = x\left(t\right) - x\left(b\right).$$

For the extension to the higher order $\rho \in (n, n+1] \;\; (n \in \mathbb{N})$, we have

Definition 1.6. [1] The left fractional derivative starting from a of a function $x : [a, +\infty) \to \mathbb{R}$ of order $\rho \in (n, n+1]$, where $x^{(n)}\left(t\right)$ exists, is defined by

$$\mathbb{D}_{a^+}^{\rho} x\left(t\right) = \mathcal{D}_{a^+}^{\rho-n} x^{(n)}\left(t\right) = \lim_{\epsilon \to 0} \frac{x^{(n)}\left(t + \epsilon\left(t-a\right)^{n+1-\rho}\right) - x^{(n)}\left(t\right)}{\epsilon}.$$

The right fractional derivative of order $\rho \in (n, n+1]$ terminating at b of x is defined by

$$\mathbb{D}_{b^-}^{\rho} x(t) = (-1)^{n+1} \mathcal{D}_{b^-}^{\rho-n} x^{(n)}(t) = (-1)^n \lim_{\varepsilon \to 0} \frac{x^{(n)}\left(t + \varepsilon (b-t)^{n+1-\rho}\right) - x^{(n)}(t)}{\varepsilon}.$$

If $\left(\mathbb{D}_{a^+}^{\rho} x\right)(t)$ exist on (a,b) then $\left(\mathbb{D}_{a^+}^{\rho} x\right)(a) = \lim_{t \to a^+} \left(\mathbb{D}_{a^+}^{\rho} x\right)(t)$.

Remark 1.4. Note that if $\rho = n+1$ then the fractional derivative of x becomes $x^{(n+1)}(t)$ and

$$\mathcal{D}_{a^+}^{\rho} x(t) = (t-a)^{n+1-\rho} x^{n+1}(t), \;\; \mathcal{D}_{b^-}^{\rho} x(t) = (-1)^n (b-t)^{n+1-\rho} x^{n+1}(t),$$

where $x^{n+1}(t) = \lim_{\varepsilon \to 0} \left[x^n(t+\varepsilon) - x^n(t)\right]/\varepsilon$.

Definition 1.7. [1] The corresponding left and right conformable fractional integral of order $\rho \in (n, n+1]$ are defined as

$$\mathbb{I}_{a^+}^{\rho} x(t) = J_{a^+}^{(n+1)}\left((t-a)^{\rho-n-1} x\right)(t) = \frac{1}{n!} \int_a^t (t-s)^n (s-a)^{\rho-n-1} x(s)\, ds,$$

$$\mathbb{I}_{b^-}^{\rho} x(t) = J_{b^-}^{(n+1)}\left((b-t)^{\rho-n-1} x\right)(t) = \frac{1}{n!} \int_t^b (s-t)^n (b-s)^{\rho-n-1} x(s)\, ds.$$

Lemma 1.3. *[1] For $\rho \in (n, n+1]$ $(n \in \mathbb{N})$, we have*

(i) *If x is n-continuous, then, for all $a < t < b$ we have*

$$\mathbb{D}_{a^+}^{\rho}\left(\mathbb{I}_{a^+}^{\rho} x(t)\right) = \mathbb{D}_{b^-}^{\rho}\left(\mathbb{I}_{b^-}^{\rho} x(t)\right) = x(t).$$

(ii) *If x is $(n+1)$-differentiable for $t > a$, then, for all $t > a$ we have*

$$\mathbb{I}_{a^+}^{\rho} \mathbb{D}_{a^+}^{\rho} x(t) = x(t) - \sum_{k=0}^{n} \frac{x^{(k)}(a)}{k!} (t-a)^k.$$

(iii) *If x is $(n+1)$-differentiable for $t < b$, then, for all $t < b$ we have*

$$\mathbb{I}_{b^-}^{\rho} \mathbb{D}_{b^-}^{\rho} x(t) = x(t) - \sum_{k=0}^{n} \frac{(-1)^k x^{(k)}(b)}{k!} (b-t)^k.$$

1.3 Some fixed point theorems

Fixed point theorems play a major role in establishing the existence theory for initial and boundary value problems. We collect here some well-known fixed point theorems which used in studying of our main results. Let the Banach space $E = C([0,1], [0,+\infty))$ with the norm

$\|x\| = \max_{t\in[0,1]} |x(t)|$. To facilitate the use of next Theorem, we provide the following definitions and theorem:

Definition 1.8. [13] Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if for all $x \in P$ and $\lambda \geq 0,\ \lambda x \in P$ and if $x,\ -x \in P$ then $x = 0$.

Definition 1.9. [13] An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Theorem 1.1. *(Ascoli-Arzelà [12]). Let the compact space E. If $\mathcal{T}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{T}$ is relatively compact.*

Theorem 1.2. *(Guo-Krasnoselskii fixed point theorem [16]). Let $P \subset E$ be a cone and Ω_1, Ω_2 are two bounded open subsets of E with $\overline{\Omega}_1 \subset \Omega_2$. Assume that $\mathcal{T} : P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ is a completely continuous operator such that either*

$$\begin{aligned} \|\mathcal{T}x\| &\geq \|x\|,\ x \in P \cap \partial\Omega_1 \text{ and } \|\mathcal{T}x\| \leq \|x\|,\ x \in P \cap \partial\Omega_2 \text{ or,} \\ \|\mathcal{T}x\| &\leq \|x\|,\ x \in P \cap \partial\Omega_1 \text{ and } \|\mathcal{T}x\| \geq \|x\|,\ x \in P \cap \partial\Omega_2. \end{aligned}$$

Then $\mathcal{T}$ has at least one fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

Theorem 1.3. *(The Banach contraction principle theorem [2, 12]). Let the Banach space E, $P \subseteq E$ a nonempty closed subset. If $\mathcal{T} : P \to P$ is a contraction mapping, then $\mathcal{T}$ has a unique fixed point in P.*

1.4 Stability concepts

This section will give a new definitions and properties of Ulam-Hyers stability for various fractional differential equations, we present four types of stability definitions, namely Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias. For the problems (P1), (P2), (P3) and (P4), we focus on the following inequalities:

Let $E = C([0,1], [0,+\infty))$ be the Banach space with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ and $P \subset E$ is a cone. For the positive real number ε and the function $\varphi \in E$, we have for all $t \in [0,1]$

$$\left|\mathcal{D}_{1^-}^{\beta}\left({}^{c}D^{\alpha}y\right)(t)-f\left(t,y\left(t\right)\right)\right|\leq\varepsilon, \tag{1.1}$$

$$\left|\mathcal{D}_{1^-}^{\beta}\left({}^{c}D^{\alpha}y\right)(t)-f\left(t,y\left(t\right)\right)\right|\leq\varphi\left(t\right), \tag{1.2}$$

$$\left|\mathcal{D}_{1^-}^{\beta}\left({}^{c}D^{\alpha}y\right)(t)-f\left(t,y\left(t\right)\right)\right|\leq\varepsilon\varphi\left(t\right), \tag{1.3}$$

$$\left|\mathcal{D}^{\beta}\mathcal{D}^{\alpha}y\left(t\right)+\lambda f\left(t,y\left(t\right)\right)\right|\leq\varepsilon, \tag{1.4}$$

$$\left|\mathcal{D}^{\beta}\mathcal{D}^{\alpha}y\left(t\right)+\lambda f\left(t,y\left(t\right)\right)\right|\leq\varepsilon\varphi\left(t\right), \tag{1.5}$$

$$\left|\mathcal{D}^{\beta}\mathcal{D}^{\alpha}y\left(t\right)+\lambda f\left(t,y\left(t\right)\right)\right|\leq\varphi\left(t\right), \tag{1.6}$$

$$\left|x\left(t\right)-\mathcal{T}\,x\left(t\right)\right|\leq\varepsilon, \tag{1.7}$$

$$\left|x\left(t\right)-\mathcal{T}\,x\left(t\right)\left(t\right)\right|\leq\varepsilon\varphi\left(t\right), \tag{1.8}$$

$$\left|x\left(t\right)-\mathcal{T}\,x\left(t\right)\right|\leq\varphi\left(t\right), \tag{1.9}$$

where the operator $\mathcal{T}\ : P\to P$ defined by

$$\mathcal{T}x\left(t\right)=x\left(t\right),\ t\in\left[0,1\right].$$

Remark 1.5. (i) Ulam-Hyers stability ⇒ generalized Ulam-Hyers stability.

(ii) Ulam-Hyers-Rassias stability ⇒ generalized Ulam-Hyers-Rassias stability.

(iii) Ulam-Hyers-Rassias stability for $\varphi\left(\cdot\right)=1$ ⇒ Ulam-Hyers stability.

1.4.1 Ulam-Hyers stability definitions for a mixed-FBVP

Definition 1.10. [35, 45] The mixed-FBVP (P1) is Ulam-Hyers stable if there exist constants $\lambda>0$ such that for each $\varepsilon>0$ and for each solution $y\in E$ of the inequality (1.1) there exists a solution $x\in E$ of the mixed-FBVP (P1) such that

$$\left|y\left(t\right)-x\left(t\right)\right|\leq\lambda\varepsilon,\ t\in\left[0,1\right].$$

Definition 1.11. [35, 45] The mixed-FBVP (P1) is generalized Ulam-Hyers stable if there exists $\theta\in\mathcal{C}\left(\mathbb{R}_{+},\mathbb{R}_{+}\right)$, $\theta\left(0\right)=0$ such that for each $\varepsilon>0$ and for each solution $y\in E$ of the inequality (1.1), there exists a solution $x\in E$ of the problem (P1) such that

$$\left|y\left(t\right)-x\left(t\right)\right|\leq\theta\left(\varepsilon\right),\ t\in\left[0,1\right].$$

Definition 1.12. [35, 45] The mixed-FBVP (P1) is Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $c>0$ such that for each $\varepsilon>0$ and for each solution $y\in E$ of the inequality (1.3) there exists a solution $x \in E$ of the problem (P1) such that

$$|y(t) - x(t)| \leq c\varepsilon\varphi(t),\ t \in [0,1].$$

Definition 1.13. [35, 45] The mixed-FBVP (P1) is generalized Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $c > 0$ such that for each solution $y \in E$ of the inequality (1.2) there exists a solution $x \in E$ of the mixed-FBVP (P1) such that

$$|y(t) - x(t)| \leq c\varphi(t),\ t \in [0,1].$$

Remark 1.6. [35, 45]

1. A function $y \in E$ is a solution of inequality (1.1) if, and only if, there exists a function $\omega \in \mathcal{C}([0,1], \mathbb{R})$ such that
   (a) $|\omega(t)| \leq \varepsilon$, $t \in [0,1]$,
   (b) $\mathcal{D}^{\beta}_{1^-}\left({}^{c}D^{\alpha}_{0^+}y\right)(t) = f(t, y(t)) + \omega(t)$, $t \in [0,1]$.
2. Also a function $y \in E$ is a solution of the inequality (1.2) if, and only if, there exist $h \in \mathcal{C}([0,1], \mathbb{R})$ such that
   (a) $|h(t)| \leq \varphi(t)$, $t \in [0,1]$,
   (b) $\mathcal{D}^{\beta}_{1^-}\left({}^{c}D^{\alpha}_{0^+}y\right)(t) = f(t, y(t)) + h(t)$, $t \in [0,1]$.
3. Similarly for (1.3) there exist a function $\Phi \in \mathcal{C}([0,1], \mathbb{R})$ such that
   (a) $|\Phi(t)| \leq \varepsilon\varphi(t)$, $t \in [0,1]$,
   (b) $\mathcal{D}^{\beta}_{1^-}\left({}^{c}D^{\alpha}_{0^+}y\right)(t) = f(t, y(t)) + \Phi(t)$, $t \in [0,1]$.

1.4.2 Ulam-Hyers stability definitions for two-CFBVP

Definition 1.14. [14] The two-CFBVP (P2) is Ulam-Hyers stable if there exists $c_f \in \mathbb{R}_+$ such that for each $\varepsilon > 0$ and for every solution $y \in E$ of the inequality (1.4) there exists a unique solution $x \in E$ of the problem (P2) with

$$\|y - x\| \leq c_f\varepsilon,\ t \in [0,1].$$

Definition 1.15. [14] The two-CFBVP (P2) is generalized Ulam-Hyers stable if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$ such that for each $\varepsilon > 0$ and for every solution $y \in E$ of the inequality (1.4), there exists a unique solution $x \in E$ of the two-CFBVP (P2) with

$$\|y - x\| \leq \theta_f(\varepsilon),\ t \in [0,1].$$

Definition 1.16. [14] The problem (P2) is Ulam-Hyers-Rassias stable with respect to φ if there exists $c_f \in \mathbb{R}_+$ such that for each $\varepsilon > 0$ and for every solution $y \in E$ of the inequality (1.5) there exists a unique solution $x \in E$ of the equations (P2) with

$$\|y - x\| \leq c_f \varepsilon \varphi(t), \ t \in [0,1].$$

Definition 1.17. [14] The two-CFBVP (P2) is generalized Ulam-Hyers-Rassias stable with respect to φ, if there exists $c_{f,\varphi} \in \mathbb{R}_+$, such that for every solution $y \in E$ of the inequality (1.6) there exists a unique solution $x \in E$ of the equations (P2) with

$$\|y - x\| \leq c_{f,\varphi} \varphi(t), \ t \in [0,1].$$

1.4.3 Ulam-Hyers stability definitions for p-Laplacian CFPs

In this subsection, we present Ulam-Hyers stability definitions for the p-Laplacian problems (P3) and (P4). In view of Definition 4.1 in [23], Definition 12 in [21] and the definitions (4, 5, 6 , 7) in [22], we give the following definitions:

Definition 1.18. The operator $\mathcal{T}$ is Ulam-Hyers stable, if there exists $c_f \in \mathbb{R}_+$, such that for each $\varepsilon > 0$, the inequality (1.7) has a unique fixed point say $x^*(t) = \mathcal{T}x^*(t)$, such that

$$|x(t) - x^*(t)| \leq \varepsilon c_f, \ t \in [0,1].$$

Definition 1.19. The operator $\mathcal{T}$ is generalized Ulam-Hyers stable, if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$ such that for each $\varepsilon > 0$, the inequality (1.7) has a unique fixed point $x^*(t)$, such that

$$|x(t) - x^*(t)| \leq \theta_f(\varepsilon), \ t \in [0,1].$$

Definition 1.20. The operator $\mathcal{T}$ is Ulam-Hyers-Rassias stable with respect to φ if there exists $c_f \in \mathbb{R}_+$ such that for each $\varepsilon > 0$, the inequality (1.8) has a unique fixed point $x^*(t)$, such that

$$|x(t) - x^*(t)| \leq c_f \varepsilon \varphi(t), \ \ t \in [0,1].$$

Definition 1.21. The operator $\mathcal{T}$ is generalized Ulam-Hyers-Rassias stable with respect to φ, if there exists $c_{f,\varphi} \in \mathbb{R}_+$, the inequality (1.9) has a unique fixed point $x^*(t)$, such that

$$|x(t) - x^*(t)| \leq c_{f,\varphi} \varphi(t), \ t \in [0,1].$$

CHAPTER 2

MIXED-FBVP WITH INTEGRAL BOUNDARY CONDITIONS

This chapter is devoted to give the positivity, existence, uniqueness and stability results of solution for the mixed-FBVP with integral boundary conditions involving both right conformable and left Caputo fractional derivatives:

$$\begin{aligned} &\mathcal{D}_{1^-}^{\beta}\left({}^{c}D_{0^+}^{\alpha}x\right)(t) = f\left(t, x\left(t\right)\right),\ t \in \left(0,1\right), \\ &x\left(0\right) = \gamma\int_0^1 x\left(t\right)dt,\ {}^{c}D_{0^+}^{\alpha}x\left(1\right) = 0, \end{aligned} \tag{P1}$$

where $\alpha,\ \beta \in \left(0,1\right],\ \gamma \in \left(0,1\right)$ and the function $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Our main tool is a Guo-Krasnoselskii fixed point theorem, The Banach contraction principle theorem and Ulam-Hyers stability. The results in this chapter are published in *"Moroccan J. of Pure and Appl. Anal."* in title *"A new class of mixed fractional differential equations with integral boundary conditions"* , see [15].

2.1 Solution form of the problem

Before we proceed to the main results, we give the equivalent solution to the problem (P1). Let the Banach space $E = C\left(\left[0,1\right], \left[0,+\infty\right)\right)$ with the norm $\left\|x\right\| = \max_{t\in[0,1]}\left|x\left(t\right)\right|$.

Theorem 2.1. *The mixed-FBVP* (P1) *has a solution* $x \in E$ *if, and only if,* x *satisfies the nonlinear and homogeneous Fredholm integral equation of the second kind,*

$$x\left(t\right) = \int_0^1 G\left(t,s\right) f\left(s, x\left(s\right)\right)ds, \tag{2.1}$$

where

$$G\left(t,s\right) = \frac{\left(1-s\right)^{\beta-1}}{\Gamma\left(\alpha+1\right)} \begin{cases} \frac{\gamma}{(1-\gamma)(\alpha+1)}\left(1-\left(1-s\right)^{\alpha+1}\right) + t^{\alpha} - \left(t-s\right)^{\alpha} & ,\ 0 \le s \le t \le 1, \\ \frac{\gamma}{(1-\gamma)(\alpha+1)}\left(1-\left(1-s\right)^{\alpha+1}\right) + t^{\alpha} & ,\ 0 \le t \le s \le 1. \end{cases} \tag{G-P1}$$

Proof. (i) First we prove the necessity. In view of Lemma (1.2, (ii)), we apply $\mathcal{I}^{\beta}_{1^-}$ on

$$\mathcal{D}^{\beta}_{1^-}\left({}^{c}D^{\alpha}_{0^+}x\right)(t)=f\left(t,x\left(t\right)\right),\ 0<t<1, \tag{A-P1}$$

and with the boundary condition

$${}^{c}D^{\alpha}_{0^+}x\left(1\right)=0, \tag{B-P1}$$

we get

$${}^{c}D^{\alpha}_{0^+}x\left(t\right)=\mathcal{I}^{\beta}_{1^-}f\left(t,x\left(t\right)\right),$$

it follows, in view of Lemma (1.1, (ii)), we apply $I^{\alpha}_{0^+}$ on above expression together with the boundary condition

$$x\left(0\right)=\gamma\int_0^1 x\left(t\right)dt, \tag{C-P1}$$

we get

$$x\left(t\right)=\gamma\int_0^1 x\left(t\right)dt+I^{\alpha}_{0^+}\mathcal{I}^{\beta}_{1^-}f\left(t,x\left(t\right)\right).$$

where

$$\begin{aligned}
I^{\alpha}_{0^+}\mathcal{I}^{\beta}_{1^-}f\left(t,x\left(t\right)\right)&=\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-r\right)^{\alpha-1}I^{\beta}_{1^-}f\left(r,x\left(r\right)\right)dr\\
&=\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-r\right)^{\alpha-1}\left[\int_r^1\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds\right]dr\\
&=\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-r\right)^{\alpha-1}\left[\int_r^t\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds\right]dr\\
&\quad+\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-r\right)^{\alpha-1}\left[\int_t^1\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds\right]dr,
\end{aligned}$$

and with Fubini theorem, we have

$$\begin{aligned} I_{0^+}^{\alpha}\mathcal{I}_{1^-}^{\beta} f\left(t,x\left(t\right)\right) &= \frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(\int_0^s\left(t-r\right)^{\alpha-1}dr\right)\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds \\ &+\frac{1}{\Gamma\left(\alpha\right)}\int_t^1\left(\int_0^t\left(t-r\right)^{\alpha-1}dr\right)\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds \\ &= \frac{1}{\Gamma\left(\alpha+1\right)}\int_0^t\left(t^{\alpha}-\left(t-s\right)^{\alpha}\right)\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds \\ &+\frac{1}{\Gamma\left(\alpha+1\right)}\int_t^1 t^{\alpha}\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds \\ &= \frac{1}{\Gamma\left(\alpha+1\right)}\int_0^1 t^{\alpha}\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds \\ &-\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^t\left(t-s\right)^{\alpha}\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds, \end{aligned}$$

which yields that

$$\begin{aligned} x\left(t\right) = \gamma\int_0^1 x\left(t\right)dt+&\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^1 t^{\alpha}\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds \\ &-\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^t\left(t-s\right)^{\alpha}\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds. \end{aligned} \tag{2.2}$$

Now, we integrate (2.2) on $[0,1]$ in both sides and using the Fubini theorem, we obtain

$$\begin{aligned} \left(1-\gamma\right)\int_0^1 x\left(t\right)dt &= \frac{1}{\Gamma\left(\alpha+1\right)}\int_0^1\left[\int_0^1 t^{\alpha}\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds\right]dt \\ &-\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^1\left[\int_0^t\left(t-s\right)^{\alpha}\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds\right]dt \\ &= \frac{1}{\Gamma\left(\alpha+1\right)}\int_0^1 t^{\alpha}dt\int_0^1\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds \\ &-\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^1\left[\int_s^1\left(t-s\right)^{\alpha}dt\right]\left(1-s\right)^{\beta-1}f\left(s,x\left(s\right)\right)ds \\ &= \frac{1}{\Gamma\left(\alpha+2\right)}\int_0^1\left[\left(1-s\right)^{\beta-1}-\left(1-s\right)^{\alpha+\beta}\right]f\left(s,x\left(s\right)\right)ds. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2) yields

$$\begin{aligned} x\left(t\right) &= \int_0^t \left[\frac{\gamma}{\left(1-\gamma\right)\left(\alpha+1\right)}\left(1-\left(1-s\right)^{\alpha+1}\right)+t^{\alpha}-\left(t-s\right)^{\alpha}\right]\frac{\left(1-s\right)^{\beta-1}}{\Gamma\left(\alpha+1\right)}f\left(s,x\left(s\right)\right)ds \\ &+\int_t^1 \left[\frac{\gamma}{\left(1-\gamma\right)\left(\alpha+1\right)}\left(1-\left(1-s\right)^{\alpha+1}\right)+t^{\alpha}\right]\frac{\left(1-s\right)^{\beta-1}}{\Gamma\left(\alpha+1\right)}f\left(s,x\left(s\right)\right)ds \\ &= \int_0^1 G\left(t,s\right)f\left(s,x\left(s\right)\right)ds. \end{aligned}$$

(ii) Now, let $x \in E$ be the solution to the integral equation (2.1).

• Let us first show that x satisfies the boundary condition (C-P1): From (2.1), function of Green (G-P1) and Fubini theorem, we obtain

$$\begin{aligned} x\left(0\right)-\gamma\int_0^1 x\left(t\right)dt &= \int_0^1 G\left(0,s\right)f\left(s,x\left(s\right)\right)ds \\ &-\gamma\int_0^1\left[\int_0^t G\left(t,s\right)f\left(s,x\left(s\right)\right)ds+\int_t^1 G\left(t,s\right)f\left(s,x\left(s\right)\right)ds\right]dt \\ &= \int_0^1 G\left(0,s\right)f\left(s,x\left(s\right)\right)ds \\ &-\gamma\int_0^1\left[\int_s^1 G\left(t,s\right)dt+\int_0^s G\left(t,s\right)dt\right]f\left(s,x\left(s\right)\right)ds. \end{aligned} \tag{2.4}$$

On the other hand and from function (G-P1), we have

$$\begin{aligned} \int_s^1 G\left(t,s\right)dt &= \frac{\gamma\left(1-s\right)}{\left(1-\gamma\right)\Gamma\left(\alpha+2\right)}\left[\left(1-s\right)^{\beta-1}-\left(1-s\right)^{\alpha+\beta}\right] \\ &+\frac{1}{\Gamma\left(\alpha+2\right)}\left[1-s^{\alpha+1}-\left(1-s\right)^{\alpha+1}\right]\left(1-s\right)^{\beta-1}, \end{aligned} \tag{2.5}$$

and

$$\int_0^s G\left(t,s\right)dt = \frac{\gamma s}{\left(1-\gamma\right)\Gamma\left(\alpha+2\right)}\left[\left(1-s\right)^{\beta-1}-\left(1-s\right)^{\alpha+\beta}\right]+\frac{1}{\Gamma\left(\alpha+2\right)}s^{\alpha+1}\left(1-s\right)^{\beta-1}. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), using Eq.(G-P1) we obtain

$$\begin{aligned} x(0)-\gamma\int_0^1 x(t)\,dt &= \int_0^1 G(0,s)\,f(s,x(s))\,ds \\ &\quad -\frac{\gamma}{(1-\gamma)\,\Gamma(\alpha+2)}\int_0^1 \left[1-(1-s)^{\alpha+1}\right](1-s)^{\beta-1} f(s,x(s))\,ds \\ &= \int_0^1 G(0,s)\,f(s,x(s))\,ds-\int_0^1 G(0,s)\,f(s,x(s))\,ds \\ &= 0. \end{aligned}$$

• Now, we show that $x\in E$ satisfies the boundary condition (B-P1): using (2.1), the function (G-P1) and definition of ${}^{c}D_{0^+}^{\alpha}$ in (1.3), we get

$$\begin{aligned} {}^{c}D_{0^+}^{\alpha}x(1) &= \frac{1}{\Gamma(1-\alpha)}\int_0^1 (1-s)^{-\alpha}\,x'(s)\,ds \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^1 (1-s)^{-\alpha}\int_0^s \frac{\partial G}{\partial s}(s,r)\,f(r,x(r))\,drds \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_0^1 (1-s)^{-\alpha}\int_s^1 \frac{\partial G}{\partial s}(s,r)\,f(r,x(r))\,drds \\ &= \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_0^1\int_0^s (1-s)^{-\alpha}\left[s^{\alpha-1}-(s-r)^{\alpha-1}\right](1-r)^{\beta-1} f(r,x(r))\,drds \\ &+ \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_0^1\int_s^1 (1-s)^{\alpha-1}\,s^{\alpha-1}(1-r)^{\beta-1} f(r,x(r))\,drds \\ &= \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_0^1\int_0^1 (1-s)^{-\alpha}\,s^{\alpha-1}(1-r)^{\beta-1} f(r,x(r))\,drds \\ &- \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_0^1\int_0^s (1-s)^{-\alpha}(s-r)^{\alpha-1}(1-r)^{\beta-1} f(r,x(r))\,drds. \end{aligned}$$

Using Fubini theorem, we obtain

$$\begin{aligned} {}^{c}D_{0^+}^{\alpha}x(1) &= \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\left[\int_0^1 (1-s)^{-\alpha}\,s^{\alpha-1}ds\right]\left[\int_0^1 (1-r)^{\beta-1} f(r,x(r))\,dr\right] \\ &- \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_0^1\left[\int_r^1 (1-s)^{-\alpha}(s-r)^{\alpha-1}\,ds\right](1-r)^{\beta-1} f(r,x(r))\,dr. \end{aligned} \tag{2.7}$$

Using the relation between the beta and Euler gamma functions (subsection 1.1.2), we have

$$B(\alpha,\beta)=\int_0^1 (1-s)^{\alpha-1}\,s^{\beta-1}ds=\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{2.8}$$

and using the change of variable $\mu = \frac{s-r}{1-r}$, we obtain

$$\int_r^1 (1-s)^{-\alpha} (s-r)^{\alpha-1} ds = B(\alpha, 1-\alpha). \tag{2.9}$$

From (2.7), (2.8) and (2.9) we obtain the boundary condition (B-P1).

• It remains to show that $x \in E$ satisfies the equation (A-P1): in view of Definition 1.3, (2.1), (G-P1) and Fubini theorem we get

$$\begin{aligned} {}^cD_{0^+}^{\alpha} x(t) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_0^s (t-s)^{-\alpha} \left[s^{\alpha-1} - (s-r)^{\alpha-1}\right] (1-r)^{\beta-1} f(r, x(r))\, drds \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_s^1 (t-s)^{-\alpha} s^{\alpha-1} (1-r)^{\beta-1} f(r, x(r))\, drds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_0^1 (t-s)^{-\alpha} s^{\alpha-1} (1-r)^{\beta-1} f(r, x(r))\, drds \\ &- \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \int_0^s (t-s)^{-\alpha} (s-r)^{\alpha-1} (1-r)^{\beta-1} f(r, x(r))\, drds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 \left[\int_0^t (t-s)^{-\alpha} s^{\alpha-1} ds\right] (1-r)^{\beta-1} f(r, x(r))\, dr \\ &- \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \left[\int_r^t (t-s)^{-\alpha} (s-r)^{\alpha-1} ds\right] (1-r)^{\beta-1} f(r, x(r))\, dr. \end{aligned} \tag{2.10}$$

Using the change of variables $\mu = \frac{s}{t}$ and $\mu = \frac{s-r}{t-r}$, we obtain

$$\int_0^t (t-s)^{-\alpha} s^{\alpha-1} ds = \int_r^t (t-s)^{-\alpha} (s-r)^{\alpha-1} ds = B(\alpha, 1-\alpha). \tag{2.11}$$

From (2.10), (2.11) and (2.8) we obtain

$${}^cD_{0^+}^{\alpha} x(t) = \int_0^1 (1-r)^{\beta-1} f(r, x(r))\, dr - \int_0^t (1-r)^{\beta-1} f(r, x(r))\, dr. \tag{2.12}$$

By applying the right conformable derivative $\mathcal{I}_{1^-}^{\beta}$ in both sides of (2.12), we obtain (A-P1). This completes the proof.

□

Now, we prove a several important properties of the Green's function (G-P1).

Lemma 2.1. *For all $t \in (0,1)$ and $s \in [0,1)$, we have*

1. $G(t,s) > 0$ *and continuous.*
2. $t^{\alpha} G(1,s) \leq G(t,s) \leq G(1,s)$.

Proof. 1. For all $t \in (0,1)$ and from (G-P1), we have:

$$\frac{\partial G(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(t^{\alpha-1} - (t-s)^{\alpha-1}\right)(1-s)^{\beta-1} & , \ 0 \le s < t < 1, \\ \\ t^{\alpha-1}(1-s)^{\beta-1} & , \ 0 < t < s < 1. \end{cases}$$

Clearly that $\frac{\partial G(t,s)}{\partial t} \geq 0$ for all $t \in (0,1]$ and $s \in [0,1)$, then $G(t,s)$ is increasing with respect to $t \in (0,1]$. Therefore, for all $t \in (0,1]$ and $s \in [0,1)$, we have

$$G(t,s) \geq G(0,s) = \left[\frac{\gamma}{(1-\gamma)\Gamma(\alpha+2)}\left(1-(1-s)^{\alpha+1}\right)\right](1-s)^{\beta-1} > 0.$$

2. Using the increasing of the Green's function $G(t,s)$ with respect to t, we obtain for all $t \in (0,1]$ and $s \in [0,1)$,

$$G(t,s) \leq G(1,s) = \frac{1}{\Gamma(\alpha+1)}\left[\frac{\gamma}{(1-\gamma)(\alpha+1)}\left(1-(1-s)^{\alpha+1}\right) + 1 - (1-s)^{\alpha}\right](1-s)^{\beta-1}.$$

On the other hand, From (G-P1) we have

$$\frac{G(t,s) - t^{\alpha}G(1,s)}{(1-s)^{\beta-1}} = \begin{cases} \frac{\gamma(1-t^{\alpha})}{(1-\gamma)\Gamma(\alpha+2)}\left[1-(1-s)^{\alpha+1}\right] + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left[(1-s)^{\alpha} - \left(1-\frac{s}{t}\right)^{\alpha}\right] & , \ s \leq t, \\ \\ \frac{\gamma(1-t^{\alpha})}{(1-\gamma)\Gamma(\alpha+2)}\left[1-(1-s)^{\alpha+1}\right] + \frac{t^{\alpha}}{\Gamma(\alpha+1)}(1-s)^{\alpha} & , \ s \geq t, \end{cases}$$

consequently, $t^{\alpha}G(1,s) \leq G(t,s) \leq G(1,s)$.

□

2.2 Main results

For the convenience of studying the existence uniqueness and stability of positive solutions of the mixed-FBVP (P1), let us define: the Banach space $E = C([0,1],[0,+\infty))$ with the norm $\|x\| = \max_{t\in[0,1]}|x(t)|$ and the cone $\mathcal{P}$ by

$$\mathcal{P} = \{x \in E, \ x(t) \geq t^{\alpha}\|x\|, \ t \in [0,1]\},$$

Furthermore, we define the operator $\mathcal{T} : E \to E$ by

$$\mathcal{T}x(t) = \int_0^1 G(t,s) f(s,x(s))\, ds,$$

where G defined in (G-P1). Obviously, by Theorem 2.1, we can conclude that any fixed point of operator $\mathcal{T}$ coincide with the solution of problem (P1).

The operator $\mathcal{T}$ has the following properties.

Lemma 2.2. *1. $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$.*

2. The operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Proof. 1. Let $x \in \mathcal{P}$. From Lemma 2.1, we have

$$\begin{aligned} \mathcal{T}x(t) &= \int_0^1 G(t,s) f(s, x(s))\, ds \\ &\geq t^\alpha \int_0^1 G(1,s) f(s, x(s))\, ds \\ &\geq t^\alpha \int_0^1 G(t,s) f(s, x(s))\, ds. \end{aligned}$$

Then, for all $t \in [0,1]$ we have

$$\mathcal{T}x(t) \geq t^\alpha \max_{t\in[0,1]} \int_0^1 G(t,s) f(s, x(s)) ds = t^\alpha \|\mathcal{T}x\|.$$

Hence $\mathcal{T}x \in \mathcal{P}$.

2. Let $\Omega_0 \subset \mathcal{P}$ be bounded, which is to say there exists a positive constant $r_0 > 0$ such that $\|x\| \leq r_0$ for all $x \in \Omega_0$. Define now

$$L_0 = \max_{t\in[0,1], x\in\Omega_0} f(t,x).$$

Then, for all $x \in \Omega_0$, the Lemma 2.1 gives that

$$\|\mathcal{T}x\| = \max_{t\in[0,1]} \int_0^1 G(t,s) f(s, x(s))\, ds \leq L_0 \int_0^1 G(1,s)\, ds,$$

which implies that $\mathcal{T}(\Omega_0)$ is bounded in $\mathcal{P}$.

For each $x \in \Omega_0$, we have from Lemma 2.1

$$\begin{aligned}\left|(\mathcal{T}x)'(t)\right| &= \left|\int_0^1 \frac{\partial G}{\partial t}(t,s)\, f(s,x(s))\, ds\right| \\ &= \frac{1}{\Gamma(\alpha)}\left(\int_0^t \left[t^{\alpha-1}-(t-s)^{\alpha-1}\right](1-s)^{\beta-1} f(s,x(s))\, ds \right. \\ &\qquad \left. + \int_t^1 t^{\alpha-1}(1-s)^{\beta-1} f(s,x(s))\, ds\right) \\ &= \frac{1}{\Gamma(\alpha)}\left(\int_0^1 t^{\alpha-1}(1-s)^{\beta-1} f(s,x(s))\, ds \right. \\ &\qquad \left. - \int_0^t (t-s)^{\alpha-1}(1-s)^{\beta-1} f(s,x(s))\, ds\right) \\ &\leq \frac{L_0}{\Gamma(\alpha)}\left(\int_0^1 (1-s)^{\beta-1} ds + \int_0^t (t-s)^{\alpha-1} ds\right) \\ &\leq \frac{(\alpha+\beta) L_0}{\beta\Gamma(\alpha+1)}.\end{aligned}$$

As consequence, for all $t_1,\ t_2 \in [0,1],\ t_1 < t_2$, we have

$$\begin{aligned}\left|(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)\right| &= \left|\int_{t_1}^{t_2} (\mathcal{T}x)'(s)\, ds\right| \\ &\leq \int_{t_1}^{t_2} \left|(\mathcal{T}x)'(s)\right| ds \\ &\leq \frac{(\alpha+\beta) L_0 \left|t_2 - t_1\right|}{\beta\Gamma(\alpha+1)}.\end{aligned}$$

Then, $\left|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)\right| \to 0$ as $t_1 \to t_2$ which implies that the set $\mathcal{T}(\Omega_0)$ is equicontinuous.

Now, from Arzelà-Ascoli theorem 1.1, we conclude that $\overline{T(\Omega)}$ is compact, i.e., $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator.

□

We give some important notations as follows:

$$\begin{aligned} f^0 = \lim_{x\to 0}\max_{t\in[0,1]} \frac{f(t,x)}{x},\ f_0 = \lim_{x\to 0}\min_{t\in[0,1]} \frac{f(t,x)}{x}, \\ f^\infty = \lim_{x\to +\infty}\max_{t\in[0,1]} \frac{f(t,x)}{x},\ f_\infty = \lim_{x\to +\infty}\min_{t\in[0,1]} \frac{f(t,x)}{x}.\end{aligned}$$

and

$$\Lambda_1 = \int_0^1 G(1,s)\, ds,\ \ \Lambda_2 = \rho^\alpha \int_\rho^{1-\rho} G(1,s)\, ds.$$

where $\rho \in \left(0, \frac{1}{2}\right)$ and the function G is defined in (G-P1).

Now, we are in position to prove the following results of problem (P1).

2.2.1 Existence results via Guo-Krasnoselskii fixed point theorem

Theorem 2.2. *Assume that, There exists $r_2 > r_1 > 0$, such that for all $x \in [r_1, r_2]$, for all $t \in [0,1]$:*

$$\frac{r_1}{\Lambda_2} \leq f\left(t,x\right) \leq \frac{r_2}{\Lambda_1}.$$

Then, the mixed-FBVP (P1) *has at least one positive solution.*

Proof. Let $x \in \mathcal{P} \cap \partial\Omega_1$, i.e., $x \in \mathcal{P}$ and $\|x\| = r_1$. Using Lemma 2.1, we get

$$\begin{aligned}\|\mathcal{T}x\| = \max_{t\in[0,1]} \int_0^1 G\left(t,s\right) f\left(s, x\left(s\right)\right) ds &\geq t^{\alpha} \int_0^1 G\left(1,s\right) f\left(s,x\left(s\right)\right) ds \\ &\geq \frac{r_1}{\Lambda_2} t^{\alpha} \int_0^1 G\left(1,s\right) ds \\ &\geq \frac{r_1}{\Lambda_2} t^{\alpha} \int_{\rho}^{1-\rho} G\left(1,s\right) ds \\ &\geq \frac{r_1}{\Lambda_2} \rho^{\alpha} \int_{\rho}^{1-\rho} G\left(1,s\right) ds = r_1,\end{aligned}$$

then $\|\mathcal{T}x\| \geq \|x\|$.

For $x \in \mathcal{P} \cap \partial\Omega_2$, i.e., $x \in \mathcal{P}$ and $\|x\| = r_2$, using Lemma 2.1, we get

$$\begin{aligned}\mathcal{T}x\left(t\right) = \int_0^1 G\left(t,s\right) f\left(s,x\left(s\right)\right) ds &\leq \int_0^1 G\left(1,s\right) f\left(s,x\left(s\right)\right) ds \\ &\leq \frac{r_2}{\Lambda_1} \int_0^1 G\left(1,s\right) ds = r_2,\end{aligned}$$

then, $\|\mathcal{T}x\| \leq \|x\|$. Applying Theorem 1.2 yields that $\mathcal{T}$ has at least one fixed point $x \in \mathcal{P} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \leq \|x\| \leq r_2$. It follows from Theorem 2.1 that the mixed-FBVP (P1) has at least one positive solution x. □

Theorem 2.3. *Assume that, the following conditions:*

$$\Lambda_1 f^0 \leq \frac{1}{2}, \ \ \Lambda_2 f_{\infty} \geq 2,$$

are fulfilled. Then, the mixed-FBVP (P1) *has at least one positive solution.*

Proof. From the definition of f^0, there exists $r_1 > 0$, such that

$$f(t,x) \leq \left(f^0 + \varepsilon\right) x, \text{ for all } t \in [0,1], \; 0 < x \leq r_1, \text{ where } \varepsilon > 0 \text{ satisfies } \Lambda_1 \varepsilon \leq \frac{1}{2}.$$

Let $x \in \mathcal{P} \cap \partial\Omega_1$, i.e., $x \in \mathcal{P}$ and $\|x\| = r_1$. Using Lemma 2.1 we obtain

$$\begin{aligned} \mathcal{T}x(t) = \int_0^1 G(t,s) f(s,x(s))\, ds &\leq \int_0^1 G(1,s) f(s,x(s))\, ds \\ &\leq \left(f^0 + \varepsilon\right) \int_0^1 G(1,s)\, x(s)\, ds \\ &\leq \left(f^0 + \varepsilon\right) \|x\| \int_0^1 G(1,s)\, ds \\ &\leq \Lambda_1 \left(f^0 + \varepsilon\right) \|x\| \\ &\leq \|x\|. \end{aligned}$$

Consequently, $\|\mathcal{T}x\| \leq \|x\|$.

By the definition of f_∞, there exists $r_3 > 0$, such that

$$f(t,x) \geq \left(f_\infty - \varepsilon\right) x, \text{ for all } t \in [0,1], \; x \geq r_3, \text{ where } \varepsilon > 0 \text{ satisfies } \Lambda_2 \varepsilon \leq 1. \tag{2.13}$$

Let $x \in \mathcal{P} \cap \partial\Omega_2$, i.e., $x \in \mathcal{P}$ and $\|x\| = r_2$ with $r_2 = \max\{2r_1, \rho^{-\alpha} r_3\}$. We have

$$x(t) \geq t^\alpha \|x\| \geq \rho^\alpha r_2 \geq r_3, \text{ for } t \in [\rho, 1-\rho],$$

and hence, by the inequality (2.13)

$$f(t,x) \geq \left(f_\infty - \varepsilon\right) x, \text{ for } t \in [\rho, 1-\rho], \; x \in \mathcal{P} \cap \partial\Omega_2 \text{ and } \Lambda_2 \varepsilon \leq 1.$$

Using Lemma 2.1, we have

$$\begin{aligned} \|\mathcal{T}x\| = \max_{t \in [0,1]} \int_0^1 G(t,s) f(s,x(s))\, ds &\geq t^\alpha \int_0^1 G(1,s) f(s,x(s))\, ds \\ &\geq t^\alpha \int_{\rho^2}^{1-\rho^2} G(1,s) f(s,x(s))\, ds, \; 0 < \rho < 1/2 \\ &\geq t^\alpha \left(f_\infty - \varepsilon\right) \int_{\rho^2}^{1-\rho^2} G(1,s)\, x(s)\, ds \\ &\geq \frac{t^{2\alpha}}{\rho^{2\alpha}} \left(f_\infty - \varepsilon\right) \|x\| \rho^{2\alpha} \int_{\rho^2}^{1-\rho^2} G(1,s)\, ds \\ &\geq \Lambda_2 \left(f_\infty - \varepsilon\right) \|x\| \geq \|x\|. \end{aligned}$$

From Theorem 1.2 the operator $\mathcal{T}$ has at least one fixed point $x \in \mathcal{P} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \leq \|x\| \leq r_2$. It follows from Theorem 2.1 that the problem (P1) has at least one positive solution x. ☐

Theorem 2.4. *Assume that, the following conditions:*

$$\Lambda_2 f_0 \geq 2,\ \Lambda_1 f^{\infty} \leq \frac{1}{4}, \tag{2.14}$$

are satisfied. Then, the mixed-FBVP (P1) *has at least one positive solution.*

Proof. From the definition of f_0, there exists $r_1 > 0$, such that

$$f\left(t,x\right) \geq \left(f_0 - \varepsilon\right) x,\ \ \text{for all}\ \ t \in [0,1]\,,\ 0 < x \leq r_1,\ \text{where } \varepsilon > 0 \text{ satisfies } \Lambda_2 \varepsilon \leq 1.$$

Let $x \in \mathcal{P} \cap \partial\Omega_1$, i.e., $x \in \mathcal{P}$ and $\|x\| = r_1$. Using Lemma 2.1, we obtain

$$\begin{aligned}
\|\mathcal{T}x\| &\geq t^{\alpha} \int_0^1 G\left(1,s\right) f\left(s, x\left(s\right)\right) ds \\
&\geq t^{\alpha} \left(f_0 - \varepsilon\right) \int_0^1 G\left(1,s\right) x\left(s\right) ds \\
&\geq t^{2\alpha} \left(f_0 - \varepsilon\right) \|x\| \int_0^1 G\left(1,s\right) ds \\
&\geq \frac{t^{2\alpha}}{\rho^{2\alpha}} \left(f_0 - \varepsilon\right) \|x\| \rho^{2\alpha} \int_{\rho^2}^{1-\rho^2} G\left(1,s\right) ds,\ t \in [\rho, 1-\rho] \\
&\geq \Lambda_2 \left(f_0 - \varepsilon\right) \|x\| \\
&\geq \|x\|\,.
\end{aligned}$$

By the definition of f^{∞}, there exists $r_4 > 0$, such that

$$f\left(t,x\right) \leq \left(f^{\infty} + \varepsilon\right) x,\ \ \text{for all}\ \ t \in [0,1]\,,\ x \geq r_4,\ \text{where } \varepsilon > 0 \text{ satisfies } \Lambda_1 \varepsilon \leq 1/4.$$

it follows, there exists $\delta > 0$, such that

$$\delta = \max_{t \in [0,1]} f\left(t, r_4\right),\ \ \text{for all}\ \ t \in [0,1]\,.$$

Then

$$f\left(t,x\right) \leq \left(f^{\infty} + \varepsilon\right) x + \delta,\ \ \text{for all}\ \ t \in [0,1]\,,\ x \geq r_4.$$

Let $x \in \mathcal{P} \cap \partial\Omega_2$, i.e., $x \in \mathcal{P}$ and $\|x\| = r_2$ with $r_2 = \max\{2r_1,\ 2\delta\Lambda_1\}$. Using Lemma 2.1, we get

$$\begin{aligned}
\mathcal{T}x(t) &= \int_0^1 G(t,s) f(s,x(s))\,ds \\
&\le \int_0^1 G(1,s) f(s,x(s))\,ds \\
&\le \int_0^1 G(1,s)\left[(f^\infty + \varepsilon)x(s) + \delta\right] ds \\
&\le (f^\infty + \varepsilon)\int_0^1 G(1,s)x(s)\,ds + \delta\int_0^1 G(1,s)\,ds \\
&\le \Lambda_1 (f^\infty + \varepsilon)\|x\| + \delta\Lambda_1 \\
&\le \frac{\|x\|}{2} + \delta\Lambda_1 \\
&\le \frac{\|x\|}{2} + \frac{r_2}{2} \\
&\le \|x\|.
\end{aligned}$$

Consequently, $\|\mathcal{T}x\| \le \|x\|$. Applying Theorem 1.2 yields that $\mathcal{T}$ has at least one fixed point $x \in \mathcal{P} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ and Theorem 2.1 ensure that the mixed-FBVP (P1) has at least one positive solution x. □

By the two Theorems 2.3 and 2.4, we directly obtain the following corollary.

Corollary 2.1. *Assume that one of the two following conditions*

- $f^0 = 0$ *and* $f_\infty = +\infty$.
- $f_0 = +\infty$ *and* $f^\infty = 0$.

is satisfied. Then, the mixed-FBVP (P1) *has at least one positive solution.*

2.2.2 Uniqueness results via Banach contraction principle theorem

Theorem 2.5. *Assume there exists* $L > 0$ *such that*

$$|f(t,x) - f(t,y)| \le L|x-y|, \ \textit{for all} \ t \in [0,1] \ \textit{and all}\ x,y \in \mathcal{P}. \tag{2.15}$$

If

$$0 < L\Lambda_1 < 1, \tag{2.16}$$

then, the mixed-FBVP (P1) *has exactly one positive solution in* $\mathcal{P}$.

Proof. Let $x, y : [0,1] \to \mathbb{R}_+$, $x \neq y$, two positive solutions of the mixed-FBVP (P1). Using (2.15) and Lemma 2.1, we get

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}y(t)| &\leq \int_0^1 G(t,s)\,|(f(s,x(s)) - f(s,y(s)))|\,ds \\ &\leq L\int_0^1 G(1,s)\,|x(s) - y(s)|\,ds \leq L\Lambda_1\,\|x-y\|\,. \end{aligned}$$

Consequently, $\|\mathcal{T}x - \mathcal{T}y\| \leq L\Lambda_1\,\|x-y\|$. By the condition (2.16), the operator $\mathcal{T}$ is a strictly contraction. From Theorem 1.3 and Theorem 2.1, the mixed-FBVP (P1) has exactly one positive solution in $\mathcal{P}$. ☐

2.2.3 Ulam-Hyers stability results

In this subsection, we present fourth types of Ulam-Hyers stability for the problem (P1). Suppose that the function f satisfying the conditions in Eq.(2.15) and Eq.(2.16).

Theorem 2.6. *The problem* (P1) *is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.*

Proof. Let $y \in E$ be any solution of the inequality (1.1), then by Remark 1.6, we have

$$\mathcal{D}_{1^-}^{\beta}\left({}^cD_{0^+}^{\alpha}y\right)(t) = f(t,y(t)) + \omega(t)\,,\ t \in [0,1]\,.$$

Using Theorem 2.1, we can write

$$y(t) = \int_0^1 G(t,s)\,f(s,y(s))\,ds + \int_0^1 G(t,s)\,\omega(s)\,ds,$$

which gives

$$\left|y(t) - \int_0^1 G(t,s)\,f(s,y(s))\,ds\right| \leq \Lambda_1\varepsilon. \tag{2.17}$$

Now, let $x \in E$ be a unique solution of the mixed-FBVP (P1), from Lemma 2.1 we have for any $t \in [0,1]$

$$\begin{aligned} |y(t) - x(t)| &= \left|y(t) - \int_0^1 G(t,s)\,f(s,x(s))\,ds\right| \\ &\leq \left|y(t) - \int_0^1 G(t,s)\,f(t,y(s))\,ds\right| \\ &\quad + \left|\int_0^1 G(t,s)\,(f(s,y(s)) - f(s,x(s)))\,ds\right|. \end{aligned}$$

From (2.17), (2.15) and Lemma 2.1 we have

$$\|y-x\| \leq \varepsilon\Lambda_1 + L\Lambda_1 \|y-x\|,$$

which further implies

$$\|y-x\| \leq \lambda\varepsilon,$$

where $\lambda = \frac{\Lambda_1}{1-L\Lambda_1} > 0$. Then, the mixed-FBVP (P1) is Ulam-Hyers stable. Moreover, if we set $\theta(\varepsilon) = \lambda\varepsilon$, then the problem (P1) is generalized Ulam-Hyers stable. □

Theorem 2.7. *Let* $\varphi : [0,1] \to \mathbb{R}_+$ *be a differentiable and increasing function such that* $\varphi(0) \neq 0$. *The mixed-FBVP* (P1) *is Ulam-Hyers-Rassias stable. Further the mixed-FBVP* (P1) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Let $y \in E$ be any solution of the inequality (1.3), then by Remark 1.6, we have

$$\mathcal{D}_{1^-}^{\beta}\left({}^{c}D_{0^+}^{\alpha}y\right)(t) = f(t,y(t)) + \Phi(t),\ t\in[0,1].$$

Using Theorem 2.1, we obtain

$$y(t) = \int_0^1 G(t,s) f(s,y(s))\,ds + \int_0^1 G(t,s)\Phi(s)\,ds.$$

From Remark 1.6, we have

$$\begin{aligned}\left|y(t) - \int_0^1 G(t,s) f(s,y(s))\,ds\right| &\leq \int_0^1 G(t,s)|\Phi(s)|\,ds \\ &\leq \varepsilon\int_0^1 G(t,s)\varphi(s)\,ds \qquad (2.18)\\ &\leq \varepsilon\left[\int_0^t G(t,s)\varphi(s)\,ds + \int_t^1 G(t,s)\varphi(s)\,ds\right].\end{aligned}$$

By the increasing function $\varphi \in E$ and Lemma 2.1, we have

$$\left.\begin{array}{r} s\leq t \Rightarrow \varphi(s)\leq\varphi(t) \\ G(t,s)\leq G(1,s)\end{array}\right\} \Rightarrow \int_0^t G(t,s)\varphi(s)\,ds \leq \varphi(t)\int_0^t G(1,s)\,ds \qquad (2.19)$$

and

$$\left.\begin{array}{r} s\leq 1 \Rightarrow \varphi(s)\leq\varphi(1) \\ G(t,s)\leq G(1,s)\end{array}\right\} \Rightarrow \int_t^1 G(t,s)\varphi(s)\,ds \leq \varphi(1)\int_t^1 G(1,s)\,ds. \qquad (2.20)$$

From (2.18),(2.19) and (2.20), we obtain

$$\left|y(t)-\int_0^1 G(t,s)\,f(s,y(s))\,ds\right| \leq \varepsilon\left[\varphi(t)\int_0^t G(1,s)\,ds+\varphi(1)\int_t^1 G(1,s)\,ds\right]. \tag{2.21}$$

Let $\mu:[0,1]\to\mathbb{R}_+$ be a function defined by:

$$\mu(t)=\varphi(t)\int_0^t G(1,s)\,ds+\varphi(1)\int_t^1 G(1,s)\,ds-\frac{\varphi(1)}{\varphi(0)}\Lambda_1\varphi(t).$$

The function μ is differentiable on $]\,0,1\,[$ and for all $t\in\,]\,0,1\,[$, we have

$$\mu'(t)=\varphi'(t)\left[\int_0^t G(1,s)\,ds-\frac{\varphi(1)}{\varphi(0)}\Lambda_1\right]+(\varphi(t)-\varphi(1))\,G(1,t).$$

The function φ is differentiable and increasing, then $\mu'(t)\leq 0$.
On other hand, we have $\mu(0)=0$. Then, from (2.21) we obtain

$$\left|y(t)-\int_0^1 G(t,s)\,f(s,y(s))\,ds\right| \leq \frac{\varphi(1)}{\varphi(0)}\varepsilon\Lambda_1\varphi(t). \tag{2.22}$$

Let $x\in E$ be a unique solution of the mixed-FBVP (P1), from Lemma 2.1 we have

$$\begin{aligned}|y(t)-x(t)| &= \left|y(t)-\int_0^1 G(t,s)\,f(s,x(s))\,ds\right|\\ &\leq \left|y(t)-\int_0^1 G(t,s)\,f(t,y(s))\,ds\right|\\ &+\left|\int_0^1 G(t,s)\,(f(s,y(s))-f(s,x(s)))\,ds\right|.\end{aligned}$$

From (2.22), (2.15) and Lemma 2.1 we have

$$\|y-x\|\leq c\varepsilon\varphi(t),$$

where

$$c=\frac{\Lambda_1\varphi(1)}{(1-L\Lambda_1)\,\varphi(0)}>0.$$

Then, the mixed-FBVP (P1) is Ulam-Hyers-Rassias stable. Consequently, From Remark 1.5 the mixed-FBVP (P1) is generalized Ulam-Hyers-Rassias stable, which completes the proof. □

2.3 Examples

In this subsection, we present an examples to explain the applicability of the main results.

Example 2.1. Consider the mixed-FBVP (P1) with $\alpha = \beta = \gamma = 1/2$.

- If $f(t,x) = (1+t)\,x\ln(1+x)$, we have
$$f^0 = 0 \ \ and \ \ f_\infty = +\infty.$$
Thus, by the first part of Corollary 2.1, we can get that the problem (P1) has at least one positive solution.
- If $f(t,x) = (2t+1)\,e^{-x}\cos x$, we have
$$f_0 = +\infty \ \ and \ \ f^\infty = 0,$$
then by the second part of Corollary 2.1, we can get that the problem (P1) has at least one positive solution.
- Consider the continuous function $f(t,x(t)) = \frac{\cos(x(t))}{t+3}$, for any $t \in [0,1]$ and any $x > 0$. Then, we have
$$|f(t,x) - f(t,y)| \leq \frac{1}{3}|x-y|, \ L = \frac{1}{3}, \ \Lambda_1 = \int_0^1 G(1,s)\,ds = \frac{28-3\sqrt{\pi}}{6\sqrt{\pi}} \text{ and } L\Lambda_1 \approx 0.71 < 1.$$
From Theorem 2.5, the mixed-FBVP (P1) has exactly one positive solution x on $[0,1]$.
Now, let $y \in E$ be a solution of inequality
$$\left|\mathcal{D}_{1^-}^{1/2}\left({}^cD_{0^+}^{1/2}y\right)(t) - \frac{\cos(y(t))}{t+3}\right| \leq \varepsilon, \ t \in [0,1],$$
then, by Theorem 2.6 the mixed-FBVP (P1) is Ulma-Hyers stable with
$$\lambda = \frac{\Lambda_1}{1-L\Lambda_1} = \frac{84-9\sqrt{\pi}}{21\sqrt{\pi}-28} > 0.$$
On the other hand, consider the inequality
$$\left|\mathcal{D}_{1^-}^{1/2}\left({}^cD_{0^+}^{1/2}y\right)(t) - \frac{\cos(y(t))}{t+3}\right| \leq \varepsilon\varphi(t), \ t \in [0,1],$$
where $\varphi(t) = e^t$. By Theorem 2.6 the problem (P1) is Ulam-Hyers-Rassias stable with
$$c = \frac{\varphi(1)\Lambda_1}{(1-L\Lambda_1)\varphi(0)} = \frac{(84-9\sqrt{\pi})e}{21\sqrt{\pi}-28} > 0.$$

Example 2.2. Consider the following mixed-FBVP

$$\begin{cases} \mathcal{D}_{1^-}^{0.8}\left({}^cD_{0^+}^{0.2}x\right)(t) = \frac{x(t)}{11\pi e^t + x(t)},\ 0 < t < 1, \\ x(0) = \frac{98}{100}\displaystyle\int_0^1 x(t)\,dt,\ {}^cD_{0^+}^{0.2}x(1) = 0. \end{cases} \tag{2.23}$$

The function $f(t, x(t)) = \frac{x(t)}{11\pi e^t + x(t)}$ is continuous for any $t \in [0,1]$ and any $x > 0$. As

$$|f(t,x) - f(t,y)| \leq \frac{1}{11\pi}|x - y|,\ L = 1/11\pi \text{ and } \Lambda_1 = \int_0^1 G(1,s)\,ds \approx 33.63 \text{ and } L\Lambda_1 \approx 0.97,$$

therefore, by Theorem 2.5, the mixed-FBVP (2.23) has exactly one positive solution x on $[0,1]$.

Let $y \in E$ be a solution of the inequality

$$\left|\mathcal{D}_{1^-}^{0.8}\left({}^cD_{0^+}^{0.2}y\right)(t) - \frac{y(t)}{11\pi e^t + y(t)}\right| \leq \varepsilon,\ t \in [0,1].$$

Using Theorem 2.6, the mixed-FBVP (2.23) is Ulma-Hyers stable with

$$\lambda = \frac{\Lambda_1}{1 - L\Lambda_1} \approx 1121.$$

Let $y \in E$ be a solution of the inequality

$$\left|\mathcal{D}_{1^-}^{0.8}\left({}^cD_{0^+}^{0.2}y\right)(t) - \frac{y(t)}{11\pi e^t + y(t)}\right| \leq \varepsilon\varphi(t),\ t \in [0,1],$$

where $\varphi(t) = e^t$. By Theorem 2.6, the mixed-FBVP (2.23) is Ulam-Hyers-Rassias stable with

$$c = \lambda\frac{\varphi(1)}{\varphi(0)} = \lambda e > 0.$$

CHAPTER 3

TWO-CFBVP WITH INTEGRAL BOUNDARY CONDITIONS

In this chapter, we discussed the existence, uniqueness and stability results of a positive solution for the two-CFBVP with integral boundary conditions:

$$\begin{aligned} &\mathcal{D}_{0^+}^{\beta}\mathcal{D}_{0^+}^{\alpha}x\left(t\right)+\lambda f\left(t,x\left(t\right)\right)=0,\ t\in\left(0,1\right),\\ &\mathcal{D}_{0^+}^{\alpha}x\left(0\right)=0,\ x\left(1\right)=\gamma\int_0^1 x\left(t\right)dt, \end{aligned} \tag{P2}$$

where $\alpha,\ \beta\in\left(0,1\right],\ \lambda>0,\ \gamma\geq0$ and the function $f:\left[0,1\right]\times\mathbb{R}_+\to\mathbb{R}_+$ is continuous. , applying the Guo-Krasnoselskii fixed point theorem, The Banach contraction principle theorem and Ulam-Hyers stability. The results in this chapter are accepted for publication in "*Stud. Univ. Babeş-Bolyai Math.*" in title "*Nonlinear two conformable fractional differential equation with integral boundary condition*" , see [14].

3.1 Solution form of the problem

Let the Banach space $E=C\left(\left[0,1\right],\left[0,+\infty\right)\right)$ with the norm $\left\|x\right\|=\max_{t\in[0,1]}\left|x\left(t\right)\right|$.
Firstly, we present the solution of the linearized equation related to the two-CFBVP (P2): let $g\in E$

$$\mathcal{D}_{0^+}^{\beta}\mathcal{D}_{0^+}^{\alpha}x\left(t\right)+\lambda g\left(t\right)=0, \tag{A-P2}$$

with the boundary conditions

$$\mathcal{D}_{0^+}^{\alpha}x\left(0\right)=0,\ x\left(1\right)=\gamma\int_0^1 x\left(t\right)dt. \tag{B-P2}$$

Lemma 3.1. *The problem represented in* (A-P2)-(B-P2) *has a unique solution x given by*

$$x\left(t\right)=\lambda\int_0^1 G\left(t,s\right)g\left(s\right)ds,$$

where

$$G(t,s)=\frac{1}{\beta}\begin{cases}\left[\frac{\beta+1-\gamma}{(\beta+1)(1-\gamma)}\left(1-s^{\beta}\right)-\left(t^{\beta}-s^{\beta}\right)\right]s^{\alpha-1}, & 0\leq s\leq t\leq 1,\\ \\ \frac{\beta+1-\gamma}{(\beta+1)(1-\gamma)}\left(1-s^{\beta}\right)s^{\alpha-1}, & 0\leq t\leq s\leq 1.\end{cases}\tag{G-P2}$$

Proof. By the continuity of g and Lemma 1.2, we apply $\mathcal{I}_{0^+}^{\beta}$ it follows $\mathcal{I}_{0^+}^{\alpha}$ on (A-P2), we obtain

$$x(t)=x(0)+\mathcal{I}_{0^+}^{\alpha}\mathcal{D}_{0^+}^{\alpha}x(0)-\lambda\mathcal{I}_{0^+}^{\alpha}\mathcal{I}_{0^+}^{\beta}g(t)\,,\ t\in[0,1]\,.$$

This, together with the boundary conditions (B-P2), give

$$x(t)=\gamma\int_0^1 x(t)\,dt+\lambda\mathcal{I}_{0^+}^{\alpha}\mathcal{I}_{0^+}^{\beta}g(1)-\lambda\mathcal{I}_{0^+}^{\alpha}\mathcal{I}_{0^+}^{\beta}g(t)\,,\ t\in[0,1]\,.\tag{3.1}$$

Now, we integrate (3.1) from 0 to 1 in both sides and by using the Fubini theorem, we get

$$\int_0^1 x(t)dt=\gamma\int_0^1 x(t)\,dt+\frac{\lambda}{\beta}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds-\frac{\lambda}{\beta(\beta+1)}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds,$$

which implies

$$\int_0^1 x(t)\,dt=\frac{\lambda}{(\beta+1)(1-\gamma)}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds.\tag{3.2}$$

Substituting (3.2) into (3.1), which yields

$$\begin{aligned}x(t) &= \frac{\lambda\gamma}{(\beta+1)(1-\gamma)}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds\\ &\quad+\frac{\lambda}{\beta}\int_0^1 s^{\alpha-1}\left(1-s^{\beta}\right)g(s)\,ds-\frac{\lambda}{\beta}\int_0^t s^{\alpha-1}\left(t^{\beta}-s^{\beta}\right)g(s)\,ds.\end{aligned}$$

□

Next, the Green function G in (G-P2) has several important properties given as follows:

Lemma 3.2. *For any (t,s) in $[0,1]\times[0,1]$ and $\gamma\in[0,1)$:*

1. *$0\leq G(t,s)$ and continuous.*
2. *$G(1,s)\leq G(t,s)\leq G(0,s)$.*
3. *$G(0,s)=G(s,s)=\frac{\beta+1-\gamma}{\gamma\beta}G(1,s)$.*

Proof. Obviously that G is positive and continuous for all $t \in (0,1)$. From

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} -t^{\beta-1}s^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ \\ 0, & 0 \leq t \leq s \leq 1, \end{cases}$$

it is evident that $\frac{\partial G(t,s)}{\partial t} \leq 0$, for $0 \leq t, s \leq 1$, then $G(t,s)$ is decreasing with respect to $t \in [0,1]$, and therefore

$$G(1,s) \leq G(t,s) \leq G(0,s), \ for \ 0 \leq t, s \leq 1.$$

A simple calculation shows that

$$\begin{aligned} G(0,s) &= \frac{\beta+1-\gamma}{\beta(\beta+1)(1-\gamma)}\left(1-s^{\beta}\right)s^{\alpha-1} = G(s,s), \\ G(1,s) &= \frac{\gamma}{(\beta+1)(1-\gamma)}\left(1-s^{\beta}\right)s^{\alpha-1} = \frac{\gamma\beta}{\beta+1-\gamma}G(0,s). \end{aligned}$$

□

3.2 Main results

For investigating the existence, uniqueness and stability of positive solutions for the two-CFBVP (P2), we define the Banach space $E = C([0,1],[0,+\infty))$ with the norm $\|x\| = \max_{t\in[0,1]}|x(t)|$. Further, we define the cone $\mathcal{P}$ in E by

$$\mathcal{P} = \left\{ x \in E, \ x(t) \geq \frac{\gamma\beta}{\beta+1-\gamma}\|x\|, \ t \in [0,1], \ \gamma \in [0,1) \right\},$$

and the operator $\mathcal{T} : E \to E$ as

$$\mathcal{T}x(t) = \lambda \int_0^1 G(t,s) f(s,x(s))\, ds,$$

under the properties of G in Lemma 3.2 and our assumptions on f, the operator is well-defined, continuous, positive and has the following properties.

Lemma 3.3. **(i)** $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$.

(ii) *The operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. **(i)** From Lemma 3.2 and the definition of the cone $\mathcal{P}$, we have

$$\begin{aligned}
\mathcal{T}x(t) &= \lambda \int_0^1 G(t,s) f(s,x(s))\,ds \\
&\geq \frac{\lambda\gamma\beta}{\beta+1-\gamma} \int_0^1 G(0,s) f(s,x(s))\,ds \\
&= \frac{\lambda\gamma\beta}{\beta+1-\gamma} \int_0^1 \max_{t\in[0,1]} G(t,s) f(s,x(s))\,ds \\
&= \frac{\lambda\gamma\beta}{\beta+1-\gamma} \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,x(s))\,ds \\
&= \frac{\gamma\beta}{\beta+1-\gamma} \|\mathcal{T}x\|,\ \textit{for all}\ t\in[0,1].
\end{aligned}$$

Then, for all $t\in[0,1]$ we have

$$\mathcal{T}x(t) \geq \frac{\gamma\beta}{\beta+1-\gamma} \|\mathcal{T}x\|.$$

Hence $\mathcal{T}x \in \mathcal{P}$.

(ii) Let $x\in\Omega_0$, then there exists a positive constant L_0 such that

$$\sup_{\|x\|\leq r_0} \max_{t\in[0,1]} f(t,x) \leq L_0,$$

then, it holds that

$$\|\mathcal{T}x(t)\| = \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) f(s,x(s))\,ds \leq \lambda L_0 \int_0^1 G(0,s)\,ds,$$

which implies that $\mathcal{T}(\Omega_0)$ is bounded. Hence, for all $t_1,\ t_2 \in [0,1],\ t_1 < t_2$ and by Lemma 3.2, we have

$$\begin{aligned}
\|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)\| &\leq \max_{t\in[0,1]} \int_{t_1}^{t_2} G(t,s) f(s,x(s))\,ds \\
&\leq L_0 \int_{t_1}^{t_2} G(0,s)\,ds \\
&= \frac{L_0\lambda(\beta+1-\gamma)}{\beta(\beta+1)(1-\gamma)} \int_{t_1}^{t_2} \left(1-s^\beta\right) s^{\alpha-1} ds \\
&\leq \frac{L_0\lambda(\beta+1-\gamma)}{\alpha\beta(\beta+1)(1-\gamma)} \left(t_2^\alpha - t_1^\alpha\right),
\end{aligned}$$

$\|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)\| \to 0$ as $t_1 \to t_2$ which implies that the set $\mathcal{T}(\Omega_0)$ is equicontinuous. By the Arzelà-Ascoli theorem 1.1, the operator $\mathcal{T}:\Omega_0 \to \Omega_0$ is compact. We thus complete the proof.

□

Lemma 3.4. *The two-CFBVP* (P2) *has a positive solution $x \in E$ if and only if it is a fixed point of $\mathcal{T}$ in $\mathcal{P}$.*

Proof. Let x be a fixed point of $\mathcal{T}$ in $\mathcal{P}$, then

$$\begin{aligned} x(t) &= \lambda \int_0^1 G(t,s) f(s, x(s)) \, ds, \ t \in [0,1], \\ &= \gamma \int_0^1 x(t) \, dt + \lambda \mathcal{I}_{0^+}^{\alpha} \mathcal{I}_{0^+}^{\beta} f(t, x(t)), \end{aligned} \tag{3.3}$$

and thus, by the continuity of f and Lemma 1.2, we obtain

$$\mathcal{D}_{0^+}^{\beta} \mathcal{D}_{0^+}^{\alpha} x(t) = \lambda f(t, x(t)).$$

Furthermore, the equality (3.3) directly implies (B-P2) Therefore, x is a positive solution of the two-CFBVP (P2). Moreover, the Lemmas 3.1 and 3.3 imply that x is a fixed point of $\mathcal{T}$ in $\mathcal{P}$. □

Before presenting our results, we present some important notations as follows:

$$\Lambda_1 = \int_0^1 G(0,s) \, ds, \ \Lambda_2 = \frac{\gamma \beta}{\beta + 1 - \gamma} \int_0^1 G(0,s) \, ds.$$

and

$$\begin{aligned} f^0 &= \lim_{x \to 0} \max_{t \in [0,1]} \frac{f(t,x)}{x}, \ f_0 = \lim_{x \to 0} \min_{t \in [0,1]} \frac{f(t,x)}{x}, \\ f^{\infty} &= \lim_{x \to +\infty} \max_{t \in [0,1]} \frac{f(t,x)}{x}, \ f_{\infty} = \lim_{x \to +\infty} \min_{t \in [0,1]} \frac{f(t,x)}{x}. \end{aligned}$$

3.2.1 Existence results via Guo-Krasnoselskii fixed point theorem

Theorem 3.1. *Assume there exists $r_2 > r_1 > 0$, such that*

$$\begin{aligned} f(t,x) &\leq \frac{r_2}{\lambda \Lambda_1}, \ x \in [0, r_2], \ t \in [0,1], \\ f(t,x) &\geq \frac{r_1}{\lambda \Lambda_2}, \ x \in [0, r_1], \ t \in [0,1], \end{aligned}$$

then the two-CFBVP (P2) *has at least one positive solution.*

Proof. By Lemma 3.2, for $x \in \mathcal{P} \cap \partial \Omega_1$, we have

$$\|\mathcal{T}x\| \geq \mathcal{T}x(t) \geq \frac{\gamma \beta}{\beta + 1 - \gamma} \int_0^1 G(0,s) \frac{r_1}{\Lambda_2} ds = r_1.$$

For $x \in \mathcal{P} \cap \partial\Omega_2$, we get

$$\|\mathcal{T}x\| = \int_0^1 G(0,s)\, f(s, x(s))\, ds \leq \int_0^1 G(0,s)\, \frac{r_2}{\Lambda_1} ds = r_2.$$

Applying Theorem 1.2 yields that $\mathcal{T}$ has at least one fixed point $x \in \mathcal{P} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \leq \|x\| \leq r_2$. It follows by Lemma 3.4 that the problem (P2) has at least one positive solution x. □

Theorem 3.2. *Let*

$$f_\infty \geq \frac{\beta + 1 - \gamma}{\gamma\beta} \ \ and \ \ f^0 \leq \frac{\gamma\beta}{\beta + 1 - \gamma}$$

are satisfied, then for each $\lambda \in \left(\frac{1}{\Lambda_1}, \frac{1}{\Lambda_2}\right)$ *the two-CFBVP* (P2) *has at least one positive solution.*

Proof. From the definition of f^0, there exists $r_1 > 0$, such that

$$f(t, x) \leq f^0 x, \ \ for\ all\ t \in [0,1], \ 0 < x \leq r_1.$$

For $x \in \mathcal{P} \cap \partial\Omega_1$, we have

$$\|\mathcal{T}x\| = \lambda \int_0^1 G(0,s)\, f(s, x(s))\, ds \leq \lambda \int_0^1 G(0,s)\, f^0 x(s)\, ds \leq \lambda f^0 \|x\| \Lambda_1 \leq \|x\|.$$

Consequently

$$\|\mathcal{T}x\| \leq \|x\|, \ x \in \mathcal{P} \cap \partial\Omega_1. \tag{3.4}$$

By the definition of f_∞, there exists $r_3 > 0$, such that

$$f(t, x) \geq f_\infty x, \ \ for\ all\ t \in [0,1], \ x \geq r_3.$$

If $x \in \mathcal{P} \cap \partial\Omega_2$ with $r_2 = \max\{2r_1, r_3\}$, then by the definition of cone $\mathcal{P}$, we have

$$\begin{aligned} \|\mathcal{T}x\| = \lambda \int_0^1 G(0,s)\, f(s, x(s))\, ds &\geq \lambda f_\infty \int_0^1 G(0,s)\, x(s)\, ds \\ &\geq \lambda \frac{\gamma\beta}{\beta + 1 - \gamma} f_\infty \|x\| \int_0^1 G(0,s)\, ds \\ &\geq \|x\|. \end{aligned}$$

Hence

$$\|\mathcal{T}x\| \geq \|x\|, \ x \in \mathcal{P} \cap \partial\Omega_2. \tag{3.5}$$

From (3.4)-(3.5) and Theorem 1.2 we assurance that the operator $\mathcal{T}$ has at least one fixed point $x \in \mathcal{P} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ with $r_1 \leq \|x\| \leq r_2$. It follows from Lemma 3.4 that the two-CFBVP (P2) has at least one positive solution x. □

Theorem 3.3. *If*

$$f_0 \geq \frac{\beta+1-\gamma}{\gamma\beta} \quad and \quad f^{\infty} \leq \frac{\gamma\beta}{2\left(\beta+1-\gamma\right)}$$

are hold, then for each $\lambda \in \left(\frac{1}{\Lambda_1}, \frac{1}{\Lambda_2}\right)$ *the two-CFBVP* (P2) *has at least one positive solution.*

Proof. From the definition of f_0, there exists $r_1 > 0$, such that

$$f\left(t,x\right) \geq f_0 x, \; for \; all \; t \in \left[0,1\right], \; 0 < x \leq r_1.$$

Further, for $x \in \mathcal{P}$ with $\|x\| = r_1$, then as previously

$$\|\mathcal{T}x\| \geq \lambda \int_0^1 G\left(0,s\right) f_0 x\left(s\right) ds \geq \lambda \frac{\gamma\beta}{\beta+1-\gamma} f_0 \|x\| \int_0^1 G\left(0,s\right) ds \geq \|x\|.$$

Hence

$$\|\mathcal{T}x\| \geq \|x\|, \; x \in \mathcal{P} \cap \partial\Omega_1.$$

By the definition of f^{∞}, there exists $r_4 > 0$, such that

$$f\left(t,x\right) \leq f^{\infty} x, \; for \; all \; t \in \left[0,1\right], \; x \geq r_4,$$

it follows, there exists $\delta > 0$, such that

$$\delta = \max_{t \in [0,1]} f\left(t, r_4\right), \; for \; all \; t \in \left[0,1\right], \; 0 < x \leq r_4.$$

Then

$$f\left(t,x\right) \leq f^{\infty} x + \delta, \; for \; all \; t \in \left[0,1\right], \; x \geq 0.$$

If $x \in \mathcal{P} \cap \partial\Omega_2$, with $r_2 = \max\left\{2r_1, \; \frac{2\gamma\beta\delta}{\beta+1-\gamma}\right\}$, we get

$$\begin{aligned} \|\mathcal{T}x\| = \lambda \int_0^1 G\left(0,s\right) f\left(s, x\left(s\right)\right) ds &\leq \lambda \int_0^1 G\left(0,s\right)\left(f^{\infty} x\left(s\right) + \delta\right) ds \\ &\leq \lambda\left(f^{\infty}\|x\| + \delta\right)\Lambda_1 \\ &\leq \|x\|. \end{aligned}$$

Thus

$$\|\mathcal{T}x\| \leq \|x\|, \; x \in \mathcal{P} \cap \partial\Omega_2.$$

Applying Theorem 1.2 yields that $\mathcal{T}$ has at least one fixed point $x \in \mathcal{P} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ and Lemma 3.4 ensure that the two-CFBVP (P2) has at least one positive solution x. □

3.2.2 Uniqueness results via Banach contraction principle theorem

Theorem 3.4. *Assume there exists* $L > 0$ *such that*

$$|f(t,x) - f(t,y)| \leq L|x-y|, \ for \ almost \ every \ t \in [0,1], \ and \ all \ x, y \in E.$$

Then, if

$$\Delta = \lambda L \Lambda_1 < 1, \tag{3.6}$$

the two-CFBVP (P2) *has exactly one positive solution defined on* $[0,1]$.

Proof. Using Lemma 1.2, we have

$$\begin{aligned}
\|\mathcal{T}x(t) - \mathcal{T}y(t)\| &\leq \lambda \int_0^1 G(0,s)\left|\left(f(s,x(s)) - f(s,y(s))\right)\right| ds \\
&\leq \lambda L \|x-y\| \int_0^1 G(0,s)\, ds \\
&= \Delta \|x-y\|.
\end{aligned}$$

By Theorem 1.3 and Lemma 3.4, there is a unique and positive x in E with $x = \mathcal{T}x$. □

3.2.3 Ulam-Hyers stability results

This subsection present fourth types of Ulam-Hyers stability for the problem (P2), namely Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias:

Theorem 3.5. *Let* (3.6) *holds, then the two-CFBVP* (P2) *is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.*

Proof. Let $y \in E$ be any solution of the inequality (1.4). Thank to Lemma 3.1, we have

$$y(t) = \lambda \int_0^1 G(t,s) f(s,y(s))\, ds,$$

which yields

$$\begin{aligned}
\left| y(t) - \lambda \int_0^1 G(t,s) f(s,y(s))\, ds \right| &\leq \frac{\varepsilon}{\beta} \int_0^t \left(t - s^\beta\right) s^{\alpha-1} ds \\
&\leq \frac{\varepsilon}{\beta} \int_0^1 \left(1 - s^\beta\right) s^{\alpha-1} ds \\
&\leq \varepsilon \Lambda_1.
\end{aligned}$$

Let $x \in E$ be the unique solution of the two-CFBVP (P2), we have for any $t \in [0,1]$

$$\begin{aligned} |y(t) - x(t)| &= \left| y(t) - \lambda \int_0^1 G(t,s) f(s, x(s))\, ds \right| \\ &= \left| y(t) - \lambda \int_0^1 G(t,s) f(s, y(s))\, ds + \lambda \int_0^1 G(t,s) f(s, y(s))\, ds \right. \\ &\quad \left. -\lambda \int_0^1 G(t,s) f(s, x(s))\, ds \right| \\ &\leq \left| y(t) - \lambda \int_0^1 G(t,s) f(t, y(s))\, ds \right| \\ &\quad +\lambda \left| \int_0^1 G(t,s) \left(f(s, y(s)) - f(s, x(s)) \right) ds \right| \\ &\leq \varepsilon \Lambda_1 + \lambda L \int_0^1 G(0,s) \left| (y(s) - x(s)) \right| ds, \end{aligned}$$

which implies

$$\|y - x\| \leq \varepsilon \Lambda_1 + \lambda L \Lambda_1 \|y - x\|,$$

on simplification it gives

$$\|y - x\| \leq \varepsilon c_f, \ where \ c_f = \frac{\Lambda_1}{1 - \lambda L \Lambda_1},$$

which completes the proof. By putting

$$\theta_f(\varepsilon) = \varepsilon c_f, \ \theta_f(0) = 0,$$

then the two-CFBVP (P2) is generalized Ulam-Hyers stable. □

Theorem 3.6. *Let* (3.6) *holds. Assume that, there exists an increasing function* $\varphi \in E$ *and there exists* $\sigma_\varphi \in \mathbb{R}_+$ *such that for any* $t \in [0,1]$

$$\mathcal{I}_{0+}^{\alpha} \mathcal{I}_{0+}^{\beta} \varphi(t) \leq \sigma_\varphi \varphi(t),$$

then the solutions of the two-CFBVP (P2) *is Ulam-Hyers-Rassias stable. Further the solutions of the considered two-CFBVP* (P2) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Similar to the proof of Theorem 3.5, let $y \in E$ be any solution of the inequality (1.5), Thank to Lemma 3.1, we obtain

$$\begin{aligned} \left| y(t) - \lambda \int_0^1 G(t,s) f(s, y(s))\, ds \right| &\leq \varepsilon I_{0+}^{\alpha} I_{0+}^{\beta} \varphi(t) \\ &\leq \varepsilon \sigma_\varphi \varphi(t). \end{aligned}$$

Let $x \in E$ be the unique solution of the two-CFBVP (P2), we have for any $t \in [0,1]$

$$\begin{aligned} |y(t) - x(t)| &\leq \left| y(t) - \lambda \int_0^1 G(t,s) f(t, y(s))\, ds \right| \\ &\quad + \lambda \left| \int_0^1 G(t,s) \left(f(s, y(s)) - f(s, x(s)) \right) ds \right| \\ &\leq \varepsilon \sigma_\varphi \varphi(t) + \lambda L \int_0^1 G(0,s) \left| (y(s) - x(s)) \right| ds, \end{aligned}$$

which implies that

$$\|y - x\| \leq c_f \varepsilon \sigma_\varphi \varphi(t)\,,\ where\ c_f = \frac{1}{1 - \lambda L \Lambda_1},$$

which completes the proof of the theorem. Moreover, if we set

$$\varphi(\varepsilon) = \varepsilon \varphi(t)\,, \varphi(0) = 0.$$

Analogously one can easily prove that the solutions of two-CFBVP (P2) are generalized Ulam-Hyers-Rassias stable. □

3.3 Examples

Example 3.1. Consider the two-CFBVP (P2) with $\beta = 1,\ \alpha = \frac{1}{2},\ \gamma = \frac{3}{4}$. By simple calculations we obtain $\frac{\gamma\beta}{\beta+1-\gamma} = \frac{3}{5}$ and

$$\begin{aligned} \Lambda_1 &= \int_0^1 G(0,s)\, ds = \frac{\beta + 1 - \gamma}{\beta(\beta+1)(1-\gamma)} \int_0^1 \left(1 - s^\beta\right) s^\alpha ds = \frac{2}{3}, \\ \Lambda_2 &= \frac{\gamma\beta}{\beta + 1 - \gamma} \int_0^1 G(0,s)\, ds = \frac{2}{5}. \end{aligned}$$

Set the continuous functions f and F, for any $t \in [0,1]$ and any $x > 0$, by

$$f(t,x) = \begin{cases} (t+1)\, x^2,\ (t,x) \in [0,1] \times (0,2]\,, \\ 2(t+1)\, x,\ (t,x) \in [0,1] \times (2,\infty)\,, \end{cases} \quad and\ \ F(t,x) = (2t+1)\left(\sin x + e^{-x}\right),$$

we have

$$f^0 = 0,\ f_\infty = 2\ \ and\ \ F_0 = \infty,\ F^\infty = 0.$$

For $\lambda \in \left(\frac{3}{2}, \frac{5}{2}\right)$, for specified function f the Theorem 3.2 (or for function F the Theorem 3.3) gives that the two-CFBVP (P2) has at least one positive solution x defined on $[0,1]$.

Example 3.2. Consider the two-CFBVP (P2) with $\beta = \frac{1}{2}, \alpha = 1, \gamma = \frac{3}{4}$ and the continuous function $f\left(t,x\right)=\frac{1}{t+2}\sin x$. As

$$\left|f\left(t,x\right)-f\left(t,y\right)\right|\leq\frac{1}{2}\left|x-y\right|\ and\ \Lambda_{1}=\frac{2}{5}.$$

For $\lambda\in\left(0,5\right)$, the Theorem 3.4 give that the two-CFBVP (P2) has exactly one positive solution x defined on $\left[0,1\right]$. Now, let

$$\left|D_{0^{+}}^{\frac{1}{2}}y'\left(t\right)+\frac{3}{5\left(t+2\right)}\sin x\right|\leq\varepsilon,\ t\in\left[0,1\right],$$

then, by Theorem 3.5 the two-CFBVP (P2) is Ulma-Hyers stable with $c_{f}=\frac{5}{11}$. On the other hand, Consider the inequality

$$\left|D_{0^{+}}^{\frac{1}{2}}y'\left(t\right)+\frac{3}{5\left(t+2\right)}\sin x\right|\leq\varepsilon t,\ t\in\left[0,1\right],$$

by Theorem 4.5 the two-CFBVP (P2) is Ulam-Hyers-Rassias stable with

$$c_{f}=\frac{1}{1-\lambda L\Lambda_{1}}=\frac{25}{22},\ \sigma_{t}=\frac{1}{\left(\alpha+1\right)\beta}=1.$$

CHAPTER 4

P-LAPLACIAN CFPS WITH INTEGRAL BOUNDARY CONDITIONS

In this chapter, we will investigate the existence, uniqueness and stability as well the positivity of solutions for two p-Laplacian CFPs with integral boundary conditions:

$$\begin{aligned}
&\mathcal{D}_{0^+}^{\beta}\left(\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)\right)(t) = -f\left(t, x\left(t\right)\right),\ t \in \left(0,1\right),\\
&\qquad \varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(0) = 0,\ \varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(1) = \lambda \int_0^1 \varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(t)\,dt,\\
&\qquad\qquad x\left(0\right) = 0,\ x\left(1\right) = \gamma \int_0^1 x\left(t\right) dt,
\end{aligned} \tag{P3}$$

and

$$\begin{aligned}
&\mathcal{D}_{0^+}^{\beta}\left(\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)\right)(t) = f\left(t, x\left(t\right)\right),\ t \in \left(0,1\right),\\
&\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(0) = 0,\ \left(\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)\right)'(0) = \mu \int_0^1 \varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(t)\,dt,\\
&x\left(0\right) = 0,\ x'\left(0\right) = \lambda \int_0^1 x\left(t\right) dt,
\end{aligned} \tag{P4}$$

where $\alpha,\ \beta \in (1,2]$, the function $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, the parameter λ and γ belong to $[1,2)$ and the p-Laplacian operator $\varphi_p\left(t\right) = t\left|t\right|^{p-2},\ p > 1$, where $\left(\varphi_p\right)^{-1} = \varphi_q,\ \frac{1}{p} + \frac{1}{q} = 1$. Our main tool is a Guo-Krasnoselskii fixed point theorem, The Banach contraction principle theorem and Ulam-Hyers stability. One example is also given to show the applicability of our results.

4.1 Solutions form of the problems

Let $E = C([0,1], [0,+\infty))$ be the Banach space with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$.
Firstly, we give the corresponding linear systems of (P3): for any given function $g \in E$, we have

$$\begin{aligned}
&\mathcal{D}_{0^+}^{\beta}\left(\varphi_p \mathcal{D}_{0^+}^{\alpha} x\right)(t) = -g(t),\ t \in (0,1),\\
&\varphi_p\left(\mathcal{D}_{0^+}^{\alpha} x\right)(0) = 0,\ \varphi_p\left(\mathcal{D}_{0^+}^{\alpha}\right) x(1) = \mu \int_0^1 \varphi_p \mathcal{D}_{0^+}^{\alpha} x(t)\, dt,\\
&x(0) = 0,\ x(1) = \lambda \int_0^1 x(t)\, dt.
\end{aligned} \tag{A-P3}$$

To get the linear solution representation of problem (P3), we decompose (A-P3) into two coupled problems:

$$\begin{aligned}
&\mathcal{D}_{0^+}^{\beta} v(t) = -g(t),\ t \in (0,1),\\
&v(0) = 0,\ v(1) = \mu \int_0^1 v(t)\, dt,
\end{aligned} \tag{B-P3}$$

and

$$\begin{aligned}
&\mathcal{D}_{0^+}^{\alpha} x(t) = \varphi_q(v)(t),\ t \in (0,1),\\
&x(0) = 0,\ x(1) = \lambda \int_0^1 x(t)\, dt.
\end{aligned} \tag{C-P3}$$

where

$$v(t) = \varphi_p\left(\mathcal{D}_{0^+}^{\alpha} x\right)(t). \tag{4.1}$$

Lemma 4.1. *The problem* (B-P3) *has a unique solution defined on* $[0,1]$ *given by*

$$v(t) = \int_0^1 H_1(t,s)\, g(s)\, ds = \varphi_p\left(\mathcal{D}_{0^+}^{\alpha} x\right)(t), \tag{4.2}$$

where

$$H_1(t,s) = \begin{cases} \left(\frac{\mu}{2-\mu} t(1-s) + 1 - t\right) s^{\beta-1}, & 0 \le s \le t \le 1, \\ \\ t\left(1 + \frac{\mu s}{2-\mu}\right)(1-s)\, s^{\beta-2}, & 0 \le t \le s \le 1. \end{cases} \tag{H-P3}$$

Proof. According to Lemma ((1.3),(ii)), applying $\mathbb{I}_{0^+}^{\beta}$ on (B-P3) gives

$$v(t) = v(0) + v'(0)\, t - \mathbb{I}_{0^+}^{\beta} g(t).$$

This, together the boundary conditions, implies

$$v(0) = 0,\ v(1) = v'(0) - \mathbb{I}_{0^+}^{\beta} g(1),$$

then

$$v'(0) = \mathbb{I}_{0^+}^{\beta} g(1) + \mu \int_0^1 x(s)\, ds.$$

Hence

$$v(t) = \mu t \int_0^1 v(s)\, ds + t \int_0^1 (1-s)\, s^{\beta-2} g(s)\, ds - \int_0^t (t-s)\, s^{\beta-2} g(s)\, ds. \tag{4.3}$$

Now we integrate (4.3) from 0 to 1 in both sides and we use Fubini's theorem, we get

$$\int_0^1 v(t)\, dt = \frac{\mu}{2} \int_0^1 v(s)\, ds + \frac{1}{2} \int_0^1 (1-s)\, s^{\beta-2} g(s)\, ds - \frac{1}{2} \int_0^1 (1-s)^2\, s^{\beta-2} g(s)\, ds.$$

Therefore

$$\int_0^1 v(t)\, dt = \frac{1}{(2-\mu)} \int_0^1 (1-s)\, s^{\beta-1} g(s)\, ds.$$

Substituting the above expression into Eq. (4.3) and with the relation (4.1), we arrive at the desire result. □

Lemma 4.2. *The problem* (A-P3) *has a unique solution*

$$x(t) = \int_0^1 G_1(t,s)\, \varphi_q \left(\int_0^1 H_1(s,\tau)\, g(\tau)\, d\tau \right) ds, \tag{4.4}$$

where

$$G_1(t,s) = \begin{cases} \left(\frac{\lambda}{2-\lambda} t(1-s) + 1 - t \right) s^{\alpha-1}, & 0 \le s \le t \le 1, \\ \\ t \left(1 + \frac{\lambda s}{2-\lambda} \right) (1-s)\, s^{\alpha-2} & 0 \le t \le s \le 1, \end{cases} \tag{G-P3}$$

where H_1 is defined by (H-P3).

Proof. In a similar way that is used to prove Lemma 4.1, we can get that boundary value problem (C-P3) has a unique solution, which is given by

$$x(t) = \int_0^1 G_1(t,s)\, \varphi_q(v)(s)\, ds.$$

Substituting (4.2) into the above expression to get (4.4). The proof is completed. □

Remark 4.1. The existence of positive solutions of (B-P3) or (C-P3) was discussed in [54], by employing a fixed point theorem in a cone, where the expression of Green functions has two parts.

Next, we give the solutions representations for linear systems of problem (P4)

Lemma 4.3. *For any given function $g \in E$, the following problem*

$$\begin{aligned} &\mathcal{D}_{0^+}^{\beta}\left(\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)\right)(t) = g(t), \ t \in (0,1), \\ &\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(0) = 0, \ \left(\varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)\right)'(0) = \mu \int_0^1 \varphi_p\left(\mathcal{D}_{0^+}^{\alpha}x\right)(t)\,dt, \\ &x(0) = 0, \ x'(0) = \lambda \int_0^1 x(t)\,dt, \end{aligned} \tag{A-P4}$$

has a unique solution, which is given by

$$x(t) = \int_0^1 G_2(t,s)\,\varphi_q\left(\int_0^1 H_2(s,\tau)\,g(\tau)\,d\tau\right)ds, \tag{4.5}$$

where

$$G_2(t,s) = s^{\alpha-2}\begin{cases} \frac{\lambda}{2-\lambda} t(1-s)^2 + (t-s), & 0 \le s \le t \le 1, \\ \\ \frac{\lambda}{2-\lambda} t(1-s)^2, & 0 \le t \le s \le 1, \end{cases} \tag{G-P4}$$

and

$$H_2(s,\tau) = \tau^{\beta-2}\begin{cases} \frac{\mu}{2-\mu} s(1-\tau)^2 + (s-\tau), & 0 \le \tau \le s \le 1, \\ \\ \frac{\mu}{2-\mu} s(1-\tau)^2, & 0 \le s \le \tau \le 1. \end{cases} \tag{H-P4}$$

Proof. By the relation (4.1), the problem (A-P4) decomposed by the following coupled BVPs:

$$\begin{aligned} &D_{0^+}^{\beta} v(t) = -g(t), \ t \in (0,1), \\ &v(0) = 0, \ v'(0) = \mu \int_0^1 v(t)\,dt, \end{aligned} \tag{B-P4}$$

and

$$\begin{aligned} &D_{0^+}^{\alpha} x(t) = \varphi_q(v)(t), \ t \in (0,1), \\ &x(0) = 0, \ x'(0) = \lambda \int_0^1 x(t)\,dt. \end{aligned} \tag{C-P4}$$

By Lemma ((1.3),(ii)), we apply the operator $\mathbb{I}_{0+}^{\beta}$ on both sides of (B-P4) combining with the boundary conditions to get

$$v\left(t\right)=\mu t\int_{0}^{1}v\left(t\right)dt+\mathbb{I}_{0+}^{\beta}g\left(t\right). \tag{4.6}$$

Now we integrate (4.6) from 0 to 1 in both sides and by using the Fubini theorem, we get

$$\int_{0}^{1}v\left(t\right)dt=\frac{1}{2-\mu}\int_{0}^{1}\left(1-s\right)^{2}s^{\beta-2}g\left(s\right)ds.$$

Substituting above expression into (4.6), which yields

$$v\left(t\right)=\frac{\mu t}{2-\mu}\int_{0}^{1}\left(1-s\right)^{2}s^{\beta-2}g\left(s\right)ds+\int_{0}^{t}\left(t-s\right)s^{\beta-2}g\left(s\right)ds. \tag{4.7}$$

By the standard way, we can get that (C-P4) has a unique solution, which is given by

$$x\left(t\right)=\int_{0}^{1}G_{2}\left(t,s\right)\varphi_{q}\left(v\right)\left(s\right)ds.$$

Substituting (4.7) into the above expression to get (4.5). The proof is completed. □

4.1.1 Properties of the Green functions

In order to investigate the properties of the Green functions G_1 in (G-P3), H_1 in (H-P3), G_2 (G-P4) and H_2 in (H-P4), we have

Lemma 4.4. *For any $(t,s)\in(0,1)\times(0,1)$ and $\mu,\lambda\in[1,2)$, the following properties are satisfied. $\forall i=1,2$:*

(i)
1. $0\leq G_i\left(t,s\right)$, *continuous and* $\frac{\partial}{\partial t}G_i\left(t,s\right)\leq\frac{2}{2-\lambda}s^{\alpha-2}$.
2. $G_1\left(t,s\right)\leq\frac{2}{\lambda}G_1\left(1,s\right)$.
3. $G_2\left(t,s\right)\leq G_2\left(1,s\right)\leq\frac{2}{\lambda}G_2\left(1,s\right)$.
4. $G_i\left(t,s\right)\leq\frac{2}{2-\lambda}s^{\alpha-2}$.
5. $t\,G_i\left(s,s\right)\leq G_i\left(t,s\right)$.

(ii)
1. $0\leq H_i\left(t,s\right)$, *continuous and* $\frac{\partial}{\partial t}H_i\left(t,s\right)\leq\frac{2}{2-\mu}s^{\beta-2}$.
2. $H_1\left(t,s\right)\leq\frac{2}{\mu}H_1\left(1,s\right)$.
3. $H_2\left(t,s\right)\leq H_2\left(1,s\right)\leq\frac{2}{\mu}H_2\left(1,s\right)$.
4. $H_i\left(t,s\right)\leq\frac{2}{2-\mu}s^{\beta-2}$.
5. $t\,H_i\left(s,s\right)\leq H_i\left(t,s\right)$.

Proof. **(i)** 1. Obviously that G_i is positive and continuous for all $t \in (0,1)$ and

$$\begin{aligned}
\frac{\partial}{\partial t} G_1(t,s) &= \begin{cases} \left(\frac{\lambda}{2-\lambda}(1-s)-1\right)s^{\alpha-1}, & 0 \le s \le t \le 1, \\ \\ \left(1+\frac{\lambda s}{2-\lambda}\right)(1-s)s^{\alpha-2} & 0 \le t \le s \le 1, \end{cases} \\
\frac{\partial}{\partial t} G_2(t,s) &= s^{\alpha-2}\begin{cases} \frac{\lambda}{2-\lambda}(1-s)^2+1, & 0 \le s \le t \le 1, \\ \\ \frac{\lambda}{2-\lambda}(1-s)^2, & 0 \le t \le s \le 1. \end{cases}
\end{aligned}$$

We have, for $s \le t$

$$\begin{gathered}
\left(\frac{\lambda}{2-\lambda}(1-s)-1\right)s^{\alpha-1} \le \frac{\lambda}{2-\lambda}s^{\alpha-2} \le \left(\frac{\lambda}{2-\lambda}+1\right)s^{\alpha-2} = \frac{2}{2-\lambda}s^{\alpha-2}, \\
\left(\frac{\lambda}{2-\lambda}(1-s)^2+1\right)s^{\alpha-2} \le \left(\frac{\lambda}{2-\lambda}+1\right)s^{\alpha-2} = \frac{2}{2-\lambda}s^{\alpha-2},
\end{gathered}$$

and for $t \le s$

$$\begin{gathered}
\left(1+\frac{\lambda s}{2-\lambda}\right)(1-s)s^{\alpha-2} \le \left(1+\frac{\lambda}{2-\lambda}\right)s^{\alpha-2} = \frac{2}{2-\lambda}s^{\alpha-2}, \\
\frac{\lambda}{2-\lambda}(1-s)^2 s^{\alpha-2} \le \frac{\lambda}{2-\lambda}s^{\alpha-2} \le \frac{2}{2-\lambda}s^{\alpha-2},
\end{gathered}$$

Hence

$$\frac{\partial}{\partial t} G_i(t,s) \le \frac{2}{2-\lambda}s^{\alpha-2}.$$

2. If $s \le t$

$$\begin{aligned}
G_1(t,s) = \left[s(1-t)+\frac{t\lambda s}{2-\lambda}(1-s)\right]s^{\alpha-2} &\le \left[s(1-s)+\frac{\lambda s}{2-\lambda}(1-s)\right]s^{\alpha-2} \\
&\le \left(1+\frac{\lambda}{2-\lambda}\right)(1-s)s^{\alpha-1} \\
&\le \frac{2}{2-\lambda}(1-s)s^{\alpha-1} \\
&= \frac{2}{\lambda}G_1(1,s),
\end{aligned}$$

and if $t \leq s$

$$\begin{aligned} G_1(t,s) = \frac{t}{s}\left(1+\frac{\lambda s}{2-\lambda}\right)(1-s)\,s^{\alpha-1} &\leq \left(1+\frac{\lambda s}{2-\lambda}\right)(1-s)\,s^{\alpha-1} \\ &\leq \left(1+\frac{\lambda}{2-\lambda}\right)(1-s)\,s^{\alpha-1} \\ &\leq \frac{2}{2-\lambda}(1-s)\,s^{\alpha-1} \\ &= \frac{2}{\lambda}G_1(1,s), \end{aligned}$$

then, it is clear that

$$\frac{2}{\lambda}G_1(1,s) = \frac{2}{2-\lambda}(1-s)\,s^{\alpha-1} \leq \frac{2}{2-\lambda}s^{\alpha-1} \leq \frac{2}{2-\lambda}s^{\alpha-2}.$$

3. On the other hand, we have $\frac{\partial G_2(t,s)}{\partial t} \geq 0$, for $0 < t, s < 1$, then $G_2(t,s)$ is increasing with respect to t, and therefore,

$$\begin{aligned} G_2(t,s) \leq G_2(1,s) &= \left(\frac{\lambda s}{2-\lambda}(1-s)+1\right)(1-s)\,s^{\alpha-2} \\ &\leq \left(\frac{\lambda}{2-\lambda}+1\right)s^{\alpha-2} = \frac{2}{2-\lambda}s^{\alpha-2}. \end{aligned}$$

4. Moreover, observe that, By the definition of G_i: for $s \leq t$

$$\begin{aligned} G_1(t,s) = \left[(1-t)+t\frac{\lambda}{2-\lambda}(1-s)\right]s^{\alpha-1} &\geq \left[(1-s)+t\frac{\lambda s}{2-\lambda}(1-s)\right]s^{\alpha-1} \\ &\geq t\left[1+\frac{\lambda s}{2-\lambda}\right](1-s)\,s^{\alpha-1} \\ &= tG_1(s,s). \end{aligned}$$

$$\begin{aligned} G_2(t,s) = \left(\frac{\lambda}{2-\lambda}t(1-s)^2+(t-s)\right)s^{\alpha-2} &\geq \frac{t\lambda}{2-\lambda}(1-s)^2\,s^{\alpha-2},\ s^{\alpha-2} \geq s^{\alpha-1} \\ &\geq \frac{t\lambda}{2-\lambda}(1-s)^2\,s^{\alpha-1} \\ &\geq tG_2(s,s). \end{aligned}$$

For $t \leq s$

$$G_1(t,s) = t\left(1+\frac{\lambda s}{2-\lambda}\right)(1-s)s^{\alpha-2} \geq t\left(1+\frac{\lambda s}{2-\lambda}\right)(1-s)s^{\alpha-1} = tG_1(s,s).$$

$$G_2(t,s) = \frac{t\lambda}{2-\lambda}(1-s)^2 s^{\alpha-2} \geq \frac{t\lambda}{2-\lambda}(1-s)^2 s^{\alpha-1} = tG_2(s,s).$$

(ii) The proof can be easily accomplished by **(i)**, so we omitted it.

□

4.2 Main results

Let $E = C([0,1],[0,+\infty))$ be the Banach space with the norm $\|x\| = \max_{t\in[0,1]}|x(t)|$.
For investigating the existence, uniqueness and stability of positive solutions for the p-Laplacian CFPs (P3) and (P4), we define the cone $\mathcal{P}$ in E by

$$\mathcal{P} = \{x \in E,\ x(t) \geq 0,\ t \in [0,1]\}.$$

Furthermore, define the operators $\mathcal{T}_i : E \to E$ as

$$\mathcal{T}_i x(t) = \int_0^1 G_i(t,s)\varphi_q\left(\int_0^1 H_i(s,\tau) f(\tau, x(\tau))\,d\tau\right)ds,\ i = 1,2,$$

under the properties of the Green functions G_i and H_i, $(i = 1,2)$ in Lemma 4.4 and the continuity and positivity of f, the operator is well defined, continuous, positive and has the following properties. Let $\mathcal{T} = \{\mathcal{T}_1,\ \mathcal{T}_2\}$, $G = \{G_1,\ G_2\}$ and $H = \{H_1,\ H_2\}$.

Lemma 4.5. **(i)** $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$.

(ii) *The operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. **(i)** It's easily to accomplish.

(ii) For the uniformly boundedness of $\mathcal{T}$: for any $(t,x) \in [0,1] \times \Omega_0$, there exists a positive constant L_0 such that

$$L_0 = \sup_{\|x\|\leq r}\max_{t\in[0,1]} f(t,x) + 1,$$

then by Lemma 4.4, it holds that

$$\begin{aligned}
\|\mathcal{T}x(t)\| &= \max_{t\in[0,1]}\int_0^1 G(t,s)\,\varphi_q\left(\int_0^1 H(s,\tau)\,f(\tau,x(\tau))\,d\tau\right)ds\\
&\le \frac{2L_0^{q-1}}{\lambda}\int_0^1 G(1,s)\,\varphi_q\left(\int_0^1 H(s,\tau)\,d\tau\right)ds\\
&\le \frac{4L_0^{q-1}}{\lambda\mu}\int_0^1 G(1,s)\,\varphi_q\left(\int_0^1 H(1,\tau)\,d\tau\right)ds\\
&< +\infty.
\end{aligned}$$

Hence, $\mathcal{T}(\Omega_0)$ is uniformly bounded.
Now for the equicontinuity of $\mathcal{T}$: for all $0\le t_1<t_2\le 1$, by Lemma 4.4, we have

$$\begin{aligned}
|\mathcal{T}x(t_2)-\mathcal{T}x(t_1)| &\le \int_0^1 |G(t_2,s)-G(t_1,s)|\,\varphi_q\left(\int_0^1 H(s,\tau)\,f(\tau,x(\tau))\,d\tau\right)ds\\
&\le \frac{2L_0^{q-1}}{\mu}\varphi_q\left(\int_0^1 H(1,\tau)\,d\tau\right)\int_0^1 |G(t_2,s)-G(t_1,s)|\,ds
\end{aligned}$$

As consequence, for all $t_1,\ t_2\in[0,1],\ t_1<t_2$, we have by Lemma 4.4

$$\begin{aligned}
|G(t_2,s)-G(t_1,s)| &= \int_{t_1}^{t_2} |G'(t,s)|\,ds\\
&\le \frac{2}{2-\lambda}\int_{t_1}^{t_2} s^{\alpha-2}ds\\
&\le \frac{2}{(2-\lambda)(\alpha-1)}\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)
\end{aligned}$$

Then $\|\mathcal{T}x(t_2)-\mathcal{T}x(t_1)\|\to 0$ as $t_1\to t_2$ which implies that the set $\mathcal{T}(\Omega_0)$ is equicontinuous.

By the Ascoli-Arzelà theorem 1.1, the operator $\mathcal{T}:\Omega_0\to\Omega_0$ is compact.

□

Lemma 4.6. *The solution of p-Laplacian CFP* (P3) *(or* (P4)*) is equivalent to a fixed point of* $\mathcal{T}$ *in* $\mathcal{P}$*.*

Proof. We use the following Lemmas 1.3, 4.2, 4.3 and 4.5 to prove the desire results. □

Next, to prove the existence, uniqueness and stability results we need the following notations and assumptions:

$$\begin{aligned}
\Lambda_1 &= \frac{\lambda\mu}{4}\left[\int_0^1 G(1,s)\,ds\ \varphi_q\left(\int_0^1 H(1,\tau)\,d\tau\right)\right]^{-1},\\
\Lambda_2 &= \left[\rho^2\int_\rho^{1-\rho} G(s,s)\,ds\ \varphi_q\left(\int_\rho^{1-\rho} H(\tau,\tau)\,d\tau\right)\right]^{-1},
\end{aligned}$$

where $\rho \in (0, 1/2)$. Assume there exists $r > r_1 > 0$ such for all $t \in [0, 1]$, such that

(H1) $f(t, x) \geq \varphi_p(\Lambda_2 r_1), \ x \in [0, r_1]$.

(H2) $f(t, x) \leq \varphi_p(\Lambda_1 r), \ x \in [0, r]$.

(H3) For any $(t, x), (t, y)$ in $[0, 1] \times [0, +\infty)$, there exists a constant $0 < L < 1$ such that

$$|f(t, x) - f(t, y)| \leq \varphi_p(\Lambda_1 L |x - y|).$$

Now we enunciate the following existence and uniqueness results for (P3) and (P4):

4.2.1 Existence results via Guo-Krasnoselskii fixed point theorem

In this subsection, we establish the existence of positive solution for the problem (P3) and (P4) by using technique of Guo-Krasnoselskii fixed point theorem.

Theorem 4.1. *Assume that (H1)-(H2) hold, then the p-Laplacian CFP* (P3) *(or* (P4)*) has at least one positive solution $x \in \mathcal{P}$, with $r_1 \leq \|x\| \leq r$.*

Proof. From Lemma 4.4, we have: for $x \in P \cap \partial\Omega_1$

$$\begin{aligned}
\mathcal{T}x(t) &= \int_0^1 G(t, s)\varphi_q\left(\int_0^1 H(s, \tau) f(\tau, x(\tau))\, d\tau\right) ds \\
&\geq \int_0^1 tG(s, s)\varphi_q\left(\int_0^1 sH(\tau, \tau) f(\tau, x(\tau))\, d\tau\right) ds \\
&\geq t\int_0^1 sG(s, s)\, ds\, \varphi_q\left(\int_0^1 H(\tau, \tau)\varphi_p(\Lambda_2 r')\, d\tau\right) \\
&\geq \Lambda_2 r'\rho^2 \int_0^1 G(s, s)\, ds\, \varphi_q\left(\int_0^1 H(\tau, \tau)\, d\tau\right) \\
&\geq \Lambda_2 r'\rho^2 \int_\rho^{1-\rho} G(s, s)\, ds\, \varphi_q\left(\int_\rho^{1-\rho} H(\tau, \tau)\, d\tau\right) \\
&\geq r_1,
\end{aligned}$$

and if $x \in \mathcal{P} \cap \partial\Omega_0$, we get

$$\begin{aligned}
\|\mathcal{T}x(t)\| &= \max_{t\in[0,1]}\left|\int_0^1 G(t, s)\varphi_q\left(\int_0^1 H(s, \tau) f(\tau, x(\tau))\, d\tau\right) ds\right| \\
&\leq \frac{4}{\lambda\mu}\int_0^1 G(1, s)\varphi_q\left(\int_0^1 H(1, \tau)\varphi_p(\Lambda_1 r)\, d\tau\right) ds \leq r.
\end{aligned}$$

So, from Theorem 1.2 and Lemma 4.6 the operator $\mathcal{T}$ has at least one positive solution $x \in \mathcal{P} \cap \left(\overline{\Omega}_0 \backslash \Omega_1\right)$ with $r_1 \leq \|x\| \leq r$. □

4.2.2 Uniqueness results via Banach contraction principle theorem

In this subsection, we establish the uniqueness of positive solution for the problem (P3) and (P4) by using Banach contraction mapping principle.

Theorem 4.2. *Let (H3) holds, then the problem* (P3) *(or* (P4)*) has exactly one positive solution in* $\mathcal{P}$.

Proof. By Lemma 4.4, we have

$$\begin{aligned}
\left|\mathcal{T}x\left(t\right)-\mathcal{T}y\left(t\right)\right| &\leq \int_0^1 G\left(t,s\right)\varphi_q\left(\int_0^1 H\left(s,\tau\right)\left|f\left(\tau,x\left(\tau\right)\right)-f\left(\tau,y\left(\tau\right)\right)\right|d\tau\right)ds\\
&\leq \frac{4}{\lambda\mu}\int_0^1 G\left(1,s\right)\varphi_q\left(\int_0^1 H\left(1,\tau\right)\varphi_p\left(\Lambda_1 L\left|x-y\right|\right)d\tau\right)ds\\
&\leq \frac{4}{\lambda\mu}\Lambda_1 L\left|x-y\right|\int_0^1 G\left(1,s\right)\varphi_q\left(\int_0^1 H\left(1,\tau\right)d\tau\right)ds\\
&\leq L\left|x-y\right|.
\end{aligned}$$

Since $0<L<1$, then the Theorem 1.3 and Lemma 4.6 ensure that the p-Laplacian CFP (P3) (or (P4)) has a unique and positive solution x in $\mathcal{P}$ with $x=\mathcal{T}x$. □

4.2.3 Ulam-Hyers stability results

In this subsection, we study four different types of Ulam stability for the p-Laplacian problems (P3) and (P4), which are Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias.

Theorem 4.3. *With the assumption (H3), the p-Laplacian CFP* (P3) *(or* (P4)*) is Ulam-Hyers stable.*

Proof. In view Definition 1.18, let x be the solution of $\mathcal{T}$ and x^* be an approximation satisfying

$$x^*\left(t\right)=\int_0^1 G\left(t,s\right)\varphi_q\left(\int_0^1 H\left(s,\tau\right)f\left(\tau,x^*\left(\tau\right)\right)d\tau\right)ds. \tag{4.8}$$

Then, we get

$$\begin{aligned}
\left|x\left(t\right)-x^*\left(t\right)\right| &= \left|x\left(t\right)-\int_0^1 G\left(t,s\right)\varphi_q\left(\int_0^1 H\left(s,\tau\right)f\left(\tau,x^*\left(\tau\right)\right)d\tau\right)ds\right|\\
&\leq \left|x\left(t\right)-\int_0^1 G\left(t,s\right)\varphi_q\left(\int_0^1 H\left(s,\tau\right)f\left(\tau,x\left(\tau\right)\right)d\tau\right)ds\right|\\
&\quad+\left|\int_0^1 G\left(t,s\right)\varphi_q\left(\int_0^1 H\left(s,\tau\right)\left(f\left(\tau,x\left(\tau\right)\right)-f\left(\tau,x^*\left(\tau\right)\right)\right)d\tau\right)ds\right|.
\end{aligned}$$

By assumption (H3) and the inequality (1.7), we have

$$\left|x-x^{*}\right| \leq \varepsilon+L\left|x-x^{*}\right|,$$

which further implies

$$\left|x-x^{*}\right| \leq c_{f} \varepsilon,$$

where $c_{f}=\frac{1}{1-L}>0.$ Hence, from Definition 1.18, the operator $\mathcal{T}$ is Ulam-Hyers stable. Consequently, the p-Laplacian CFP (P3) (or (P4)) is Ulam-Hyers stable. Moreover, if we set

$$\theta_{f}(\varepsilon)=\varepsilon c_{f},\ \theta_{f}(0)=0,$$

then (P3) and (P4) are generalized Ulam-Hyers stable. □

Theorem 4.4. *Let (H3) holds. the p-Laplacian CFP* (P3) *(or* (P4)*) is Ulam-Hyers-Rassias stable.*

Proof. In view Definition 1.20, let $x^{*}(t)$ the approximation in (4.8), then, we get

$$\begin{aligned}\left|x(t)-x^{*}(t)\right| \leq & \left|x(t)-\int_{0}^{1} G(t, s) \varphi_{q}\left(\int_{0}^{1} H(s, \tau) f(\tau, x(\tau)) d \tau\right) d s\right| \\ & +\left|\int_{0}^{1} G(t, s) \varphi_{q}\left(\int_{0}^{1} H(s, \tau)\left(f(\tau, x(\tau))-f\left(\tau, x^{*}(\tau)\right)\right) d \tau\right) d s\right| .\end{aligned}$$

By assumption (H3) and the inequality (1.8), we have

$$\left|x-x^{*}\right| \leq \varepsilon \varphi(t)+L\left|x-x^{*}\right|,$$

which further implies

$$\left|x-x^{*}\right| \leq \varepsilon \varphi(t) c_{f},$$

where $c_{f}=\frac{1}{1-L}>0.$ Hence, from Definition 1.20 the operator $\mathcal{T}$ is Ulam-Hyers-Rassias stable. Consequently, the p-Laplacian CFP (P3) (or (P4)) is Ulam-Hyers-Rassias stable. Moreover, if we set

$$\varphi(\varepsilon)=\varepsilon \varphi(t), \varphi(0)=0,$$

then (P3) (or (P4)) is generalized Ulam-Hyers-Rassias stable. □

4.3 Example

Example 4.1. Let $\beta = \alpha = 2,\ \lambda = \mu = 1,\ \rho = 1/4,\ p = 2$ and the continuous function $f(t,x) = (t+1)\,x(t)$, we can be checked easily that

$$\left|f(t,x) - f(t,y)\right| \leq 2\left|x-y\right|.$$

For the suggested problem (P4), we have

$$\Lambda_1 = \left[2\int_0^1 \left(s(1-s)+1\right)(1-s)\,ds\right]^{-2} = \frac{36}{49}.$$

For the suggested problem (P3), we have

$$\Lambda_1 = \left[2\int_0^1 s(1-s)\,ds\right]^{-2} = \frac{1}{9}.$$

On the other hand, the assumption (H3):

$$\left|f(t,x) - f(t,y)\right| \leq \Lambda_1 L\left|x-y\right| \leq 2\left|x-y\right|,$$

is verified for $0 < L < 1$, for specified function f the Theorem 3.4 gives that the (P3) and (P4) has exactly one positive solution and the Theorems 4.3 and 4.4 assure that this solution is Ulma-Hyers stable and Ulam-Hyers-Rassias stable.

General conclusion

In this thesis, we have posed three types of fractional problems with integral boundary conditions:

- ☞ (P1): Mixed fractional boundary value problem involving both right conformable and left Caputo-type fractional derivatives.
- ☞ (P2): Fractional boundary value problem involving two conformable fractional derivatives.
- ☞ (P3) and (P4): Nonlinear p-Laplacian operators involving conformable fractional derivatives.

Three important aspects of the fractional problems have been studied. They are :

- ✓ **Existence results:** based a positively diminishing Greens function, we use the Guo-Krasnoselskii fixed point theorem to prove existence of positive solution for the considered problems.
- ✓ **Uniqueness results:** we have proved results for uniqueness by Banach contraction principle theorem.
- ✓ **Hyers-Ulam stability results:** we use Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias to prove stability results.

For these aims, each problem from the four suggested problems is converted into an integral equation via Green function. The Green function properties are analyzed. Then, some classical fixed-point theorems are used for establish the existence and uniqueness of positive solution. Hyers-Ulam stability of the proposed problem is also studied. For the application of the results, an expressive examples are included.
The future work in the thesis may include many aspects for the conformable fractional problems with integral boundary conditions:

- ✓ figure out more fixed point theorems.
- ✓ presenting a new numerical methods for solving.
- ✓ try to solve the conformable fractional problems with higher order $\alpha \in (n, n+1]\,,\ n \in \mathbb{N}^*$.
- ✓ searching for methods of resolution of partial differential equations of fractional order in the sense of conformable fractional derivative.

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ملخص: في هذه الأطروحة، درسنا آربعة مسائل لثلاثة آنواع من المعادلات التفاضلية الكسرية بشروط حدية تكاملية. تم تقديم حلول هذه المسائل على شكل معادلات تكاملية غير خطية باستعمال دالة غرين مناسبة أين قمنا بدراسة خصائصها وتحديد الطريقة المناسبة لمعالجة المسائل المقترحة. أثبتنا ايجابية و وجود و وحدانية الحلول لهذه المسائل باستخدام بعض نظريات النقطة الثابتة. بعد ذلك، اعطينا نتائج جديدة تتعلق باستقرار حلول المسائل المقترحة بمفهوم إلام- هايرس. عدة أمثلة تم تقديمها للتأكد من فعالية النظريات المستعملة في هذه الاطروحة.

كلمات مفتاحية: ألمعادلات التفاضلية الكسرية، شروط حدية تكاملية، نظريات النقطة الثابتة، استقرار إلام-هايرس.

In this thesis, we have studied four problems of three types of fractional differential equations with integral boundary conditions:

- ✓ At first, their solutions are presented in form of nonlinear integral equations by using the Green function, which her properties are analyzed to determine the appropriate way to solve the proposed problems.
- ✓ Secondly, we proved the positivity, existence and uniqueness of solutions to the proposed problems by using some fixed theorems.
- ✓ Next, we give a new results concerning the stability of fractional equations in Ulam-Hyres sense to prove the stability of solutions for proposed problems.
- ✓ Finally, several examples are presented to confirm the effectiveness of some utilized theorems.

Keywords: Fractional differential equations, Integral boundary conditions, Fixed point theorems, Ulam-Hyers stability.

Dans cette thèse, nous avons étudié quatre problèmes de trois types d'équations différentielles fractionnaires avec des conditions aux limites intégrales:

- ✓ Premièrement, nous représentons leurs solutions qui sont en forme d'équations intégrales non linéaires utilisant la fonction de Green, dont les propriétés sont étudiées pour déterminer la manière appropriée pour résoudre les problèmes proposés.
- ✓ Deuxièmement, nous avons prouvé la positivité, l'existence et l'unicité des solutions aux problèmes proposés basée à certains théorèmes de point fixe.
- ✓ Ensuite, nous avons donné de nouveaux résultats concernant la stabilité des équations fractionnaires au sens d'Ulam-Hyers pour prouver la stabilité des solutions aux problèmes proposés.
- ✓ Enfin, plusieurs exemples sont donnés pour confirmer l'efficacité de certains théorèmes utilisés.

Mots-Clés: Équation différentielles fractionnaires, Conditions aux limites intégrales, Théorèmes de point fixe, Stabilité d'Ulam-Hyers.