PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESERACH

Mohamed Boudiaf University of M'sila
Faculty of Mathematics and Informatics
Department of Mathematics

Master of Mathematics

Mathematics and Informatics
Specialty: Mathematics
Option : Algebra and Discrete Mathematics

Theme

Planar Graphs and Their Applications

Persented by :
M^r SAIDI Haroun

On : 28/06/2022.

in front of the jury :

Mr. N. Ghadbane	M.C.A,	University of M'sila	**President**.
Mr. D. Mihoubi	Prof,	University of M'sila	**Sypervisor**.
Mr. L. Heboub	M.A.A,	University of M'sila	**Examinator**.

University years: 2021/2022.

Acknowledgements

Praise and thanks be to allah who enabled me to complete this research. Which crowned my academic career..I want to thank those who have had the greatest credit for being who I am now. My dear parants they spared no effort for me and made every effort to achieve my succes.. Thank you to all the teachers who supervised my education.. Thanks to the teachers of the mathematics departement.. And a special thanks to the professor who suppervised my memo Dr.MIHOUBI. Thank you for his valuable advice and guidance, it is an honor for me to be supervised by a scholar like him..I thank my comrades, my family and friends and every one who favored by me..I hope this work will be useful and a source of pride for the University of Msila and the Algerian University Generaly.

Contents

List of Symbols

Notation	Name
$G = (V, E)$	Graph.
$V(G)$	The set of vertices of graph G.
$E(G)$	The set of edges of graph G.
$n(G)$	The order of G.
$m(G)$	The size of G.
$G \sim H$	G Isomorph to H .
$D(V, E)$	Directed graph (Digraph).
$N(v)$	The set of neighbors.
$deg(v)$	The degree of vertex v.
K_5	The complete graph of degree 5.
$K_{3.3}$	The complete bipartite graph 3 ver 3
$\|V\|$	The cardinality of the set V
$\|E\|$	The cardinality of the set E
a_{ij}^k	The inpute of matix A^k.
$T = (V, E)$	Tree.
$T - v$	The tree T except v.
$G + \{u, v\}$	The graph G plus the edge $\{u, v\}$.
$G(P)$	The graph of polyhedra P.

Introduction

During the 18th century, a puzzle appeared in the city of Königsberg related to the Briegle River, which passes through the city center and divides it into four lands with seven bridges. The topic of this puzzle was about finding a road through the city that passes through each bridge only once.

In 1736 the swiss mathematician Leonhard Euler realised that solving this puzzle did not require precise measurements of lengths, angles and areas, or even the design of the city itself. All that matters is how you connect the dots (vertices) together in his model shown in the following figure:

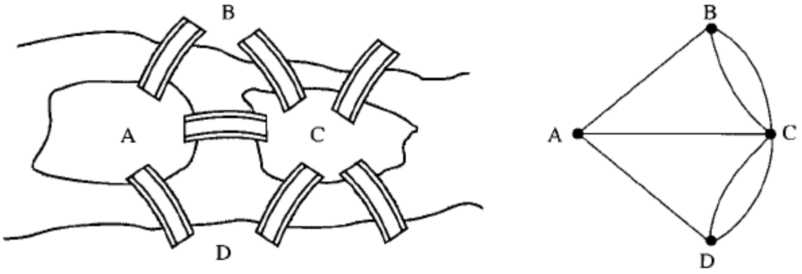

Figure 0.1: The seven bridges of Königsberg

In the end, Euler, with a new look, came to solve this puzzle by observing the relationship between lines and points.

The application of this idea extended after that to reach Topology, Geometry, Chemistry, Cartography and Computer science in our time and other fields, to be known later as the graph theory.

In graph theory, a planar graphs are one of the most important classes due to thier many applications in various sciences.

In this project we will study some applications of this concept in fields Chemistry, Geometry and Cartography. In the first chapter, we will introduce the graph and some concepts for it. After that we give some types and classes of graphs such as K_5 and $K_{3.3}$ in the end we introduce a very important class is called the trees. In the second chapter, we will introduce the concept of planar graph and the very important theorem of it which called Euler's Formula and some corolleries about it and geve a characterizations of planar graphs. In the last chapter, we will give the application of Euler's formula and handshaking lemma to solve some mathimatics problems such as proof that there exist only five regular polyhedra.

Chapter 1

Introduction to Graph Theory

The graph theory is one of very important concepts in combinatorics and discret mathematics, In this chapter we give an introduction to graph theory througt basic definitions and examples.

1.1 General concepts of graph theory

1.1.1 Graph, Subgraph, Isomorphisme

Definition 1.1 (Binary relation)

Let V be a set. A relation E in V is a subset of the Cartesian product $V \times V$. We write the fact that $(u, v) \in E$ as uEv.

Example 1.2

Let $V = \{a, b, c, d\}$ be a set. Then the set $E \subset (V \times V)$ such that

$E = \{(a, a), (a, b), (b, c), (c, d)\}$ is a binary relation in V.

Definition 1.3 (Graph)

A graph G is a finite nonempty set V of element is called the vertices, and a relation $E \subseteq (V \times V)$ i.e, (a set E of 2 element subset of V) is called the set of edges, we can denote this with $G = (V, E)$.

The graph is represented as a picture drawn in plane, every vertex is represented by small circles and every edge is represented by lines, Figure 1.1.

The cardinality of V is called the order of G denote by $|V|$ or $n(G)$, and the cardinality of E is called the size of G denote by $|E|$ or $m(G)$.

Example 1.4

Let $G = (V, E)$ *be a graph defined by the set* $V = \{v_1, v_2, v_3, v_4\}$ *of vertices and the set* E=$\{e_1 = \{v_1, v_2\}, e_2 = \{v_1, v_3\}, e_3 = \{v_2, v_3\}, e_4 = \{v_2, v_4\}, e_5 = \{v_3, v_4\}\}$ *of edges .*

The representation of this graph as shown in Figure 1.1

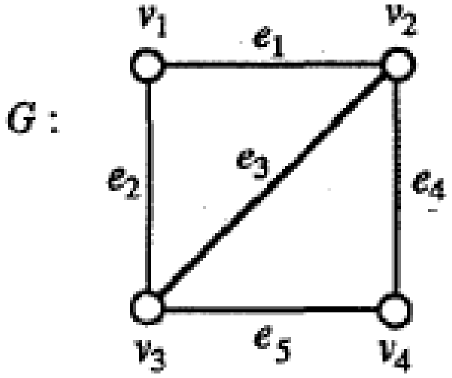

Figure 1.1: Graph

Considering the concepts of order and size, some particuler cases arise including:

- If $n(G) = 0$ then G is called Null graph.
- If $m(G) = 0$ then G is called Empty graph.
- If $n(G) = 1$ then G is called Trivial graph.

Terminologies 1.5

We give some terms about topic of graph.

Adjacent vertices: *Two vertices u and v are said to be adjacent if there exist an edge* $\{u, v\}$ *in the graph* $G = (V, E)$.

Incident: *An edge e and a vertex v are said to be incident if there exist a vertex u such that* $e = \{u, v\}$.

Parallal edges: *If there is more than one edge between the same pair of vertices, then the edges are termed as parallal edges.*

Loop: *A loop is an edge (or arc) that joins a vertex to itself. Mathematically* $e = \{v, v\}$.

Pandant vertex: *A vertex* v *in graph* G *is said to be pandant vertex if there is only one edge is incident with it.*

Adjacence Matrix 1.6

Typically, the graphs are represented as a pictures, but there existe an other representation by use of matrices.

The $n \times n$ *adjacence matrix* $A = [a_{ij}]$ *such that*

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{if } \{v_i, v_j\} \notin E \end{cases}$$

Example 1.7

The matrix A *is adjacent matrix of graph* G *in Figure 1.1*

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Definition 1.8 (Subgraph)

Let $G = (V, E)$ *be a graph, A graphe* $H = (V', E')$ *is subgraph of* G *if* $V' \subseteq V$ *and* $E' \subseteq E$.

Some particuler cases arise including:

- *If* $V' = V$ *and* $E' = E$ *then* H *is called Improper subgraph of* G.
- *If* $V' \subsetneq V$ *and* $E' \subsetneq E$ *then* H *is called Proper subgraph of* G.
- *If* $V' = V$ *and* $E' \subsetneq E$ *then* H *is called Spanning subgraph.*

Example 1.9

Let $G = (V, E)$ *and* $H = (V', E')$ *be the following graphs:*

$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *and*

$E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{3, 9\}, \{4, 7\}, \{5, 6\}, \{5, 7\}, \{6, 8\}, \{6, 9\}, \{7, 8\}, \{8, 9\}\}$

$V' = \{1, 2, 3, 4, 6, 7, 8\}$ *and* $E' = \{\{1, 2\}, \{2, 3\}, \{2, 6\}, \{3, 6\}, \{4, 7\}, \{6, 8\}, \{7, 8\}\}$.

Notice that $V' \subset V$ *and* $E' \subset E$ *and so* H *is a subgraph of* G, *as shown in Figure 1.2.*

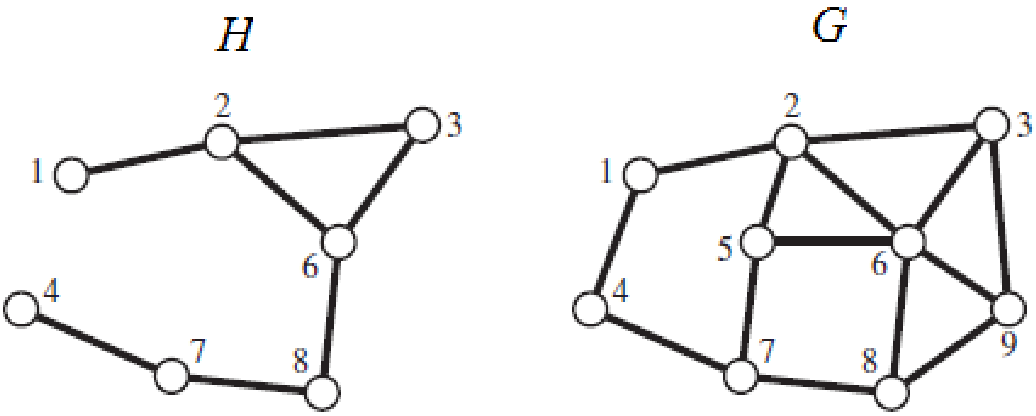

Figure 1.2: H is Subgraphof G

Consider the graphs G *and* H' *in Figure 1.3. The graph* H' *is not subgraph of* G*, because* $\{2, 7\} \in H'$ *but* $\{2, 7\} \notin G$

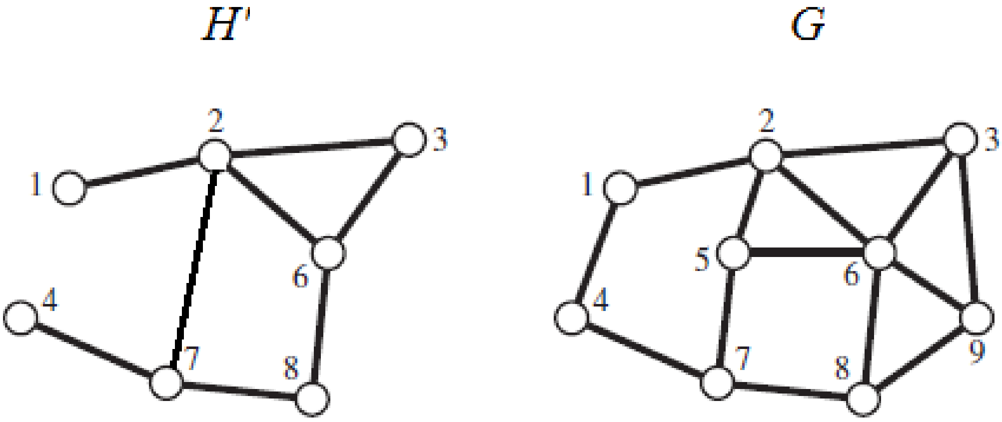

Figure 1.3: H' is nonsubgraph of G

Definition 1.10 (Isomorphisme)

Two simple graphs $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *are isomorphic if there is a bijection* $\varphi : V_1 \to V_2$*, which preserves the adjacency .*

i.e, if $\{u, v\}$ *is an edge in* E_1*, then* $\{\varphi(u), \varphi(v)\}$ *is an edge in* E_2 *and we write* $G_1 \simeq G_2$*.*

Example 1.11

Consider the graphs G *and* H *of to be order* 6 *and size* 6 *Figure 1.4*

The mapping $\varphi : V_1 \longrightarrow V_2$ *defined by:*

$\varphi(a) = 5, \varphi(b) = 2, \varphi(c) = 1, \varphi(d) = 4, \varphi(e) = 3, \varphi(h) = 6$ *is an isomorphisme, then* $G \sim H$

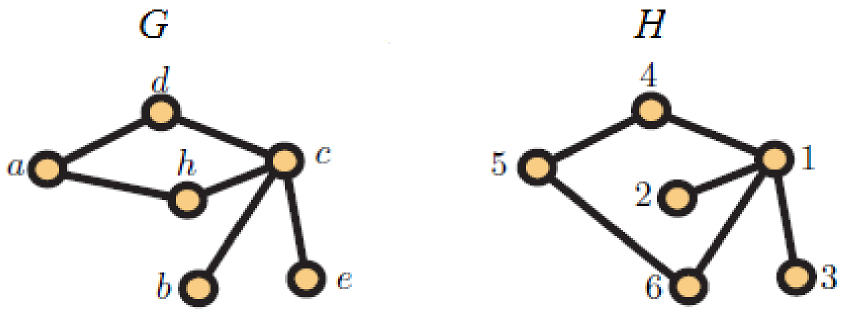

Figure 1.4: Isomorphic graphs

Consider the graphs G *and* H *in Figure 1.5. The graph* G *is nonisomorphic to* H*.*

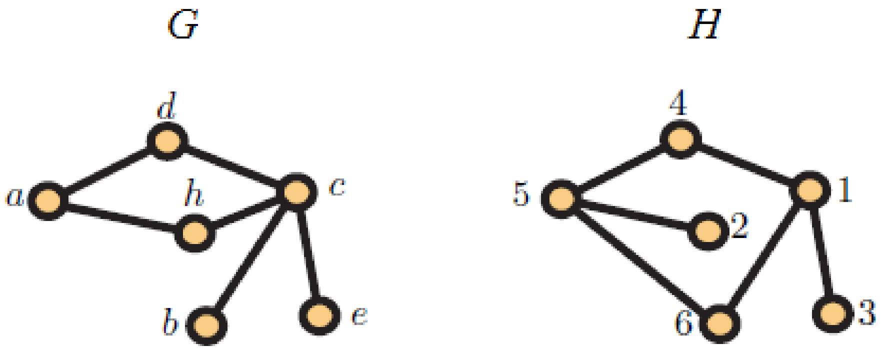

Figure 1.5: nonisomorphic graphs

1.1.2 Types and Classes of graphs

Types of graphs

Definition 1.12 (Simple graph)

A simple graph $G = (V, E)$ *is a graph has no self loop or parallel edges*

Example 1.13

Consider the graphs G *and* H *in Figure 1.6*

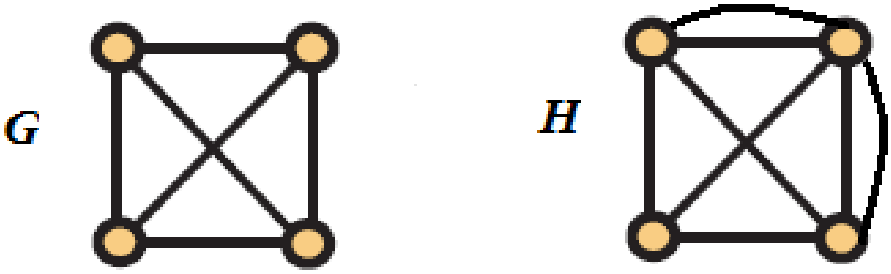

Figure 1.6: G is simple graph and H nonsimple graph

The graph H *is not simple graph because there exist parallel edges, whereas the graph* G *is a simple graph*

Definition 1.14 (Directed graph)

A directed graph (Digraph) D *is a finite nonempty set* V *of vertices and antisymmetric relation* $E \subseteq V \times V$ *is called the set of Arcs or directed edges, we can denote this with* $D = (V, E)$

Example 1.15

The graph D *in Figure 1.7 is a Digraph*

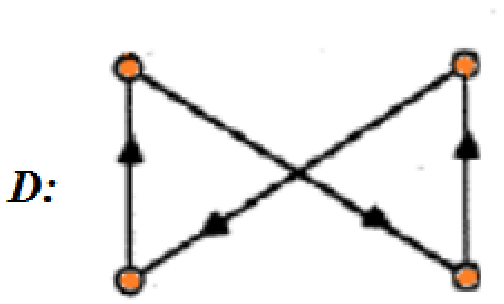

Figure 1.7: Directed Graph

Definition 1.16 (Multigraph)

A graph $G = (V, E)$ is known as a multigraph if it contains parallel edges, i.e, two or more edges between a paire of vertices. It is be noted that every simple graph is a multigraph but the converse is not true

Example 1.17

Consider the graph G in Figure 1.8

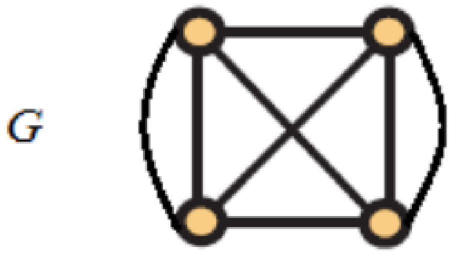

Figure 1.8: Multigraph

G is a multigraph because there are parallel edges.

Definition 1.18 (Pseudograph)

A graph $G = (V, E)$ is known as a pseudograph if we allow both parallel edges and loops. it is be noted that every simple graph and multigraph are pseudograph but the converse is not true

Example 1.19

Consider the graph G in Figure 1.9

G is a pseudograph because it has parallel edges and loops

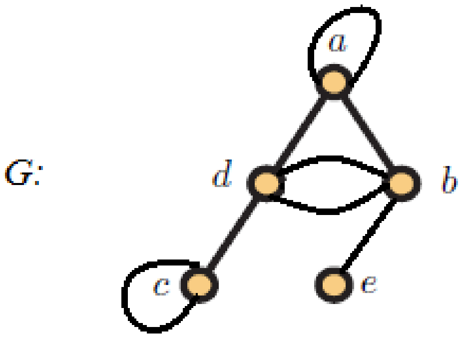

Figure 1.9: Pseudograph

Definition 1.20

Let $G = (V, E)$ *be a graph, and let* v *be a vertex of* G*. the set* $N(v) = \{u : u \text{ is adjacent to } v\}$ *is called the set of a neighbours of a vertex* v*.*

the cardinality of $N(v)$ *is called the degree of* v*, we can denote this with* $deg(v)$*. A vertex is called even or odd according to whether its degree is even or odd. A vertex of degree* 0 *in* G *is called isolated vertex and a vertex of degree* 1 *is called pendant vertex.*

Theorem 1.21 (Handshaking Lemma)

Let G *be a graph of order* n *and size* m *then*

$$\sum_{i=1}^{n} deg(v_i) = 2m$$

Proof.

Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex.

Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end.

Thus the sum of the degrees equal twice the number of edges.

$$\sum_{i=1}^{n} deg(v_i) = 2m$$

□

Corollary 1.22

In any graph, there is an even number of odd vertices.

Proof.

Let V_1 be the set of odd vertices of G and let V_2 be the set of even vertices of G, with $V_1 \cup V_2 = V$ and $V_1 \bigcap V_2 = \varPhi$.

By use Theorem 1.7

$$\sum_{v\in V} deg(v) \quad = \quad \sum_{v\in V_1} deg(v) + \sum_{v\in V_2} deg(v) = 2m$$

then

$$\sum_{v\in V_1} deg(v) = 2m - \sum_{v\in V_2} deg(v)$$

Clearly that $\sum_{v\in V_2 deg(v)}$ is even then $\sum_{v\in V_2 deg(v)} = 2k$ such that $k \in \mathbb{Z}$

$$\sum_{v\in V_1} deg(v) = 2m - 2k$$

then

$$\sum_{v\in V_1} deg(v) = 2(m-k)$$

Thus, the numbre of odd vertices is even □

Classes of graphs

Definition 1.23 (Regular graph)

Let $G = (V, E)$ be a simple graph, we say that G is r-regular graph (of degree r) if $deg(v) = r$ for all $v \in V$

Example 1.24

Consider the graph G in Figure 1.10

In G we have ($deg(1) = deg(2) = ... = deg(5) = 2$) then the graph G is 2-regular

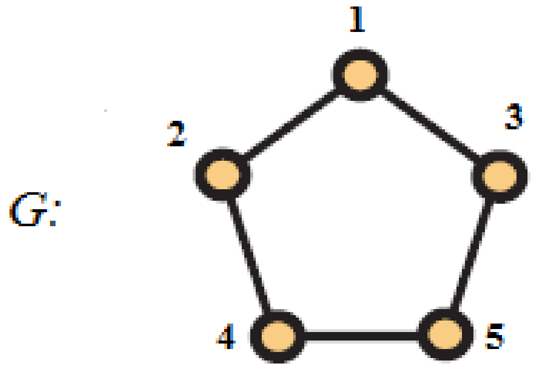

Figure 1.10: 2-regular Graph

Definition 1.25 (Complete graph)

Let $G = (V, E)$ be a simple graph, we say that G is complete graph of order n if every two of its vertices are adjacent.

Remark 1.26

A complete graph G of order n and size m is therefore a regular graph of degree $n - 1$ having $m = \dfrac{n(n-1)}{2}$, we denote this graph by K_n.

Example 1.27

The complete graphs K_n with $n \leqslant 5$ in Figure 1.11.

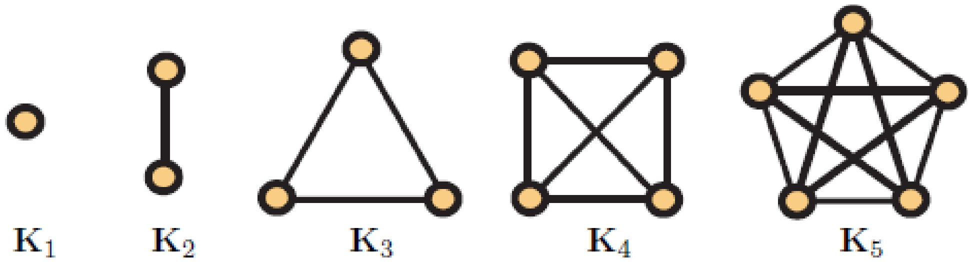

Figure 1.11: The completes graphs K_n with $n \leqslant 5$

Definition 1.28 (Bipartite graph)

A graph G is a bipartite graph if it is possible to partition V into two disjoint subsets V_1 and V_2 called partite sets such that every edges of E joins a vertex of V_1 to a vertex of V_2 i.e, $E \subseteq (V_1 \times V_2)$.

Remark 1.29

If G is an r-regular bipartite graph, $r > 1$, with partite sets V_1 and V_2 then $|V_1| = |V_2|$ this follows since its size $m = r|V_1| = r|V_2|$

Definition 1.30 (Complete bipartite graph)

A complete bipartite graph G is a bipartite graph with partite sets V_1 and V_2 having the added property that : If $u \in V_1$ and $v \in V_2$, then $\{u, v\} \in E$.

We can denote this by $K_{s.t}$, such that $s = |V_1|$ and $t = |V_2|$.

Example 1.31

Consider the bipartite graphs G_1 and G_2 as shown in Figure 1.12

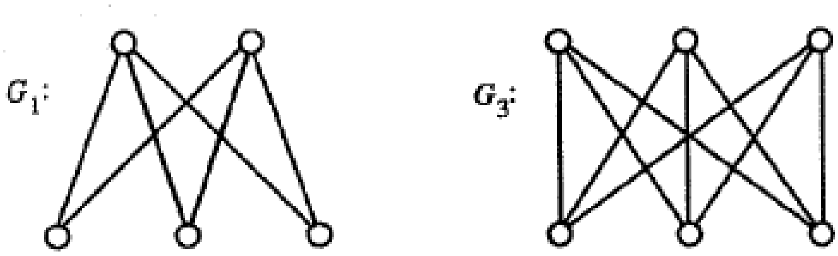

Figure 1.12: Complete bipartite Graph

G_1 is called $K_{2.3}$, and G_2 is called $K_{3.3}$

1.1.3 Path, Cycle and connectivity

Definition 1.32 (Walk)

Let $G = (V, E)$ *be a simple graph. A walk is a finite sequence* $W : v_1, v_2, ..., v_k$ *of vertices such that:* $\{v_i, v_j\} \in E$, *with* $1 \leqslant i, j \leqslant k$

Definition 1.33 (Path)

Let $W : v_1, v_2, ..., v_k$ *be a walk, if the vertices of* W *are distinct then* W *is called path of length* $l = k - 1$

Example 1.34

Consider the graph in the Figure 1.13. The following sequences of vertices are walks with lengths l_i*:*

$W_1 : 1, 2, 3, 6, 2, 1, 5 \quad /l_1 = 6.$

$W_2 : 1, 5, 6, 3, 2, 6, 4, 9 \quad /l_2 = 7.$

Consider the graph in the Figure 1.13. The following sequences of vertices are paths with lengths l_i*:*

$P_1 : 1, 2, 3, 4, 9 \quad /l_1 = 4.$

$P_2 : 1, 5, 6, 3, 7, 9 \quad /k_2 = 5.$

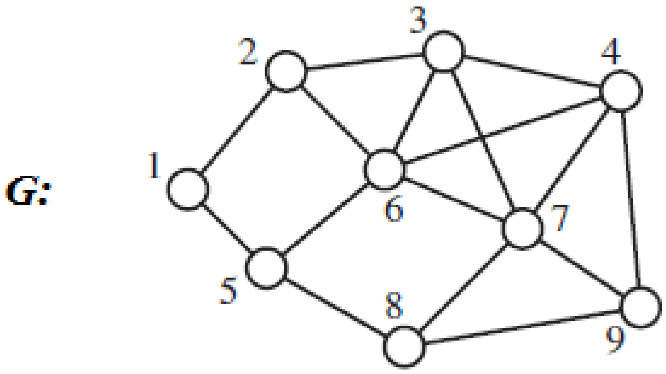

Figure 1.13: Walk, Path and Cycle

Remark 1.35

Every path is a walk, but conversely not necessarily true.

Every path is a subgraph, but conversely not necessarily true.

Theorem 1.36

Let G *be a simple graph and* A *is the adjacency matrix of* G*, then* a_{ij} *entry of* A^k *whith* $k \geqslant 1$ *is the numbre of different walk frome* v_i *to* v_j *of length* k *in the graph* G*.*

Proof.

The proof is by induction on k.

basic case The theorem is obviously true for $k = 1$ because there existe a walk of length 1 if and only if $\{v_i, v_j\} \in E$.

Induction hypothesis Suppose that the theorem is true for k (i.e, a_{ij}^k is the number of different walks of length k in G).

Let $A^{k+1} = [a_{ij}^{k+1}]$, since $A^{k+1} = A^k A$ we have

$$a_{ij}^k = \sum_{l=1}^{n} .a_{il}^k a_{li} \tag{1.1}$$

Every walk of length $k+1$ in G consists of a walk of length k, where v_l is adjacent to v_j, followed by the edge $\{v_l, v_j\}$ and the vertex v_j. Thus by the inductive hypothesis and 1.1, the theorem is proved.

□

Definition 1.37 (Cycle)

Let $P : v_1, v_2, ..., v_k$ *be a path of a simple graph* G*, if* $v_1 = v_k$ *then* P *is called a cycle of length* k*.*

A cycle is even if its length is even and odd if its length is odd.

Example 1.38

Consider the graph in the Figure 1.13. The following sequences of vertices are cycles with *lengths* k_i*:*

$L_1 = 1, 2, 6, 5, 1 \qquad /k_1 = 4.$

$L_2 = 1, 2, 6, 7, 8, 5, 1 \qquad /k_2 = 6.$

$L_3 = 2, 3, 6, 2 \qquad /k_3 = 3.$

Definition 1.39 (Connectd graph)

Let $G = (V, E)$ *be a graph. If for any two distinct elements* u *and* v *of* V *there is a path* P *from* u *to* v *then* G *is called connected graph.*

Example 1.40

Consider the graphs G *and* H *in the Figure 1.14, the graph* G *is connected and* H *is not connected.*

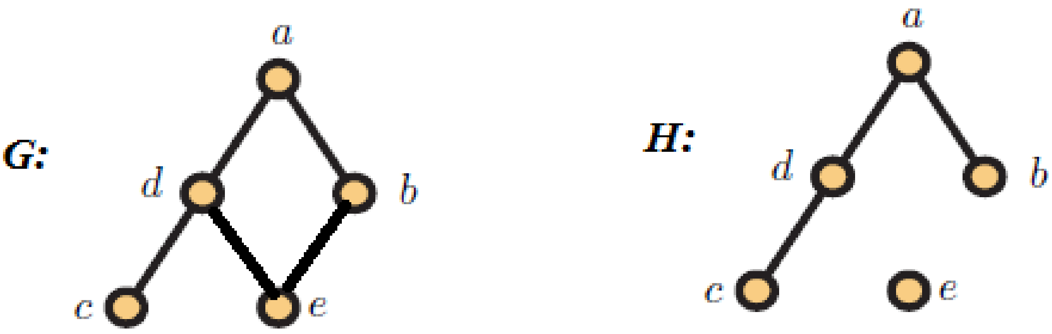

Figure 1.14: Connected graph and nonconnected

Theorem 1.41

If a nontrivial graph G *is bipartite then it has no odd cycle.*

Proof.

Let $G = (V, E)$ be a bipartite graph.

Suppose that $C : v_1, v_2, ..., v_k, v_1$ is odd cycle of graph G, with $v_i \in V_1$ if i is odd and $v_i \in V_2$ if i is even.

$\Rightarrow v_k \in V_1$ because C is odd cycle.

$\Rightarrow \{v_k, v_1\} \notin E$ (because G is bipartite graph), then we have a contradiction.

Thus the graph G has no odd cycle. $\square$

1.2 Trees

Definition 1.42 (Tree)

Let $G = (V, E)$ be a simple graph, if G is connected graph contains no cycle then G is called a tree denoted by $T = (V, E)$.

Example 1.43

Consider the graphs (a), (b) and (c) in Figure 1.15.

The graphs (a) and (b) are a trees but the graph (c) is not tree because it has an 3-cycle

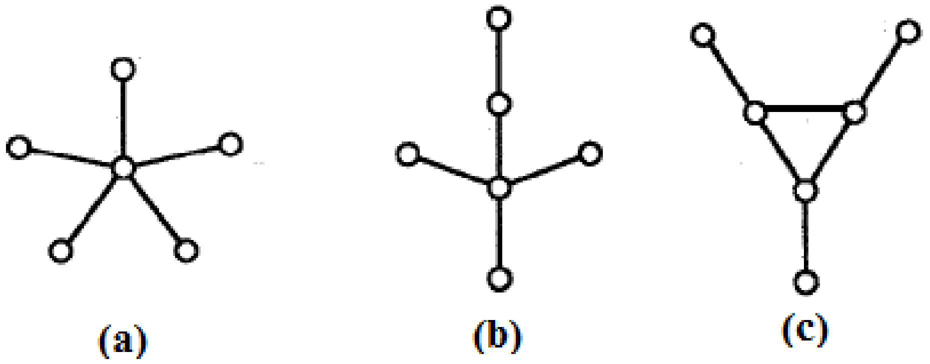

Figure 1.15: Tree and nontree

Theorem 1.44

A tree with n vertices has exactly $n - 1$ edges.

Proof.

The proof is by induction on the number of vertices in the graph.

basic case: The theorem is obviously true for all trees on one vertex, because every graph with one vertex (trivial graph) has no edges.

Induction hypothesis: Suppose the theorem is true for all trees on n vertices (i.e, all trees with n vertices has $n - 1$ edges.

Let T be a tree with $n + 1$ vertices and let v be an pendant vertex of T.

The graph $T' = T - v$ is a tree of order n, so by induction hypotesis T' has $n - 1$ edges.

Since T has one more vertex than T', then T has n edges.

Thus every tree of n vertices has $n-1$ edges.

□

Chapter 2

planar graphs

In this chapter, we intrduce the concept of a planar graph and study some properties and characterization.

2.1 What is a planar graph ?

In this section, we give the notion of planar graph and some basic definitions.

Definition 2.1 (Planar graph)

Let $G = (V, E)$ be a simple graph, we say that G is planar if it can be embedded in the plan, i.e, it can be drawn in such a way that no edges cross each other. Embedding a graph in the plane is equivalent to embedding it on the sphere.

Example 2.2

The graph $G_1 = K_{2.3}$ of Figure 2.1 is planar, although, as drawn, it is not plane; however, $G_2 = K_{2.3}$ is both planar and plane. The graph $G_3 = K_{3.3}$ is nonplanar.
This last statement will be proved presently.

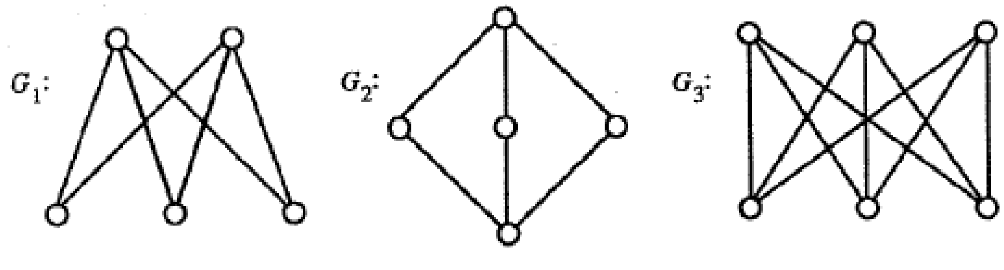

Figure 2.1: Planar and nonplanar graph

Definition 2.3

Let π *be a plane and let* $G = (V, E)$ *be a planar graph embadded in* π*, then the edges of* G *divides the plane* π *into pieces is called faces or region. The number of edges bounding for face is called the degree of face*

Definition 2.4 (Dual)

Let G *be a planar graph. A dual graph* G^* *is a multigraph verify the following:*

Corresponding to each face f *of* G *there is a vertex* f^* *of* G^* *and corresponding to each edge* e *of* G *there is an edge* e^* *of* G^*

Two vertices f^* *and* g^* *are joind by the edge* e^* *in* G^* *if and only if thier corresponding face* f *and* g *are separated by the* e *in* G*.*

Lemma 2.5

Let G *be a planar graph of* n *vertices,* m *edges and* r *faces. The sum of the degrees of the faces in a crossing-free embedding of* G *in the plane equals* $2m$

$$\sum_{f \in F} deg(f) = 2m$$

Proof.

Let G_* be the dual of G, then by definition

$$\sum_{f \in F} deg(f) = \sum_{f^* \in V^*} deg(f^*)$$

By Hanshaking lemma

$$\sum_{f \in F} deg(f) = \sum_{f_* \in V^*} deg(f_*) = 2m$$

□

Definition 2.6 (Maximal planar)

Let $G = (V, E)$ *be a planar graph, we say that* G *is maximal planar if for evry pair* u, v *of nonadjacent vertices of* G *the graph* $G + \{u, v\}$ *is nonplanar*

Example 2.7

Consider the planar graph G *in figure 2.2*

The graph G *is maximal planar of* 5 *vertices.*

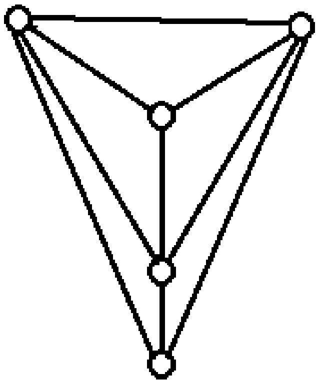

Figure 2.2: Maximal planar

2.2 Euler's Formula

The formula devloped by Euler play an important rol in planar graphs, In this section we will give the Euler's formula and some corolleries about it.

Theorem 2.8 (Euler's Formula)

Let $G = (V, E)$ be a connected planar graph of n vertices, m edges and r faces, then

$$n - m + r = 2 \tag{2.1}$$

Proof.

Let $G = (V, E)$ be a connected planar graph have n vertices, m edges and r faces

- If the graph G is a tree, then G have n vertices, $m = n - 1$ edges and $r = 1$ faces then

  $$\begin{aligned} n - m + r &= n - (n - 1) + 1 \\ &= 2 \end{aligned}$$

  Thus the theorem is true

- If the graph G is not a tree then G has at least one cycle

  The proof is by induction on the number of edges m in the graph

Basic case: The theorem is obviously true for all graphs on one edge, because every graph of one edge has 2 vertices and two region.

$$\begin{aligned} n-m+r &= 2-1+1 \\ &= 2 \end{aligned}$$

Induction hypothesis: Suppose that theorem is true for all connected planar graph has m edges.

We want to show that the theorem is true for $G' = (V, E - \{e\})$ with e is a cycle edge, then G' has n vertices, $m-1$ edges and $r-1$ faces

$$\begin{aligned} n-(m-1)+(r-1) &= n-m+1+r-1 \\ &= n-m+r=2 \end{aligned}$$

□

Corollary 2.9

- *Let G be a connected planar graph with at least two edges. Then*

$$m \leqslant 3n-6$$

- *If G does not contain a 3-cycle, then*

$$m \leqslant 2n-4$$

Proof.

- The proof of the first inequality

  We have by lemma 2.5, the sum of the face degrees is $2m$.

  Otherwise, every face has degree at least 3, so the sum of the face degrees is at least $3r$. Therefore we have: $2m \leqslant 3r \Rightarrow r \leqslant \frac{2}{3}m$

  By Euler's formula

$$\begin{aligned} r = 2-n+m \leqslant \frac{2}{3}m &\Rightarrow 2-n+\frac{1}{3}m \leqslant 0 \\ &\Rightarrow m \leqslant 3n-6 \end{aligned}$$

- The proof of the second inequality

  We have by lemma 2.5, the sum of the face degrees is $2m$.

  Otherwise, every face has degree at least 4, so the sum of the face degrees is at least $4r$. Therefore we have: $2m \leqslant 4r \Rightarrow r \leqslant \frac{1}{2}m$

  By Euler's formula

$$\begin{aligned} r = 2 - n + m \leqslant \frac{1}{2}m \ &\Rightarrow \ 2 - n + \frac{1}{2}m \leqslant \\ &\Rightarrow \ m \leqslant 2n - 4 \end{aligned}$$

□

Remark 2.10

If $m = 3n - 6$*, then the graph* G *is maximal planar graph*

Theorem 2.11

Every planar graph has at least one vertex of degree 5 *or less*

Proof.

Let G be a planar graph of order n and size m, then

$$m \leqslant 3n - 6 \ \Rightarrow \ \sum_{i=1}^{n} deg(v_i) = 2m \leqslant 6n - 12 \tag{2.2}$$

By contradiction suppose that $deg(v) \geqslant 6$ for all $v \in G$, then

$$\sum_{i=1}^{n} deg(v_i) \geqslant 6n \tag{2.3}$$

From 2.2 and 2.3

$$6n \leqslant \sum_{i=1}^{n} deg(v_i) \leqslant 6n - 12$$

So, we have a contradiction

Thus, every planar graph has at least one vertex of degree 5 or less □

2.3 Characterizations of planar graphs

The two graphs K_5 and $K_{3.3}$, that play an important role in the study of planar graphs. In this section, we will use the crollery 2.9 to prove that K_5 and $K_{3.3}$ are nonplanar.

Theorem 2.12

The complete graph K_5 is nonplanar.

Proof.

Suppose that: the complete graph K_5 is a planar graph. By corollery 2.9 we have

$$10 = m \leqslant 3n - 6 = 9 \tag{2.4}$$

We have a contradiction. Thus the complete graph K_5 is nonplanar. □

Theorem 2.13

The complete bipartite graph $K_{3.3}$ is nonplanar.

This theorem is used to prove the impossibility of connecting the three huts with Water, Gas and Elictricity in the puzzle of three utilities, as shown in Figure 2.3

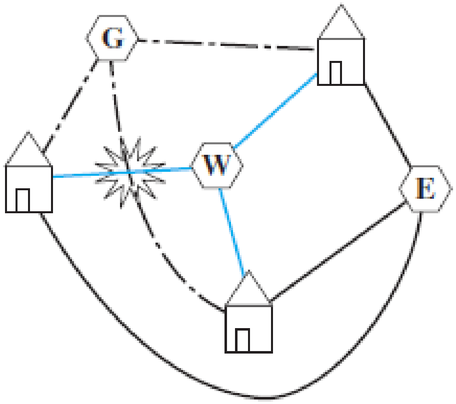

Figure 2.3: The puzzle of three utilities

Proof.

Suppose that: the complete bipartite graph $K_{3.3}$ is planar graph. Since $K_{3.3}$ has no odd cycle, then $K_{3.3}$ has no 3-cycle.

Applying the second part of corollery

$$9 = m \leqslant 2n - 4 = 2(6) - 4 = 8$$

We have a contradiction.Thus the complete bipartite graph $K_{3.3}$ is nonplanar. □

Definition 2.14 (Subdivision)

An elementary subdivision of a nonempty graph G is a graph obtained from G by removing some edge $e = \{u, v\}$ and adding a new vertex wand edges $\{u, w\}$ and $\{w, v\}$. A subdivision of G is a graph obtained from G by a succession of elementary subdivisions.

Example 2.15

Consider the graphs G_1, G_2 and G_3 in Figure 2.4. The graphs G_1 and G_2 are subdivisions of G_3.

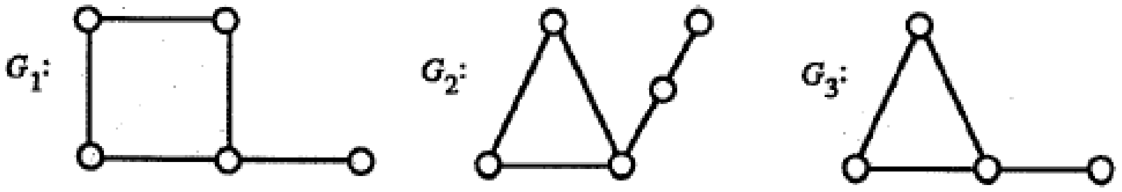

Figure 2.4: Subdivisions

Theorem 2.16 (Kuratowski)

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3.3}$ as a subgraph.

Kuratowski's Theorem is a marvelous characterization of planarity. But the proof was very long and very complicated. in this part we will just give an example.

Example 2.17

Consider the Petersen graph (Figure 2.5 (a)) is nonplanar since it contains the subgraph of (Figure 2.5 (b)) that is a subdivision of $K_{3.3}$ *(Figure 2.5 (c)).*

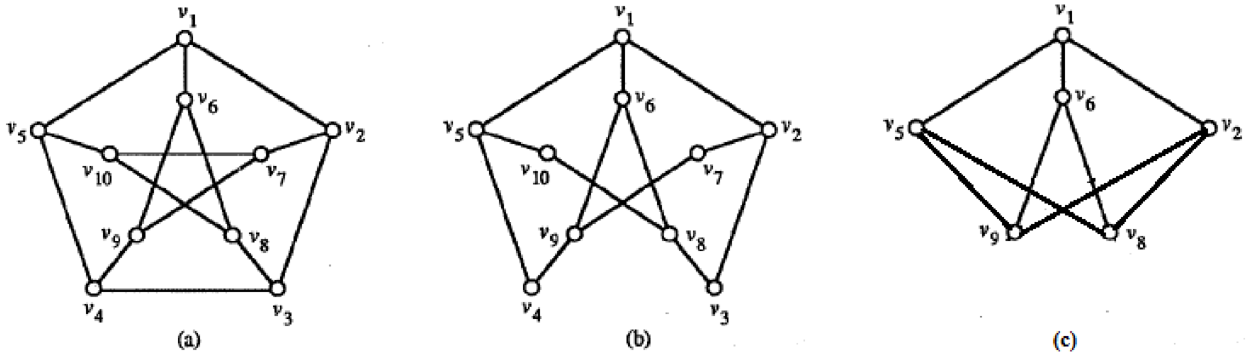

Figure 2.5: The Petersen graph and a subgraph that is a subdivision of $K_{3.3}$

Chapter 3

Applications

3.1 Organic Chemistry

The graph theory has wide uses in organic chemistry. In this section, we will give an examples of using the graph theory to drawing the carboxylic acids molecules and checking of some formulas.

Molecules are consiste up of a number of atoms that are chained together by chemical bonds. We can descripe this by a graph, whose vertices are the atoms and the edges are the bonds.

A carboxylic acid molecules contains carbon and hydrogen atoms and the carboxyl group ($COOH$). Moreeover, the carboxyl group and each hydrogen atom is bonded with only singel carbon atom and each carbon atom is bonded with 4 hydrogen atoms or 3 hydrogen and the carboxil group (i.e, $deg(H) = 1$, $deg(COOH) = 1$, $deg(C) = 4$).

For example: an ethanoic acid molecule has one carbon, three hydrogen and the carboxyl group, thus, the ethanoic acid molecule is represented by the molecular formula CH_3COOH which can be represent with the following graph:

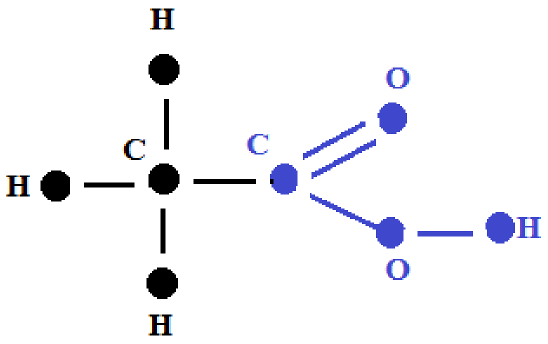

Figure 3.1: Acid ethanoic molecule

So, the question is: can there exist a caboxylic acid molecule with the following molecular formulas ?

- C_2H_5COOH
- $C_7H_{12}COOH$

For answer to this question, we will applied the handshaking lemma, we obtain

The first part

$$\begin{aligned}\sum_{v\in V} deg(v) = 2m \quad &\Rightarrow \quad 2deg(C) + 5deg(H) + deg(COOH) = 2m\\ &\Rightarrow \quad 2(4) + 5(1) + 1 = 2m\\ &\Rightarrow \quad 14 = 2m\\ &\Rightarrow \quad m = 7\end{aligned}$$

So, the handshaking lemma is satisfied

Thus C_2H_5COOH is a carboxylic acid has 7 covalent bond is called Propionic acid as shown in Figure 3.2

The second part:

$$\begin{aligned}\sum_{v\in V} deg(v) = 2m \quad &\Rightarrow \quad 7deg(C) + 12deg(H) + deg(COOH) = 2m\\ &\Rightarrow \quad 7(4) + 12(1) + 1 = 2m\\ &\Rightarrow \quad 41 = 2m\end{aligned}$$

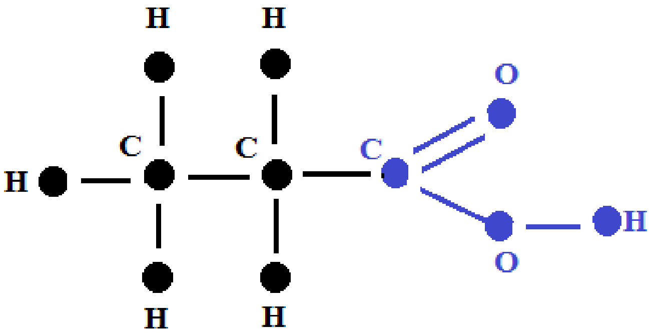

Figure 3.2: Acid propionic molecule

This is impossible. so, the handshaking lemma is not satisfied.

Thus $C_7H_{12}COOH$ is not carboxylic acid

Similarly, we can check a molecules belongs to the family of corboxylic acid or not.

3.2 Regular polyhedra

A regular polyhedron is a three-dimensional geometrical shape whose surface consists of plane faces that are polygone. In this section, we give an application of planar graph to proof that there are exactly five regular polyhedra.

Firstly we need proof that at least one face of every regular polyhedra is of degree $k = 3, 4$ or 5, then

We will defined a planar graph $G(P)$ is a representation of a regular polyhedron P in the sphere, is called the associated planar graph. The planar graph $G(P)$ whose vertices and edges are the vertices and edges of P, denote the number of vertices, edges and faces of the associated graph $G(P)$ by V, E and F, respectively. According to Euler's formula (sometimes called Euler's polyhedron formula because it used in connection with regular polyhedra in three-space).

$$V - E + F = 2$$

We will represent the number of vertices of degree k by V_k and the number of faces (regions) of degree k by F_k. Then by (handshaking lemma) and (lemma 2.5)

$$2E = \sum_{k \geqslant 3} kV_k = \sum_{k \geqslant 3} kF_k \tag{3.1}$$

Assume that $F_3 = F_4 = F_5 = 0$, so by Equation 3.1

$$2E = \sum_{k \geqslant 6} kF_k \geqslant \sum_{k \geqslant 6} 6F_k = 6 \sum_{k \geqslant 6} F_k = 6F$$

Hance $E \geqslant 3F$

$$2E = \sum_{k \geqslant 3} kV_k \geqslant \sum_{k \geqslant 3} 6V_k = 3 \sum_{k \geqslant 6} V_k = 3F$$

By Euler's formula $V - E + F = 2 \Rightarrow \frac{2}{3}E - E + \frac{1}{3}E \geqslant 2 \Rightarrow 0 \geqslant 2$.

This is a contradiction, so ther existe at least one face of degree $k = 3, 4, 5$ in every polyhedron

Now, we will proof that there are exactly five regular polyhedra.

Proof.

Let P be a regular polyhedron and let $G(P)$ be an associated planar graph.

By Euler's formula $2 = V - E + F$, then

$$\begin{aligned} -8 &= 4V - 4E + 4F \\ &= 2E + 2E - 4V + 4F \\ &= \sum_{k \geqslant 3} kF_k + \sum_{k \geqslant 3} kV_k - 4\sum_{k \geqslant 3} V_k + 4\sum_{k \geqslant 3} F_k \\ &= \sum_{k \geqslant 3} (k-4)F_k + \sum_{k \geqslant 3} (k-4)V_k \end{aligned}$$

Since P is regular, then there exist an integers $s \geqslant 3$ and $t \geqslant 3$ such that $F = F_s$ and $V = V_t$. So $-8 = (s-4)F_s + (t-4)V_t$.

On the other hand, we note that $3 \leqslant s \leqslant 5$ and $3 \leqslant t \leqslant 5$ and $sF_s = 2E = tV_t$.

This gives us nine cases to consider

Case1: If $s = 3$ and $t = 3$, then

$$\begin{cases} -8 = -F_3 - V_3 \\ 3F_3 = 3V_3 \end{cases}$$

Thus $F_3 = V_3 = 4$ and P is the tetrahedron.

Case2: If $s = 3$ and $t = 4$, then

$$\begin{cases} -8 = -F_3 \\ 3F_3 = 4V_4 \end{cases}$$

Thus $F_3 = 8$ $V_4 = 6$ and P is the octahedron.

Case3: If $s = 3$ and $t = 5$, then

$$\begin{cases} -8 = -F_3 + V_5 \\ 3F_3 = 5V_5 \end{cases}$$

Thus $F_3 = 20$ $V_5 = 12$ and P is the icosahedron.

Case4: If $s = 4$ and $t = 3$, then

$$\begin{cases} -8 = -V_3 \\ 4F_4 = 3V_3 \end{cases}$$

Thus $F_4 = 8$ $V_3 = 6$ and P is the cube.

Case5: If $s = 4$ and $t = 4$, this is impossible since $-8 \neq 0$

Case6: If $s = 4$ and $t = 5$, this is impossible since $V_5 \neq -8$

Case7: If $s = 5$ and $t = 3$, then

$$\begin{cases} -8 = F_5 - V_3 \\ 5F_5 = 3V_3 \end{cases}$$

Thus $F_5 = 12$ $V_3 = 20$ and P is the dodecahedron.

Case8: If $s = 5$ and $t = 4$, this is impossible since $F_5 \neq -8$

Case9: If $s = 5$ and $t = 5$, this is impossible since $F_5 = V_5 \neq -8$

Hence there are exactly five regular polyhedra shown in Figure 3.3

□

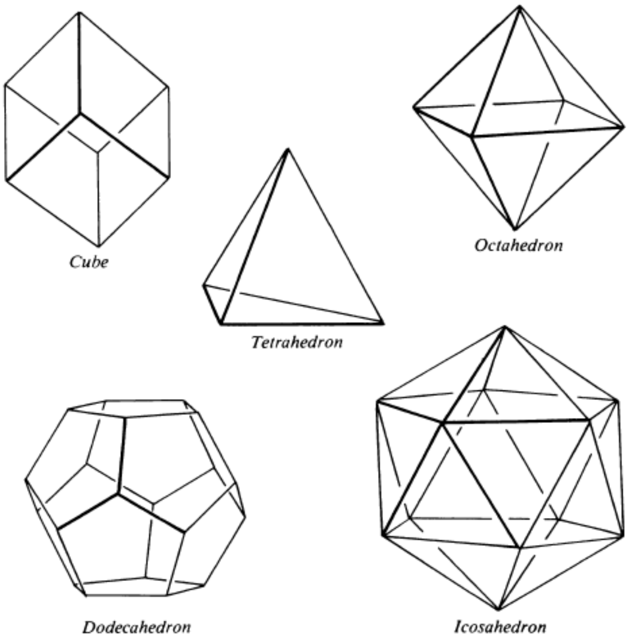

Figure 3.3: The five regular polyhedra

The symmetry groups of the regular polyhedra play an important role in the determination of all the finite groups of isometries.

3.3 The five colours problem

The four color conjecture is one of the most complex mathimatical problems in history, it is states that "A map on a plane or on the surface of a sphere can be colored with no more than four colors so that no two adjacent countries have the same color".

In 1989, the two scientists, Appel and Haken, prove the correctencess of the conjecture, but the proof was very long and very complicated.
In this section, we prove a simpler version of the four colors conjecture is called the five colors problem.

Firstly, we will defined a graph G, whose vertices are the countries and the edges are the borders of those countries. We note that the graph G is planar. Shown in Figure 3.4

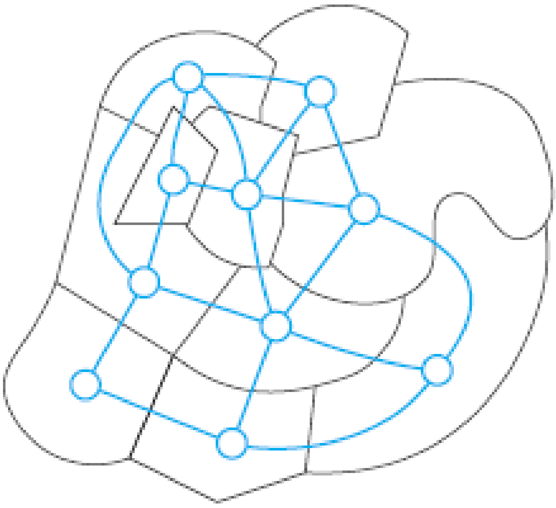

Figure 3.4: The planar graph of map

So, the theorem becomes as follows: the vertices of planar graph can be colored with no more than five colors such that no two adjacent vertices have the same color

Proof.

The proof is by induction on the number of vertices in the graph.

Basic case: The theorem is obviously true for all graphs on five or fewer vertices, because we can give each vertex a separate color

Induction hypothesis: Suppose the theorem is true for all graphs on n vertices (i.e, all planar graphs with n vertices are five-colorable).

Let G be a planar graph with $n + 1$ vertices. By Theorem 2.11 G contains a vertex v with $deg(v) \leqslant 5$

Let $G' = G - v$, notice that G' is planar graph has n vertices. By hypothesis G' is five-colorable.

We want to extend this coloring to G by giving v a color. Consider $N(v) = \{u_1, u_2, u_3, u_4, u_5\}$ the set of neighbors of v. If among the neighbors of v there are only four different colors, then there is a left over color that we can assign to v.

The problem has been reduced to the case where $d(v) = 5$ and all five of its neighbors are different colors. There is no way to extend this coloring to v whatever color we might choose for v would be the same color as one of its neighbors. So to extend the coloring to vertex v. We nedd to prove that $N(v)$ is four-colorable.

Then, there exist two vertices $x, y \in N(v)$ such that there is no path between x and y because we would then have K_5 as a subdivision of G (K_5 is non planar)as shown in figure 3.5.

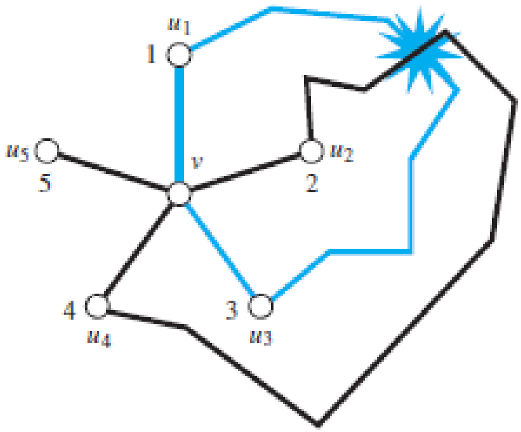

Figure 3.5:

We combaine v, x and y into one vertex which we call u, we now obtain a graph H in the plane with two less vertices than G. By induction hypothesis H is five-colorable.

We color all vertices of G using the same coloring as for H, except for v, x and y, which are colored as follows. Give x and y the same color as u. Give v a color different from all the colors used on the four vertices u, u_3, u_4 and u_5. (There must be one since we have five colors available.).

Thus the vertex of planar graph can be colored by five color

Finally, the theorem is proved

□

3.4 Electrical circuits

One of the important applications of planar graphs and thier uses in computer sience is the design and construction of electrical circuits.

The objective of using planar graphs is to reduce or minimazing the amount of nonplanarity and its major design criterion.

Example 3.1

Consider the two electrical circuits as shown in the following Figure.

The circuit (a) is nonplanar but the circuit (b) is planar electrical circuit.

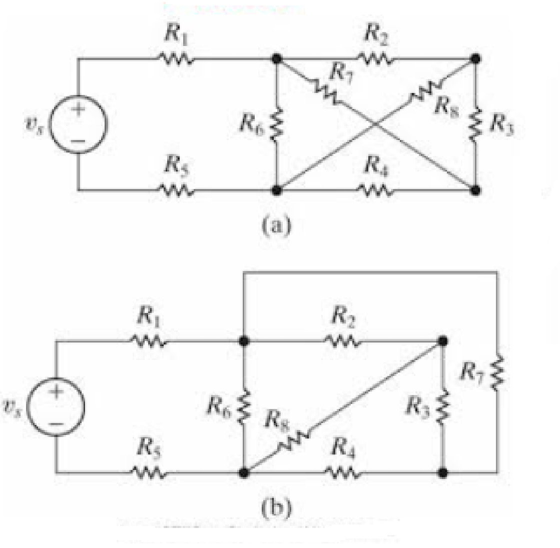

Figure 3.6: planar and nonplanar electrical circuits

3.5 Conclusion

In this project, we present the solutions of some mathematical problems (The number of regular polyhedra, The five colors brobleme, ...) using the concept of planar graphs. In the first chapter, we define the graph and some concepts for it. After that give some types and a very important classes of graph called K_5 and $K_{3.3}$. In the second chapter, we introduce the concept of planar graph and the very important theorem of it which called Euler's Formula and some corolleries about it and geve a characterizations of planar graphs. In the last chapter, we will give the application of Euler's formula to proof that there exist only five regular polyhedra, and solved some mathematics and chemical problems.

Bibliography

[Ref.01] Alan Slomson, An introduction to combinatorics, Chapman and Hall (1991). London. Now York. Tokyo. Melbourne. Madras

[Ref.02] C.L.Liu, Introduction to combinatorial Mathematics, McGraw-Hill, 1968.

[Ref.03] D.P.Acharjya Sveakumar, Fundamental Approach to Discrete Mathematics, New Age International Publishers

[Ref.04] Edward A.Bender and S.Gill Williamson, Foundations of Combinatorics with Applications, Dover in 2006

[Ref.05] Edward R and Scheinerman, Mathematics A Discrete Introduction (Third Edition), Brooks/Cole in 2013

[Ref.06] G.Chartrand and L.Lesniak, Graphs and Diagraphs, Chapman and Hall/CRC(1979). Boca Raton. London. Now York. Washington D.C.

[Ref.07] George E.Martin, Trasformation Geometry An Introduction to Symmetry, Springer-Verlag New York Heidelberg Berlin, 1982.

[Ref.08] Geral Berman and K.D.Fryer, Introduction to Combinatorics, Academic Press, INC. New York. 1972.

[Ref.09] J.A.Bondy and U.S.R.Murty, GRAPH THEORY WITH APPLICATIONS, The Macmillan Press Ltd, 1976.

[Ref.10] Joe L.Mott and Abraham Kandel and Theodore P.Baker, Discret Mathematics for Computer scientists and Mathematicse, Prentice, Hall of India Private Limited in 2008

ملخص

نقدم في هذه المذكرة تطبيقات الرسوم البيانية في تحديد عدد وشكل متعددات السطوح المنتظمة إضافة إلى إستخدامها في حل بعض المشاكل الرياضية كمعضلة الالوان الخمسة.

Abstract

In this project, we present the applications of planar graphs in determining the number and shape of regular polyhedra, in addition to thier use in solving some mathematical problems such as the five-color problem.

Résume

Dans ce mémoire, nous présentons les applications des graphes planaires dans la détermination du nombres et de la forme des polèdres réguliers, ainsi que leur utilisation dans la résolution de certains problémes mathématique.