

## GENERALIZATIONS OF REID INEQUALITY

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**ABSTRACT.** In this paper, we improve the famous Reid inequality related to linear operators. Some monotony results for positive operators are also established with a different approach from what is known in the existing literature. Lastly, Reid's (and Halmos-Reid's) inequalities are extended to unbounded operators.

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### 1. Introduction

Let  $H$  be a complex Hilbert space, and let  $B(H)$  denote the algebra of all bounded, linear operators on  $H$ . We assume that the reader is familiar with notions and result on  $B(H)$ . We do recall a few definitions and results though:

- (1) Let  $A \in B(H)$ . We say that  $A$  is positive (we then write  $A \geq 0$ ) if
 
$$\langle Ax, x \rangle \geq 0 \quad \text{for all } x \in H.$$
- (2) For every positive operator  $A \in B(H)$ , there is a unique positive  $B \in B(H)$  such that  $B^2 = A$ . We call  $B$  the positive square root of  $A$ .
- (3) The absolute value of  $A \in B(H)$  is defined to be the (unique) positive square root of the positive operator  $A^*A$ . We denote it by  $|A|$ .
- (4) If  $A \geq B \geq 0$ , then  $\sqrt{A} \geq \sqrt{B}$ . (This is a particular case of the so-called Heinz inequality.)
- (5) We say that an operator  $A$  in  $B(H)$  is hyponormal if  $AA^* \leq A^*A$ , or equivalently, if  $\|A^*x\| \leq \|Ax\|$  for all  $x$  in  $H$ .

We also need the following lemma:

**LEMMA 1.1** (See, for example, [10]). *If  $A, B \in B(H)$ , then  $\|Ax\| \leq \|Bx\|$  for all  $x$  in  $H$  if and only if there exists a contraction  $K$  in  $B(H)$  such that  $A = KB$ .*

The inequality of Reid which first appeared in [8] is recalled next:

**THEOREM 1.2.** *Let  $A, K \in B(H)$  be such that  $A$  is positive and  $AK$  is self-adjoint. Then*

$$|\langle AKx, x \rangle| \leq \|K\| \langle Ax, x \rangle \quad \text{for all } x \in H.$$

**Remark 1.** As shown in, for example, [5], Reid's inequality is equivalent to the operator monotony of the positive square root on the set of positive operators.

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Halmos in [2] improved the inequality by replacing  $\|K\|$  by  $r(K)$  where  $r(K)$  is the usual spectral radius. We shall call this the Halmos-Reid inequality. Other generalizations of Theorem 1.2 are known in the literature from which we cite only [4] and [5].

In an earlier version of this paper (see [7]), the corresponding author showed the following:

**THEOREM 1.3.** *Let  $A, K \in B(H)$  be such that  $A$  is positive and  $AK$  is normal. Then*

$$|\langle AKx, x \rangle| \leq \|K\| \langle Ax, x \rangle \quad \text{for all } x \in H.$$

Theorem 1.3 above shows that we can replace the hypothesis that the operator  $AK$  is self-adjoint in Reid’s inequality by the hypothesis that  $AK$  is normal. Is the result true if  $AK$  is assumed to be hyponormal? The following counterexample shows that the result is not true even if we assume that the operator  $AK$  is quasinormal. Hence we cannot replace the hypothesis that the operator  $AK$  is self-adjoint in Reid’s inequality by the hypothesis that  $AK$  is hyponormal, or even subnormal.

**Example 1.4.** Let  $S$  be the shift operator on  $\ell^2$ . Setting  $A = SS^*$ , we see that  $A \geq 0$ . Now, take  $K = S$  (and so  $\|K\| = 1$ ). It is clear that  $AK = SS^*S = S$  is quasinormal. If Reid’s inequality held, then we would have

$$|\langle Sx, x \rangle| \leq \langle SS^*x, x \rangle = \|S^*x\|^2$$

for each  $x \in \ell^2$ . This inequality clearly fails to hold for all  $x$ . Indeed, taking  $x = (2, 1, 0, 0, \dots)$ , we see that

$$|\langle Sx, x \rangle| = 2 \leq \|S^*x\|^2 = 1$$

which is absurd.

The good news is that Reid’s inequality can yet be improved as it holds if  $AK$  is co-hyponormal, that is, if  $(AK)^*$  is hyponormal. The proof, however, relies on the following result:

**LEMMA 1.5** ([4: cf. Theorem 3.2]). *Let  $A \in B(H)$  be hyponormal. Then*

$$|\langle Ax, x \rangle| \leq \langle |A|x, x \rangle.$$

This and some interesting consequences may be found in Section 2.

In Section 3 we treat Reid’s (and Halmos-Reid) inequality for unbounded operators. Fortunately, the latter inequality does hold for co-hyponormal unbounded operators as well. The proof is a little more technical and so it seems appropriate to recall a couple of definitions here (other notions on unbounded operators are assumed, cf. [9]):

**DEFINITION 1.** Let  $T$  and  $S$  be unbounded positive self-adjoint operators. We say that  $S \geq T$  if  $D(S^{\frac{1}{2}}) \subseteq D(T^{\frac{1}{2}})$  and  $\|S^{\frac{1}{2}}x\| \geq \|T^{\frac{1}{2}}x\|$  for all  $x \in D(S^{\frac{1}{2}})$ .

**Remark 2.** Heinz inequality is valid for positive unbounded operators as well. See, e.g. [9].

Now, we recall the definition of an unbounded hyponormal operator.

**DEFINITION 2.** A densely defined operator  $T$  with domain  $D(T)$  is called hyponormal if

$$D(T) \subset D(T^*) \quad \text{and} \quad \|T^*x\| \leq \|Tx\| \quad \text{for all } x \in D(T).$$

## 2. Main results: The bounded case

**THEOREM 2.1.** *Let  $A, K \in B(H)$  be such that  $A$  is positive and  $(AK)^*$  is hyponormal. Then*

$$|\langle AKx, x \rangle| \leq \|K\| \langle Ax, x \rangle \quad \text{for all } x \in H.$$

Proof. It is clear that

$$|(AK)^*|^2 = AKK^*A \leq \|K\|^2 A^2$$

or simply

$$|(AK)^*| \leq \|K\|A$$

after passing to square roots.

Now, for all  $x \in H$

$$|\langle AKx, x \rangle| = |\langle x, (AK)^*x \rangle| = |\overline{\langle (AK)^*x, x \rangle}| = |\langle (AK)^*x, x \rangle|.$$

Since  $(AK)^*$  is hyponormal, Lemma 1.5 combined with  $|(AK)^*| \leq \|K\|A$  give

$$|\langle AKx, x \rangle| = |\langle (AK)^*x, x \rangle| \leq |\langle (AK)^*|x, x \rangle| \leq \|K\|\langle Ax, x \rangle$$

and this marks the end of the proof. □

**THEOREM 2.2.** *Theorem 2.1 can be deduced from the operator monotony of the positive square root on the set of positive operators, and vice versa.*

Proof. Let  $A, B \in B(H)$ .

(1) Assume that  $0 \leq A \leq B$ . Let  $x \in H$ . Since  $A \leq B$ , we easily see that:

$$\|\sqrt{A}x\|^2 \leq \|\sqrt{B}x\|^2.$$

So, by Lemma 1.1, we know that  $\sqrt{A} = K\sqrt{B}$  for some contraction  $K \in B(H)$ . Since  $\sqrt{A}$  is self-adjoint, it follows that  $K\sqrt{B}$  too is self-adjoint (hence co-hyponormal!). As  $\sqrt{B} \geq 0$ , then by Reid's inequality (Theorem 2.1) we obtain:

$$\langle \sqrt{A}x, x \rangle = \langle \sqrt{B}K^*x, x \rangle \leq \langle \sqrt{B}x, x \rangle \quad \text{or} \quad \sqrt{A} \leq \sqrt{B},$$

as required.

(2) The other implication, which uses the fact that the square root is increasing, has already been presented in the proof of Theorem 2.1. □

The idea of the first part of the proof of the preceding result may be used to produce new proofs of other known results on monotony.

**THEOREM 2.3.** *Let  $A, B \in B(H)$ . If  $0 \leq A \leq B$  and if  $A$  is invertible, then  $B$  is invertible and  $B^{-1} \leq A^{-1}$ .*

Proof. As in the proof of Theorem 2.2, we know that  $\sqrt{A} = K\sqrt{B}$  for some contraction  $K \in B(H)$ . Since  $\sqrt{A}$  is invertible (as  $A$  is), it follows that  $I = (\sqrt{A})^{-1}K\sqrt{B}$ , i.e. the self-adjoint  $\sqrt{B}$  is left invertible and so  $\sqrt{B}$  or simply  $B$  is invertible (cf. [1]) and

$$(\sqrt{B})^{-1} = (\sqrt{A})^{-1}K = K^*(\sqrt{A})^{-1}$$

by the self-adjointness of both  $(\sqrt{B})^{-1}$  and  $(\sqrt{A})^{-1}$ .

Let  $x \in H$ . Then (since  $K^*$  too is a contraction)

$$\langle B^{-1}x, x \rangle = \|(\sqrt{B})^{-1}x\|^2 = \|K^*(\sqrt{A})^{-1}x\|^2 \leq \|(\sqrt{A})^{-1}x\|^2 = \langle A^{-1}x, x \rangle,$$

as needed. □

The following modification of Lemma 1.1 makes proofs in case of commutativity very simple.

**LEMMA 2.4.** *Suppose that  $A$  and  $B$  are self-adjoint operators in  $B(H)$ . Then  $BA \geq 0$  and  $\|Ax\| \leq \|Bx\|$  for all  $x$  in  $H$  if and only if there exists a positive contraction  $K$  such that  $A = KB$ .*

Proof. (1) “ $\Leftarrow$ ”: Let  $x \in H$ . Then

$$0 \leq \langle KBx, Bx \rangle = \langle Ax, Bx \rangle = \langle BAx, x \rangle.$$

Hence  $BA \geq 0$ . Also  $\|Ax\| = \|KBx\| \leq \|Bx\|$ , for each  $x$  in  $H$ .

(2) “ $\Rightarrow$ ”: Since  $BA \geq 0$ , it follows that  $BA$  is self-adjoint, i.e.,  $AB = BA$ . As a consequence,  $\ker A$  reduces  $A$  and  $B$ , and the restriction of  $A$  to  $\ker A$  is the zero operator on  $\ker A$ . Hence, we can assume that  $A$  is injective. Therefore, because  $\ker B \subset \ker A = \{0\}$ , we see that  $B^{-1}$  is self-adjoint and densely defined. Set  $K_0 = AB^{-1}$ . Then  $K_0$  is densely defined and

$$\|K_0(Bx)\| = \|AB^{-1}Bx\| = \|Ax\| \leq \|Bx\| \quad \text{for all } x \in H,$$

signifying that  $K_0$  is a contraction with a unique contractive extension  $K$  to the whole  $H$ . Since

$$\langle K_0(Bx), Bx \rangle = \langle Ax, Bx \rangle = \langle BAx, x \rangle \geq 0 \quad \text{for all } x \in H,$$

we see that  $K$  is positive as well. Clearly

$$KBx = K_0(Bx) = Ax \quad \text{for all } x \in H,$$

and this completes the proof. □

**Remark 3.** By scrutinizing the previous proof, we see that we can replace each word “positive” by “self-adjoint” in the statement and in the proof. Hence we also have:

**PROPOSITION 2.5.** *Suppose that  $A$  and  $B$  are self-adjoint operators in  $B(H)$ . Then  $BA$  is self-adjoint and  $\|Ax\| \leq \|Bx\|$  for all  $x$  in  $H$  if and only if there exists a self-adjoint contraction  $K$  such that  $A = KB$ .*

The next result is known. Its proof is a simple application of Lemma 2.4.

**COROLLARY 2.6.** *Let  $A, B \in B(H)$  be positive and commuting. Then*

$$0 \leq A \leq B \implies A^2 \leq B^2.$$

Proof. Since  $AB \geq 0$ , we know by Lemma 2.4 that  $\sqrt{A} = K\sqrt{B}$  for some positive contraction  $K \in B(H)$  and  $K\sqrt{B} = \sqrt{BK}$ . Hence

$$A = K\sqrt{BK}\sqrt{B} = K^2B.$$

So for all  $x \in H$ :

$$\|Ax\|^2 = \|K^2Bx\|^2 \leq \|Bx\|^2$$

or merely

$$\langle A^2x, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \leq \|Bx\|^2 = \langle B^2x, x \rangle,$$

as required. □

**Remark 4.** The previous proof could even be shortened by directly using Reid’s inequality. As is presented, it can be given in courses which do not cover Reid’s inequality.

By invoking the functional calculus of positive operators, we know that we can define  $A^\alpha$  for any (real)  $\alpha > 0$  (we may allow  $\alpha = 0$ ) whenever  $A \geq 0$ . We also know that if  $B$  is also positive and it commutes with  $A$ , then (using the spectral theorem or else)

$$(AB)^\alpha = A^\alpha B^\alpha.$$

As a generalization of Corollary 2.6, we have

**PROPOSITION 2.7.** *Let  $A, B \in B(H)$  be positive and commuting. Then*

$$0 \leq A \leq B \implies A^\alpha \leq B^\alpha \quad \text{for any } \alpha \in (0, \infty).$$

To prove it, we need the following perhaps known result:

**LEMMA 2.8.** *Let  $A \in B(H)$  be such that  $A \geq 0$ . Then*

$$\|A^\alpha\| = \|A\|^\alpha \quad \text{for any } \alpha \in (0, \infty).$$

*Proof.* One of the ways of seeing this is to use the spectral radius theorem and the fact that  $A^\alpha$  is positive. □

Now, we give the proof of Proposition 2.7:

*Proof of Proposition 2.7.* Since  $AB \geq 0$ , we know by Lemma 2.4 that  $\sqrt{A} = K\sqrt{B}$  for some positive contraction  $K \in B(H)$  and  $K\sqrt{B} = \sqrt{B}K$ . Hence

$$A = K\sqrt{B}K\sqrt{B} = K^2B = BK^2.$$

Ergo

$$A^\alpha = (BK^2)^\alpha = B^\alpha K^{2\alpha} \quad (\text{because } K^2B = BK^2).$$

Finally, for any  $x \in H$ , Reid's inequality (and a glance at Lemma 2.8) allows us to write:

$$\langle A^\alpha x, x \rangle = \langle B^\alpha K^{2\alpha} x, x \rangle \leq \|K^{2\alpha}\| \langle B^\alpha x, x \rangle = \|K\|^{2\alpha} \langle B^\alpha x, x \rangle \leq \langle B^\alpha x, x \rangle,$$

as desired. □

**Remark 5.** Proposition 2.7 is known to be true even for non-commuting positive operators  $A$  and  $B$  for  $\alpha$  in the interval  $[0, 1]$  and known not to be true, in general, for non-commuting positive operators  $A$  and  $B$  when  $\alpha > 1$ .

**Remark 6.** Also, we have used operator-theoretic techniques to establish our results about inequalities of commuting bounded self-adjoint operators. It is worth noticing that these results follow also easily in the following manner: The operators generate an abelian  $C^*$ -algebra. This algebra can be realized as algebra of continuous functions, order and norms remain valid (for continuous functions these results are obviously true).

### 3. Main results: The unbounded case

We start with the next practical result (probably known):

**PROPOSITION 3.1.** *Let  $T$  be a closed hyponormal operator. Then*

$$T^*T \geq TT^*.$$

*Proof.* First, since  $T$  is closed, both  $T^*T$  and  $TT^*$  are self-adjoint and positive (cf. [9]). As in the bounded case, write  $|T| = (T^*T)^{\frac{1}{2}}$ . Then it is known that

$$D(|T|) = D(T) \subseteq D(T^*) = D(|T^*|).$$

Finally, for all  $x \in D(T)$ , we have

$$\| |T^*| x \| = \| T^* x \| \leq \| T x \| = \| |T| x \|.$$

Therefore, according to Definition 1, we have  $T^*T \geq TT^*$ , as required. □

**Remark 7.** By Definition 1 above and some trivial observations,  $T^*T \geq TT^*$  clearly implies that  $T$  is hyponormal.

Now, we prove the analogue of Lemma 1.5 for unbounded operators. The proof uses the Generalized Cauchy-Schwarz Inequality for unbounded self-adjoint positive operators.

**THEOREM 3.2.** *Let  $T$  be a closed hyponormal operator. Then*

$$|\langle Tx, x \rangle| \leq \langle |T|x, x \rangle \quad \text{for all } x \in D(T).$$

*Proof.* Let  $T = U|T|$  be the polar decomposition of  $T$  where  $U$  is partial isometry. Remember that (see e.g. [3])

$$|T^*| = U|T|U^*$$

and

$$U^*U|T| = |T|.$$

By Proposition 3.1,  $TT^* \leq T^*T$ . Hence by Heinz inequality (“unbounded” version), we infer that  $|T^*| \leq |T|$ .

Now, to show the required inequality, let  $x \in D(T)$ . Then, we have

$$\begin{aligned} |\langle Tx, x \rangle|^2 &= |\langle U|T|x, x \rangle|^2 = |\langle |T|x, U^*x \rangle|^2 \\ &\leq \langle |T|x, x \rangle \langle |T|U^*x, U^*x \rangle \\ &= \langle |T|x, x \rangle \langle U|T|U^*x, x \rangle = \langle |T|x, x \rangle \langle |T^*|x, x \rangle \\ &\leq \langle |T|x, x \rangle \langle |T|x, x \rangle \quad (\text{as } T \text{ is hyponormal}). \end{aligned}$$

Accordingly,

$$|\langle Tx, x \rangle| \leq \langle |T|x, x \rangle,$$

as required. □

We are ready to give the “unbounded” analogue of Theorem 2.1.

**THEOREM 3.3.** *Let  $K \in B(H)$  and let  $A$  be a positive self-adjoint not necessarily bounded operator such that  $(AK)^*$  is hyponormal. Then*

$$|\langle AKx, x \rangle| \leq \|K\| \langle Ax, x \rangle \quad \text{for all } x \in D(A).$$

The proof of the preceding theorem is based on the following simple lemma.

**LEMMA 3.4.** *Let  $K \in B(H)$  and let  $A$  be a self-adjoint not necessarily bounded operator on  $H$ . If  $AK$  is densely defined, then*

$$|(AK)^*| \leq \|K\| A.$$

*Proof.* First observe that  $AK$  is closed since  $A$  is closed and  $K$  is bounded and observe that

$$D(A) = D(K^*A) \subseteq D((AK)^*) = D(|(AK)^*|).$$

We have

$$\| |(AK)^*| x \|^2 = \| (AK)^* x \|^2 = \| K^*Ax \|^2 \leq \|K\|^2 \|Ax\|^2,$$

for each  $x$  in  $D(A)$ . It follows that

$$|(AK)^*|^2 \leq \|K\|^2 A^2.$$

By the Heinz inequality, it follows that

$$|(AK)^*| \leq \|K\| A \quad \square$$

We are ready to give a proof of Theorem 3.3:

*Proof of Theorem 3.3.* First, since  $(AK)^*$  is hyponormal, we clearly have

$$D(A) \subseteq D((AK)^*) \subset D((AK)^{**}) = D(AK)$$

because  $AK$  is also closed.

Lemma 3.4 then yields

$$|(AK)^*| \leq \|K\| A.$$

Therefore, for all  $x \in D(A)$

$$\begin{aligned} |\langle AKx, x \rangle| &= |\langle x, (AK)^* x \rangle| = \overline{|\langle (AK)^* x, x \rangle|} \\ &= |\langle (AK)^* x, x \rangle| \\ &\leq \langle (AK)^* | x, x \rangle \quad ((AK)^* \text{ is hyponormal}). \\ &\leq \|K\| \langle Ax, x \rangle \end{aligned}$$

and this marks the end of the proof. □

In the end, we give the generalization of Halmos-Reid inequality. The proof, which uses a standard argument (cf. [2] or [6]), is more technical in the unbounded case.

**THEOREM 3.5.** *Let  $K$  be a bounded operator, and let  $A$  be an unbounded self-adjoint positive operator such that  $K^*A \subseteq AK$ , then*

$$|\langle AKx, x \rangle| \leq r(K) \langle Ax, x \rangle \quad \text{for all } x \in D(A),$$

where  $r(K)$  denotes the spectral radius of  $K$ .

*Proof.* Since  $K^*A \subseteq AK$ , we can get by induction that

$$K^{*n}A \subseteq AK^n \text{ or merely } (AK^n)^* \subseteq AK^n$$

for all  $n$ . Hence  $(AK^n)^*$  is hyponormal. We also observe that as  $A^{\frac{1}{2}}A^{\frac{1}{2}} = A$ , then  $D(A) \subset D(A^{\frac{1}{2}})$ .

Now, let us prove the following key result:

$$|\langle AKx, x \rangle| \leq \langle AK^{2n}x, x \rangle^{\frac{1}{2n}} \langle Ax, x \rangle^{\frac{2^n-1}{2^n}}$$

We use a proof by induction. For  $n = 1$ , we have for all  $x \in D(A)$ :

$$\begin{aligned} |\langle AKx, x \rangle| &\leq |\langle A^{\frac{1}{2}}A^{\frac{1}{2}}Kx, x \rangle| \\ &= |\langle A^{\frac{1}{2}}Kx, A^{\frac{1}{2}}x \rangle| \quad \text{because } A^{\frac{1}{2}}Kx \in D(A^{\frac{1}{2}}). \\ &\leq \|A^{\frac{1}{2}}Kx\| \|A^{\frac{1}{2}}x\| \\ &= \langle A^{\frac{1}{2}}Kx, A^{\frac{1}{2}}Kx \rangle^{\frac{1}{2}} \langle Ax, x \rangle^{\frac{1}{2}} \\ &= \langle A^{\frac{1}{2}}A^{\frac{1}{2}}Kx, Kx \rangle^{\frac{1}{2}} \langle Ax, x \rangle^{\frac{1}{2}} \\ &= \langle AKx, Kx \rangle^{\frac{1}{2}} \langle Ax, x \rangle^{\frac{1}{2}} \\ &= \langle K^*AKx, x \rangle^{\frac{1}{2}} \langle Ax, x \rangle^{\frac{1}{2}} \\ &= \langle AK^2x, x \rangle^{\frac{1}{2}} \langle Ax, x \rangle^{\frac{1}{2}} \end{aligned}$$

Using a similar argument we can prove it for  $n + 1$  if it holds for  $n$ .

Therefore, for all  $n$  (and all  $x \in D(A)$ )

$$|\langle AKx, x \rangle| \leq \langle AK^{2n}x, x \rangle^{\frac{1}{2n}} \langle Ax, x \rangle^{\frac{2^n-1}{2^n}}.$$

Since  $(AK^{2n})^*$  is hyponormal, we might apply Theorem 3.3 to get

$$\begin{aligned} |\langle AKx, x \rangle| &\leq \langle AK^{2n}x, x \rangle^{\frac{1}{2n}} \langle Ax, x \rangle^{\frac{2^n-1}{2^n}} \\ &\leq \|K^{2n}\|^{\frac{1}{2n}} \langle Ax, x \rangle^{\frac{1}{2n}} \langle Ax, x \rangle^{\frac{2^n-1}{2^n}} = \|K^{2n}\|^{\frac{1}{2n}} \langle Ax, x \rangle \end{aligned}$$

whichever  $n$ . Passing to the limit finally yields

$$|\langle AKx, x \rangle| \leq r(K) \langle Ax, x \rangle \quad \text{for all } x \in D(A),$$

as needed. □

**COROLLARY 3.6.** *Let  $K$  be a bounded operator, and let  $A$  be an unbounded self-adjoint positive operator such that  $AK$  is self-adjoint, then*

$$|\langle AKx, x \rangle| \leq r(K) \langle Ax, x \rangle \quad \text{for all } x \in D(A).$$

Proof. The proof simply follows from

$$K^*A \subset (AK)^* = AK. \quad \square$$

## 4. Conclusion

The new bounded version of Reid's inequality is equivalent to a bunch of other properties. Indeed, since we have shown that it is equivalent to the fact that the square root is increasing on the set of positive operators, we may call on [5] to list other equivalent conditions.

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