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Theme

On the summability of nonlinear mappings

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In front of the jury composed of :

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Dedication

To those who planted in me the love of knowledge and learning, and whose prayers lit my path... To my beloved parents, the source of love,strength,and sacrifice. Without the grace of God and their unwavering support, I would not be where I am today. To my wife who believed in me, supported me and also my daughters **Miral** and **Abrar** To my brothers and sisters, for their constant encouragement and comforting presence throughout every stage of this journey To everyone who played a role in this achievement...

Contents

Notaions

$\mathbb{K}$	The field of real or complex numbers.
p'	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p'} = 1$
E^*	The topological dual of E.
B_E	The closed unit ball of E
$L(E;F)$	The set of all linear operators.
$\mathcal{I}$	The ideal of all linear operator.
$\mathcal{K}$	The set of all compact linear operators.
$\mathcal{W}$	The set of all weakly compact linear operators.
$\mathcal{L}_f$	The set of all finite rank linear operators.
$\Pi_p(X,Y)$	The ideal linear p-summing
$\mathcal{I}_{Lip}$	The ideal of Lipschitz operators.
Lip_0	The set of all Lipschitz operators that vanish at 0.
$X^\#$	The Lipschitz dual of the pointed metric space X.
$\mathcal{M}(X)$	The linear space of all molecules on the metric space X.
$m_{xx'}$	The molecule defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$ for $x, x' \in X$, where χ_A is the characteristic function of the set A.
Æ(X)	The Arens-Eells space of X.
T^*	The adjoint operator of T.
$T^\#$	The Lipschitz adjoint operator of T.
T_L	The linearization of the operator T.
$\mathcal{L}_f$	The set of all finite rank linear operators.
Lip_{0F}	The set of Lipschitz finite rank operators.
Π_p	The set of all linear p-summing operators $(1 \leq p < \infty)$.
Π_p^L	The set of all Lipschitz p-summing operator $(1 \leq p < \infty)$.
$\mathcal{H}(U,F)$	The space of all holomorphic mappings from U into F.
$\mathcal{H}_v^\infty(U,F)$	The space of weighted holomorphic mappings
$\Pi_p^{\mathcal{H}_v^\infty}(U,F)$	The set of all p-summing weighted holomorphic mappings from U into F $(1 \leq p \leq \infty)$.

Introduction

The concept of p-summing operators—first studied by Grothendieck for $p = 1$ [16] and later extended by Pietsch to all $p > 0$ [24]—has profoundly shaped Banach space theory (see, e.g., [11, 12, 26]). Its success has spurred analogous notions of p-summability in various nonlinear settings: multilinear and polynomial mappings [1, 2, 13, 22], Lipschitz functions [14, 28], and holomorphic mappings [18, 21].

In this memory , we concentrate on two leading examples of such nonlinear operator ideals:

The first one is the classes of **Lipschitz p-summing operators**, was introduced by Farmer and jhonson for $1 \leq p < \infty$ (see [14]) and Manaf Adnan Saleh Saleh for $0 < p < 1$. Recall that a mapping $T : X \longrightarrow Y$ between two pointed metric spaces is *Lipschitz p-summing* if there exists a constant $C \geq 0$ such that for all $(x_i)_{i\leq n}, (x'_i)_{i\leq n}$ in X and all $(a_i)_{i\leq n} \subset \mathbb{R}^+$

$$\left(\sum_{i=1}^{n} a_i d(T(x_i), T(x'_i))^p\right)^{\frac{1}{p}} \leq C \sup_{f\in B_{X^\#}} \left(\sum_{i=1}^{n} a_i |f(x_i) - f(x'_i)|^p\right)^{\frac{1}{p}}.$$

Farmer and Johnson in [14] proved basic fundamental properties and leave to interested readers a list of open problems (*what results about p-summing operators have analogues for Lipschitz p-summing operators?*). In this part we present tow fundamental theorems, namely Pietsch's Domination Theorem and Pietsch's Factorization Theorem.

The second one is the classes of **p-summing weighted holomorphic mappings**. witch introduced recently by M. G. Cabrera-Padilla, A. Jiménez-Vargas and A. Keten Çopur, Let E and F be complex Banach spaces, and let $U \subset E$ be an open set equipped with a positive continuous weight $v\colon U \to (0, \infty)$. We denote by $\mathcal{H}(U, F)$ the space of all holomorphic maps $U \to F$, the weighted supremum norm

$$\|f\|_v = \sup_{x\in U} v(x)\, \|f(x)\|.$$

The corresponding Banach space of bounded weighted holomorphic functions is

$$\mathcal{H}_v^\infty(U,F) = \{ f \in \mathcal{H}(U,F) : \|f\|_v < \infty \},$$

with the special case $F = \mathbb{C}$ written $\mathcal{H}_v^\infty(U)$. For a detailed overview of these spaces see Bonet's survey [4]. Given $1 \le p \le \infty$, a mapping $f \in \mathcal{H}(U,F)$ is said to be p-summing weighted holomorphic if there is a constant $C \geq 0$ such that

$$\begin{aligned} \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|f(x_i)\|^p\right)^{\frac{1}{p}} &\le C \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, |g(x_i)|^p\right)^{\frac{1}{p}} \qquad \text{if} \quad 1 \le p < \infty, \\ \max_{1\le i\le n} |\lambda_i|\, v(x_i)\, \|f(x_i)\| &\le C \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\max_{1\le i\le n} |\lambda_i|\, v(x_i)\, |g(x_i)|\right) \qquad \text{if} \quad p = \infty, \end{aligned}$$

for any $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. We denote by $\pi_p^{\mathcal{H}_v^\infty}(f)$ the infimum of all constants C satisfying the inequality above, and by $\Pi_p^{\mathcal{H}_v^\infty}(U,F)$ the set of all p-summing weighted holomorphic mappings from U into F.
In this part, we show that $\Pi_p^{\mathcal{H}_v^\infty}(U,F)$ form an injective Banach ideal in the weighted holomorphic category, and we present natural analogues of Pietsch's Domination and Factorization Theorems.

This memory is divided into four chapters as follows ,
The first chapter is a preliminaries which contains some basic concepts and fundamentals of p-Summing Operators
In chapter two we study the ideal of *Lipschitz p-summing.* Domination and Factorization theorems related to this classes are given. The inclusion theorem is also studied.
The third chapter is devoted to study the class p-summing weighted holomorphic mappings. We prove that it is an injective Banach ideal of weighted holomorphic mappings which. Pietsch Domination Theorem, Pietsch Factorization Theorem are presented.
In the last chapter we study the duality of the classes of p-summing weighted holomorphic mappings, under a suitable version $d_{p^*}^{\mathcal{H}_v^\infty}$ of the Chevet–Saphar tensor norms.

Chapter 1

Preliminaries

In this chapter, we present the fundamental theory of p-summing operators based on two key references: the monograph Absolutely Summing Operators by J. Diestel, H. Jarchow, and A. Tonge [12], and A. Pietsch's book Operator Ideals [25].

1.1 Fundamentals of p-Summing Operators

Definition 1.1 *Let $1 \leq p < \infty$. A linear operator $T : E \longrightarrow F$ is said to be p-summing if there exists a constant $C \geq 0$ such that for all finite sequence $(x_i)_{1\leq i\leq n}$ in E*

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*}\leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)^{\frac{1}{p}}. \tag{1.1}$$

The infimum of all such constants $C \geq 0$ is denoted by $\pi_p(T)$. The collection of all p-summing operators between E and F is denoted by $\Pi_p(E, F)$.

Theorem 1.1 *[12, Page 39] If $1 \leq p \leq q < \infty$, then $\Pi_p(E, F) \subset \Pi_q(E, F)$. Moreover, $\pi_q(T) \leq \pi_p(T)$ for every $u \in \Pi_p(E, F)$.*

The following basic result about p-summing operators is due to A. Pietsch, and it characterizes the p-summability by means of a domination theorem.

Theorem 1.2 *(Pietsch Domination Theorem) [12, page 44]*
Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E, F)$. Then T is p-summing if and only if there exist a constant C and a regular Borel probability measure μ on B_{E^*} (with the weak star topology) so that

$$\|T(x)\| \leq C \int_{B_{E^*}} |\langle x, x^*\rangle|^p \, d\mu(x^*), \qquad x \in E. \tag{1.2}$$

In this case, $\pi_p(T)$ is the least of all the constants C such that (1.2) holds.

In order to adapt the previous result into a factorization theorem, we present basic examples of p-summing linear operators.

Example 1.1 *see [12, Example 2.9 (b),(d)]*
(1) Let K be a compact Hausdorff space, let μ be a positive regular Borel measure on K, and let $1 \leq p < \infty$. The canonical inclusion

$$J_p : C(K) \longrightarrow L_p(\mu),$$

is p-summing with $\pi_p(J_p) = \|J_p\| = \mu(K)^{\frac{1}{p}}$.
(2) Let (Ω, Σ, μ) be a finite measure space and let $1 \leq p < \infty$. The formal inclusion map

$$I_{\infty,p} : L_\infty(\mu) \longrightarrow L_p(\mu),$$

is p-summing, with $\pi_p(I_{\infty,p}) = \mu(\Omega)^{\frac{1}{p}}$.

We denote by i_E the isometric embedding $E \longrightarrow C(B_{E^*})$ given by $i_E(x) = \langle x, . \rangle$.

Corollary 1.1 *[12, page 45] (Pietsch Factorization Theorem)*
Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E, F)$. The following are equivalent
(i) T is p-summing.
(ii) There exist a regular Borel probability measure μ on B_{E^*} (with the weak star topology), a closed subspace E_p of $L_p(\mu)$ and a linear continuous operator $\widetilde{u} : E_p \longrightarrow F$ such that $J_p \circ i_E(E) \subset E_p$ and $\widetilde{u} \circ J_p \circ i_E(x) = T(x)$ for all $x \in E$.
In other words, if $\overline{J_p}$ is the map $i_E(E) \longrightarrow E_p$ induced by J_p, then the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[d, "i_E"'] & F \\
i_E(E) \arrow[r, "\overline{J_p}"] \arrow[d, phantom, "\cap"] & E_p \arrow[u, "\widetilde{T}"'] \arrow[d, phantom, "\cap"] \\
C\left(B_{E^*}\right) \arrow[r, "J_p"] & L_p(\mu).
\end{tikzcd}$$

In addition, we may choose μ and $\widetilde{T}$ so that $\left\|\widetilde{T}\right\| = \pi_p(T)$.

1.2 Linear operator ideals

1.2.1 Finite rank operator

Recall that a linear operator $T \in \mathcal{L}(E, F)$ is said to have finite rank if $T(E)$ is a finite dimensional subspace of F. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(E, F)$. An operator has rank one if and only it has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle \, y$$

i.e. if $u \in \mathcal{L}_f(E, F)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$ (see [25, Page 25]).

Remark 1.1 *1. It is not hard to establish that* $T = x^* \otimes y$ *is in* $\Pi_p(X, Y)$,*with* $\pi_p(u) = \|x^*\| \cdot \|y\|$. *clearly* $\|x^*\| \cdot \|y\| = \|T\| \leq \pi_p(T)$, *so we only need to check that* $\pi_p \leq \|x^*\| \cdot \|y\|$. *but this follows from*

$$(\sum_{k=1}^{m} \|T(x)_k\|^p)^{\frac{1}{p}} = \|x^*\| \cdot \|y\| \cdot (\sum_{k=1}^{m} |\langle \frac{x^*}{\|x^*\|}, x_k \rangle|^p)^{\frac{1}{p}} \leq \|x^*\| \cdot \|y\| \cdot \|(x^k)_1^m\|_p^{weak}$$

which is valide for all choices of finitely many vectors $x_1, ..., x_m$ *from* X

2. Let $T \in \mathcal{L}(X, Y)$ *have finite rank. then* T *is* p*-summing for every* $1 \leq p < \infty$
To see why this is so, take $y_1, ...y_n$ *to be a basis for* $T(x)$. *then we can find* $x_1^*, ...x_n^* \in X$ *with* $T(x) = \sum_{k=1}^{n} \langle x_k^*, x \rangle y_k$ *for all* $x \in X$ *this exhibits* T *as a sum of rank one operators and so as a member of the vector space* $\Pi_p(X, Y)$

1.2.2 Linear operator ideals

Definition 1.2 *An operator ideal* $\mathcal{I}$ *is a subclass of the class* $\mathcal{L}$ *of all continuous linear operators between Banach spaces such that for all Banach spaces* E *and* F *its components* $\mathcal{I}(E, F) := \mathcal{L}(E, F) \cap \mathcal{I}$ *satisfy:*

(i) $\mathcal{I}(E, F)$ *is a linear subspace of* $\mathcal{L}(E, F)$ *which contains the finite rank operators.*

(ii) The ideal property: if $v \in \mathcal{L}(G, E)$*,* $u \in \mathcal{I}(E, F)$ *and* $w \in \mathcal{L}(F, H)$*, then the composition* $w \circ v \circ u$ *is in* $\mathcal{I}(G, H)$*.*
If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^{+}$ *satisfies:*
(i') $(\mathcal{I}(E, F), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F,
(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$*,*
(iii') If $v \in \mathcal{L}(G, E)$*,* $u \in \mathcal{I}(E, F)$ *and* $w \in \mathcal{L}(F, H)$*,*

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\| \, ,$$

then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ *is called a normed (Banach) operator ideal.*

The operator ideal $\mathcal{I}$ is said to be closed if each $\mathcal{I}(E, F)$ is a closed subspace of $\mathcal{L}(E, F)$ for the sup norm.

Definition 1.3 *(injective operator ideal)*
A normed operator ideal $(\mathcal{I}, \|.\|_{\mathcal{I}})$ *is said to be injective if for every metric injection* $i : F \hookrightarrow G$ *and every* $u \in \mathcal{L}(E, F)$ *it follows from* $i \circ u \in \mathcal{I}(E, G)$ *that* $u \in \mathcal{I}(E, F)$*. Moreover*

$$\|i \circ u\|_{\mathcal{I}} = \|u\|_{\mathcal{I}} \, ,$$

The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one [25, Theorem 1.2.2].

Proposition 1.1 ***(Ideal proprty)*** *Let* $1 \leq p < \infty$ *and let* $v \in \Pi_p(X, Y)$*. then the composition of* v *with any bounded linear operator is p-summing. More spesifically, if* X_0 *and* Y_0 *are banach space then, regardless of how we choose* $u \in \mathcal{L}(X, Y)$ *and* $w \in \mathcal{L}(X_0, X)$*, we always have* $uvw \in \Pi_p(X_0, Y_0)$ *with* $\pi_p(uvw) \leq \|u\| \cdot \pi_p(v) \cdot \|w\|$*.*

Remark 1.2 *If* X_0 *is a subspace of* X *and* $v : X \to Y$ *is p-summing, then the restriction map* $v|_{X_0} : X \to Y$ *is also p-summing, with* $\pi_p(v|_{X_0}) \leq \pi_p(v)$*. This follows from the ideal property when we take* $u : X_0 \to X$ *to be the inclusion map and set* w *to be the identity operator on* Y*.*

Proposition 1.2 ***(Injectivity of*** Π_p***:)*** *If* $i : Y \to Y_0$ *is isometric, then* $v \in \Pi_p(X, Y)$ *if and only if* $iv \in \Pi_p(X, Y_0)$ *in this case, we even have* $\pi_p(iv) = \pi_p(v)$

Proposition 1.3 (Π_p, π_p) *is an injective Banach operator ideal.*

1.3 Ideal of compact and weakly compact linear operators.

We say that a bounded linear operator $T : E \to F$ is compact (weakly compact) if $T(B_E)$ is relatively compact (respectively, relatively weakly compact)in F. By $\mathcal{K}(E,F)$ and $\mathcal{W}(E,F)$ we denote the sets of compact linear operators and weakly compact linear operators from E to F, respectively.

Schauder theorem about the compactness of the adjoint of a compact linear operator between two Banach spaces is well known. We will remind it as we need in the sequel of this thesis.

Theorem 1.3 *Let $T \in \mathcal{L}(E,F)$. Then T is compact if and only if T^* is compact.*

The weakly version is due to Gantmacher:

Theorem 1.4 *Let $T \in \mathcal{L}(E,F)$. Then T is weakly compact if and only if T^* is weakly compact.*

Proposition 1.4 *([25]) The classes $\mathcal{K}, \mathcal{W}$ consitute closed injective Banach operator ideals, where the ideal norm is the operator norm.*

Theorem 1.5 *Let $1 \leq p < \infty$.*

(1) Every p-summing operator between Banach spaces is weakly compact.

(2) If E is a reflexive space, then every p-summing operator between E and F is compact.

Chapter 2

Lipschitz p-summing operators

2.1 Metric space

Definition 2.1 *Recall that a metric or distance on a non empty set X is a function*

$$d : X \times X \longrightarrow \mathbb{R}_+$$

with the following properties:

1. (i) *(Positivity) For all* $x, y \in X$, $d(x, y) \geq 0$ *with equality if and only if* $x = y$.
2. (ii) *(Symmetry) For all* $x, y \in X$, $d(x, y) = d(y, x)$.
3. (iii) *(Triangle inequality) For all* $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

The set X equipped with the distance d *is called a metric space.*

2.2 Lipschitz space (Lip)

In the category of metric spaces, the natural morphisms are **Lipschitz functions**.

Definition 2.2 *A function* $T : X \to Y$ *between two metric spaces* (X, d_X) *and* (Y, d_Y) *is called* ***Lipschitz*** *if there exists a constant* $C > 0$ *such that:*

$$\forall x, y \in X, \quad d_Y(T(x), T(y)) \leq C\, d_X(x, y). \tag{2.1}$$

The smallest such constant C is called the **Lipschitz constant** of T, denoted $\mathrm{Lip}(T)$, and is given by:

$$\mathrm{Lip}(T) = \sup \left\{ \frac{d_Y(T(x), T(y))}{d_X(x, y)} \;\middle|\; x, y \in X,\, x \neq y \right\}.$$

2.3 The space Lip_0

Recall that a pointed metric space X is a metric space with a bas point in X denoted by 0.

Definition 2.3 *The Lipschitz space $Lip_0(X, E)$ is the Banach space of all Lipschitz mappings T from the pointed metric space X to the Banach space E that vanish at 0, under the Lipschitz norm given by*

$$Lip\,(T) := \sup_{x \neq y} \frac{\|T(x) - T(x')\|}{d(x, x')} \tag{2.1}$$

For $E = \mathbb{K}$, we designate $X^{\#} = Lip_0(X, \mathbb{K})$.

Example 2.1 *Let $[0, 1]$ be equipped with the usual metric, and let the distinguished element be $e = 0$. For every $f \in L^1[0, 1]$, we define:*

$$F(t) = \int_0^t f(s)\,ds.$$

For all $a, b \in [0, 1]$, with $a \leq b$, we have:

$$|F(b) - F(a)| = \left| \int_a^b f(s)\,ds \right| \leq \|f\|_1 (b - a),$$

where $\|f\|_1$ is the L^1-norm of f.
Thus, F is Lipschitz, and:

$$Lip(F) \leq \|f\|_1.$$

Moreover, since $F(0) = 0$, it follows that $F \in Lip_0[0, 1]$.

2.4 Molecule space

Now we are going to present some concepts about the space of molecules, the reader can see [29] or [15] for more details.

Definition 2.4 *A molecule on X is a scalar valued function m on X with finite support that satisfies*

$$\sum_{x \in X} m\,(x) = 0.$$

By $\mathcal{M}(X)$ we denote the linear space of all molecules on X.

1. For $x, x' \in X$ the molecule $m_{xx'}$ is defined by

$$m_{xx'} = 1_{\{x\}} - 1_{\{x'\}},$$

where 1_A is the characteristic function of the set A.

2. For $m \in \mathcal{M}(X)$ we write

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^{n} |\lambda_j| \, d(x_j, x'_j), \; m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j} \right\},$$

where the infimum is taken over all representations of the molecule m.

2.5 The predual of $X^{\#}$

Definition 2.5 *A Banach space Z has a predual if there is a Banach space E and an isometric isomorphic operator $S : Z \longrightarrow E^*$, in this case we say that E is the predual of Z and Z is the dual of E.*

Theorem 2.1 *The map $Q_X : X^{\#} \longrightarrow Æ(X)^*$ defined by*

$$Q_X(f) = f_L, \text{ where } f_L(m) = \sum_{x \in X} f(x) m(x),$$

establish an isometric isomorphic between $X^{\#}$ and $Æ(X)^*$. (see [29, Theorem 2.2.2]).

2.6 The linearization of a Lipschitz operator

The Banach space $Æ(X)$ has some remarkable properties. We mention the following ones.

Theorem 2.2 *([29, Theorem 2.2.4])*
Let X be a pointed metric space and E be a Banach space and let $T : X \longrightarrow E$ be a Lipschitz map which preserves the base point; that is $T(0) = 0$. Then there is a unique bounded linear map T_L :

Æ$(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$ that is, the diagram

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "\delta_X"'] & & E, \\
& Æ(X)\,, \arrow[ur, "T_L"'] &
\end{tikzcd}$$

commutes. Furthermore $\|T_L\| = Lip(T)$

The operator T_L is referred to as the linearization of T. The correspondence

$$T \longleftrightarrow T_L$$

establishes an isomorphism between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}(Æ\,(X)\,, E)$.

Theorem 2.3 *([9, Lemma 3.1])*
Let X,Y two pointed metric spaces and let $T : X \longrightarrow Y$ be a Lipschitz map which preserves the base point. Then there is a unique bounded linear map $\hat{T}$:Æ$(X) \longrightarrow$Æ(Y) such that $\widehat{T}\delta_X = \delta_Y T$ that is, the diagram

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "\delta_X"] & Y \arrow[d, "\delta_Y"] \\
Æ(X) \arrow[r, "\hat{T}"] & Æ(Y)
\end{tikzcd}$$

commutes. Furthermore $\|\hat{T}\| = Lip(T)$.

Sawashima [27] defined the Lipschitz adjoint (or dual) of $T \in Lip_0(X, Y)$ as the continuous linear operator

$$\begin{aligned} T^\# : Y^\# &\longrightarrow \quad X^\# \\ g &\longmapsto T^\#(g) = g \circ T \end{aligned}$$

2.7 Lipschitz p-summing operators

This section based on the article of J. D. Farmer and W. B. Johnson, Lipschitz p-summing operators (see[14]).

Definition 2.6 *A mapping $T \in Lip_0(X, Y)$ is Lipschitz p-summing if there exists a constant*

$C \geq 0$ *such that for all* $(x_i)_{i\leq n}, (x'_i)_{i\leq n}$ *in* X *and all* $(a_i)_{i\leq n} \subset \mathbb{R}^+$

$$\sum_{i=1}^{n} a_i d(T(x_i), T(x'_i))^p \leq C^p \sup_{f \in B_{X^\#}} \sum_{i=1}^{n} a_i |f(x_i) - f(x'_i)|^p.$$

The infimum of all such constants $C \geq 0$ *is denoted by* $\pi_p^L(T)$. *This class of mappings is denoted by* $\Pi_p^L(X, Y)$. *Thanks to an argument detailed in [14] the scalars* $a_1, \ldots, a_n$ *can be removed from the definition.*

2.7.1 Pietsch factorization theorem

The Theorem characterizing the Lipschitz p*-summing operators is the following.*

Theorem 2.4 *Let* $1 \leq p < \infty$. *The following properties are equivalent for a mapping* $T \in Lip_0(X, Y)$ *and a positive constant* C.

1. $\pi_p^L(T) \leq C$.

2. *There is a probability* μ *on* $B_{X^\#}$ *such that*

$$d(T(x), T(y)) \leq C \left(\int_{B_{X^\#}} |f(x) - f(y)|^p \, d\mu(f) \right)^{\frac{1}{p}}.$$

2.7.2 Pietsch domination theorem

Theorem 2.5 1. $T \in \Pi_p^L(X, Y)$

2. *For some (or any) isometric embedding* J *of* Y *into a* 1*-injective space* Z, *there is a factorization*

$$\begin{tikzcd}
L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "B"] \\
X \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & Z
\end{tikzcd}$$

with μ *a probability and* $Lip(A) \cdot Lip(B) \leq C$.

(Pietsch factorization theorem).

The domination theorem immediately implies the monotonicity of the Lipschitz $p-$*summing norm. Then we have the following theorem.*

Theorem 2.6 *If* $1 \leq p \leq q < \infty$, *then* $\Pi_p^L(X,Y) \subset \Pi_q^L(X,Y)$. *Moreover,* $\pi_p^L(T) \leq \pi_q^L(T)$.

For a linear operator $T \in \mathcal{L}(E,F)$ *it is clear that* $\pi_p^L(T) \leq \pi_p(T)$. *J.D. Farmer and W.B. Johnson proved that the reverse inequality is true. This justifies that the notion of Lipschitz* p*-summing operator is really a generalization of the concept of linear* p*-summing operator.*

Theorem 2.7 *[14, Theorem 2] Let* T *be a bounded linear operator from* E *into* F *and* $1 \leq p < \infty$. *Then* $\pi_p^L(T) = \pi_p(T)$

Chapter 3

p-Summing weighted holomorphic mappings

This chapter based on the paper: p-Summing weighted holomorphic mappings introduced recently by M. G. Cabrera-Padilla, A. Jiménez-Vargas and A. Keten Çopur (see [7])

3.1 Holomorphic mappings

In this section we give some definitions and properties related to Holomorphic mappings.

Definition 3.1 *Let* $f : \Omega \longrightarrow \mathbb{C}$ *be a function defined on a non-empty open set* Ω.

1. *We say that* f *is differentiable at* $z \in \Omega$ *if there exists* $l \in \mathbb{C}$ *such that*

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}^*}} \frac{f(z+h) - f(z)}{h} = l.$$

   We denote $l = f'(z)$ *and call it the derivative of* f *at* z.

2. *We say that* f *is holomorphic on* Ω *if* f *is differentiable at every point of* Ω. *We denote by* $H(\Omega)$ *the set of holomorphic functions on* Ω.

Example 3.1

1. *The function* $f(z) = z^n$ *is holomorphic on* $\mathbb{C}$, *for every* $n \in \mathbb{N}$.
2. *The function* $f(z) = \bar{z}$ *is not differentiable at any point of* $\mathbb{C}$ *because*

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \bar{z}}{h} = \frac{\bar{h}}{h} = \frac{\cos(-\theta_h) + i\sin(-\theta_h)}{\cos(\theta_h) + i\sin(\theta_h)}$$

   and the limit $\lim_{h\to 0} \frac{\bar{h}}{h}$ *does not exist (as the value depends on the direction of approach).*

Notation: *Let* E *and* F *be complex Banach spaces and let* U *be an open subset of* E. *Let* $\mathcal{H}(U, F)$ *be the space of all holomorphic mappings from* U *into* F.

3.2 The ideal of weighted holomorphic mappings

Definition 3.2 *A weight v on U is a (strictly) positive continuous function. The space of weighted holomorphic mappings, denoted by $\mathcal{H}_v^\infty(U,F)$, is the Banach space of all mappings $f \in \mathcal{H}(U,F)$ such that*

$$\|f\|_v := \sup\{v(x)\|f(x)\| : x \in U\} < \infty,$$

under the weighted supremum norm $\|\cdot\|_v$. In particular, $\mathcal{H}_v^\infty(U) := \mathcal{H}_v^\infty(U,\mathbb{C})$.

By $\mathcal{G}_v^\infty(U)$ we denote the space of all linear functionals on $\mathcal{H}_v^\infty(U)$ whose restriction to $B_{\mathcal{H}_v^\infty(U)}$ is continuous for the compact-open topology see [3, 5, 6],.

Theorem 3.1 *[3, 5, 6] Let U be an open set of a complex Banach space E and let v be a weight on U.*

1. *$\mathcal{G}_v^\infty(U)$ is a closed subspace of $\mathcal{H}_v^\infty(U)^*$, and the mapping $J_v\colon \mathcal{H}_v^\infty(U) \to \mathcal{G}_v^\infty(U)^*$, given by $J_v(g)(\phi) = \phi(g)$ for $\phi \in \mathcal{G}_v^\infty(U)$ and $g \in \mathcal{H}_v^\infty(U)$, is an isometric isomorphism.*
2. *For each $x \in U$, the functional $\delta_x\colon \mathcal{H}_v^\infty(U) \to \mathbb{C}$, defined by $\delta_x(f) = f(x)$ for $f \in \mathcal{H}_v^\infty(U)$, is in $\mathcal{G}_v^\infty(U)$, and there exists $g_x \in B_{\mathcal{H}_v^\infty(U)}$ such that $g_x(x) = \|\delta_x\| := \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} |g(x)|$.*
3. *The mapping $\Delta_v\colon U \to \mathcal{G}_v^\infty(U)$ given by $\Delta_v(x) = \delta_x$ is in $\mathcal{H}_v^\infty(U,\mathcal{G}_v^\infty(U))$ with $\|\Delta_v\|_v \leq 1$.*
4. *For every complex Banach space F and every mapping $f \in \mathcal{H}_v^\infty(U,F)$, there exists a unique operator $T_f \in \mathcal{L}(\mathcal{G}_v^\infty(U),F)$ such that $T_f \circ \Delta_v = f$. Furthermore, $\|T_f\| = \|f\|_v$.*
5. *The correspondence $f \mapsto T_f$ is an isometric isomorphism from $\mathcal{H}_v^\infty(U,F)$ onto $\mathcal{L}(\mathcal{G}_v^\infty(U),F)$*

3.3 p-Summing weighted holomorphic mappings

The concept of p-Summing weighted holomorphic mappings introduced recently by M. G. Cabrera-Padilla, A. Jiménez-Vargas and A. Keten Çopur as a generalization of linear p-Summing mappings to the non linear case

Definition 3.3 *Given* $1 \leq p \leq \infty$, *a mapping* $f \in \mathcal{H}(U,F)$ *is said to be* p*-summing weighted holomorphic if there is a constant* $C \geq 0$ *such that*

$$\left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \, \|f(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \, |g(x_i)|^p\right)^{\frac{1}{p}} \qquad \text{if} \quad 1 \leq p < \infty,$$

$$\max_{1 \leq i \leq n} |\lambda_i| \, v(x_i) \, \|f(x_i)\| \leq C \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\max_{1 \leq i \leq n} |\lambda_i| \, v(x_i) \, |g(x_i)|\right) \qquad \text{if} \quad p = \infty,$$

for any $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ *and* $x_1, \ldots, x_n \in U$. *We denote by* $\pi_p^{\mathcal{H}_v^\infty}(f)$ *the infimum of all constants* C *satisfying the inequality above, and by* $\Pi_p^{\mathcal{H}_v^\infty}(U,F)$ *the set of all* p*-summing weighted holomorphic mappings from* U *into* F.

Remark 3.1 $\Pi_p^{\mathcal{H}_v^\infty}(U,F) \subseteq \mathcal{H}_v^\infty(U,F)$ *with* $\|f\|_v \leq \pi_p^{\mathcal{H}_v^\infty}(f)$ *for all* $f \in \Pi_p^{\mathcal{H}_v^\infty}(U,F)$. *Indeed, for such a function* f, *note that*

$$v(x)\|f(x)\| = (v(x)^p \|f(x)\|^p)^{\frac{1}{p}} \leq \pi_p^{\mathcal{H}_v^\infty}(f) \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} (v(x)^p \, |g(x)|^p)^{\frac{1}{p}} \leq \pi_p^{\mathcal{H}_v^\infty}(f)$$

for all $x \in U$.

Definition 3.4 *[8] A Banach ideal of weighted holomorphic mappings (in short, a Banach weighted holomorphic ideal) is an assignment* $\left[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right]$ *which associates with every pair* (U,F), *where* U *is an open subset of a complex Banach space* E *and* F *is a complex Banach space, a set* $\mathcal{I}^{\mathcal{H}_v^\infty}(U,F) \subseteq \mathcal{H}_v^\infty(U,F)$ *and a function* $\|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} : \mathcal{I}^{\mathcal{H}_v^\infty}(U,F) \to \mathbb{R}_0^+$ *satisfying the properties:*

(P1) $\left(\mathcal{I}^{\mathcal{H}_v^\infty}(U,F), \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}\right)$ *is a Banach space with* $\|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \geq \|f\|_v$ *for all* $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$.

(P2) For any $h \in \mathcal{H}_v^\infty(U)$ *and* $y \in F$, *the map* $h \cdot y \colon x \mapsto h(x)y$ *from* U *to* F *is in* $\mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$ *with* $\|h \cdot y\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|h\|_v \|y\|$.

(P3) The ideal property: if V *is an open subset of* E *such that* $V \subseteq U$, $h \in \mathcal{H}(V,U)$ *with* $c_v(h) := \sup_{x \in V}(v(x)/v(h(x))) < \infty$, $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$ *and* $T \in \mathcal{L}(F,G)$, *where* G *is a complex Banach space, then* $T \circ f \circ h \in \mathcal{I}^{\mathcal{H}_v^\infty}(V,G)$ *with* $\|T \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} \leq \|T\| \|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} c_v(h)$.

A Banach weighted holomorphic ideal $[\mathcal{I}^{\mathcal{H}_v^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}]$ *is called:*

(I) Injective if for any mapping $f \in \mathcal{H}_v^\infty(U,F)$, *any complex Banach space* G *and any isometric linear embedding* $\iota \colon F \to G$, *we have that* $f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,F)$ *with* $\|f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}} = \|\iota \circ f\|_{\mathcal{I}^{\mathcal{H}_v^\infty}}$ *whenever* $\iota \circ f \in \mathcal{I}^{\mathcal{H}_v^\infty}(U,G)$.

Proposition 3.1 $[\Pi_p^{\mathcal{H}_v^\infty}, \pi_p^{\mathcal{H}_v^\infty}]$ *is an injective Banach weighted holomorphic ideal.*

Proof 3.1 *We will prove the case* $1 \leq p < \infty$*, and for* $p = \infty$ *a similar proof works.*
(P1) Let $n \in \mathbb{N}$*,* $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ *and* $x_1, \ldots, x_n \in U$*. If* $f \in \Pi_p^{\mathcal{H}_v^\infty}(U, F)$ *and* $\pi_p^{\mathcal{H}_v^\infty}(f) = 0$*, then* $\|f\|_v = 0$ *by Remark 3.1, and so* $f = 0$*. Given* $f_1, f_2 \in \Pi_p^{\mathcal{H}_v^\infty}(U, F)$*, one has*

$$\begin{aligned}\left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|(f_1+f_2)(x_i)\|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^n \left(|\lambda_i|\, v(x_i)\, \|f_1(x_i)\| + |\lambda_i|\, v(x_i)\, \|f_2(x_i)\|\right)^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p \|f_1(x_i)\|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p \|f_2(x_i)\|^p\right)^{\frac{1}{p}} \\ &\leq \left(\pi_p^{\mathcal{H}_v^\infty}(f_1) + \pi_p^{\mathcal{H}_v^\infty}(f_2)\right) \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, |g(x_i)|^p\right)^{\frac{1}{p}},\end{aligned}$$

and thus $f_1 + f_2 \in \Pi_p^{\mathcal{H}_v^\infty}(U, F)$ *with* $\pi_p^{\mathcal{H}_v^\infty}(f_1+f_2) \leq \pi_p^{\mathcal{H}_v^\infty}(f_1) + \pi_p^{\mathcal{H}_v^\infty}(f_2)$*.*
Let $\lambda \in \mathbb{C}$ *and* $f \in \Pi_p^{\mathcal{H}_v^\infty}(U, F)$*. An easy calculation yields*

$$\begin{aligned}\left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|(\lambda f)(x_i)\|^p\right)^{\frac{1}{p}} &= |\lambda| \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|f(x_i)\|^p\right)^{\frac{1}{p}} \\ &\leq |\lambda|\, \pi_p^{\mathcal{H}_v^\infty}(f) \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, |g(x_i)|^p\right)^{\frac{1}{p}},\end{aligned}$$

and so $\lambda f \in \Pi_p^{\mathcal{H}_v^\infty}(U, F)$ *with* $\pi_p^{\mathcal{H}_v^\infty}(\lambda f) \leq |\lambda| \pi_p^{\mathcal{H}_v^\infty}(f)$*. Hence* $\pi_p^{\mathcal{H}_v^\infty}(\lambda f) = 0 = |\lambda|\, \pi_p^{\mathcal{H}_v^\infty}(f)$ *if* $\lambda = 0$*. For* $\lambda \neq 0$*, it is clear that* $\pi_p^{\mathcal{H}_v^\infty}(f) = \pi_p^{\mathcal{H}_v^\infty}(\lambda^{-1}(\lambda f)) \leq |\lambda|^{-1}\, \pi_p^{\mathcal{H}_v^\infty}(\lambda f)$*, hence* $|\lambda|\, \pi_p^{\mathcal{H}_v^\infty}(f) \leq \pi_p^{\mathcal{H}_v^\infty}(\lambda f)$*, and, consequently,* $\pi_p^{\mathcal{H}_v^\infty}(\lambda f) = |\lambda|\, \pi_p^{\mathcal{H}_v^\infty}(f)$*. In this way,* $\left(\Pi_p^{\mathcal{H}_v^\infty}(U, F), \pi_p^{\mathcal{H}_v^\infty}\right)$ *is a normed space.*
To show that it is a Banach space, let $(f_n)_{n\geq 1}$ *be a sequence in* $\Pi_p^{\mathcal{H}_v^\infty}(U, F)$ *such that* $\sum_n \pi_p^{\mathcal{H}_v^\infty}(f_n)$ *converges. Since* $\|f_n\|_v \leq \pi_p^{\mathcal{H}_v^\infty}(f_n)$ *for all* $n \in \mathbb{N}$ *and* $(\mathcal{H}_v^\infty(U, F), \|\cdot\|_v)$ *is a Banach space, then* $\sum_n f_n$ *converges in* $(\mathcal{H}_v^\infty(U, F), \|\cdot\|_v)$ *to a function* $f \in \mathcal{H}_v^\infty(U, F)$*. Given* $m \in \mathbb{N}$*,* $x_1, \ldots, x_m \in U$

and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$, *we have*

$$\begin{aligned}\left(\sum_{k=1}^{m} |\lambda_k|^p v(x_k)^p \left\|\sum_{i=1}^{n} f_i(x_k)\right\|^p\right)^{\frac{1}{p}} &\leq \pi_p^{\mathcal{H}_v^\infty}\left(\sum_{i=1}^{n} f_i\right) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{k=1}^{m} |\lambda_k|^p v(x_k)^p |g(x_k)|^p\right)^{\frac{1}{p}} \\ &\leq \sum_{i=1}^{n} \pi_p^{\mathcal{H}_v^\infty}(f_i) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{k=1}^{m} |\lambda_k|^p v(x_k)^p |g(x_k)|^p\right)^{\frac{1}{p}}\end{aligned}$$

for all $n \in \mathbb{N}$. *Taking limits with* $n \to \infty$, *we deduce*

$$\left(\sum_{k=1}^{m} |\lambda_k|^p v(x_k)^p \left\|\sum_{i=1}^{\infty} f_i(x_k)\right\|^p\right)^{\frac{1}{p}} \leq \sum_{i=1}^{\infty} \pi_p^{\mathcal{H}_v^\infty}(f_i) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{k=1}^{m} |\lambda_k|^p v(x_k)^p |g(x_k)|^p\right)^{\frac{1}{p}},$$

using that, for any $x \in U$, *one has*

$$\left\|\sum_{i=1}^{n} v(x)f_i(x) - v(x)f(x)\right\| \leq \left\|\sum_{i=1}^{n} f_i - f\right\|_v$$

for all $n \in \mathbb{N}$. *Hence* $f \in \Pi_p^{\mathcal{H}_v^\infty}(U,F)$ *with* $\pi_p^{\mathcal{H}_v^\infty}(f) \leq \sum_{n=1}^{\infty} \pi_p^{\mathcal{H}_v^\infty}(f_n)$. *Furthermore,*

$$\pi_p^{\mathcal{H}_v^\infty}\left(f - \sum_{i=1}^{n} f_i\right) = \pi_p^{\mathcal{H}_v^\infty}\left(\sum_{i=n+1}^{\infty} f_i\right) \leq \sum_{i=n+1}^{\infty} \pi_p^{\mathcal{H}_v^\infty}(f_i)$$

for all $n \in \mathbb{N}$, *and so* f *is the* $\pi_p^{\mathcal{H}_v^\infty}$*-limit of the series* $\sum_n f_n$. *This proves the completeness of* $\pi_p^{\mathcal{H}_v^\infty}$.

(P2) Let $h \in \mathcal{H}_v^\infty(U)$ *and* $y \in F$. *Note that* $h \cdot y \in \mathcal{H}_v^\infty(U,F)$ *with* $\|h \cdot y\|_v = \|h\|_v \|y\|$. *We can suppose* $h \neq 0$ *and obtain*

$$\begin{aligned}\left(\sum_{i=1}^{n} |\lambda_i|^p v(x_i)^p \|(h\cdot y)(x_i)\|^p\right)^{\frac{1}{p}} &= \|y\| \|h\|_v \left(\sum_{i=1}^{n} |\lambda_i|^p v(x_i)^p \left|\left(\frac{h}{\|h\|_v}\right)(x_i)\right|^p\right)^{\frac{1}{p}} \\ &\leq \|y\| \|h\|_v \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_i|^p v(x_i)^p |g(x_i)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Thus $h \cdot y \in \Pi_p^{\mathcal{H}_v^\infty}(U,F)$ *with* $\pi_p^{\mathcal{H}_v^\infty}(h \cdot y) \leq \|h\|_v \|y\|$. *Conversely, note that* $\|h\|_v\|y\| \leq \pi_p^{\mathcal{H}_v^\infty}(h \cdot y)$

by Remark 3.1.

(P3) Let V *be an open subset of* E *such that* $V \subseteq U$, $h \in \mathcal{H}(V,U)$ *with* $c_v(h) < \infty$, $f \in \Pi_p^{\mathcal{H}_v^\infty}(U,F)$ *and* $T \in \mathcal{L}(F,G)$. *Observe that* $T \circ f \circ h \in \mathcal{H}_v^\infty(V,G)$. *For any* $n \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ *and* $x_1,\ldots,x_n \in V$, *we obtain*

$$\left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|(T\circ f\circ h)(x_i)\|^p\right)^{\frac{1}{p}} \leq \|T\| \left(\sum_{i=1}^n |\lambda_i|^p \frac{v(x_i)^p}{v(h(x_i))^p} v(h(x_i))^p \|f(h(x_i))\|^p\right)^{\frac{1}{p}}$$

$$\leq \|T\| \pi_p^{\mathcal{H}_v^\infty}(f) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p \frac{v(x_i)^p}{v(h(x_i))^p} v(h(x_i))^p |g(h(x_i))|^p\right)^{\frac{1}{p}}$$

$$= \|T\| \pi_p^{\mathcal{H}_v^\infty}(f) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p v(x_i)^p |(g\circ h)(x_i)|^p\right)^{\frac{1}{p}}$$

$$\leq \|T\| \pi_p^{\mathcal{H}_v^\infty}(f) c_v(h) \sup_{0\neq g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p v(x_i)^p ||g||_v^p \left| \left(\frac{g\circ h}{c_v(h)||g||_v}\right)(x_i)\right|^p\right)^{\frac{1}{p}}$$

$$\leq \|T\| \pi_p^{\mathcal{H}_v^\infty}(f) c_v(h) \sup_{g_0\in B_{\mathcal{H}_v^\infty(V)}} \left(\sum_{i=1}^n |\lambda_i|^p v(x_i)^p |g_0(x_i)|^p\right)^{\frac{1}{p}},$$

since $g \circ h \in \mathcal{H}_v^\infty(V)$ *with* $\|g\circ h\|_v \leq c_v(h)\|g\|_v$. *So,* $T\circ f\circ h \in \Pi_p^{\mathcal{H}_v^\infty}(V,G)$ *and* $\pi_p^{\mathcal{H}_v^\infty}(T\circ f\circ h) \leq \|T\|\, \pi_p^{\mathcal{H}_v^\infty}(f) c_v(h)$.

(I) Let $f \in \mathcal{H}_v^\infty(U,F)$ *and* $\iota\colon F \to G$ *be an into linear isometry such that* $\iota\circ f \in \Pi_p^{\mathcal{H}_v^\infty}(U,G)$. *Given* $n \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ *and* $x_1,\ldots,x_n \in U$, *we have*

$$\left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|f(x_i)\|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|(\iota\circ f)(x_i))\|^p\right)^{\frac{1}{p}}$$

$$\leq \pi_p^{\mathcal{H}_v^\infty}(\iota\circ f) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, |g(x_i)|^p\right)^{\frac{1}{p}},$$

and so $f \in \Pi_p^{\mathcal{H}_v^\infty}(U,F)$ *with* $\pi_p^{\mathcal{H}_v^\infty}(f) \leq \pi_p^{\mathcal{H}_v^\infty}(\iota\circ f)$. *The opposite inequality follows from (P3).*

3.4 Composition weighted holomorphic ideal

Definition 3.5 *[8] Given an operator ideal* $\mathcal{I}$*, a mapping* $f \in \mathcal{H}_v^\infty(U,F)$ *belongs to the composition ideal* $\mathcal{I} \circ \mathcal{H}_v^\infty$*, and we write* $f \in \mathcal{I} \circ \mathcal{H}_v^\infty(U,F)$*, if there are a complex Banach space* G*, an operator* $T \in \mathcal{I}(G,F)$ *and a mapping* $g \in \mathcal{H}_v^\infty(U,G)$ *such that* $f = T \circ g$*.*

Proposition 3.2 *For* $1 \leq p < \infty$*, the Banach ideal of* p*-summing weighted holomorphic mappings* $\Pi_p^{\mathcal{H}_v^\infty}$ *and the corresponding composition ideal* $\Pi_p \circ \mathcal{H}_v^\infty$ *do not coincide.*

Proof 3.2 *Let* $1 \leq p < \infty$*. Clearly,* $\Delta_v \in \Pi_p^{\mathcal{H}_v^\infty}(\mathbb{C}, \mathcal{G}_v^\infty(\mathbb{C}))$ *with* $\pi_p^{\mathcal{H}_v^\infty}(\Delta_v) \leq 1$*. On the other hand, the equality* $\mathrm{id}_{\mathcal{G}_v^\infty(\mathbb{C})} \circ \Delta_v = \Delta_v$ *and the uniqueness of the linearization shows that* $T_{\Delta_v} = \mathrm{id}_{\mathcal{G}_v^\infty(\mathbb{C})}$ *(the identity map on the infinite-dimensional Banach space* $\mathcal{G}_v^\infty(\mathbb{C})$*). Hence* $T_{\Delta_v} \notin \Pi_p(\mathcal{G}_v^\infty(\mathbb{C}), \mathcal{G}_v^\infty(\mathbb{C}))$ *by Weak Dvoretzky–Rogers Theorem [12, 2.18]. Finally, [8, Theorem 2.7] asserts that* $\Delta_v \notin \Pi_p \circ \mathcal{H}_v^\infty(\mathbb{C}, \mathcal{G}_v^\infty(\mathbb{C}))$*.*

3.5 Pietsch domination

Let $\mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$ *be the set of all Borel regular probability measures* μ *on the compact set* $(B_{\mathcal{H}_v^\infty(U)}, w^*)$*.*

Theorem 3.2 *Let* $1 \leq p < \infty$ *and* $f \in \mathcal{H}_v^\infty(U,F)$*. The following statements are equivalent:*

(i) f *is* p*-summing weighted holomorphic.*

(ii) There is a constant $C \geq 0$ *and a measure* $\mu \in \mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$ *such that*

$$\|f(x)\| \leq C \left(\int_{B_{\mathcal{H}_v^\infty(U)}} |g(x)|^p \, d\mu(g) \right)^{\frac{1}{p}} \qquad (x \in U).$$

In this case, $\pi_p^{\mathcal{H}_v^\infty}(f)$ *is the infimum of all constants* $C \geq 0$ *satisfying the inequality and, in fact, this infimum is attained.*

Proof 3.3 $(i) \Rightarrow (ii)$*: In order to apply an unified abstract version of Piestch Domination Theorem stated in [23], consider the functions* $S\colon \mathcal{H}_v^\infty(U,F) \times U \times \mathbb{C} \to [0,\infty[$ *and* $R\colon B_{\mathcal{H}_v^\infty(U)} \times$

$U \times \mathbb{C} \to [0,\infty[$ *defined, respectively, by*

$$S(h,x,\lambda) = |\lambda|\, v(x)\, \|h(x)\|\,,$$
$$R(g,x,\lambda) = |\lambda|\, v(x)\, |g(x)|\,.$$

Note first that for any $x \in U$ *and* $\lambda \in \mathbb{C}$*, the function* $R_{x,\lambda}\colon B_{\mathcal{H}_v^\infty(U)} \to [0,\infty[$*, given by*

$$R_{x,\lambda}(g) = R(g,x,\lambda),$$

is continuous. For every $n \in \mathbb{N}$*,* $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ *and* $x_1,\dots,x_n \in U$*, we have*

$$\begin{aligned}\left(\sum_{i=1}^{n} S(f,x_i,\lambda_i)^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{n} |\lambda_i|^p\, v(x_i)^p\, \|f(x_i)\|^p\right)^{\frac{1}{p}} \\ &\le \pi_p^{\mathcal{H}_v^\infty}(f) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_i|^p\, v(x_i)^p\, |g(x_i)|^p\right)^{\frac{1}{p}} \\ &= \pi_p^{\mathcal{H}_v^\infty}(f) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} R(g,x_i,\lambda_i)^p\right)^{\frac{1}{p}},\end{aligned}$$

and therefore f *is RS-abstract* p*-summing. Hence, by applying [23, Theorem 3.1], there is a measure* $\mu \in \mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$ *such that*

$$S(f,x,\lambda) \le \pi_p^{\mathcal{H}_v^\infty}(f) \left(\int_{B_{\mathcal{H}_v^\infty(U)}} R(g,x,\lambda)^p \; d\mu(g)\right)^{\frac{1}{p}}$$

for all $x \in U$ *and* $\lambda \in \mathbb{C}$*, and therefore*

$$\|f(x)\| \le \pi_p^{\mathcal{H}_v^\infty}(f) \left(\int_{B_{\mathcal{H}_v^\infty(U)}} |g(x)|^p \, d\mu(g)\right)^{\frac{1}{p}}$$

for all $x \in U$*.*

$(ii) \Rightarrow (i)$: Given $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $x_1, \dots, x_n \in U$, we have

$$\begin{aligned}\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \, \|f(x_i)\|^p &\leq C^p \sum_{i=1}^{n} \int_{B_{\mathcal{H}_v^\infty(U)}} |\lambda_i|^p \, v(x_i)^p \, |g(x_i)|^p \, d\mu(g) \\ &= C^p \int_{B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \, |g(x_i)|^p \right) d\mu(g) \\ &\leq C^p \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_i|^p \, v(x_i)^p \, |g(x_i)|^p \right).\end{aligned}$$

Hence $f \in \Pi_p^{\mathcal{H}_v^\infty}(U, F)$ with $\pi_p^{\mathcal{H}_v^\infty}(f) \leq C$, and thus $\pi_p^{\mathcal{H}_v^\infty}(f) \leq \inf\{C \geq 0$ satisfying $(ii)\}$.

3.6 Pietsch factorization

We consider the following mappings. Given $\mu \in \mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$ and $1 \leq p < \infty$, $I_{\infty,p} \colon L_\infty(\mu) \to L_p(\mu)$ and $j_\infty \colon C(B_{\mathcal{H}_v^\infty(U)}) \to L_\infty(\mu)$ denote the formal inclusion operators. Consider also the mapping $\iota_U \colon U \to C(B_{\mathcal{H}_v^\infty(U)})$ defined by

$$\iota_U(x)(g) = g(x) \quad (x \in U,\ g \in B_{\mathcal{H}_v^\infty(U)}),$$

and the isometric linear embedding $\kappa_F \colon F \to \ell_\infty(B_{F^*})$ given by

$$\langle \kappa_F(y), y^* \rangle = y^*(y) \quad (y^* \in B_{F^*},\ y \in F).$$

We now construct the following useful mapping.

Lemma 3.1 *Let $\mu \in \mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$. Then $j_\infty \circ \iota_U \in \Pi_p^{\mathcal{H}_v^\infty}(U, L_\infty(\mu))$ with $\pi_p^{\mathcal{H}_v^\infty}(j_\infty \circ \iota_U) \leq 1$ for any $1 \leq p < \infty$.*

Proof 3.4 *Clearly, $j_\infty \circ \iota_U \in \mathcal{H}(U, L_\infty(\mu))$. Let $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $x_1, \dots, x_n \in U$. By Theorem 3.1, for each $i = 1, \dots, n$, there exists $g_{x_i} \in B_{\mathcal{H}_v^\infty(U)}$ such that $|g_{x_i}(x_i)| = \|\iota_U(x_i)\|_\infty :=$*

$\sup_{g\in B_{\mathcal{H}_v^\infty(U)}} |g(x_i)|$. *Given* $1 \leq p < \infty$, *one has*

$$\begin{aligned}\left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|j_\infty(\iota_U(x_i))\|_{L_\infty(\mu)}^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|\iota_U(x_i)\|_\infty^p\right)^{\frac{1}{p}}\\ &= \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, |g_{x_i}(x_i)|^p\right)^{\frac{1}{p}}\\ &\leq \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, |g(x_i)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

We are now ready to state the announced result.

Theorem 3.3 *Let* $1 \leq p < \infty$ *and* $f \in \mathcal{H}_v^\infty(U, F)$. *The following assertions are equivalent:*

(i) f *is* p*-summing weighted holomorphic.*

(ii) There exist a measure $\mu \in \mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$, *an operator* $T \in \mathcal{L}(L_p(\mu), \ell_\infty(B_{F^*}))$ *and a mapping* $h \in \Pi_p^{\mathcal{H}_v^\infty}(U, L_\infty(\mu))$ *such that the following diagram commutes:*

$$\begin{tikzcd} L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[d, "T"] \\ U \arrow[u, "h"] \arrow[r, "f"] & F \arrow[r, "\kappa_F"] & \ell_\infty(B_{F^*}) \end{tikzcd}$$

In this case, $\pi_p^{\mathcal{H}_v^\infty}(f) = \inf\left\{\|T\|\, \pi_p^{\mathcal{H}_v^\infty}(h)\right\}$, *where the infimum is taken over all factorizations of* $\kappa_F \circ f$ *as in* (ii), *and this infimum is attained.*

Proof 3.5 $(i) \Rightarrow (ii)$: *If* $f \in \Pi_p^{\mathcal{H}_v^\infty}(U, F)$, *Theorem 3.2 yields a measure* $\mu \in \mathcal{P}(B_{\mathcal{H}_v^\infty(U)})$ *such that*

$$\|f(x)\| \leq \pi_p^{\mathcal{H}_v^\infty}(f) \left(\int_{B_{\mathcal{H}_v^\infty(U)}} |g(x)|^p\, d\mu(g)\right)^{\frac{1}{p}} \qquad (x \in U).$$

By Lemma 3.1, the mapping $h := j_\infty \circ \iota_U$ *is in* $\Pi_p^{\mathcal{H}_v^\infty}(U, L_\infty(\mu))$ *with* $\pi_p^{\mathcal{H}_v^\infty}(h) \leq 1$. *Take the space* $S_p := \overline{\mathrm{lin}}(I_{\infty,p}(h(U))) \subseteq L_p(\mu)$ *and the operator* $T_0 \in \mathcal{L}(S_p, \ell_\infty(B_{F^*}))$ *defined by*

$$T_0(I_{\infty,p}(h(x))) = \kappa_F(f(x)) \qquad (x \in U).$$

Using Theorem 3.1 and denoting $I_n = \{i \in \{1, \ldots, n\} \colon \alpha_i \neq 0\}$, *note that* $\|T_0\| \leq \pi_p^{\mathcal{H}_v^\infty}(f)$ *since*

$$\begin{aligned}
\left\| T_0 \left(\sum_{i=1}^n \alpha_i I_{\infty,p}(h(x_i)) \right) \right\|_\infty &= \left\| \sum_{i=1}^n \alpha_i T_0(I_{\infty,p}(h(x_i))) \right\|_\infty = \left\| \sum_{i=1}^n \alpha_i \kappa_F(f(x_i)) \right\|_\infty \\
&\leq \sum_{i=1}^n |\alpha_i| \, \|\kappa_F(f(x_i))\|_\infty = \sum_{i=1}^n |\alpha_i| \, \|f(x_i)\| \\
&\leq \pi_p^{\mathcal{H}_v^\infty}(f) \sum_{i=1}^n |\alpha_i| \left(\int_{B_{\mathcal{H}_v^\infty(U)}} |g(x_i)|^p \, d\mu(g) \right)^{\frac{1}{p}} \\
&= \pi_p^{\mathcal{H}_v^\infty}(f) \sum_{i=1}^n |\alpha_i| \left(\int_{B_{\mathcal{H}_v^\infty(U)}} |J_v(g)(\delta_{x_i})|^p \, d\mu(g) \right)^{\frac{1}{p}} \\
&\leq \pi_p^{\mathcal{H}_v^\infty}(f) \sum_{i=1}^n |\alpha_i| \left(\int_{B_{\mathcal{H}_v^\infty(U)}} \|g\|_v^p \, \|\delta_{x_i}\|^p \, d\mu(g) \right)^{\frac{1}{p}} \\
&\leq \pi_p^{\mathcal{H}_v^\infty}(f) \sum_{i=1}^n |\alpha_i| \, |g_{x_i}(x_i)| = \pi_p^{\mathcal{H}_v^\infty}(f) \sum_{i=1}^n |\alpha_i| \, g_{x_i}(x_i) \\
&= \pi_p^{\mathcal{H}_v^\infty}(f) \sum_{i \in I_n} \alpha_i \left(\frac{\overline{\alpha_i}}{|\alpha_i|} g_{x_i} \right)(x_i) = \pi_p^{\mathcal{H}_v^\infty}(f) \left| \sum_{i \in I_n} \alpha_i \left(\frac{\overline{\alpha_i}}{|\alpha_i|} g_{x_i} \right)(x_i) \right| \\
&\leq \pi_p^{\mathcal{H}_v^\infty}(f) \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left| \sum_{i=1}^n \alpha_i g(x_i) \right| = \pi_p^{\mathcal{H}_v^\infty}(f) \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left| \sum_{i=1}^n \alpha_i \iota_U(x_i)(g) \right| \\
&= \pi_p^{\mathcal{H}_v^\infty}(f) \left\| \sum_{i=1}^n \alpha_i \iota_U(x_i) \right\|_\infty = \pi_p^{\mathcal{H}_v^\infty}(f) \left\| \sum_{i=1}^n \alpha_i h(x_i) \right\|_{L_\infty(\mu)} \\
&= \pi_p^{\mathcal{H}_v^\infty}(f) \left\| \sum_{i=1}^n \alpha_i I_{\infty,p}(h(x_i)) \right\|_{L_p(\mu)}
\end{aligned}$$

for any $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ *and* $x_1, \ldots, x_n \in U$. *Since the Banach space* $\ell_\infty(B_{F^*})$ *is injective (see [12, p. 45]), there is a* $T \in \mathcal{L}(L_p(\mu), \ell_\infty(B_{F^*}))$ *such that* $T|_{S_p} = T_0$ *with* $\|T\| = \|T_0\|$. *Consequently,* $\kappa_F \circ f = T \circ I_{\infty,p} \circ h$ *with* $\|T\| \, \pi_p^{\mathcal{H}_v^\infty}(h) \leq \pi_p^{\mathcal{H}_v^\infty}(f)$.

$(ii) \Rightarrow (i)$: *We can write* $\kappa_F \circ f = T \circ I_{\infty,p} \circ h$ *as in* (ii). *Let* $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ *and*

$x_1, \ldots, x_n \in U$. We have

$$\begin{aligned}\left(\sum_{i=1}^{n} |\lambda_i|^p\, v(x_i)^p\, \|f(x_i)\|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{n} |\lambda_i|^p\, v(x_i)^p\, \|\kappa_F(f(x_i))\|_\infty^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^{n} |\lambda_i|^p\, v(x_i)^p\, \left\|T(I_{\infty,p}(h(x_i)))\right\|_\infty^p\right)^{\frac{1}{p}} \\ &\leq \|T\| \left(\sum_{i=1}^{n} |\lambda_i|^p\, v(x_i)^p\, \left\|I_{\infty,p}(h(x_i))\right\|_{L_p(\mu)}^p\right)^{\frac{1}{p}} \\ &= \|T\| \left(\sum_{i=1}^{n} |\lambda_i|^p\, v(x_i)^p\, \|h(x_i)\|_{L_\infty(\mu)}^p\right)^{\frac{1}{p}} \\ &\leq \|T\|\, \pi_p^{\mathcal{H}_v^\infty}(h) \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_i|^p\, v(x_i)^p\, |g(x_i)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Hence $f \in \Pi_p^{\mathcal{H}_v^\infty}(U,F)$ with $\pi_p^{\mathcal{H}_v^\infty}(f) \leq \|T\|\, \pi_p^{\mathcal{H}_v^\infty}(h)$. Taking the infimum over all factorizations of $\kappa_F \circ f$ as in (ii), it follows that $\pi_p^{\mathcal{H}_v^\infty}(f) \leq \inf\left\{\|T\|\, \pi_p^{\mathcal{H}_v^\infty}(h)\right\}$.

3.7 Inclusion relations

The next result establishes some inclusion relations between classes of p-summing weighted holomorphic mappings in terms of p,

Proposition 3.3 *If $1 \leq p < q \leq \infty$, then*

$$(\Pi_p^{\mathcal{H}_v^\infty}(U,F), \pi_p^{\mathcal{H}_v^\infty}) \leq (\Pi_q^{\mathcal{H}_v^\infty}(U,F), \pi_q^{\mathcal{H}_v^\infty}) \leq (\Pi_\infty^{\mathcal{H}_v^\infty}(U,F), \pi_\infty^{\mathcal{H}_v^\infty}) = (\mathcal{H}_v^\infty(U,F), \|\cdot\|_v).$$

Proof 3.6 *Let $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in U$. We first prove the coincidence. Let $f \in \Pi_\infty^{\mathcal{H}_v^\infty}(U,F)$. For all $x \in U$, we have*

$$v(x)\|f(x)\| \leq \pi_\infty^{\mathcal{H}_v^\infty}(f) \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} v(x)\,|g(x)| \leq \pi_\infty^{\mathcal{H}_v^\infty}(f).$$

Hence $f \in \mathcal{H}_v^\infty(U,F)$ with $\|f\|_v \leq \pi_\infty^{\mathcal{H}_v^\infty}(f)$. Conversely, given $f \in \mathcal{H}_v^\infty(U,F)$, Theorem 3.1 *provides*

$$\begin{aligned}|\lambda_i|\,v(x_i)\,\|f(x_i)\| &= |\lambda_i|v(x_i)\|T_f(\delta_{x_i})\| \leq \|T_f\|\,|\lambda_i|\,v(x_i)\|\delta_{x_i}\| \\ &= \|f\|_v|\lambda_i|v(x_i)\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}|g(x_i)| = \|f\|_v\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}|\lambda_i|v(x_i)\,|g(x_i)|\end{aligned}$$

for all $i \in \{1,\ldots,n\}$, *hence*

$$\max_{1\leq i\leq n}|\lambda_i|\,v(x_i)\,\|f(x_i)\| \leq \|f\|_v\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\max_{1\leq i\leq n}|\lambda_i|\,v(x_i)\,|g(x_i)|\right),$$

and so $f \in \Pi_\infty^{\mathcal{H}_v^\infty}(U,F)$ *with* $\pi_\infty^{\mathcal{H}_v^\infty}(f) \leq \|f\|_v$.

We now prove the first inequality. Let $f \in \Pi_p^{\mathcal{H}_v^\infty}(U,F)$. *Assume first* $q < \infty$. *Taking*

$$\beta_i = |\lambda_i|^{\frac{q}{p}}\,v(x_i)^{\frac{q}{p}-1}\,\|f(x_i)\|^{\frac{q}{p}-1} \qquad (i=1,\ldots,n),$$

we have

$$\begin{aligned}\left(\sum_{i=1}^n|\lambda_i|^q\,v(x_i)^q\,\|f(x_i)\|^q\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n|\beta_i|^p\,v(x_i)^p\,\|f(x_i)\|^p\right)^{\frac{1}{p}} \\ &\leq \pi_p^{\mathcal{H}_v^\infty}(f)\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^n|\beta_i|^p\,v(x_i)^p\,|g(x_i)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Since $q/p > 1$ *and* $(q/p)^* = q/(q-p)$, *Hölder inequality yields*

$$\begin{aligned}\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^n|\beta_i|^p\,v(x_i)^p\,|g(x_i)|^p\right)^{\frac{1}{p}} &= \sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^n(|\lambda_i|\,v(x_i)\,\|f(x_i)\|)^{q-p}\,(|\lambda_i|\,v(x_i)\,|g(x_i)|)^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^n|\lambda_i|^q\,v(x_i)^q\,\|f(x_i)\|^q\right)^{\frac{1}{p}-\frac{1}{q}}\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^n|\lambda_i|^q\,v(x_i)^q\,|g(x_i)|^q\right)^{\frac{1}{q}},\end{aligned}$$

and thus we obtain

$$\left(\sum_{i=1}^n|\lambda_i|^q\,v(x_i)^q\,\|f(x_i)\|^q\right)^{\frac{1}{q}} \leq \pi_p^{\mathcal{H}_v^\infty}(f)\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^n|\lambda_i|^q\,v(x_i)^q\,|g(x_i)|^q\right)^{\frac{1}{q}}.$$

This shows that $f \in \Pi_q^{\mathcal{H}_v^\infty}(U,F)$ *with* $\pi_q^{\mathcal{H}_v^\infty}(f) \leq \pi_p^{\mathcal{H}_v^\infty}(f)$ *if* $q < \infty$*. For* $q = \infty$*, the same follows from the coincidence and Proposition 3.1.*

CHAPTER 4

Duality

4.1 Banach-valued $\mathcal{H}_v^\infty$-molecules

In this section we study the duality of the spaces of p-summing weighted holomorphic mappings. (see [7]).

Proposition 4.1 *Let* $x \in U$ *and* $y \in F$*. Then the function* $v(x)\delta_x \otimes y\colon \mathcal{H}_v^\infty(U, F^*) \to \mathbb{C}$ *defined by*

$$(v(x)\delta_x \otimes y)(f) = \langle v(x)f(x), y\rangle \qquad (f \in \mathcal{H}_v^\infty(U, F^*)),$$

belongs to $\mathcal{H}_v^\infty(U, F^*)^*$ *and* $\|v(x)\delta_x \otimes y\| = v(x)\|\delta_x\|\|y\|$.

Proof 4.1 *Clearly,* $v(x)\delta_x \otimes y$ *is linear. For any* $f \in \mathcal{H}_v^\infty(U, F^*)$*, we have*

$$\begin{aligned} |(v(x)\delta_x \otimes y)(f)| &= |\langle v(x)f(x), y\rangle| \le v(x)\|f(x)\|\|y\| = v(x)\left\|T_f(\delta_x)\right\|\|y\| \\ &\le v(x)\left\|T_f\right\|\|\delta_x\|\|y\| = \|f\|_v v(x)\|\delta_x\|\|y\|, \end{aligned}$$

and thus $v(x)\delta_x \otimes y \in \mathcal{H}_v^\infty(U, F^*)^*$ *with* $\|v(x)\delta_x \otimes y\| \le v(x)\|\delta_x\|\|y\|$*. For the converse inequality, take* $y^* \in B_{F^*}$ *such that* $|\langle y^*, y\rangle| = \|y\|$ *and consider* $g_x \cdot y^* \in \mathcal{H}_v^\infty(U, F^*)$*. Since* $\|g_x \cdot y^*\|_v = \|g_x\|_v\|y^*\| \le 1$*, we deduce that*

$$\begin{aligned} \|v(x)\delta_x \otimes y\| &\ge |(v(x)\delta_x \otimes y)(g_x \cdot y^*)| = v(x)|\langle (g_x \cdot y^*)(x), y\rangle| \\ &= v(x)|\langle g_x(x)y^*, y\rangle| = v(x)|g_x(x)||\langle y^*, y\rangle| = v(x)\|\delta_x\|\|y\|. \end{aligned}$$

The elements of the following tensor product space could be referred to as F*-valued* $\mathcal{H}_v^\infty$*-molecules on* U*.*

Definition 4.1 *Let* E *and* F *be complex Banach spaces, let* U *be an open subset of* E *and let* v

be a weight on U. *Define the linear space*

$$\operatorname{lin}((v\Delta_v)(U)) \otimes F := \operatorname{lin}\{v(x)\delta_x \otimes y \colon x \in U,\, y \in F\} \subseteq \mathcal{H}_v^\infty(U, F^*)^*.$$

Each element $\gamma \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$ *can be expressed in a form (not necessarily unique):*

$$\gamma = \sum_{i=1}^{n} \lambda_i v(x_i)\delta_{x_i} \otimes y_i,$$

where $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $x_i \in U$ *and* $y_i \in F$ *for* $i = 1, \dots, n$; *and its action as a functional on a mapping* $f \in \mathcal{H}_v^\infty(U, F^*)$ *comes given by*

$$\gamma(f) = \sum_{i=1}^{n} \lambda_i v(x_i) \langle f(x_i), y_i \rangle .$$

4.1.1 Projective norm

Given two linear spaces E *and* F, *the tensor product space* $E \otimes F$, *equipped with a norm* α, *is usually denoted by* $E \otimes_\alpha F$, *and the completion of* $E \otimes_\alpha F$ *by* $E \widehat{\otimes}_\alpha F$. *Consider the projective norm* π *on* $u \in E \otimes F$, *defined by*

$$\pi(u) = \inf\left\{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| : n \in \mathbb{N},\, x_1, \dots, x_n \in E,\, y_1, \dots, y_n \in F,\, u = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

where the infimum is taken over all such representations of u.

We now show that the projective norm and the operator canonical norm coincide on the space of F*-valued* $\mathcal{H}_v^\infty$*-molecules on* U.

Proposition 4.2 *Let* $\gamma \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$. *Then* $\|\gamma\| = \pi(\gamma)$, *where*

$$\|\gamma\| = \sup\left\{ |\gamma(f)| : f \in \mathcal{H}_v^\infty(U, F^*), \|f\|_v \leq 1 \right\}$$

and

$$\pi(\gamma) = \inf\left\{ \sum_{i=1}^{n} |\lambda_i| \, v(x_i) \, \|\delta_{x_i}\| \, \|y_i\| : \gamma = \sum_{i=1}^{n} \lambda_i v(x_i)\delta_{x_i} \otimes y_i \right\}.$$

Proof 4.2 *Let* $\sum_{i=1}^{n} \lambda_i v(x_i)\delta_{x_i} \otimes y_i$ *be a representation of* γ*. Since* $\gamma \in \mathcal{H}_v^\infty(U, F^*)^*$ *and*

$$\begin{aligned} |\gamma(f)| &= \left| \sum_{i=1}^{n} \lambda_i v(x_i) \langle f(x_i), y_i \rangle \right| \leq \sum_{i=1}^{n} |\lambda_i| \, v(x_i) \, \|f(x_i)\| \, \|y_i\| \\ &= \sum_{i=1}^{n} |\lambda_i| \, v(x_i) \, \|T_f(\delta_{x_i})\| \, \|y_i\| \leq \|f\|_v \sum_{i=1}^{n} |\lambda_i| \, v(x_i) \, \|\delta_{x_i}\| \, \|y_i\| \end{aligned}$$

for all $f \in \mathcal{H}_v^\infty(U, F^*)$*, we have that* $||\gamma|| \leq \sum_{i=1}^{n} |\lambda_i| \, v(x_i) \, \|\delta_{x_i}\| \, \|y_i\|$*. Since this holds for each representation of* γ *as above, it is deduced that* $||\gamma|| \leq \pi(\gamma)$*.*

To prove that $\pi(\gamma) \leq ||\gamma||$*, suppose by contradiction that* $\|\gamma\| < \pi(\gamma)$*. Note that* $\gamma \neq 0$ *and consider the set* $B = \{\mu \in \operatorname{lin}((v\Delta_v)(U)) \otimes F \colon \pi(\mu) \leq \|\gamma\|\}$*. Clearly,* B *is a closed convex subset of* $\operatorname{lin}((v\Delta_v)(U)) \otimes_\pi F$*. Applying the Hahn–Banach Separation Theorem to* B *and* $\{\gamma\}$*, we can take a functional* $\eta \in (\operatorname{lin}((v\Delta_v)(U)) \otimes_\pi F)^*$ *with* $\|\eta\| = 1$ *such that*

$$\|\gamma\| = \sup\{\operatorname{Re}(\eta(\mu)) \colon \mu \in B\} < \operatorname{Re}(\eta(\gamma)).$$

Define $f_\eta \colon U \to F^*$ *by*

$$\langle f_\eta(x), y \rangle = \eta\left(\delta_x \otimes y\right) \qquad (x \in U, \ y \in F).$$

We now show that f_η *is holomorphic. By [20, Exercise 8.D], it suffices to prove that for each* $y \in F$*, the function* $f_{\eta,y} \colon U \to \mathbb{C}$*, defined by*

$$f_{\eta,y}(x) = \eta(\delta_x \otimes y) \qquad (x \in U),$$

is holomorphic. For it, let $a \in U$ *and since* $\Delta_v \colon U \to \operatorname{lin}((v\Delta_v)(U))$ *is holomorphic, there exists* $D\Delta_v(a) \in \mathcal{L}(E, \operatorname{lin}((v\Delta_v)(U)))$ *such that*

$$\lim_{x \to a} \frac{\delta_x - \delta_a - D\Delta_v(a)(x-a)}{\|x-a\|} = 0.$$

Define the function $T(a) \colon E \to \mathbb{C}$ *by*

$$T(a)(x) = \eta(D\Delta_v(a)(x) \otimes y) \qquad (x \in E).$$

Clearly, $T(a)$ is linear and

$$|T(a)(x)| = |\eta(D\Delta_v(a)(x) \otimes y)| \leq \|\eta\| \, \pi\left(D\Delta_v(a)(x) \otimes y\right) \\ \leq \|D\Delta_v(a)(x)\| \, \|y\| \leq \|D\Delta_v(a)\| \, \|x\| \, \|y\|$$

for all $x \in E$, hence $T(a) \in E^*$. Since

$$f_{\eta,y}(x) - f_{\eta,y}(a) - T(a)(x-a) = \eta(\delta_x \otimes y) - \eta(\delta_a \otimes y) - \eta(D\Delta_v(a)(x-a) \otimes y) \\ = \eta\left((\delta_x - \delta_a - D\Delta_v(a)(x-a)) \otimes y\right)$$

for all $x \in U$, it follows that

$$\lim_{x \to a} \frac{f_{\eta,y}(x) - f_{\eta,y}(a) - T(a)(x-a)}{\|x-a\|} = \lim_{x \to a} \eta\left(\frac{\delta_x - \delta_a - D\Delta_v(a)(x-a)}{\|x-a\|} \otimes y\right) = 0.$$

Thus $f_{\eta,y}$ is holomorphic at a with $Df_{\eta,y}(a) = T(a)$, as required.
Given $x \in U$, we have

$$v(x)|\langle f_\eta(x), y\rangle| = v(x)\,|\eta\,(\delta_x \otimes y)| \leq v(x)\,\|\eta\|\,\pi(\delta_x \otimes y) \leq v(x)\,\|\delta_x\|\,\|y\| \leq \|y\|$$

for all $y \in F$, and so $v(x)\left\|f_\eta(x)\right\| \leq 1$. Therefore $f_\eta \in \mathcal{H}_v^\infty(U, F^*)$ and $\left\|f_\eta\right\|_v \leq 1$. Furthermore, $\mu(f_\eta) = \eta(\mu)$ for all $\mu \in \mathrm{lin}((v\Delta_v)(U)) \otimes F$. Then $\|\gamma\| \geq |\gamma(f_\eta)| \geq \mathrm{Re}(\gamma(f_\eta)) = \mathrm{Re}(\eta(\gamma))$, and we arrive at a contradiction.

4.1.2 p-Chevet–Saphar $\mathcal{H}_v^\infty$-norms

The p-Chevet–Saphar norms d_p on the tensor product of two Banach spaces $E \otimes F$ are well known (see, for example, [26, Section 6.2]).
Our study of the duality of the spaces of p-summing weighted holomorphic mappings requires the introduction of the following $\mathcal{H}_v^\infty$-variants of such norms.

Definition 4.2 *Let E and F be complex Banach spaces, let U be an open subset of E, let v be a* *weight on* U *and let* $1 \leq p \leq \infty$. *For* $\gamma \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$, *define:*

$$d_1^{\mathcal{H}_v^\infty}(\gamma) = \inf\left\{\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\max_{1\leq i\leq n}|\lambda_i|\,v(x_i)\,|g(x_i)|\right)\right)\left(\sum_{i=1}^n \|y_i\|\right)\right\},$$

$$d_p^{\mathcal{H}_v^\infty}(\gamma) = \inf\left\{\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^n |\lambda_i|^{p^*}\,v(x_i)^{p^*}\,|g(x_i)|^{p^*}\right)^{\frac{1}{p^*}}\right)\left(\sum_{i=1}^n \|y_i\|^p\right)^{\frac{1}{p}}\right\} \quad (1 < p < \infty),$$

$$d_\infty^{\mathcal{H}_v^\infty}(\gamma) = \inf\left\{\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^n |\lambda_i|\,v(x_i)\,|g(x_i)|\right)\right)\left(\max_{1\leq i\leq n}\|y_i\|\right)\right\},$$

where the infimum is taken over all such representations of γ *as* $\sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i} \otimes y_i$.

Motivated by the analogue concept for tensor product spaces, we introduce the following.

Definition 4.3 *A norm* α *on* $\operatorname{lin}((v\Delta_v)(U)) \otimes F$ *is said to be a reasonable* $\mathcal{H}_v^\infty$*-crossnorm if it enjoys the following properties:*

(i) $\alpha(v(x)\delta_x \otimes y) = v(x)\,\|\delta_x\|\,\|y\|$ *for all* $x \in U$ *and* $y \in F$,

(ii) For every $g \in \mathcal{H}_v^\infty(U)$ *and* $y^* \in F^*$, *the linear functional* $g \otimes y^* \colon \operatorname{lin}((v\Delta_v)(U)) \otimes F \to \mathbb{C}$ *defined by* $(g \otimes y^*)(v(x)\delta_x \otimes y) = v(x)g(x)y^*(y)$ *is bounded on* $\operatorname{lin}((v\Delta_v)(U)) \otimes_\alpha F$ *with* $\|g \otimes y^*\| \leq \|g\|_v\,\|y^*\|$.

Theorem 4.1 $d_p^{\mathcal{H}_v^\infty}$ *is a reasonable* $\mathcal{H}_v^\infty$*-crossnorm on* $\operatorname{lin}((v\Delta_v)(U)) \otimes F$ *for any* $1 \leq p \leq \infty$.

Proof 4.3 *We will only prove it for* $1 < p < \infty$. *The other cases follow similarly.*
Let $\gamma \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$ *and let* $\sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i} \otimes y_i$ *be a representation of* γ. *Clearly,* $d_p^{\mathcal{H}_v^\infty}(\gamma) \geq 0$. *Given* $\lambda \in \mathbb{C}$, *since* $\sum_{i=1}^n (\lambda\lambda_i)v(x_i)\delta_{x_i} \otimes y_i$ *is a representation of* $\lambda\gamma$, *we have*

$$\begin{aligned} d_p^{\mathcal{H}_v^\infty}(\lambda\gamma) &\leq \left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^n |\lambda\lambda_i|^{p^*}\,v(x_i)^{p^*}\,|g(x_i)|^{p^*}\right)^{\frac{1}{p^*}}\right)\left(\sum_{i=1}^n \|y_i\|^p\right)^{\frac{1}{p}} \\ &= |\lambda|\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^n |\lambda_i|^{p^*}\,v(x_i)^{p^*}\,|g(x_i)|^{p^*}\right)^{\frac{1}{p^*}}\right)\left(\sum_{i=1}^n \|y_i\|^p\right)^{\frac{1}{p}}. \end{aligned}$$

If $\lambda = 0$, *we obtain* $d_p^{\mathcal{H}_v^\infty}(\lambda\gamma) = 0 = |\lambda|\,d_p^{\mathcal{H}_v^\infty}(\gamma)$. *For* $\lambda \neq 0$, *since the preceding inequality holds for every representation of* γ, *we deduce that* $d_p^{\mathcal{H}_v^\infty}(\lambda\gamma) \leq |\lambda|\,d_p^{\mathcal{H}_v^\infty}(\gamma)$. *For the converse* *inequality, note that* $d_p^{\mathcal{H}_v^\infty}(\gamma) = d_p^{\mathcal{H}_v^\infty}(\lambda^{-1}(\lambda\gamma)) \leq |\lambda^{-1}| d_p^{\mathcal{H}_v^\infty}(\lambda\gamma)$ *by using the proved inequality, thus* $|\lambda|\, d_p^{\mathcal{H}_v^\infty}(\gamma) \leq d_p^{\mathcal{H}_v^\infty}(\lambda\gamma)$ *and hence* $d_p^{\mathcal{H}_v^\infty}(\lambda\gamma) = |\lambda|\, d_p^{\mathcal{H}_v^\infty}(\gamma)$.

We now prove the triangular inequality of $d_p^{\mathcal{H}_v^\infty}$. *Let* $\gamma_1, \gamma_2 \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$ *and let* $\varepsilon > 0$. *If* $\gamma_1 = 0$ *or* $\gamma_2 = 0$, *there is nothing to prove. Assume* $\gamma_1 \neq 0 \neq \gamma_2$. *We can choose representations*

$$\gamma_1 = \sum_{i=1}^{n} \lambda_{1,i} v(x_{1,i}) \delta_{x_{1,i}} \otimes y_{1,i}, \qquad \gamma_2 = \sum_{i=1}^{m} \lambda_{2,i} v(x_{2,i}) \delta_{x_{2,i}} \otimes y_{2,i},$$

so that

$$\left(\sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_{1,i}|^{p^*} v(x_{1,i})^{p^*} |g(x_{1,i})|^{p^*} \right)^{\frac{1}{p^*}} \right) \left(\sum_{i=1}^{n} \|y_{1,i}\|^p \right)^{\frac{1}{p}} \leq d_p^{\mathcal{H}_v^\infty}(\gamma_1) + \varepsilon$$

and

$$\left(\sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{m} |\lambda_{2,i}|^{p^*} v(x_{2,i})^{p^*} |g(x_{2,i})|^{p^*} \right)^{\frac{1}{p^*}} \right) \left(\sum_{i=1}^{m} \|y_{2,i}\|^p \right)^{\frac{1}{p}} \leq d_p^{\mathcal{H}_v^\infty}(\gamma_2) + \varepsilon.$$

Fix arbitrary $r, s \in \mathbb{R}^+$ *and define*

$$\lambda_{3,i} v(x_{3,i}) \delta_{x_{3,i}} = \begin{cases} r^{-1} \lambda_{1,i} v(x_{1,i}) \delta_{x_{1,i}} & \text{if } i = 1, \ldots, n, \\ s^{-1} \lambda_{2,i-n} v(x_{2,i-n}) \delta_{x_{2,i-n}} & \text{if } i = n+1, \ldots, n+m, \end{cases}$$

$$y_{3,i} = \begin{cases} r y_{1,i} & \text{if } i = 1, \ldots, n, \\ s y_{2,i-n} & \text{if } i = n+1, \ldots, n+m. \end{cases}$$

It is clear that $\gamma_1 + \gamma_2 = \sum_{i=1}^{n+m} \lambda_{3,i} v(x_{3,i}) \delta_{x_{3,i}} \otimes y_{3,i}$ *and thus we have*

$$d_p^{\mathcal{H}_v^\infty}(\gamma_1 + \gamma_2) \leq \left(\sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n+m} |\lambda_{3,i}|^{p^*} v(x_{3,i})^{p^*} |g(x_{3,i})|^{p^*} \right)^{\frac{1}{p^*}} \right) \left(\sum_{i=1}^{n+m} \|y_{3,i}\|^p \right)^{\frac{1}{p}}.$$

An easy verification gives

$$\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^{n+m}|\lambda_{3,i}|^{p^*}v(x_{3,i})^{p^*}|g(x_{3,i})|^{p^*}\right)^{\frac{1}{p^*}}\right)^{p^*}$$

$$\leq\left(r^{-1}\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^{n}|\lambda_{1,i}|^{p^*}v(x_{1,i})^{p^*}|g(x_{1,i})|^{p^*}\right)^{\frac{1}{p^*}}\right)^{p^*}+\left(s^{-1}\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^{m}|\lambda_{2,i}|^{p^*}v(x_{2,i})^{p^*}|g(x_{2,i})|^{p^*}\right)^{\frac{1}{p^*}}$$

and

$$\sum_{i=1}^{n+m}\|y_{3,i}\|^p=r^p\sum_{i=1}^{n}\|y_{1,i}\|^p+s^p\sum_{i=1}^{m}\|y_{2,i}\|^p.$$

Using Young's Inequality, it follows that

$$\begin{aligned}d_p^{\mathcal{H}_v^\infty}(\gamma_1+\gamma_2)&\leq\frac{1}{p^*}\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^{n+m}|\lambda_{3,i}|^{p^*}v(x_{3,i})^{p^*}|g(x_{3,i})|^{p^*}\right)^{\frac{1}{p^*}}\right)^{p^*}+\frac{1}{p}\sum_{i=1}^{n+m}\|y_{3,i}\|^p\\&\leq\frac{r^{-p^*}}{p^*}\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^{n}|\lambda_{1,i}|^{p^*}v(x_{1,i})^{p^*}|g(x_{1,i})|^{p^*}\right)^{\frac{1}{p^*}}\right)^{p^*}+\frac{r^p}{p}\sum_{i=1}^{n}\|y_{1,i}\|^p\\&+\frac{s^{-p^*}}{p^*}\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^{m}|\lambda_{2,i}|^{p^*}v(x_{2,i})^{p^*}|g(z_{2,i})|^{p^*}\right)^{\frac{1}{p^*}}\right)^{p^*}+\frac{s^p}{p}\sum_{i=1}^{m}\|y_{2,i}\|^p.\end{aligned}$$

Since r,s were arbitrary in $\mathbb{R}^+$, taking above

$$\begin{aligned}r&=(d_p^{\mathcal{H}_v^\infty}(\gamma_1)+\varepsilon)^{-\frac{1}{p^*}}\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^{n}|\lambda_{1,i}|^{p^*}v(x_{1,i})^{p^*}|g(x_{1,i})|^{p^*}\right)^{\frac{1}{p^*}}\right),\\s&=(d_p^{\mathcal{H}_v^\infty}(\gamma_2)+\varepsilon)^{-\frac{1}{p^*}}\left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}}\left(\sum_{i=1}^{m}|\lambda_{2,i}|^{p^*}v(x_{2,i})^{p^*}|g(x_{2,i})|^{p^*}\right)^{\frac{1}{p^*}}\right),\end{aligned}$$

we obtain that $d_p^{\mathcal{H}_v^\infty}(\gamma_1+\gamma_2)\leq d_p^{\mathcal{H}_v^\infty}(\gamma_1)+d_p^{\mathcal{H}_v^\infty}(\gamma_2)+2\varepsilon$, and thus $d_p^{\mathcal{H}_v^\infty}(\gamma_1+\gamma_2)\leq d_p^{\mathcal{H}_v^\infty}(\gamma_1)+d_p^{\mathcal{H}_v^\infty}(\gamma_2)$ by the arbitrariness of ε. Hence $d_p^{\mathcal{H}_v^\infty}$ is a seminorm. To prove that it is a norm, note *first that*

$$\begin{aligned}
\pi(\gamma) &\leq \sum_{i=1}^{n} |\lambda_i|\, v(x_i)\, \|\delta_{x_i}\|\, \|y_i\| \\
&= \sum_{i=1}^{n} |\lambda_i|\, v(x_i)\, |g_{x_i}(x_i)|\, \|y_i\| \\
&\leq \left(\sum_{i=1}^{n} |\lambda_i|^{p^*}\, v(x_i)^{p^*}\, |g_{x_i}(x_i)|^{p^*}\right)^{\frac{1}{p^*}} \left(\sum_{i=1}^{n} \|y_i\|^{p}\right)^{\frac{1}{p}} \\
&\leq \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^{n} |\lambda_i|^{p^*}\, v(x_i)^{p^*}\, |g(x_i)|^{p^*}\right)^{\frac{1}{p^*}} \left(\sum_{i=1}^{n} \|y_i\|^{p}\right)^{\frac{1}{p}},
\end{aligned}$$

by applying Hölder's Inequality, and therefore $\pi(\gamma) \leq d_p^{\mathcal{H}_v^\infty}(\gamma)$ *by taking the infimum over all representations of* γ*. Now, if* $d_p^{\mathcal{H}_v^\infty}(\gamma) = 0$*, then* $\pi(\gamma) = 0$ *and thus* $\gamma = 0$ *by Proposition 4.2. Finally, we show that* $d_p^{\mathcal{H}_v^\infty}$ *is a reasonable* $\mathcal{H}_v^\infty$*-crossnorm on* $\mathrm{lin}((v\Delta_v)(U)) \otimes F$*. First, given* $x \in U$ *and* $y \in F$*, we have*

$$v(x)\, \|\delta_x\|\, \|y\| = \|v(x)\delta_x \otimes y\| = \pi(v(x)\delta_x \otimes y) \leq d_p^{\mathcal{H}_v^\infty}(v(x)\delta_x \otimes y)$$

by Propositions 4.1 and 4.2 and by proved above, and, conversely, one has

$$d_p^{\mathcal{H}_v^\infty}(v(x)\delta_x \otimes y) \leq \left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}} v(x)\, |g(x)|\right) \|y\| = v(x)\|\delta_x\|\|y\|.$$

Second, given $g \in \mathcal{H}_v^\infty(U)$ *and* $y^* \in F^*$,

$$\begin{aligned}
|(g \otimes y^*)(\gamma)| &= \left| \sum_{i=1}^n \lambda_i (g \otimes y^*)(v(x_i)\delta_{x_i} \otimes y_i) \right| = \left| \sum_{i=1}^n \lambda_i v(x_i) g(x_i) y^*(y_i) \right| \\
&\leq \sum_{i=1}^n |\lambda_i| \, v(x_i) \, |g(x_i)| \, |y^*(y_i)| = \sum_{i=1}^n |\lambda_i| \, v(x_i) \, |J_v(g)(\delta_{x_i})| \, |y^*(y_i)| \\
&\leq \|g\|_v \, \|y^*\| \sum_{i=1}^n |\lambda_i| \, v(x_i) \, \|\delta_{x_i}\| \, \|y_i\| = \|g\|_v \, \|y^*\| \sum_{i=1}^n |\lambda_i| \, v(x_i) \, |g_{x_i}(x_i)| \, \|y_i\| \\
&\leq \|g\|_v \, \|y^*\| \left(\sum_{i=1}^n |\lambda_i|^{p^*} v(x_i)^{p^*} |g_{x_i}(x_i)|^{p^*} \right)^{\frac{1}{p^*}} \left(\sum_{i=1}^n \|y_i\|^p \right)^{\frac{1}{p}} \\
&\leq \|g\|_v \, \|y^*\| \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^{p^*} v(x_i)^{p^*} |g(x_i)|^{p^*} \right)^{\frac{1}{p^*}} \left(\sum_{i=1}^n \|y_i\|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Taking infimum over all the representations of γ, *we deduce that* $|(g \otimes y^*)(\gamma)| \leq \|g\|_v \, \|y^*\| \, d_p^{\mathcal{H}_v^\infty}(\gamma)$. *Hence* $g \otimes y^* \in (\operatorname{lin}((v\Delta_v)(U)) \otimes_{d_p^{\mathcal{H}_v^\infty}} F)^*$ *with* $\|g \otimes y^*\| \leq \|g\|_v \, \|y^*\|$.

We now compute $d_p^{\mathcal{H}_v^\infty}$ *with a simpler formula for* $p = 1$ *and* $p = \infty$. *In fact, the* 1*-Chevet–Saphar* $\mathcal{H}_v^\infty$*-norm coincides with the projective norm.*

Proposition 4.3 *For* $\gamma \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$, *we have*

$$d_1^{\mathcal{H}_v^\infty}(\gamma) = \inf \left\{ \sum_{i=1}^n |\lambda_i| \, v(x_i) \, \|\delta_{x_i}\| \, \|y_i\| \right\}$$

and

$$d_\infty^{\mathcal{H}_v^\infty}(\gamma) = \inf \left\{ \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i| \, v(x_i) \, |g(x_i)| \, \|y_i\| \right) \right\},$$

where the infimum is taken over all such representations of γ *as* $\sum_{i=1}^n \lambda_i v(x_i) \delta_{x_i} \otimes y_i$.

Proof 4.4 *Let* $\gamma \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$ *and let* $\sum_{i=1}^n \lambda_i v(x_i) \delta_{x_i} \otimes y_i$ *be a representation of* γ. *We* *have*

$$\begin{aligned}\pi(\gamma) &\leq \sum_{i=1}^n |\lambda_i|\, v(x_i)\, \|\delta_{x_i}\|\, \|y_i\| = \sum_{i=1}^n |\lambda_i|\, v(x_i) \left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}} |g(x_i)| \right) \|y_i\| \\ &\leq \sum_{i=1}^n \max_{1\leq i\leq n} \left(|\lambda_i|\, v(x_i) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} |g(x_i)| \right) \|y_i\| = \left(\max_{1\leq i\leq n} \left(|\lambda_i|\, v(x_i) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} |g(x_i)| \right) \right) \left(\sum_{i=1}^n \|y_i\| \right) \\ &= \left(\sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\max_{1\leq i\leq n} |\lambda_i|\, v(x_i)\, |g(x_i)| \right) \right) \left(\sum_{i=1}^n \|y_i\| \right)\end{aligned}$$

and therefore $\pi(\gamma) \leq d_1^{\mathcal{H}_v^\infty}(\gamma)$. *Conversely, since* $d_1^{\mathcal{H}_v^\infty}$ *is a crossnorm, we have*

$$d_1^{\mathcal{H}_v^\infty}(\gamma) \leq \sum_{i=1}^n |\lambda_i|\, v(x_i) d_1^{\mathcal{H}_v^\infty}(\delta_{x_i} \otimes y_i) = \sum_{i=1}^n |\lambda_i|\, v(x_i)\, \|\delta_{x_i}\|\, \|y_i\|\,,$$

and thus $d_1^{\mathcal{H}_v^\infty}(\gamma) \leq \pi(\gamma)$.

On the other hand, we have

$$\sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|\, v(x_i)\, |g(x_i)|\, \|y_i\| \right) \leq \left(\max_{1\leq i\leq n} \|y_i\| \right) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|\, v(x_i)\, |g(x_i)| \right),$$

and taking the infimum over all representations of γ *gives*

$$\inf \left\{ \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|\, v(x_i)\, |g(x_i)|\, \|y_i\| \right) : \gamma = \sum_{i=1}^n \lambda_i v(x_i) \delta_{x_i} \otimes y_i \right\} \leq d_\infty^{\mathcal{H}_v^\infty}(\gamma).$$

Conversely, we can assume without loss of generality that $y_i \neq 0$ *for all* $i \in \{1, \ldots, n\}$ *and since* $\gamma = \sum_{i=1}^n \lambda_i v(x_i)\, \|y_i\|\, \delta_{x_i} \otimes (y_i / \|y_i\|)$, *we obtain*

$$d_\infty^{\mathcal{H}_v^\infty}(\gamma) \leq \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|\, v(x_i)\, \|y_i\|\, |g(x_i)| \right),$$

and taking the infimum over all representations of γ*, we conclude that*

$$d_\infty^{\mathcal{H}_v^\infty}(\gamma) \le \inf \left\{ \sup_{g \in B_{\mathcal{H}_v^\infty}} \left(\sum_{i=1}^n |\lambda_i| \, v(x_i) \, |g(x_i)| \, \|y_i\| \right) : \gamma = \sum_{i=1}^n \lambda_i v(x_i) \delta_{x_i} \otimes y_i \right\}.$$

4.1.3 Duality

Given $p \in [1, \infty]$*, we will show that the dual of the space* $\operatorname{lin}((v\Delta_v)(U)) \widehat{\otimes}_{d_{p^*}^{\mathcal{H}_v^\infty}} F$ *can be canonically identified as the space of* p*-summing weighted holomorphic mappings from* U *to* F^**.*

Theorem 4.2 *For* $1 \le p \le \infty$*, the space* $(\Pi_p^{\mathcal{H}_v^\infty}(U, F^*), \pi_p^{\mathcal{H}_v^\infty})$ *is isometrically isomorphic to* $(\operatorname{lin}((v\Delta_v)(U)) \widehat{\otimes}_{d_{p^*}^{\mathcal{H}_v^\infty}} F)^*$*, via the mapping* $\Lambda \colon \Pi_p^{\mathcal{H}_v^\infty}(U, F^*) \to (\operatorname{lin}((v\Delta_v)(U)) \widehat{\otimes}_{d_{p^*}^{\mathcal{H}_v^\infty}} F)^*$ *defined by*

$$\Lambda(f)(\gamma) = \sum_{i=1}^n \lambda_i v(x_i) \, \langle f(x_i), y_i \rangle$$

for $f \in \Pi_p^{\mathcal{H}_v^\infty}(U, F^*)$ *and* $\gamma = \sum_{i=1}^n \lambda_i v(x_i) \delta_{x_i} \otimes y_i \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$*. Furthermore, its inverse comes given by*

$$\left\langle \Lambda^{-1}(\varphi)(x), y \right\rangle = \varphi(\delta_x \otimes y)$$

for $\varphi \in (\operatorname{lin}((v\Delta_v)(U)) \widehat{\otimes}_{d_{p^*}^{\mathcal{H}_v^\infty}} F)^*$*,* $x \in U$ *and* $y \in F$*.*

Proof 4.5 *We prove it for* $1 < p < \infty$*. The cases* $p = 1$ *and* $p = \infty$ *can be proved similarly. Let* $f \in \Pi_p^{\mathcal{H}_v^\infty}(U, F^*)$ *and let* $\Lambda_0(f) \colon \operatorname{lin}((v\Delta_v)(U)) \otimes F \to \mathbb{C}$ *be the linear functional given by*

$$\Lambda_0(f)(\gamma) = \sum_{i=1}^n \lambda_i v(x_i) \, \langle f(x_i), y_i \rangle$$

for $\gamma = \sum_{i=1}^n \lambda_i v(x_i) \delta_{x_i} \otimes y_i \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$*. Note that* $\Lambda_0(f) \in (\operatorname{lin}((v\Delta_v)(U)) \otimes_{d_{p^*}^{\mathcal{H}_v^\infty}} F)^*$

with $\|\Lambda_0(f)\| \leq \pi_p^{\mathcal{H}_v^\infty}(f)$ *since*

$$\begin{aligned}
|\Lambda_0(f)(\gamma)| &= \left|\sum_{i=1}^n \lambda_i v(x_i) \langle f(x_i), y_i\rangle\right| \leq \sum_{i=1}^n |\lambda_i|\, v(x_i)\, \|f(x_i)\|\, \|y_i\| \\
&\leq \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, \|f(x_i)\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|y_i\|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq \pi_p^{\mathcal{H}_v^\infty}(f) \sup_{g\in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p\, v(x_i)^p\, |g(x_i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|y_i\|^{p^*}\right)^{\frac{1}{p^*}},
\end{aligned}$$

and taking infimum over all the representations of γ, *we deduce that* $|\Lambda_0(f)(\gamma)| \leq \pi_p^{\mathcal{H}_v^\infty}(f) d_{p^*}^{\mathcal{H}_v^\infty}(\gamma)$. *Since* γ *was arbitrary, then* $\Lambda_0(f)$ *is continuous on* $\mathrm{lin}((v\Delta_v)(U)) \otimes_{d_{p^*}^{\mathcal{H}_v^\infty}} F$ *with* $\|\Lambda_0(f)\| \leq \pi_p^{\mathcal{H}_v^\infty}(f)$.

Hence there is a unique continuous function $\Lambda(f)\colon \mathrm{lin}((v\Delta_v)(U))\widehat{\otimes}_{d_{p^*}^{\mathcal{H}_v^\infty}} F \to \mathbb{C}$ *that extends* $\Lambda_0(f)$. *Further,* $\Lambda(f)$ *is linear and* $\|\Lambda(f)\| = \|\Lambda_0(f)\|$. *Let* $\Lambda\colon \Pi_p^{\mathcal{H}_v^\infty}(U, F^*) \to (\mathrm{lin}((v\Delta_v)(U))\widehat{\otimes}_{d_{p^*}^{\mathcal{H}_v^\infty}} F)^*$ *be the mapping so defined.*

Clearly, $\Lambda_0\colon \Pi_p^{\mathcal{H}_v^\infty}(U, F^*) \to (\mathrm{lin}((v\Delta_v)(U)) \otimes_{d_{p^*}^{\mathcal{H}_v^\infty}} F)^*$ *is linear. To show its injectivity, if* $f \in \Pi_p^{\mathcal{H}_v^\infty}(U, F^*)$ *and* $\Lambda_0(f) = 0$, *then* $v(x)\langle f(x), y\rangle = \Lambda_0(f)(v(x)\delta_x \otimes y) = 0$ *for all* $x \in U$ *and* $y \in F$, *hence* $f(x) = 0$ *for all* $x \in U$ *and thus* $f = 0$. *Then* Λ *is also linear and injective. Indeed, let* $\phi \in \mathrm{lin}((v\Delta_v)(U))\widehat{\otimes}_{d_{p^*}^{\mathcal{H}_v^\infty}} F$ *and let* $\{\gamma_n\}$ *be a sequence in* $\mathrm{lin}((v\Delta_v)(U)) \otimes_{d_{p^*}^{\mathcal{H}_v^\infty}} F$ *such that* $d_{p^*}^{\mathcal{H}_v^\infty}(\gamma_n - \phi) \to 0$ *when* $n \to \infty$. *Given* $a, b \in \mathbb{C}$ *and* $f, g \in \Pi_p^{\mathcal{H}_v^\infty}(U, F^*)$, *an easy calculation shows that*

$$\Lambda(af + bg)(\gamma_n) = \Lambda_0(af + bg)(\gamma_n) = (a\Lambda_0(f) + b\Lambda_0(g))(\gamma_n) = (a\Lambda(f) + b\Lambda(g))(\gamma_n)$$

for all $n \in \mathbb{N}$, *and taking limits with* $n \to \infty$, *we have that* $\Lambda(af + bg)(\phi) = (a\Lambda(f) + b\Lambda(g))(\phi)$. *Hence* Λ *is linear. For the injectivity of* Λ, *note that if* $f \in \Pi_p^{\mathcal{H}_v^\infty}(U, F^*)$ *and* $\Lambda(f) = 0$, *then* $\Lambda_0(f) = 0$ *which implies that* $f = 0$ *by the injectivity of* Λ_0.

To prove that Λ *is a surjective isometry, let* $\varphi \in (\mathrm{lin}((v\Delta_v)(U))\widehat{\otimes}_{d_{p^*}^{\mathcal{H}_v^\infty}} F)^*$ *and define* $f_\varphi\colon U \to F^*$ *by*

$$\langle f_\varphi(x), y\rangle = \varphi(\delta_x \otimes y) \qquad (x \in U,\ y \in F).$$

As in the proof of Proposition 4.2, it is similarly proved that $f_\varphi \in \mathcal{H}_v^\infty(U, F^*)$ *with* $\|f_\varphi\|_v \leq \|\varphi\|$.

We now prove that $f_\varphi \in \Pi_p^{\mathcal{H}_v^\infty}(U, F^*)$. *Fix* $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ *and* $x_1, \ldots, x_n \in U$. *Let* $\varepsilon > 0$. *For each* $i \in \{1, \ldots, n\}$, *there exists* $y_i \in F$ *with* $\|y_i\| \leq 1 + \varepsilon$ *such that* $\langle f_\varphi(x_i), y_i \rangle = \|f_\varphi(x_i)\|$. *Clearly, the function* $T \colon \mathbb{C}^n \to \mathbb{C}$, *defined by*

$$T(t_1, \ldots, t_n) = \sum_{i=1}^{n} t_i \lambda_i v(x_i) \left\| f_\varphi(x_i) \right\|, \quad \forall (t_1, \ldots, t_n) \in \mathbb{C}^n,$$

is linear and continuous on $(\mathbb{C}^n, ||\cdot||_{p^*})$ *with* $\|T\| = \left(\sum_{i=1}^n |\lambda_i|^p \, v(x_i)^p \left\| f_\varphi(x_i) \right\|^p \right)^{\frac{1}{p}}$. *For any* $(t_1, \ldots, t_n) \in \mathbb{C}^n$ *with* $||(t_1, \ldots, t_n)||_{p^*} \leq 1$, *we have*

$$\begin{aligned} |T(t_1, \ldots, t_n)| &= \left| \varphi \left(\sum_{i=1}^n t_i \lambda_i v(x_i) \delta_{x_i} \otimes y_i \right) \right| \leq \|\varphi\| \, d_{p^*}^{\mathcal{H}_v^\infty} \left(\sum_{i=1}^n \lambda_i v(x_i) \delta_{x_i} \otimes t_i y_i \right) \\ &\leq \|\varphi\| \left(\sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p \, v(x_i)^p \, |g(x_i)|^p \right)^{\frac{1}{p}} \right) \left(\sum_{i=1}^n \|t_i y_i\|^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq (1 + \varepsilon) \, \|\varphi\| \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p \, v(x_i)^p \, |g(x_i)|^p \right)^{\frac{1}{p}}, \end{aligned}$$

therefore

$$\left(\sum_{i=1}^n |\lambda_i|^p \, v(x_i)^p \left\| f_\varphi(x_i) \right\|^p \right)^{\frac{1}{p}} \leq (1 + \varepsilon) \, \|\varphi\| \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p \, v(x_i)^p \, |g(x_i)|^p \right)^{\frac{1}{p}},$$

and since ε *was arbitrary, we have*

$$\left(\sum_{i=1}^n |\lambda_i|^p \, v(x_i)^p \left\| f_\varphi(x_i) \right\|^p \right)^{\frac{1}{p}} \leq \|\varphi\| \sup_{g \in B_{\mathcal{H}_v^\infty(U)}} \left(\sum_{i=1}^n |\lambda_i|^p \, v(x_i)^p \, |g(x_i)|^p \right)^{\frac{1}{p}},$$

and we conclude that $f_\varphi \in \Pi_p^{\mathcal{H}_v^\infty}(U, F^*)$ *with* $\pi_p^{\mathcal{H}_v^\infty}(f_\varphi) \leq \|\varphi\|$.
Finally, for any $\gamma = \sum_{i=1}^n \lambda_i v(x_i) \delta_{x_i} \otimes y_i \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$, *we get*

$$\Lambda(f_\varphi)(\gamma) = \sum_{i=1}^n \lambda_i v(x_i) \left\langle f_\varphi(x_i), y_i \right\rangle = \sum_{i=1}^n \lambda_i v(x_i) \varphi(\delta_{x_i} \otimes y_i) = \varphi \left(\sum_{i=1}^n \lambda_i v(x_i) \delta_{x_i} \otimes y_i \right) = \varphi(\gamma).$$

Hence $\Lambda(f_\varphi) = \varphi$ *on a dense subspace of* $\operatorname{lin}((v\Delta_v)(U)) \widehat{\otimes}_{d_{p^*}^{\mathcal{H}_v^\infty}} F$ *and we conclude that* $\Lambda(f_\varphi) = \varphi$.

This also justifies the last statement of the theorem. Moreover, $\pi_p^{\mathcal{H}_v^\infty}(f_\varphi) \leq \|\varphi\| = \|\Lambda(f_\varphi)\|$.

In light of Theorem 4.2 and taking into account Propositions 3.3, 4.2 and 4.3, we can identify the space $\mathcal{H}_v^\infty(U, F^*)$ *with the dual space of* $\operatorname{lin}((v\Delta_v)(U))\widehat{\otimes}F \subseteq \mathcal{H}_v^\infty(U, F^*)^*$.

Corollary 4.1 *The space* $(\mathcal{H}_v^\infty(U, F^*), \|\cdot\|_v)$ *is isometrically isomorphic to* $(\operatorname{lin}((v\Delta_v)(U))\widehat{\otimes}_{\|\cdot\|}F)^*$, *via the mapping* $\Lambda\colon \mathcal{H}_v^\infty(U, F^*) \to (\operatorname{lin}((v\Delta_v)(U))\widehat{\otimes}_{\|\cdot\|}F)^*$ *given by*

$$\Lambda(f)(\gamma) = \sum_{i=1}^{n} \lambda_i v(x_i) \langle f(x_i), y_i \rangle$$

for $f \in \mathcal{H}_v^\infty(U, F^*)$ *and* $\gamma = \sum_{i=1}^n \lambda_i v(x_i)\delta_{x_i} \otimes y_i \in \operatorname{lin}((v\Delta_v)(U)) \otimes F$, *with inverse*

$$\left\langle \Lambda^{-1}(\varphi)(x), y \right\rangle = \varphi(\delta_x \otimes y)$$

for $\varphi \in (\operatorname{lin}((v\Delta_v)(U))\widehat{\otimes}_{\|\cdot\|}F)^*$, $x \in U$ *and* $y \in F$.

Bibliography

[1] D. Achour and L. Mezrag, On the Cohen strongly p-summing multilinear operators, J. Math. Anal. Appl. ***327*** *(2007), no. 1, 550–563.*

[2] J. C. Angulo-López and M. Fernández-Unzueta, Lipschitz p-summing multilinear operators, J. Funct. Anal. ***279*** *(2020), no. 4, 108572, 20 pp.*

[3] K. D. Bierstedt and W. H. Summers, Biduals of weighted Banach spaces of analytic functions, J. Austral. Math. Soc. Ser. A ***54*** *(1993), no. 1, 70–79.*

[4] J. Bonet, Weighted Banach spaces of analytic functions with sup-norms and operators between them: a survey, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2022) 116:184.

[5] J. Bonet, P. Domanski and M. Lindström, Weakly Compact Composition Operators on Weighted Vector–Valued Banach Spaces of Analytic Mappings, Ann. Acad. Sci. Fenn. Ser. A I. Math. ***26*** *(2001), 233–248.*

[6] J. Bonet and M. Friz, Weakly Compact Composition Operators on Locally Convex Spaces, Math. Nachr. ***245*** *(2002), 26–44.*

[7] M. G. Cabrera-Padilla, A. Jiménez-Vargas and A. Keten Çopur, p-Summing weighted holomorphic mappings, https://arxiv.org/abs/2501.15863v1.

[8] M. G. Cabrera-Padilla, A. Jiménez-Vargas and A. Keten Çopur, Weighted holomorphic mappings associated with p-compact type sets, https://arxiv.org/submit/5814632, to appear in Bull. Malays. Math. Sci. Soc.

[9] N. J. Kalton, Spaces of Lipschitz and Hölder functions and their applications, Collect. Math. 55 (2004) 171–217.

[10] W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński, Factoring weakly compact operators, J. Funct. Anal. ***17*** *(1974), 311–327.*

[11] *A. Defant and K. Floret, Tensor norms and operator ideals. North-Holland Mathematics Studies, vol. 176, North-Holland Publishing Co., Amsterdam, 1993.*

[12] *J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995.*

[13] *V. Dimant, Strongly p-summing multilinear mappings, J. Math. Anal. Appl.* ***278*** *(2003) 182–193.*

[14] *J. D. Farmer and W. B. Johnson, Lipschitz p-summing operators, Proc. Amer. Math. Soc.* ***137*** *(2009), no. 9, 2989–2995.*

[15] *R. F. Arens and J. Eels Jr., On embedding uniform and topological spaces, Pacific J. Math. 6 (1956) 397-403.*

[16] *A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Memoirs American Mathematical Society 16, Providence, Rhode Island 1955.*

[17] *M. Gupta and D. Baweja, Weighted spaces of holomorphic functions on Banach spaces and the approximation property, Extracta Math.* ***31*** *(2016), no. 2, 123–144.*

[18] *M. C. Matos, Absolutely summing holomorphic mappings, An. Acad. Brasil. Cienc.* ***68*** *(1996) 1–13.*

[19] *B. Maurey, Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L_p, Soc. Math. France, Asterisque 11, Paris, 1974.*

[20] *J. Mujica, Complex Analysis in Banach spaces, Dover Publications, 2010.*

[21] *D. Pellegrino, Strongly almost summing holomorphic mappings, J. Math. Anal. Appl.* ***287*** *(2003), no. 1, 244–252.*

[22] *D. Pellegrino, P. Rueda and E. A. Sánchez-Pérez, Surveying the spirit of absolute summability on multilinear operators and homogeneous polynomials, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* ***110*** *(2016), no. 1, 285–302.*

[23] *D. Pellegrino and J. Santos, A general Pietsch Domination Theorem, J. Math. Anal. Appl.* ***375*** *(2011), 371–374.*

[24] A. Pietsch, Absolut p-summierende Abbildungen in normierten Räumen, Studia Math. ***28*** *(1967), 333–353.*

[25] A. Pietsch, Operator ideals. Deutsch. Verlag Wiss., Berlin, 1978; North-Holland, Amsterdam-London-New York-Tokyo, 1980.

[26] R. A. Ryan, Introduction to tensor products of Banach spaces, Series: Springer Monographs in Mathematics, Springer, 2002.

[27] I. Sawashima, Methods of Lipschitz duals, Lecture Notes Ec. Math Sust, 419, Springer Verlag (1975) 247–259.

[28] K. Saadi, Some properties for Lipschitz strongly p-summing operators, J. Math. Anal. Appl. ***423*** *(2015) 1410–1426.*

[29] N. Weaver, Lipschitz Algebras, World Scientific Publishing Co., Singapore, 1999.

Summary and Keywords

ملخص

نتناول في هدا العمل مثالين مهمين من المثاليات للمؤثرات غير الخطية: الأول هو المؤثرات الليبشيتزية p الجمعية، للعالمين: *J. D. Farmer and W. B. Johnson,* المقدمة سنة 2009 . المثالي الثاني يتعلق بالمؤثرات الهولومورفية الموزونة p الجمعية، المدروسة حديثا من طرف *M. G. Cabrera-Padilla, A. Jiménez-Vargas and A. Keten Çopur* ، سنة 2024 في كل من هاتين الفئتين، نثبت نظريتي بيتش للهيمنة ومبرهنة التفكيك.

أخيرا، من خلال تزويد حاصل الضرب العنصري بالنظيم المناسب من نوع *Chevet–Saphar* نحدد الفضاء الثنائي للدوال الهولومورفية الموزونة p الجمعية،

الكلمات المفتاحية:مؤثر p جمعي، مؤثر ذي بعد منتهي ، المؤثرات الليبشيزية p الجمعية , الدوال الهولومورفية الموزونة، مبرهنة الهيمنة ومبرهنة التفكيك لبيتش

Abstract

This work focuses on two examples of nonlinear operator ideals: the ideal of Lipschitz p-summing operators and the ideal of p-summing weighted holomorphic mappings. The First one was introduced by Farmer and Johnson in 2009 , and the second one by M. G. Cabrera-Padilla, A. Jiménez-Vargas and A. Keten Çopurin 2024 . In each category, we establish the corresponding Pietsch Domination and Factorization theorems. Finally, by equipping a suitable tensor product with an appropriate Chevet–Saphar–type norm, we identify the dual of the space of p-summing weighted holomorphic mappings.

keywords: *p-summing operators, Lipschitz p-summing operators, Weighted holomorphic mappings, Pietsch Domination and Factorization theorems. .*

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