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Faculty of Mathematics and Informatics
Departement of Mathematics

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Theme

Application of Potential Flow theory to Free Surface Flow Probleme

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Research Plan

1. **Introduction:** Overview of potential flow theory and its relevance to free surface flow problems.

2. **Chapter.1.:Basic Concepts**

   (a) Derivation of governing equation :

      i. The continuity equation

      ii. Bernoulli's equation

      iii. Boundary conditions for free surface problems

   (b) Analytical solutions for simple potential flow proplems.

3. **Chapter.2.:Numerical Approaches and Applications**

   (a) Numerical Methods: Exploration of series truncation and integral method for solving potential flow and free surface flow problems.

4. **Chapter.3.:Some examples and Results**

   (a) Applications: Some examples, such as flow around obstacles and wave propagation.

   (b) Results:Comparison of analytical and numerical results.

   (c) Conclusion: Real-world applications and suggestions for future work.

Introducation

In fluid dynamics, potential flow or irrotational flow refers to a description of a fluid flow with no vorticity in it, (inviscid) fluid and with no vorticity present in the flow. Potential flow describes the velocity field as the gradient of a scalar function: $\vec{v} = \vec{grad}\phi$, where: ϕ is the velocity potential. As a result, a potential flow is characterized by an irrotational velocity field, which is a valid approximation for several applications. The irrotationality of a potential flow is due to the curl of the gradient of a scalar always being equal to zero.

In the case of an incompressible flow, the velocity potential satisfies Laplace's equation, and potential theory is applicable. However, potential flows also have been used to describe compressible flows and Hele-Shaw flows. The potential flow approach occurs in the modeling of both stationary as well as non stationary flows. Applications of potential flow include: the outer flow field for aerofoils, water waves, electroosmotic flow, and groundwater flow. For flows with strong vorticity effects, the potential flow approximation is not applicable. In flow regions where vorticity is known to be important, such as wakes and boundary layers, potential flow theory is not able to provide reasonable predictions of the flow [1]. Fortunately, there are often large regions of a flow where the assumption of irrotationality is valid, which is why potential flow is used for various applications, such as flow around aircraft, groundwater flow, acoustics, water waves, and electroosmotic flow [2][3].

Moreover, the study of free surface flow problems also relies on the application of potential flow theory, especially when the interface between the liquid and the air is subject to time-dependent variations. In such problems, potential flow provides a powerful tool to model the behavior of the free surface and to predict its evolution under different flow conditions.[4]

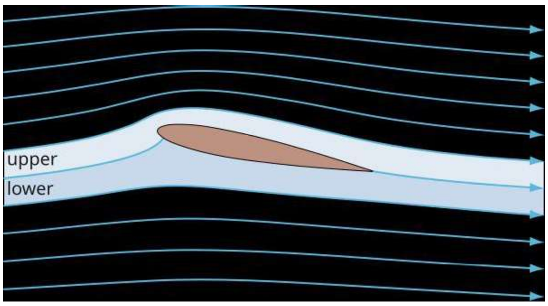

Figure 1: Potential-flow streamlines around airfoil [1]

In this work, we are interested in how the potential flow theory can be applied to free-surface flows. And what its parctical applications in solving real-world problems, and what the challenges arise in its implementation?

This memory is organized into three chapters.

We started in the first chapter by introducing the main definitions and concepts in this section. We then derived the governing equations (the continuity equation and Bernoulli's equation) and presented their numerical solutions. We also provided examples illustrating the application of each of them.

The second chapter presents a numerical solution using the Series Truncation Method and the Boundary Element Method (BEM).

In the third chapter, we present a study of two practical examples: one related to flow around and another to wave propagation.

We concluded this work with a conclusion in which we discussed the obtained results.

Chapter.1.:Basic Concepts

• Basic Concepts:

Before proceeding with this chapter, let's first introduce some fundamental definitions terms:

Fluid:

It is a substance that can flow, has no fixed shape, is affected by pressure and shear forces, and includes both liquids and gases.

Flow:

is the movement of the fluid in a specific medium, and it can be regular or turbulent.

Potential:

A mathematical function used to describe fluid motion when vorticity is zero.

Free Surface:

It is the boundary interface between a fluid and another medium where the pressure equals the external pressure.

Incompressible:

It is a fluid with constant density that does not change volume under pressure variations or external forces when in motion.

Inviscid Fluid:

It is a fluid used in mathematical modeling to simplify motion analysis, considered ideal with no viscosity due to neglecting internal friction effects.
These definitions form the foundation for the upcoming concepts in this chapter.

I.Derivation of governing equation:

1.Definition:

Governing equations are a set of mathematical equations that describe the behavior of fluid flow in a given system. in the context of potential flow theory and free surface flows, these equations include:[5]

- the continuity equation
- Bernoulli's equation
- Laplace's equation

1.1.continuity equation:

1.1.1.Defintion:

The continuity equation is a differential equation used to describe the flow of a conserved physical quantity(i.e.,it expresses the principle of mass conservation in fluid flow: meaning that the amount of fluid entering a given volume must equal the amount leaving it, provided there is no generation or loss of mass). It applies to the study of mass, semiconductor physics, relativity theory electromagnetism and quantum mechanics.[7][8][9]

1.1.2.General form of the continuity equation:

The general form of the equation written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

where:

ρ: Fluid density ($\mathrm{kg/m^3}$).

$t : time(s)$.

$v : velocity.$

$\nabla\cdot$: Divergence operator.

1.1.3.Derivation of continuity equation:

Continuity equation represents that the product of cross-sectional area of the pipe and the fluid speed at any point along the pipe is always constant. This product is equal to the volume the volume flow per second or simply the flow arte. The continuity equation is given as:

$$R = AV$$

Where:

R: is the volume flow art.

A: is the flow area.

V: is the flow velocity.

Assumtion:

are the assumtions of continuity equation:

- the tube is having a single entry and single exist
- the fluid flowing in the tube is non-viscons
- the flow is incompressible
- the fluid flow is steady

Derivation:

a.Continuity equation in 1D

Consider the following diagram:

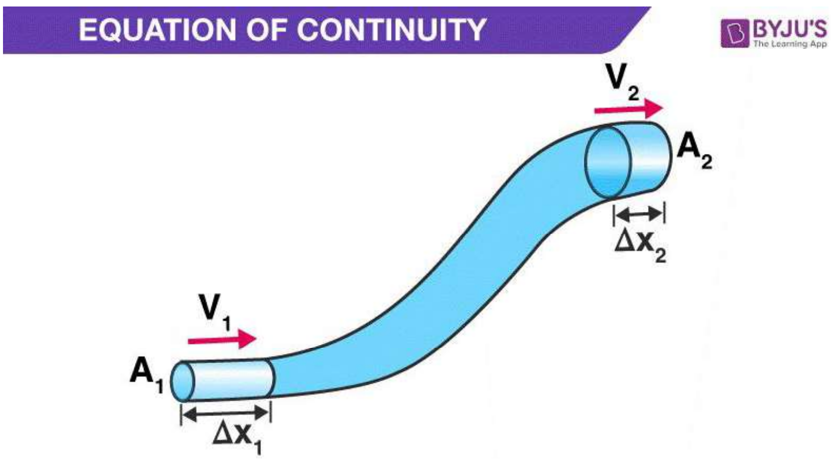

Figure 2: variable diameter pipe: continuity equation, according to the continuity equation, as the cross-sectional area decreases, the flow velocity increases. [6]

Now,consider the fluid flows for a short interval of time in the tube. So, assume that short interval of time as Δt. In this time the fluid will cover a distance of Δx_1 with a velocity V_1 at the lower end of the pipe. [6] At this time, the distance covered by the fluid will be:

$$\Delta x_1 = V_1 \Delta t \tag{1}$$

Now,at the lower and of the pipe,the volume of the fluid that will flow into the pipe will be:

$$v = A_1 \Delta x_1 = A_1 V_1 \Delta t \tag{2}$$

It is know that (mass(m)=Density(ρ)*volume(v)). So , the mass of the fluid in Δx_1 region will be:

$$\Delta m_1 = Density * volume \Rightarrow \Delta m_1 = \rho A_1 V_1 \tag{3}$$

Now, the mass flux has to be calculated at the lower end. Mass flux is simply defined as the mass of fluid per unit time passing through any cross-sectional area. For the lower end with cross-sectional area A_1, mass flux will be:

$$\frac{\Delta m_1}{\Delta t} = \rho A_1 V_1 \tag{4}$$

Simplarly, the mass flux at the upper end will be:

$$\frac{\Delta m_2}{\Delta t} = \rho A_2 V_2 \tag{5}$$

Here, V_2 is the velocity of the fluid through the upper end of the pipe i.e. through Δx_2,in Δt time and A_2, is the cross-sectional area of the upper end. In this, the density of the fluid between the lower end of the pipe and the upper end of the pipe remains the same with time as the flow is steady. So, the mass flux at the lower end the pipe i.e;

$$equation_2 = equation_3 \tag{6}$$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \tag{7}$$

This can be written in a more general formas:

$$\rho A v = constant \tag{8}$$

The equation proves the low of coservation of mass in fluid dyanimcs. Also, if the fluid is incompressible, the density will remain constant for steady flow. So: ρ_1=ρ_2

Thus, equation 4 can be now written as: $A_1V_1=A_2V_2$

This equation can be written in general form as: $Av=constant$

Now, if R is the volume rate, the above equation can be expressed as:[6] $R=Av=constant$

This was the derivation of continuity equation

b.Continuity equation in cylindrical coordinates(3D)

Following is the continuity equation in cylindrical coordinates:[6]

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho u)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0 \tag{9}$$

c.Incompressible Flow continuity equation:

Follwing is the continuity equation for incompressible flow as the density $\rho=constant$, and is independent of space and time, we get $\nabla.v=0$

d.Steady Flow continuity equation(incompressible):

Following is this continuity equation in cylindrical coordinates:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0 \tag{10}$$

1.1.4.Effect of the continuity equation on potential flows and free surface

a.In potential flow:

- In potential flow, the flow is irrotational ($\nabla * u = 0$) meaningthat the velocity field is drived from a velocity potential function ($u = \nabla\phi$) .

b.In free surface flow:

- Free surface flows are characterized by a moving boundary between the fluid and air (or onther fluid), where the surface shape change due to flow forces.
- the continuity equation ensures that the velocity at the free surface matches the motion of the surface its self, leding to a kinematic condition:

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}=w \tag{11}$$

  where:

  h: is the free surface height.

u, v ,w: are the velocity coponent is three direction.

1.1.5.Boundary condition for free surface

At the free surface there are two boundary conditions:[10]

- **The kinematic condion:** which relates the notion of the free surface interface to the fluid velocities at the free surface.
- **The dynamic condition:**which is concerned with the force balance at the free surface.

- **Steps to apply the kinematic condition to the continuity equation**

a.Write the cotinuity equation: For an icompressible fluid, the continuity equation is:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{12}$$

b.Express the vertical velocity at the free surface: At the free surface: $z = h(x, y, t)$, we use the relations:

$$v_z = \frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x} + v_y \frac{\partial h}{\partial y} \tag{13}$$

c.Subtitute into the continuity equation: We substitute$v_z$into the continuity equation at :$z = h(x, y, t)$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial}{\partial z}(\frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x} + v_y \frac{\partial h}{\partial y}) \tag{14}$$

d.Obtain the differential equation: After simplification, we get the equation that describes the evolution of the free surface:

$$\frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x} + v_y \frac{\partial h}{\partial y} = -\int_0^h \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) dz \tag{15}$$

The equation determines how the free surface changes in respones to the flow.[21]

1.1.6.Analytical solution of the continuity equation in two dimensions

For an incomperssible (ρ constant) and two-dimensional (x,y) flow the continuity equation given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{16}$$

Where:

u: is the velocity component in the x-direction.

v: is the velocity component in the y-direction.

1.1.6.1.Solution using the stream function:

We defin the stream function ψ(x,y) such that: $u = \frac{\partial\psi}{\partial y}$ $v = -\frac{\partial\psi}{\partial x}$ subtituting into the continuity equation:

$$\frac{\partial}{\partial x}(\frac{\partial\psi}{\partial y}) + \frac{\partial}{\partial y}(-\frac{\partial\psi}{\partial x}) = 0 \qquad (17)$$

Rearraning the terms, we obtien:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad (18)$$

This laplace's equation which governs potential flow and is used to derive analytical solution.[11]

1.1.7.Exemples of the application of the continuity equation

a.Boold flow in boold vessels

- Where a boold vessel narrows due to fat accumulation, the boold velocity increases to maintain a constant flow rate for this reason, doctors rely on the continuity equation to study boold flow in arteries and veins. [14]

b.Irrigation and agricultural systeme

- In irrigation systeme, pipe with fifferent diametes are used to transport water. During this process, the continuity equation is applied to ensure that water reaches the fields with an appropriate amount and peed. [15]

1.2.Euler's equation:

1.2.1.Defintion:

The Euler equations are a set of partial differential equations (PDEs) that describe the motion of fluids (whether gases or liquids) in the case where the flow is inviscid (i.e., there is no internal friction) and without thermal conduction.

1.2.2.The generale form of Euler's equation :

The general form of the equation written as:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = 0 \tag{19}$$

where:

- $\mathbf{U}$: vector of conserved quantities (e.g., mass, momentum, and energy)
- $\mathbf{F}(\mathbf{U})$: flux vector
- t: time

Starting from Euler's momentum equation for an inviscid flow, and assuming steady and incompressible conditions, Bernoulli's equation can be obtained.
We begin with the Euler momentum equation for an inviscid flow:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

Next, we take the dot product of both sides with the velocity vector u:

$$\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\mathbf{u} \cdot \nabla p + \mathbf{u} \cdot \mathbf{g}$$

Using the identity:

$$\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = \frac{D}{Dt}\left(\frac{1}{2}|\mathbf{u}|^2\right)$$

And letting the gravitational potential be $\phi = gz$ the equation becomes:

$$\frac{D}{Dt}\left(\frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \phi\right) = 0 \tag{20}$$

This implies that the following quantity is conserved along a streamline:

$$\frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + gz = \text{const}$$

This is the Bernoulli equation, valid for steady, incompressible, and inviscid flows along a streamline.[30]

1.2.Bernoulli's equation:

1.2.1.Defintion:

Bernoulli's equation is an equation of notion for incompressible flow based an newton's second law of motion adated for fluid and Euler's equation of motion. Which is the expresses the principle of energy conservation in fluid flow. It states that the sum of kinetic energy, pressure energy and potential energy remains constant along a streamline of an ideal fluid (invixid and incompressible).[16]

1.2.2.The generale form of Bernoulli's equation :

The general form of the equation written as:

$$P + \frac{1}{2}\rho v^2 + \rho g h = conctant \tag{21}$$

Where:

P: pressure(the pressure of the fluid (Pa or N/m^2).

ρ: density (the density of the fluid kg/m^3).

v: velocity (the velocity of the fluid m/s).

g: gravitational acceleration (the acceleration ducto gravity m/s^2, with a value of 9.81 m/s^2 on Earth).

h: height (the height above a relevence level m).

1.2.3.Derivation of Bernoulli's equation:

Consider a pipe with varying diameter and height throigh which an incomperssible fluid is flowing. The reltion ship between the areas of corss-section A, the flow speed v, hight from the ground y and pressure p at two different point 1 and 2 are given in the figure below.[17]

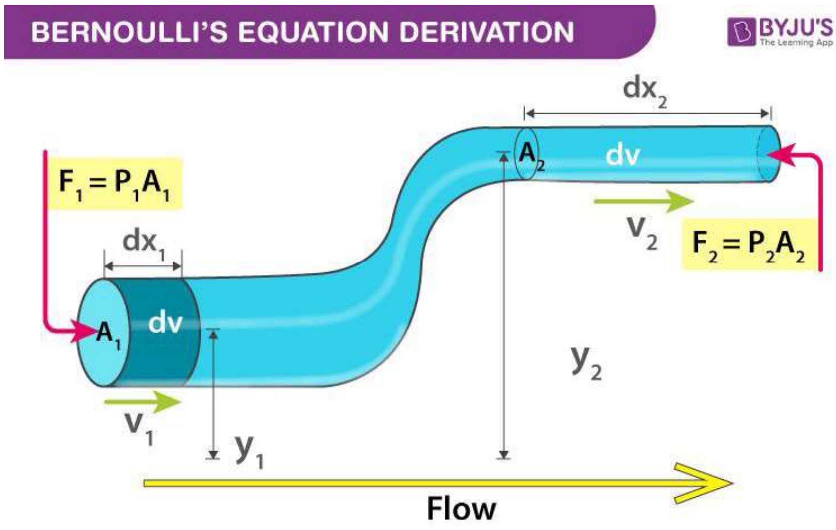

Figure 3: Bernoulli's equation applied to a pipe with varying diameter: pressure and velocity distributio. [17]

Assumpation:

- The density of the incomperssible fluid remains constant at both points.
- The energy of the fluid is conerved as there no viscous in the fluid therefore, the work done:

$$\text{Work done} = \Delta\,(\text{Kinetic Energy} + \text{Potential Energy} + \text{Flow Work}) \tag{22}$$

On the fluid is given as:

$$dw = F_1 dx_1 - F_2 dx_2 \tag{23}$$

$$dw = P_1 A_1 dx_1 - P_2 A_2 dx_2 \tag{24}$$

$$dw = P_1 dv - P_2 dv = (P_1 - P_2)dv \tag{25}$$

We know that the work done on the fluid was dne to the convervation of chang in gravitational potential energy and change in kinetic energy the change in kinetic energy of the fluid is given as:[17]

$$dk = \frac{1}{2}m_2 V_2^2 - \frac{1}{2}m_1 V_1^2 = \frac{1}{2}\rho dv(V_2^2 - V_1^2) \tag{26}$$

The change in potential energy is given as:

$$du = m_2 g y_2 - m_1 g y_1 = \rho dv g(y_2 - y_1) \tag{27}$$

Therefore, the energy equation is given as:

$$dw = dk + du \tag{28}$$

$$(P_1 - P_2)dv = \frac{1}{2}\rho dv(V_2^2 - V_1^2) + \rho dvg(y_2 - y_1) \tag{29}$$

$$(P_1 - P_2) = \frac{1}{2}\rho(V_2^2 - V_1^2) + \rho g(y_2 - y_1) \tag{30}$$

Rearranging the above equation, we get:

$$P_1 + \frac{1}{2}\rho V_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho V_2^2 + \rho g y_2 \tag{31}$$

This is Bernoulli's equation.

1.2.4.Effect of Bernoulli's equation on potential flows and free-surface flows:

- The relation ship between pressure engrgy, kinetic energy, and potential energy in any fluid flows is determined by Bernoulli's equation.
- In potential flows, the fluid is used in this case to understand pressure change around solid objects, such as wings and hydrodynamic structures.
- In free-surface flows, wave height, velocity changes, and pressure distribution are analyzed using Bernoulli's equation, which helps in designing hydraulic systems and controling water flow.

  In general, the equation applies to systeme where the fluid surface in unconfined, such as rivers, open channels, and surface waves.
- Considering atmospheric pressure as constant, when applying Bernoulli's equation between tow points on free-surface fluid, a relation ship between velocity and height is established, expressed by the following equation:

$$P_a tm + \frac{1}{2}\rho v^2 + \rho g h = constant \tag{32}$$

- **Steps to apply the dynamic condition to the Bernoulli's equation**

a.Write the general Bernoulli's equation: For an inviscid, incompressible fluid along a streamline, Bernoulli's equation:

$$P + \frac{1}{2}\rho v^2 + \rho g z = constant \tag{33}$$

b.Express the Dynamic condition at the free surface: At the free surface $z = h(x, y, t)$, there is no shear stress (since the surface is in constant with air), so the pressure at the free surface equal the atmospheric pressure P_atm.
Appling this to Bernoulli's equation, we get:

$$P_atm + \frac{1}{2}\rho V^2 + \rho gh = constant \tag{34}$$

c.Obtain the differntial equation: Since atmospheric pressure is constant, we derive the equation describing the free surface evolution:

$$\frac{\partial h}{\partial t} + \frac{1}{2g}\frac{\partial v^2}{\partial t} = 0 \tag{35}$$

This equation hiks the variation of the free surface height to the flow velocity.[20]

1.2.5.Analyttical solution of Bernoulli's equation and derivation of flow velocity

- Bernoulli's equation expresses energy conservation in an comperssible iniviscid fluid and is written as:

$$P + \frac{1}{2}\rho v^2 + \rho gz = constant \tag{36}$$

- We apply Bernoulli's equation between two points:

  point 1: at the fluid surface in the tank:

  Pressure: $P_1 = P_tam$(atmospheric pressure).

  velocity: $v_1 = 0$(fluid in mearly stationary at the surface).

  Height: $z_1 = h$(measured from the outlet).

  point 2: at the outlet of the tank.

  Pressure: $P_2 = P_atm$ (since the outlet is exposed to air).

  velocity: v_2 (to be determined).

  Height: $z_2 = 0$ (outlet level is the reference).

- Since atmospheric pressure is equal at both point, it cancels out:

$$\frac{0}{2} + gh = \frac{1}{2}v^2 + g(0) \Rightarrow gh = \frac{1}{2}v^2 \tag{37}$$

- Multipling both sides by 2:

$$v_2^2 = 2gh \tag{38}$$

taking the square root: $\mathrm{v}_2 = \sqrt{2gh}$

This is known as torricelli's Low, which given the velocity of fluid exiting from a hole in a tank.

1.2.6.Exemples of the application of Bernoulli's equation

a.Explanation of how airplance gererate lift : The air passing over he oirplance wing move faster compared to the air below it. This directly affects the pressure, causing it to dercase on the upper surface. The pressure difference creates a lift force that helps the air plane continure fling.

b.Calculating the speed of fluid flow: Bernoulli's equation is used to determine the velocity of fluids in pipes and reservoirs. The pressure difference help calculate the flow speed at any point in the system.

c.Measuring fluid flow in pipes and open channels: When water flows the rough a pipe with a marrow section and a wider section, its velocity changes, increasing in the narrow section to maintain the flow rate due to the decreasz in pressure. One exemples of this is the venturimetre, one of many derives that rely on Bernoulli's equation to measure the flow rate at two different points using the pressure difference .

Chapter.2.:Numerical Approaches and Applications

• Numerical Method

I.Series truncation method

1.Definition:

An infinte in which all terms beyond the n_{max} terms (for some integer n_{max} have been discarded (set to 0). [19]

2.General formale for truncation:

If we have an infinite series of the form:

$$s = \sum_{n=0}^{\infty} a_n \tag{39}$$

Truncating the series at a certain term n_{max} gives:

$$s = \sum_{n=0}^{n_{max}} a_n \tag{40}$$

3.Definition of tylor series:

Taylor series is the polynomial or a function of an infinite sum of termes. And the general formula for the tylor series is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{41}$$

The truncating of tylor series is :

$$f(x) = \sum_{n=0}^{n_{max}} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{42}$$

3.1.Tylor series can be used to expand the function around a reference point $(\mathbf{x}_0, y_0) as fol$

As an example, we consider the velocity potential function ϕ which can be expanded as follows:

$$\phi(x,y) = \phi(x_0,y_0)+\frac{\partial\phi}{\partial x}(x-x_0)+\frac{\partial\phi}{\partial y}(y-y_0)+\frac{1}{2}[\frac{\partial^2\phi}{\partial x^2}(x-x_0)^2+2\frac{\partial^2\phi}{\partial x\partial y}(x-x_0)(y-y_0)+\frac{\partial^2\phi}{\partial y^2}(y-y_0)^2]......... \tag{43}$$

We truncate the taylor series expressed using the velocity potential function at the first-order term, to obtain the following relation ship: (equation 32).

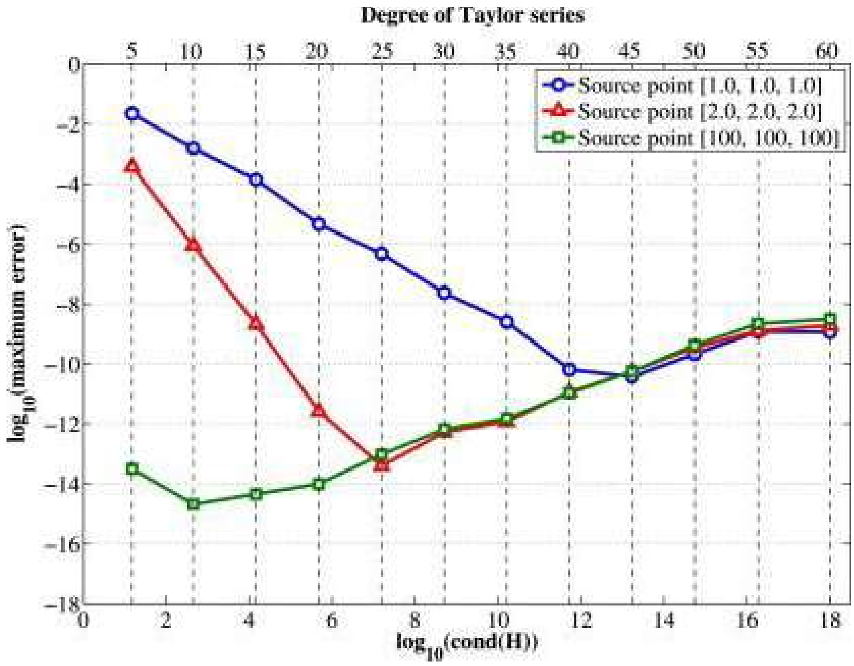

Figure 4: Approximation of the function by taylor series expansion (Figure produced by the author using MATLAB software.).

3.2.potential flow:

Since the velocity function $\phi(x,y)$ satisfais Laplace's equation in potential flow, we apply this equation to the previously obtained expansion, and Laplace's equation written as:[18]

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{44}$$

By substituting the truncated velocity potential function at the second term into Laplace's equation and simplification, we obtien :

.First derivatives:

$$\begin{cases} \frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial x_0} + \frac{\partial^2\Phi}{\partial x^2}(x-x_0) + \frac{\partial^2\Phi}{\partial x\partial y}(y-y_0) \\ \\ \frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial y_0} + \frac{\partial^2\Phi}{\partial y^2}(y-y_0) + \frac{\partial^2\Phi}{\partial x\partial y}(x-x_0) \end{cases} \tag{45}$$

.Second deriatives:

$$\begin{cases} \frac{\partial^2\Phi}{\partial x^2}(x,y) = \frac{\partial^2\Phi}{\partial x^2}(x_0,y_0) + \frac{\partial^3\Phi}{\partial x^3}(x_0,y_0)(x-x_0) + \frac{\partial^3\Phi}{\partial x^2\partial y}(x_0,y_0)(y-y_0) \\ \\ \frac{\partial^2\Phi}{\partial y^2}(x,y) = \frac{\partial^2\Phi}{\partial y^2}(x_0,y_0) + \frac{\partial^3\Phi}{\partial y^3}(x_0,y_0)(y-y_0) + \frac{\partial^3\Phi}{\partial x\partial y^2}(x_0,y_0)(x-x_0) \end{cases} \tag{46}$$

.After adding the two equations , we find ...

$$\frac{\partial^2\Phi}{\partial x^2}(x_0,y_0)+\frac{\partial^2\Phi}{\partial y^2}(x_0,y_0)+O((x-x_0)^2,(y-y_0)^2)=0 \tag{47}$$

Then, we find that both sides of the equation are equal, confirming the expansion satisfies Laplace's equation.

.Result: This means that the potential function, after the truncated expansion, still satisfies Laplace's equation, which implies that the flow remains potential. this indicates that it conforms to the conditions of the Laplacian. [18]

3.3.Free surface :

For irrotational flow, the fluid velocity is given by the gradient of a velocity potential which satisfies Laplace's equation. Finally, Bernoulli's equation can be used to satisfies the dynamic boundray condition at the surface. And it satisfies the following free surface condition

.The kinematic condition: the kinemiatic condition is that the normal coponents of the fluid velocites must match:

$$n.u_1 = n.u_2 \tag{48}$$

.The dynamic condition: dynamic consideration require that the pressure jump acrors the interface is balanced by the interfacial force due to surface tension:[22]

$$p_2 - p_1 = T_k \tag{49}$$

II.Boundray element methode (BEM) in potential flow

1.Definition:

The boundray element methode (BEM) is a numerical technique used to solve partial differential equation (PDEs) by transforming them into integral equation that are defined only on the boundray of the domain. unlike the finite element methode (FEM) and finite difference methode (FDM), which require discretization of the boundray, reducing the number of unknows and coputational copmlexity.

question: Why do we use the boundary element methode (BEM) in potential flow ...

Becouse:

a.Reduces the number of required element: BEM only requires discretizing the boundaries instead of meshing the entire domain like FEM, reducing computational effort.

b.Suitable for solving Laplace's equation: BEM converts Laplace's equation into integral equation, making numerical solution easier.

c.Effectire for unbounded domain: ideal for solving flow around sumberged bodies since it does not require a large internal mesh like FEM.

d.High numerical accuracy: BEM does not impose approximations inside the domain, leading to more accurate solutions compared to some other methodes.

e.Lower computational requirements: since calculations are performed only on boundaries, matrix sizes are smaller and faster to solve than in FEM.

2.Mathematical model of BEM in potential flow:

Laplace's equation is transformed into an integral equation using green's function the general BEM equation is:[23]

$$\phi(x) = \int_\Gamma G(x,\xi) q(\xi)\, d\Gamma - \int_\Gamma \frac{\partial G(x,\xi)}{\partial n} \phi(\xi)\, d\Gamma \tag{50}$$

where:

- $G(x,\xi)$: is Green's function for Laplace's equation.
- $q(\xi)$: represents the normal derivative of the potential on the boundary.
- $\phi(\xi)$: is the potential on the boundary.
- Γ: represents the boundary where BEM is applied.

3.Basic steps to apply BEM :

- Choosing an apprapiate green's function for the probleme.
- Transforming the partial differential equation into an integral equation using the above formulation.
- Discrtization the boundary into numerical integration for each elements.

- Solving the resulting linear systeme to determine the vaules of ϕ and q on the boundary.[23]

4.Steps to solve a potential flow probleme using the boundary element methode (BEM)

To properly present the numerical solution in your thesis, follow these structued steps: We aim to solve Laplace's equation in potential flow using the boundary element methode (BEM).[24]

- **Governing equation:**

$$\nabla^2 \phi = 0 \tag{51}$$

- **Integral formulation:**

$$c(P)\phi(P) = \int_\Gamma \left[\phi(Q)\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n} \right] d\Gamma \tag{52}$$

$$G(P,Q) = \frac{1}{2\pi} \ln \frac{1}{|P-Q|} \tag{53}$$

Discretization of the Boundary

The boundary is divided into N elements, where numerical integration is performed for each element.

After discretization, the system becomes: [28]

$$\left(\sum_{j=1}^{N} G_{ij}\phi_j \right) = \sum_{j=1}^{N} H_{ij}q_j$$

or:

$$[A]\{x\} = \{B\} \quad \Rightarrow \quad \{x\} = [A]^{-1}\{B\}$$

Where:

A: The coefficient matrix which contains values resulting from the boundary integral in BEM.

x: The unknown variables, such as the potential ϕ or the normal derivatives q.

B: The boundary conditions matrix, containing known values from the boundary conditions.

Example: LU Decomposition Method

In this section, we apply the LU decomposition method to solve the linear system obtained from the BEM formulation. The objective is to determine the unknown boundary values by decomposing the coefficient matrix into lower and upper triangular matrices and solving the resulting equations step by step.[24]

We need to solve the system:

$$[A]\{x\} = \{B\}$$

Where:

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 15 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

LU Decomposition

Factorizing $A = L \cdot U$:

Lower Triangular Matrix (L):

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 \\ 0 & -0.2667 & 1 & 0 \\ 0 & 0 & -0.3 & 1 \end{bmatrix}$$

Upper Triangular Matrix (U):

$$U = \begin{bmatrix} 4 & -1 & 0 & 0 \\ 0 & 3.75 & -1 & 0 \\ 0 & 0 & 3.7333 & -1 \\ 0 & 0 & 0 & 2.7 \end{bmatrix}$$

Step 1: Solve $L \cdot Y = B$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 \\ 0 & -0.2667 & 1 & 0 \\ 0 & 0 & -0.3 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

Step-by-step calculations:

$$y_1 = 15$$
$$y_2 = 10 + 0.25 \times 15 = 13.75$$
$$y_3 = 10 + 0.2667 \times 13.75 = 13.6667$$
$$y_4 = 10 + 0.3 \times 13.6667 = 14.1$$

Step 2: Solve $U \cdot X = Y$

$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ 0 & 3.75 & -1 & 0 \\ 0 & 0 & 3.7333 & -1 \\ 0 & 0 & 0 & 2.7 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 15 \\ 13.75 \\ 13.6667 \\ 14.1 \end{bmatrix}$$

Step-by-step calculations:

$$x_4 = \frac{14.1}{2.7} = 5.2222$$
$$x_3 = \frac{13.6667 + x_4}{3.7333} = 5.0008$$
$$x_2 = \frac{13.75 + x_3}{3.75} = 5.0003$$
$$x_1 = \frac{15 + x_2}{4} = 5.8809$$

Final Result

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5.8809 \\ 5.0003 \\ 5.0008 \\ 5.2222 \end{bmatrix}$$

III.Boundary element methode (BEM) in free surface

question: Why do we use the boundary element methode (BEM) in free surface flow ...

Becouse:

We apply the boundary element methode (BEM) to free surface to free surface flow for the same reasons as in potential flow, but there are some minor differences, the most notoble being:

- The free surface changes orer time, requiring continuous updates.
- Additional kinematic and dyanimc conditions to control surface movement.
- Effects of gravity and pressure on the potential (ϕ).
- The need to trak boundary changes at each numerical step.

1.Mathematical model of BEM in free surface flow

The mathematical formulation of the boundary element methode (BEM) in free surface is as follows:[25]

$$c(\epsilon)\psi(\epsilon) = \int_\Gamma \left[\psi(x)\frac{\partial G(\epsilon, x)}{\partial n}\right] d\Gamma - \int_\Gamma \left[q(x)G(\epsilon, x)\right] d\Gamma \tag{54}$$

where

- ψ: is the stream fubction.
- q: is the normal flux on the boundary.
- G(ϵ,x): is the fundamental solutio to Laplace's equation.

- c(ϵ): depends on the location of the point ϵ on the boundary Γ.

Addionally, on the free surface the Bernoulli's dynamic condition is applied:

$$B = \frac{g}{2}h + \frac{1}{2}(\frac{\partial\psi}{\partial n})^2 \tag{55}$$

2.Basic steps to apply BEM

- Applying the Bernoulli's dynamic condition on the condition on the free surface

$$B = \frac{g}{2}h + \frac{1}{2}(\frac{\partial\psi}{\partial n})^2 \tag{56}$$

- Using an iterative methode to determine the free surface location: the free surface position is unknow, so an iterative newton-Rophson methode is used to upadet it.[25]

The correction equation is given by:

$$\mathbf{B}\begin{bmatrix}\Delta\mathbf{B}\\ \Delta\mathbf{S}\end{bmatrix} = \left(\mathbf{J}_B^T\mathbf{J}_B\right)^{-1}\left[\mathbf{J}_B^T\Delta\mathbf{R}\right]$$

Adding pseudo-nodes to improve stability.

Using a damping factor (α) for stability control. The free surface update equation includes a damping factor for control the convergence rate:

$$\mathbf{R}^{k+1} = \mathbf{R}^{k} + \alpha \Delta \mathbf{R}$$

Recomputing the $[\Delta B], [\Delta S]$ matrix when necessary.

BEM is used to solve a free surface flow problem using BEM and Laplace's equation.[25]

Laplace's equation is used to ensure potential flow:

$$\nabla^2 \phi = 0$$

On the free surface, the dynamic Bernoulli's condition is applied to determine its location:

$$B = \frac{\partial \phi}{\partial t} + \frac{1}{2} \left(\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right)$$

Using Green's theorem, Laplace's equation is transformed into the boundary integral equation:

$$c(\xi)\phi(\xi) = \int_{\Gamma} \phi(\eta) \frac{\partial G}{\partial n}(\eta, \xi) d\Gamma(\eta) - \int_{\Gamma} \frac{\partial \phi}{\partial n}(\eta) G(\eta, \xi) d\Gamma(\eta)$$

The domain is divided into linear or curved elements.

On the free surface, pseudo-nodes are added to improve numerical stability.

Applying the boundary conditions:

- On fixed boundaries, conditions such as $\phi = 0$ or $q = 0$ are applied.
- On the free surface, the free surface height h is determined using the Bernoulli's conditions.

The boundary integral equation is converted into a system of algebraic equations which is solved using methods like:

- LU Decomposition
- Gaussian Elimination
- Iterative solvers

- The free surface is updated iteratively using the Newton-Raphson method:

$$\mathbf{R}^{k+1} = \mathbf{R}^{k} + \alpha \Delta \mathbf{R}$$

-Analysies of results and variation :

Chapter.3.:Some examples and Results

1.Flow around hydrofil:

flow around a hydrofil with a free surface.

In order to illustrate the behavior of the flow around a hydrofoil, two illustrative images are presented below. The first image represents a visual appearance of a calm water surface, highlighting light reflections and surface ripples. The second image shows a numerical simulation of the flow around the hydrofoil, displaying streamlines, separation zones, and pressure distribution.[26]

Figure 5: Artistic representation of a calm water surface to illustrate the visual characteristics of a free surface flow (Google Images (educational use)).

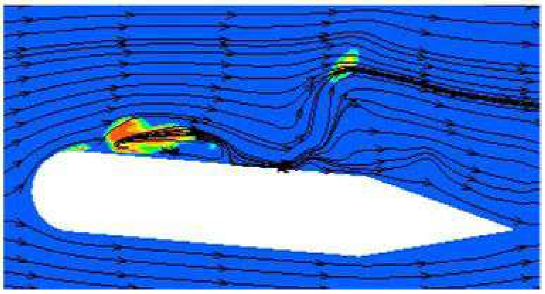

Figure 6: Numerical simulation of flow around a hydrofoil showing streamlines and pressure distribution (Google Images (educational use)).

Study of the Effect of Free Surface on the Hydrodynamic Forces of a Submerged Hydrofoil

In this exercise, we study the effect of the free surface on the hydrodynamic forces acting on a submerged hydrofoil. The flow is assumed to be inviscid and irrotational, and thus the velocity potential Φ satisfies Laplace's equation:

$$\Delta\Phi = 0$$

1. Bernoulli's Dynamic Condition

$$P + \frac{1}{2}\rho V^2 + \rho g h = \text{constant}$$

Where:

- P is the pressure,
- V is the velocity,
- g is the gravitational acceleration,
- h is the elevation of the free surface.

2. Neumann-Kelvin Condition on the Free Surface

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on } z = \eta$$

3. Numerical Resolution Steps

- Discretize the body surface and free surface into boundary elements (panels).
- Use Green's identity to compute the potential on each panel.
- Apply boundary integral equations.
- Solve for the velocity potential Φ.
- Apply boundary conditions on the hydrofoil and free surface.
- Compute hydrodynamic forces (lift and drag).
- Use Bernoulli's equation to calculate pressure, then compute lift and drag coefficients:

$$C_L = \frac{2L}{\rho V^2 S} \qquad C_D = \frac{2D}{\rho V^2 S}$$

4.Result analysis:

a.The liftnand darg coefficients are compared at different depths

b.The effect of aspect ratio on the free surface flow is studied .

5. Numerical Results

Depth (m)	Φ (m^2/s)	Pressure (Pa)	C_L	C_D
0.5	2.5	17405.00	1.358	0.139
1.0	5.0	22310.00	1.785	0.183
1.5	7.5	27915.00	2.177	0.218
2.0	10.0	32150.00	2.570	0.257
2.5	12.5	37250.00	2.962	0.296
3.0	15.0	41930.00	3.354	0.335
3.5	17.5	46835.00	3.747	0.375
4.0	20.0	51740.00	4.139	0.414
4.5	22.5	56645.00	4.532	0.453
5.0	25.0	61550.00	4.924	0.492

6. Graphical Representation

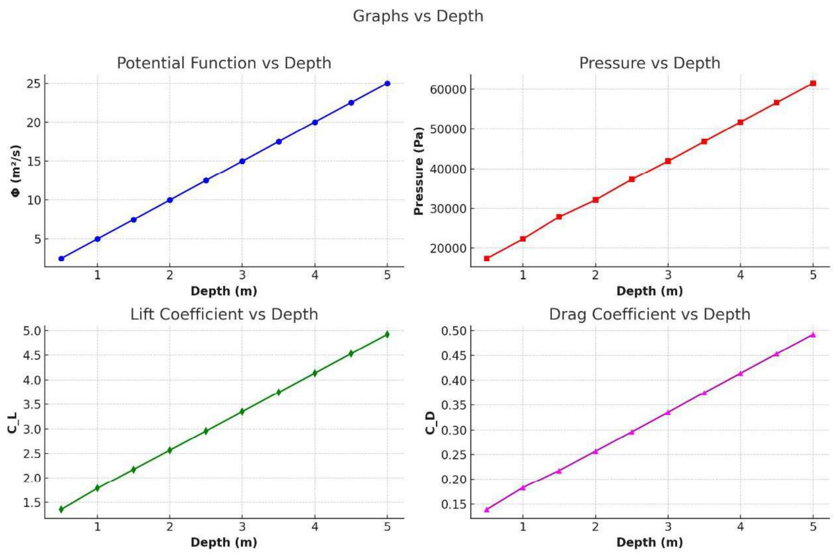

Figure 7: Flow parameters versus depth: a comparative analysis (Figure produced by the author using MATLAB software).

7.Results analysis :

-Potential function ϕ increases with depth.

-Pressure P increases with depth due to gvavitational effect.

-Lift coefficient C_L increases with depth, indicating that the free surface rednces effect shallow depths.

-Darge coefficient C_D is lower than C_L but follows the same increasing trend.

Conculusion:

- When the hydrofoil is close to the free surface, the lift coefficient decreases due to unsteady flow effects.
- As the depth increases, the values approach the infinite fluid case, where the free surface effect diminishes.
- At low aspect rations the free surface has a stronger impact on the flow.

2.Wave propagation:

Wave propagation in water channel with variable depth.[27]

a.Channel wave characteristic:

-channel legth: 150 meters

-channel width: 5 meters

-intial water depth: variable (1.5 m at the beginning, increasing to 2.5 m at the end) -geerated wave: due to a sudden water relese at the starting point -initial wave speed: calculated using the lineare wave equation:

$$c = \sqrt{gh} \tag{57}$$

where:

-:$g = 9.81 m/s^2$ (gravitational acceleration)

-:h is the variable depth: assuming the wave starts is a regoin where $h = 1.5m$

$c = \sqrt{9.81 * 1.5} = \sqrt{14.715} = 3.84 m/s$

b.Factors affecting wave propagation:

-Depth variation: as the wave advance, it encounters a deeper region(2.5m), which may cause wave acceleration.

-Bed roughness: assume a manning coefficient n=0.015 (smooth sandy bed).

-Wave reflection: if a soild wall is present at the channels end, part of the wave will reflect backward.

c.Mathematical modeling using saint-venant equation:

The saint-venant (shallow water) equation are suitable for modeling wave propagation in open channels:

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0 \tag{58}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = \frac{-f}{\rho} \tag{59}$$

where :

-h: is the water height.

-u: is the flow velocity.

-g: is gravity.

-f: is the frication effect.

d.Numerical simulation:

(matlab)

- **Calculation exemple:**

  Determing the wave position after 5 second.

  Assuming the wave moves in a region with a depth of $2m$

$c = \sqrt{9.81 * 2} = 4.43m/s$

$x = c * t = 4.43 * 5 = 22.15m$

e.Result analysis:

-Initially, the wave movesat $3.84m/s$. -Upon reaching the deeper region, its speed increases to $4.43m/s$. -If a barrier existsat the channel's end, the wave will portially reflect.

f.Graphical representation of results:

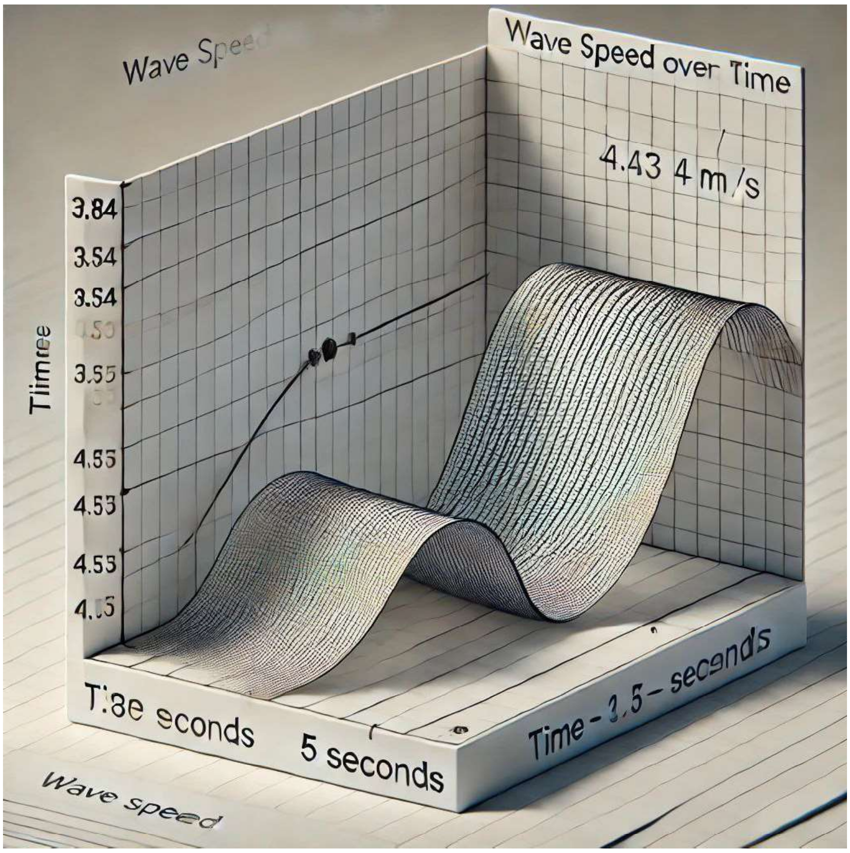

Figure 8: Graphical representation of wave motion results (3D plot created by the author based on personal data/input).

1. Results

- Analytical solutions provided a general understanding of the flow in ideal conditions.
- Numerical solutions, especially using the Boundary Element Method (BEM), gave more accurate results in complex situations.
- **For submerged body flow:**
  * As the depth increases, the lift and drag coefficients also increase.
  * The effect of the free surface becomes negligible at greater depths.
- **For wave propagation:**
  * Wave speed increases with water depth.
  * The free surface has a significant influence on wave behavior.

2. Conclusion and Future Work

Potential flow theory effectively models surface flows, especially when viscosity and turbulence are negligible. The BEM method is efficient and computationally light. Future work may include the effects of viscosity and turbulence in the model. Additionally, studying the role of surface tension in small-scale flows is important. Artificial intelligence can be used to enhance flow prediction. Finally, the study can be extended to 3-D cases or problems with moving boundaries.

Abstract:

This thesis addresses the application of potential flow theory} to free surface flow problems, using both analytical and numerical solutions. The research focuses on solving the fundamental equations (continuity, Bernoulli, Laplace) under specific boundary conditions, with particular emphasis on the Boundary Element Method (BEM) as an accurate and computationally efficient tool. This study provides practical insights for the fields of hydraulic engineering and aerodynamics.

Ce mémoire traite de l'application de la théorie de l'écoulement potentiel} aux problèmes d'écoulement à surface libre, en utilisant des solutions analytiques et numériques. La recherche se concentre sur la résolution des équations fondamentales (continuité, Bernoulli, Laplace) sous certaines conditions aux limites, avec un accent particulier sur la méthode des éléments de frontière (BEM) comme outil précis et efficace sur le plan computationnel. Cette étude offre des perspectives applicables dans les domaines de l'ingénierie hydraulique et de l'aérodynamique.

تتناول هذه المذكرة تطبيق نظرية الجريان المحتمل على مشاكل الجريان ذات السطح الحر ,وذلك باستخدام حلول تحليلية وعددية. يركز البحث على حل المعادلات الأساسية (معادلة الاستمرارية، برنولي، لابلاس) تحت شروط حدودية معينة، مع التركيز بشكل خاص على طريقة العناصر الحدية (BEM) كأداة دقيقة وفعالة حسابياً. تقدم هذه الدراسة رؤى قابلة للتطبيق في مجالي الهندسة الهيدروليكية والديناميكا الهوائية.

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